Characterization of the Computable Real Numbers by Means of Primitive Recursive Functions

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Abstract. One usually defines the notion of a computable real number by using recursive functions. However, there is a simple way due to A. Mostowski to characterize the computable real numbers by using only primitive recursive functions. We prove Mostowski's result differently and apply it to get other simple characterizations of this kind. For instance, a real number is shown to be computable if and only if it belongs to all members of some primitive recursive sequence of nested intervals with rational end points and with lengths arbitrarily closely approaching 0.

Introduction

Let $\mathbb{N}$ and $\mathbb{Q}$ be the set of the non-negative integers and the set of the rational numbers, respectively. A function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ is said to be recursive if it can be represented in the form

$$A(n) = \frac{u(n) - v(n)}{w(n) + 1}\ , \qquad (1)$$

where u, v, w are recursive functions from $\mathbb{N}$ into $\mathbb{N}$ (in the case when the values of A are non-negative the second term in the numerator can be omitted).[1] The notion of a primitive recursive function from $\mathbb{N}$ into $\mathbb{Q}$ is defined in a similar way, namely one must replace "recursive" by "primitive recursive" in the above definition. If we regard a function A from $\mathbb{N}$ into $\mathbb{N}$ as a function from $\mathbb{N}$ into $\mathbb{Q}$ then the above notions are clearly equivalent to the ordinary recursiveness and to the ordinary primitive recursiveness of A, respectively. Of course, we can treat quite similarly also those $\mathbb{Q}$-valued functions that depend on several natural arguments.

One usually defines the notion of a computable (or recursive) real number by using recursive functions.[2] Any of the definitions implies the following statement (according to [2], it can be attributed to S. Mazur): a real number α is

[1] A definition using effective enumeration of the set $\mathbb{Q}$ can also be found in the literature.

[2] Here are several of the places where one can find some (mutually equivalent) definitions of this kind: the paper [4], § 12 of [9], Exercise 15-34 in [5], Lemma 4.2.1 and Exercise 4.2.1 in [10] (other relevant references can be found in Section II.4 of [3]).

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computable if and only if there is a recursive function A from $\mathbb{N}$ into $\mathbb{Q}$ that satisfies for any n in $\mathbb{N}$ the inequality

$$|A(n) - \alpha| \leq \frac{1}{n+1} \tag{2}$$

It is not permissible to replace "recursive" by "primitive recursive" in this statement (cf. Appendix 1). Nevertheless A. Mostowski showed in [2] that such a replacement is possible if we allow the right-hand side of (2) to be a suitable primitive recursive function from $\mathbb{N}$ into $\mathbb{Q}$ and to depend on the choice of the number α (unfortunately the paper [2] has been not known to us when writing the preliminary version [8] of the present paper). We shall give here another proof of Mostowski's result; certain issues related to this result will be also studied.

It will be useful in Section 2 to consider the notion of primitive recursiveness also for partial functions from $\mathbb{N}$ into $\mathbb{Q}$. We adopt the following definition: a partial function from $\mathbb{N}$ into $\mathbb{Q}$ is called *primitive recursive* if it is the restriction of some primitive recursive total function from $\mathbb{N}$ into $\mathbb{Q}$ to some primitive recursive subset of $\mathbb{N}$. The requirement for a partial function A from $\mathbb{N}$ into $\mathbb{Q}$ to be primitive recursive is equivalent to its representability in the form

$$A(n) = \frac{u(n) - v(n)}{w(n)}\ , \tag{3}$$

where u, v and w are primitive recursive functions from $\mathbb{N}$ into $\mathbb{N}$ and it is assumed that the domain of A is $\{n \mid w(n) \neq 0\}$.

