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WEIGHTED APPROXIMATION BY KANTOROVICH TYPE MODIFICATION OF MEYER-KÖNIG AND ZELLER OPERATOR

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We investigate the weighted approximation of functions in L_p-norm by Kantorovich modifications of the classical Meyer-König and Zeller operator, with weights of type $(1-x)^\alpha$, $\alpha \in \mathbb{R}$. By defining an appropriate K-functional we prove direct theorems for them.

Keywords: Meyer-König and Zeller operator, K-functional, direct theorem, moduli of smoothness.

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1. INTRODUCTION

In order to approximate unbounded functions in uniform norm in $[0,1)$, Meyer-König and Zeller (see [15]) introduced a new operator by the formula

$$M_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\Big(\frac{k}{n+k}\Big), \tag{1.1}$$

where

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}. \tag{1.2}$$

But this operator cannot be used to approximate functions in L_p-norm because it is not bounded operator in L_p. Some kind of modification is needed. In this paper we investigate the weighted approximation of functions in L_p-norm by Kantorovich modifications of the classical Meyer-König and Zeller (MKZ) operator.

In 1930, Kantorovich [13] suggested a modification of the classical Bernstein operator, replacing the function values by mean values. Analogously, Totik [16] introduced Kantorovich type modification of MKZ operator:

$$\tilde{M}_n^*(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(n+k)(n+k+1)}{n} \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k+1}} f(u)\, du,$$

and proved direct and converse theorems of weak type in terminology of Ditzian and Ivanov [4] for it. Although this definition looks as the most natural one, the operator $\tilde{M}_n^*$ is not a contraction, hence it is not very suitable for approximating functions in L_p-norm for $p < \infty$.

In [14] Müller defined a Kantorovich modification of MKZ operator in a slightly different way, so that the resulting operator is a contraction:

$$\tilde{M}_n(f;x) = \tilde{M}_n f(x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(n+k+1)(n+k+2)}{n+1} \int_{\frac{k}{n+k+1}}^{\frac{k+1}{n+k+2}} f(u)\, du. \quad (1.3)$$

Recently, in [11] by introducing an appropriate K-functional the first author proved a direct theorem for the operators $\tilde{M}_n(f;x)$. Our goal in this paper is to extend this result for the case of weighted approximation of functions in L_p-norm by $\tilde{M}_n(f;x)$ operator.

Let us introduce some notations. For the sake of simplicity and brevity of our presentation we set

$$\gamma_{n,k} = \frac{(n+k+1)(n+k+2)}{n+1}, \qquad \Delta_{n,k} = \Big[\frac{k}{n+k+1}, \frac{k+1}{n+k+2}\Big]. \quad (1.4)$$

Then, the Kantorovich modification of MKZ operator (1.3) takes the form

$$\tilde{M}_n(f;x) = \sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \int_{\Delta_{n,k}} f(u)\, du.$$

The weights under consideration in our survey are

$$w(x) = (1-x)^{\alpha}, \qquad \alpha \in \mathbb{R}. \quad (1.5)$$

By $\varphi(x) = x(1-x)^2$ we denote the weight which is naturally related to the second derivative of MKZ operator. The usual first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^k g(x) = g^{(k)}(x)$ for every $k \in \mathbb{N}$.

We define a differential operator $\tilde{D}$ by the formula

$$\tilde{D} = \frac{d}{dx}\Big(\varphi(x)\frac{d}{dx}\Big) = D\varphi D.$$

The space $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$. For $1 \le p \le \infty$ and weight function $w(x)$ as in (1.5) we set

$$L_p(w) = \{f : wf \in L_p[0,1)\},$$

$$W_p(w) = \begin{cases} \big\{f : f, Df \in AC_{loc}(0,1),\ w\tilde{D}f \in L_p[0,1),\ \lim\limits_{x\to 0^+,1^-} \varphi(x)Df(x) = 0\big\}, & \alpha < 0, \\ \big\{f : f, Df \in AC_{loc}(0,1),\ w\tilde{D}f \in L_p[0,1),\ \lim\limits_{x\to 0^+} \varphi(x)Df(x) = 0\big\}, & \alpha \ge 0, \end{cases}$$

$$L_p(w) + W_p(w) = \big\{f : f = f_1 + f_2,\ f_1 \in L_p(w),\ f_2 \in W_p(w)\big\}.$$

Also, we define a K-functional $\tilde{K}_w(f,t)_p$ for $t > 0$ by

$$\tilde{K}_w(f,t)_p = \inf\big\{\|w(f-g)\|_p + t\|w\tilde{D}g\|_p : f - g \in L_p(w),\ g \in W_p(w)\big\}. \tag{1.6}$$

Our main result is the following theorem.

Theorem 1. *For $1 \le p \le \infty$, w defined by* (1.5), *$\tilde{M}_n$ defined by* (1.3), *and the K-functional given by* (1.6) *there exists a positive constant C such that for every $n > |\alpha|$, $n \in \mathbb{N}$, and for all functions $f \in L_p(w) + W_p(w)$ there holds*

$$\|w(\tilde{M}_n f - f)\|_p \le C\tilde{K}_w\Big(f, \frac{1}{n}\Big)_p. \tag{1.7}$$

Remark 1. Converse theorem remains an open problem even for the non-weighted case, i.e., for $w(x) = 1$ in (1.5).

