ON AN EQUATION INVOLVING FRACTIONAL POWERS
WITH PRIME NUMBERS OF A SPECIAL TYPE

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We consider the equation \( [p_1^c] + [p_2^c] + [p_3^c] = N \), where \( N \) is a sufficiently large integer, and \([t]\) denotes the integer part of \( t \). We prove that if \( 1 < c < \frac{17}{16} \), then it has a solution in prime numbers \( p_1, p_2, p_3 \) such that each of the numbers \( p_1 + 2, p_2 + 2, p_3 + 2 \) has at most \( \left\lfloor \frac{95}{17 - 16c} \right\rfloor \) prime factors, counted with their multiplicities.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In 1937 I. M. Vinogradov [16] proved that for every sufficiently large odd integer \( N \) the equation

\[
p_1 + p_2 + p_3 = N
\]  

(1.1)

has a solution in prime numbers \( p_1, p_2, p_3 \).

Analogous problem was considered in 1952 by Piatetski-Shapiro [9]. If \( H(c) \) denotes the least integer \( s \) such that the diophantine inequality

\[
|p_1^c + \cdots + p_s^c - N| < \varepsilon,
\]

has a solution in primes \( p_1, \ldots, p_s \), where \( c > 1 \) is not an integer, \( \varepsilon > 0 \) is small, and \( N \) is large real number, then Piatetski-Shapiro proved that

\[
\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4.
\]

He also proved that if $1 < c < 3/2$, then $H(c) \leq 5$. In 1992, Tolev [14] established that if $1 < c < \frac{15}{14}$, then the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < N^{-\kappa}$$

has a solution in prime numbers $p_1, p_2, p_3$ for certain $\kappa = \kappa(c) > 0$. Several improvements were made and the strongest of them is due to Baker and Weingartner [1], who improved Tolev’s result with $1 < c < \frac{16}{17}$.

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In 1995, M. B. Laporta and D. I. Tolev [7] considered the equation

$$[p_1^c] + [p_2^c] + [p_3^c] = N, \quad (1.2)$$

where $c \in \mathbb{R}$, $c > 1$, $N \in \mathbb{N}$ and $[t]$ denotes the integer part of $t$. They showed that if $1 < c < \frac{17}{16}$ and $N$ is a sufficiently large integer, then the equation (1.2) has a solution in prime numbers $p_1, p_2, p_3$.

For any natural number $r$, let $P_r$ denote the set of $r$-almost primes, i.e. the set of natural numbers having at most $r$ prime factors counted with multiplicities. There are many papers devoted to the study of problems involving primes and almost primes. For example, in 1973 J. R. Chen [4] established that there exist infinitely many primes $p$ such that $p + 2 \in P_2$. In 2000 Tolev [12] proved that for every sufficiently large integer $N \equiv 3 \pmod{6}$ the equation (1.1) has a solution in prime numbers $p_1, p_2, p_3$ such that $p_1 + 2 \in P_2$, $p_2 + 2 \in P_5$, $p_3 + 2 \in P_7$. Thereafter this result was improved by Matomäki and Shao [8], who showed that for every sufficiently large integer $N \equiv 3 \pmod{6}$ the equation (1.1) has a solution in prime numbers $p_1, p_2, p_3$ such that $p_1 + 2, p_2 + 2, p_3 + 2 \in P_2$.

Recently Tolev [15] established that if $N$ is sufficiently large, $E > 0$ is an arbitrarily large constant and $1 < c < \frac{17}{16}$, then the inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E}$$

has a solution in prime numbers $p_1, p_2, p_3$, such that each of the numbers $p_1 + 2, p_2 + 2, p_3 + 2$ has at most $\left[\frac{95}{17 - 10c}\right]$ prime factors, counted with their multiplicities.

In this paper, we prove the following

**Theorem 1.1.** Suppose that $1 < c < \frac{17}{16}$. Then for every sufficiently large $N$ the equation (1.2) has a solution in prime numbers $p_1, p_2, p_3$, such that each of the numbers $p_1 + 2, p_2 + 2, p_3 + 2$ has at most $\left[\frac{95}{17 - 10c}\right]$ prime factors, counted with their multiplicities.

