We prove two-sided estimates for the best (i.e., the smallest possible) constant $c_n(\alpha)$ in the Markov inequality
\[ \| p_n' \|_{w_\alpha} \leq c_n(\alpha) \| p_n \|_{w_\alpha}, \quad p_n \in \mathcal{P}_n. \]
Here, $\mathcal{P}_n$ stands for the set of algebraic polynomials of degree $\leq n$, $w_\alpha(x) := x^\alpha e^{-x}$, $\alpha > -1$, is the Laguerre weight function, and $\| \cdot \|_{w_\alpha}$ is the associated $L_2$-norm,
\[ \| f \|_{w_\alpha} = \left( \int_0^\infty |f(x)|^2 w_\alpha(x) \, dx \right)^{1/2}. \]

Our approach is based on the fact that $c_n^{-2}(\alpha)$ equals the smallest zero of a polynomial $Q_n$, orthogonal with respect to a measure supported on the positive axis and defined by an explicit three-term recurrence relation. We employ computer algebra to evaluate the seven lowest degree coefficients of $Q_n$ and to obtain thereby bounds for $c_n(\alpha)$. This work is a continuation of a recent paper [5], where estimates for $c_n(\alpha)$ were proven on the basis of the four lowest degree coefficients of $Q_n$.

Keywords: Markov type inequalities, Laguerre polynomials, three-term recurrence relation, Newton identities, computer algebra.

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$w_\alpha(x) := x^\alpha e^{-x}$, where $\alpha > -1$, be the Laguerre weight function, and $\| \cdot \|_{w_\alpha}$ be the associated $L_2$-norm,

$$\|f\|_{w_\alpha} = \left( \int_0^\infty |f(x)|^2 w_\alpha(x) \, dx \right)^{1/2}.$$  

We study the best constant $c_n(\alpha)$ in the Markov inequality in this norm

$$\|p_n'\|_{w_\alpha} \leq c_n(\alpha)\|p_n\|_{w_\alpha}, \quad p_n \in \mathcal{P}_n, \quad (1.1)$$

namely the constant

$$c_n(\alpha) := \sup_{p_n \in \mathcal{P}_n} \frac{\|p_n'\|_{w_\alpha}}{\|p_n\|_{w_\alpha}}.$$ 

Before formulating our results, let us give a brief account on the results known so far.

It is only the case $\alpha = 0$ where the best Markov constant is known, namely, Turán [9] proved that

$$c_n(0) = \left( 2 \sin \frac{\pi}{4n + 2} \right)^{-1}.$$ 

Dörfler [2] showed that $c_n(\alpha) = O(n)$ for every fixed $\alpha > -1$ by proving the estimates

$$c_n^2(\alpha) \geq \frac{n^2}{(\alpha + 1)(\alpha + 3)} + \frac{(2\alpha^2 + 5\alpha + 6)n}{3(\alpha + 1)(\alpha + 2)(\alpha + 3)} + \frac{\alpha + 6}{3(\alpha + 2)(\alpha + 3)}, \quad (1.2)$$

$$c_n^2(\alpha) \leq \frac{n(n + 1)}{2(\alpha + 1)}, \quad (1.3)$$

see [3] for a more accessible source. In the same paper, [3], Dörfler proved for the asymptotic constant

$$c(\alpha) := \lim_{n \to \infty} \frac{c_n(\alpha)}{n}, \quad (1.4)$$

that

$$c(\alpha) = \frac{1}{j_{\nu,1}^{(\alpha-1)/2,1}}, \quad (1.5)$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(z)$.

Nikolov and Shadrin obtained in [5] the following result:

**Theorem A ([5, Theorem 1]).** For all $\alpha > -1$ and $n \in \mathbb{N}$, $n \geq 3$, the best constant $c_n(\alpha)$ in the Markov inequality (1.1) admits the estimates

$$\frac{2(n + \alpha/2)(n - \alpha + 1)}{(\alpha + 1)(\alpha + 5)} < c_n^2(\alpha) < \frac{(n + 1)(n + 2(\alpha + 1))}{(\alpha + 1)(\alpha + 3)(\alpha + 5)}^{1/3}, \quad (1.6)$$

where for the left-hand inequality it is additionally assumed that $n > (\alpha + 1)/6$. 

Theorem A implies some inequalities for the asymptotic Markov constant $c(\alpha)$ and, through (1.5), inequalities for $J_{\nu,1}$, the first positive zero of the Bessel function $J_{\nu}$ (see [5, Corollaries 1, 3]). It was also shown in [5, Theorem 2] that $c(\alpha) = \mathcal{O}(\alpha^{-1})$, which indicates that the upper estimate for $c_n(\alpha)$ in Theorem A, though rather good for moderate $\alpha$, is not optimal.

In a recent paper [7] Nikolov and Shadrin proved an upper bound for $c_n(\alpha)$ which is of the correct order with respect to both $n$ and $\alpha$ as they tend to infinity.

**Theorem B ([7, Theorem 1.1]).** For all $n \in \mathbb{N}$, $n \geq 3$, the best constant $c_n(\alpha)$ in the Markov inequality (1.1) satisfies the inequality

$$c_n^2(\alpha) \leq \frac{4n(n + 2 + \frac{3(\alpha + 1)}{4})}{\alpha^2 + 10\alpha + 8}, \quad \alpha \geq 2. \quad (1.7)$$

As a consequence of Theorem B and Dörfler’s lower bound (1.2) for $c_n(\alpha)$ Nikolov and Shadrin showed that

$$c_n^2(\alpha) \geq \frac{n(n + \alpha + 3)}{(\alpha + 1)(\alpha + 8)}, \quad n \geq 3, \alpha \geq 2.$$

**Corollary C ([7, Corollary 1.1]).** For all $\alpha \geq 2$ and $n \geq 3$ the best constant $c_n(\alpha)$ in the Markov inequality (1.1) satisfies

$$\frac{2n(n + \alpha + 3)}{3(\alpha + 1)(\alpha + 8)} \leq c_n^2(\alpha) \leq \frac{4n(n + \alpha + 3)}{(\alpha + 1)(\alpha + 8)}. \quad (1.8)$$

In addition, Nikolov and Shadrin found the limit value of $(\alpha + 1)c_n^2(\alpha)$ as $\alpha \to -1$, and proved asymptotic inequalities for $\alpha c_n^2(\alpha)$ as $\alpha \to \infty$.

**Corollary D ([7, Corollary 1.2]).** The best constant $c_n(\alpha)$ in the Markov inequality (1.1) satisfies:

(i) $\lim_{\alpha \to -1}(\alpha + 1)c_n^2(\alpha) = \frac{n(n + 1)}{2}$;

(ii) $\frac{2n}{3} \leq \lim_{\alpha \to \infty} \alpha c_n^2(\alpha) \leq 3n$.