Problems on characterization of weighted K-functionals by moduli of smoothness were considered by Draganov and Ivanov in [6, 7, 9]. Particularly, they characterized the K-functional

$$K_w(f,t)_p = \inf\big\{\|w(f-g)\|_p + t\|w\varphi D^2 g\|_p : g, Dg \in AC_{loc}(0,1),\ f-g, \varphi D^2 g \in L_p(w)\big\}. \tag{1.8}$$

In this paper we also show that the same moduli of smoothness can be used for computing the K-functional $\tilde{K}_w(f,t)_p$. So, we prove the next statement.

Theorem 2. *For $1 < p < \infty$ and w, $\tilde{K}_w(f,t)_p$, $K_w(f,t)_p$, defined by* (1.5), (1.6) *and* (1.8), *respectively, there exists a positive constant C such that for all $f \in L_p(w) + W_p(w)$ there holds*

$$\tilde{K}_w(f,t)_p \le C\big(K_w(f,t)_p + tE_0(f)\big), \tag{1.9}$$

where $E_0(f) = \inf_{c\in\mathbb{R}} \|w(f-c)\|_p$ is the best weighted approximation to f by a constant.

Remark 2. For $p = 1$ and $p = \infty$ new moduli are needed. Also, a problem on characterization of the K-functional $\tilde{K}_w(f,t)_p$ arises, but it is not the subject of our survey here.

Henceforth, the constant C will always be an absolute positive constant, which means it does not depend on f and n. Also, it may be different on each occurrence. The relation $\theta_1(f,t) \sim \theta_2(f,t)$ means that there exists a constant $c \geq 1$, independent of f and t, such that

$$c^{-1}\theta_1(f,t) \leq \theta_2(f,t) \leq c\,\theta_1(f,t).$$

2. AUXILIARY RESULTS

In this section we present some properties of the operators M_n, $\tilde{M}_n$, basis functions $m_{n,k}$ (see [1, 10, 12]), and prove auxiliary lemmas that we need further.

The operators M_n and $\tilde{M}_n$ are linear positive operators with $\|M_n f\|_\infty \leq \|f\|_\infty$ and $\|\tilde{M}_n\|_1 = 1$. Moreover,

$$\|\tilde{M}_n\|_p \leq 1, \qquad 1 \leq p \leq \infty, \tag{2.1}$$

$$M_n(1;x) = 1, \qquad M_n(t-x;x) = 0, \tag{2.2}$$

$$\tilde{M}_n(1;x) = 1. \tag{2.3}$$

A direct integration yields the identity:

$$\int_0^1 m_{n,k}(x)dx = \frac{1}{\gamma_{n,k}}. \tag{2.4}$$

We shall need the next three properties of the functions $\{m_{n,k}\}_{k=0}^{\infty}$, defined by (1.2) (for proofs, see e.g., [11]).

Lemma 1. *If $n \in \mathbb{N}$, then*

$$\frac{1}{1-x} = \frac{1}{n+1}\sum_{k=0}^{\infty}(n+k+1)m_{n,k}(x), \qquad x \in [0,1). \tag{2.5}$$

Lemma 2. *If $n \in \mathbb{N}$, then*

$$\sum_{k=1}^{n}\frac{(1-x)^k}{k} = \sum_{k=0}^{\infty} m_{n,k}(x)\sum_{j=1}^{n}\frac{1}{k+j}, \qquad x \in [0,1). \tag{2.6}$$

Lemma 3. *There exists an absolute constant C such that for every $n \in \mathbb{N}$ the following inequality holds true:*

$$\left|\ln(1-x) + \sum_{k=0}^{\infty} m_{n,k}(x)\sum_{j=1}^{k+1}\frac{1}{n+j}\right| \leq \frac{C}{n}, \qquad x \in [0,1). \tag{2.7}$$

In [16, Lemma 3] Totik proved that for $1 \le p < \infty$,

$$\|(1-x)Df(x)\|_p \le C\big(\|f\|_p + \|\varphi D^2 f\|_p\big). \tag{2.8}$$

In order to prove our main results we need a few additional lemmas.

Lemma 4. *For every integer ν there exists a constant $C = C(\nu)$, such that*

$$\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu} m_{n,k}(x) \le C(1-x)^{\nu}, \qquad x\in[0,1), \tag{2.9}$$

for all $n > |\nu|$, $n \in \mathbb{N}$.

Proof. We have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu} m_{n,k}(x)\\
&\quad = \sum_{k=0}^{\infty}\Big(\frac{n+1}{n+k+1}\Big)^{\nu}\binom{n+k}{k}x^k(1-x)^{n+1}\\
&\quad = (1-x)^{\nu}\sum_{k=0}^{\infty}\frac{(n+1)^{\nu}\,(n+k-\nu+1)\cdots(n+k)}{(n-\nu+1)\cdots n\,(n+k+1)^{\nu}}\,m_{n-\nu,k}(x)\\
&\quad \le (1-x)^{\nu}\sum_{k=0}^{\infty}C(\nu)\,m_{n-\nu,k}(x)\\
&\quad = C(\nu)(1-x)^{\nu}.
\end{aligned}$$

□

Lemma 5. *For every $\alpha\in\mathbb{R}$ there exists a constant $C = C(\alpha)$, such that the following inequality is satisfied:*

$$\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\alpha} m_{n,k}(x) \le C(1-x)^{\alpha}, \qquad x\in[0,1), \tag{2.10}$$

for all $n > |\alpha|$, $n \in \mathbb{N}$.