1 Total Approximations and Localizations

Definition 1. *Let A and E be (total) functions from $\mathbb{N}$ into $\mathbb{Q}$. The pair (A, E) is called a* **total approximation** *of a given real number α if*

$$|A(n) - \alpha| \leq E(n)$$

for any n in $\mathbb{N}$ and there are numbers arbitrarily close to 0 among the values of E. The pair (A, E) is called **primitive recursive** *if both A and E are primitive recursive.*

Theorem 1. *A real number is computable if and only if it has a primitive recursive total approximation.*

Proof. Let α be a real number. If (A, E) is a primitive recursive total approximation of α then the function $s : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$s(n) = \min\left\{t \mid t \in \mathbb{N}, E(t) \leq \frac{1}{n+1}\right\}$$

is recursive and we have

$$|A(s(n)) - \alpha| \leq \frac{1}{n+1}$$

for any n in $\mathbb{N}$, hence α is computable. For the other direction of the proof suppose that α is computable. Then there is a recursive function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ that satisfies the inequality (2) for any n in $\mathbb{N}$. We shall find now a surjective primitive recursive function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $A(f(i))$ is a primitive recursive function of i (this could be done also by an easy application of the lemma from [2]). Firstly we represent the function A in the form (1) and we choose a system of four unary primitive recursive functions in $\mathbb{N}$ that enumerates the set of all quadruples of the form $(n, u(n), v(n), w(n))$, where $n \in \mathbb{N}$. Then we take as f the first one of these four functions. Clearly for all n in $\mathbb{N}$ we have the inequality

$$|A(f(i)) - \alpha| \leq \frac{1}{f(i)+1} \ ,$$

hence the pair

$$\left(\lambda i.A(f(i)), \lambda i.\frac{1}{f(i)+1}\right) \tag{4}$$

is a primitive recursive total approximation of α. □

Remark 1. The way of reasoning in the above proof can be used also in certain more complicated other situations. In the concrete situation considered here, however, a simplification is possible, namely the second part of the proof can replaced by the following somewhat shorter reasoning. Let α be a computable real number. Then consider the set of all quadruples (u, v, w, k) of natural numbers satisfying the inequality

$$\left|\frac{u-v}{w+1} - \alpha\right| < \frac{1}{k+1} \ . \tag{5}$$

This set is not empty and it is recursively enumerable.[3] Hence it can be enumerated by a system U, V, W, K of four primitive recursive functions in $\mathbb{N}$, and the corresponding pair

$$\left(\lambda n.\frac{U(n)-V(n)}{W(n)+1}, \lambda n.\frac{1}{K(n)+1}\right)$$

is a primitive recursive total approximation of α.

Suppose we have some primitive recursive total approximations (A, E) and (B, F) of two given real numbers α and β and we want to find primitive recursive total approximations of the numbers $\alpha + \beta$ and $\alpha\beta$. This turns out to be a little

[3] The recursive enumerability in question could be considered well-known. Still let us mention the following way to see it: we take a recursive function A from $\mathbb{N}$ into $\mathbb{Q}$ satisfying (2) for any n in $\mathbb{N}$, and we observe that (5) holds if and only if

$$\left|\frac{u-v}{w+1} - A(n)\right| + \frac{1}{n+1} < \frac{1}{k+1}$$

for some some n in $\mathbb{N}$.

troublesome in the general case, since, roughly speaking, the values of $E(n)$ and $F(n)$ are not obliged to become small for one and the same values of n. To overcome this problem we could use certain special kinds of total approximations.

Definition 2. *Let (A, E) be a total approximation of a given real number. We call (A, E)* **acceptable** *if the sequence $E(0), E(1), E(2), \ldots$ converges to 0,*[4] *and* **stable** *if the function E is monotonically decreasing.*[5]

As it is easily seen, if (A, E) and (B, F) are acceptable total approximations of the real numbers α and β then $(A+B, E+F)$ and $(AB, |B|E+|A|F+EF)$ are acceptable total approximations of $\alpha+\beta$ and $\alpha\beta$, respectively. Of course, if (A, E) and (B, F) are primitive recursive then so are $(A+B, E+F)$ and $(AB, |B|E+|A|F+EF)$.

Clearly any stable total approximation is acceptable. If (A, E) and (B, F) are stable primitive recursive total approximations of the real numbers α and β then the primitive recursive total approximation $(A+B, E+F)$ of the number $\alpha+\beta$ is a stable one, and the pair $(AB, (|B(0)|+F(0))E+(|A(0)|+E(0))F+EF)$ is a stable primitive recursive total approximation of the number $\alpha\beta$ (unfortunately the stability of (A, E) and (B, F) does not always guarantee the stability of the other total approximation of $\alpha\beta$ considered above).

