Discrete Linear Orderings and Fraïssé Games

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Abstract

Fraïssé games are used to prove some axiomatizations of the order types $(\omega, <) \quad (\omega, <)$, $(\ 0, \ldots, m \quad 1 \ , <)$, $(\omega \ \ m, <)$, $(\omega \ \ m + q, <)$ and $(\omega^m, <)$.

We consider structures of the signature ((2),∅,0). The language $\mathcal{L}$ that belongs to this signature has two binary relation symbols, $\sqsubset$ and $=$, and does not have function symbols nor individual constants.

We consider formulas $\theta(x_0 \ldots x_{k-1})$ in $\mathcal{L}$, where all the free variables of θ are among $x_0 \ldots x_{k-1}$, that we abbreviate by $\overrightarrow{x}$. For every x we define two mappings L_x and R_x between such formulas in a way that the formulas $L_x(\theta)$ and $R_x(\theta)$ have free variables among $x, x_0 \ldots x_{k-1}$. Sometimes we use a second notation:

$$\theta \ \ ^x(x, \overrightarrow{x}) \equiv L_x(\theta(\overrightarrow{x})), \text{ and } \theta^x \ \ (x, \overrightarrow{x}) \equiv R_x(\theta(\overrightarrow{x})).$$

If $\mathfrak{A} = (A, <)$ is a structure of the given signature, for each element $b \in A$ we define $A^{<b}$ to be the set $\{a \in A \mid a < b\}$, and $\mathfrak{A}^{<b}$ to be the structure $(A^{<b}, <)$ where the relation $<$ is restricted to the set $A^{<b}$. We want to have the following property:

$$\mathfrak{A} \models \theta \ \ ^x[b, a_0 \ldots a_{k-1}] \text{ if and only if } \mathfrak{A}^{<b} \models \theta[a_0 \ldots a_{k-1}],$$

for all $b \in A$ and for all $a_0 \ldots a_{k-1} \in A^{<b}$. And we want the analogous property for the mapping R_x, i.e. if $A^{>b} = \{a \in A \mid b < a\}$ and $\mathfrak{A}^{>b} = (A^{>b}, <)$, then $\mathfrak{A} \models \theta^x \ \ [b, a_0 \ldots a_{k-1}]$ if and only if $\mathfrak{A}_b \models \theta[a_0 \ldots a_{k-1}]$, where $b \in A$ and $a_0 \ldots a_{k-1} \in A^{>b}$.

Definition
We define the mappings L_x and R_x by induction on the formula θ :

- If θ is a basic formula or the negation of a basic formula, then $L(x, \theta) \equiv R(x, \theta) \equiv \theta$.
- If $\theta(\overrightarrow{x})$ is the formula $\theta_1(\overrightarrow{x}) \wedge \theta_2(\overrightarrow{x})$, then

  $L(x, \theta) \equiv L(x, \theta_1) \wedge L(x, \theta_2), \quad R(x, \theta) \equiv R(x, \theta_1) \wedge R(x, \theta_2)$.
  We define in the same way the image of the disjunction: $\theta(\overrightarrow{x})$ is the formula $\theta_1(\overrightarrow{x}) \vee \theta_2(\overrightarrow{x})$.

- If $\theta(\overrightarrow{x})$ is the formula $\exists y\ \varphi(y, \overrightarrow{x})$, then

$$L(x, \theta(\overrightarrow{x})) \equiv \exists y (y \sqsubset x \wedge L(x, \varphi(y, \overrightarrow{x}))),$$
$$R(x, \theta(\overrightarrow{x})) \equiv \exists y (x \sqsubset y \wedge R(x, \varphi(y, \overrightarrow{x}))).$$

- If $\theta(\overrightarrow{x})$ is the formula $\forall y\ \varphi(y, \overrightarrow{x})$, then

$$L(x, \theta(\overrightarrow{x})) \equiv \forall y (y \sqsubset x \rightarrow L(x, \varphi(y, \overrightarrow{x}))),$$
$$R(x, \theta(\overrightarrow{x})) \equiv \forall y (x \sqsubset y \rightarrow R(x, \varphi(y, \overrightarrow{x}))).$$

Remark: The bounding of the quantifiers in the formula does not change its quantifier depth: $QD(L_x(\theta)) = QD(\theta) = QD(R_x(\theta))$.

1 The order type $(\omega, <) \oplus (\omega, <)$

The first goal is to find an axiomatization of the structure $(\mathbb{N}, <) \oplus (\mathbb{N}, <)$.
We define the following sentence:

$$\varphi \equiv \begin{array}{ll} \forall x_0 \big(\neg(x_0 \sqsubset x_0)\big) \wedge & \text{(irreflexibility)} \\ \forall x_0 \forall x_1 \forall x_2 \big(x_0 \sqsubset x_1 \sqsubset x_2 \rightarrow x_0 \sqsubset x_2\big) \wedge & \text{(transitivity)} \\ \forall x_0 \forall x_1 \big(x_0 \sqsubset x_1 \vee x_0 = x_1 \vee x_1 \sqsubset x_0\big) \wedge & \text{(linearity)} \\ \forall x_0 \exists x_1 \big(x_0 \sqsubset x_1 \wedge \neg \exists x_2 (x_0 \sqsubset x_2 \sqsubset x_1)\big) \wedge & \text{(immediate successor)} \\ \exists x_0 \exists x_1 \left(\begin{array}{l} x_0 \sqsubset x_1 \wedge \neg \exists x_2 (x_2 \sqsubset x_0) \wedge \\ \quad \forall x_2 \left(\begin{array}{l} \neg(x_0 = x_2) \wedge \neg(x_1 = x_2) \longleftrightarrow \\ \exists x_3 \big(x_3 \sqsubset x_2 \wedge \neg \exists x_4 (x_3 \sqsubset x_4 \sqsubset x_2)\big) \end{array} \right) \end{array} \right) & \end{array}$$

Definition: Let $\mathfrak{A} = (A, <)$ be a structure with signature ((2),$\emptyset$,0). We define *distance* between two elements of this structure as follows:
for all $a, b \in A$, such that $a < b$, $d(a, b) = 1+$ the number of elements between a and b, i.e.

$$\begin{cases} d(a,b) = 1 + \|\{c \in A \mid a < c < b\}\| & \text{, for } a < b \\ d(a,a) = 0 \end{cases}$$

Notation:

$$\psi(x_0, x_1) \equiv \left(\begin{array}{l} x_0 \sqsubset x_1 \wedge \neg \exists x_2 (x_2 \sqsubset x_0) \wedge \\ \quad \forall x_2 \left(\begin{array}{l} \neg(x_0 = x_2) \wedge \neg(x_1 = x_2) \longleftrightarrow \\ \exists x_3 \big(x_3 \sqsubset x_2 \wedge \neg \exists x_4 (x_3 \sqsubset x_4 \sqsubset x_2)\big) \end{array} \right) \end{array} \right).$$

If $\mathfrak{A} \models \varphi$ then $\mathfrak{A} \models \exists x_0 \exists x_1 \psi(x_0, x_1)$ and therefore there exist exactly two elements (we will denote them by $0_0^{\mathfrak{A}}$ and $0_1^{\mathfrak{A}}$) such that $\mathfrak{A} \models \psi[0_0^{\mathfrak{A}}, 0_1^{\mathfrak{A}}]$.

