A Subrecursive Refinement of the Fundamental Theorem of Algebra

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Abstract. Let us call an *approximator* of a complex number α any sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ of rational complex numbers such that

$$|\gamma_t - \alpha| \leq \frac{1}{t+1}, \quad t = 0, 1, 2, \ldots$$

Denoting by $\mathbb{N}$ the set of the natural numbers, we shall call a *representation* of α any 6-tuple of functions $f_1, f_2, f_3, f_4, f_5, f_6$ from $\mathbb{N}$ into $\mathbb{N}$ such that the sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ defined by

$$\gamma_t = \frac{f_1(t) - f_2(t)}{f_3(t) + 1} + \frac{f_4(t) - f_5(t)}{f_6(t) + 1} i, \quad t = 0, 1, 2, \ldots ,$$

is an approximator of α. For any representations of the members of a finite sequence of complex numbers, the concatenation of these representations will be called a representation of the sequence in question (thus the representations of the sequence have a length equal to 6 times the length of the sequence itself). By adapting a proof given by P. C. Rosenbloom we prove the following refinement of the fundamental theorem of algebra: for any positive integer N there is a 6-tuple of computable operators belonging to the second Grzegorczyk class and transforming any representation of any sequence $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of N complex numbers into the components of some representation of some root of the corresponding polynomial $P(z) = z^N + \alpha_{N-1} z^{N-1} + \cdots + \alpha_1 z + \alpha_0$.

Keywords: Fundamental theorem of algebra, Rosenbloom's proof, computable analysis, computable operator, second Grzegorczyk class.

1 Introduction

In the paper [4] a proof is given of the fact that for any positive integer N and any complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ the polynomial

$$P(z) = z^N + \alpha_{N-1} z^{N-1} + \cdots + \alpha_1 z + \alpha_0 \tag{1}$$

has at least a root in the complex plane, and the proof is constructive in some sense. Making use of the notion of approximator considered in the abstract,

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we may describe the constructive character of the proof as follows: the proof shows implicitly (after some small changes) that for any positive integer N there is a computable procedure for transforming any approximators of any $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ into some approximator of some root of the corresponding polynomial $P(z)$.[1] Clearly the following more rigorous formulation of this can be given, where F consists of all total mappings of $\mathbb{N}$ into $\mathbb{N}$: for any positive integer N there are recursive operators $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ with domain F^{6N} such that whenever an element $\bar{f}$ of F^{6N} is a representation of some N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, then $\Gamma_k(\bar{f}),\ k = 1, 2, 3, 4, 5, 6$, belong to F and form a representation of some root of the corresponding polynomial $P(z)$.[2]

The present paper is devoted to the fact that one can replace the words "recursive operators" in the above formulation by "computable operators belonging to the second Grzegorczyk class" (the fact was established in the first author's master thesis [3] written under the supervision of the second author).

2 The Notion of Computable Operator of the Second Grzegorczyk Class

For any natural number k let F_k be the set of all total k-argument functions in the set $\mathbb{N}$ (thus $F_1 = F$). For any natural numbers n and k we shall consider operators acting from F^n into F_k. The ones among them that are computable operators of the second Grzegorczyk class will be called $\mathcal{E}^2$-*computable operators* for short. The class of these operators can be defined by means of a natural extension of a definition of the class of functions $\mathcal{E}^2$ from the hierarchy introduced in [1] (such a step would be similar to the extension in [2] of the definition of $\mathcal{E}^3$ by introducing the notion of elementary recursive functional). Roughly speaking, we can use the same initial functions and the same ways of construction of new functions as in the definition of $\mathcal{E}^2$, except that we must add to the initial functions also the function arguments of the operator and to consider only constructions that are uniform with respect to these arguments. Skipping the details of the definition[3], we note the following properties of the $\mathcal{E}^2$-computable operators, where $\bar{f}$ is used as an abbreviation for the n-tuple $f_1, \ldots, f_n$ of functions from F.

1. *For any k-argument function g belonging to the class $\mathcal{E}^2$ the mapping $\lambda\bar{f}.g$ of F^n into F_k is an $\mathcal{E}^2$-computable operator.*
2. *The mappings $\lambda\bar{f}.f_j$, $j = 1, \ldots, n$, of F^n into F are $\mathcal{E}^2$-computable operators.*

[1] By certain continuity reasons, a dependence of this root not only on the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$, but also on the choice of their approximators, cannot be excluded in the case of $N > 1$.