Proof. Let ν be the smallest positive integer such that $\nu \ge |\alpha|$. Then, by Hölder's inequality it follows that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\alpha} m_{n,k}(x)\\
&\quad \le \left(\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu\,\mathrm{sign}\,(\alpha)} m_{n,k}(x)\right)^{|\alpha|/\nu}\left(\sum_{k=0}^{\infty}m_{n,k}(x)\right)^{1-|\alpha|/\nu}.
\end{aligned}$$

Applying Lemma 4 we obtain

$$\left(\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu\ \mathrm{sign}(\alpha)} m_{n,k}(x)\right)^{|\alpha|/\nu} \le \big(C(1-x)^{\nu\ \mathrm{sign}(\alpha)}\big)^{|\alpha|/\nu} = C(\alpha)(1-x)^{\alpha}.$$

Therefore,

$$\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\alpha} m_{n,k}(x) \le C(\alpha)(1-x)^{\alpha}$$

and the lemma is proved. □

The next lemma is a weighted variant of (2.1).

Lemma 6. *Let $1\le p\le\infty$ and $\alpha\in\mathbb{R}$. Then, there exists an absolute constant C such that for all $n>|\alpha|$, $n\in\mathbb{N}$, and $f\in L_p(w)$, we have*

$$\|w\tilde{M}_n f\|_p \le C\|wf\|_p. \tag{2.11}$$

Proof. First we prove (2.11) for $p=1$ and $p=\infty$. Then, by applying Riesz-Thorin theorem we obtain the estimation for every $1<p<\infty$.

The case $p=1$. We have

$$\begin{aligned}
\|w\tilde{M}_n f\|_1 &= \int_0^1 w(x)\left|\sum_{k=0}^{\infty}\gamma_{n,k}\, m_{n,k}(x)\int_{\Delta_{n,k}} f(t)\,dt\right| dx\\
&\le \int_0^1 w(x)\left[\sum_{k=0}^{\infty}\gamma_{n,k}\, m_{n,k}(x)\int_{\Delta_{n,k}} \frac{|(wf)(t)|}{w(t)}\,dt\right] dx\\
&\le C\int_0^1 \left[\sum_{k=0}^{\infty}\gamma_{n,k}\,\frac{w(x)}{w\big(\frac{k}{n+k+1}\big)}\, m_{n,k}(x)\int_{\Delta_{n,k}} |(wf)(t)|\,dt\right] dx\\
&= C\int_0^1 \sum_{k=0}^{\infty}\left(\frac{1-x}{1-\frac{k}{n+k+1}}\right)^{\alpha} a_{n,k} m_{n,k}(x)\,dx,
\end{aligned}$$

where we set

$$a_{n,k}=\gamma_{n,k}\int_{\Delta_{n,k}} |(wf)(t)|\,dt.$$

Let $\nu=\lceil|\alpha|\rceil$ be the smallest positive integer such that $\nu\ge|\alpha|$. Applying Hölder's inequality twice we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{1-x}{1-\frac{k}{n+k+1}}\right)^{\alpha} a_{n,k} m_{n,k}(x)\\
&\quad\le \left[\sum_{k=0}^{\infty}\left(\frac{1-x}{1-\frac{k}{n+k+1}}\right)^{\nu\ \mathrm{sign}(\alpha)} a_{n,k} m_{n,k}(x)\right]^{|\alpha|/\nu}\left[\sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x)\right]^{1-|\alpha|/\nu},
\end{aligned}$$

thus

$$\|w\tilde{M}_n f\|_1 \le C \left\| \sum_{k=0}^{\infty} \left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\nu\ \mathrm{sign}(\alpha)} a_{n,k} m_{n,k}(x) \right\|_1^{|\alpha|/\nu} \times \left\| \sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) \right\|_1^{1-|\alpha|/\nu}. \tag{2.12}$$

Now, we estimate the first nonconstant multiplier in the right-hand side of inequality (2.12). Let $\ell = \nu\ \mathrm{sign}(\alpha)$. For every integer number ℓ we have

$$\left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\ell} m_{n,k}(x) = \frac{(n+k+1)^{\ell}\,(n+1)\cdots(n+\ell)}{(n+k+1)\cdots(n+k+\ell)\,(n+1)^{\ell}}\, m_{n+\ell,k}(x) \le C(\ell) m_{n+\ell,k}(x),$$

hence

$$\sum_{k=0}^{\infty} \left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\ell} a_{n,k}\, m_{n,k}(x) \le C(\ell) \sum_{k=0}^{\infty} a_{n,k}\, m_{n+\ell,k}(x).$$

Then, by (2.4),

$$\begin{aligned} \left\| \sum_{k=0}^{\infty} \left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\ell} a_{n,k} m_{n,k}(x) \right\|_1 &\le C \left\| \sum_{k=0}^{\infty} a_{n,k}\, m_{n+\ell,k}(x) \right\|_1 \\ &\le C \sum_{k=0}^{\infty} a_{n,k} \|m_{n+\ell,k}(x)\|_1 = C \sum_{k=0}^{\infty} \frac{a_{n,k}}{\gamma_{n+\ell,k}} \\ &= C \sum_{k=0}^{\infty} \frac{\gamma_{n,k}}{\gamma_{n+\ell,k}} \int_{\Delta_{n,k}} |(wf)(t)|\, dt \\ &\le C \sum_{k=0}^{\infty} \int_{\Delta_{n,k}} |(wf)(t)|\, dt = C\|wf\|_1. \end{aligned}$$