Theorem 2. *A real number is computable if and only if it has a stable primitive recursive total approximation.*

Proof. In view of Theorem 1, it is sufficient to show that each primitive recursive total approximation of a real number can be transformed into a stable one. Let (A, E) be a primitive recursive total approximation of a real number α. We set

$$\begin{aligned} E'(n) &= \min\{E(i) \mid 0 \leq i \leq n\} , \\ k(n) &= \min\{i \mid 0 \leq i \leq n,\ E(i) = E'(n)\} , \\ A'(n) &= A(k(n)) . \end{aligned}$$

Then (A', E') is a stable primitive recursive total approximation of α. □

One often uses intervals with rational end points for the localization of a real number.

[4] This condition is not a good one from a constructive point of view. From such a point of view it would be preferable to impose the stronger requirement that $E(0), E(1), E(2), \ldots$ effectively converges to 0, i.e. to require the existence of a recursive function $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ such that $E(n) \leq 1/(k+1)$ whenever $k, n \in \mathbb{N}$ and $n \geq \nu(k)$ (such a function surely exists in the case of a stable primitive recursive total approximation considered later). A relatively simple equivalent requirement using primitive recursive functions instead of recursive ones is given in Appendix 3.

[5] Only stable primitive recursive total approximations have been considered in the preliminary version [8] of this paper, and they have been called primitive recursive approximations there.

Definition 3. *Let A_0 and A_1 be (total) functions from $\mathbb{N}$ into $\mathbb{Q}$, and let α be a real number. The pair (A_0, A_1) is called a* **total localization** *of α if*

$$A_0(n) \leq \alpha \leq A_1(n)$$

for any n in $\mathbb{N}$ and the set $\{A_1(n) - A_0(n) \mid n \in \mathbb{N}\}$ contains numbers arbitrarily close to 0.

Definition 4. *A total localization (A_0, A_1) of a given real number is called* **nested** *if the function A_0 is monotonically increasing and the function A_1 is monotonically decreasing.*[6]

The next two theorems characterize the computable real numbers in the terms of primitive recursive total localizations.

Theorem 3. *A real number is computable if and only if it has a primitive recursive total localization.*

Proof. We apply Theorem 1. If (A_0, A_1) is a primitive recursive total localization of a real number α then the corresponding pair

$$\left(\frac{A_1 + A_0}{2}, \frac{A_1 - A_0}{2}\right) \tag{6}$$

is a primitive recursive total approximation of α, hence α is computable. Conversely, if α is computable and (A, E) is a primitive recursive total approximation of α, then the corresponding pair $(A - E, A + E)$ is a primitive recursive total localization of α. □

Theorem 4. *A real number is computable if and only if it has a nested primitive recursive total localization.*

Proof. It is sufficient to show how to transform any primitive recursive total localization of a real number into a nested one. Let α be a real number and let (A_0, A_1) be a primitive recursive total localization of α. If we define functions A'_0 and A'_1 from $\mathbb{N}$ into $\mathbb{Q}$ by setting

$$A'_0(n) = \max\{A_0(i) \mid 0 \leq i \leq n\} \ , \ \ A'_1(n) = \min\{A_1(i) \mid 0 \leq i \leq n\} \ ,$$

then (A'_0, A'_1) will be a nested primitive recursive total localization of α. □

Remark 2. If α is a real number and (A_0, A_1) is a nested primitive recursive total localization of α, then the pair (6) is a stable primitive recursive total approximation of α. This can be used for proving Theorem 2 in another way.

[6] In [8] only nested primitive recursive total localizations have been considered, and no term for them has been introduced there.

Remark 3. We succeeded to characterize the computable real numbers by means of primitive recursive functions mainly because every recursively enumerable set has a primitive recursive enumeration (cf. the proof of Theorem 1 and especially Remark 1). One could replace the primitive recursive functions in the above characterizations by functions elementary in Kalmár's sense [1] or by lower elementary functions in Skolem's sense [6], and also by elementary definable ones in the sense of [7][7] (all these elementariness notions can be extended to functions from $\mathbb{N}$ into $\mathbb{Q}$ similarly to the extension of recursiveness and primitive recursiveness).

2 Partial Approximations and Localizations

The primitive recursive total approximations and total localizations of the computable real numbers have certain drawbacks. The next example illustrates this for the case of approximations, but the situation is similar also in the case of localizations.