[2] This statement holds also for a more usual notion of approximator based on the inequality $|\gamma_t - \alpha| \le 2^{-t}$ instead of the inequality $|\gamma_t - \alpha| \le \frac{1}{t+1}$ (cf. for example the approach to computable analysis by Cauchy representations in [6]). However, the main result of the present paper would be not valid in that case, as it can be seen by means of an easy application of Liouville's approximation theorem.

[3] See, however, the remark on the next page.

3. *Whenever* Γ_0 *is an* $\mathcal{E}^2$*-computable operator from* F^n *into* F_m, $\Gamma_1, \ldots, \Gamma_m$ *are* $\mathcal{E}^2$*-computable operators from* F^n *into* F_k, *and* Γ *is the mapping of* F^n *into* F_k *defined by*
$$\Gamma(\bar{f})(x_1, \ldots, x_k) = \Gamma_0(\bar{f})(\Gamma_1(\bar{f})(x_1, \ldots, x_k), \ldots, \Gamma_m(\bar{f})(x_1, \ldots, x_k)),$$
then Γ *is also an* $\mathcal{E}^2$*-computable operator.*
4. *Whenever* Γ_0 *is an* $\mathcal{E}^2$*-computable operator from* F^n *into* F_m, Γ_1 *is an* $\mathcal{E}^2$*-computable operator from* F^n *into* F_{m+2}, Γ_2 *is an* $\mathcal{E}^2$*-computable operator from* F^n *into* F_{m+1}, *the mapping* Γ *of* F^n *into* F_{m+1} *is defined by*
$$\Gamma(\bar{f})(0, x_1, \ldots, x_m) = \Gamma_0(\bar{f})(x_1, \ldots, x_m),$$
$$\Gamma(\bar{f})(t+1, x_1, \ldots, x_m) = \Gamma_1(\bar{f})(\Gamma(\bar{f})(t, x_1, \ldots, x_m), t, x_1, \ldots, x_m),$$
and for all $\bar{f}, t, x_1, \ldots, x_m$ *the inequality*
$$\Gamma(\bar{f})(t, x_1, \ldots, x_m) \le \Gamma_2(\bar{f})(t, x_1, \ldots, x_m)$$
holds, then Γ *is also an* $\mathcal{E}^2$*-computable operator.*
5. *Whenever* Γ_0 *is an* $\mathcal{E}^2$*-computable operator from* F^n *into* F_{m+1}, *and the mapping* Γ *of* F^n *into* F_{m+1} *is defined by*
$$\Gamma(\bar{f})(t, x_1, \ldots, x_m) = \min\{\, s \mid s = t \vee \Gamma_0(\bar{f})(s, x_1, \ldots, x_m) = 0 \,\},$$
then Γ *is also an* $\mathcal{E}^2$*-computable operator.*
6. *Whenever* Γ_0 *is an* $\mathcal{E}^2$*-computable operator from* F^m *into* F_k, $\Gamma_1, \ldots, \Gamma_m$ *are* $\mathcal{E}^2$*-computable operators from* F^n *into* F_{l+1}, *and* Γ *is the mapping of* F^n *into* F_{k+l} *defined by*
$$\Gamma(\bar{f})(x_1, \ldots, x_k, y_1, \ldots, y_l) = \Gamma_0(\lambda t.\Gamma_1(\bar{f})(y_1, \ldots, y_l, t), \ldots, \lambda t.\Gamma_m(\bar{f})(y_1, \ldots, y_l, t))(x_1, \ldots, x_k),$$
then Γ *is also an* $\mathcal{E}^2$*-computable operator.*
7. *If* Γ *is an* $\mathcal{E}^2$*-computable operator from* F^n *into* F_k, *and the functions* $f_1, \ldots, f_n$ *belong to Grzegorczyk class* $\mathcal{E}^m$, *where* $m \ge 2$, *then the function* $\Gamma(\bar{f})$ *also belongs to* $\mathcal{E}^m$.