A combination of Theorems A and B implies bounds for $c(\alpha)$ defined in (1.4):

**Corollary E ([7, Corollary 1.3]).** The asymptotic Markov constant $c(\alpha)$ satisfies

$$\frac{2}{(\alpha + 1)(\alpha + 5)} < c^2(\alpha) < \begin{cases} \frac{1}{4} \frac{\sqrt{(\alpha + 1)(\alpha + 3)(\alpha + 5)}}{4}, & -1 < \alpha \leq \alpha^*, \\ \frac{1}{\alpha^2 + 10\alpha + 8}, & \alpha > \alpha^*, \end{cases}$$

where \( \alpha^* \approx 43.4 \).

The ratio of the upper and the lower bound for \( c(\alpha) \) in Corollary E is less than \( \sqrt{2} \) for all \( \alpha > -1 \).

In this paper we investigate the best Markov constant \( c_n(\alpha) \) following the approach from [5]. It is known (see Proposition 1 below) that \( c_{n-2}(\alpha) \) is equal to the smallest zero of a polynomial \( Q_n \), which is orthogonal with respect to a measure supported on \( \mathbb{R}_+ \). Since \( \{Q_n\}_{n \in \mathbb{N}} \) are defined by an explicit three-term recurrence relation, one can evaluate (at least theoretically) as many coefficients of \( Q_n \) as necessary. With the assistance of Wolfram’s Mathematica we find the seven lowest degree coefficients of the polynomial \( Q_n \), and thereby the six highest degree coefficients of \( R_n \), the monic polynomial reciprocal to \( Q_n \). Then we apply a simple technique for estimating the largest zero \( x_n \) of \( R_n \) on the basis of its \( k \) highest degree coefficients, \( 3 \leq k \leq 6 \), thus obtaining lower and upper bounds for \( c_n^*(\alpha) \). Our main result in this paper is:

**Theorem 1.** For \( 3 \leq k \leq 6 \) and for all \( n \geq k \), the best constant \( c_n(\alpha) \) in the Markov inequality (1.1) admits the estimates

\[
\xi_{n,k}(\alpha) \leq c_n(\alpha) \leq \eta_{n,k}(\alpha), \quad \alpha > -1, \tag{1.9}
\]

where

\[
\xi_{n,3}^2(\alpha) = \frac{2n(n + 3(\alpha + 1))}{(\alpha + 1)(\alpha + 5)} \tag{1.10},
\]

\[
\eta_{n,3}^2(\alpha) = \frac{(n + 1)(n + 2(\alpha + 1))}{(\alpha + 1)[(\alpha + 3)(\alpha + 5)]^{1/3}} \tag{1.11},
\]

\[
\xi_{n,4}^2(\alpha) = \frac{5\alpha + 17}{2(\alpha + 1)(\alpha + 3)(\alpha + 7)} \tag{1.12},
\]

\[
\eta_{n,4}^2(\alpha) = \frac{(5\alpha + 17)^{1/2}(n + 1)(n + 3(\alpha + 1))}{(\alpha + 1)(\alpha + 3)^{1/2}[2(\alpha + 5)(\alpha + 7)]^{1/4}} \tag{1.13},
\]

\[
\xi_{n,5}^2(\alpha) = \frac{2(7\alpha + 31)n(n + 25(\alpha + 1))}{(\alpha + 1)(\alpha + 9)(5\alpha + 17)} \tag{1.14},
\]

\[
\eta_{n,5}^2(\alpha) = \frac{(7\alpha + 31)^{1/3}(n + 1)(n + 4(\alpha + 1))}{(\alpha + 1)(\alpha + 3)^{2/5}[(\alpha + 5)(\alpha + 7)(\alpha + 9)]^{1/5}} \tag{1.15},
\]

\[
\xi_{n,6}^2(\alpha) = \frac{(21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073)n(n + 2(\alpha + 1))}{(\alpha + 1)(\alpha + 3)(\alpha + 5)(\alpha + 11)(7\alpha + 31)} \tag{1.16},
\]

\[
\eta_{n,6}^2(\alpha) = \frac{(21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073)^{1/6}(n + 1)(n + 5(\alpha + 1))^{1/11}}{(\alpha + 1)(\alpha + 3)^{1/2}(\alpha + 5)^{1/3}[(\alpha + 7)(\alpha + 9)(\alpha + 11)]^{1/6}} \tag{1.17}.
\]

Remark 1. For $3 \leq k \leq 6$, the pair $(\zeta_{n,k}(\alpha), \tau_{n,k}(\alpha))$ of bounds for $c_n(\alpha)$ is deduced with the use of the $k$ highest degree coefficients of the polynomial $R_n$ (and (1.11) is also proved in [5]). Generally, the bounds for $c_n(\alpha)$ obtained with larger $k$ are better, though some exceptions are observed for small $n$ and $\alpha$.

Clearly, inequalities (1.9) imply bounds for the asymptotic Markov constant $c(\alpha)$. Here, it is not difficult to prove that the larger $k$, the better the implied lower and upper bounds for $c(\alpha)$, hence the best bounds for $c(\alpha)$ are obtained from (1.9) with $k = 6$.

Thus, Theorem 1 yields an improvement of the estimates for the asymptotic Markov constant $c(\alpha)$ in Corollary E.

Corollary 1. The asymptotic Markov constant $c(\alpha) = \lim_{n \to \infty} n^{-1} c_n(\alpha)$ satisfies the inequalities

$$\zeta(\alpha) < c(\alpha) < \tau(\alpha),$$

where

$$\zeta(\alpha) := \frac{21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073}{(\alpha + 1)(\alpha + 3)(\alpha + 5)(\alpha + 11)(7\alpha + 31)},$$

and

$$\tau^2(\alpha) := \begin{cases} \left( \frac{(21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073)^{1/6}}{(\alpha + 1)(\alpha + 3)^{1/2}(\alpha + 5)^{1/3}[(\alpha + 7)(\alpha + 9)(\alpha + 11)]^{1/6}} \right), & -1 < \alpha \leq \alpha^*, \\ \frac{4}{\alpha^2 + 10\alpha + 8}, & \alpha > \alpha^*, \end{cases}$$

with $\alpha^* \approx 172$.

It is worth noticing that the ratio of the upper and the lower bound for $c(\alpha)$ in Corollary 1 does not exceed $2\sqrt[3]{3} \approx 1.1547$ for all $\alpha > -1$.

Theorem 1, in particular inequality (1.16), implies an improvement of the lower bound in Corollary D(ii).

