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AUTOMORPHISMS OF ALGEBRAS  
AND BOCHNER'S PROPERTY  
FOR DISCRETE VECTOR ORTHOGONAL POLYNOMIALS

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We construct new families of discrete vector orthogonal polynomials that have the property to be eigenfunctions of some difference operator. They are extensions of Charlier, Meixner and Kravchuk polynomial systems. The ideas behind our approach lie in the studies of bispectral operators. We exploit automorphisms of associative algebras which transform elementary (vector) orthogonal polynomial systems which are eigenfunctions of a difference operator into other systems of this type. While the extension of Charlier polynomials is well known it is obtained by different methods. The extension of Meixner polynomial system is new.

**Keywords:** vector orthogonal polynomials, finite recurrence relations, bispectral problem.

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## 1. INTRODUCTION

The present paper is a continuation of [16] but could be read independently. Both papers are devoted to vector orthogonal polynomials with Bochner's property.

S. Bochner [7] has classified all systems of orthogonal polynomials  $P_n(x)$ ,  $n = 0, \dots$ , that are also eigenfunctions of a second order differential operator

$$L(x, \partial_x) = A(x)\partial_x^2 + B(x)\partial_x + C(x) \quad (1.1)$$

with eigenvalues  $\lambda_n$ . Here the coefficients  $A, B, C$  of the differential equation do not depend on the index  $n$ .

A similar problem was solved by O. Lancaster [19] and P. Lesky [21], although earlier E. Hildebrandt [15] has found all needed components of the proof. For more information see the excellent review article by W. Al-Salam [1].

The statement of Lancaster's theorem is that all polynomial systems with such properties are the discrete orthogonal polynomials of Hahn, Meixner, Charlier and Kravchuk.<sup>1</sup>

In recent times there is much activity in generalizations and versions of the classical result of Bochner as well as their discrete counterparts. The first one was the generalization by H. L. Krall [18]. He classified all order 4 differential operators which have a family of orthogonal polynomials as eigenfunctions. Later many authors found new families of orthogonal polynomials that are eigenfunctions of a differential operator (see, e.g. [12, 13]).

The classical discrete orthogonal polynomials are also a source from which new orthogonal polynomials have been obtained. In particular A. Durán and M. de la Iglesia [9] have obtained extensions of the classical polynomial systems of Hahn, Meixner and Charlier.

An important role in some of these generalizations plays the ideology of the bispectral problem which was initiated in [8]. Translating the Bochner and Krall results (cf. [11]) into this language already gives a good basis to continue investigations. We formulate it for the case of discrete orthogonal polynomials. Let us introduce the function  $\psi(x, n) = P_n(x)$ . Denote by  $D$  the shift operator acting on functions of  $x$  as  $Df(x) = f(x + 1)$ . Also let  $T$  be the shift operator in  $n$ , i.e.  $Th(n) = h(n + 1)$ . Recall that the orthogonality condition, due to a classical theorem by Favard-Shohat is equivalent to the well known 3-terms recurrence relation

$$xP_n = P_{n+1} + \gamma_0(n)P_n + \gamma_1(n)P_{n-1}, \quad (1.2)$$

where  $\gamma_j(n)$  are constants, depending on  $n$ . Here we use the polynomials normalized by the condition that their highest order coefficient is 1.

If we write the right-hand side of the 3-term recurrence relation as a difference operator  $\Lambda(n)$  acting on the variable  $n$  then the 3-term recurrence relation can be written as

$$\Lambda(n)\psi(x, n) = x\psi(x, n).$$

On the other hand we want  $\psi(x, n)$  to be an eigenfunction of a difference operator  $L$  in  $x$ :

$$L\psi(x, n) = \lambda(n)\psi(x, n).$$

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<sup>1</sup>Below when speaking about orthogonality we mean orthogonality with respect to a nondegenerate functional, which does not need to be positive definite.