Remark. The properties 1–4 can be used as the clauses of an inductive definition of the notion of $\mathcal{E}^2$-computable operator. Moreover, in such a case one can reduce the property 1 to its instances when g is the function $\lambda xy.\,(x+1)\cdot(y+1)$ or some of the functions $\lambda x_1 \ldots x_k.\, x_j$, $j = 1, \ldots, k$. In order to eliminate the not effectively verifiable domination requirement in the clause corresponding to property 4, one could omit this requirement and replace the right-hand side of the second equality by the expression
$$\min\{\Gamma_1(\bar{f})(\Gamma(\bar{f})(t, x_1, \ldots, x_m), t, x_1, \ldots, x_m), \Gamma_2(\bar{f})(t, x_1, \ldots, x_m)\}.$$

3 $\mathcal{E}^2$-Computable Functions in the Set of the Complex Numbers

Let $\mathbb{C}$ be the set of the complex numbers. A function φ from $\mathbb{C}^N$ into $\mathbb{C}$ will be called *$\mathcal{E}^2$-computable* if six $\mathcal{E}^2$-computable operators $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ from F^{6N} into F exist such that, whenever an element $\bar{f}$ of F^{6N} is a representation of some N-tuple $\zeta_1, \ldots, \zeta_N$ of complex numbers, then the corresponding 6-tuple $\Gamma_1(\bar{f}), \Gamma_2(\bar{f}), \Gamma_3(\bar{f}), \Gamma_4(\bar{f}), \Gamma_5(\bar{f}), \Gamma_6(\bar{f})$ is a representation of the complex number $\varphi(\zeta_1, \ldots, \zeta_N)$. The following properties are evident or easily provable.[4]

1. *All functions φ from $\mathbb{C}^N$ into $\mathbb{C}$ that have the form $\varphi(z_1, \ldots, z_N) = z_j$ with $j \in \{1, \ldots, N\}$ are $\mathcal{E}^2$-computable.*
2. *For any rational complex number γ the constant function $\varphi(z_1, \ldots, z_N) = \gamma$ is $\mathcal{E}^2$-computable.*
3. *The functions $\lambda z.\, \bar{z}$, $\lambda z_1 z_2.\, z_1 + z_2$ and $\lambda z_1 z_2.\, z_1 \cdot z_2$ are $\mathcal{E}^2$-computable.*
4. *If φ is an $\mathcal{E}^2$-computable function from $\mathbb{C}^m$ into $\mathbb{C}$, and $\psi_1, \ldots, \psi_m$ are $\mathcal{E}^2$-computable function from $\mathbb{C}^N$ into $\mathbb{C}$ then the function θ defined by*
$$\theta(z_1, \ldots, z_N) = \varphi(\psi_1(z_1, \ldots, z_N), \ldots, \psi_m(z_1, \ldots, z_N))$$
is also $\mathcal{E}^2$-computable.
5. *The real-valued function $\lambda z.\, |z|^2$ is $\mathcal{E}^2$-computable.*
6. *For any given positive integer N, the value of a polynomial $P(z)$ of the form (1) as well as the corresponding value of $|P(z)|^2$ are $\mathcal{E}^2$-computable functions of the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ and the argument z.*
7. *For any given positive integer N, if $P(z)$ is an arbitrary polynomial of the form (1), and α is an arbitrary complex number, then the coefficients of the polynomial that is the quotient of $P(z) - P(\alpha)$ and $z - \alpha$ are $\mathcal{E}^2$-computable functions of $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ and α.*
8. *If φ is an $\mathcal{E}^2$-computable function from $\mathbb{C}^{N+1}$ into $\mathbb{C}$, then there are $\mathcal{E}^2$-computable operators $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ from F^{6N} into F_7 such that, whenever an element $\bar{f}$ of F^{6N} is a representation of some N-tuple $\zeta_1, \ldots, \zeta_N$ of complex numbers, then for any natural numbers $y_1, y_2, y_3, y_4, y_5, y_6$ the 6-tuple of the functions*
$$\lambda u.\, \Gamma_j(\bar{f})(u, y_1, y_2, y_3, y_4, y_5, y_6), \quad j = 1, 2, 3, 4, 5, 6,$$
is a representation of the complex number
$$\varphi\left(\zeta_1, \ldots, \zeta_N, \frac{y_1 - y_2}{y_3 + 1} + \frac{y_4 - y_5}{y_6 + 1} i\right).$$

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[4] The property 5 can be derived from the properties 3, 4 and the equality $|z|^2 = z \cdot \bar{z}$. The properties 3, 4 and 5 imply the property 6, and it implies the property 7. The proof of the property 8 makes use of property 6 from section 2 (with $k = 1$, $l = 6$) and of the fact that for any natural numbers $y_1, y_2, y_3, y_4, y_5, y_6$ the 6-tuple of the constant functions $\lambda t.\, y_1, \lambda t.\, y_2, \lambda t.\, y_3, \lambda t.\, y_4, \lambda t.\, y_5, \lambda t.\, y_6$ is a representation of the rational complex number
$$\frac{y_1 - y_2}{y_3 + 1} + \frac{y_4 - y_5}{y_6 + 1} i.$$

Remark. Except for the case of $N = 1$, there is no $\mathcal{E}^2$-computable function φ from $\mathbb{C}^N$ into $\mathbb{C}$ such that for any complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ the number $\varphi(\alpha_0, \alpha_1, \ldots, \alpha_{N-1})$ is a root of the corresponding polynomial (1). This follows from the non-existence of a continuous function with such a property.

