DIOPHANTINE APPROXIMATION BY PRIME NUMBERS
OF A SPECIAL FORM

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We show that for \( B > 1 \) and for some constants \( \lambda_i, i = 1, 2, 3 \) subject to certain assumptions, there are infinitely many prime triples \( p_1, p_2, p_3 \) satisfying the inequality

\[
|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \left[ \log(\max p_j) \right]^{-B}
\]

and such that \( p_1 + 2, p_2 + 2 \) and \( p_3 + 2 \) have no more than 8 prime factors. The proof uses Davenport-Heilbronn adaptation of the circle method together with a vector sieve method.

**Keywords:** Rosser’s weights, vector sieve, circle method, almost primes, diophantine inequality

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1. INTRODUCTION

The famous prime twins conjecture states that there exist infinitely many primes \( p \) such that \( p + 2 \) is a prime too. This hypothesis is still not proved but there are established many approximations to this result.

Throughout, \( P_r \) will stand for an integer with no more than \( r \) prime factors, counted with their multiplicities. In 1973 Chen [2] showed that there are infinitely many primes \( p \) with \( p + 2 = P_2 \).

Here are some examples of problems, concerning primes \( p \) with \( p + 2 = P_r \) for some \( r \geq 2 \).

In 1937, Vinogradov [16] proved that every sufficiently large odd \( n \) can be represented as a sum

\[
p_1 + p_2 + p_3 = n
\]
of primes $p_1, p_2, p_3$. In 2000 Peneva [10] and Tolev [14] looked for representations (1.1) with primes $p_i$, subject to $p_i + 2 = P_r$, for some $r_i \geq 2$. It was established in [14] that if $n$ is sufficiently large and $n \equiv 3 \pmod{6}$, then (1.1) has a solution in primes $p_1, p_2, p_3$

$$p_1 + 2 = P_2, \quad p_2 + 2 = P_5, \quad p_3 + 2 = P_7.$$ 

In 1947 Vinogradov [17] established that if $0 < \theta < 1/5$, then there are infinitely many primes $p$ satisfying the inequality

$$||\alpha p + \beta|| < p^{-\theta}. \quad (1.2)$$

In 2007 Todorova and Tolev [13] proved that if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\beta \in \mathbb{R}$ and $0 < \theta \leq 1/100$, then there are infinitely many primes $p$ with $p + 2 = P_4$, satisfying the inequality (1.2). Latter Matomäki [8] proved a Bombieri-Vinogradov type result for linear exponential sums over primes and showed that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$.

The present paper is devoted to another popular problem for primes $p_i$, which is studied under the additional restrictions $p_i + 2 = P_r$, for some $r_i \geq 2$. According to R. C. Vaughan’s [18], there are infinitely many ordered triples of primes $p_1, p_2, p_3$ with

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi + \delta}$$

for $\xi = 1/10, \delta > 0$ and some constants $\lambda_j, j = 1, 2, 3, \eta$, subject to the following restrictions:

$$\lambda_i \in \mathbb{R}, \lambda_i \neq 0, i = 1, 2, 3; \quad (1.3)$$

$$\lambda_1, \lambda_2, \lambda_3 \text{ not all of the same sign; } \quad (1.4)$$

$$\lambda_1 / \lambda_2 \in \mathbb{R} \setminus \mathbb{Q}; \quad (1.5)$$

$$\eta \in \mathbb{R}. \quad (1.6)$$

Latter the upper bound for $\xi$ was improved and the strongest published result is due to K. Matomäki with $\xi = 2/9$.

Here we prove the following result:

**Theorem 1.** Let $B$ be an arbitrary large and fixed. Then under the conditions (1.3), (1.4), (1.5), (1.6) there are infinitely many ordered triples of primes $p_1, p_2, p_3$ with

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < [\log(\max p_j)]^{-B} \quad (1.7)$$

and

$$p_1 + 2 = P_8', \quad p_2 + 2 = P_8'', \quad p_3 + 2 = P_8'''.$$
2. NOTATIONS

By \( p \) and \( q \) we always denote primes. By \( \varphi(n), \mu(n), \Lambda(n) \) we denote Euler’s function, Möbius’ function and Mangoldt’s function, respectively. We denote by \( \tau(n) \) the number of the natural divisors of \( n \). The notations \((m_1, m_2)\) and \([m_1, m_2]\) stand for the greatest common divisor and the least common multiple of \( m_1, m_2 \), respectively. Instead of \( m \equiv n \pmod{k} \) we write for simplicity \( m \equiv n(k) \). As usual, \([y]\) denotes the integer part of \( y \), 
\[ e(y) = e^{2\pi iy}, \]
\[ \theta(x, q, a) = \sum_{\substack{p \leq x \\atop p \equiv a \pmod{q}}} \log p; \]
\[ E(x, q, a) = \theta(x, q, a) - \frac{x}{\varphi(q)}; \] (2.1)

For positive \( A \) and \( B \) we write \( A \asymp B \) instead of \( A \ll B \ll A \).

Let \( q_0 \) be an arbitrary positive integer and \( X \) be such that
\[ q_0^2 = \frac{X}{(\log X)^A}, \quad A \geq 5; \] (2.2)
\[ \varepsilon = \frac{1}{(\log X)^{B+1}}, \quad B > 1 \text{ is arbitrary large}; \] (2.3)
\[ H = \frac{1000 \log X}{\varepsilon}; \] (2.4)
\[ \Delta = \frac{(\log X)^{A+1}}{X}; \] (2.5)
\[ D = \frac{X^{1/3}}{(\log X)^A}; \] (2.6)
\[ z = X^\alpha, \quad 0 < \alpha < 1/4; \] (2.7)
\[ P(z) = \prod_{2 < p \leq z} p; \]
\[ S_k(\alpha) = \sum_{\substack{\lambda_0 X < p \leq X \\atop p+2 \equiv 0 \pmod{k}}} \lambda (\alpha p) \log p, \quad 0 < \lambda_0 < 1. \] (2.8)

The restrictions on \( A, \lambda_0 \) and the value of \( \alpha \) will be specified latter.