Lemma

Let $\mathfrak{A} = (A, <_A)$ and $\mathfrak{B} = (B, <_B)$ be models of φ.
Let $f \in LI_0(\mathfrak{A}, \mathfrak{B})$ be a local isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Let us denote $a^{(0)} :=$

$$\begin{cases} \min\{a_i \,|\, a_i \in Dom(f)\} & \text{, if } f \neq \emptyset \\ 0_0^{\mathfrak{A}} & \text{, otherwise} \end{cases}$$

and

$$a^{(1)} := \begin{cases} \min\{a_i \,|\, a_i \in Dom(f) \;\&\; 0_1^{\mathfrak{A}} \leq_A a_i\} & \text{, if } \exists a \in Dom(f)(0_1^{\mathfrak{A}} \leq_A a) \\ 0_1^{\mathfrak{A}} & \text{, otherwise} \end{cases}$$

Then for every natural number $p \geq 2$,

$f \in LI_p(\mathfrak{A}, \mathfrak{B})$ if and only if

$$\left(\begin{array}{l} \forall n \leq 2^{(p-1)} \; \forall a_i, a_j \in Dom(f) \\ \qquad \big(d(a_i, a_j) = n \Leftrightarrow d(f(a_i), f(a_j)) = n\big) \\ \qquad\qquad \& \\ \forall n \leq 2^{(p-1)} + p - 2 \\ \qquad \Big(a^{(0)} \in Dom(f) \Rightarrow \; \big(d(0_0^{\mathfrak{A}}, a^{(0)}) = n \Leftrightarrow d(0_0^{\mathfrak{B}}, f(a^{(0)})) = n\big)\Big) \\ \qquad\qquad \& \\ \forall n \leq 2^{(p-1)} + p - 4 \\ \qquad \Big(a^{(1)} \in Dom(f) \Rightarrow \; \big(d(0_1^{\mathfrak{A}}, a^{(1)}) = n \Leftrightarrow d(0_1^{\mathfrak{B}}, f(a^{(1)})) = n\big)\Big) \end{array} \right)$$

The three conjuncts of the right side formalize the following three properies:

- "to preserve distances less than or equal to2^{p-1}",
- "to preserve the distances to the first element 0_0, which are less than or equal to $2^{(p-1)} + p - 2$" and
- "to preserve the distances to the second zero 0_1, which are less than or equal to $2^{(p-1)} + p - 4$".

We will need the direction ($\Leftarrow$) only.

Proof: Induction on p.

$p = 2$.

($\Rightarrow$) $f \in LI_2(\mathfrak{A}, \mathfrak{B})$

- It is easy to verify that $d(a_i, a_j) = d(f(a_i), f(a_j))$ when one of these distances is ≤ 2.
- If $a^{(0)} \in Dom(f)$ and one of $d(0_0^{\mathfrak{A}}, a^{(0)})$ and $d(0_0^{\mathfrak{B}}, f(a^{(0)}))$ is ≤ 2, it is easy to verify that they are equal.
- If $a^{(1)} \in Dom(f)$, it is easy to verify that $a^{(1)} = 0_1^{\mathfrak{A}}$ if and only if $f(a^{(1)}) = 0_1^{\mathfrak{B}}$:
  Assume for example that $a^{(1)} = 0_1^{\mathfrak{A}}$ and $d(0_1^{\mathfrak{B}}, f(a^{(1)}) \neq 0$. We have two cases:
  If $0_1^{\mathfrak{B}} < f(0_1^{\mathfrak{A}})$, then for $b \in B$ - the predecessor of $f(0_1^{\mathfrak{A}})$, there exists $a \in A$, such that $f \cup \{(a, b)\} \in LI_1(\mathfrak{A}, \mathfrak{B})$ and therefore $a < 0_1^{\mathfrak{A}}$. Then

for $S_a \in A$ - the successor of a, that is between a and $0_1^{\mathfrak{A}}$, there exists $c \in B$, such that $f \cup \{(a,b)\} \cup \{(S_a, c)\} \in LI_0(\mathfrak{A}, \mathfrak{B})$ and therefore c is between b and its successor $f(0_1^{\mathfrak{A}})$, which is a contradiction.

In the case $f(0_1^{\mathfrak{A}}) < 0_1^{\mathfrak{B}}$ we take the predecessor of $f(0_1^{\mathfrak{A}})$.

In the other direction the proof is similar.

($\Leftarrow$) The local isomorphism f preserves the distances for $p = 2$. We can prove $f \in LI_2(\mathfrak{A}, \mathfrak{B})$. using the fact that $f \in LI_1(\mathfrak{A}, \mathfrak{B})$ if and only if both $\forall n \leq 1$ $(d(a_i, a_j) = n \Leftrightarrow d(f(a_i), f(a_j)) = n)$ and $f(a) = 0_0^{\mathfrak{B}} \Leftrightarrow a = 0_0^{\mathfrak{A}}$, where $a_i, a_j, a \in Dom(f)$.

The interesting case is when we choose $a \in A$, such that $0_0^{\mathfrak{A}} < a < a^{(0)}$ and $d(a, a^{(0)}) > 1$. Then we have to find $b \in B$, such that $0_0^{\mathfrak{B}} < b$ and $d(b, f(a^{(0)}) > 1$. This is possible, because otherwise $d(0_0^{\mathfrak{B}}, f(a^{(0)})) \leq 2$, which is a contradiction, since $d(0_0^{\mathfrak{A}}, a^{(0)}) > 2$.

IH - assume it is true for some $p \geq 2$. We have to prove it for $p+1$.

($\Rightarrow$) Let $f \in LI_{p+1}(\mathfrak{A}, \mathfrak{B})$.

1. Suppose $d(a_i, a_j) = n \leq 2^p$ for some $a_i, a_j \in Dom(f)$ and $n > 0$, the case $n = 0$ is easy.

   Take an element c of A, such that $d(a_i, c) = \left[\frac{n}{2}\right]$. Then $d(c, a_j) = \left[\frac{n-1}{2}\right] + 1$. There exists $e \in B$, such that $f \cup \{(c, e)\} \in LI_p(\mathfrak{A}, \mathfrak{B})$. Therefore $f(a_i) \leq e \leq f(a_j)$. Since $\left[\frac{n}{2}\right] \leq 2^{(p-1)}$ and $\left[\frac{n-1}{2}\right] + 1 \leq 2^{(p-1)}+1$, from the IH it follows that $d(f(a_i), e) = d(a_i, c)$ and $d(e, f(a_j)) = d(c, a_j)$ and therefore $d(f(a_i), f(a_j)) = d(f(a_i), e) + d(e, f(a_j)) = n$. In the other direction the proof is similar.

2. Suppose $a^{(0)} \in Dom(f)$ and $d(0_0^{\mathfrak{A}}, a^{(0)}) \leq 2^{(p+1)-1} + (p+1) - 2 = 2^p + p - 1$.

   We have two subcases:

   - $d(0_0^{\mathfrak{A}}, a^{(0)}) < 2^p + p - 1$.
     Now we can choose $c \in A$, such that $d(c, a^{(0)}) \leq 2^{(p-1)}$ and $d(0_0^{\mathfrak{A}}, c) \leq 2^{(p-1)} + p - 2$. There exists $e \in B$, such that $f \cup \{(c, e)\} \in LI_p(\mathfrak{A}, \mathfrak{B})$. From IH it follows that $d(0_0^{\mathfrak{A}}, c) = d(0_0^{\mathfrak{B}}, e)$ and $d(c, a^{(0)}) = d(e, f(a^{(0)}))$ and therefore $d(0_0^{\mathfrak{A}}, a^{(0)}) = d(0_0^{\mathfrak{B}}, f(a^{(0)}))$.
   - $d(0_0^{\mathfrak{A}}, a^{(0)}) = 2^p + p - 1$.
     Assume that $d(0_0^{\mathfrak{A}}, a^{(0)}) \neq d(0_0^{\mathfrak{B}}, f(a^{(0)}))$. Then $d(0_0^{\mathfrak{B}}, f(a^{(0)})) > 2^p + p - 1$. Now we can choose $e \in B$, such that $d(0_0^{\mathfrak{B}}, e) > 2^{(p-1)} + p - 2$ and $d(e, f(a^{(0)})) > 2^{(p-1)}$. But there exists $c \in A$, such that $f \cup \{(c, e)\} \in LI_p(\mathfrak{A}, \mathfrak{B})$ and from IH it follows that $d(0_0^{\mathfrak{A}}, c) > 2^{(p-1)} + p - 2$ and $d(c, a^{(0)}) > 2^{(p-1)}$. Then $d(0_0^{\mathfrak{A}}, (a^{(0)}) > 2^p + p - 1$, which is a contradiction.

3. Suppose $a^{(1)} \in Dom(f)$ and $d(0_1^{\mathfrak{A}}, a^{(1)}) \leq 2^{(p+1)-1} + (p+1) - 4 = 2^p + p - 3$. The proof is similar.