We note that the integer $\left[\frac{95}{17 - 10c}\right]$ is equal to 95 if $c$ is close to 1 and it is large if $c$ is close to $\frac{17}{16}$.

To prove Theorem 1.1 we combine ideas developed by Laporta and Tolev [7] and Tolev [15]. First we apply a version of the vector sieve and then the circle method. In section 4 we find an asymptotic formula for the integrals $\Gamma_1'$ and $\Gamma_4'$ (defined by...
In section 5 we estimate $\Gamma_1''$ and $\Gamma_4''$ (defined by (3.12) and (3.15) respectively) and we then complete the proof of Theorem 1.1.

2. NOTATION AND SOME LEMMAS

We use the following notations: with $\{t\} = t - \lfloor t \rfloor$ we denote the fractional part of $t$. With $||t||$ we denote the distance from $t$ to the nearest integer. As usual with $\mu(n)$, $\varphi(n)$ and $\Lambda(n)$ we denote respectively, Möbius' function, Euler’s function and von Mangoldt’s function. Also $e(t) = e^{2\pi it}$.

We use Vinogradov’s notation $A \ll B$, which is equivalent to $A = O(B)$. If we have simultaneously $A \ll B$ and $B \ll A$, then we shall write $A \asymp B$.

We reserve $p, p_1, p_2, p_3$ for prime numbers. By $\epsilon$ we denote an arbitrarily small positive number, which is not necessarily the same in the different formulae.

With $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ we will denote respectively the set of natural numbers, the set of integer numbers and the set of real numbers.

Now we quote some lemmas, which shall be used later.

**Lemma 2.1.** Suppose that $D \in \mathbb{R}, D > 4$. There exist arithmetical functions $\lambda^\pm (d)$ (Rosser’s functions of level $D$) with the following properties:

1. For any positive integer $d$ we have
   $$|\lambda^\pm (d)| \leq 1, \quad \lambda^\pm (d) = 0 \text{ if } d > D \text{ or } \mu(d) = 0.$$

2. If $n \in \mathbb{N}$, then
   $$\sum_{d \mid n} \lambda^-(d) \leq \sum_{d \mid n} \mu(d) \leq \sum_{d \mid n} \lambda^+(d).$$

3. If $z \in \mathbb{R}$ is such that $z^2 \leq D \leq z^3$ and if
   $$P(z) = \prod_{2 < p < z} p, \quad B = \prod_{2 < p < z} \left(1 - \frac{1}{p - 1}\right),$$
   $$N^\pm = \sum_{d \mid P(z)} \lambda^\pm (d) \varphi (d), \quad s_0 = \frac{\log D}{\log z},$$
   then we have
   $$B \leq N^+ \leq B \left(F(s_0) + O \left( (\log D)^{\frac{1}{2}} \right) \right),$$
   $$B \geq N^- \geq B \left(f(s_0) + O \left( (\log D)^{\frac{1}{2}} \right) \right),$$
   where $F(s)$ and $f(s)$ satisfy
   $$f(s) = 2e^\gamma s^{-1} \log (s - 1), \quad F(s) = 2e^\gamma s^{-1} \text{ for } 2 \leq s \leq 3.$$  

Here $\gamma$ is Euler’s constant.
**Proof.** See Greaves [5, Chapter 4, Theorem 3]. □

**Lemma 2.2.** Suppose that $\Lambda_i, \Lambda_i^\pm$ are real numbers satisfying $\Lambda_i = 0$ or 1, $\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+$, $i = 1, 2, 3$. Then

$$\Lambda_1\Lambda_2\Lambda_3 \geq \Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+. \quad (2.5)$$