Since $\sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) = \tilde{M}_n(wf;x)$ and $\|\tilde{M}_n(wf)\|_1 \le \|wf\|_1$ by (2.1), then for the last multiplier in the right-hand side of (2.12) we obtain the inequality $\|\sum_{k=0}^{\infty} a_{n,k} m_{n,k}\|_1 \le \|wf\|_1$. Therefore,

$$\|w\tilde{M}_n f\|_1 \le C\|wf\|_1^{|\alpha|/\nu} \|wf\|_1^{1-|\alpha|/\nu} = C\|wf\|_1$$

and the proof of the estimate (2.11) for $p = 1$ is complete.

The case $p = \infty$. We obtain

$$\left| w(x) \sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \int_{\Delta_{n,k}} f(t)\, dt \right| \le w(x) \sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \int_{\Delta_{n,k}} \frac{|(wf)(t)|}{w(t)}\, dt$$
$$\le C w(x) \sum_{k=0}^{\infty} \frac{\gamma_{n,k}\, m_{n,k}(x)}{w\big(\frac{k}{n+k+1}\big)} \int_{\Delta_{n,k}} |(wf)(t)|\, dt$$
$$\le C w(x) \sum_{k=0}^{\infty} m_{n,k}(x)\, \frac{\|wf\|_\infty}{w\big(\frac{k}{n+k+1}\big)}$$
$$= C w(x) \|wf\|_\infty \sum_{k=0}^{\infty} \Big(1 - \frac{k}{n+k+1}\Big)^{-\alpha} m_{n,k}(x).$$

Now, by Lemma 5 we have

$$\sum_{k=0}^{\infty} \Big(1 - \frac{k}{n+k+1}\Big)^{-\alpha} m_{n,k}(x) \le C(1-x)^{-\alpha}.$$

Hence,

$$\|w\tilde{M}_n f\|_\infty \le C w(x) \|wf\|_\infty (1-x)^{-\alpha} = C\|wf\|_\infty,$$

which proves (2.11) in the case $p = \infty$.

Finally, the inequality (2.11) follows for all $1 \le p \le \infty$ by the Riesz-Thorin interpolation theorem. □

The crucial result in our investigation is the following Jackson type inequality.

Lemma 7. *Let $1 \le p \le \infty$ and $\alpha \in \mathbb{R}$. Then there exists an absolute constant C, such that for all $n > |\alpha|$, $n \in \mathbb{N}$, and $f \in W_p(w)$, the following estimate holds true:*

$$\big\|w(\tilde{M}_n f - f)\big\|_p \le \frac{C}{n} \big\|w\tilde{D} f\big\|_p. \tag{2.13}$$

(Let us note that the lemma implies that $\tilde{M}_n f - f \in L_p(w)$ for $f \in W_p(w)$.)

Proof. Let us set

$$\phi(x) = \ln \frac{x}{1-x} + \frac{1}{1-x}, \qquad x \in (0,1),$$

with $\phi'(x) = \frac{1}{x(1-x)^2} = \frac{1}{\varphi(x)} > 0$, i.e., $\phi(x)$ is an increasing function. Then we have

$$f(t) = f(x) + \varphi(x)[\phi(t) - \phi(x)] D f(x) + \int_x^t [\phi(t) - \phi(u)] \tilde{D} f(u)\, du, \qquad t \in (0,1).$$

Applying the operator $\tilde{M}_n$ to both sides of the latter equality and multiplying by $w(x)$ we obtain

$$w(x)\big(\tilde{M}_n f(x) - f(x)\big) = w(x)\varphi(x)Df(x)[\tilde{M}_n\phi(x) - \phi(x)] \\ + w(x)\tilde{M}_n\bigg(\int_x^{(\cdot)}[\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du; x\bigg). \quad (2.14)$$

First we prove the lemma for $p = 1$ and $p = \infty$. Then we apply the Riesz-Thorin theorem to obtain (2.13) for every $1 < p < \infty$.

The case $p = 1$. In order to prove that

$$\big\| w\varphi Df\,[\tilde{M}_n\phi - \phi]\big\|_1 \le \frac{C}{n}\|w\tilde{D}f\|_1 \quad (2.15)$$

for all weights (1.5), we shall make use of the estimate

$$\big\|\tilde{M}_n\phi - \phi\big\|_1 \le \frac{C}{n} \quad (2.16)$$

(see [11, Proof of Theorem 1] for a complete proof).

Let $\alpha > 0$ be fixed. Then, for all $n > \alpha$ and $f \in W_1(w)$ we have

$$\varphi(x)Df(x) = \int_0^x (\varphi Df)'(u)\,du = \int_0^x \tilde{D}f(u)\,du, \qquad x \in (0,1).$$

Hence,

$$|w(x)\varphi(x)Df(x)| \le w(x)\int_0^x |\tilde{D}f(u)|\,du \le \int_0^x |(w\tilde{D}f)(u)|\,du \le \int_0^1 |(w\tilde{D}f)(u)|\,du,$$

i.e.,

$$|w(x)\varphi(x)Df(x)| \le \|w\tilde{D}f\|_1, \qquad x \in (0,1).$$

Thus,

$$\big\|w\varphi Df\,[\tilde{M}_n\phi - \phi]\big\|_1 \le \|w\tilde{D}f\|_1\,\big\|\tilde{M}_n\phi - \phi\big\|_1$$

and (2.15) follows from (2.16).