Hence we can formulate the discrete version of Bochner-Krall problem as follows.

*Find all systems of orthogonal polynomials  $P_n(x)$  (with respect to some functional  $u$ ) which are eigenvalues of a difference operator.*

We also use some ideas relevant to the studies of bispectral operators. Before explaining them and the main results of the present paper let us introduce one more concept which is central for us. This is the notion of vector orthogonal polynomials (VOP), introduced by J. van Iseghem [24]. Let  $\{P_n(x)\}$  be a family of monic polynomials such that  $\deg P_n = n$ . A theorem of P. Maroni [22]  $\{P_n(x)\}$  gives an equivalent condition, which we use as definition.

**Definition 1.1.** We will say that the set of polynomials  $(P_n)$  are Vector Orthogonal polynomials (VOP) iff they satisfy a  $d+2$ -term recurrence relation,  $d \geq 1$ , of the form

$$xP_n(x) = P_{n+1} + \sum_{j=0}^d \gamma_j(n)P_{n-j}(x) \quad (1.3)$$

with constants (independent of  $x$ )  $\gamma_j(n)$ ,  $\gamma_d(n) \neq 0$ .

In the last 20-30 years there is much activity in the study of vector orthogonal polynomials and the broader class of multiple orthogonal polynomials.

Applications of the VOP include the simultaneous Padé approximation problem [2] and random matrix theory [2, 6]. The VOP can be obtained from general multiple orthogonal polynomials under some restrictions upon their parameters.

One problem that deserves attention is to find vector orthogonal analogs of the classical orthogonal polynomials. Several authors [3, 14] have found multiple orthogonal polynomials, that share a number of properties with the classical orthogonal polynomials - they have a raising operators, Rodrigues type formulas, Pearson equations for the weights, etc. However one of the features of the classical orthogonal polynomials - a differential or difference operator for which the polynomials are eigenfunctions is missing. Sometimes this property is relaxed to the property that the polynomials satisfy linear differential/difference equation, whose coefficients may depend on the index of the polynomial, see [20].

In the present paper we are looking for polynomials  $P_n(x)$ ,  $n = 0, 1, \dots$  that are eigenfunctions of a difference operator  $L(x, D)$  with eigenvalues depending on the variable  $n$  (the index) and which at the same time are eigenfunctions of a difference operator in  $n$ , i.e. finite-term recurrence relation with an eigenvalues, depending only on the variable  $x$ . Hence we find families  $\{P_n(x)\}$ ,  $n = 0, 1, \dots$  of discrete VOP that possess Bochner's property - they are simultaneously eigenfunctions of two discrete operators:

$$L(x, D)P_n(x) = \lambda(n)P_n(x), \quad \Lambda(n, T)P_n(x) = xP_n(x).$$

Our main results include an extension of Meixner polynomials. We construct systems of vector orthogonal polynomials  $\{P_n(x)\}$  which are eigenfunctions of a difference operator  $L(x, D)$ . It is different from the family found in [10] and [5] except for the first member. Our approach uses ideas of the bispectral theory from [4] but does not use Darboux transformations, which is usually the case, see e.g. [12, 13, 9]. We use methods introduced in [4]. Also a well known extension of Charlier polynomials (see [5, 25]) is presented. The reason to repeat it is that our construction is a new one in comparison to the techniques of [5, 25]. However, there are some similarities with [25]. The authors also use automorphisms of algebras and make a beautiful connection with representation theory. Our construction is simpler and quite straightforward. The same method was recently applied to extensions of Hermite and Laguerre polynomials as well as to a family that has no classical analog [16].

The methods from the present paper and [16] can be applied to various versions of vector orthogonal polynomials as well as to matrix, multivariate, etc. This will be done elsewhere.

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## 2. ELEMENTS OF BISPECTRAL THEORY

The following introductory material is mainly borrowed from [4]. Below we present the difference-difference version of the general bispectral problem which is suitable in the set-up of discrete orthogonal polynomial sequences.