3. OUTLINE OF THE PROOF

We notice that if \((p+2, P(z)) = 1\), then \( p + 2 = P_{1/\alpha}\). Our aim is to prove that for a specific (as large as possible) value of \( \alpha \) there exists a sequence \( X_1, X_2, \ldots \rightarrow \infty \) and primes \( p_i \in (\lambda_0 X_j, X_j], \quad i = 1, 2, 3 \) with \( |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon \) and \( p_i + 2 = P_{1/\alpha}, \quad i = 1, 2, 3 \). In such a way, we get an infinite sequence of triples of primes \( p_1, p_2, p_3 \) with the desired properties.
Our method goes back to Vaughan [18], but we also use the Davenport-Heilbronn adaptation of the circle method (see [19, ch. 11]) combined with a vector sieve similar to that one from [15].

We choose a function \( v \) such that
\[
\begin{align*}
    v(x) &= 1 & \text{for } |x| \leq \varepsilon/2; \\
    0 < v(x) < 1 & \text{for } \varepsilon/2 < |x| < \varepsilon; \\
    v(x) &= 0 & \text{for } |x| \geq \varepsilon,
\end{align*}
\]
and \( v(x) \) has derivatives of sufficiently large order.

So if
\[
\sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} v(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \lambda_1 \Lambda_1 \Lambda_2 \Lambda_3 \log p_1 \log p_2 \log p_3 > 0,
\]
then the number of the solutions of (1.7) in primes \( p_i \in (\lambda_0 X, X] \), \( p_i + 2 = P[1/\alpha] \), \( i = 1, 2, 3 \), is positive.

Let \( \lambda^\pm(d) \) be the lower and upper bounds Rosser’s weights of level \( D \), hence
\[
|\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \text{ if } d \geq D \text{ or } \mu(d) = 0.
\]
For further properties of Rosser’s weights we refer to [5], [6].

Let \( \Lambda_i = \sum_{d \mid (p_i + 2, P(z))} \mu(d) \) be the characteristic function of primes \( p_i \), such that \( (p_i + 2, P(z)) = 1 \) for \( i = 1, 2, 3 \). Then from (3.2) we obtain the condition
\[
\sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} v(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \Lambda_1 \Lambda_2 \Lambda_3 \log p_1 \log p_2 \log p_3 > 0.
\]

To set up a vector sieve, we use the lower and the upper bounds
\[
\Lambda_i^\pm = \sum_{d \mid (p_i + 2, P(z))} \lambda^\pm(d), \quad i = 1, 2, 3.
\]

From the linear sieve we know that \( \Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+ \) (see [1, Lemma 10]). Moreover, we have the simple inequality
\[
\Lambda_1 \Lambda_2 \Lambda_3 \geq \Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^- + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+,
\]
alogous to the one in [1, Lemma 13]. Using (3.4) we get
\[
\sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} v(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \\
\times (\Lambda_1^- \Lambda_2^+ \Lambda_3^- + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+) \log p_1 \log p_2 \log p_3 > 0.
\]
Let $\Upsilon(x) = \int_{-\infty}^{\infty} v(t)e(-tx)dt$ be the Fourier transform of the function $v$ defined in (3.1). Then

$$|\Upsilon(x)| \leq \min \left( \frac{3\varepsilon}{2}, \frac{1}{\pi|x|}, \frac{1}{k \pi|x|} \left( \frac{k}{2\pi|x|\varepsilon/4} \right)^k \right),$$

(3.7)

for all $k \in \mathbb{N}$ - see [11].

We substitute the function $v(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)$ in (3.6) with its Fourier transform:

$$\sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \log p_1 \log p_2 \log p_3 \times \int_{-\infty}^{\infty} \Upsilon(t)e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t)\Lambda_1 \Lambda_2 \Lambda_3 dt > 0. \quad (3.8)$$

Our next argument is based on the following consequence of (3.8).

**Lemma 1.** If the following integral is positive,

$$\Gamma(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t) \log p_1 \log p_2 \log p_3 \log p_1 \log p_2 \log p_3 \times (\Lambda_1^+ \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2 \Lambda_1^+ \Lambda_2^- \Lambda_3^+) \ dt \quad (3.9)$$

then the number of the solutions of (1.7) in primes $p_i \in (\lambda_0 X, X]$, $p_i + 2 = P_{[1/\alpha]}$, $i = 1, 2, 3$, is positive. Here

$$\Gamma_1(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \log p_1 \log p_2 \log p_3 \times e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t)\Lambda_1^- \Lambda_2^+ \Lambda_3^+ \ dt;$$

$$\Gamma_2(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \log p_1 \log p_2 \log p_3 \times e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t)\Lambda_1^+ \Lambda_2^- \Lambda_3^+ \ dt;$$

$$\Gamma_3(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \log p_1 \log p_2 \log p_3 \times e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t)\Lambda_1^+ \Lambda_2^+ \Lambda_3^- \ dt;$$

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\[
\Gamma_4(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \\
\times e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t) \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \, dt.
\]

We shall estimate \( \Gamma_1(X) \), the remaining integrals \( \Gamma_2(X), \Gamma_3(X), \Gamma_4(X) \) can be treated in a similar way. Changing the order of summation we obtain

\[
\Gamma_1(X) = \int_{-\infty}^{\infty} \Upsilon(t)e(\eta t)L^-(\lambda_1 t, X)L^+(\lambda_2 t, X)L^+(\lambda_3 t, X)dt, 
\] (3.10)

where

\[
L^\pm(t, X) = \sum_{d | P(z)} \lambda^\pm(d) \sum_{\lambda_0 X < p \leq X \atop p + 2 \equiv 0(d)} e(pt) \log p. 
\] (3.11)