Similarly, let $\alpha < 0$ be fixed. Then, for all $n > -\alpha$ we have $-n < \alpha < 0$ and for $f \in W_1(w)$, we consecutively obtain

$$\varphi(x)Df(x) = \int_x^1 (\varphi Df)'(u)\,du = \int_x^1 \tilde{D}f(u)\,du, \qquad x \in (0,1),$$

$$|w(x)\varphi(x)Df(x)| \le w(x)\int_x^1 |\tilde{D}f(u)|\,du \le \int_x^1 |(w\tilde{D}f)(u)|\,du \le \int_0^1 |(w\tilde{D}f)(u)|\,du,$$

i.e.,

$$|w(x)\varphi(x)Df(x)| \le \|w\tilde{D}f\|_1, \qquad x \in (0,1).$$

Hence, (2.16) yields (2.15).

Therefore, for arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$ and $f \in W_1(w)$ the estimate (2.15) holds true for $n > |\alpha|$. The case $\alpha = 0$ was considered by the first author in [11].

Now, we estimate the L_1-norm of the second summand in the right-hand side of (2.14). More precisely, we will prove

$$\left\| w(x)\tilde{M}_n\left(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du\,;x \right) \right\|_1 \leq \frac{C}{n} \|w\tilde{D}f\|_1. \tag{2.17}$$

Having in mind (1.4), for $x \in (0,1)$ we have

$$\begin{aligned}
&\left| w(x)\tilde{M}_n\left(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du\,;x \right) \right| \\
&\quad \leq w(x) \sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \int_{\Delta n,k} \left(\int_x^t [\phi(t) - \phi(u)] \frac{|(w\tilde{D}f)(u)|}{w(u)}\,du \right) dt \\
&\quad \leq Cw(x) \sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \\
&\qquad \times \left(\frac{1}{w\left(\frac{k}{n+k+1}\right)} + \frac{1}{w(x)} \right) \int_{\Delta n,k} \left(\int_x^t [\phi(t) - \phi(u)] |(w\tilde{D}f)(u)| du \right) dt \\
&\quad \leq C \sum_{k=0}^{\infty} \left(\frac{w(x)}{w\left(\frac{k}{n+k+1}\right)} + 1 \right) b_{n,k}\, m_{n,k}(x),
\end{aligned}$$

where

$$b_{n,k} = \gamma_{n,k} \int_{\Delta n,k} \left(\int_x^t [\phi(t) - \phi(u)] |(w\tilde{D}f)(u)| du \right) dt.$$

Let ν be the smallest positive integer such that $\nu \geq |\alpha|$. Applying twice Hölder's inequality we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{w(x)}{w\left(\frac{k}{n+k+1}\right)}\, b_{n,k} m_{n,k}(x) &\leq \left[\sum_{k=0}^{\infty} \left(\frac{w(x)}{w\left(\frac{k}{n+k+1}\right)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k}(x) \right]^{|\alpha|/\nu} \\
&\quad \times \left[\sum_{k=0}^{\infty} b_{n,k} m_{n,k}(x) \right]^{1-|\alpha|/\nu},
\end{aligned}$$

thus

$$\begin{aligned}
&\left\| w(x)\tilde{M}_n\left(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du\,; \right) \right\|_1 \\
&\quad \leq C \left\| \sum_{k=0}^{\infty} \left(\frac{w(x)}{w\left(\frac{k}{n+k+1}\right)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k} \right\|_1^{|\alpha|/\nu} \left\| \sum_{k=0}^{\infty} b_{n,k} m_{n,k} \right\|_1^{1-|\alpha|/\nu}.
\end{aligned} \tag{2.18}$$

For estimation of the last factor in (2.18) we apply the estimate from [11] (see Proof of Theorem 1, Case 1, therein), by simply replacing $\tilde{D}f$ with $w\tilde{D}f$. So, we obtain

$$\Big\| \sum_{k=0}^{\infty} b_{n,k} m_{n,k} \Big\|_1 \le \frac{C}{n} \| w\tilde{D}f \|_1. \tag{2.19}$$

Next, we focus on the estimating of the other multiplier in (2.18). Clearly,

$$\sum_{k=0}^{\infty} \left(\frac{w(x)}{w\big(\frac{k}{n+k+1}\big)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k}(x) = \sum_{k=0}^{\infty} \Big(\frac{(1-x)(n+k+1)}{n+1} \Big)^{\nu \, \mathrm{sign}(\alpha)} b_{n,k} m_{n,k}(x).$$

Let us set for simplicity $\ell = \nu \, \mathrm{sign}(\alpha) = \lceil |\alpha| \rceil \, \mathrm{sign}(\alpha)$. We have

$$\begin{aligned} \Big(\frac{(1-x)(n+k+1)}{n+1} \Big)^{\ell} m_{n,k}(x) &= \frac{(n+k+1)^{\ell} \, (n+1) \cdots (n+\ell)}{(n+k+1) \cdots (n+k+\ell) \, (n+1)^{\ell}} \, m_{n+\ell,k}(x) \\ &\le C(\ell) \, m_{n+\ell,k}(x) \\ &\le C(\ell) \, \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \, m_{n+\ell,k}(x). \end{aligned}$$

Observe that the constant $C(\ell)$ depends only on α.