($\Leftarrow$) Suppose that for the local isomorphism f the distance conditions for $p+1$ hold. From IH it follows that $f \in LI_p(\mathfrak{A}, \mathfrak{B})$. We have to prove that $f \in LI_{p+1}(\mathfrak{A}, \mathfrak{B})$. Take for example $a \in A$. We have the following cases:

- $a \in Dom(f)$.
  Therefore there exists $b \in B$, $b = f(a)$, such that $f \cup \{(a,b)\} = f \in LI_p(\mathfrak{A}, \mathfrak{B})$.
- $a' < a < a''$, for some $a', a'' \in Dom(f)$, s.t. between them there are no elements from $Dom(f)$.
  - $d(a', a) \leq 2^{(p-1)}$ and $d(a, a') \leq 2^{(p-1)}$.
    Then $d(a', a'') \leq 2^p$ and we can choose $b \in B$, such that $d(f(a'), b) = d(a', a)$ and $d(b, f(a'')) = d(a, a'')$. From IH it follows that $f \in LI_{p+1}(\mathfrak{A}, \mathfrak{B})$.
  - $d(a', a) \leq 2^{(p-1)}$ and $d(a, a'') > 2^{(p-1)}$.
    We can choose $b \in B$, such that $d(f(a'), b) = d(a', a)$. If $d(b, f(a'')) \leq 2^{(p-1)}$, then $d(f(a'), f(a'')) \leq 2^p$, which is a contradiction.
  - $d(a', a) > 2^{(p-1)}$ and $d(a, a'') \leq 2^{(p-1)}$.
    This case is similar to the previous, except that it is possible $a' \leq 0_1^{\mathfrak{A}} \leq a$. Then $d(0_1^{\mathfrak{B}}) \geq 2^{(p-1)}$ and we can choose $b \in B$, such that $d(b, f(a'')) = d(a, a'')$. If $d(0_1^{\mathfrak{A}}, a) \leq 2^{(p-1)} + p - 4$, then $d(0_1^{\mathfrak{A}}, a'') \leq 2^p + p - 4 < 2^p + p - 3$ and therefore (IH) $d(0_1^{\mathfrak{A}}, a'') = d(0_1^{\mathfrak{B}}, f(a''))$, then $d(0_1^{\mathfrak{A}}, a) = d(0_1^{\mathfrak{A}}, b)$ and $f \in LI_{p+1}(\mathfrak{A}, \mathfrak{B})$.
  - $d(a', a) > 2^{(p-1)}$ and $d(a, a'') > 2^{(p-1)}$. Analogous.
- $0_0^{\mathfrak{A}} < a < a^{(0)}$. Analogous.
- $\max\{a_i \mid a_i \in Dom(f)\} < a$. This case is easy.

Notation: By $(\omega \cdot 2, <')$ we denote the structure $(\mathbb{N}, <) \oplus (\mathbb{N}, <)$ with domain $\omega \cdot 2 = \mathbb{N} \cup \{x' \mid x \in \mathbb{N}\}$ and the following relation: for all $x, y \in \mathbb{N}$, such that $x < y$,

$$x <' y \quad \text{and} \quad x' <' y' \quad \text{and} \quad x <' x'.$$

Theorem: $(\mathbb{N}, <) \oplus (\mathbb{N}, <)$ is a prime model of φ.

Proof: Let $\mathfrak{A} = (A, <_A)$ be a model of φ. Define a function f from $\omega \cdot 2$ to A, as follows: $f(0) := 0_0^{\mathfrak{A}}$, $f(0') := 0_1^{\mathfrak{A}}$ and for each $x > 0$ in $\mathbb{N}$, $f(x+1) :=$ the $<_A$-successor of $f(x)$ and $f((x+1)') :=$ the $<_A$-successor of $f(x')$.
From the Lemma it follows that every finite subset of f belongs to every $LI_p((\omega \cdot 2, <'), \mathfrak{A})$, and therefore f is an elementary embedding from $(N, <')$ into $\mathfrak{A}$.

□

From the last theorem it follows that for every model $\mathfrak{A}$ of φ, for every sentence ψ, if $(\mathbb{N},<)\oplus(\mathbb{N},<) \models \psi$ then $\mathfrak{A} \models \psi$, and therefore φ is an axiom for the structure $(\mathbb{N},<) \oplus (\mathbb{N},<)$.

2 Finite structures $\mathfrak{A}_m = (\{0, \ldots, m-1\}, <)$

Consider structures $\mathfrak{A}_m = (A_m, <)$, where $A_m = \{0, \ldots, m-1\}$.
We want to prove that for all $m, n \geq 1$ and for all $p \geq 0$,

$$\mathfrak{A}_m \equiv_p \mathfrak{A}_n \text{ if and only if } \Big((m = n) \text{ or } (m \neq n \text{ and } 2^p - 2 < \min(m,n))\Big).$$

Proof:
($\Rightarrow$) We want to prove that if $m < n$ and $2^p - 2 \geq m$, then $\mathfrak{A}_m \not\equiv_p \mathfrak{A}_n$, finding sentence φ_m, with $QD(\varphi_m) \leq p$ and such that $\mathfrak{A}_n \models \varphi_m$, but $\mathfrak{A}_m \not\models \varphi_m$.

We can easily define sentences $DiffEl_m$, such that $\mathfrak{A}_m \models \neg DiffEl_m$ and $\mathfrak{A}_n \models DiffEl_m$ for all $n > m$, ($DiffEl_m$ says 'there exist at least $m+1$ different elements in the structure'):

$DiffEl_0 \equiv \exists x_0(x_0 = x_0)$
$DiffEl_{m+1} \equiv \exists x_0 \ldots \exists x_{m+1}(x_0 \sqsubset x_1 \wedge x_1 \sqsubset x_2 \wedge \ldots \wedge x_m \sqsubset x_{m+1})$,

but $QD(DiffEl_m) = m + 1$. It suffices to find sentences φ_m, such that:
(a) $QD(\varphi_m) = \mu p[2^p - 2 \geq m]$; and (b) $\mathfrak{A}_n \models \varphi_m \leftrightarrow DiffEl_m$, for all n.

Now define φ_m by induction on m as follows:
$\varphi_0 \equiv \psi_0 \equiv \exists x_0(x_0 = x_0)$ and $\varphi_1 \equiv \psi_1 \equiv \exists x_0 \exists x_1(x_0 \sqsubset x_1)$, and for $m \geq 0$,

$$\varphi_{m+2} \equiv \exists x\big(\varphi^{x}_{[\frac{m}{2}]}(x) \wedge \varphi^{x}_{[\frac{m+1}{2}]}(x)\big), \text{ i.e.}$$

$\varphi_{2k+2} \equiv \exists x(\varphi_k{}^x(x) \wedge \varphi^x_k\ (x))$
$\varphi_{2k+3} \equiv \exists x(\varphi_k{}^x(x) \wedge \varphi^x_{k+1}(x))$

It is easy to check (b), i.e. $\mathfrak{A}_n \models \varphi_m \leftrightarrow DiffEl_m$, for all n.
We prove (a), i.e. $QD(\varphi_m) = \mu p[2^p - 2 \geq m]$, by induction on m :

- for $m = 0$ and $m = 1$,
  $QD(\varphi_0) = 1 = \mu p[2^p - 2 \geq 0]$ and $QD(\varphi_1) = 2 = \mu p[2^p - 2 \geq 1]$.
- IH for smaller than $m \geq 2$.
  Remark:
  $QD(\varphi_m) \leq QD(\varphi_{m+1})$.