Let us split \( \Gamma_1(X) \) into three integrals,

\[
\Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X), 
\] (3.12)

where

\[
\Gamma_1^{(1)}(X) = \int_{|t| \leq \Delta} \Upsilon(t)e(\eta t)L^-(\lambda_1 t, X)L^+(\lambda_2 t, X)L^+(\lambda_3 t, X)dt, 
\] (3.13)

\[
\Gamma_1^{(2)}(X) = \int_{\Delta < |t| < H} \Upsilon(t)e(\eta t)L^-(\lambda_1 t, X)L^+(\lambda_2 t, X)L^+(\lambda_3 t, X)dt, 
\] (3.14)

\[
\Gamma_1^{(3)}(X) = \int_{|t| \geq H} \Upsilon(t)e(\eta t)L^-(\lambda_1 t, X)L^+(\lambda_2 t, X)L^+(\lambda_3 t, X)dt. 
\] (3.15)

Here the functions \( \Delta = \Delta(X) \) and \( H = H(X) \) are defined in (2.5) and (2.4).

We estimate \( \Gamma_1^{(3)}(X), \Gamma_1^{(1)}(X), \Gamma_1^{(2)}(X) \), respectively, in the sections 4, 5, 6. In section 7 we complete the proof of the theorem.

4. UPPER BOUND FOR \( \Gamma_1^{(3)}(X) \).

**Lemma 2.** For the integral \( \Gamma_1^{(3)}(X) \), defined by (3.15), we have

\[
\Gamma_1^{(3)}(X) \ll 1.
\]
Proof. From (2.8) and (3.11) it follows that

$$|L^\pm(t, X)| \leq \sum_{d|P(z)} |\lambda^\pm(d)| |S_d(t)|.$$  

For $|S_d(t)|$ we use the trivial estimate

$$|S_d(t)| \leq \sum_{n \leq X} \log X \leq \log X \left(\frac{X}{d} + 1\right) \ll \frac{X \log X}{d} + \log X.$$  

Combining with (3.3) we obtain

$$L^\pm(t, X) \ll \sum_{d \leq D} \log X \left(\frac{X}{d} + 1\right) \ll X(\log X)^2 \quad (4.1)$$

Bearing in mind that $|\Upsilon(t)| \leq \frac{1}{\pi t} \left(\frac{k}{2\pi t \varepsilon/4}\right)^k$ (see (3.7)), from (4.1) and (3.15) one concludes that

$$\Gamma^{(3)}_1(X) \ll X^3(\log X)^6 \int_H^1 \frac{1}{t} \left(\frac{k}{2\pi t \varepsilon/4}\right)^k \, dt = \frac{X^3(\log X)^6}{k} \left(\frac{2k}{\pi \varepsilon H}\right)^k \quad (4.2)$$

The choice $k = \lfloor \log X \rfloor$ provides $\log X - 1 < k \leq \log X$ and by (2.4) it follows

$$\left(\frac{2k}{\pi \varepsilon H}\right)^k \ll \left(\frac{\log X}{\varepsilon 1000 \log X}\right)^{\log X} \ll \frac{1}{X \log 1000} \quad (4.3)$$

Finally, (4.2) and (4.3) imply

$$\Gamma^{(3)}_1(X) \ll 1. \quad (4.4)$$

5. ASYMPTOTIC FORMULA FOR $\Gamma^{(1)}_1(X)$.

We will derive the main term of the integral $\Gamma_1(X)$ from $\Gamma^{(1)}_1(X)$. Making use of (2.8), one expresses the sums (3.11) as

$$L^\pm(t, X) = \sum_{d|P(z)} \lambda^\pm(d) S_d(t). \quad (5.1)$$

We change the order of summation and integration in (3.13) to obtain

$$\Gamma^{(1)}_1(X) = \sum_{d|P(z)} \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3)$$

$$\times \int \left| t \right| \left| t \right| \Upsilon(t) e(\eta t) S_{d_1}(\lambda_1 t) S_{d_2}(\lambda_2 t) S_{d_3}(\lambda_3 t) \, dt. \quad (5.2)$$
Let
\[ S_i = S_{d_i}(\lambda_i t) , \quad (5.3) \]
\[ I_i = I_{d_i}(\lambda_i t) = \frac{1}{\varphi(d_i)} \int_{\lambda_0 X}^{X} e(\lambda_i t y) dy , \quad (5.4) \]
\[ R_i = R_{d_i} = (1 + \Delta X) \max_{y \in [\lambda_0 X, X]} |E(y, d_i, -2)| , \quad (5.5) \]
where \( E(x, q, a) \) is defined by (2.1). Using (2.6), it is not difficult to prove the estimate
\[ S_i \ll X \log X d_i . \quad (5.6) \]
From the inequality \( \frac{n}{\varphi(n)} \leq e^\gamma \log \log n \) (see [4, §XVIII, Theorem 328]) we get the following estimate for \( |I_i| \):
\[ |I_i| \leq \frac{X}{\varphi(d_i)} \ll \frac{X \log \log X}{d_i} \ll \frac{X \log X}{d_i} . \quad (5.7) \]

Our aim is to separate the main part of the sum (5.2).

As the first step, we replace the product \( S_1 S_2 S_3 \) by \( I_1 I_2 I_3 \), as far as the integral over \( I_1 I_2 I_3 \) is easier to be estimated. We use the identity
\[ S_1 S_2 S_3 = I_1 I_2 I_3 + (S_1 - I_1) I_2 I_3 + S_1 (S_2 - I_2) I_3 + S_1 S_2 (S_3 - I_3) . \quad (5.8) \]

Let \( 2 \nmid k \). Applying Abel’s transform to \( S_k(\alpha) \), one gets
\[ S_k(\alpha) = - \int_{\lambda_0 X}^{X} \sum_{\substack{\lambda_0 X < p \leq \lambda \leq \epsilon \\ p+2 \equiv 0 (k)}} \log p \cdot \frac{d}{dt} e(\alpha t) dt + e(\alpha X) \sum_{\substack{\lambda_0 X < p \leq X \\ p+2 \equiv 0 (k)}} \log p . \]