4 On Rosenbloom's Proof of the Fundamental Theorem of Algebra

P. C. Rosenbloom's proof in [4] of the fundamental theorem of algebra makes use of an analogue of Cauchy's theorem from the theory of analytic functions. In its complete form the result obtained in the proof of this analogue can be formulated as follows (see Lemma 2, Theorems 1, 2 and Corollary 1 in [4]).

Theorem R. *Let N be a positive integer, $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ be complex numbers, $P(z)$ be the corresponding polynomial (1), and ε be a positive real number. If*

$$A = \max\{|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{N-1}|, 1\}, \quad \gamma = \binom{N+1}{[(N+1)/2]},$$

a is a real number not less than $5NA$, $K = 2^{(3N/2)+6}\gamma^3 A^3 a^{3N+3}$, and n is an integer greater than K/ε^3, then

$$\left|P\left(\frac{(u+vi)a}{n}\right)\right| < \varepsilon$$

for some integers u and v with $|u| \le n$, $|v| \le n$.

The further presentation in [4] goes through the following statement (its formulation here coincides with the original one up to inessential details).

Lemma 3. *Let N be a positive integer, $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ be complex numbers, $P(z)$ be the corresponding polynomial (1), and ε be a positive real number less than 1. Then we can find points $z_1, \ldots, z_N$ such that*

$$|P(z_j)| < \varepsilon, \quad j = 1, \ldots, N,$$

and such that if $|P(z)| < \delta$, where $\varepsilon \le \delta < 1$, then

$$\min_{1 \le j \le N} |z_j - z| < 2\delta^{1/2^N}.$$

The proof of the lemma in the paper (after the elimination of a small problem[5]) can be adapted to the needs of the present paper. However, a strengthening of the lemma is possible that is more convenient for us, namely by adding

[5] The problem is in the induction used for actually proving a strengthening of the lemma with the factor $2^{1-1/2^N}$ in place of 2 in the last inequality. Namely the inequality $(2\delta)^{1/2} < 1$ is needed for being able to use the inductive hypothesis at the final step, but the assumption $\delta < 1$ is not sufficient for the truth of this inequality. Fortunately, as the first author observed, this problem can be eliminated by replacing the inequality $\delta < 1$ in the formulation of the lemma with the inequality $\delta < 2$.

the words "with rational coordinates" after the phrase "we can find points $z_1, \dots, z_N$". We shall prove constructively even the slightly stronger statement with $2\delta^{1/2^{N-1}}$ instead of $2\delta^{1/2^N}$.

LEMMA 3$'$. *Let N be a positive integer, $\alpha_0, \alpha_1, \dots, \alpha_{N-1}$ be complex numbers, $P(z)$ be the corresponding polynomial (1), and ε be a positive real number less than 1. Then we can find rational complex numbers $z_1, \dots, z_N$ such that $|P(z_j)| < \varepsilon$, $j = 1, \dots, N$, and such that if $|P(z)| < \delta$, where $\varepsilon \le \delta < 1$, then*

$$\min_{1 \le j \le N} |z_j - z| < 2\delta^{1/2^{N-1}}.$$

Proof. Our reasoning will be very close to the proof of Lemma 3 in [4]. We see as there that all complex numbers z with $|P(z)| \le 1$ satisfy the inequality

$$|z| < 1 + N \max_{0 \le k < N} |\alpha_k|.$$

If $N = 1$ then we take a rational complex number z_1 such that $|z_1 + \alpha_0| < \varepsilon$. Clearly $|P(z_1)| < \varepsilon$, and if $|P(z)| < \delta$, where $\varepsilon \le \delta < 1$, then

$$|z_1 - z| \le |z_1 + \alpha_0| + |z + \alpha_0| < \varepsilon + \delta \le 2\delta = 2\delta^{1/2^{N-1}}$$

