For details see Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys., 18 (1969), p. 161-166.

Montel spaces of Hölder continuous functions

by

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A continuous function $f: [0, 1] \to \mathbf{R}$ is *α-Hölder-continuous* ($0 < \alpha \leqslant 1$) if

$$\|f\|_\alpha := \sup \left\{ \frac{|f(s)-f(t)|}{|s-t|^\alpha}, \; s, t \in [0, 1] \right\} < \infty.$$

The space $C_\alpha[0, 1]$ of all functions f with $f(0) = 0$ and $\|f\|_\alpha < \infty$ is a Banach space with the norm $\| \; \|_\alpha$. Pełczyński and Ciesielski have shown, that the space $C_\alpha[0, 1]$ is isomorphic to l^∞. Let $\alpha \in (0, 1]$ and

$$H_{\alpha-}[0, 1] = \operatorname*{proj}_{\beta<\alpha} C_\beta[0, 1] = \bigcap_{\beta<\alpha} C_\beta[0, 1].$$

THEOREM 1. *The space $H_{\alpha-}[0, 1]$ is a Montel space.*

Let (s_α) be the following sequence space:

$$(s_\alpha) = \{\xi : p_\beta(\xi) = \sup |\xi_n| n^\beta < \infty, \; \beta < \alpha\}.$$

(s_α) is a non-nuclear Montel space.

THEOREM 2. *The spaces $H_{\alpha-}[0, 1]$ and (s_α) are isomorphic.*

THEOREM 3. *The functions φ_n,*

$$\varphi_n(t) = \int_0^t \chi_n(\tau) \, d\tau,$$

$\{\chi_n\}$ the Haar system, form an unconditional basis in $H_{\alpha-}[0, 1]$.

Let $\alpha \in [0, 1]$ and

$$H_{\alpha+}[0, 1] = \operatorname*{ind}_{\beta>\alpha} C_\beta[0, 1] = \bigcup_{\beta>\alpha} C_\beta[0, 1].$$

THEOREM 4. *The space $H_{\alpha+}[0, 1]$ is a Montel space.*

Let $l_n^\infty = \{\xi : \sup |\xi_n| n^\beta < \infty\}$ and $(\sigma_\alpha) = \bigcup_{\beta>\alpha} l_\beta^\infty$.

(σ_α) is a non-nuclear Montel space.

THEOREM 5. *The spaces $H_{\alpha+}[0, 1]$ and (σ_α) are isomorphic.*

THEOREM 6. *The functions φ_n form an unconditional basis in $H_{\alpha+}[0, 1]$.*

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Linear operators with sufficiently many a priori given eigenvectors

by

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Let X be a real Banach space and let T be a Hausdorff locally convex topology in X. The unit ball S of X is assumed to be T-bounded and T-closed.

Let M be a set of non-zero elements of X such that the T-closed convex hull of $M \cup (-M)$ is equal to S. We denote by $\mathscr{A}(X, T, M)$ the set of all linear mappings A of X into X which satisfy the following two conditions:

(a) All the elements of M are eigenvectors of A.

(b) The restriction of A on S is T-continuous.

The set $\mathscr{A}(X, T, M)$ is not empty: at least the operators A which have the form $Ax = \lambda x$ (with constant real λ) belong to it. More interesting examples can be found in [2] and in other papers of Tagamlitzki.

Let $\mathscr{L}(X)$ be the Banach algebra of all bounded linear mappings of the Banach space X into itself.

THEOREM I. *$\mathscr{A}(X, T, M)$ is a closed subalgebra of $\mathscr{L}(X)$.*

To each A belonging to $\mathscr{A}(X, T, M)$ we make to correspond the real-valued function $A^\wedge(x)$ which is defined for every $x \in M$ by means of the equation

$$Ax = A^\wedge(x) x.$$

Let $\mathscr{A}^\wedge(X, T, M)$ be the set of all such functions and let $C(T, M)$ be the Banach algebra of all bounded T-continuous real-valued functions defined on M.