Using (2.1), we have
\[ S_k(\alpha) = - \int_{\lambda_0 X}^{X} \left[ \frac{t - \lambda_0 X}{\varphi(k)} + E(t, k, -2) - E(\lambda_0 X, k, -2) \right] \frac{d}{dt} e(\alpha t) dt \]
\[ + \left[ \frac{X - \lambda_0 X}{\varphi(k)} + E(X, k, -2) - E(\lambda_0 X, k, -2) \right] e(\alpha X) \]
\[ = \frac{1}{\varphi(k)} \left[ - \int_{\lambda_0 X}^{X} (t - \lambda_0 X) \frac{d}{dt} e(\alpha t) dt + (X - \lambda_0 X) e(\alpha X) \right] \]
\[ + \mathcal{O} \left( \int_{\lambda_0 X}^{X} \max_{y \in [\lambda_0 X, X]} |E(y, k, -2)||\alpha| dt \right) + \mathcal{O} \left( \max_{y \in [\lambda_0 X, X]} |E(y, k, -2)| \right) , \]

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whence
\[ S_k(\alpha) = \frac{1}{\varphi(k)} \int_{\lambda_0 X}^{X} e(\alpha t)dt + \mathcal{O}\left( \max_{y \in (\lambda_0 X, X]} |E(y, k, -2)| (1 + |\alpha|X) \right). \]

Let \(|\alpha| \leq \Delta\). Then from (5.3), (5.4) and (5.5) we obtain
\[ S_i = I_i + \mathcal{O}(R_i), \quad i = 1, 2, 3. \tag{5.9} \]

From (5.5) - (5.9) it follows that
\[ S_1S_2S_3 - I_1I_2I_3 \ll (X \log X)^2 (1 + \Delta X) \left( \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_1, -2)|}{d_2d_3} + \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_2, -2)|}{d_1d_3} + \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_3, -2)|}{d_1d_2} \right). \]

Using (5.2) and the above inequality one gets
\[ \Gamma_1^{(1)}(X) = M^{(1)} + \mathcal{O}(R^{(1)}), \tag{5.10} \]

where
\[ M^{(1)} = \sum_{d_i \mid P(z), \ i=1,2,3} \lambda^{-}(d_1)\lambda^{+}(d_2)\lambda^{+}(d_3) \int_{|t| \leq \Delta} \Upsilon(t)e(\eta t)I_1(\lambda_1 t)I_2(\lambda_2 t)I_3(\lambda_3 t)dt, \tag{5.11} \]
\[ R^{(1)} = (X \log X)^2 (1 + \Delta X) \sum_{d_i \mid P(z), \ i=1,2,3} \left( \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_1, -2)|}{d_2d_3} + \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_2, -2)|}{d_1d_3} + \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_3, -2)|}{d_1d_2} \right) \int_{|t| \leq \Delta} |\Upsilon(t)| dt. \]

Let us estimate \( R^{(1)} \). Since \(|\Upsilon(t)| \leq \frac{3\varepsilon}{2} \) (see (3.7)), we find
\[ \int_{|t| \leq \Delta} |\Upsilon(t)| dt \ll \varepsilon \Delta. \]

Then using (3.3) we obtain
\[ R^{(1)} \ll \varepsilon \Delta (X \log X)^2 (1 + \Delta X) \sum_{d_i \leq D, \ i=1,2,3} \left( \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_1, -2)|}{d_2d_3} + \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_2, -2)|}{d_1d_3} + \frac{\max_{y \in (\lambda_0 X, X]} |E(y, d_3, -2)|}{d_1d_2} \right) \]
\[ \ll \varepsilon \Delta (1 + \Delta X) X^2 (\log X)^4 \sum_{d \leq D, \ 2 \mid d} \max_{y \in (\lambda_0 X, X]} |E(y, d, -2)|. \tag{5.12} \]
We shall use the following well-known result.

**Theorem 2 (Bombieri - Vinogradov).** For any $A > 0$ the following inequality is fulfilled (see [3, ch.28]):

$$\sum_{q \leq X^{1/2}/(\log X)^{C+5}} \max_{y \leq X} \max_{(a, q) = 1} E(y, q, a) \ll \frac{X}{(\log X)^C}.$$ 

We apply the above theorem with $C = 4A + 5$ to the last sum in (5.12). Using (2.6) and (2.5) we obtain

$$R^{(1)} \ll \varepsilon \Delta (1 + \Delta X) X^2 (\log X)^4 \frac{X}{(\log X)^{4A+5}} \ll \frac{\varepsilon \Delta^2 X^4}{(\log X)^{4A+1}}. \quad (5.13)$$

Then from (5.10) and (5.13) it follows

$$\Gamma^{(1)}_1(X) - M^{(1)} \ll \frac{\varepsilon \Delta^2 X^4}{(\log X)^{4A+1}}. \quad (5.14)$$

As a second step we represent $M^{(1)}$ in the form

$$M^{(1)} = \sum_{d_i | P(z)} \frac{\lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3)}{\varphi(d_1) \varphi(d_2) \varphi(d_3)} B(X) + R, \quad (5.15)$$

where

$$B(X) = \int_{-\infty}^{\infty} \Upsilon(t) e(\eta t) \left( \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} e(t \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3) dy_1 dy_2 dy_3 \right) dt, \quad (5.16)$$

$$R \ll \left| \int_{\Delta} \Upsilon(t) e(\eta t) \left( \int_{\lambda_0 X}^{X} e(\lambda_1 t y_1) dy_1 \int_{\lambda_0 X}^{X} e(\lambda_2 t y_2) dy_2 \int_{\lambda_0 X}^{X} e(\lambda_3 t y_3) dy_3 \right) dt \right|$$