Suppose now $N > 1$, and the statement of Lemma 3$'$ is true for $N - 1$. Let (as in the original proof) $\varepsilon_1 = \varepsilon / C$, where $C = 3 + NA + (N-1)NA(1+NA)^{N-1}$, and $A = \max\{|\alpha_0|, |\alpha_1|, \dots, |\alpha_{N-1}|, 1\}$. Clearly $\varepsilon_1 < \varepsilon$. By Theorem R (applied with some rational number a) we find a rational complex number z_1 such that $|P(z_1)| < \varepsilon_1$, hence $|P(z_1)| < \varepsilon < 1$, and therefore $|z_1| < 1 + NA$. Now

$$P(z) = P(z_1) + (z - z_1)P_1(z),$$

where

$$P_1(z) = \sum_{m=0}^{N-1} \beta_m z^m, \quad |\beta_m| = \left| \sum_{k=m+1}^{N} \alpha_k z_1^{k-m-1} \right| < NA(1+NA)^{N-1}.$$

By the inductive assumption we can find rational complex numbers $z_2, \dots, z_N$ such that $|P_1(z_j)| < \varepsilon_1, \ j = 2, \dots, N$, and

$$\min_{2 \le j \le N} |z_j - z| < 2\delta^{1/2^{N-2}},$$

whenever $|P_1(z)| < \delta$ and $\varepsilon_1 \le \delta < 1$. If j is any of the numbers $2, \dots, N$, then $|P_1(z_j)| < 1$, hence $|z_j| < 1 + (N-1)NA(1+NA)^{N-1}$, and therefore

$$|P(z_j)| \le |P(z_1)| + (|z_j| + |z_1|)|P_1(z_j)| < \varepsilon_1 + (|z_j| + |z_1|)\varepsilon_1 < C\varepsilon_1 = \varepsilon.$$

Now let $|P(z)| < \delta$, where $\varepsilon \le \delta < 1$. Then

$$|z_1 - z||P_1(z)| = |P(z_1) - P(z)| \le |P(z_1)| + |P(z)| < \varepsilon + \delta \le 2\delta \le 2\delta^{1/2^{N-1}}\delta^{1/2},$$

hence $|z_1 - z| < 2\delta^{1/2^{N-1}}$ or $|P_1(z)| < \delta^{1/2}$. In the case of $|P_1(z)| < \delta^{1/2}$ we have the inequality

$$\min_{2\le j\le N} |z_j - z| < 2(\delta^{1/2})^{1/2^{N-2}} = 2\delta^{1/2^{N-1}}$$

since $\varepsilon_1 < \delta < \delta^{1/2} < 1$. Therefore in both cases

$$\min_{1\le j\le N} |z_j - z| < 2\delta^{1/2^{N-1}}. \qquad \square$$

The concluding part of Rosenbloom's proof is in his Theorem 3. Making use of Lemma 3′ instead of Lemma 3, we can strengthen Theorem 3 by constructing a sequence of rational complex numbers converging to a root of the polynomial. In the original proof Lemma 3 is applied with values of ε of the form 2^{-n2^N}, $n = 1, 2, \ldots$, and this leads to the inequality $|z_{n+1} - z_n| < 2^{1-n}$ for the members of the constructed sequence $z_1, z_2, \ldots$ Of course this can be done also through Lemma 3′, and the rate of the convergence is quite good. Unfortunately the exponential dependence of 2^{-n2^N} on n is an obstacle to realize such a construction of the sequence by means of $\mathcal{E}^2$-computable operators. Therefore it is appropriate to change the construction. Namely an inequality $|z_{n+1} - z_n| < (n+1)^{-2}$ would still give an admissible rate of convergence, and this inequality can be achieved by using values of ε of the form $2^{-2^{N-1}}(n+1)^{-2^N}$, $n = 1, 2, \ldots$ (since these values are used also as values of δ when $n+1$ is considered instead of n, and we have $2\delta^{1/2^{N-1}} = (n+1)^{-2}$ for $\delta = 2^{-2^{N-1}}(n+1)^{-2^N}$).

5 Construction of the Needed $\mathcal{E}^2$-Computable Operators

We shall first formulate three theorems, and then we shall sketch their proofs. The first two of these theorems (corresponding to Theorem R and to Lemma 3′ from the previous section) describe the major preliminary steps in the construction of the $\mathcal{E}^2$-computable operators needed to get the promised refinement of the fundamental theorem of algebra. The third theorem is the refinement itself.