**Proof.** The proof is similar to the proof of Lemma 13 in [2]. □

**Lemma 2.3.** Suppose that $x, y \in \mathbb{R}$ and $M \in \mathbb{N}$, $M \geq 3$. Then

$$e(\{x\} - y) = \sum_{|m| \leq M} c_m e(my) + O\left(\min \left(1, \frac{1}{M||y||}\right)\right),$$

where

$$c_m = \frac{1 - e(-x)}{2\pi i (x + m)}.$$ (2.6)

**Proof.** Proof can be find in Buriev [3, Lemma 12]. □

**Lemma 2.4.** Consider the integral

$$I = \int_a^b e(f(x))dx,$$

where $f(x)$ is real function with continuous second derivative and monotonous first derivative. If $|f'(x)| \geq h > 0$ for all $x \in [a, b]$, then $I \ll h^{-1}$.

**Proof.** See [10, Lemma 4.3]. □

3. BEGINNING OF THE PROOF

Let $\eta, \delta, \xi$ and $\mu$ be positive real numbers depending on $c$. We shall specify them later. Now we only assume that they satisfy the conditions

$$\xi + 3\delta < \frac{12}{25}, \quad 2 < \frac{\delta}{\eta} < 3, \quad \mu < 1.$$ (3.1)

We denote

$$X = N^\frac{\xi}{2}, \quad z = X^\eta, \quad D = X^\delta, \quad \Delta = X^{\xi-c}$$ (3.2)

and

$$P(z) = \prod_{2<p<z} p.$$ (3.3)
Consider the sum
\[
\Gamma = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3).
\] (3.4)

If we prove the inequality
\[
\Gamma > 0,
\] (3.5)
then equation (1.2) would have a solution in primes \(p_1, p_2, p_3\) satisfying conditions in the sum \(\Gamma\). Suppose that \(p_i + 2\) has \(l\) prime factors, counted with multiplicities. From (3.2), (3.3) and \((p_i + 2, P(z)) = 1\) we have
\[
X + 2 \geq p_i + 2 \geq z^l = X^{\eta l}
\]
and then \(l \leq \frac{1}{\eta}\). This means that \(p_i + 2\) has at most \([\eta^{-1}]\) prime factors counted with multiplicities. Therefore, to prove Theorem 1.1 we have to establish (3.5) for an appropriate choice of \(\eta\).

For \(i = 1, 2, 3\) we define
\[
\Lambda_i = \sum_{d \mid (p_i + 2, P(z))} \mu(d) = \begin{cases} 1 & \text{if } (p_i + 2, P(z)) = 1, \\ 0 & \text{otherwise}. \end{cases}
\] (3.6)

Then we find that
\[
\Gamma = \sum_{\mu X < p_1, p_2, p_3 \leq X} \Lambda_1 \Lambda_2 \Lambda_3 (\log p_1)(\log p_2)(\log p_3).
\]

We can write \(\Gamma\) as
\[
\Gamma = \sum_{\mu X < p_1, p_2, p_3 \leq X} \Lambda_1 \Lambda_2 \Lambda_3 (\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1^2] + [p_2^2] + [p_3^2] - N))d\alpha.
\]

Suppose that \(\lambda^\pm(d)\) are the Rosser functions of level \(D\). Let also denote
\[
\Lambda_i^\pm = \sum_{d \mid (p_i + 2, P(z))} \lambda^\pm(d), \quad i = 1, 2, 3.
\] (3.7)

Then from Lemma 2.1, (3.6) and (3.7) we find that
\[
\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+.
\]

We use Lemma 2.2 and find that
\[
\Gamma \geq \Gamma_1 + \Gamma_2 + \Gamma_3 - 2\Gamma_4.
\]

where $\Gamma_1, \ldots, \Gamma_4$ are the contributions coming from the consecutive terms of the right-hand side of (2.5). We have $\Gamma_1 = \Gamma_2 = \Gamma_3$ and

$$
\Gamma_1 = \sum_{\mu X < p_1, p_2, p_3 \leq X} \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ (\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1^c] + [p_2^c] + [p_3^c] - N)) \, d\alpha,
$$

$$
\Gamma_4 = \sum_{\mu X < p_1, p_2, p_3 \leq X} \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ (\log p_1)(\log p_2)(\log p_3) \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\alpha([p_1^c] + [p_2^c] + [p_3^c] - N)) \, d\alpha.
$$