$$\times \sum_{d_i | P(z)} \frac{|\lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3)|}{\varphi(d_1) \varphi(d_2) \varphi(d_3)}.$$
From (5.15) and (5.17) we obtain
\[
M^{(1)} = B(X) \sum_{d_i \mid P(z)} \frac{\lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3)}{\varphi(d_1) \varphi(d_2) \varphi(d_3)} + O\left(\frac{\varepsilon \log^3 X}{\Delta^2}\right)
\]
and from (5.14) we have
\[
\Gamma^{(1)}_1(X) = B(X) \sum_{d_1 \mid P(z)} \frac{\lambda^-(d_1)}{\varphi(d_1)} \sum_{d_2 \mid P(z)} \frac{\lambda^+(d_2)}{\varphi(d_2)} \sum_{d_3 \mid P(z)} \frac{\lambda^+(d_3)}{\varphi(d_3)}
+ O\left(\frac{\varepsilon \log^3 X}{\Delta^2}\right) + O\left(\frac{\varepsilon \Delta^2 X^4}{(\log X)^{4A+1}}\right).
\]
The function \(\Delta\) defined by (2.5) is such that
\[
\varepsilon \log^3 X \Delta^2 = \varepsilon \Delta^2 X^4 = \frac{\varepsilon \Delta^2 X^4}{(\log X)^{4A+1}}.\]
Therefore, using (2.3), (2.5) and (5.18), we find
\[
\Gamma^{(1)}_1(X) = B(X) \sum_{d_1 \mid P(z)} \frac{\lambda^-(d_1)}{\varphi(d_1)} \sum_{d_2 \mid P(z)} \frac{\lambda^+(d_2)}{\varphi(d_2)} \sum_{d_3 \mid P(z)} \frac{\lambda^+(d_3)}{\varphi(d_3)}
+ O\left(\frac{X^2}{(\log X)^{2A+B}}\right).
\]
Let
\[
G^\pm = \sum_{d \mid P(z)} \frac{\lambda^\pm(d)}{\varphi(d)}.
\]
Then from (5.19) and (5.20) it follows
\[
\Gamma^{(1)}_1(X) = B(X)G^-(G^+)^2 + O\left(\frac{X^2}{(\log X)^{2A+B}}\right).
\]
We conclude this section with the following lemma:

**Lemma 3.** If (1.3), (1.4) hold and
\[
\lambda_0 < \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16}\right),
\]
then \(B(X)\) defined by (5.16) satisfies
\[
B(X) \gg \varepsilon X^2,
\]
and the constant in “\(\gg\)” depends only on \(\lambda_1, \lambda_2\) and \(\lambda_3\).
Proof. Let us consider $B(X)$. We change the order of integration and use that $\Upsilon(t)$ is Fourier’s transform of $v(t)$ to obtain

$$B(X) = \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} v(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta) dy_1 dy_2 dy_3.$$ 

From the definition (3.1) of $v$ follows the inequality

$$B(X) \geq \iiint_V dy_1 dy_2 dy_3 = B_1(X),$$

where

$$V = \{|\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta| < \varepsilon/2, \lambda_0 X \leq y_j \leq X, j = 1, 2, 3\}.$$

Since $\lambda_1, \lambda_2, \lambda_3$ are not all of the same sign, we may assume that $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 < 0$. We substitute $\lambda_1 y_1 = z_1, \lambda_2 y_2 = z_2, \lambda_3 y_3 = -z_3$, then

$$B_1(X) = \frac{1}{\lambda_1 \lambda_2 |\lambda_3|} \iint_{V'} dz_1 dz_2 dz_3 = B_1(X),$$

with $V' = \{(z_1, z_2, z_3) : |z_1 + z_2 - z_3 + \eta| < \varepsilon/2, \lambda_0 |\lambda_j| X \leq z_j \leq |\lambda_j| X, j = 1, 2, 3\}$. Set

$$\xi_1 = \frac{2\lambda_0 |\lambda_3|}{\lambda_1}, \quad \xi_2 = \frac{2\lambda_0 |\lambda_3|}{\lambda_2}, \quad \xi'_1 = 2\xi_1, \quad \xi'_2 = 2\xi_2,$$

$$\lambda_0 < \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16}\right).$$

Then $\lambda_0 < \xi_1 < \xi'_1 < 1, \quad \lambda_0 < \xi_2 < \xi'_2 < 1,$

$$\lambda_0 \lambda_1 X < \xi_1 \lambda_1 X < z_1 < \xi'_1 \lambda_1 X < \lambda_1 X,$$

$$\lambda_0 \lambda_2 X < \xi_2 \lambda_2 X < z_2 < \xi'_2 \lambda_2 X < \lambda_2 X,$$

$$\lambda_0 |\lambda_3| X < z_1 + z_2 - \varepsilon/2 + \eta < z_3 < z_1 + z_2 + \varepsilon/2 + \eta < |\lambda_3| X,$$

and from (5.22), (5.23) and (5.24) there follows

$$B(X) \geq B_1(X) \gg \int_{\xi_1 \lambda_1 X}^{\xi'_1 \lambda_1 X} \left(\int_{\xi_2 \lambda_2 X}^{\xi'_2 \lambda_2 X} \left(\int_{z_1 + z_2 - \varepsilon/2 + \eta}^{z_1 + z_2 + \varepsilon/2 + \eta} dz_3\right) dz_2\right) dz_1$$

$$= \varepsilon (\xi'_2 - \xi_2) \lambda_2 X (\xi'_1 - \xi_1) \lambda_1 X = 4\lambda_0^2 \lambda_3^2 \varepsilon X^2$$

$$\gg \varepsilon X^2.$$
6. UPPER BOUND FOR $\Gamma_1^{(2)}(X)$.

We shall use (2.6) and the following lemma:

**Lemma 4 ([13, Lemma 1], [15, Lemma 12]).** Suppose $\alpha \in \mathbb{R}\setminus \mathbb{Q}$ with a rational approximation $\frac{a}{q}$ satisfying $\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2}$, where $(a,q) = 1, q \geq 1, a \neq 0$.