Corollary 2. The best constant $c_n(\alpha)$ in the Markov inequality (1.1) satisfies:

$$\frac{6n}{7} \leq \lim_{\alpha \to \infty} \alpha c_n^2(\alpha) \leq 3n.$$
In Section 4 we give some final remarks and conclusions, and formulate two conjectures concerning the asymptotic behavior of the best Markov constant and the coefficients of the characteristic polynomial $R_n$.

## 2. PRELIMINARIES

### 2.1. AN ORTHOGONAL POLYNOMIAL RELATED TO $c_n(\alpha)$

It is well-known that the squared best constant in a Markov-type inequality in $L_2$-norm is equal to the largest eigenvalue of a related positive definite $n \times n$ matrix $A_n$, thus the problem of finding the best Markov constant is equivalent to evaluating the largest eigenvalue of $A_n$. Perhaps, a less known fact is that for a wide class of $L_2$-norms, the inverse matrix $A_n^{-1}$ is tri-diagonal, see [1, Sect. 2].

In the particular case of the $L_2$-norm induced by the Laguerre weight function $w_\alpha$ this connection is given by the following proposition:

**Proposition 1** ([3, p. 85]). The quantity $c_n^2(\alpha)$ is equal to the smallest zero of the polynomial $Q_n(x) = Q_n(x, \alpha)$, which is defined recursively by

\[
Q_{n+1}(x) = (x - d_n)Q_n(x) - \lambda_n^2 Q_{n-1}(x), \quad n \geq 0;
\]

\[
Q_{-1}(x) := 0, \quad Q_0(x) := 1;
\]

\[
d_0 := 1 + \alpha, \quad d_n := 2 + \frac{\alpha}{n+1}, \quad n \geq 1;
\]

\[
\lambda_0 > 0 \text{ arbitrary, } \lambda_n^2 := 1 + \frac{\alpha}{n}, \quad n \geq 1.
\]

By Favard’s theorem, for any $\alpha > -1$, $\{Q_n(x, \alpha)\}_{n=0}^{\infty}$ form a system of monic orthogonal polynomials. Since $Q_n$ is the characteristic polynomial of the inverse of a positive definite matrix (which is also positive definite), it follows that all the zeros of $Q_n$ are positive (and distinct). Consequently, $\{Q_n\}_{n=0}^{\infty}$ are orthogonal with respect to a measure supported on $\mathbb{R}_+$.

By Proposition 1, we have

\[
Q_{n+1}(x) = \left( x - 2 - \frac{\alpha}{n+1} \right) Q_n(x) - \left( 1 + \frac{\alpha}{n} \right) Q_{n-1}(x), \quad n \geq 1, \tag{2.1}
\]

\[
Q_0(x) = 1, \quad Q_1(x) = x - \alpha - 1. \tag{2.2}
\]

If we write $Q_n$ in the form

\[
Q_n(x) = x^n - a_{n-1,n} x^{n-1} + a_{n-2,n} x^{n-2} - \cdots + (-1)^n a_{0,n},
\]

then

\[
a_{0,n} = \left( \frac{n + \alpha}{n} \right), \quad n \in \mathbb{N}_0, \tag{2.3}
\]
with the convention that the right-hand side is equal to 1 for \( n = 0 \). The proof is by induction with respect to \( n \). For \( n = 0, 1 \), (2.3) follows from (2.2). Assuming (2.3) is true for all \( m \leq n \), we verify it for \( m = n + 1 \) by putting \( x = 0 \) in (2.1) and using the induction hypothesis:

\[
(-1)^{n+1}a_{0,n+1} = \left(2 + \frac{\alpha}{n+1}\right)(-1)^{n+1}\left(\frac{n+\alpha}{n}\right) + \left(1 + \frac{\alpha}{n}\right)(-1)^{n}\left(\frac{n-1+\alpha}{n-1}\right)
\]

\[
= (-1)^{n+1}\left(\frac{n+1+\alpha}{n}\right).
\]

Now, instead of \( \{Q_n\}_{n=0}^{\infty} \), we consider the sequence of orthogonal polynomials \( \{\tilde{Q}_n\}_{n=0}^{\infty} \) normalized so that \( \tilde{Q}_n(0) = 1 \), \( n \in \mathbb{N}_0 \), i.e.,

\[
Q_n(x) = (-1)^n\left(\frac{n+\alpha}{n}\right)\tilde{Q}_n(x), \quad n \in \mathbb{N}_0.
\]

It follows from (2.1) and (2.2) that \( \{\tilde{Q}_n\}_{n=0}^{\infty} \) are determined by

\[
\left(1 + \frac{\alpha}{n+1}\right)\tilde{Q}_{n+1}(x) = \left(2 + \frac{\alpha}{n+1} - x\right)\tilde{Q}_n(x) - \tilde{Q}_{n-1}(x), \quad n \geq 1,
\]

(2.4)

\[
\tilde{Q}_0(x) = 1, \quad \tilde{Q}_1(x) = 1 - \frac{x}{\alpha+1}.
\]

(2.5)

Writing \( \tilde{Q}_n \) in the form

\[
\tilde{Q}_n(x) = 1 - A_{1,n}x + A_{2,n}x^2 - \cdots + (-1)^nA_{n,n}x^n
\]

and rewriting (2.4) as

\[
\tilde{Q}_{n+1}(x) - \tilde{Q}_n(x) = \frac{n+1}{n+\alpha+1}(\tilde{Q}_n(x) - \tilde{Q}_{n-1}(x)) + \frac{n+1}{n+\alpha+1}x\tilde{Q}_n(x), \quad n \in \mathbb{N},
\]

we deduce the following recurrence relation for the evaluation of the coefficients \( \{A_{i,m}\} \):

\[
A_{i,n+1} - A_{i,n} = \frac{n+1}{n+\alpha+1}(A_{1,n} - A_{i,n-1}) + \frac{n+1}{n+\alpha+1}A_{i-1,n}, \quad n \geq \ell \geq 1,
\]

(2.6)

with \( A_{0,n} = 1 \) and \( A_{1,1} = \frac{1}{\alpha+1} \).