Theorem 1. *For any positive integer N there are $\mathcal{E}^2$-computable operators $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ from F^{6N} into F such that, whenever an element $\bar{f}$ of F^{6N} is a representation of an N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, and $P(z)$ is the polynomial (1) corresponding to this N-tuple, then*

$$\left| P\left(\frac{\Gamma_1(\bar{f})(t) - \Gamma_2(\bar{f})(t)}{\Gamma_3(\bar{f})(t) + 1} + \frac{\Gamma_4(\bar{f})(t) - \Gamma_5(\bar{f})(t)}{\Gamma_6(\bar{f})(t) + 1} i \right) \right| < \frac{1}{t+1}, \quad t = 0, 1, 2, \ldots$$

Theorem 2. *For any positive integer N there are $\mathcal{E}^2$-computable operators $\Gamma_{1j}, \Gamma_{2j}, \Gamma_{3j}, \Gamma_{4j}, \Gamma_{5j}, \Gamma_{6j}$, $j = 1, 2, \ldots, N$, from F^{6N} into F such that, whenever an element $\bar{f}$ of F^{6N} is a representation of an N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, and $P(z)$ is the polynomial (1) corresponding to this N-tuple, then for any natural number t and*

$$z_j = \frac{\Gamma_{1j}(\bar{f})(t) - \Gamma_{2j}(\bar{f})(t)}{\Gamma_{3j}(\bar{f})(t) + 1} + \frac{\Gamma_{4j}(\bar{f})(t) - \Gamma_{5j}(\bar{f})(t)}{\Gamma_{6j}(\bar{f})(t) + 1} i, \quad j = 1, 2, \ldots, N,$$

the inequalities $|P(z_j)| < (t+1)^{-1}$, $j = 1, 2, \ldots, N$, *hold, and*

$$\min_{1\le j\le N} |z_j - z| < 2\delta^{1/2^{N-1}}$$

for all δ *and* z *satisfying the inequalities* $(t+1)^{-1} \le \delta < 1$, $|P(z)| < \delta$.

THEOREM 3. *For any positive integer* N *there are* $\mathcal{E}^2$*-computable operators* $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ *from* F^{6N} *into* F *such that, whenever an element* $\bar{f}$ *of* F^{6N} *is a representation of an* N*-tuple* $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ *of complex numbers, and* $P(z)$ *is the polynomial (1) corresponding to this* N*-tuple, then the 6-tuple of the functions* $\Gamma_1(\bar{f}), \Gamma_2(\bar{f}), \Gamma_3(\bar{f}), \Gamma_4(\bar{f}), \Gamma_5(\bar{f}), \Gamma_6(\bar{f})$ *is a representation of some root of* $P(z)$.

The proof of Theorem 1 is based on the statement of Theorem R and does not use any details from its proof. Let N be a positive integer. One easily constructs $\mathcal{E}^2$-computable operators Δ_1 and Δ_2 from F^{6N} into F_0 such that, whenever an element $\bar{f}$ of F^{6N} is a representation of an N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, and $P(z)$ is the polynomial (1) corresponding to this N-tuple, the natural number $\Delta_1(\bar{f})$ is not less than the number $5NA$ from Theorem R for the given numbers $N, \alpha_0, \alpha_1, \ldots, \alpha_{N-1}$, and the natural number $\Delta_2(\bar{f})$ is not less than the number K for the given numbers $N, \alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ and for $a = \Delta_1(\bar{f})$. In this situation, if t is an arbitrary natural number then an application of Theorem R with $\varepsilon = \frac{1}{2(t+1)}$, $a = \Delta_1(\bar{f})$, $n = 8(t+1)^3\Delta_2(\bar{f}) + 1$ and with the substitution $u = r - n$, $v = s - n$ allows concluding that

$$\left| P\left(\frac{r\Delta_1(\bar{f}) - \Delta_1'(\bar{f})(t)}{\Delta_2'(\bar{f})(t) + 1} + \frac{s\Delta_1(\bar{f}) - \Delta_1'(\bar{f})(t)}{\Delta_2'(\bar{f})(t) + 1} i \right) \right|^2 < \frac{1}{4(t+1)^2} \tag{2}$$

for some natural numbers r and s not greater than $\Delta_2'(\bar{f})(t)$, where Δ_1' and Δ_2' are the mappings of F^{6N} into F defined by

$$\Delta_2'(\bar{f})(t) = 8(t+1)^3\Delta_2(\bar{f}), \quad \Delta_1'(\bar{f})(t) = (\Delta_2'(\bar{f})(t) + 1)\Delta_1(\bar{f})$$