Let $D$ be defined by (2.6), $\xi(d)$ be complex numbers defined for $d \leq D$ and $\xi(d) \ll 1$. If

$$L(X) = \sum_{d \leq D} \xi(d) \sum_{X/2 < p \leq X \atop p+2 \equiv 0 (d)} e(\alpha p) \log p, \quad (6.1)$$

then we have

$$L(X) \ll (\log X)^{37} \left( \frac{X}{q^{1/4}} + \frac{X}{(\log X)^{A/2}} + X^{3/4} q^{1/4} \right). \quad (6.2)$$

Let us consider any sum $L^\pm(\alpha, X)$ denoted by (3.11). We represent it as sum of finite number of sums of the type

$$L(\alpha, Y) = \sum_{d \leq D} \xi(d) \sum_{Y/2 < p \leq Y \atop p+2 \equiv 0 (d)} e(\alpha p) \log p,$$

where

$$\xi(d) = \begin{cases} \lambda^\pm(d), & \text{if } d \mid P(z), \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$L^\pm(\alpha, X) \ll \max_{\lambda_0 X \leq Y \leq X} L(\alpha, Y).$$

If

$$q \in \left[ (\log X)^A, \frac{X}{(\log X)^A} \right], \quad (6.3)$$

then from the above lemma for the sums $L(\alpha, Y)$ we get

$$L(\alpha, Y) \ll \frac{Y}{(\log Y)^{A/4 - 37}}. \quad (6.4)$$

Therefore

$$L^\pm(\alpha, X) \ll \max_{\lambda_0 X \leq Y \leq X} \frac{Y}{(\log Y)^{A/4 - 37}} \ll \frac{X}{(\log X)^{A/4 - 37}}. \quad (6.5)$$

Let

$$V(t, X) = \min \{ |L^\pm(\lambda_1 t, X)|, |L^\pm(\lambda_2 t, X)| \}. \quad (6.6)$$

We shall need the following result:

Lemma 5. Let \( t, X, \lambda_1, \lambda_2 \in \mathbb{R} \),
\[
|t| \in (\Delta, H),
\]
(6.5)
where \( \Delta \) and \( H \) are defined by (2.5) and (2.4), let \( \lambda_1, \lambda_2 \) satisfy (1.5) and \( V(t, X) \) be defined by (6.4). Then there exists a sequence of real numbers \( X_1, X_2, \ldots \) with \( \lim X_n = \infty \) such that
\[
V(t, X_j) \ll \frac{X_j}{(\log X_j)^{A/4-3/7}}, \quad j = 1, 2, \ldots.
\]
(6.6)

Proof. Our goal is to prove that there exists a sequence \( X_1, X_2, \ldots \to \infty \) such that for every \( j \in \mathbb{N} \) at least one of the numbers \( \lambda_1 t \) and \( \lambda_2 t \), with \( t \) fulfilling (6.5), can be approximated by rational numbers with denominators satisfying (6.2). Then the proof follows from (6.3) and (6.4).

Since \( \frac{\lambda_1}{\lambda_2} \in \mathbb{R}/\mathbb{Q} \) then, by [12, Corollary 1B], there exist infinitely many fractions \( \frac{a_0}{q_0} \) with arbitrary large denominators such that
\[
\left| \frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0} \right| < \frac{1}{q_0^2}, \quad (a_0, q_0) = 1.
\]
(6.7)
Let \( q_0 \) be sufficiently large and \( X \) be such that \( q_0^2 = \frac{X}{(\log X)^A} \) (see (2.2)). Let us notice that there exist \( a_1, q_1 \in \mathbb{Z} \) such that
\[
\left| \lambda_1 t - \frac{a_1}{q_1} \right| < \frac{1}{q_1 q_0^2}, \quad (a_1, q_1) = 1, \quad 1 < q_1 < q_0^2, \quad a_1 \neq 0.
\]
(6.8)
The Dirichlet theorem (see [7, ch.10, §1]) implies the existence of integers \( a_1 \) and \( q_1 \) satisfying the first three conditions in (6.8). If \( a_1 = 0 \), then \( |\lambda_1 t| < \frac{1}{q_1 q_0^2} \) and from (6.5) it follows
\[
\lambda_1 \Delta < \lambda_1 |t| < \frac{1}{q_0^2}, \quad q_0^2 < \frac{1}{\lambda_1 \Delta}.
\]
From the last inequality, (2.2) and (2.5), one obtains
\[
\frac{X}{(\log X)^A} < \frac{X}{\lambda_1 (\log X)^{A+1}},
\]
which is impossible for large \( q_0 \), respectively, for a large \( X \). So \( a_1 \neq 0 \). By analogy there exist \( a_2, q_2 \in \mathbb{Z} \), such that
\[
\left| \lambda_2 t - \frac{a_2}{q_2} \right| < \frac{1}{q_2 q_0^2}, \quad (a_2, q_2) = 1, \quad 1 < q_2 < q_0^2, \quad a_2 \neq 0.
\]
(6.9)
If \( q_i \in [\log X]^A, \frac{X}{(\log X)^A} \) for \( i = 1 \) or \( i = 2 \), then the proof is completed.

From (2.2), (6.8) and (6.9) we have

\[
q_i \leq \frac{X}{(\log X)^A} = q_0^2, \quad i = 1, 2.
\]

Thus it remains to prove that the case

\[
q_i < (\log X)^A, \quad i = 1, 2
\]

is impossible. Let \( q_i < (\log X)^A, i = 1, 2 \). From (6.8), (6.9) and (6.10) it follows that

\[
1 \leq |a_i| \leq \frac{1}{q_0^2} + q_i \lambda_i |t| < \frac{1}{q_0^2} + q_i \lambda_i H,
\]

where \( \lambda_i = \frac{a_i}{a_i} \left( \lambda_i t - \frac{a_i}{q_i} \right), \quad i = 1, 2 \).