Since, by Proposition 1, \( c_{n-2}^2(\alpha) \) is equal to the smallest zero of \( \tilde{Q}_n \), it follows that \( c_{n}^2(\alpha) \) equals the largest zero of the reciprocal polynomial of \( \tilde{Q}_n \),

\[
R_n(x) = x^n\tilde{Q}_n(1/x).
\]

(2.7)

The above observations allow us to reformulate Proposition 1 in the following equivalent form:
Proposition 2. The squared best Markov constant \( c_n^2(\alpha) \) is equal to the largest zero of the polynomial

\[
R_n(x) = x^n - A_{1,n} x^{n-1} + A_{2,n} x^{n-2} - \cdots + (-1)^n A_{n,n} = 0.
\]

The coefficients of \( R_n \) are evaluated recursively by the following procedure:

- \( A_{1,1} = \frac{1}{\alpha+1} \);
- Set \( A_{0,m} = 1, \ m = 0, \ldots, n \);
- For \( i = 1 \) to \( n \):
  1. Find the sequence \( \{D_{i,m}\}_{m=i-1}^n \) as solution of the recurrence equation

\[
D_{i,m+1} = \frac{m+1}{m+\alpha+1} D_{i,m} + \frac{m+1}{m+\alpha+1} A_{i-1,m}
\]

with the initial condition \( D_{i,i-1} = 0 \);
  2. Evaluate

\[
A_{i,n} = \sum_{m=i}^n D_{i,m}.
\]

2.2. POLYNOMIALS WITH POSITIVE ROOTS: BOUNDS FOR THE LARGEST ZERO

Let \( P \) be a monic polynomial of degree \( n \) with zeros \( \{x_i\}_{i=1}^n \),

\[
P(x) = \prod_{i=1}^n (x - x_i) = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^n b_n.
\]

The coefficients \( b_r = b_r(P), \ r = 1, \ldots, n \), are given by the elementary symmetric functions of \( \{x_i\}_{i=1}^n \),

\[
b_r = s_r = s_r(P) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}, \quad r = 1, \ldots, n.
\]

It is well known that the elementary symmetric functions \( \{s_r\} \) and the Newton functions (sums of powers of \( x_i \))

\[
p_r = p_r(P) = \sum_{i=1}^n x_i^r, \quad r = 1, 2, 3, \ldots,
\]

are connected by the Newton identities:

\[
p_r + \sum_{i=1}^{r-1} (-1)^i p_{r-i} s_i + (-1)^r s_r = 0, \quad \text{if } 1 \leq r \leq n,
\]

\[
p_r + \sum_{i=1}^n (-1)^i p_{r-i} s_i = 0, \quad \text{if } r > n.
\]
For a proof, see e.g. [10] or [4].

Our interest in the Newton functions is motivated by the fact that they provide tight bounds for the largest zero of a polynomial whose roots are all positive. For any such polynomial \( P \), we set

\[
\ell_k(P) := \frac{p_k(P)}{p_{k-1}(P)}, \quad u_k(P) := \left[ p_k(P) \right]^{1/k}, \quad k \in \mathbb{N},
\]

with the convention that \( p_0(P) := \deg(P) \).

**Proposition 3.** Let \( P(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^{n-1} b_{n-1} x + (-1)^n b_n \) be a polynomial with positive zeros \( x_1 \leq x_2 \leq \cdots \leq x_n \).

Then the largest zero \( x_n \) of \( P \) satisfies the inequalities

\[
\ell_k(P) \leq x_n < u_k(P), \quad k \in \mathbb{N}. \tag{2.13}
\]

Moreover, the sequence \( \{\ell_k(P)\}_{k=1}^{\infty} \) is monotonically increasing while the sequence \( \{u_k(P)\}_{k=1}^{\infty} \) is monotonically decreasing, and

\[
\lim_{k \to \infty} \ell_k(P) = \lim_{k \to \infty} u_k(P) = x_n. \tag{2.14}
\]

**Proof.** For \( i = 1, \ldots, n-1 \), we set \( a_i := \frac{x_i}{x_n} \), then \( 0 < a_i \leq 1 \). Now both inequalities (2.13) and the limit relations (2.14) readily follow from the representations

\[
\ell_k(P) = \frac{a_1^k + \cdots + a_{n-1}^k + 1}{a_1^k + \cdots + a_{n-1}^k + 1} x_n, \quad u_k(P) = (a_1^k + \cdots + a_{n-1}^k + 1)^{1/k} x_n.
\]

The monotonicity of the sequence \( \{\ell_k(P)\}_{k=1}^{\infty} \) follows easily from Cauchy-Bouniakowsky’s inequality. Indeed, we have

\[
\left( \sum_{i=1}^{n} x_i^k \right)^2 = \left( \sum_{i=1}^{n} \frac{x_i^{k-1}}{x_i} x_i^{k+1} \right)^2 \leq \left( \sum_{i=1}^{n} x_i^{k-1} \right) \left( \sum_{i=1}^{n} x_i^{k+1} \right),
\]

whence \( p_k^2(P) \leq p_{k-1}(P) p_{k+1}(P) \), and consequently

\[
\ell_k(P) = \frac{p_k(P)}{p_{k-1}(P)} \leq \frac{p_{k+1}(P)}{p_k(P)} = \ell_{k+1}(P).
\]

To prove monotonicity of the sequence \( \{u_k(P)\}_{k=1}^{\infty} \), we recall that \( 0 < a_i \leq 1 \) and therefore \( a_i^{k+1} \leq a_i^k \). We have

\[
(a_1^{k+1} + \cdots + a_{n-1}^{k+1} + 1)^{1/(k+1)} < (a_1^{k+1} + \cdots + a_{n-1}^{k+1} + 1)^{1/k} \leq (a_1^k + \cdots + a_{n-1}^k + 1)^{1/k},
\]

which yields

\[
\quad u_{k+1}(P) < u_k(P). \quad \square
\]

3. COMPUTER ALGEBRA ASSISTED PROOF OF THE RESULTS

Here we give the algorithms, the source code and the results of the computer algebra assisted proof of estimates (1.10)-(1.17) in Theorem 1. While the case \( k = 3 \) and to a certain extent \( k = 4 \) could be studied by hand, it seems impossible to provide similar calculations for larger \( k \). We implement the idea from [5] for estimating \( c_n(\alpha) \) using \( k = 3 \) highest degree coefficients of the polynomial \( R_n(x) \) and with the assistance of Wolfram’s Mathematica v. 10 software we investigate the cases \( k = 4, 5, 6 \), as well. Software based on the algorithms described below failed with calculations for \( k > 6 \).

Henceforth, we write the polynomial \( R_n \) from (2.7) and (2.8) in the form

\[
R_n(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} + \cdots + (-1)^n b_n.
\]

3.1. LOWER BOUNDS FOR \( c_n(\alpha) \)

We apply Proposition 3 to estimate the largest zero \( x_n = c_n^2(\alpha) \) of the polynomial \( R_n(x) \) from below,

\[
x_n \geq \ell_k(R_n) = \frac{p_k(R_n)}{p_{k-1}(R_n)}, \quad k = 3, 4, 5, 6,
\]

and then with the help of computer algebra obtain a further estimation of the form

\[
\ell_k(R_n) \geq c n(n + \sigma(\alpha + 1)),
\]

with the optimal (i.e., the largest possible) constants \( c = c(k) \) and \( \sigma = \sigma(k) \).

