The Computable Sets

**Definition.** Let $A \subseteq \mathbb{N}^n$. The set $A$ is *computable* if the function

$$
\chi_A(x) \sim \begin{cases} 
0 & \text{if } x \in A, \\
1 & \text{if } x \notin A.
\end{cases}
$$

is computable.

**Theorem.** The computable sets are closed under the operations union, intersection and taking the complement. Cartesian product of computable sets is computable.

**Theorem.** If $A$ is a computable subset of $\mathbb{N}^{n+1}$ then either of the sets

$$
\{ (y, \bar{x}) \mid (\exists z < y)((z, \bar{x}) \in A) \} \text{ and } \{ (y, \bar{x}) \mid (\forall z < y)((z, \bar{x}) \in A) \}
$$

is computable.
The Semi-computable Sets - Definitions

**Definition.** Let $A \subseteq \mathbb{N}^n$. The set $A$ is *semi-computable* if the function $c_A(x) \simeq \begin{cases} 0 & \text{if } x \in A, \\ \neg! & \text{if } \not\in A. \end{cases}$ is computable.

**Theorem.** A set $A$ is semi-computable if and only if $A = \text{dom}(\varphi)$ for some computable function $\varphi$.

**Theorem.** A set $A \subseteq \mathbb{N}^n$ is semi-computable if and only if there exist a primitive recursive function $\rho$ of $n + 1$ arguments such that

$$(\forall \bar{x})(\bar{x} \in A \iff (\exists y)(\rho(y, \bar{x}) = 0)).$$
The Semi-computable Sets - Properties

**Theorem.** Union and intersection of semi-computable sets is semi-computable.

**Theorem.** Let $A \subseteq \mathbb{N}^n$ be semi-computable. Then either of the following sets is semi-computable:

\[
\{(y, \bar{x})| (\exists z < y)( (z, \bar{x}) \in A)\},
\{(y, \bar{x})| (\forall z < y)( (z, \bar{x}) \in A)\},
\{\bar{x}| (\exists z)( (z, \bar{x}) \in A)\}.
\]

**Theorem.** *(Post)* A set $A$ is computable if and only if both $A$ and $\bar{A}$ are semi-computable.
The Semi-computable Sets - Properties

**Theorem.** A partial function $\varphi$ is computable if and only if the graph $G_\varphi$ of $f$ is semi-computable.

**Theorem.** Let $A$ be a non empty semi-computable set of natural numbers. Then there exists a primitive recursive function $g$ such that $A = \text{range}(g)$.

**Corollary.** A set $A$ of natural numbers is semi-computable if and only if $A = \text{range}(\varphi)$ for some computable function $\varphi$. 
Theorem. Let $A_1, \ldots, A_k$ be mutually disjoint semi-computable subsets of $\mathbb{N}^n$. Let $f_1, \ldots, f_k$ be computable functions of $n$ arguments. Then the function $g$ defined below is computable:

$$g(\bar{x}) \simeq \begin{cases} 
  f_1(\bar{x}) & \text{if } \bar{x} \in A_1, \\
  f_2(\bar{x}) & \text{if } \bar{x} \in A_2, \\
  \ldots & \ldots \\
  f_k(\bar{x}) & \text{if } \bar{x} \in A_k, \\
  \text{undefined} & \text{otherwise}.
\end{cases}$$
The set $K$

**Definition.** For $n \geq 1$ and $a \in \mathbb{N}$, set $W_a^n = \text{dom}(\varphi_a^n)$.

**Theorem.**

1. *The sequence $W_0^n, \ldots, W_a^n, \ldots$ is an enumeration of the semi-computable subsets of $\mathbb{N}^n$.***

2. $(\forall a)(\forall \bar{x} \in \mathbb{N}^m)(\forall \bar{y} \in \mathbb{N}^n)((\bar{x}, \bar{y}) \in W_{a+m}^n \iff \bar{x} \in W_{S_n^m(a,\bar{x})}^n)$.

3. *For $n \geq 1$ set $U_n = \text{dom}(\Phi_n)$. Then

   $(\forall a)(\forall \bar{x} \in \mathbb{N}^n)(\bar{x} \in W_a^n \iff (a, \bar{x}) \in U_n)$.

**Definition.** $K = \{x|x \in W_x\}$. 
Properties of $K$

**Theorem.**
1. The set $K$ is semi-computable.
2. The set $\bar{K}$ is not semi-computable.
3. The set $K$ is not computable.

**Definition.** Given two sets $A$ and $B$ of natural numbers set $A \leq_m B$ if there exists a total computable $h$ such that

$$(\forall x)(x \in A \iff h(x) \in B).$$

Note that if $A \leq_m B$ then computability of $B$ implies computability of $A$ and semi-computability of $B$ implies semi-computability of $A$.

**Theorem.** If $A$ is a semi-computable set of natural numbers then $A \leq_m K$. 

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Introduction to Computability Theory II
**Problem.** Show that the semi-computable sets are not closed under the operations taking the complement and unbounded universal quantification.

**Problem.** Show that for any infinite semi-computable set $A$ of natural numbers there exists a one to one computable function $f$ such that $\text{range}(f) = A$.

**Problem.** Show that an infinite set $A$ of natural numbers is computable if and only if $A$ is range of some strongly monotonically increasing total computable function.
Problems

**Problem.** Let $A \subseteq \mathbb{N}^{n+1}$ be semi-computable. Show that there exist a computable function $f$ of $n$ arguments such that

1. $!f(\bar{x}) \iff (\exists y)((\bar{x}, y) \in A)$.
2. $G_f \subseteq A$.

**Problem.** Show that there exist two disjoint semi-computable sets $A$ and $B$ of natural numbers which are recursively unseparable, i.e. there is no computable set $C$ such that $A \subseteq C$ and $B \subseteq \bar{C}$.