Hence, we get

$$
\Gamma \geq 3\Gamma_1 - 2\Gamma_4. \tag{3.8}
$$

Let us first consider $\Gamma_1$. We have

$$
\Gamma_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} e(-N\alpha) L^-(\alpha) L^+(\alpha) \sqrt{2} \, d\alpha, \tag{3.9}
$$

where

$$
L^\pm(\alpha) = \sum_{\mu X < p \leq X} (\log p) e(\alpha([p^c])) \sum_{d|\mu X < p \leq X} \lambda^\pm(d).
$$

Changing the order of summation we get

$$
L^\pm(\alpha) = \sum_{d|\mu X < p \leq X} \lambda^\pm(d) \sum_{p + 2z \equiv 0 \pmod{d}} (\log p) e(\alpha(p^c)).
$$

We divide the integral from (3.9) into two parts:

$$
\Gamma_1 = \Gamma_1' + \Gamma_1'', \tag{3.10}
$$

where

$$
\Gamma_1' = \int_{|\alpha| < \Delta} e(-N\alpha) L^-(\alpha) L^+(\alpha) \sqrt{2} \, d\alpha, \tag{3.11}
$$

$$
\Gamma_1'' = \int_{\Delta < |\alpha| \leq \frac{1}{2}} e(-N\alpha) L^-(\alpha) L^+(\alpha) \sqrt{2} \, d\alpha, \tag{3.12}
$$

with $\Delta$ defined by (3.2). Similarly, for $\Gamma_4$ we have

$$
\Gamma_4 = \Gamma_4' + \Gamma_4'', \tag{3.13}
$$
where
\[
\Gamma_4' = \int_{|\alpha|<\Delta} e(-N\alpha)L^+(\alpha)^3 d\alpha, 
\]
(3.14)
\[
\Gamma_4'' = \int_{\Delta<|\alpha|<\frac{1}{2}} e(-N\alpha)L^+(\alpha)^3 d\alpha, 
\]
(3.15)
and $\Delta$ is defined by (3.2).

4. THE INTEGRALS $\Gamma_1'$ AND $\Gamma_4'$

We shall find an asymptotic formula for the integrals $\Gamma_1'$ and $\Gamma_4'$ defined by (3.11) and (3.14), respectively. The arithmetic structure of the Rosser weights $\lambda^d(d)$ is not important here, so we consider a sum of the form
\[
L(\alpha) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < p \leq X \atop p + 2 \equiv 0 (\text{mod} \ d)} (\log p) e(\alpha[p^2]), 
\]
(4.1)
where $\lambda(d)$ are real numbers satisfying
\[
|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \text{ if } 2 | d \text{ or } \mu(d) = 0. 
\]
(4.2)
It is easy to see that
\[
L(\alpha) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < p \leq X \atop p + 2 \equiv 0 (\text{mod} \ d)} (\log p) e(\alpha p^2 + O(|\alpha|)) 
\]
\[
= \sum_{d \leq D} \lambda(d) \sum_{\mu X < p \leq X \atop p + 2 \equiv 0 (\text{mod} \ d)} (\log p) e(\alpha p^2)(1 + O(|\alpha|)) 
\]
\[
= \overline{L}(\alpha) + O(\Delta X (\log X)), 
\]
(4.3)
where
\[
\overline{L}(\alpha) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < p \leq X \atop p + 2 \equiv 0 (\text{mod} \ d)} (\log p) e(\alpha p^2). 
\]
For $\overline{L}(\alpha)$ we use the asymptotic formula from Lemma 10 in [15]. From (3.1) and (3.2) we see that, when $|\alpha| < \Delta$, then for every constant $A > 0$, we have
\[
\overline{L}(\alpha) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(\alpha) + O(X (\log X)^{-A}), 
\]
(4.4)

where

\[ I(\alpha) = \int_{\mu X}^{X} e(\alpha t^c)dt. \]  