Example 1. There is no pair of primitive recursive operators that transform each stable primitive recursive total approximation of a non-zero computable real number into a primitive recursive total approximation of its reciprocal.[8] In fact, suppose there is a pair of primitive recursive operators with this property. Let $t : \mathbb{N} \longrightarrow \mathbb{N}$ be a recursive function such that t has a primitive recursive graph, but t is not primitive recursive. For any k and n in $\mathbb{N}$ let us set

$$a_k = \frac{1}{t(k)+1}, \quad A_k(n) = \frac{1}{\min\{t(k), n\}+1}, \quad E(n) = \frac{1}{n+1} .$$

Then for any k in $\mathbb{N}$ the pair (A_k, E) is a stable primitive recursive total approximation of the number a_k. Since $A_k(n)$ is primitive recursive also as a function of both k and n, the assumption we made allows us to conclude the existence of $A'_k(n)$ and $E'_k(n)$ that are primitive recursive as functions of k and n and satisfy the condition

$$|A'_k(n) - (t(k)+1)| \leq E'_k(n)$$

for all k and n in $\mathbb{N}$.[9] In particular, we shall have

$$|A'_k(0) - (t(k)+1)| \leq E'_k(0)$$

[7] The elementary definable functions form a subclass of the lower elementary ones, but we unfortunately do not know whether the two classes are different.

[8] We skip the details concerning the notion of a primitive recursive operator acting on $\mathbb{Q}$-valued functions of natural arguments. Hopefully it would be enough to mention that a reduction to ordinary primitive recursive operators is possible through replacing the $\mathbb{Q}$-valued functions by triples of $\mathbb{N}$-valued functions in accordance with the representation (1).

[9] One may use the fact that for any primitive recursive operator its extensions in the sense of [9], § 11, Subsection 3, preserve the primitive recursiveness (actually we need the following instance of the mentioned fact: if Γ is a primitive recursive operator acting on pairs of functions from $\mathbb{N}$ into $\mathbb{Q}$ and transforming them again into such functions, then the function $\lambda kn.\Gamma(A_k, E)(n)$ is primitive recursive).

and this provides us with a primitive recursive upper bound of the function t. On the other hand, such an upper bound cannot exist due to the choice of that function.

In order to avoid the indicated drawbacks, we shall consider now partial approximations and partial localizations of the real numbers. We start with the case of approximations.

Definition 5. *Let A and E be functions from some subset D of $\mathbb{N}$ into $\mathbb{Q}$. The pair (A, E) is called a* **(partial) approximation** *of a given real number α if*

$$|A(n) - \alpha| \leq E(n)$$

for any n in D and there are numbers arbitrarily close to 0 among the values of E. The pair (A, E) is called **primitive recursive** *if both A and E are primitive recursive.*[10]

Theorem 5. *A real number is computable if and only if it has a primitive recursive approximation.*

Proof. Let α be a real number. Similarly to the proof of Theorem 1, if (A, E) is a primitive recursive approximation of α then the function $s : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$s(n) = \min\left\{t \mid t \in \mathrm{dom}(E), E(t) \leq \frac{1}{n+1}\right\}$$

is recursive and we have

$$|A(s(n)) - \alpha| \leq \frac{1}{n+1}$$

for any n in $\mathbb{N}$, hence α is computable. On the other hand, if α is computable then it has a primitive recursive total approximation (A, E) by Theorem 1, and clearly (A, E) is a primitive recursive approximation of α. $\square$

Remark 4. The application of Theorem 1 in the above proof brings a certain non-uniformity in it, since the primitive recursive total approximation constructed in the proof of Theorem 1 do not depend in a primitive recursive way on the Gödel number of the given recursive function A from $\mathbb{N}$ into $\mathbb{Q}$ satisfying (2).[11] Nevertheless, Theorem 5 can be proved without introducing the non-uniformity in question. This can be done as follows. As in the proof of Theorem 1, when we suppose that α is a computable real number, we take a recursive function A from $\mathbb{N}$ into $\mathbb{Q}$ satisfying (2), represent A in the form (1) with recursive u, v, w and consider the set of all quadruples of the form $(n, u(n), v(n), w(n))$, where $n \in \mathbb{N}$.

[10] According to the definition of primitive recursiveness for partial functions from $\mathbb{N}$ into $\mathbb{Q}$, this entails the primitive recursiveness of the set D.