(clearly Δ_1' and Δ_2' are also $\mathcal{E}^2$-computable operators). By the properties 6 and 8 from Section 3, there are $\mathcal{E}^2$-computable operators Γ and Δ from F^{6N} to F_4 such that, whenever an element $\bar{f}$ of F^{6N} is a representation of an N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, and $P(z)$ is the polynomial (1) corresponding to this N-tuple, then the absolute value of the difference

$$\frac{\Gamma(\bar{f})(u, r, s, t)}{\Delta(\bar{f})(u, r, s, t) + 1} - \left| P\left(\frac{r\Delta_1(\bar{f}) - \Delta_1'(\bar{f})(t)}{\Delta_2'(\bar{f})(t) + 1} + \frac{s\Delta_1(\bar{f}) - \Delta_1'(\bar{f})(t)}{\Delta_2'(\bar{f})(t) + 1} i \right) \right|^2$$

is not greater than $(u+1)^{-1}$ for any r, s, t, u in $\mathbb{N}$. With $u = 4(t+1)^2 - 1$ we get that

$$\frac{\Gamma(\bar{f})(4(t+1)^2 - 1, r, s, t)}{\Delta(\bar{f})(4(t+1)^2 - 1, r, s, t)) + 1} < \frac{1}{2(t+1)^2} \tag{3}$$

for all natural numbers r, s, t satisfying the inequality (2), and

$$\left|P\left(\frac{r\Delta_1(\bar{f}) - \Delta_1'(\bar{f})(t)}{\Delta_2'(\bar{f})(t)+1} + \frac{s\Delta_1(\bar{f}) - \Delta_1'(\bar{f})(t)}{\Delta_2'(\bar{f})(t)+1}i\right)\right|^2 < \frac{3}{4(t+1)^2} < \frac{1}{(t+1)^2}$$

for any r, s, t in $\mathbb{N}$ that satisfy (3). It is a routine work (making use of the property 5 from Section 2) to construct two $\mathcal{E}^2$-computable operators Δ_3 and Δ_4 from F^{6N} to F such that $\Delta_3(\bar{f})$ and $\Delta_4(\bar{f})$ transform any natural number t into natural numbers r and s not greater than $\Delta_2'(\bar{f})(t)$ and satisfying (3), whenever such r and s exist. Then Theorem 1 will hold with $\Gamma_2 = \Gamma_5 = \Delta_1'$, $\Gamma_3 = \Gamma_6 = \Delta_2'$ and $\Gamma_1(\bar{f})(t) = \Delta_3(\bar{f})(t)\Delta_1(\bar{f})$, $\Gamma_4(\bar{f})(t) = \Delta_4(\bar{f})(t)\Delta_1(\bar{f})$.

The proof of Theorem 2 is actually an operator refinement of the one of Lemma $3'$ and follows closely it. In the case of $N = 1$ we, roughly speaking, use rational approximations of the number $-\alpha_0$ that can be constructed by means of the representation $\bar{f}$ of α_0. For the inductive step, we suppose the existence of the needed $6(N-1)$-tuple of $\mathcal{E}^2$-computable operators for the case of polynomials of degree $N-1$, and use them to construct the needed $6N$-tuple of ones for polynomials of degree N, making use also of the $\mathcal{E}^2$-computable operators from Theorem 1 for this case and of the properties 7 and 8 from Section 3.

Of course the operators $\Gamma_{1j}, \Gamma_{2j}, \Gamma_{3j}, \Gamma_{4j}, \Gamma_{5j}, \Gamma_{6j}$, $j = 1, 2, \ldots, N$, from Theorem 2 are used in the proof of Theorem 3. For any element $\bar{f}$ of F^{6N} and any natural number n we set

$$\gamma_{n,j}^{\bar{f}} = \frac{\Gamma_{1j}(\bar{f})(t_n) - \Gamma_{2j}(\bar{f})(t_n)}{\Gamma_{3j}(\bar{f})(t_n)+1} + \frac{\Gamma_{4j}(\bar{f})(t_n) - \Gamma_{5j}(\bar{f})(t_n)}{\Gamma_{6j}(\bar{f})(t_n)+1}i, \quad j = 1, 2, \ldots N, \quad (4)$$