(4.5)

Hence from (3.2), (4.3) and (4.4) we see that

\[ L(\alpha) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(\alpha) + O(X(\log X)^{-A}). \]  

(4.6)

From (2.1) and (4.6) we find

\[ L(\pm) = N(\pm) + O(X(\log X)^{-A}), \quad \text{for } |\alpha| < \Delta. \]  

(4.7)

Let

\[ M(\pm) = N(\pm)I(\alpha). \]  

(4.8)

It is easy to see that

\[ N(\pm) \ll \log X. \]  

(4.9)

We use (4.7), (4.8) and the identity

\[ L^{-}(L^{+})^2 = (L^{-}M^{-})(L^{+})^2 + (L^{+}M^{+})M^{-}L^{+} + (L^{+}M^{+})M^{-} + M^{+}M^{-} = (M^{+})^2 \]  

to find that

\[ |L^{-}(L^{+})^2 - M^{-}(M^{+})^2| \ll X(\log X)^{-A} \left( |L^{+}|^2 + |M^{+}|^2 \right). \]  

(4.10)

Let

\[ B = \int_{|\alpha| < \Delta} e(-N\alpha)M^{-}(\alpha)(M^{+}(\alpha))^2d\alpha. \]  

(4.11)

From (3.11), (4.9) – (4.11) we have

\[ \Gamma_{1} - B \ll X(\log X)^{2-A} \left( \int_{|\alpha| < \Delta} |L^{+}(\alpha)|^2d\alpha + \int_{|\alpha| < \Delta} |I(\alpha)|^2d\alpha \right). \]

We need the next lemma, which is an analog of Lemma 11 in [15].

**Lemma 4.5.** If \( \Delta \leq X^{1-c} \), then for the sum \( L(\alpha) \) defined by (4.1) and for the integral \( I(\alpha) \) defined by (4.5) we have

\[ \int_{|\alpha| < \Delta} |L(\alpha)|^2d\alpha \ll X^{2-c}(\log X)^6, \]

\[ \int_{|\alpha| < \Delta} |I(\alpha)|^2d\alpha \ll X^{2-c}(\log X)^6, \]

\[ \int_{|\alpha| < 1} |L(\alpha)|^2d\alpha \ll X(\log X)^5. \]
Proof. The proof is similar to the proof of Lemma 11 in [15].

Hence
\[ \Gamma_1' - B \ll X^{3-c} (\log X)^{8-A}. \] (4.12)

Consider now the integral
\[ B_1 = \int_{-\infty}^{\infty} e(-N\alpha) I(\alpha)^3 d\alpha. \] (4.13)

Using the method in Lemma 5.6.1 in [11] we find
\[ B_1 \gg X^{3-c}. \] (4.14)

For \( I(\alpha) \) we apply Lemma 2.4 and see that \( I(\alpha) \ll |\alpha|^{-1} X^{1-c} \). Then from (3.2), (4.8), (4.11) and (4.13) we find
\[ |N^{-}(N^+)^2 B_1 - B| \ll (\log X)^3 \int_{|\alpha| > \Delta} |I(\alpha)|^3 d\alpha \ll (\log x)^3 X^{3-c-2\xi}. \] (4.15)

If \( A = 12 \), then using (4.12) and (4.15) we find
\[ \Gamma_1' = N^{-}(N^+)^2 B_1 + O(X^{3-c}(\log X)^{-4}). \] (4.16)

We proceed with \( \Gamma_4' \) in the same way and prove that
\[ \Gamma_4' = (N^+)^3 B_1 + O(X^{3-c}(\log X)^{-4}). \] (4.17)

5. ESTIMATION OF INTEGRALS \( \Gamma_1'' \) AND \( \Gamma_4'' \) AND COMPLETION OF THE PROOF

In this section we consider the integrals \( \Gamma_1'' \) and \( \Gamma_4'' \) defined by (3.12) and (3.15) respectively. We shall show that \( \Gamma_1'' \) and \( \Gamma_4'' \) are small enough. Now we assume that
\[ \xi = \frac{16c - 5}{32}, \quad \delta = \frac{17 - 16c}{32}. \] (5.1)