[11] The absence of such a primitive recursive dependence is not a defect of the proof of the mentioned theorem and this can be shown by appropriately using the construction from Example 1.

For the new proof, we find a natural number e such that (n, x, y, z) belongs to this set if and only if some t in $\mathbb{N}$ satisfies the condition

$$T_4(e, n, x, y, z, t) ,$$

where T_4 is the primitive recursive predicate from Kleene's Normal Form Theorem for the four argument partial recursive functions (the number e can be constructed from the Gödel number of A by an appropriate application of the *s-m-n* Theorem). Let D be the set of the natural numbers i that satisfy the condition

$$T_4(e, (i)_0, (i)_1, (i)_2, (i)_3, (i)_4)$$

(the notations $(i)_j$ have their usual meaning from Recursive Function Theory). If f is the restriction of the function $\lambda i.(i)_0$ to the set D then the corresponding pair (4) is a primitive recursive approximation of α.

The next example shows the existence of a pair of primitive recursive operators that transform each primitive recursive approximation of a non-zero computable real number into a primitive recursive approximation of its reciprocal.[12]

Example 2. Let (A, E) be a primitive recursive approximation of a real number α that is distinct from 0. For all n belonging to the common domain of A and E and satisfying the condition $A(n) \neq 0$ we have

$$\left|\frac{1}{A(n)} - \frac{1}{\alpha}\right| = \frac{|\alpha - A(n)|}{|A(n)||\alpha|} \leq \frac{E(n)}{|A(n)||\alpha|} .$$

Let D be the set of the elements n of the mentioned common domain that satisfy the stronger inequality $|A(n)| - E(n) > 0$ (since $|A(n)| - E(n) \geq |\alpha| - 2E(n)$, at least the elements n with $E(n) < |\alpha|/2$ belong to D). Then for all n in D the inequality

$$\left|\frac{1}{A(n)} - \frac{1}{\alpha}\right| \leq \frac{E(n)}{|A(n)|(|A(n)| - E(n))}$$

holds. If A' and E' are, respectively, the restrictions to D of the functions

$$\lambda n.\frac{1}{A(n)}, \quad \lambda n.\frac{E(n)}{|A(n)|(|A(n)| - E(n))}$$

then (A', E') is a primitive recursive approximation of $1/\alpha$.

Similarly to the case of total approximations, we introduce the notions of acceptable and of stable approximation.

Definition 6. *Let (A, E) be an approximation of a given real number, and let D be the common domain of the functions A and E. We call this approximation* **acceptable** *if D contains all sufficiently large natural numbers and $E(n)$*

[12] The needed notion of a primitive recursive operator can be again reduced to the ordinary notion of such operator, this time using the representation (3) of the considered $\mathbb{Q}$-valued functions.

converges to 0 *when* n *tends to infinity.*[13] *The approximation* (A, E) *is called* **stable** *if the set* D *contains* $n+1$*, whenever it contains* n*, and the function* E *is monotonically decreasing.*

Again all stable approximations of a real number are acceptable approximations of it. Note that the approximation (A', E') from Example 2 is acceptable if (A, E) is acceptable, but (A', E') is not necessarily stable when (A, E) is stable.

Theorem 6. *A real number is computable if and only if it has a stable primitive recursive approximation.*

Proof. We use Theorem 5. The "if"-direction follows immediately from it. The other direction of the proof is a slight modification of the corresponding part of the proof of Theorem 2. Namely, we show that each primitive recursive approximation of a real number can be transformed into a stable one. Let (A, E) be a primitive recursive approximation of a real number α, and let D be the common domain of the functions A and E. We denote by D' the set of those numbers from $\mathbb{N}$ that belong to D or are greater than some element of D. Let the functions $E' : D' \longrightarrow \mathbb{Q}$, $k : D' \longrightarrow \mathbb{N}$ and $A' : D' \longrightarrow \mathbb{Q}$ be defined by setting

$$\begin{aligned} E'(n) &= \min\{E(i) \mid 0 \leq i \leq n,\, i \in D\} \ , \\ k(n) &= \min\{i \mid 0 \leq i \leq n,\, i \in D,\, E(i) = E'(n)\} \ , \\ A'(n) &= A(k(n)) \ . \end{aligned}$$