where $t_n = 2^{2^{N-1}}(n+1)^{2^N} - 1$. Then, for any element $\bar{f}$ of F^{6N}, we define a sequence $j_0^{\bar{f}}, j_1^{\bar{f}}, j_2^{\bar{f}}, \ldots$ of integers from the set $\{1, 2, \ldots, N\}$ in the following recursive way: we set $j_0^{\bar{f}} = 1$ and, whenever $j_n^{\bar{f}}$ is already defined, we set $j_{n+1}^{\bar{f}}$ to be the first $j \in \{1, 2, \ldots, N\}$ such that $j = N$ or $\left|\gamma_{n+1,j}^{\bar{f}} - \gamma_{n,j_n^{\bar{f}}}^{\bar{f}}\right| < (n+1)^{-2}$. Finally, we set

$$\Gamma_k(\bar{f})(n) = \Gamma_{kj_n^{\bar{f}}}(\bar{f})(t_n), \quad k = 1, 2, 3, 4, 5, 6, \ n = 0, 1, 2, \ldots \quad (5)$$

The $\mathcal{E}^2$-computability of the constructed operators is easily verifiable, thus it remains only to prove the other property formulated in Theorem 3. Let an element $\bar{f}$ of F^{6N} be a representation of an N-tuple $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of complex numbers, and $P(z)$ be the polynomial (1) corresponding to this N-tuple. Then for any natural number n the inequalities $|P(\gamma_{n,j}^{\bar{f}})| < (t_n+1)^{-1}$, $j = 1, 2, \ldots, N$, hold, and whenever $|P(z)| < \delta$, where $(t_{n+1}+1)^{-1} \leq \delta < 1$, then

$$\min_{1 \leq j \leq N} |\gamma_{n+1,j}^{\bar{f}} - z| < 2\delta^{1/2^{N-1}}.$$

By the first part of this statement $\lim_{n\to\infty} P(\gamma_{n,j_n^{\bar{f}}}^{\bar{f}}) = 0$. Applying the second one with $z = \gamma_{n,j_n^{\bar{f}}}^{\bar{f}}$, $\delta = (t_n+1)^{-1}$, we see that $|\gamma_{n+1,j_{n+1}^{\bar{f}}}^{\bar{f}} - \gamma_{n,j_n^{\bar{f}}}^{\bar{f}}| < (n+1)^{-2}$, since

$2\delta^{1/2^{N-1}} = (n+1)^{-2}$ for this value of δ. To complete the proof, it is sufficient to use the equality (4) with $j = j_n^{\bar{f}}$, as well as the equality (5) and the inequality $(n+1)^{-2} + (n+2)^{-2} + \cdots + (n+p)^{-2} \leq (n+1)^{-1}$.

6 Some Corollaries from Theorem 3

By induction on N one easily proves

Corollary 1. *For any positive integer N there are $\mathcal{E}^2$-computable operators $\Gamma_{1j}, \Gamma_{2j}, \Gamma_{3j}, \Gamma_{4j}, \Gamma_{5j}, \Gamma_{6j},\ j = 1, 2, \ldots, N$, from F^{6N} into F such that, whenever an element $\bar{f}$ of F^{6N} is a representation of an N-tuple of complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$, and $P(z)$ is the polynomial (1) corresponding to this N-tuple, then the N-tuples $\Gamma_{1j}(\bar{f}), \Gamma_{2j}(\bar{f}), \Gamma_{3j}(\bar{f}), \Gamma_{4j}(\bar{f}), \Gamma_{5j}(\bar{f}), \Gamma_{6j}(\bar{f}),\ j = 1, 2, \ldots, N$, are representations of some complex numbers $z_1, z_2, \ldots, z_N$ with the property that for all z the equality $P(z) = (z - z_1)(z - z_2) \cdots (z - z_N)$* holds.

This implies the following statement (derivable also from Theorem 2.5 of [5]).

Corollary 2. *If an N-tuple of complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ has a representation consisting of functions from Grzegorczyk class $\mathcal{E}^m$, where $m \geq 2$, then any root of the corresponding polynomial (1) has a representation consisting of functions from the same class $\mathcal{E}^m$.*

Remark. Neither of the indicated two proofs of Corollary 2 yields an interpretation of the existential statement in the conclusion via recursive operators using as input an $\mathcal{E}^m$-representation of the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ and an arbitrary representation of the considered root. The existence of such operators (even of $\mathcal{E}^2$-computable ones) was additionally shown by the second author.

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