It is obvious that for \( \Gamma_1'' \) defined by (3.12) we have
\[ \Gamma_1'' \ll \max_{\Delta \leq |\alpha| \leq \frac{1}{2}} |L^{-}(\alpha)| \int_{0}^{1} |L^{+}(\alpha)|^2 d\alpha. \]

We use Lemma 4.5 and find that
\[ \Gamma_1'' \ll X (\log X)^5 \max_{\Delta \leq |\alpha| \leq \frac{1}{2}} |L^{-}(\alpha)|. \] (5.2)

From (4.1) we see that

\[ L(\alpha) = L_1(\alpha) + O\left(X^{\frac{1}{2}+\varepsilon}\right), \]

(5.3)

where

\[ L_1(\alpha) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < n \leq X \atop n + 2 \equiv 0(\text{mod } d)} \Lambda(n)e(\alpha[n^c]). \]

Let \( M = X^\kappa \) for some \( \kappa \), which will be specified later. Now for \( L_1(\alpha) \) we apply Lemma 2.3 with parameters \( x = \alpha, y = n^c \) and \( M \) (note that \( |t| = t - \{t\} \)). We obtain

\[ L_1(\alpha) = \sum_{|m| \leq M} c_m \sum_{d \leq D} \lambda(d) \sum_{\mu X < n \leq X \atop n + 2 \equiv 0(\text{mod } d)} \Lambda(n)e((\alpha + m)n^c) + O\left(X^{\varepsilon} \sum_{\mu X < n \leq X} \min\left(1, \frac{1}{M||n^c||}\right)\right). \]

(5.4)

We need the following

**Lemma 5.6.** Suppose that \( D, \Delta \) are defined by (3.2) and \( \xi, \delta \) are specified by (5.1). Suppose also that \( \lambda(d) \) satisfy (4.2) and \( c_m \) are defined by (2.6). Then

\[ \max_{\Delta \leq \kappa \leq M+1} \left| \sum_{|m| \leq M} c_m \sum_{d \leq D} \lambda(d) \sum_{\mu X < n \leq X \atop n + 2 \equiv 0(\text{mod } d)} \Lambda(n)e(\alpha n^c) \right| \ll (X^{rac{1}{2}} + X^{\frac{1}{2}+\varepsilon} + X^{\frac{1}{2}+\varepsilon} D\frac{1}{2} + X^{\frac{1}{2}+\varepsilon} D\frac{1}{2} M^2 + X^2 + X^{1-\varepsilon} D\frac{1}{2} \Delta^{-\frac{1}{2}} + X^{1-\varepsilon} \Delta^{-\frac{1}{2}}). \]

**Proof.** See Lemma 15 in [15]. \( \square \)

We also need the following result.

**Lemma 5.7.** One has

\[ \sum_{\mu X < n \leq X} \min\left(1, \frac{1}{M||n^c||}\right) \ll X^{\varepsilon} \left(X M^{-1} + M^{\frac{1}{2}} X^{\frac{1}{2}}\right). \]

(5.5)

**Proof.** From [13, Lemma 5.2.3] we know that the Fourier series

\[ \min\left(1, \frac{1}{M||n^c||}\right) = \sum_{k \in \mathbb{N}} b_M(k)e(kn^c), \]

has Fourier coefficients satisfying

\[ |b_M(k)| \leq \begin{cases} \frac{4\log M}{M^2} & \text{if } k \in \mathbb{Z}, \\ \frac{M^2}{4} & \text{if } k \in \mathbb{Z}, k \neq 0. \end{cases} \]

(5.7)
From (5.6) we get

\[ \sum_{\mu X < n \leq X} \min\left(1, \frac{1}{M||n||} \right) = \sum_{\mu X < n \leq X} \sum_{k \in \mathbb{N}} b_M(k) e(kn^c). \]  

