On the saturation of the Kantorovich sampling operators in variable exponent Lebesgue spaces

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Abstract

The purpose of this note is to correct an absurd mistake in [3, Theorem 3.4(i)], which lead to a mistake in the description of the trivial class of the family of the Kantorovich sampling operators in variable exponent Lebesgue spaces. In addition, we extend slightly the results in the abovementioned paper dropping the assumption that the exponent function is finite.

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Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lebesgue integrable function and $\chi : \mathbb{R} \to \mathbb{R}$. Bardaro, Butzer, Stens and Vinti [1] introduced the Kantorovich-type sampling operators

$$(1) \qquad (S_w^\chi f)(x) := \sum_{k\in\mathbb{Z}} \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du\, \chi(wx - t_k), \quad x \in \mathbb{R}, \quad w > 0,$$

where $\{t_k\}_{k\in\mathbb{Z}}$ is a sequence of reals such that $\theta \le \theta_k := t_{k+1} - t_k \le \Theta$ for all $k \in \mathbb{Z}$ with some constants $\theta, \Theta > 0$. It is assumed that both f and χ satisfy certain assumptions, which provide the convergence of the series in (1) at least almost everywhere in $\mathbb{R}$.

We consider approximation in variable exponent Lebesgue spaces. Let us recall their definition.

Let $p : \mathbb{R} \to [1, +\infty]$ be Lebesgue measurable. As it is customary in this setting, to emphasize that p is generally non-constant, we will write $p(\cdot)$. It is the exponent function of the variable exponent Lebesgue space.

We set

$$p_* := \operatorname*{ess\,inf}_{x\in\mathbb{R}} p(x), \quad p^* := \operatorname*{ess\,sup}_{x\in\mathbb{R}} p(x)$$

and

$$\mathbb{R}_\infty^{p(\cdot)} := \{x \in \mathbb{R} : p(x) = +\infty\}.$$

Next, for a Lebesgue measurable function f on $\mathbb{R}$, we set

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}\setminus\mathbb{R}_\infty^{p(\cdot)}} |f(x)|^{p(x)}dx + \operatorname*{ess\,sup}_{x\in\mathbb{R}_\infty^{p(\cdot)}} |f(x)|.$$

The variable exponent Lebesgue space $L_{p(\cdot)}(\mathbb{R})$ is the set of all Lebesgue measurable functions f on $\mathbb{R}$, for which there exists $\lambda > 0$ such that

$$\rho_{p(\cdot)}(f/\lambda) < \infty.$$

It is a Banach space with the norm (see e.g. [2, Theorem 2.25])

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\}.$$

As is known, if $f \in L_{p(\cdot)}(\mathbb{R})$, then f is locally Lebesgue integrable (see e.g. [2, Corollary 2.27]); hence the integral terms in (1) are defined.

In [3, Theorems 1.2 and 1.3] we established a direct estimate and a matching two-term strong converse one for $\|S_w^\chi f - f\|_{p(\cdot)}$ by means of moduli of smoothness in $L_{p(\cdot)}(\mathbb{R})$ with finite exponent $p(\cdot)$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Crucial in the proofs was the boundedness of the Hardy-Littlewood maximal operator, which follows from $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$ regardless of whether $p(\cdot)$ is finite or not. As usual, $LH(\mathbb{R})$ denotes the space of the log-Hölder continuous functions on $\mathbb{R}$. We refer to [2, Definition 2.2] for its definition.

The assumption that $p(\cdot)$ is finite was made only for convenience and is not actually related to the proof of [3, Theorems 1.2 and 1.3] and, hence, their validity. That is also true for the assertions concerning the saturation class and rate of $\{S_w^\chi\}_{w>0}$ stated in [3, Theorem 1.5]. The description of its trivial class, however, needs clarification; moreover, the statement concerning it in [3, Theorem 1.5] is not quite correct—it holds for finite $p(\cdot)$ only if, in addition, $p^* < \infty$.

Below, we give the complete and correct description of the saturation property of the family $\{S_w^\chi\}_{w>0}$. In its statement, $W_{p(\cdot)}^r(\mathbb{R})$ is the variable exponent Sobolev space

$$W_{p(\cdot)}^r(\mathbb{R}) := \{f \in L_{p(\cdot)}(\mathbb{R}) : f \in AC_{loc}^{r-1}(\mathbb{R}),\ f^{(r)} \in L_{p(\cdot)}(\mathbb{R})\},$$

$M_\alpha(\chi)$ is the discrete absolute moment of χ of order $\alpha \ge 0$ w.r.t. $\{t_k\}_{k\in\mathbb{Z}}$, which is given by

$$M_\alpha(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |t_k - u|^\alpha |\chi(u - t_k)|,$$

and $|S|$ is the Lebesgue measure of the set $S \subseteq \mathbb{R}$ provided that it exists.

Theorem 1. *Let $r \in \mathbb{N}_+$ and $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\chi \in C^{r+1}(\mathbb{R})$ be such that:*

(i) $M_{r+1}(\chi), M_r(\chi^{(r+1)}) < \infty$,

(ii) $\sum_{k\in\mathbb{Z}} \chi(u - t_k) \equiv 1$,

(iii) $\displaystyle\sum_{\ell=0}^{j}\binom{j+1}{\ell}\sum_{k\in\mathbb{Z}}\theta_k^{j-\ell}(t_k-u)^\ell\chi(u-t_k)\equiv 0,\quad j=1,\ldots,r-1,$ *if* $r\geq 2,$

(iv) $\displaystyle\sum_{\ell=0}^{r}\binom{r+1}{\ell}\sum_{k\in\mathbb{Z}}\theta_k^{r-\ell}(t_k-u)^\ell\chi(u-t_k)\equiv \mathrm{const}\neq 0.$

In addition, let the series

$$\sum_{k\in\mathbb{Z}}|t_k-u|^{r-1}|\chi(u-t_k)|\quad \textit{and}\quad \sum_{k\in\mathbb{Z}}|t_k-u|^{r-1}|\chi^{(r+1)}(u-t_k)|$$

be uniformly convergent on the compact intervals of $\mathbb{R}$.