We shall make use of the following operator defined by

$$\tilde{M}_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} \gamma_{n+\ell,k} \, m_{n+\ell,k}(x) \int_{\Delta_{n,k}} f(u) \, du. \tag{2.20}$$

Then,

$$\sum_{k=0}^{\infty} \left(\frac{w(x)}{w\big(\frac{k}{n+k+1}\big)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k}(x) \le C \tilde{M}_{n,\alpha} \left(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)] |(w\tilde{D}f)(u)| \, du; \, x \right). \tag{2.21}$$

In order to estimate the L_1-norm of the right-hand side in (2.21) we follow an approach applied, e.g., in [2, pp. 41–43]. The proof in our case is much more complicated, because the operator $\tilde{M}_{n,\alpha}$ does not preserve the constant functions. More precisely, it has the properties

$$\| \tilde{M}_{n,\alpha} \|_1 = 1, \qquad \tilde{M}_{n,\alpha}(1;x) = \sum_{k=0}^{\infty} \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \, m_{n+\ell,k}(x).$$

Let us write the operator $\tilde{M}_{n,\alpha}$ from (2.20) in the form

$$\tilde{M}_{n,\alpha}(f;x) = \int_0^1 K_n(x,t) f(t) \, dt,$$

where $K_n(\cdot,\cdot)$ is the related kernel. Introducing the functions

$$\phi_1(x) = \ln x, \qquad \phi_2(x) = -\ln(1-x), \qquad \phi_3(x) = \frac{1}{1-x},$$

we have $\phi(x) = \phi_1(x) + \phi_2(x) + \phi_3(x)$ and for $j = 1, 2, 3$,

$$\begin{aligned}
&\tilde{M}_{n,\alpha}\Big(\int_x^{(\cdot)} [\phi_j(\cdot) - \phi_j(u)] |(w\tilde{D}f)(u)|\, du; x\Big) \\
&\quad = \int_0^x K_n(x,t) \int_x^t [\phi_j(t) - \phi_j(u)] |(w\tilde{D}f)(u)|\, du\, dt \\
&\qquad + \int_x^1 K_n(x,t) \int_x^t [\phi_j(t) - \phi_j(u)] |(w\tilde{D}f)(u)|\, du\, dt.
\end{aligned}$$

Then, by Fubini's theorem we obtain:

$$\begin{aligned}
&\Big\|\tilde{M}_{n,\alpha} \int_x^{(\cdot)} [\phi(\cdot) - \phi(u)] |(w\tilde{D}f)(u)|\, du\Big\|_1 \\
&\quad = \int_0^1 |(w\tilde{D}f)(u)| \sum_{j=1}^{3} \Big(\int_u^1 \tilde{M}_{n,\alpha}\big([\phi_j(u) - \phi_j(\cdot)]_+; x\big)\, dx \\
&\qquad\qquad + \int_0^u \tilde{M}_{n,\alpha}\big([\phi_j(\cdot) - \phi_j(u)]_+; x\big)\, dx\Big)\, du. \qquad (2.22)
\end{aligned}$$

To estimate the right-hand side of (2.22) we need estimations for the expressions in the sum for each of the functions ϕ_j, $j = 1, 2, 3$.

First, for ϕ_1, using

$$\begin{aligned}
\int_0^1 \tilde{M}_{n,\alpha}\big([\phi_1(u) - \phi_1(\cdot)]_+; x\big)\, dx &= \|\tilde{M}_{n,\alpha}\big([\phi_1(u) - \phi_1(\cdot)]_+; x\big)\|_1 \\
&\le \|[\phi_1(u) - \phi_1(x)]_+\|_1 \\
&= \int_0^u (\phi_1(u) - \phi_1(x))\, dx,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_u^1 \tilde{M}_{n,\alpha}\big([\phi_1(u) - \phi_1(\cdot)]_+; x\big)\, dx + \int_0^u \tilde{M}_{n,\alpha}\big([\phi_1(\cdot) - \phi_1(u)]_+; x\big)\, dx \\
&\quad = \int_0^1 \tilde{M}_{n,\alpha}\big([\phi_1(u) - \phi_1(\cdot)]_+; x\big)\, dx - \int_0^u \tilde{M}_{n,\alpha}\big([\phi_1(u) - \phi_1(\cdot)]_+; x\big)\, dx \\
&\qquad + \int_0^u \tilde{M}_{n,\alpha}\big([\phi_1(\cdot) - \phi_1(u)]_+; x\big)\, dx
\end{aligned}$$

$$\le \int_0^u (\phi_1(u)-\phi_1(x))\,dx + \int_0^u \tilde{M}_{n,\alpha}\big([\phi_1(\cdot)-\phi_1(u)]_+ - [\phi_1(u)-\phi_1(\cdot)]_+\,;x\big)\,dx$$
$$= u\phi_1(u) - \int_0^u \phi_1(x)\,dx + \int_0^u \tilde{M}_{n,\alpha}(\phi_1;x)\,dx - \phi_1(u)\int_0^u \tilde{M}_{n,\alpha}(1;x)\,dx$$
$$= \int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x)-\phi_1(x)\big)\,dx - \phi_1(u)\int_0^u \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx. \tag{2.23}$$