**Problem.** Let $A, B \subseteq \mathbb{N}^n$. Show that the semi-computable sets $A$ and $B$ are computable if and only if $A \cup B$ and $A \cap B$ are computable.
The Halting Problem

**Theorem.** The following sets are not computable:

- \( H = \{ (a, x) | \neg \varphi_a(x) \} \).
- \( O = \{ a | \varphi_a = \lambda x.0 \} \).
- \( E = \{ (a, b) | \varphi_a = \varphi_b \} \).
- \( T = \{ a | \varphi_a \text{ is totally defined} \} \).

**Definition.** Let \( \mathcal{A} \) be a class of unary computable functions. The index set \( I_\mathcal{A} \) of \( \mathcal{A} \) consists of all natural numbers \( a \) such that \( \varphi_a \in \mathcal{A} \).

The class \( \mathcal{A} \) is trivial if \( \mathcal{A} = \emptyset \) or \( \mathcal{A} \) contains all unary computable functions.

**Theorem.** *(Rice)* If \( \mathcal{A} \) is a non trivial class of unary computable functions then the set \( I_\mathcal{A} \) is not computable.
The Rice-Shapiro Theorem

**Theorem.** Let $A$ be a class of unary computable functions with a semi-decidable index set. Then for all unary computable functions $f$ the following equivalence holds:

$$f \in A \iff (\exists \theta \subseteq f)(\theta \text{ is finite and } \theta \in A).$$

**Corollary.** The index set of either of the following classes is not semi-computable.

1. $\{\varphi | \varphi \text{ is finite}\}$.
2. $\{\varphi | \varphi \text{ is total and computable}\}$.
3. For any computable function $\varphi$ the singleton $\{\varphi\}$. 
Given a monotone sequence $f_0 \subseteq f_1 \cdots \subseteq f_k \cdots$ of partial functions of $n$ arguments, denote by $\bigcup_n f_n$ the function $g$, where

$$g(\bar{x}) \simeq y \iff (\exists k)(f_k(\bar{x}) \simeq y).$$

**Theorem.** The function $g$ is the least upper bound of the sequence $\{f_k\}$ with respect to the partial ordering $\subseteq$, i.e. the following two conditions are satisfied:

1. $g$ is an upper bound of $\{f_k\}$,
2. If $h$ is an upper bound of $\{f_k\}$ then $g \subseteq h$.

**Proposition.** If $\theta \subseteq \bigcup_n f_n$ is finite then for some $k$, $\theta \subseteq f_k$. 
Notation. Finite function will be denoted by \( \theta \).

**Definition.** A mapping \( \Gamma : F_n \to F_m \) is called *compact operator* if for all \( f \in F_n \), for all \( \bar{x} \in \mathbb{N}^m \) and all \( y \),

\[
\Gamma(f)(\bar{x}) \simeq y \iff (\exists \theta)(\theta \subseteq f \land \Gamma(\theta)(\bar{x}) \simeq y).
\]

**Theorem.** Every compact operator is monotone, i.e.

\[
(\forall f, g \in F_n)(f \subseteq g \implies \Gamma(f) \subseteq \Gamma(g)).
\]

**Theorem.** Suppose that \( \Gamma \) and \( \Delta \) are compact operators mapping \( F_n \) into \( F_m \). Let \( (\forall \theta \in F_n)(\Gamma(\theta) = \Delta(\theta)) \). Then \( \Gamma = \Delta \).
Theorem. Let $\Gamma$ be a compact mapping of $F_n$ into $F_m$. Then for every monotone sequence $\{f_k\}$ of elements of $F_n$,

$$\Gamma\left(\bigcup f_k\right) = \bigcup_{k} \Gamma(f_k).$$

Theorem. (Knaster and Tarski) Let $\Gamma : F_n \to F_n$ be a compact operator. There exists a partial function $f \in F_n$ satisfying the following conditions:

1. $\Gamma(f) = f$.
2. $(\forall g \in F_n)(\Gamma(g) \subseteq g \Rightarrow f \subseteq g)$.

The function $f$ having the above properties is called least fixed point of $\Gamma$. 
**Definition.** An operator $\Gamma$ is *recursive* if it is effective and compact.

**Theorem.** *(Myhill-Shepherdson)* Let $\mathcal{E} : \mathcal{F}_n \to \mathcal{F}_m$ be an effective mapping. There exists a recursive operator $\Gamma$ such that for all computable functions $f \in \mathcal{F}_n$, $\Gamma(f) = \mathcal{E}(f)$.

**Theorem.** *(First Recursion Theorem)* Let $\Gamma$ be a recursive operator mapping $\mathcal{F}_n$ into $\mathcal{F}_n$. Then the least fixed point of $\Gamma$ is computable.
Problem. Let $\mathcal{E}$ be an effective mapping. Show that if $\varphi \subseteq \psi$ are computable then $\mathcal{E}(\varphi) \subseteq \mathcal{E}(\psi)$

Problem. Let $\mathcal{E}$ and $\mathcal{D}$ be effective mappings of $\mathcal{F}_n$ into $\mathcal{F}_m$ such that for all finite functions $\theta$, $\mathcal{E}(\theta) = \mathcal{D}(\theta)$. Show that $\mathcal{E}(\varphi) = \mathcal{D}(\varphi)$ for all computable functions $\varphi$ of $n$ arguments.

Problem. Show that for every effective mapping $\mathcal{E} : \mathcal{F}_n \to \mathcal{F}_n$ there exists a computable function $\varphi$ of $n$ arguments such that $\mathcal{E}(\varphi) = \varphi$ and for all computable functions $\psi$, if $\mathcal{E}(\psi) \subseteq \psi$ then $\varphi \subseteq \psi$. 