**Algorithm 1** Estimating \( c_n(\alpha) \) from below

**Input:** \( k \in \{3, 4, 5, 6\} \) – the number of the highest degree coefficients of \( R_n(x) \)

**Step 1.** Express the power sums \( p_k(R_n) \) and \( p_k(R_n) \) in terms of \( \{b_i\}_{i=1}^k \)

**Step 2.** Find coefficients \( \{b_i\}_{i=1}^k \) in terms of \( n \) and \( \alpha \) using Proposition 2

**Step 3.** Find a proper value \( \sigma \) for parameter \( s \) in \( p_k - c n(n + s(\alpha + 1))p_{k-1} \), where \( c \) is the coefficient of \( n^2 \) in the quotient \( p_k/p_{k-1} \)

**Step 4.** Represent the numerator of \( f = p_k - c n(n + s(\alpha + 1))p_{k-1} \) in powers of \( n \) and \( (\alpha + 1) \)

**Step 5.** Estimate from below the expression \( f \) to prove that \( f \geq 0 \)

**Step 1:** Let \( \{x_i\}_{i=1}^n \) be all the zeros of the polynomial \( R_n(x) \) from (2.7). In order to express a power sum \( p_r = \sum_{i=1}^r x_i^r \), \( 1 \leq r \leq n \), by \( \{b_i\}_{i=1}^r \), we apply the direct formula

\[
p_r = \begin{vmatrix}
  b_1 & 1 & 0 & \cdots & 0 \\
  2b_2 & b_1 & 1 & \cdots & 0 \\
  3b_3 & b_2 & b_1 & \cdots & 0 \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  rb_r & b_{r-1} & b_{r-2} & \cdots & b_1
\end{vmatrix}
\]
which easily follows from the Newton identities (2.11).

Below is the code of the programme and the results for \( k = 1, \ldots, 6 \):

\[
\begin{align*}
\text{Step 2:} & \quad \text{We find coefficients } \{b_i\}_{i=1}^k \text{ of the polynomial } R_n(x) \text{ using Proposition 2. The source and the results for } k = 1, \ldots, 6 \text{ follow below:} \\
\text{Step 3:} & \quad \text{The quotient } p_k/p_{k-1} \text{ is a quadratic polynomial in } n, \text{ and we denote by } c \text{ its leading coefficient.}
\end{align*}
\]
The goal of this step is to find a proper value (say $\sigma$) for parameter $s$ in the expression

$$f_s = p_k - c n (n + s (\alpha + 1)) p_{k-1},$$

such that $f_s \geq 0$ for all admissible $\alpha$ and $n$. For a fixed $k$ quantity $f_s$ depends on $\alpha$, $n$, and $s$. It is a polynomial of degree $2k - 1$ in $n$ and a rational function in $\alpha$. Let us write the numerator of $f_s$ in the form

$$\sum_{i=1}^{2k-1} \sum_{j=0}^d \mu_{i,j}(s)(\alpha + 1)^{d-j} a^{2k-i}.$$

The highest order coefficients in $\sum_j \mu_{i,j}(s)(\alpha + 1)^{d-j}$ are linear functions in $s$ of the form $A_i - B_i s$, with $A_i > 0$ and $B_i > 0$. We denote their zeros by $s_i$ for each $i$ and set $\sigma = \min_i s_i$. Since we seek estimates valid for all $\alpha > -1$, our choice of $\sigma$ guarantee that for $\alpha$ sufficiently large the inequality $\sum_j \mu_{i,j}(s)(\alpha + 1)^{d-j} > 0$ holds true.

The code is as follows:

```plaintext
t = PolynomialQuotient[pk, p[n, n];
c = Factor[Coefficient[t, n, 2]];
s = p_k - c n (n + s (\alpha + 1)) p_{k-1};
um = Numerator[Together[Factor[s, \alpha]]];
Dolgs = Factor[Coefficient[num, n, 1]];
num = Normal[Series[gs, \alpha, -1, Exponent[gs, \alpha]]];
sols = Solve[Coefficient[num, \alpha, Exponent[gs, \alpha]] == 0, \alpha, Reals];
s[i] = Select[sols, \alpha[2 k - 1, -1]]
\sigma = Min[Table[s[i]], \{2, 2 k - 1\}];
```

Table 1 gives results for the optimal values of $c$ and $\sigma$ for $k = 3, 4, 5, 6$.

Table 1: The optimal values of $c$ and $\sigma$ in the lower bounds for $c_n^2(\alpha)$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$c$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$\frac{(\alpha + 1)(\alpha + 5)}{5\alpha + 17}$</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2(\alpha + 1)(\alpha + 3)(\alpha + 7)}{2(7\alpha + 31)}$</td>
<td>$\frac{25}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{(\alpha + 1)(\alpha + 9)(5\alpha + 17)}{21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073}$</td>
<td>$\frac{84}{2}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{(\alpha + 1)(\alpha + 3)(\alpha + 5)(\alpha + 11)(7\alpha + 31)}{7}$</td>
<td></td>
</tr>
</tbody>
</table>

Step 4: We set

$$f = p_k - c n (n + \sigma (\alpha + 1)) p_{k-1} =: \frac{\varphi(n, \alpha)}{\psi(\alpha)}$$

66

with $c$ and $\sigma$ determined in Step 3. Here, $\varphi(n, \alpha)$ is a bivariate polynomial in $n$ and $\alpha$, and $\psi(\alpha)$ is a polynomial in $\alpha$. More precisely, $\varphi(n, \alpha)$ has degree $2k - 1$ in $n$, and degree $d$ in $\alpha$ which our programme calculates for each fixed $k$.

Note that $\psi(\alpha) > 0$ for $\alpha > -1$ since it is a product of powers of $\alpha + j$, $j \geq 1$ and multipliers $A\alpha + B$, $0 < A < B$. Therefore, sign $f = \text{sign } \varphi$.

We expand $\varphi(n, \alpha)$ in the form

$$\varphi(n, \alpha) = \sum_{i=1}^{2k-1} \sum_{j=0}^{d} \mu_{i,j} (\alpha + 1)^{d-j} n^{2k-i},$$

where $M = \left(\mu_{i,j}\right)_{i=1,j=0}^{2k-1,d}$ and all entries $\mu_{i,j}$ are integer numbers.

The source for computation of the matrix $M$ is listed below.