For  $i = 1, 2$ , let  $\Omega_i$  be two subsets of  $\mathbb{C}$  such that  $\Omega_1$  is invariant under the translation operator

$$D: x \mapsto x + 1, x \in \Omega_1$$

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<sup>2</sup>See, e.g. EMS NEWSLETTER, <http://www.ems-ph.org/journals/newsletter/pdf/2015-12-98.pdf>

and its inverse  $D^{-1}$ , while  $\Omega_2$  is invariant under the translation operator

$$T: n \mapsto n + 1$$

and its inverse  $T^{-1}$ .

A difference operator on  $\Omega_1$  is a finite sum of the form

$$\sum_{k \in \mathbb{Z}} c_k(x) D^k,$$

where  $c_k: \Omega_1 \rightarrow \mathbb{C}$  are some functions in  $x$ . In the same way we define difference operators on  $\Omega_2$  to be finite sums of the form

$$\sum_{k \in \mathbb{Z}} s_k(n) T^k,$$

where  $s_k: \Omega_1 \rightarrow \mathbb{C}$  are functions in  $n$ .

By  $\mathcal{B}_1$  we denote an algebra with unit, consisting of difference operators  $L(x, D)$  in the variable  $x$ . By  $\mathcal{B}_2$  we denote an algebra of difference operators  $\Lambda(n, T)$ . Denote by  $\mathcal{M}$  the space of functions on  $\Omega_1 \times \Omega_2$ . The space  $\mathcal{M}$  is naturally equipped with the structure of bimodule over the algebra of difference operators  $L(x, D)$  on  $\Omega_1$  and the difference operators  $\Lambda(n, T)$  on  $\Omega_2$ .

Assume that there exists an algebra map  $b: \mathcal{B}_1 \rightarrow \mathcal{B}_2$  and an element  $\psi \in \mathcal{M}$  such that

$$P\psi = b(P)\psi, \quad \forall P \in \mathcal{B}_1.$$

We call  $\psi \in \mathcal{M}$  a *discrete-discrete bispectral function*, i.e., if there exist difference operators  $L(x, D)$  and  $\Lambda(n, T)$  on  $\Omega_1$  and  $\Omega_2$ , and functions

$$\theta(x) \quad \text{and} \quad \lambda(n),$$

such that

$$\begin{aligned} L(x, D)\psi(x, n) &= \lambda(n)\psi(x, n), \\ \Lambda(n, T)\psi(x, n) &= \theta(x)\psi(x, n), \end{aligned} \tag{2.1}$$

on  $\Omega_1 \times \Omega_2$ . In fact, as we would be interested in VOP, we will consider only the case when  $\theta(x) \equiv x$ . We will assume that  $\psi(x, n)$  is a nonsplit function of  $x$  and  $n$  in the sense that it satisfies the condition

(\*\*) there are no nonzero difference operators  $L(x, \partial_x)$  and  $\Lambda(n, T)$  that satisfy one of the above conditions with  $f(n) \equiv 0$  or  $\theta(x) \equiv 0$ .

The assumption (\*\*) implies that the map  $b: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ , given by  $b(P(x, \partial_x)) := S(n, T)$  is a well defined algebra anti-isomorphism. Let us introduce the subalgebras  $K_i$   $i = 1, 2$  of  $\mathcal{B}_i$  to be the algebras of functions in  $x$  (respectively in  $n$ ). The algebra

$$A_1 := b^{-1}(K_2)$$

consists of the bispectral operators corresponding to  $\psi(x, z)$  (i.e., difference operators in  $x$  having the properties (2.1)) and the algebra

$$A_2 := b(K_1)$$

consists of the bispectral operators corresponding to  $\psi(x, n)$ , i.e. difference operators in  $n$  having the properties (2.1)).