Analogously, for ϕ_j, $j=2,3$, we obtain

$$\int_u^1 \tilde{M}_{n,\alpha}\big([\phi_j(u)-\phi_j(\cdot)]_+\,;x\big)\,dx + \int_0^u \tilde{M}_{n,\alpha}\big([\phi_j(\cdot)-\phi_j(u)]_+\,;x\big)\,dx$$
$$\le \int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_j;x)-\phi_j(x)\big)\,dx - \phi_j(u)\int_u^1 \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx. \tag{2.24}$$

Since for $x,u\in(0,1)$,

$$|\tilde{M}_{n,\alpha}(1;x)-1| = \Big|\sum_{k=0}^{\infty}\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\,m_{n+\ell,k}(x)-1\Big| \le \frac{C}{n},$$
$$|u\phi_1(u)|\le C, \qquad |(1-u)\phi_2(u)|\le C, \qquad |(1-u)\phi_3(u)|\le C,$$

then

$$\begin{aligned}\Big|\phi_1(u)\int_0^u \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx\Big| &\le \frac{C}{n},\\ \Big|\phi_j(u)\int_u^1 \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx\Big| &\le \frac{C}{n}, \qquad j=2,3.\end{aligned} \tag{2.25}$$

<u>*1. Estimation of* $\big|\int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x)-\phi_1(x)\big)\,dx\big|$</u>. We have

$$\int_{\Delta_{n,k}} \phi_1(t)\,dt = \frac{k+1}{n+k+2}\ln\frac{k+1}{n+k+2} - \frac{k}{n+k+1}\ln\frac{k}{n+k+1} - \frac{1}{\gamma_{n,k}},$$

and for $x\in(0,1)$,

$$\phi_1(x) = -\sum_{k=1}^{n+\ell}\frac{(1-x)^k}{k} - \sum_{k=n+\ell+1}^{\infty}\frac{(1-x)^k}{k}.$$

By Lemma 2,

$$\sum_{k=1}^{n+\ell}\frac{(1-x)^k}{k} = \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\sum_{i=1}^{n+\ell}\frac{1}{k+i},$$

and therefore

$$\begin{aligned}
&\Big|\int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x)-\phi_1(x)\big)\,dx\Big| \\
&\quad= \Big|\int_0^u \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\Big[\gamma_{n+\ell,k}\int_{\Delta_{n,k}}\phi_1(t)\,dt+\sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big]dx+\int_0^u \sum_{k=n+\ell+1}^{\infty}\frac{(1-x)^k}{k}\,dx\Big| \\
&\quad\le \Big|\int_0^u \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\Big[\gamma_{n+\ell,k}\int_{\Delta_{n,k}}\phi_1(t)\,dt+\sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big]dx\Big|+\frac{C}{n}.
\end{aligned}$$

For $k \ge 1$,

$$\begin{aligned}
\ln\frac{k+1}{n+k+2} &= -\ln\prod_{i=1}^{n+1}\frac{k+i+1}{k+i} = -\sum_{i=1}^{n+1}\ln\Big(1+\frac{1}{k+i}\Big) \\
&= -\sum_{i=1}^{n+1}\Big[\frac{1}{k+i}-\frac{1}{2(k+i)^2}+\mathcal{O}\Big(\frac{1}{(k+i)^3}\Big)\Big],
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{n+1}\frac{1}{(k+i)^2} &= \sum_{i=1}^{n+1}\Big[\frac{1}{(k+i)(k+i+1)}+\mathcal{O}\Big(\frac{1}{(k+i)^3}\Big)\Big] \\
&= \frac{n+1}{(k+1)(n+k+2)}+\sum_{i=1}^{n+1}\mathcal{O}\Big(\frac{1}{(k+i)^3}\Big),
\end{aligned}$$

hence

$$\ln\frac{k+1}{n+k+2} = -\sum_{i=1}^{n+1}\frac{1}{k+i}+\frac{n+1}{2(k+1)(n+k+2)}+\mathcal{O}\Big(\frac{1}{k^2}\Big).$$

Since

$$\frac{k+1}{n+k+2}\,\mathcal{O}\Big(\frac{1}{k^2}\Big)=\mathcal{O}\Big(\frac{1}{k^2}\Big),$$

then

$$\frac{k+1}{n+k+2}\,\ln\frac{k+1}{n+k+2} = -\frac{k+1}{n+k+2}\sum_{i=1}^{n+1}\frac{1}{k+i}+\frac{n+1}{2(n+k+2)^2}+\mathcal{O}\Big(\frac{1}{k^2}\Big).$$

Similarly,

$$\frac{k}{n+k+1}\,\ln\frac{k}{n+k+1} = -\frac{k}{n+k+1}\sum_{i=0}^{n}\frac{1}{k+i}+\frac{n+1}{2(n+k+1)^2}+\mathcal{O}\Big(\frac{1}{k^2}\Big).$$