Step 5: If there are coefficients $\mu_{i,j} < 0$ we need additional arguments to verify that $f \geq 0$ for all $\alpha > -1$ and $n \geq k$. In a case some of coefficients $\mu_{i,j} < 0$ we apply the next step of the algorithm.

The results for $k = 3, 4, 5, 6$ are given together with the estimates from Step 5.

The procedure described below checks recursively all coefficients $\lambda_{i,j}$ and makes the corresponding estimations. We need not introduce a new matrix after each iteration, but only replace a pair of elements in a column of $\Lambda$ with new entries in such a manner that the value of the function

$$\Phi(\Lambda) = \sum_{i=1}^{2k-1} \sum_{j=0}^{d} \lambda_{i,j} (\alpha + 1)^{d-j} n^{2k-i},$$

decreases. At the end of the procedure we get a matrix $\Lambda$ satisfying $0 \leq \Lambda \leq M$ (in the sense that $0 \leq \lambda_{i,j} \leq \mu_{i,j}$ for all $i,j$) and therefore

$$0 \leq \Phi(\Lambda) \leq \Phi(M) = \varphi(n, \alpha).$$
Suppose that $\lambda_{i,j} < 0$ for some pair of indices $i, j$. Then we set
\[ h := \min\{i - \eta : \lambda_{\eta,j} > 0, \ 1 \leq \eta \leq i - 1\} \quad \text{and} \quad \delta := \frac{\lambda_{i,j}}{k^{i-h}} \quad (\delta < 0). \]

If $\lambda_{h,j} + \delta \geq 0$, for $n \geq k$ we have
\[
(\lambda_{h,j} + \delta) n^{2k-h} + 0 n^{2k-i} = \left(\lambda_{h,j} + \frac{\lambda_{i,j}}{k^{i-h}}\right) n^{2k-h} + \lambda_{i,j} n^{2k-i} - \lambda_{h,j} n^{2k-h} + \lambda_{i,j} n^{2k-i}.
\]

Otherwise, if $\lambda_{h,j} + \delta < 0$, for $n \geq k$ we have
\[
0 n^{2k-h} + (\lambda_{h,j} k^{i-h} + \lambda_{i,j}) n^{2k-i} = \lambda_{h,j} n^{2k-i} - \lambda_{i,j} n^{2k-i} + \lambda_{h,j} n^{2k-h} + \lambda_{i,j} n^{2k-i}.
\]

So, replacing only two elements in $\Lambda$,
\[
\begin{cases}
\lambda_{h,j} := \lambda_{h,j} + \lfloor \delta \rfloor & \text{and} \ \lambda_{i,j} := 0, \quad \text{if} \ \lambda_{h,j} + \delta \geq 0, \\
\lambda_{i,j} := \lambda_{h,j} k^{i-h} + \lambda_{i,j} & \text{and} \ \lambda_{h,j} := 0, \quad \text{otherwise},
\end{cases}
\]
we obtain that
\[
\lambda_{h,j}(\alpha + 1)^{d+1-j} n^{2k-h} + \lambda_{i,j}(\alpha + 1)^{d+1-j} n^{2k-i}
\]
decreasess for the new values of $\lambda_{h,j}$ and $\lambda_{i,j}$, and hence $\Phi(\Lambda)$ also decreases.

Applying recursively the above iteration process for $i = 2k - 1, 2k - 2, \ldots, 1$ and $j = 0, 1, \ldots, d$ we finally obtain a matrix $\Lambda$ satisfying $0 \leq \Lambda \leq M$. Then $\varphi(n, \alpha) \geq 0$, $f \geq 0$ and therefore
\[
c_p^2(\alpha) \geq \frac{p_k}{p_{k-1}} \geq c n(n + \sigma(\alpha + 1))
\]
for the optimal $c$ and $\sigma$ evaluated in Step 3. For $k = 3, 4, 5, 6$ we obtain estimates (1.10), (1.12), (1.14), and (1.16), respectively.

The following source implements the procedure described in Step 5.

```plaintext
\begin{verbatim}
\lambda = \mu;
For[i=2k-1, i > 1, i--]
  For[j=1, j < i+1, j++]
    b = \!\!\textbf{if} \!\! \textbf{First}[\!\! \textbf{FirstPosition}[\textbf{Positive}[\lambda[i+1:-1,j]], \textbf{True}]]
    \delta = \lambda[i,j] / \!\! (k^{i-b})
    \!\!\textbf{if} \!\! \lambda[i,j]+\delta > 0 \!\! \lambda[i,j] = \lambda[i,j]+\textbf{Floor}[\delta], \lambda[i,j] = 0,
    \!\!\textbf{if} \!\! k^{i-j} > 0 \!\! \lambda[i,j] = \lambda[i,j]+0 \!\! \lambda[i,j] = 0,
    \!\!\textbf{if} \!\! k^{i-j} > 0 \!\! \lambda[i,j] = \lambda[i,j]+1 \!\! \lambda[i,j] = 0, i = i+1]
Print[\!\! \lambda = '; \!\! \textbf{MatrixForm}[\lambda]]
Print[\!\! \textbf{M} = '; \!\! \textbf{MatrixForm}[\!\! \textbf{M}]]
\end{verbatim}
```

Next, we give matrices $M$ from Step 4 and $\Lambda$ from Step 5 obtained with Mathematica.

Case $k = 3$:
This partial case needs a special attention as we have to assume strict inequality $n > k$, i.e., $n \geq 4$, to obtain estimate (1.10). This causes a minor modification in Step 5 of Algorithm 1, namely, replacement of $k^{-h}$ with $(k+1)^{-h}$. Namely, we determine $\delta := \lambda_{h,j}/(k+1)^{-h}$ and set
\[
\begin{align*}
\lambda_{h,j} & := \lambda_{h,j} + [\delta] \\
\lambda_{j} & := 0, \quad \text{if } \lambda_{h,j} + \delta \geq 0, \\
\lambda_{i} & := \lambda_{h,j} (k+1)^{-h} + \lambda_{j} \quad \text{and } \lambda_{h,j} := 0, \quad \text{otherwise}.
\end{align*}
\]
Matrices $M$ and $\Lambda$ in this case are
\[
\Lambda = \begin{pmatrix}
0 & -4 & 225 & 360 \\
0 & 0 & 390 & 510 & 720 \\
0 & 0 & 0 & 0 & 0 \\
2100 & 46515 & 120645 & 2404465 & 10159765 & 20026720 & 25810890 & 16625700 & 19890000 \\
2756 & 106120 & 876330 & 2582090 & 7614630 & 17567550 & 18060000 & 6300000 & 0 \\
0 & 11060 & 662604 & 2635840 & 6125776 & 11212880 & 7413000 & 0 & 0
\end{pmatrix},
M = \begin{pmatrix}
4 & -4 & 225 & 360 \\
0 & 0 & 390 & 510 & 720 \\
0 & 0 & 0 & 0 & 0 \\
19024 & 130709 & 3459019 & 84677472 & 334863520 & 1237510250 & 24837120 & 1892110 & 5670000 \\
15330 & 229110 & 1642830 & 6282570 & 16699200 & 24837120 & 1892110 & 5670000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Although there is a negative element of $\Lambda$, from $4(\alpha + 1)^2 - 4(\alpha + 1) + 225 \geq 0$ for all $\alpha > -1$ we conclude that $4(\alpha + 1)^3 - 4(\alpha + 1)^2 + 225(\alpha + 1) + 360 > 0$ and consequently $\Phi(\Lambda) \geq 0$ for $n \geq 4$.