For the goals of VOP we are interested in the case when, for any fixed  $n$ , the function  $\psi(x, n)$  defining the map  $b$  is a polynomial in  $x$ .

Let  $\mathcal{R}_1$  be the algebra spanned over  $\mathbb{C}$  by the operator  $\hat{x}$  (multiplication by  $x$ ),  $D$  and  $D^{-1}$ . Needless to say, the commutation relations in  $\mathcal{R}_1$

$$[D, x] = D, \quad [D^{-1}, x] = -D^{-1}, \quad [D, D^{-1}] = 0$$

play a crucial role.

In the same way we define another algebra  $\mathcal{R}_2$ , using the operators  $T$ , its inverse  $T^{-1}$  and the operator  $n$  of multiplication by the variable  $n$ . Finally the module  $\mathcal{M}$  is a linear space of bivariate functions  $f(x, n)$ , where  $x$  and  $n$  are discrete variables. Next we define a subalgebra  $\mathcal{B}_1 \subset \mathcal{R}_1$  as follows. Introduce the operators  $\Delta = D - 1$  and  $\nabla = D^{-1} - 1$ .  $\mathcal{B}_1$  will be spanned by the generators  $\Delta, L = -x\nabla, \hat{x}$ . It would also be convenient to introduce the element  $f = \hat{x} - L$ .

$$[\Delta, L] = \Delta, \quad [L, f] = f, \quad [\Delta, f] = 1. \tag{2.2}$$

□

In what follows we use the notation of the falling factorial:

$$(x)_k = x(x-1)\dots(x-k+1) \text{ for } k \in \mathbb{N}, \text{ and } (x)_0 = 1.$$

We notice that the notation  $(x)_k$  is quite often used with a different meaning but here we will use it only in the above sense. Let  $\psi(x, n) := S_n(x) := (x)_n$ . Obviously

$$LS(x, n) = nS(x, n), \quad (T + n)S(x, n) = xS(x, n).$$

In this way we can define the anti-automorphism  $b$  by

$$\begin{cases} b(f) = T \\ b(L) = n \\ b(\Delta) = nT^{-1}. \end{cases} \tag{2.3}$$

The image of  $\mathcal{B}_1$  under the map  $b$  will be the algebra  $\mathcal{B}_2$ .

Next, following [4], we recall how to construct new bispectral operators from already known ones. The method is quite general and does not depend on the specific form of the operators. First, we remind the reader that for an operator

$L \in \mathcal{B}$  it is said that  $\text{ad}_L$  acts locally nilpotently on  $\mathcal{B}$  when for any element  $a \in \mathcal{B}$  there exists  $k \in \mathbb{N}$ , such that

$$\text{ad}_L^k(a) = 0.$$

We formulate the simple observation from [4], needed in the present paper, in a form suitable for the discrete VOP.

**Proposition 2.1.** *Let  $\mathcal{B}_1, \mathcal{B}_2$  be unital algebras with the properties described above. Let  $L \in \mathcal{B}_1$  such that  $\text{ad}_L : \mathcal{B}_1 \rightarrow \mathcal{B}_1$  is a locally nilpotent operator and let  $b : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  be a bispectral involution. Suppose that, for any fixed  $n$ ,  $e^L \psi(x, n)$  is a polynomial in  $x$  of degree  $n$ . Define a new map  $b' : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  via the new polynomial function  $\psi'(x, n) := e^{\text{ad}_L} \psi(x, n)$ . Then  $b' : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  is a bispectral anti-involution.*

### 3. CHARLIER TYPE VECTOR ORTHOGONAL POLYNOMIALS

Here the algebras  $\mathcal{B}_i$  are the ones defined in the previous section. Let  $P(X)$  be a polynomial of degree  $d \geq 1$  without a free term. We define the automorphism  $\sigma : \mathcal{B}_1 \rightarrow \mathcal{B}_1$  by

$$\sigma = e^{\text{ad}_{P(\Delta)}}.$$

Let us compute explicitly its action on the generators.