Then (A', E') is a stable primitive recursive approximation of α. □

Definition 7. *Let* A_0 *and* A_1 *be functions from some subset* D *of* $\mathbb{N}$ *into* $\mathbb{Q}$*, and let* α *be a real number. The pair* (A_0, A_1) *is called a* **(partial) localization** *of* α *if*

$$A_0(n) \leq \alpha \leq A_1(n)$$

for any n *in* D *and there are numbers arbitrarily close to* 0 *among the values of the function* $A_1 - A_0$*.*

Definition 8. *Let* (A_0, A_1) *be a localization of a given real number, and let* D *be the common domain of* A_0 *and* A_1*. We call this localization* **nested** *if* D *contains* $n+1$*, whenever it contains* n*, the function* A_0 *is monotonically increasing and the function* A_1 *is monotonically decreasing.*

Theorem 7. *A real number is computable if and only if it has a primitive recursive localization.*

Proof. We apply Theorem 5 in the same way as we applied Theorem 1 for the proof of Theorem 3. □

[13] An effective version of this requirement is the following one: there is a recursive function $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ such that $n \in D$ and $E(n) \leq 1/(k+1)$ whenever $k, n \in \mathbb{N}$ and $n \geq \nu(k)$.

Theorem 8. *A real number is computable if and only if it has a nested primitive recursive localization.*

Proof. It is sufficient to show how to transform any primitive recursive localization of a real number into a nested one. Let α be a real number and let (A_0, A_1) be a primitive recursive localization of α. Let D be the common domain of A_0 and A_1. We define a set D' as in the proof of Theorem 6 and then we define functions A'_0 and A'_1 from D' into $\mathbb{Q}$ by setting

$$A'_0(n) = \max\{A_0(i) \mid 0 \leq i \leq n,\, i \in D\} \ ,$$
$$A'_1(n) = \min\{A_1(i) \mid 0 \leq i \leq n,\, i \in D\} \ .$$

Then the pair (A'_0, A'_1) is a nested primitive recursive localization of α. □

3 Co-approximations

There is a way to do almost the same as with partial approximations, but without using partial functions.

Definition 9. *Let A and H be (total) functions from $\mathbb{N}$ into $\mathbb{Q}$, and let α be a real number. The pair (A, H) is called a* **co-approximation** *of α if*

$$H(n)|A(n) - \alpha| \leq 1$$

for any n in $\mathbb{N}$, all values of H are non-negative and there are arbitrarily large among them.

Definition 10. *Let (A, H) be a co-approximation of a given real number. We call (A, H)* **normal** *if all values of H belong to $\mathbb{N}$,* **acceptable** *if $H(n)$ diverges to infinity when n tends to infinity,*[14] *and* **stable** *if the function H is monotonically increasing.*[15]

Clearly all stable co-approximations of a real number are acceptable.

Theorem 9. *A real number is computable if and only if it has a primitive recursive co-approximation.*

Proof. We shall use Theorem 5. Let α be a real number. Suppose α is computable and take a primitive recursive approximation (A, E) of α. Let D be the common domain of A and E. We may assume without a loss of generality that all values of E are distinct from 0. Now denote by A' any total primitive recursive extension of A and by H the function from $\mathbb{N}$ into $\mathbb{Q}$ defined as follows:

$$H(n) = \begin{cases} 1/E(n) & \text{if } n \in D, \\ \quad 0 & \text{otherwise.} \end{cases}$$

[14] An effective version of this can be also considered, namely: there is a recursive function $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ such that $H(n) \geq k$ whenever $k, n \in \mathbb{N}$ and $n \geq \nu(k)$.

[15] Only primitive recursive stable normal co-approximations of a real number have been studied in [8] under the name primitive recursive representations of this number.

Then (A', H) is a primitive recursive co-approximation of α. Conversely, suppose α has a primitive recursive co-approximation (A', H). Let D be the set of all n in $\mathbb{N}$ such that $H(n) \neq 0$, and let A be the restriction of A' to D. Then $(A, 1/H)$ is a primitive recursive approximation of α. □

Remark 5. There is no function $H : \mathbb{N} \longrightarrow \mathbb{Q}$ such that every computable real number has a primitive recursive co-approximation with second member H (cf. Appendix 2, where an even stronger statement is proved). From here, taking into account the proof of Theorem 9, we see the non-existence of a function $E : \mathbb{N} \longrightarrow \mathbb{Q}$ such that every computable real number has a primitive recursive approximation with second member E.