By a direct verification one can see that inequality (1.10) holds also in the case $n = k = 3$.

Case $k = 4$:

\[
\Lambda = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
10200 & 72480 & 323700 & 1413060 & 3602340 & 4340700 & 19890000 \\
8715 & 30891 & 339695 & 2625289 & 7966210 & 13275700 & 12707100 & 5670000 & 0 \\
0 & 0 & 112060 & 4777900 & 3435000 & 0 & 0 & 0 & 0
\end{pmatrix},
M = \begin{pmatrix}
4 & -4 & 225 & 360 \\
0 & 0 & 390 & 510 & 720 \\
0 & 0 & 0 & 0 & 0 \\
2100 & 46515 & 120645 & 2404465 & 10159765 & 20026720 & 25810890 & 16625700 & 19890000 \\
2800 & 106120 & 876330 & 2582090 & 7614630 & 17567550 & 18060000 & 6300000 & 0 \\
0 & 11960 & 722904 & 2635840 & 6125776 & 11212880 & 7413000 & 0 & 0
\end{pmatrix}.
\]

Case $k = 5$:

\[
\Lambda = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
M = \begin{pmatrix}
4 & -4 & 225 & 360 \\
0 & 0 & 390 & 510 & 720 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Case $k = 6$:

We apply Proposition 3 to estimate the largest zero $x_n = c_n^2(\alpha)$ of the polynomial $R_n(x)$ from above,

$$x_n \leq u_k(R_n) = p_k(R_n)^{1/k}, \quad k = 3, 4, 5, 6.$$

Then with the assistance of computer algebra we obtain a further estimation of the form

$$u_k(R_n) \leq c^{1/k} (n + 1)(n + \sigma(\alpha + 1)),$$

with the optimal (i.e., the smallest possible) constants $c = c(k)$ and $\sigma = \sigma(k)$.

The algorithm is analogous to Algorithm 1, and the code has only a few differences which are specified later.

**Algorithm 2** Estimating $c_n(\alpha)$ from above

**Input:** $k \in \{3, 4, 5, 6\}$ – the number of the highest degree coefficients of $R_n(x)$

**Step 1.** Express the power sum $p_k(R_n)$ in terms of $\{b_i\}_{i=1}^k$.

**Step 2.** Find $\{b_i\}_{i=1}^k$ in terms of $n$ and $\alpha$ using Proposition 2.

**Step 3.** Find a proper value $\sigma$ for parameter $s$ in the expression $c(n + 1)^k(n + s(\alpha + 1))^k - p_k$, where $c$ is the coefficient of $n^{2k}$ in $p_k$.

**Step 4.** Represent the numerator of $f = c(n + 1)^k(n + s(\alpha + 1))^k - p_k$ in powers of $n$ and $(\alpha + 1)$.

**Step 5.** Estimate from below the expression $f$ to prove that $f \geq 0$.

---

Step 1: The same as in Algorithm 1.

Step 2: Identical to that in Algorithm 1.
Step 3: The only differences with Algorithm 1 are that we set \( c \) to be the coefficient of \( n^2k \) in \( p_k \) and

\[
f_s = c(n + 1)^k(n + s(\alpha + 1))^k - p_k.
\]

The highest order coefficients in \( \sum_j \mu_{i,j}(s)(\alpha + 1)^{d-j} \) are functions in \( s \) of the form \( A_i s^\nu - B_i \), with \( A_i > 0 \) and \( B_i \geq 0 \). We denote their non-negative zeros by \( s_i \) for each \( i \) and choose \( \sigma = \max_i s_i \).

The results for \( k = 3, 4, 5, 6 \) obtained by symbolic computations are given in Table 2.

Table 2: The optimal values of \( c \) and \( \sigma \) in the upper bounds for \( c^2_2(\alpha) \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( c )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>((\alpha + 1)^3(\alpha + 3)(\alpha + 5))</td>
<td>( \frac{5\alpha + 17}{3} )</td>
</tr>
<tr>
<td>5</td>
<td>(2(\alpha + 1)^4(\alpha + 3)^2(\alpha + 5)(\alpha + 7))</td>
<td>( \frac{7(7\alpha + 31)}{4} )</td>
</tr>
<tr>
<td>6</td>
<td>((\alpha + 1)^6(\alpha + 3)^4(\alpha + 5)(\alpha + 7)(\alpha + 9))</td>
<td>( \frac{9(21\alpha^3 + 299\alpha^2 + 1391\alpha + 2073)}{5} )</td>
</tr>
</tbody>
</table>

Step 4: With \( c \) and \( \sigma \) determined in the previous Step 3 we set

\[ f = c(n + 1)^k(n + \sigma(\alpha + 1))^k - p_k =: \frac{\varphi(n, \alpha)}{\psi(\alpha)}. \]

The rest of the source has no difference with Step 4 of Algorithm 1.

Step 5: The same as in Algorithm 1. Using the same recursive procedure we find a matrix \( A \) satisfying \( 0 \leq A \leq M \). Then \( \varphi(n, \alpha) \geq 0 \), \( f \geq 0 \) and therefore

\[ c^2_n(\alpha) \leq p_k \leq c(n + 1)^k(n + \sigma(\alpha + 1))^k \]

for the corresponding \( c \) and \( \sigma \) evaluated in Step 3. For \( k = 3, 4, 5, 6 \) we obtain estimations (1.11), (1.13), (1.15), and (1.17), respectively.

The matrices \( M \) from Step 4 and \( A \) from Step 5 obtained with Mathematica are given below.