(5.8)

Changing the order of summation in last formula we obtain

\[ \sum_{\mu X < n \leq X} \min\left(1, \frac{1}{M||n||} \right) = \sum_{k \in \mathbb{N}} b_M(k) H(k), \]

where

\[ H(k) = \sum_{\mu X < n \leq X} e(kn^c). \]

Now using (5.7) and (5.8) and the identity

\[ |H(k)| = |H(-k)| \]

we find

\[ \sum_{\mu X < n \leq X} \min\left(1, \frac{1}{M||n||} \right) \ll X \log M + \sum_{1 \leq k \leq M} |H(k)| \sum_{k > M} \frac{|H(k)|}{k^2}. \]

(5.9)

If \( \theta(x) = kx^c \), then \( \theta''(x) = c(c-1)kx^{c-2} \approx kX^{c-2} \) uniformly for \( x \in [\mu X, X] \).

Hence, we can apply Van der Corput’s theorem (see [6, Chapt. 1, Theorem 5]) to obtain

\[ H(k) \ll k^\frac{1}{2} X^{\frac{c}{2}} + k^{-\frac{1}{2}} X^{1-\frac{c}{2}}. \]

(5.10)

Hence from (5.9) and (5.10) we prove (5.5). \( \Box \)

When combining Lemma 5.6, Lemma 5.7 and (5.3) – (5.4) we find that

\[ \max_{\Delta \leq \alpha \leq M+1} |L(\alpha)| \ll x^\varepsilon \left( X^{\frac{1}{2}+\frac{\delta}{2}} D M^{\frac{1}{2}} + X^{1-\frac{c}{2}} M^{-\frac{1}{2}} + X^{\frac{1}{2}} M^{\frac{1}{2}} + X^{1-\frac{1}{2}} D M^{\frac{1}{2}} + X^{1-\frac{c}{2}} \Delta^{-\frac{1}{2}} + XM^{-1} \right). \]

Then from last formula, (3.2) and (5.2) we find

\[ \Gamma_1'' \ll x^\varepsilon \left( X^{\frac{1}{2}+\frac{\delta}{2}+\frac{\kappa}{2}} + X^{\frac{1}{2}+\frac{\kappa}{2}} + X^{\frac{1}{2}} + X^{2+\frac{1}{2}-\frac{\kappa}{2}} + X^{2-\kappa} \right). \]

(5.11)

If we choose \( \kappa = \frac{56c-5}{56} \), then from (5.1) and (5.11) we conclude that if \( 1 < c < \frac{17}{16} \) then

\[ \Gamma_1'' \ll X^{3-\varepsilon}. \]

From (3.8), (3.10), (3.13) and (4.14) – (4.17) we conclude that

\[ \Gamma \geq |3N^- - 2N^+| (N^+)^3 B_1 + O(X^{3-\varepsilon}(\log x)^{-1}). \]

(5.12)

Now we shall find a lower bound for the difference \( 3N^- - 2N^+ \). It is easy to see that

\[ B \approx (\log X)^{-1}. \]

(5.13)
From (2.2) and (2.3) we see that
\[ 3N^- - 2N^+ \geq B(3f(s_0) - F(s_0)) + O\left(\log X\right)^{-\frac{4}{3}}, \]
where \( s_0 \) is defined by (2.1) and \( F(s) \) and \( f(s) \) are defined by (2.4). If we choose \( s_0 = 2.95 \), then from (2.1), (3.2) and (5.1) we find
\[ \eta = \frac{\delta}{2.95} = \frac{17 - 16c}{94.4}, \]
and also from (2.4) we find \( 3f(s_0) - F(s_0) > 0. \)

Now from (2.2), (4.14), (5.12) and (5.13) we obtain
\[ \Gamma \gg X^{3-\epsilon}(\log X)^{-3}. \]
Therefore \( \Gamma > 0 \) and this proves Theorem 1.1. \( \square \)

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6. REFERENCES


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