**Lemma 3.1.** *The automorphism  $\sigma$  acts on the generators as*

$$\begin{cases} \sigma(f) = f + P'(\Delta) \\ \sigma(L) = L + P'(\Delta)\Delta \\ \sigma(\Delta) = \Delta. \end{cases}$$

*Proof.* Starting with the relation  $[\Delta, f] = 1$  we prove by induction that for each  $m$

$$[\Delta^m, f] = m\Delta^{m-1}. \quad (3.1)$$

Really for  $m = 1$  it is obvious. Assuming (3.1) is verified for  $m = j - 1$ , we have for  $j = m$

$$\begin{aligned} [\Delta^m, f] &= \Delta^m f - f \Delta^m = \Delta \Delta^{m-1} f - f \Delta^m \\ &= \Delta(f \Delta^{m-1} + (m-1)\Delta^{m-1}) - f \Delta^m \\ &= [\Delta, f] \Delta^{m-1} + (m-1)\Delta^{m-1} = m\Delta^{m-1}. \end{aligned}$$

Hence

$$e^{\text{ad}_{P(\Delta)}}(f) = f + P'(\Delta),$$

as the rest of the terms vanish.

To prove the second formula we start with the identity  $[\Delta, L] = \Delta$ . By induction we see that

$$[\Delta^m, L] = m\Delta^m.$$

This proves the second formula. The last formula is obvious.  $\square$

A direct consequences of the lemma is

**Corollary 3.1.** *The image of  $x$  under the automorphism  $\sigma$  is:*

$$\sigma(x) = x + P'(\Delta)(1 + \Delta). \quad (3.2)$$

*Proof.* We use  $x = f + L$ . Hence

$$\sigma(f + L) = f + P'(\Delta) + L + P'(\Delta)\Delta = x + P'(\Delta)(1 + \Delta).$$

$\square$

Let us define the anti-involution  $b_1 = b(\sigma^{-1})$ . Below we use that

$$\sigma^{-1} = \sum_{j=0}^{\infty} \frac{(-\text{ad}_{P(\Delta)})^j}{j!}.$$

Also we define the difference operator

$$L_1 = \sigma(L) = -x\nabla + P'(\Delta)\Delta. \quad (3.3)$$

From Lemma 3.1 and Corollary 3.1 it follows almost immediately that

**Lemma 3.2.** *The anti-involution  $b_1$  acts as*

$$\begin{cases} b_1(x) = T + n + P'(nT^{-1})(1 + nT^{-1}) \\ b_1(L_1) = n \\ b_1(\Delta) = nT^{-1}. \end{cases}$$

*Proof.* We have

$$b_1(x) = b(\sigma^{-1}(x)) = b(x + P'(\Delta)(1 + \Delta)) = T + n + P'(nT^{-1})(1 + nT^{-1}).$$

Next,

$$b_1(L_1) = b(\sigma^{-1} \circ \sigma(L)) = b(L) = n$$

Finally,

$$b_1(\Delta) = b(\Delta) = nT^{-1}.$$

□

Let us define the "wave function"

$$C_n^P(x) = e^{P(\Delta)}\psi(x, n) = \sum_{j=0}^{\infty} \frac{P(\Delta)^j(x)_n}{j!}. \quad (3.4)$$

Notice that the operator  $\Delta$  reduces the degrees of the polynomials. The same is true for  $P(\Delta)$  (we recall that  $P(X)$  has no free term). This shows that the sum (3.4) is finite and for this reason  $C_n^P(x)$  is a polynomial.

Let us write explicitly  $P(\Delta)$  as

$$P(\Delta) = \sum_{j=1}^d \beta_j \Delta^j.$$

We will list the basic properties of the polynomials  $C_n^P(x)$  in terms of the polynomial  $P(\Delta)$  in the next theorem.