Theorem 10. *A real number is computable if and only if it has a normal stable primitive recursive co-approximation.*

Proof. One direction of the proof is clear from Theorem 9. For the other direction suppose α is a computable real number. By Theorem 6, there is a stable primitive recursive approximation (A, E) of α. We may assume that all values of E are distinct from 0. Then the construction from the proof of Theorem 9 is applicable, and it is easy to see that the primitive recursive co-approximation (A', H) obtained by it is now a stable one. For any n in $\mathbb{N}$ let us set $h(n) = [H(n)]$, where $[r]$ denotes the greatest integer not exceeding r. Then the pair (A', h) is a normal stable primitive recursive co-approximation of α. □

Appendix 1

Let $t : \mathbb{N} \longrightarrow \{0, 1\}$ be a recursive function, and let

$$\alpha = \sum_{i=0}^{\infty} \frac{t(i)}{4^i} \ . \tag{7}$$

Clearly α is a computable real number. We shall show now a way of computing the values of t on the base of arbitrary sufficiently close rational approximations of α. Namely, whenever $m \in \mathbb{N}$, $r \in \mathbb{Q}$ and

$$|r - \alpha| \leq \frac{1}{4^{m+1}} \ , \tag{8}$$

the following equality holds:

$$t(m) = \left[\frac{[2 \cdot 4^m r + 1/2] \bmod 4}{2} \right] \ . \tag{9}$$

In fact, the equality

$$2\alpha = \sum_{i=0}^{\infty} \frac{2t(i)}{4^i}$$

implies that

$$[2 \cdot 4^m \alpha] \bmod 4 = 2t(m) \ .$$

On the other hand, the inequality (8) implies that

$$2 \cdot 4^m \alpha \leq 2 \cdot 4^m r + 1/2 \leq 2 \cdot 4^m \alpha + 1 \ ,$$

hence

$$[2 \cdot 4^m r + 1/2] = [2 \cdot 4^m \alpha] + d \ ,$$

where $d = 0$ or $d = 1$. From this equality and the previous one we get

$$[2 \cdot 4^m r + 1/2] \bmod 4 = 2t(m) + d \ ,$$

and from here the equality (9) follows.

Suppose now a primitive recursive function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ satisfies for any n in $\mathbb{N}$ the inequality (2). Then we can satisfy (8) by taking

$$r = A(4^{m+1} - 1) \ ,$$

and the equality (9) with this choice of r leads to the conclusion that t is primitive recursive. Hence if we construct the real number α by using a recursive function $t : \mathbb{N} \longrightarrow \{0, 1\}$ that is not primitive recursive, then there will be no primitive recursive function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ satisfying for any n in $\mathbb{N}$ the inequality (2).

Appendix 2

Let $H : \mathbb{N} \longrightarrow \mathbb{Q}$ be an unbounded non-negative primitive recursive function. We shall construct a computable real number α such that for any primitive recursive function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ the function $\lambda n.H(n)|A(n) - \alpha|$ is unbounded (the result from Appendix 1 can be obtained as a special case of this if we take $H(n) = n + 1$). For the construction of the number α we choose a ternary recursive function z in $\mathbb{N}$ such that any binary primitive recursive function in $\mathbb{N}$ can be obtained from z by substituting some constant for its first argument. Then we define a binary recursive function s and a unary recursive function t in $\mathbb{N}$ as follows:

$$s(k, m) = \min \left\{ n \mid H(n) \geq k \cdot 4^{m+1} \right\} \ , \tag{10}$$

$$t(i) = \overline{\mathrm{sg}}\, z((i)_0, i, s((i)_1, i)) \ , \tag{11}$$

where $\overline{\mathrm{sg}}\, l = 0$ for any l in $\mathbb{N} \backslash \{0\}$, $\overline{\mathrm{sg}}\, 0 = 1$. Making use of the function t, we define the real number α by means of equality (7) from Appendix 1. Suppose that for some primitive recursive function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ the function $\lambda n.H(n)|A(n) - \alpha|$ is bounded. Let k be a positive integer such that $H(n)|A(n) - \alpha| \leq k$ for all n in $\mathbb{N}$. From (10) we conclude that for any m in $\mathbb{N}$ the inequality (8) from Appendix 1 will be satisfied with $r = A(s(k, m))$, hence also the equality (9) will hold with this value of r. The last fact can be written in the form

$$t(m) = f(m, s(k, m)) \ , \tag{12}$$

if we set

$$f(m,n) = \left\lfloor \frac{[2 \cdot 4^m A(n) + 1/2] \bmod 4}{2} \right\rfloor .$$