  1. $m = 2k + 2$. Let $p := QD(\varphi_{2k+2})$ and $q := QD(\varphi_k)$. Then $p = q + 1$. From IH it follows that $2^q - 2 \geq k$ and therefore $2^q \geq k + 2$, then $2^p = 2^{q+1} \geq 2k + 2 + 2 = m + 2$, i.e. $2^p \geq m + 2$.
     It remains to prove that p is the least, i.e. $p = \mu t[2^t - 2 \geq m]$. Assume there is smaller p' s.t. $2^{p'} \geq m + 2$. Then $p' \leq q$ and therefore $2^q \geq m + 2 = 2k + 4$, then $2^{q-1} - 2 \geq k$, but from IH (for $k < m$) q is the least such that $2^q - 2 \geq k$, contradiction.
  2. $m = 2k + 3$. Let $p := QD(\varphi_{2k+3})$ and $q := QD(\varphi_{k+1})$. Then $p = q + 1$. From IH it follows that $2^q - 2 \geq k + 1$, i.e $2^q \geq k + 3$, then $2^p = 2^{q+1} \geq 2k + 6 = m + 3$, then $2^p \geq m + 2$. Assume there is smaller p' s.t. $2^{p'} \geq m + 2$. Again $p' \leq q$ and therefore $2^q \geq m + 2 = 2k + 5$, then $2^q \geq 2k + 6$, then $2^{q-1} \geq k + 3$, i.e. $2^{q-1} - 2 \geq k + 1$, but from IH (for

$k+1 < m$) it follows that q is the least with this property, which is a contradiction.

Therefore if $m \neq n$ and $2^p - 2 \geq \min(m,n)$ then $\mathfrak{A}_m \not\equiv_p \mathfrak{A}_n$.

($\Leftarrow$) We want to prove that if $m < n$ and $2^p \leq 1 + m$, then $\mathfrak{A}_m \equiv_p \mathfrak{A}_n$, which is equivalent to $\emptyset \in LI_p(\mathfrak{A}_m, \mathfrak{A}_n)$. By induction on $m \geq 1$.

- $m = 1$.
  Since $2^p \leq 2$, $0 \leq p \leq 1$ and then $\mathfrak{A}_1 \equiv_p \mathfrak{A}_n$, for $n > 1$.

- $m > 1$. IH for smaller than m.
  We have to prove that:

  - for all $a \in A_m$ there is $b \in A_n$, such that $\{(a,b)\} \in LI_{p-1}(\mathfrak{A}_m, \mathfrak{A}_n)$;

  - for all $b \in A_n$ there is $a \in A_m$, such that $\{(a,b)\} \in LI_{p-1}(\mathfrak{A}_m, \mathfrak{A}_n)$.

  For the first we consider different cases for a :

  1. for $0 \leq a \leq \dfrac{m-1}{2}$, take $b = a$.
     $\{(a,a)\} \in LI_{p-1}(\mathfrak{A}_m, \mathfrak{A}_n)$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{A}_{m-a-1}, \mathfrak{A}_{n-a-1})$, which follows from $2^{p-1} \leq m-a$, (from IH for $m-a-1 < m$).
     Assume $2^{p-1} > m-a$.
     Since $2^p \leq m+1$, $2^{p-1} \leq \dfrac{m+1}{2} = \dfrac{m}{2} + \dfrac{1}{2}$. Then $\dfrac{m+1}{a} > m-a$.
     Therefore $m+1 > 2m-2a$, then $2a > m-1$, then $a > \dfrac{m-1}{2}$, which is a contradiction.
  2. for $\dfrac{m-1}{2} \leq a \leq m-1$, take $b = a+n-m$,
     i.e. such that the distances $d(a, m-1) = d(b, n-1)$, $b \in A_n$, since $0 \leq a+n-m \leq n-1$. Again $\{(a,b)\} \in LI_{p-1}(\mathfrak{A}_m, \mathfrak{A}_n)$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{A}_a, \mathfrak{A}_b)$, which follows from $2^{p-1} \leq a+1$, (from IH for $a < m$).
     Assume $2^{p-1} > a+1$, then $2^p > 2a+2$. Since $a \geq \dfrac{m-1}{2}$, $2a \geq m-1$. Therefore $2^p > m-1+2 = m+1$, which is a contradiction.

  In order to prove the second, we consider cases for b :

  1. for $0 \leq b \leq \dfrac{m-1}{2}$, take $a = b$.
     The proof is the same as in the case for a.

2. for $n-1-\frac{m-1}{2} \leq b \leq n-1$, take $a=b+m-n$,
i.e. such that $d(a,m-1)=d(b,n-1)$. Then $\frac{m-1}{2} \leq a \leq m-1$ and the proof is the same as in the case for a.

3. for $\frac{m-1}{2} < b < n-1-\frac{m-1}{2}$, take $a=\left[\frac{m-1}{2}\right] \in A_m$. Then
$m-2 \leq 2a \leq m-1$, $a<b$ and $m-a-1 \leq n-b-1$,
$m \leq 2b \leq 2n-m-1$.
Therefore $\{(a,b)\} \in LI_{p-1}(\mathfrak{A}_m, \mathfrak{A}_n)$ if and only if
$$\emptyset \in LI_{p-1}(\mathfrak{A}_a, \mathfrak{A}_b) \text{ and } \emptyset \in LI_{p-1}(\mathfrak{A}_{m-a-1}, \mathfrak{A}_{n-b-1}),$$
which follows from $\left(2^{p-1} \leq a+1 \text{ and } 2^{p-1} \leq m-a\right)$,
since IH for $a+1<m$ and IH for $m-a-1<m$.
Assume $2^{p-1} > a+1$.
Then $2^p > 2a+2 >= m$. But $2^p \leq m+1$, then $m+1 \geq 2^p > m$.
Then $2^p = m+1$ and $\left[\frac{m-1}{2}\right] = \frac{m-1}{2}$, then $2a=m-1$, therefore $2^p > 2a+2 = m+1$, which is a contradiction.
Assume $2^{p-1} > m-1$.
Then $2^p > 2m-2a \geq 2m-m+1 = m+1$, i.e. $2^p > m+1$, but $2^p \leq m+1$, which is a contradiction.
Therefore $\left(2^{p-1} \leq a+1 \text{ and } 2^{p-1} \leq m-a\right)$.

□

3 The order types $\mathfrak{B}_m = (\omega \cdot m, <)$

Consider structures $\mathfrak{B}_m = (B_m, <)$ with $m \geq 1$ and $B_m = \omega \cdot m = \{k_i \mid k \in \mathbb{N} \,\&\, 0 \leq i \leq m-1\}$ and $k_i < l_j \Leftrightarrow (i < j \text{ or } i = j \,\&\, k < l)$.
We want to prove that for all $m, n \geq 1$ and for all $p \geq 0$,

$$\mathfrak{B}_m \equiv_p \mathfrak{B}_n \text{ if and only if } \Big((m = n) \text{ or } (m \neq n \text{ and } 2^{p-2} \leq \min(m, n)\,)\Big).$$

Proof:
$(\Rightarrow)$ We want to prove that if $m < n$ and $2^{p-2} > m$, then $\mathfrak{B}_m \not\equiv_p \mathfrak{B}_n$, finding sentence ψ_m, with $QD(\psi_m) \leq p$ and such that $\mathfrak{B}_n \models \psi_m$, but $\mathfrak{B}_m \not\models \varphi_m$. Define a formula $Wip(x)$ with $QD(Wip) = 2$, which says that the element x does not have immediate predecessor, but has a predecessor, as follows:

$$Wip(x) \equiv \exists y(y \sqsubset x) \wedge \forall y(y \sqsubset x \rightarrow \exists z(y \sqsubset z \wedge z \sqsubset x)).$$

Again we can define sentences $DiffWipEl_m$, for $m \geq 1$, such that
$\mathfrak{B}_m \models \neg DiffWipEl_m$ and $\mathfrak{B}_n \models DiffWipEl_m$ for all $n > m$, ($DiffWipEl_m$ says "there exist at least m different Wip-elements"):

$$DiffWipEl_m \equiv \exists x_0 \ldots \exists x_{m-1}(Wip(x_0) \wedge \ldots \wedge Wip(x_{m-1}) \wedge x_0 \sqsubset \ldots \sqsubset x_{m-1}).$$

$QD(DiffWipEl_m) = m+2$. It suffices to find sentences ψ_m, such that (a) $QD(\psi_m) = \mu p[2^{p-2} - 1 \geq m]$ and (b) $\mathfrak{B}_n \models \psi_m \leftrightarrow DiffWipEl_m$, for all n.