Then the approximation process $\{S_w^\chi\}_{w>0}$ *is saturated in* $L_{p(\cdot)}(\mathbb{R})$ *with order* $O(w^{-r})$, *its saturation class is* $W^r_{p(\cdot)}(\mathbb{R})$ *and its trivial class consists of*

(i) the functions which are equal to 0 *a.e. if* $p^*<\infty$ *and* $|\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}|=\infty$,

(ii) the functions which are equal to a const *a.e. if* $p^*=\infty$ *or* $|\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}|<\infty$.

In particular, if $p(\cdot)$ is finite, then the trivial class of $\{S_w^\chi\}_{w>0}$ consists of the functions which are equal to 0 a.e. if $p^*<\infty$, and of the functions which are equal to a const a.e. if $p^*=\infty$.

As we already pointed out, the assertions concerning the saturation class and rate follow by the same argument as in the case of the additional assumption that $p(\cdot)$ is finite. Those about the trivial class follow from a property of the moduli of smoothness used in [3]. The moduli are given by

$$\Omega_r(f,t)_{p(\cdot)}:=\sup_{0<h\leq t}\left\|\frac{1}{h}\int_0^h\Delta_u^r f\,du\right\|_{p(\cdot)}$$

and

$$\overline{\Omega}_r(f,t)_{p(\cdot)}:=\sup_{0<h\leq t}\left\|\frac{1}{h}\int_0^h|\Delta_u^r f|\,du\right\|_{p(\cdot)},$$

where $\Delta_u^r f$ is the forward finite difference of order r and step u of f, that is, $\Delta_u f(x):=f(x+u)-f(x)$, $x,u\in\mathbb{R}$, and $\Delta_u^r:=\Delta_u(\Delta_u^{r-1})$.

Now, let us state their property that directly implies the last part of Theorem 1.

Proposition 2. *Let* $r\in\mathbb{N}_+$, $p(\cdot)$ *be an exponent function on* $\mathbb{R}$ *such that* $p_*>1$ *and* $1/p(\cdot)\in LH(\mathbb{R})$. *Let* $f\in L_{p(\cdot)}(\mathbb{R})$ *and* $\Omega_r(f,t)_{p(\cdot)}=o(t^r)$. *Then:*

(i) $f=0$ *a.e. if* $p^*<\infty$ *and* $|\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}|=\infty$,

(ii) $f=\mathrm{const}$ *a.e. if* $p^*=\infty$ *or* $|\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}|<\infty$.

The assertions remain valid with $\overline{\Omega}_r(f,t)_{p(\cdot)}$ in place of $\Omega_r(f,t)_{p(\cdot)}$.

Assertion (i) corrects [3, Theorem 3.4(i)].

Proof. By virtue of [3, Theorem 3.2], either $\Omega_r(f,t)_{p(\cdot)} = o(t^r)$ or $\overline{\Omega}_r(f,t)_{p(\cdot)} = o(t^r)$ implies that f coincides a.e. with an algebraic polynomial of degree at most $r-1$. The function f needs to be in $L_{p(\cdot)}(\mathbb{R})$, that is, $\rho_{p(\cdot)}(f/\lambda) < \infty$ with some $\lambda > 0$. Since $p(x) \ge 1$ on $\mathbb{R}$ and at least one of the sets $\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}$ or $\mathbb{R}_\infty^{p(\cdot)}$ is not bounded, then $f = c \in \mathbb{R}$ a.e.

It remains to determine those $c \in \mathbb{R}$ for which

$$\int_{\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}} \left|\frac{c}{\lambda}\right|^{p(x)} dx < \infty \text{ with some } \lambda > 0. \tag{2}$$

If $p^* < \infty$, then $p(x)$ is bounded above on $\mathbb{R}$. Consequently, if $|\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}| < \infty$, then any $c \in \mathbb{R}$ satisfies (2), whereas if $|\mathbb{R}\backslash\mathbb{R}_\infty^{p(\cdot)}| = \infty$, then only $c = 0$ does.

Let $p^* = \infty$. As it follows from the definition of the log-Hölder continuity, $1/p(x)$ tends to a nonnegative real r_∞ as $x \to \pm\infty$ (we apply [2, Definition 2.2] with $r(x) = 1/p(x)$). Since $p^* = \infty$, then $r_\infty = 0$ and

$$p(x) \ge \frac{1}{C_\infty} \log(e + |x|), \quad x \in \mathbb{R},$$

with some $C_\infty > 0$.

Now, we see that any $c \in \mathbb{R}$ satisfies (2) with $\lambda > 0$ such that

$$\frac{|c|}{\lambda} \le \frac{1}{e^{2C_\infty}}.$$

Indeed, then

$$\left|\frac{c}{\lambda}\right|^{p(x)} \le \frac{1}{(e+|x|)^2}, \quad x \in \mathbb{R},$$

hence $\rho_{p(\cdot)}(c/\lambda) < \infty$. ◻

References

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