Case \( k = 3 \):

\[
A = \begin{pmatrix}
0 & 0 & 0 & 1500 & 3300 \\
0 & 115 & 1885 & 4170 & 4233 \\
32 & 598 & 3026 & 6360 & 0 \\
96 & 979 & 2143 & 850 & 0 \\
96 & 624 & 1098 & 0 & 0
\end{pmatrix}, \quad
M = \begin{pmatrix}
0 & 0 & 0 & 1500 & 3300 \\
0 & 115 & 1885 & 4170 & 4650 \\
32 & 598 & 3026 & 6360 & 0 \\
96 & 979 & 2143 & 1560 & -1950 \\
96 & 624 & 1098 & 0 & 0
\end{pmatrix}
\]
## 4. CONCLUDING REMARKS

1. In our computer algebra approach for derivation of bounds for the best Markov constant $c_n(\alpha)$ we perform some optimization with respect to parameter $s$.
Our motivation for searching lower bounds for \( c^2_n(\alpha) \) with a factor depending on \( n \) of the special form \( n(n + \sigma(\alpha + 1)) \) is Corollary D(ii).

An interesting observation about the lower bounds \( \zeta_{n,k}(\alpha) \) in Theorem 1 is that they imply
\[
\frac{k n}{k + 1} = \lim_{\alpha \to \infty} \alpha \frac{c^2_{n,k}(\alpha)}{c^2_n(\alpha)} \leq \lim_{\alpha \to \infty} \alpha c^2_{n,k}(\alpha), \quad 3 \leq k \leq 6
\]
(the lower bound in Corollary 2 follows from the case \( k = 6 \)). This observation and Proposition 3 give rise for the following

**Conjecture 1.** The best Markov constant \( c_n(\alpha) \) satisfies:
\[
\lim_{\alpha \to \infty} \alpha c^2_n(\alpha) = n.
\]

We also performed a search for lower bounds for \( c^2_n(\alpha) \) with a factor depending on \( n \) of the form \((n + 1)(n + \sigma(\alpha + 1))\). Such a choice is reasonable, as the resulting lower bounds preserve the limit relation in Corollary D(i). The optimal value then is \( \sigma = -1/3 \) (the same for all \( k \), \( 3 \leq k \leq 6 \), and we obtain lower bounds as in Theorem 1 with \( n(n + \sigma(\alpha + 1)) \) replaced by \((n + 1)(n - (\alpha + 1)/3)\). These lower bounds make sense only for \( n > (\alpha + 1)/3 \), and are better than those in Theorem 1 only for \( \alpha \) close to \(-1\).

**2.** The bounds \( \left( \zeta_{n,k}(\alpha), \zeta_{n,k}(\alpha) \right) \) (3 \( \leq k \leq 6 \)) in Theorem 1 imply bounds \( (\ell_k(\alpha), u_k(\alpha)) \) (occurring in the middle columns of Tables 1 and 2) for the asymptotic Markov constant \( c(\alpha) \), and the bounds deduced with a larger \( k \) are superior. While the lower bounds \( \ell_k(\alpha) \) are of the correct order \( O(\alpha^{-1}) \) as \( \alpha \to \infty \), for the upper bound \( u_k(\alpha) \) we have \( u_k(\alpha) = O(\alpha^{-1+1/k}) \) as \( \alpha \to \infty \), \((3 \leq k \leq 6)\). The ratio
\[
\rho_k(\alpha) := \frac{u_k(\alpha)}{\ell_k(\alpha)}, \quad 3 \leq k \leq 6,
\]
tends to 1 as \( \alpha \to -1 \), which indicates that for moderate \( \alpha \) the bounds \( \ell_k(\alpha) \) and \( u_k(\alpha) \) are rather tight. This observation is clearly seen in the particular case \( \alpha = 0 \), where, according to Turán’s result, we have \( c(0) = \frac{2}{\pi} \). We give the lower and the upper bounds for \( c(0) \) and the overestimation factors in Table 3.

**3.** Another interesting observation, concerning the coefficients of \( R_n \) inspires the following

**Conjecture 2.** For every fixed \( k \in \mathbb{N} \), the coefficient \( b_{k,n} \), \( n > k \), of the polynomial \( R_n(x) = x^n - b_{1,n} x^{n-1} + b_{2,n} x^{n-2} - \cdots + (-1)^n b_{n,n} \), satisfies
\[
b_{k,n} = \frac{n^{2k}}{2^k k!(\alpha + 1)(\alpha + 2)(\alpha + 2k - 1)} + O(n^{2k-1}). \quad (4.1)
\]

Conjecture 2 is verified with our computer algebra approach for \( 1 \leq k \leq 6 \), but so far we do not have a proof for the general case. Having (4.1) proved,
we could try to find the explicit form of $d_k$, the coefficient of $n^{2k}$ in Newton’s function $p_k(R_n)$, and consequently to obtain two sequences $\{\ell_k\}$ and $\{u_k\}$ defined by $\ell_k = \sqrt{d_k/d_{k-1}}$ and $u_k = \sqrt[3]{d_k}$ which converge monotonically from below and from above, respectively, to $c(\alpha)$, the sharp asymptotic Markov constant.

Table 3: The lower and the upper bounds for the asymptotic Markov constant $c(0)$ and the overestimation factors.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\ell_k(0)$</th>
<th>$u_k(0)$</th>
<th>$\frac{c(0)}{\ell_k(0)}$</th>
<th>$\frac{u_k(0)}{c(0)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\sqrt{\frac{2}{5}} \approx 0.6324553$</td>
<td>$\sqrt{\frac{3}{5}} \approx 0.63677321$</td>
<td>1.006584242</td>
<td>1.00024103</td>
</tr>
<tr>
<td>4</td>
<td>$\sqrt{\frac{17}{47}} \approx 0.63620901$</td>
<td>$\sqrt{\frac{21}{63}} \approx 0.63663212$</td>
<td>1.00064564</td>
<td>1.00001939</td>
</tr>
<tr>
<td>5</td>
<td>$\sqrt{\frac{62}{185}} \approx 0.63657580$</td>
<td>$\sqrt[10]{\frac{31}{29}} \approx 0.63662085$</td>
<td>1.00006906</td>
<td>1.00000170</td>
</tr>
<tr>
<td>6</td>
<td>$\sqrt[12]{\frac{2073}{5115}} \approx 0.63661494$</td>
<td>$\sqrt[12]{\frac{2073}{467775}} \approx 0.63661987$</td>
<td>1.00000757</td>
<td>1.00000015</td>
</tr>
</tbody>
</table>

Although the ratios $\rho_k$, $3 \leq k \leq 6$, satisfy $\rho_k(\alpha) \to \infty$ as $\alpha \to \infty$, they grow rather slowly. For instance, $\rho_6(\alpha) < 2$ for $\alpha < 140000$, see Figure 1.

Figure 1: The graph of $\rho_6(\alpha) < 2$.

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5. REFERENCES


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