**Theorem 3.1.** *The polynomials  $C_n^P(x)$  have the following properties:*

(i) *They satisfy  $d + 2$ -term recurrence relation*

$$xC_n^P(x) = C_{n+1}^P(x) + n(1 + \beta_1)C_n^P(x) + \sum_{j=1}^d [(j+1)\beta_{j+1} + j\beta_j](n)_j C_{n-j}^P,$$

where  $\beta_{d+1} = 0$ .

(ii) *They are eigenfunctions of the difference operator  $L_1$  (3.3)*

$$L_1 C_n^P(x) = nC_n^P(x).$$

(iii) *They have a lowering operator*

$$\Delta C_n^P(x) = nC_{n-1}^P(x)$$

*Proof.* (i) From Lemma 3.1 we have that

$$xC_n^P(x) = \left\{ T + n + P'(nT^{-1})(1 + nT^{-1}) \right\} C_n^P.$$

Let us work out the expression  $E = P'(nT^{-1})(1 - nT^{-1})C_n^P$ . We have

$$\begin{aligned} E &= \sum_{j=1}^d j\beta_j(nT^{-1})^{j-1}C_n^P + \sum_{j=1}^d j\beta_j(nT^{-1})^jC_n^P \\ &= \sum_{j=0}^d [(j+1)\beta_{j+1} + j\beta_j](nT^{-1})^jC_n^P. \end{aligned}$$

Using that  $(nT^{-1})^j = (n)_j T^{-j}$  we obtain

$$E = \sum_{j=1}^d [(j+1)\beta_{j+1} + j\beta_j](n)_j C_{n-j}^P + \beta_1 C_n^P.$$

(ii) From the definitions of  $L_1$  and  $C_n^P(x)$  we obtain

$$L_1 C_n^P(x) = e^P L e^{-P} e^P \psi(x, n) = e^P n \psi(x, n) = n C_n^P(x).$$

(iii) follows directly from Lemma 3.2. □

#### 4. MEIXNER TYPE VECTOR ORTHOGONAL POLYNOMIALS

We use the notation of the previous section  $\hat{x}$ ,  $\Delta$ ,  $L = -x\nabla$  to define an algebra  $\mathcal{B}_1$  of discrete operators. It will be spanned by the operators  $\hat{x}$ ,  $L$ ,  $G = (L + \beta)\Delta$ , where  $\beta$  is a constant. Again it would be convenient to work with the element  $f = \hat{x} - L$ . They satisfy the following commutation relations

$$\begin{cases} [L, f] = f \\ [G, f] = 2L + \beta \\ [G, L] = G. \end{cases} \quad (4.1)$$

Also the wave function would be the same as in the previous section, namely  $\psi(x, n) = (x)_n$ . It has the properties:

$$\begin{cases} L\psi(x, n) = n\psi(x, n) \\ x\psi(x, n) = \psi(x, n+1) + n\psi(x, n) \\ G\psi(x, n) = (n-1+\beta)\psi(x, n-1). \end{cases}$$

We sum up these properties in terms of the following anti-involution  $b$ .

$$\begin{cases} b(L) = n \\ b(f) = T \\ b(G) = n(n-1+\beta)T^{n-1}. \end{cases}$$

The algebra  $\mathcal{B}_2$  will be the image  $b(\mathcal{B}_1)$ . This gives our initial bispectral problem.

Let  $P(X)$  be a polynomial of degree  $d \geq 1$  without a free term. We define the automorphism  $\sigma : \mathcal{B}_1 \rightarrow \mathcal{B}_1$  by

$$\sigma = e^{\text{ad}_{P(G)}}.$$

In the next lemma we compute it on the generators.