Since the function f defined by the above equality is primitive recursive, there is a j in $\mathbb{N}$ such that

$$f(m,n) = z(j,m,n) \tag{13}$$

for all m and n in $\mathbb{N}$. From (12) and (13) we get

$$t(2^j \cdot 3^k) = z(j, 2^j \cdot 3^k, s(k, 2^j \cdot 3^k)) ,$$

and this contradicts the definition (11) of the function t.

Appendix 3

Let E be a non-negative function from $\mathbb{N}$ into $\mathbb{Q}$ (or even into the set of the real numbers). We shall show that the sequence $E(0), E(1), E(2), \ldots$ effectively converges to 0 if and only if there is a monotonically increasing unbounded primitive recursive function $h : \mathbb{N} \longrightarrow \mathbb{N}$ such that $h(n)E(n) \leq 1$ for any n in $\mathbb{N}$. One direction of the proof is obvious. For the other one suppose that $E(0), E(1), E(2), \ldots$ effectively converges to 0 and choose a recursive function $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ such that $E(n) \leq 1/(k+1)$ whenever $k, n \in \mathbb{N}$ and $n \geq \nu(k)$. Let f be a ternary primitive recursive function in $\mathbb{N}$ such that the equality $m = \nu(k)$ holds if and only if $f(m,k,i) = 0$ for some i. For any n in $\mathbb{N}$ let S_n be the set (possibly empty) of all k in $\mathbb{N}$ such that $k \leq n$ holds and some numbers m and i not exceeding n satisfy $f(m,k,i) = 0$. Then we set $h(n) = k+1$ for the greatest k in S_n if S_n is not empty, and we set $h(n) = 0$ otherwise. The function h is evidently primitive recursive. The inequality $h(n)E(n) \leq 1$ holds for any n in $\mathbb{N}$, because $(k+1)E(n) \leq 1$ whenever $f(m,k,i) = 0$ and $n \geq m$. Since S_n is a subset of S_{n+1} for any n in $\mathbb{N}$, it is clear that h is monotonically increasing. To show that h is unbounded, consider an arbitrary k in $\mathbb{N}$, set $m = \nu(k)$, consider some i satisfying $f(m,k,i) = 0$ and choose an integer n satisfying the inequalities $n \geq m$, $n \geq k$, $n \geq i$. Then $k \in S_n$, hence $h(n) \geq k+1$.

It can be seen in a similar way that for any non-negative function H defined on $\mathbb{N}$ the sequence $H(0), H(1), H(2), \ldots$ effectively diverges to infinity if and only if there is a monotonically increasing unbounded primitive recursive function $h : \mathbb{N} \longrightarrow \mathbb{N}$ such that $H(n) \geq h(n)$ for any n in $\mathbb{N}$.

Some Concluding Remarks

The authors interest in simple definitions of computability of real numbers arose from the pedagogical problem how to teach undergraduate students this notion (some of the obtained characterizations of the computable real numbers have been recently used by the author in a lecture course for such students). Hopefully the presented approach could be useful also for closer connecting Computable

Analysis with the problems of numerical computations (estimates of proximity often play a crucial role there and one looks for possibly simple approximation processes and corresponding estimates). The results formulated in Remark 5 point also to the possibility of providing the set of the computable real numbers with some hierarchy concerning the degree of their computability. There are several different ways to introduce such an hierarchy. For example, a real number β could be said to be **less or equally computable** than a real number α if the set of the second members of the primitive recursive approximations of β is a subset of the corresponding set for α (the mentioned results show that there is not a least one among the computable real numbers with respect to the quasi-ordering introduced in this way). Of course, analogues of this can be considered also with using, say, lower elementary functions instead of primitive recursive ones.

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