We define ψ_m by induction on $m \geq 1$,

$$\left|\begin{array}{ll} \psi_1 \equiv \exists x_0 Wip(x_0), & QD(\psi_1) = 3 \\ \psi_2 \equiv \exists x_0 \exists x_1(x_0 \sqsubset x_1 \wedge Wip(x_0) \wedge Wip(x_1)), & QD(\psi_2) = 4. \end{array}\right.$$

for $m > 1$, $\varphi_{m+1} \equiv \exists\big(Wip(x) \wedge \psi_{[\frac{m}{2}]}^{\,x} \wedge \psi_{[\frac{m+1}{2}]}^{\,x}\big)$, i.e.

$$\left|\begin{array}{l} \psi_{2k+1} \equiv \exists x(Wip(x) \wedge {\psi_k}^{x}(x) \wedge \psi_k^{x}\;\,(x)) \\ \psi_{2k+2} \equiv \exists x(Wip(x) \wedge {\psi_k}^{x}(x) \wedge \psi_{k+1}^{x}(x)) \end{array}\right. \quad \text{for } k \geq 1.$$

It is easy to check (b), i.e. $\mathfrak{B}_n \models \psi_m \leftrightarrow DiffWipEl_m$, for all n.
We prove (a), i.e. $QD(\psi_m) = \mu p[2^{p-2} - 1 \geq m]$, by induction on m :

- $m = 1$
  $\mu p[2^{p-2} - 1 \geq 1] = \mu p[2^{p-2} \geq 2] = \mu p[p - 2 \geq 1] = 3 = QD(\psi_1)$.
- $m = 2$
  $\mu p[2^{p-2} - 1 \geq 2] = \mu p[2^{p-2} \geq 3] = \mu p[p - 2 \geq 2] = 4 = QD(\psi_2)$.
- $m > 2$. IH for smaller than m.
  Since the function $f(p) = \mu p[2^{p-2} \geq m]$ is monotone, we have:
  1. $m = 2k + 1$, for $k \geq 1$.
     $QD(\psi_{2k+1}) = 1 + QD(\psi_k) \stackrel{IH}{=} 1 + \mu q[2^{q-2} - 1 \geq k] = 1 + \mu q[2^{q-2} \geq k + 1] = \mu p[2^{p-3} \geq k + 1] = \mu p[2^{p-2} \geq 2k + 2] = \mu p[2^{p-2} > 2k + 1] = \mu p[2^{p-2} \geq 2k + 1]$.

2. $m = 2k + 2$, for $k \geq 1$.
$QD(\psi_{2k+2}) = 1 + QD(\psi_{k+1}) \stackrel{IH}{=} 1 + \mu q[2^{q-2} - 1 \geq k+1] = 1 + \mu q[2^{q-2} \geq k + 2] = \mu p[2^{p-3} \geq k + 2] = \mu p[2^{p-2} \geq 2k + 4] = \mu p[2^{p-2} > 2k + 3] = \mu p[2^{p-2} \geq 2k + 3] = \mu p[2^{p-2} \geq m + 1]$.

Therefore if $n \neq m$ and $2^{p-2} > m$, then $\mathfrak{B}_m \not\equiv_p \mathfrak{B}_n$.

($\Leftarrow$) We want to prove that if $m < n$ and $2^{p-2} \leq m$, then $\mathfrak{B}_m \equiv_p \mathfrak{B}_n$, which is equivalent to $\emptyset \in LI_p(\mathfrak{B}_m, \mathfrak{B}_n)$. By induction on $m \geq 1$.

- $m = 1$.
Since $2^{p-2} \leq 1$, $0 \leq p \leq 2$ and then $\mathfrak{B}_1 \equiv_p \mathfrak{B}_n$, for $n > 1$.
- $m > 1$. IH for smaller than m.
Let $2^{p-2} \leq m$. We have to prove that:
  - for all $a \in \omega \cdot m$ there is $b \in \omega \cdot n$, such that $\{(a, b)\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$;
  - for all $b \in \omega \cdot m$ there is $a \in \omega \cdot n$, such that $\{(a, b)\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$.

  First we prove that for the wip-elements, $0_i^{\mathfrak{B}_m}$ and $0_j^{\mathfrak{B}_n}$, with $0 < i \leq m - 1$ and $0 < j \leq n - 1$, i.e. those elements for which $\mathfrak{B}_m \models Wip[0_i^{\mathfrak{B}_m}]$. Using that, the winning strategy for the second player in p moves, for the other elements of $\omega \cdot m$ and $\omega \cdot n$ can be expressed, since:
  $\{(0_0^{\mathfrak{B}_m}, 0_0^{\mathfrak{B}_n})\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$;
  And for $a \in \mathfrak{B}_m$, such that $0_i^{\mathfrak{B}_m} \leq a < 0_{i+1}^{\mathfrak{B}_m}$ and $b \in \mathfrak{B}_n$, such that $0_j^{\mathfrak{B}_n} \leq a < 0_{j+1}^{\mathfrak{B}_n}$ and $d(0_i^{\mathfrak{B}_m}, a) = d(0_j^{\mathfrak{B}_n}, b)$,
  $\{(a, b)\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$ if and only if $\{(0_i^{\mathfrak{B}_m}, 0_j^{\mathfrak{B}_n})\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$.

  So first we consider different cases for $a = 0_i^{\mathfrak{B}_m} \in \mathfrak{B}_m$, for which we take $b = 0_j^{\mathfrak{B}_n} \in \mathfrak{B}_n$, as follows:

  1. For $1 \leq i \leq \left[\dfrac{m-1}{2}\right]$, take $j = i$. We have $1 \leq 2i \leq m - 1$ and $b = 0_i^{\mathfrak{B}_n}$.
  Therefore $\{(a, b)\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{B}_{m-i}, \mathfrak{B}_{n-i})$, which follows from $2^{p-3} \leq m - i$, since IH for $m - i < m$.
  Assume $2^{p-3} > m - i$, i.e. $m - i + 1 \leq 2^{p-3} \leq \dfrac{m}{2}$, then $2m - 2i + 2 \leq m$, then $m + 2 \leq 2i \leq m - 1$, which is a contradiction.
  2. For $\left[\dfrac{m-1}{2}\right] < i \leq m - 1$, take $j = n - m + i$.
  We have $m \leq 2i$ and $b = 0_{n-m+i}^{\mathfrak{B}_n}$. Therefore $\{(a, b)\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{B}_i, \mathfrak{B}_j)$, which follows from $2^{p-3} \leq i$, since IH for $i < m$.
  Assume $2^{p-3} > i$, i.e. $i + 1 \leq 2^{p-3}$, but $2^{p-2} \leq m$. Then $2i + 2 \leq 2^{p-2} \leq m$, i.e. $m + 2 \leq m$, contradiction.

Now consider cases for $b = 0_j^{\mathfrak{B}_n} \in \mathfrak{B}_n$, for which we take $a = 0_i^{\mathfrak{B}_m} \in \mathfrak{B}_m$, as follows (the proof for the first two cases is the same):

1. For $1 \leq j \leq \left[\dfrac{m-1}{2}\right]$, take $i = j$.
2. For $\left[\dfrac{m-1}{2}\right] + n - m < j \leq n - 1$, take $i = j - n + m$.
3. For $\left[\dfrac{m-1}{2}\right] + 1 \leq j \leq n - m + \left[\dfrac{m-1}{2}\right]$, take $i = \left[\dfrac{m-1}{2}\right] + 1$.
   Then we have $m + 1 \leq 2j \leq 2n - m - 2$ and $m \leq 2i \leq m + 1$.
   $\{(a, b)\} \in LI_{p-1}(\mathfrak{B}_m, \mathfrak{B}_n)$ if and only if
   $$\Big(\emptyset \in LI_{p-1}(\mathfrak{B}_i, \mathfrak{B}_j) \text{ and } \emptyset \in LI_{p-1}(\mathfrak{B}_{m-i}, \mathfrak{B}_{n-j})\Big).$$
   Since $j < n - m + \left[\dfrac{m-1}{2}\right] + 1$, $j < n + i - m$. Then $m - i < n - j$. We have $i \leq j$.
   It suffices to prove ($2^{p-3} \leq i$ and $2^{p-3} \leq m - i$), since from IH for $i < m$ and $m - i < m$ it will follow that $\emptyset \in LI_{p-1}(\mathfrak{B}_{m-i}, \mathfrak{B}_{n-j})$.
   Assume $2^{p-3} > i$, then $2^{p-2} \geq 2i + 2 \geq m + 2$, but $2^{p-2} \leq m$, then $m \geq m + 2$, contradiction.
   Assume $2^{p-3} > m - i$. Then $2^{p-2} \geq 2m - 2i + 2 \geq m + 1$. But $2^{p-2} \leq m$, then $m \geq m + 1$, contradiction.