THEOREM II. *The correspondence $A \to A^\wedge$ is an isometric and isomorphic mapping of $\mathscr{A}(X, T, M)$ into $C(T, M)$.*

COROLLARY 1. *The algebra $\mathscr{A}(X, T, M)$ is commutative.*

COROLLARY 2. *$\mathscr{A}(X, T, M)$ is a closed subalgebra of $C(T, M)$.*

THEOREM III. *If R is a closed ideal in $\mathscr{A}(X, T, M)$, A is an operator belonging to $\mathscr{A}(X, T, M)$ and $A^2 \in R$, then $A \in R$.*

COROLLARY 3. *If A is an operator belonging to $\mathscr{A}(X, T, M)$, x is an element of X and $A^2 x = 0$, then $Ax = 0$.*

THEOREM IV. *If A is an operator belonging to $\mathscr{A}(X, T, M)$, then the following 3 conditions are equivalent:*

(A) *The range of A is equal to X.*

(B) *The inverse operator* A^{-1} *exists and belongs to* $\mathscr{A}(X, T, M)$.

(C) *There exists a real number* $\varrho > 0$ *such that* $|A^{\wedge}(x)| \geqslant \varrho$ *for every* x *in* M.

COROLLARY 4. *If* M *is* T*-compact, then for every operator belonging to* $\mathscr{A}(X, T, M)$ *the Fredholm alternative holds.*

The proofs of all these results are essentially the same as in [1], where these statements are proved under the restriction that S is T-complete.

References

[1] D. Skordev, Acad. Bulgare des Sci., Bull. de l'Institut de math., 10 (1969) (to appear).

[2] Y. Tagamlitzki, Ann. de l'Univers. de Sofia, Fac. des sci. phys. et math., 48 (1953/54), livre 1, I partie, p. 69-85.

Some questions concerning support points

by

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We begin with an example of a closed bounded convex set in a Fréchet space which can not be supported at any of its points by a non-trivial continuous linear functional; this answers a question of Klee and Phelps. We then show that if C is a closed bounded convex set in a Fréchet space and F is the set of all linear functionals on the Fréchet space which are continuous on C, then some support-point theorems involving functionals in F can be obtained which parallel the support-point theorems of Bishop and Phelps for Banach spaces. As an application we extend a recent theorem of Krein-Milman type due to Lindenstrauss and Bessaga-Pełczyński.

When the containing linear space is assumed only to be complete and locally convex, not necessarily Fréchet, we have some weaker support theorems, which we mention briefly.

The detailed discussion will be presented in the paper "Support points in locally convex spaces", to appear in Duke Math. Journal.

On the equivalence of weak and Schauder basis

by

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Let E denote a locally convex topological vector space (LCTVS). A *weak* (or *strong*) basis in E is a sequence f_k such that every element f of E can be expressed uniquely in the form

$$f = \sum_{k=1}^{\infty} c_k(f) f_k$$

with convergence in $\sigma(E, E')$ (or in the original topology of E). The sequence is said to be a (weak or strong) *Schauder basis* if the $c_k(f)$ are continuous linear forms on E.

Moreover, let us call a *strictly netted* space an LCTVS E in which there exists a family of absolutely convex sets $e_{n_1, \ldots, n_k}$, $n_1, \ldots, n_k$, $k = 1, 2, \ldots$, which satisfy the following conditions:

(a) $$E = \bigcup_{n_1=1}^{\infty} e_{n_1}, \ldots, \quad e_{n_1, \ldots, n_k} = \bigcup_{n_{k+1}=1}^{\infty} e_{n_1, \ldots, n_{k+1}}, \ldots,$$

(b) for any sequence n_k, there exists a sequence $\lambda_k > 0$ such that, if $\mu_k \in [0, \lambda_k]$ and $g_k \in e_{n_1, \ldots, n_k}$, the series $\sum \mu_k g_k$ converges in E and

$$\sum_{k=k_0}^{\infty} \mu_k g_k \in e_{n_1, \ldots, n_{k_0}}, \forall k_0.$$

Such spaces verify a general form of the closed graph theorem and have numerous examples and wide permanence properties (see [3]).

We prove the following theorem:

If E is bornological, sequentially complete and strictly netted, then any weak basis in E is a strong Schauder basis.

This theorem was proved for Fréchet spaces by C. Bessaga and A. Pełczyński (see [1]).

As examples of convenient E which are not Fréchet spaces, let us point the classical spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}'$, $\mathscr{S}'$, ... (see [2]).

References

[1] C. Bessaga i A. Pełczyński, *Własności baz w przestrzeniach typu B_0*, Prace Matem. 3 (1959), p. 123-142.

[2] L. Schwartz, *Sur le théorème du graphe fermé*, C. R. Paris 263 (1966), p. 602-605.

[3] De Wilde, *Réseaux dans les espaces linéaires à semi-normes*, Thèse.