From (6.8), (6.9) and (6.10) we obtain

\[
|\Sigma_i| = \frac{q_i}{|a_i|} \cdot \frac{1}{q_i q_0^2} = \frac{1}{|a_i| q_0^2} \leq \frac{1}{q_0^2}, \quad i = 1, 2.
\]

We have

\[
\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathcal{O} \left( \frac{1}{q_0^2} \right)}{1 + \mathcal{O} \left( \frac{1}{q_0^2} \right)} = \frac{a_1 q_2}{a_2 q_1} \left( 1 + \mathcal{O} \left( \frac{1}{q_0^2} \right) \right).
\]

Thus \( \frac{a_1 q_2}{a_2 q_1} = \mathcal{O}(1) \) and

\[
\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + \mathcal{O} \left( \frac{1}{q_0^2} \right).
\]

Therefore, both fractions \( \frac{a_0}{q_0} = \frac{a_1 q_2}{a_2 q_1} \) approximate \( \frac{\lambda_1}{\lambda_2} \). Using (6.9), (6.10) and inequality (6.11) with \( i = 2 \) we obtain

\[
|a_2 q_1| < 1 + \frac{1000(\log X)^{2A+1} \lambda_2}{\varepsilon} \ll (\log X)^{2A+B+2} < \frac{q_0}{\log X}, \quad (6.14)
\]
so $|a_2|q_1 \neq q_0$ and the fractions $\frac{a_0}{q_0}$ and $\frac{a_1q_2}{a_2q_1}$ are different. On using (6.14) we obtain

$$\left| \frac{a_0}{q_0} - \frac{a_1q_2}{a_2q_1} \right| = \left| \frac{a_0a_2q_1 - a_1q_2q_0}{|a_2q_1q_0|} \right| \geq \frac{1}{|a_2q_1q_0|} \gg \frac{\log X}{q_0^2}. \quad (6.15)$$

On the other hand, from (6.7) and (6.13) we have

$$\left| \frac{a_0}{q_0} - \frac{a_1q_2}{a_2q_1} \right| \leq \frac{a_0}{q_0} - \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_2} - \frac{a_1q_2}{a_2q_1} \ll \frac{1}{q_0^2},$$

which contradicts (6.15). Therefore (6.10) can not happen. Let $q_0^{(1)}$, $q_0^{(2)}$, ... be an infinite sequence of values of $q_0$, satisfying (6.7). Then using (2.2) one gets an infinite sequence $X_1$, $X_2$, ... of values of $X$, such that at least one of the numbers $\lambda_1t$ and $\lambda_2t$ can be approximated by rational numbers with denominators, satisfying (6.2). The proof of Lemma 5 is completed.

Let us estimate the integral $\Gamma_1^{(2)}(X_j)$, defined by (3.14). Using $|\Upsilon(t)| \leq \frac{3\varepsilon}{2}$ (see (3.7)), (6.4) and estimate (6.6), we find

$$\Gamma_1^{(2)}(X_j) \ll \varepsilon \int_{\Delta < |t| < H} V(t, X_j) \left[ |L^{-}(\lambda_1t, X_j)L^{+}(\lambda_3t, X_j)| + |L^{+}(\lambda_2t, X_j)L^{+}(\lambda_3t, X_j)| \right] dt$$

$$\ll \varepsilon \int_{\Delta < |t| < H} V(t, X_j) \left( |L^{-}(\lambda_1t, X_j)|^2 + |L^{+}(\lambda_2t, X_j)|^2 + |L^{+}(\lambda_3t, X_j)|^2 \right) dt$$

$$\ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-3/7}} \max_{1 \leq k \leq 3} \int_{\Delta < |t| < H} |L^{\pm}(\lambda_k t, X_j)|^2 dt.$$

Since the above integral has the same value over the positive and the negative $t$, one gets

$$\Gamma_1^{(2)}(X_j) \ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-3/7}} \max_{1 \leq k \leq 3} \mathcal{I}_k, \quad (6.16)$$

where

$$\mathcal{I}_k = \int_{\Delta}^H |L^{\pm}(\lambda_k t, X_j)|^2 dt.$$

In order to estimate $\mathcal{I}_k$, let $y = |\lambda_k|t$, $dt = \frac{1}{|\lambda_k|}dy$.

Using $|L^{\pm}(y, X_j)|^2 \geq 0$ one obtains

$$\mathcal{I}_k \leq \frac{1}{|\lambda_k|} \int_0^{[|\lambda_k|H] + 1} |L^{\pm}(y, X_j)|^2 dy.$$

From (3.11) it follows

$$|L^{\pm}(y, X_j)|^2 = \sum_{\substack{d \downarrow (p(z)) \quad i = 1, 2 \quad d_1 | p(z) \quad d_2 \downarrow \mathcal{L}(d_1) \mathcal{L}(d_2) \quad \lambda_{0 X_j < p_1, p_2 \leq X_j} \quad \lambda_{p_1 + 2 \equiv 0(d_1) \quad p_2 + 2 \equiv 0(d_2)}} e((p_1 - p_2)y) \log p_1 \log p_2.$$
Then
\[
\mathcal{I}_k \leq \frac{1}{|\lambda_k|} \sum_{d_i \mid P(z)} \lambda^\pm(d_1)\lambda^\pm(d_2) \times \sum_{\lambda_0 X_j < p \leq X_j: p_{1+2 \equiv 0(d_1)} \neq p_{2+2 \equiv 0(d_2)}} \log p_1 \log p_2 \int_0^{[\lambda_k|H]+1} e((p_1 - p_2)y)dy.
\] (6.17)

Since \(e(my), m \in \mathbb{Z}\) is periodical with period 1, there holds
\[
\int_0^{[\lambda_k|H]+1} e((p_1 - p_2)y)dy = \left([\lambda_k|H] + 1\right) \int_0^1 e((p_1 - p_2)y)dy. \tag{6.18}
\]