□

4 The order types $\mathfrak{C}_{m,q} = (\omega \cdot m + q, <)$

Consider structures $\mathfrak{C}_{m,q} = (C_{m,q}, <)$, where for $m \geq 1$ and $q \geq 0$, $C_{m,q} = \omega \cdot m + q$, and $\omega \cdot m + q = (\omega \cdot m) \cup \{\omega \cdot m, \omega \cdot m + 1, \ldots \omega \cdot m + (q-1)\}$. The elements of a structure $\mathfrak{C}_{m,q}$ will be denoted $k_i^{\mathfrak{C}_{m,q}}$ for $k_i^{\mathfrak{C}_{m,q}} \in \omega \cdot m$, i.e. $0 \leq i \leq m-1$, and $k^{\mathfrak{C}_{m,q}}$ for $\omega \cdot m + k$. Therefore $k_i^{\mathfrak{C}_{m,q}} < l_j^{\mathfrak{C}_{m,q}} \Leftrightarrow (i < j$ or $i = j \,\&\, k < l)$, and $k_i^{\mathfrak{C}_{m,q}} < k^{\mathfrak{C}_{m,q}}$, for all $k^{\mathfrak{C}_{m,q}}$ and all $k_i^{\mathfrak{C}_{m,q}} \in \omega \cdot m$.

Having two structures $\mathfrak{C}_{m,q}$ and $\mathfrak{C}_{n,r}$, we want to find for which p, $\mathfrak{C}_{m,q} \equiv_p \mathfrak{C}_{n,r}$.

4.1. First consider the case $m = n$ and $q < r$. We want to prove

$$\mathfrak{C}_{m,q} \equiv_p \mathfrak{C}_{m,r} \Leftrightarrow (q = 0 \text{ and } p \leq 1) \text{ or } (q > 0 \text{ and } 2^p < q + 5).$$

Proof:
($\Rightarrow$) We will need the following formulas:

$Gst(x) \equiv \forall y(y = x \vee y \sqsubset x)$, "$x$ is the greatest element"; $QD(Gst) = 1$.

$Wip^*(x) \equiv \forall y(y \sqsubset x \to \exists z(y \sqsubset z \sqsubset x))$, "$x$ has no immediate predecessor"; $QD(Wip^*) = 2$.

$Tail(x) \equiv \forall y(x \sqsubseteq y \to \neg Wip^*(y))$, "all the elements greater than or equal to x have immediate predecessor"; $QD(Tail) = 3$.

$DiffEl_q$, defined in Section 2, "there are at least $q+1$ different elements"; we have defined formulas φ_q, such that $\mathfrak{A}_t \models \varphi_q \leftrightarrow DiffEl_q$, for all t, and $QD(\varphi_q) = \mu p[2^p \geq q+2]$. But still $\mathfrak{C}_{s,t} \models \varphi_q \leftrightarrow DiffEl_q$, for all s and t, so here we may assume that

$$QD(DiffEl_q) = \mu p[2^p \geq q+2].$$

We define formulas Ψ_q by induction on q as follows:

$$\left| \begin{array}{lll} \Psi_0 \equiv \exists x Gst(x), & & QD(\Psi_0) = 2, \\ \Psi_q \equiv \forall x(Wip^*(x) \to DiffEl^x_{q-1}(x)), & \text{for } 1 \leq q \leq 3, & QD(\Psi_q) = 3, \\ \Psi_q \equiv \exists x(\Psi^x_{[\frac{q}{2}]-2}(x) \wedge DiffEl^x_{[\frac{q+1}{2}]}(x) \wedge Tail(x)), & \text{for } q > 3. & \end{array} \right.$$

For these formulas we can prove the following properties:

(i) $QD(\Psi_q) = \mu p[2^p \geq q+5]$, for $q > 0$.

(ii) $\mathfrak{C}_{m,q} \not\models \Psi_q$, but for all s and all $t > q$, $\mathfrak{C}_{s,t} \models \Psi_q$.

(i) The proof is by induction on q, using $\left[\frac{q}{2}\right] + \left[\frac{q+1}{2}\right] = q$.

- $1 \leq q \leq 3$; $QD(\Psi_q) = 3$ and $\mu p[2^p \geq q+5] = 3$ for $1 \leq q \leq 3$;
- $q > 3$;
  IH for $t < q$, i.e. for all t, such that $0 < t < q$, $QD(\Psi_t) = \mu p[2^p \geq t+5]$.
  Therefore $QD(\Psi_q) = 1 + \max\{QD(\Psi_{[\frac{q}{2}]-2}), QD(DiffEl_{[\frac{q+1}{2}]}), QD(Tail)\}$.
  For $q > 3$, $0 \leq \left[\frac{q}{2}\right] - 2 < q$. Therefore
  (for $q = 4$ we cannot apply the IH, but the following equalities are still valid)

$$\begin{aligned} QD(\Psi_q) &= 1+\max\Big\{\mu p\big[2^p \geq [\tfrac{q}{2}]+3\big], \mu p\big[2^p \geq [\tfrac{q+1}{2}]+2\big], 3\Big\} = \\ &= 1+\mu p\big[2^p \geq [\tfrac{q}{2}]+3\big] = \mu p\big[2^{p-1} \geq [\tfrac{q}{2}]+3\big] = \\ &= \mu p\big[2^p \geq 2\cdot[\tfrac{q}{2}]+6\big] = \mu p\big[2^p \geq 2[\tfrac{q}{2}]+5\big] = \\ &= \mu p\big[2^p \geq q+5\big]. \end{aligned}$$

(ii) The proof is by induction on q.

($\Leftarrow$) The case $q = 0$ and $p \leq 1$ is easy to check.
If $q > 0$ and $2^p \leq q+4$ we have to prove that $\emptyset \in LI_p(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$, i.e. there is a winning strategy for the second player for a game with p moves. The proof is by induction on q.

- $q = 1$, Therefore $p \leq 2$. It is easy to check that $\emptyset \in LI_p(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$.
- $q > 1$,
  IH for $t < q$, i.e. for all t, such that $0 < t < q$, if $2^p < t+5$, then $\emptyset \in LI_p(\mathfrak{C}_{m,t}, \mathfrak{C}_{n,r})$, where $m = n$ and $t < r$.
  Let $2^p \leq q+4$.
  We shall prove
  (a) $\forall a \in C_{m,q} \exists b \in C_{m,r}$ s.t. $\{(a,b)\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$; and
  (b) $\forall b \in C_{m,r} \exists a \in C_{m,q}$ s.t. $\{(a,b)\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$.

  The cases $a = k_i^{\mathfrak{C}_{m,q}}$ where $0 \leq i \leq m-1$, and $b = k_i^{\mathfrak{C}_{m,r}}$ where $0 \leq i \leq m-1$ for (a) and (b) resp. (the first player chooses element from the part $\omega \cdot m$ and the second player answers with the same element from the other structure) are trivial, since $\{(k_i^{\mathfrak{C}_{m,q}}, k_i^{\mathfrak{C}_{m,q}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$.

  Consider the case when the first player chooses element $\omega \cdot m + i$ from the tail, i.e. an element a from the set $\{0^{\mathfrak{C}_{m,q}} \ldots (q-1)^{\mathfrak{C}_{m,q}}\}$ or b from $\{0^{\mathfrak{C}_{m,r}} \ldots (r-1)^{\mathfrak{C}_{m,r}}\}$.