**Lemma 4.1.** *The automorphism  $\sigma$  acts on the generators as*

$$\begin{cases} \sigma(f) = f + (2L + \beta)P'(G) + P''(G)G + P'^2(G)G \\ \sigma(L) = L + P'(G)G \\ \sigma(G) = G. \end{cases}$$

*Proof.* Let us start with the second formula. We have  $[G, L] = G$ . Then by induction we find

$$[G^m, L] = mG^m \tag{4.2}$$

Hence

$$\text{ad}_{P(G)}L = P'(G)G.$$

which proves the second formula. Next we prove by induction that for each  $m$

$$[G^m, f] = 2 \sum_{j=0}^{m-1} G^j LG^{m-1-j} + m\beta G^{m-1}.$$

We use the above formula (4.2) in the form  $G^jL = LG^j + jG^j$  to transform the first sum into

$$\sum_{j=0}^{m-1} G^j LG^{m-1-j} = 2 \sum_{j=0}^{m-1} (L + j)G^{m-1}$$

This shows that

$$[G^m, f] = m(2L + \beta)G^{m-1} + m(m-1)G^{m-1},$$

which yields

$$\text{ad}_{P(G)}(f) = (2L + \beta)P'(G) + P''(G)G.$$

Now we easily compute  $\text{ad}_P^2(f)$ :

$$\text{ad}_P^2(f) = [P(G), 2LP'(G)] = 2P'^2(G)G.$$

From the expressions for  $\text{ad}_P^j, j = 0, 1, 2$  we obtain the first identity.

The last identity is obvious. □

We define the anti-involution  $b_1 = b(\sigma^{-1})$ . Our bispectral operator  $L_1$  will be given by

$$L_1 = \sigma(L) = -x\nabla + P'(G)G. \tag{4.3}$$

The next lemma computes the action of  $b_1$  on the needed elements.

**Lemma 4.2.** *The anti-involution  $b_1$  acts as*

$$\begin{cases} b_1(x) = T + n - [P'(u)(2n + \beta) + P''(u)u + P'(u)u - P'^2(u)u]_{|u=n(n-1+\beta)T^{-1}} \\ b_1(L_1) = n \\ b_1(G) = n(n - 1 + \beta)T^{-1}. \end{cases}$$

*Proof.* The last two identities are direct consequences of the definitions of  $L_1$  and  $G$  together with the formulas for  $b$ . The more involved first identity follow from the last two and Lemma 4.1. Really, we have

$$\begin{aligned} b_1(x) &= b_1(L) + b_1(f) = b(L - P'(G)G) + \\ &+ b(f - [(2L + \beta)P'(G) + P''(G)G] + P'^2(G)G) \\ &= b(f + L) - b([(2L + \beta + G)P'(G) + P''(G)G] - P'^2(G)G)). \end{aligned}$$

after which we put the expressions for  $b(G)$ ,  $b(f)$  and  $b(L)$ . □

We come to the definition of the VOP, i.e. the "wave function"

$$M_n^P(x) = e^{P(G)}\psi(x, n) = \sum_{j=0}^{\infty} \frac{P(G)^j(x)_n}{j!}. \quad (4.4)$$

We assume that  $P(G) = \alpha G^m + \dots$  with  $\alpha \neq 0$ .

Notice that the operator  $G$  reduces the degrees of the polynomials by one unit. This shows that the sum (4.4) is finite (we recall that  $P(X)$  has no free term) and for this reason  $M_n^P(x)$  is a polynomial.

The basic properties of the polynomials  $M_n^P(x)$  are listed in the following theorem

**Theorem 4.1.** *Let  $\beta \notin \mathbb{N}$ . Then the polynomials  $M_n^P(x)$  have the following properties:*

(i) *They satisfy the recurrence relation*

$$\begin{aligned} xM_n^P(x) &= M_{n+1}^P(x) + nM_n^P(x) \\ &- [P'(u)(2n + \beta + u) + P''(u)u - P'^2(u)u]_{|u=n(n-1+\beta)T^{-1}} M_n^P. \end{aligned}$$