From
\[
\int_0^1 e((p_1 - p_2)y)dy = \begin{cases} 
1, & \text{if } p_1 = p_2, \\
0, & \text{if } p_1 \neq p_2,
\end{cases}
\]
(6.18) and (6.17) one gets
\[
\mathcal{I}_k \leq \frac{[\lambda_k|H] + 1}{|\lambda_k|} \sum_{d_i \mid P(z)} \lambda^\pm(d_1)\lambda^\pm(d_2) \sum_{\lambda_0 X_j < p \leq X_j: p_{1+2 \equiv 0(d_1)} \neq p_{2+2 \equiv 0(d_2)}} (\log p)^2.
\]

From the last inequality and using (3.3) we find
\[
\mathcal{I}_k \ll H (\log X_j)^2 \sum_{d_i \leq D} \sum_{\mu(d_i) \neq 0, i = 1, 2} 1. \tag{6.19}
\]

Let \(d = (d_1, d_2), k_i = \frac{d_i}{d}, [d_1, d_2] = dk_1k_2\). Since \(\mu(d_i) \neq 0, i = 1, 2\), then \((d, k_i) = 1, i = 1, 2\). Now from (2.4), (2.6) and (6.19) we obtain
\[
\mathcal{I}_k \ll \frac{(\log X_j)^3}{\varepsilon} \sum_{d \leq D} \sum_{k_i \leq \frac{D}{d}} \sum_{\lambda_0 X_j < n \leq X_j: n_{p+2 \equiv 0([d_1, d_2])}} 1
\ll \frac{(\log X_j)^3}{\varepsilon} \sum_{d \leq D} \sum_{k_i \leq \frac{D}{d}} \frac{X_j}{dk_1k_2}
\ll \frac{X_j(\log X_j)^3}{\varepsilon} \sum_{d \leq D} \frac{1}{d} \left(\sum_{l \leq \frac{d}{D}} \frac{1}{l}\right)^2 \ll X_j(\log X_j)^6 / \varepsilon.
\]

From the last inequality and using (6.16) we get
\[
\Gamma_1^{(2)}(X_j) \ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-\frac{37}{43}}} - \frac{X_j^2}{(\log X_j)^{A/4-\frac{37}{43}}},
\] (6.20)

Summarizing, from (3.12), (4.4), (5.21) and (6.20) we obtain
\[
\Gamma_1(X_j) = B(X_j)G^-(G^+)^2 + \mathcal{O}\left(\frac{X_j^2}{(\log X_j)^{A/4-\frac{37}{43}}}\right).
\] (6.21)

7. PROOF OF THEOREM 1.

Since the sums \(\Gamma_2(X_j), \Gamma_3(X_j)\) and \(\Gamma_4(X_j)\) are estimated in the same fashion as \(\Gamma_1(X_j)\), we obtain from (3.9) and (6.21)
\[
\Gamma(X_j) \geq B(X_j)W(X_j) + \mathcal{O}\left(\frac{X_j^2}{(\log X_j)^{A/4-\frac{37}{43}}}\right),
\] (7.1)

where
\[
W(X_j) = 3(G^+)^2\left(G^- - \frac{2}{3}G^+\right).
\] (7.2)

Let \(f(s)\) and \(F(s)\) are the lower and the upper functions of the linear sieve. We know that if
\[
s = \frac{\log D}{\log z} = \frac{1}{3\alpha}, \quad 2 < s < 3
\] (7.3)

then
\[
F(s) = 2e^\gamma s^{-1}, \quad f(s) = 2e^\gamma s^{-1} \log(s - 1)
\] (7.4)

(see [1, Lemma 10]). Using (5.20) and [1, Lemma 10], we get
\[
\mathcal{F}(z)\left(f(s) + \mathcal{O}(\log X)^{-1/3}) \right) \leq G^- \leq \mathcal{F}(z) \leq G^+
\] (7.5)
\[
\leq \mathcal{F}(z)\left(F(s) + \mathcal{O}(\log X)^{-1/3}) \right).
\]

Here,
\[
\mathcal{F}(z) = \prod_{2 < p \leq z} \left(1 - \frac{1}{p-1}\right) \approx \frac{1}{\log X},
\] (7.6)

see Mertens formula [9, ch.9, §9.1, Theorem 9.1.3] and (2.7). To estimate \(W(X_j)\) from below, we shall use the inequalities (see (7.5))
\[
G^- - \frac{2}{3}G^+ \geq \mathcal{F}(z)\left(f(s) - \frac{2}{3}F(s) + \mathcal{O}(\log X)^{-1/3}) \right),
\] (7.7)
\[
G^+ \geq \mathcal{F}(z).
\]
Let $X = X_j$. Then from (7.2) and (7.7) it follows

$$W(X_j) \geq 3F^3(z)\left(f(s) - \frac{2}{3}F(s) + O((\log X)^{-1/3})\right).$$

We choose $s = \frac{\log D}{\log z} = 2.994$. Then

$$f(s) - \frac{2}{3}F(s) \geq 0.0000001,$$

and from (7.3) we get $\frac{1}{\alpha} = 8.982$. From (2.3), (7.1), (7.6), (7.8) and Lemma 3 we obtain:

$$\Gamma(X_j) \gg \frac{X_j^2}{(\log X_j)^{B+4}} + \frac{X_j^2}{(\log X_j)^{A/4-43}}.$$

We choose $A \geq 4B + 192$. Then

$$\Gamma(X_j) \gg \frac{X_j^2}{(\log X_j)^{B+4}}.$$

Finally, we note that if $\Gamma_0(X_j)$ is the number of the triples $p_i \in [\lambda_0 X_j, X_j]$, $p_i + 2 = P_8$, $i = 1, 2, 3$, satisfying (1.7), then there exists a positive constant $c$ such that

$$\Gamma_0(X_j) \geq \frac{1}{(\log X_j)^3} \Gamma(X_j) \geq \frac{cX_j^2}{(\log X_j)^{B+7}}$$

and for every prime factor $q$ of $p_i + 2$, $i = 1, 2, 3$ we have $q \geq X^{0.1113}$. That completes the proof of Theorem 1.

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8. REFERENCES


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