  – For $a = k^{\mathfrak{C}_{m,q}}$ such that $0 \leq k \leq [\frac{q}{2}] - 2$, take $b = k^{\mathfrak{C}_{m,r}} \in C_{m,r}$.
    Then we have $\{(a,b)\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$ if and only if $\{(k^{\mathfrak{C}_{m,q}}, k^{\mathfrak{C}_{m,r}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$ iff $\emptyset \in LI_{p-1}(\mathfrak{A}_{q-k-1}, \mathfrak{A}_{r-k-1})$, where $\mathfrak{A}_l$ denote the finite structures $(\{0, \ldots, l-1\}, <)$, defined in Section 2. We have proved that $\emptyset \in LI_{p-1}(\mathfrak{A}_{q-k-1}, \mathfrak{A}_{r-k-1})$ iff $2^{p-1} \leq 1 + (q-k-1)$, i.e. iff $2^{p-1} \leq q-k$.
    Since $0 \leq k \leq [\frac{q}{2}] - 2$ and $2^p \leq q+4$, we have $2k \leq q-4$, then $2q - 2k \geq q+4 \geq 2^p$, therefore $2^{p-1} \leq q-k$.

    For $b = l^{\mathfrak{C}_{m,r}}$, such that $0 \leq l \leq [\frac{q}{2}] - 2$, take $a = l^{\mathfrak{C}_{m,q}}$, and we have proved $\{(l^{\mathfrak{C}_{m,q}}, l^{\mathfrak{C}_{m,r}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$.

– In the cases where the distance between the chosen element and the end of the structure (the greatest element) is less than or equal to $[\frac{q+1}{2}]$, the second player chooses and element having the same distance to the greatest element.

For $a = k^{\mathfrak{C}_{m,q}}$, such that $[\frac{q}{2}] - 1 \leq k \leq q-1$, take $b = l^{\mathfrak{C}_{m,q}}$, such that $l = r - q + k$. Therefore $\{(a,b)\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$ if and only if $\{(k^{\mathfrak{C}_{m,q}}, l^{\mathfrak{C}_{m,r}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$ if and only if $\big(\emptyset \in LI_{p-1}(\mathfrak{C}_{m,k}, \mathfrak{C}_{m,l})$ and $\emptyset \in LI_{p-1}(\mathfrak{A}_{q-k-1}, \mathfrak{A}_{r-l-1})\big)$ iff $\emptyset \in LI_{p-1}(\mathfrak{C}_{m,k}, \mathfrak{C}_{m,l})$, since $q-k-1 = r-l-1$.
Since $[\frac{q}{2}] - 1 \leq k \leq q-1$ and $2^p \leq q+4$, we have $2k \geq q-3$, i.e. $2[\frac{q}{2}] \geq q-1$, then $2^p \leq q+4 \leq 2k+7$, i.e. $2^p \leq 2k+6$, then $2^{p-1} \leq k+3 < k+4$. Then from the IH it follows that $\emptyset \in LI_{p-1}(\mathfrak{C}_{m,k}, \mathfrak{C}_{m,l})$.

For $b = l^{\mathfrak{C}_{m,r}}$, such that $r - [\frac{q+1}{2}] - 1 \leq l \leq r-1$, take $a = k^{\mathfrak{C}_{m,q}}$, where $k = l - r + q$. Therefore $[\frac{q}{2}] - 1 \leq k \leq q-1$, and we have already proved that $\{(k^{\mathfrak{C}_{m,q}}, l^{\mathfrak{C}_{m,r}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$.

– For $b = l^{\mathfrak{C}_{m,r}}$, such that $[\frac{q}{2}] - 1 \leq l \leq r - [\frac{q+1}{2}] - 2$, take $a = k^{\mathfrak{C}_{m,q}}$, where $k = [\frac{q+1}{2}] - 2$.
Therefore $\{(a,b)\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$ if and only if $\{(k^{\mathfrak{C}_{m,q}}, l^{\mathfrak{C}_{m,r}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$ if and only if $\big(\emptyset \in LI_{p-1}(\mathfrak{C}_{m,k}, \mathfrak{C}_{m,l})$ and $\emptyset \in LI_{p-1}(\mathfrak{A}_{q-k-1}, \mathfrak{A}_{r-l-1})\big)$ iff $\big(2^{p-1} \leq k+4$ and $2^{p-1} \leq q-k\big)$, by IH and the result in Section 2, since $k < l$ and $q-k-1 < r-l-1$.
1) $2^{p-1} \leq q-k = [\frac{q}{2}] + 2$ if and only if $2^p \leq 2[\frac{q}{2}] + 4$ iff $2^p \leq q+4$, the latter is our assumption.
2) Since $2^p \leq q+4$, we have $2^p \leq 2[\frac{q+1}{2}] + 4$, therefore $2^{p-1} \leq [\frac{q+1}{2}] + 2 \leq k+4$.
Therefore $\{(a,b)\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,r})$.

This is the end of the proof for the first case, where $m = n$ and $q < r$.

□

4.2. Now consider the case $m \leq n$ and $q = r$. We want to prove

$$\mathfrak{C}_{m,q} \equiv_p \mathfrak{C}_{n,q} \Leftrightarrow \Big(2^{p-2} \leq m+1 \,\&\, \big(q \leq 3 \Leftrightarrow 2^{p-2} \leq m\big)\Big).$$

Proof:

($\Rightarrow$) Let $\mathfrak{C}_{m,q} \equiv_p \mathfrak{C}_{n,q}$, for $m < n$, and assume that $m+1 < 2^{p-2}$ and ($m < 2^{p-1}$ if $q \leq 3$). In order to get a contradiction we need to find formulas $\Phi_{m,q}$, such that $QD(\Phi_{m,q}) \leq p$ and $\mathfrak{C}_{m,q} \not\models \Phi_{m,q}$, but $\mathfrak{C}_{n,q} \models \Phi_{m,q}$.

We have defined sentences $DiffWipEl_m$ (see Section 3), such that $\mathfrak{B}_n \models \psi_m \leftrightarrow DiffWipEl_m$, for all n, for some sentences ψ_m with $QD(\psi_m) = \mu p[2^{p-2} \geq m+1]$. But still $\mathfrak{C}_{s,q} \models \psi_m \leftrightarrow DiffWipEl_m$, for all $s \geq 1$, and we may assume that $QD(DiffWipEl_m) = \mu p[2^{p-2} \geq m+1]$.

Define $DiffWipEl^*_{m,q} \equiv DiffWipEl_{m+1}$, "there are at least $m+1$ different Wip-elements". Therefore:

(a) $QD(DiffWipEl^*_{m,q}) = \mu p[2^{p-2} \geq m+2]$; and

(b) $\mathfrak{C}_{m,q} \not\models DiffWipEl^*_{m,q}$ and $\mathfrak{C}_{n,q} \models DiffWipEl^*_{m,q}$, since $m < n$.

In Section 2 we defined sentences $DiffEl_q$ "there are at least $q+1$ different elements" and we may assume that $QD(DiffEl_q) = \mu p[2^p \geq q+2]$, since they are still equivalent to formulas with this quantifier depth in the structures $\mathfrak{C}_{s,q}$, for all $s \geq 1$ and q. Now define sentences $\chi_{m,q}$, that are equivalent to $\exists x_0 \ldots \exists x_{m-1}\big(Wip(x_0) \wedge \ldots \wedge Wip(x_{m-1}) \wedge DiffEl^{x_{m-1}}_{q-1}\ (x_{m-1})\big)$ in any $\mathfrak{C}_{s,t}$, i.e. saying "there are at least m different Wip-elements and at least q different elements after the last", as follows:

$$\begin{aligned}
\chi_{m,0} &\equiv DiffWipEl_{m-1}, \\
\chi_{1,q} &\equiv \exists x(Wip(x) \wedge DiffEl^x_{q-1}(x_0)), \text{ for } q > 0, \\
\chi_{m+1,q} &\equiv \exists x\big(Wip(x) \wedge DiffWipEl^{\,x}_{[\frac{m}{2}]}(x) \wedge \chi^x_{[\frac{m+1}{2}],q}(x)\big), \text{ for } q > 0,
\end{aligned}$$

where the formula Wip is defined in Section 3 and $QD(Wip) = 2$.
Therefore

(c) $QD(\chi_{m,q}) = \mu p[2^{p-2} \geq m+1]$, for $q \leq 3$; and

(d) $\mathfrak{C}_{m,q} \not\models \chi_{m,q}$ and $\mathfrak{C}_{n,q} \models \chi_{m,q}$, since $m < n$.