(ii) *They are eigenfunctions of the difference operator  $L_1$  (4.3)*

$$L_1 M_n^P(x) = nM_n^P(x).$$

(iii) *The operator  $G$  acts on them as lowering operator*

$$GM_n^P(x) = n(n - 1 + \beta)M_{n-1}^P(x).$$

*Proof.* The proof is similar to the proof Theorem 3.1 and follows easily from Lemma 4.2 . Therefore it is omitted.  $\square$

**Remark 4.1.** Notice that the above polynomial system is well defined for all values of  $\beta$  but it is not always VOP. For example, when  $\beta = -N$ ,  $N \in \mathbb{N}$  and  $d = 1$  we come to Kravchuk system of orthogonal polynomials, which contains only finite number of members. Similar situation occurs when  $d > 1$ . This is discussed in the next section.

## 5. EXAMPLES

**Example 5.1.** For the definitions and properties of orthogonal polynomials we follow mainly [17].

Let  $P(\Delta) = -a\Delta$ . We find that

$$L_1 = -a\Delta - x\nabla.$$

This is the difference operator which has Charlier polynomials as eigenfunctions. The latter can be defined according to our scheme by

$$C_n^P = e^{-a\Delta}(x)_n = \sum_{j=0}^n \frac{(-a)^j \Delta^j(x)_n}{j!} = \sum \frac{(-a)^j (-n)_j(x)_{n-j}}{j!}.$$

We see that these are the normed Charlier polynomials denoted in [17] by  $p_n(x)$ . This example, together with the construction of Appell polynomials in [16], motivates the name "Charlier-Appell polynomials" for general  $P$ .

**Example 5.2.** In the second example we take the algebra from section 4. Let us take  $P(G) = \alpha G$ ,  $G = (-x\nabla + \beta)\Delta$ . Then

$$L_1 = \alpha(-x\nabla + \beta)\Delta - x\nabla.$$

Let  $c$  be a constant,  $c \neq 0, 1$ . Put  $\hat{L} = (c - 1)L_1$ . Take the constant  $\alpha$  to be

$$\alpha = \frac{c}{c - 1}$$

and  $\beta$  to be different from a negative integer. We obtain exactly the Meixner operator

$$\hat{L} = c(x + \beta)\Delta + x\nabla.$$

as given in [17]. It has eigenvalues  $(c - 1)n$ .

In case  $\beta = -N$ ,  $N \in \mathbb{N}$  we obtain Kravchuk polynomials  $K_0, \dots, K_N$ , which form a finite set of orthogonal polynomials.

**Example 5.3.** Let us again use the settings from section 4. We present here the simplest new example. Let us take  $P(G) = G^2/2$ . Then the new polynomials

$$M_n^P(x) = \sum_{j=0}^{\infty} \frac{(G^2)^j(x)_n}{j!}$$

are eigenfunctions of the operator  $L_1$

$$L_1 M_n^P(x) = (-x\nabla + (x\nabla\Delta + \beta\Delta)^2)M_n^P(x) = -nM_n^P(x)$$

The recurrence relation reads

$$\begin{aligned} xM_n^P &= M_{n+1}^P + nM_n^P - n(n + \beta - 1)(2n - 1 + \beta)M_{n-1}^P \\ &\quad - (n)_2(n + \beta)_2M_{n-2}^P + (n)_3(n + \beta)_3M_{n-3}^P. \end{aligned}$$

**Example 5.4. Kravchuk-like polynomials.** In this example we investigate the case when  $\beta = -N$ ,  $N \in \mathbb{N}$ . We take  $P(G) = G^2/2$ . The recurrence relation is as above.

We see that the polynomials satisfy 5-term recurrence relation. However the coefficient at  $M_{n-3}$  is zero for  $n = N$ , thus violating the condition of P. Maroni's theorem [22]. This shows that the vector orthogonality is valid only for the polynomials  $M_n, n = 0, \dots, N$ . The situation with the general polynomial  $P(G)$  is similar.

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