The property (c) can be proved by induction on m :

- $m = 1$,
  $QD(\chi_{1,q}) = 1 + \max(2, \mu p[2^p \geq q+1]) = 3$, for $q \leq 3$.
- IH for $m \geq 1$. Therefore
  $QD(\chi_{m+1,q}) = 1 + \max\{2, \mu p[2^{p-2} \geq [\frac{m}{2}]+1], QD(\chi_{[\frac{m+1}{2}],q})\} = 1 + \max(\mu p[2^{p-2} \geq [\frac{m}{2}]+1], \mu p[2^{p-2} \geq [\frac{m+1}{2}]+1]) = 1 + \mu p[2^{p-2} \geq [\frac{m+1}{2}]+1] = \mu p[2^{p-3} \geq [\frac{m+1}{2}]+1] = \mu p[2^{p-2} \geq],$
  since $2^{p-2} \geq 2[\frac{m+1}{2}]+2 \Leftrightarrow 2^{p-2} \geq m+2$.

Define $\Phi_{m,q}$ as follows: $\left| \begin{array}{ll} \Phi_{m,q} \equiv \chi_{m,q}, & \text{for } q \leq 3; \\ \Phi_{m,q} \equiv DiffWipEl^*_{m,q}, & \text{for } q > 3. \end{array} \right.$ Therefore for $q \leq 3$, $QD(\Phi_{m,q}) = \mu p[2^{p-2} \geq m+1]$ and for $q > 3$, $QD(\Phi_{m,q}) = \mu p[2^{p-2} \geq m+2]$, i.e.$QD(\Phi_{m,q} \leq p$, and $\mathfrak{C}_{m,q} \not\models \Phi_{m,q}$, but $\mathfrak{C}_{n,q} \models \Phi_{m,q}$, which is a contradiction.

($\Rightarrow$)Let $2^{p-2} \leq m+1 \,\&\, \big(q \leq 3 \Leftrightarrow 2^{p-2} \leq m\big)$. We want to prove that $\mathfrak{C}_{m,q} \equiv_p \mathfrak{C}_{n,q}$.

If $2^{p-2} \leq m$, then (see Section 3) for any q the second player has a winning strategy for a game with p moves (for the elements from the tail chooses the correspondent elements from the tail of the other structure, and for the elements from $\omega \cdot m$ use the winning strategy, described in Section 3).

Therefore it suffices to consider the case $2^{p-2} = m+1$ and $q \geq 4$, using induction on m. We may assume that $m = 2k+1$.

It is easy to verify the statement for $m = 1$, where $p = 3$.

Let $m > 1$ and assume (IH) the claim is true for smaller than m. We have to prove that for every $a \in C_{m,q}$ there is $b \in C_{n,q}$, (and for every $b \in C_{n,q}$ there is $a \in C_{m,q}$), such that $\{(a,b)\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{n,q})$. Consider the cases for the Wip-elements (the others are analogous), i.e. the first player chooses an element $a = 0_i^{\mathfrak{C}_{m,q}}$ (or $b = 0_j^{\mathfrak{C}_{n,q}}$), then the second player chooses an element $b = 0_j^{\mathfrak{C}_{n,q}}$ (resp. $a = 0_i^{\mathfrak{C}_{m,q}}$), depending on i :

- $1 \leq j \leq [\frac{m-1}{2}] + 1$, i.e. $1 \leq j \leq k+1$.
  Take $i = j$. Therefore $\{(0_i^{\mathfrak{C}_{m,q}}, 0_i^{\mathfrak{C}_{n,q}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{n,q})$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{C}_{m-i,q}, \mathfrak{C}_{n-i,q})$, which follows by the IH, if $2^{p-3} \leq m - i + 1$.
  Assume $2^{p-3} \geq m - i + 2$, then $2^{p-2} = m + 1 \geq 2m - 2i + 4$, then $2i \geq m + 3$, then $[\frac{m-1}{2}] = k$, therefore $2i \geq 2k+4$, but $i \leq [\frac{m-1}{2}]+1 = k+1$, contradiction.

- $n - [\frac{m}{2}] + 1 \leq j \leq n - 1$, i.e. $n - k + 1 \leq j \leq n - 1$,
  then $[\frac{m-1}{2}] \leq i \leq m - 1$, i.e. $k \leq i \leq 2k$.
  Take $i = m - n + j$. Then $\{(0_i^{\mathfrak{C}_{m,q}}, 0_i^{\mathfrak{C}_{n,q}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{n,q})$ if and only if $\emptyset \in LI_{p-1}(\mathfrak{B}_i, \mathfrak{B}_j)$, iff $2^{p-3} \leq i$, since $i < j$ (see Section 3). The case where the first player chooses $[\frac{m-1}{2}] \leq i \leq m - 1$, is the same if the second takes $j = n - m + i$.

- $[\frac{m-1}{2} + 2 \leq j \leq n - [\frac{m}{2}]]$, i.e. $k + 2 \leq j \leq n - k$.
  Take $i = [\mathfrak{m} - \imath 2]+1$, i.e. $i = k+1$. Therefore $\{(0_i^{\mathfrak{C}_{m,q}}, 0_i^{\mathfrak{C}_{n,q}})\} \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{n,q})$ if and only if
  (left part) $\emptyset \in LI_{p-1}(\mathfrak{B}_i, \mathfrak{B}_j)$ and (right part) $\emptyset \in LI_{p-1}(\mathfrak{C}_{m,q}, \mathfrak{C}_{m,q})$
  if and only if $2^{p-3} \leq i$ and $2^{p-3} \leq m - i + 1$, which is easy to check:
  $2^{p-3} \leq i$ iff $2^{p-3} \leq k+1$ iff $2k+2 = 2^{p-2} \leq 2k+2$; and $2^{p-3} \leq m-i+1 = k+1$ iff $2^{p-2} \leq 2k+2$.

□

We have solved the problem $\omega \cdot m + q \equiv_p \omega \cdot n + r$ for the cases where $q = r$ or $m = n$. It remains the case where $m \neq n$ and $q \neq r$, which we do not consider here.

5 The order types $\mathfrak{N}_m = (\omega^m, <)$

Consider structures $\mathfrak{N}_m = (\omega^m, <)$, where
$\omega^m = \{(x_0, \ldots, x_{m-1}) \,|\, x_0, \ldots, x_{m-1} \in \mathbb{N}\}$. We want to find for which p, $(\omega^m, <) \equiv_p (\omega^n, <)$?

Here we prove only that for $m < n$,

$$\text{If } (\omega^m, <) \equiv_p (\omega^n, <) \text{ then } p \leq 2m.$$

Proof:
Define formulas D_m by induction as follows:

$$\left|\begin{array}{ll} D_0(x) & \equiv x = x \\ D_{m+1}(x) & \equiv \exists y\big(y \sqsubset x \wedge D_m(y)\big) \wedge \\ & \qquad\qquad \forall y\big(y \sqsubset x \wedge D_m(y) \rightarrow \exists z\big(y \sqsubset z \sqsubset x \wedge D_m(z)\big)\big). \end{array}\right.$$

Define $\varphi_m \equiv \exists x D_m(x)$.
One can prove by induction that $QD(D_m) = 2m$ and
(a) $QD(\varphi_m) = 2m + 1$, and
(b) $(\omega^m, <) \not\models \varphi_m$, but $(\omega^n, <) \models \varphi_m$, since $n > m$.
If we assume that $\omega^m \equiv_p \omega^n$ and $p \geq 2m + 1$, then $QD(\varphi_m) \leq p$, which is a contradiction.

The other direction, i.e. the question whether for all $m < n$ and $p \leq 2m$, $(\omega^m, <) \equiv_p (\omega^n, <)$ remains unsolved. However $2m$ seems to be very large upper bound. $\square$