Therefore,

$$\int_{\Delta_{n,k}} \phi_1(t)\,dt = \frac{k}{n+k+1}\sum_{i=0}^{n}\frac{1}{k+i} - \frac{k+1}{n+k+2}\sum_{i=1}^{n+1}\frac{1}{k+i}$$
$$- \frac{n+1}{2}\Big[\frac{1}{(n+k+1)^2} - \frac{1}{(n+k+2)^2}\Big] + \mathcal{O}\Big(\frac{1}{k^2}\Big) - \frac{1}{\gamma_{n,k}}$$
$$= -\frac{1}{\gamma_{n,k}}\sum_{i=0}^{n}\frac{1}{k+i} + \mathcal{O}\Big(\frac{1}{k^2}\Big).$$

Now, we have

$$\big|\tilde{M}_{n,\alpha}(\phi_1;x) - \phi_1(x)\big| \le m_{n+\ell,0}(x)\Big|\ln(n+2) + 1 - \sum_{i=1}^{n+\ell}\frac{1}{i}\Big|$$
$$+ \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\sum_{i=1}^{n}\frac{1}{k+i} - \sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big| + \frac{C}{n}.$$

From

$$\Big|\ln(n+2) + 1 - \sum_{i=1}^{n+\ell}\frac{1}{i}\Big| \le C, \qquad \|m_{n+\ell,0}\|_1 \le \frac{C}{n},$$

it follows

$$\Big\|m_{n+\ell,0}(x)\Big|\ln(n+2) + 1 - \sum_{i=1}^{n+\ell}\frac{1}{i}\Big|\Big\|_1 \le \frac{C}{n}.$$

Moreover,

$$\sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\sum_{i=1}^{n}\frac{1}{k+i} - \sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big|$$
$$\le \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1\Big|\sum_{i=1}^{n}\frac{1}{k+i} + \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\sum_{i=n+1}^{n+\ell}\frac{1}{k+i}.$$

Now, the inequalities

$$\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1\Big| \le \frac{C}{n}, \qquad \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\sum_{i=n+1}^{n+\ell}\frac{1}{k+i} \le \frac{C}{n}\sum_{k=1}^{\infty} m_{n+\ell,k}(x) \le \frac{C}{n},$$

yield

$$\sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\sum_{i=1}^{n}\frac{1}{k+i} - \sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big| \le \frac{C}{n}\sum_{k=1}^{\infty} m_{n+\ell,k}(x)\sum_{i=1}^{n}\frac{1}{k+i} + \frac{C}{n}.$$

By Lemma 2 we obtain

$$\sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k+i} \le \sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n+\ell} \frac{1}{k+i} \le |\ln x|.$$

Therefore,

$$\Big| \int_0^u \sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k+i}\, dx \Big| \le \Big| \int_0^u \ln x\, dx \Big| \le \Big| \int_0^1 \ln x\, dx \Big| \le C,$$

and we conclude that

$$\Big| \int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x) - \phi_1(x) \big)\, dx \Big| \le \frac{C}{n}. \tag{2.26}$$

<u>*2. Estimation of* $\big| \int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x) \big)\, dx \big|$</u>. We have

$$\int_{\Delta_{n,k}} \phi_2(t)\, dt = \frac{n+1}{n+k+2} \ln \frac{n+1}{n+k+2} - \frac{n+1}{n+k+1} \ln \frac{n+1}{n+k+1} + \frac{1}{\gamma_{n,k}},$$

$$\begin{aligned} \gamma_{n,k} \int_{\Delta_{n,k}} \phi_2(t)\, dt &= 1 - (n+k+1) \ln \Big(1 + \frac{1}{n+k+1} \Big) - \ln \frac{n+1}{n+k+1} \\ &= \ln \frac{n+k+1}{n+1} + \mathcal{O}\Big(\frac{1}{n+k} \Big), \end{aligned}$$

hence,

$$M_{n,\alpha}(\phi_2;x) = \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\, \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \Big[\ln \frac{n+k+1}{n+1} + \mathcal{O}\Big(\frac{1}{n+k} \Big) \Big].$$

Applying Lemma 3 we obtain

$$\Big| \phi_2(x) - \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| \le \frac{C}{n},$$

and then

$$\big| \tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x) \big| \le \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\, \ln \frac{n+k+1}{n+1} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n}.$$

Taking into account that

$$\ln \frac{n+k+1}{n+1} = \sum_{i=1}^{k} \ln \Big(1 + \frac{1}{n+i} \Big) = \sum_{i=1}^{k} \frac{1}{n+i} + \sum_{i=1}^{k} \mathcal{O}\Big(\frac{1}{(n+i)^2} \Big)$$

and

$$\sum_{i=1}^{k} \frac{1}{(n+i)^2} \le \frac{C}{n},$$

we estimate

$$\begin{aligned}
\big|\tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x)\big| &\le \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n} \\
&\le \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \sum_{i=1}^{k} \frac{1}{n+i} \\
&\quad + \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n}.
\end{aligned}$$

Since

$$\Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \le \frac{C}{n}\,,$$

it follows that

$$\begin{aligned}
&\sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \sum_{i=1}^{k} \frac{1}{n+i} \le \frac{C}{n} \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k} \frac{1}{n+i} \\
&\quad \le \frac{C}{n} \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n} \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i}.
\end{aligned}$$

Observe that

$$\sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| \le \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{\ell} \frac{1}{n+i} \le \frac{C}{n}.$$

We recall that $\ell = \lceil |\alpha| \rceil \operatorname{sign}(\alpha)$ and $C = C(\alpha)$, i.e. C is an absolute constant for a fixed α. Then, by Lemma 3 we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \sum_{i=1}^{k} \frac{1}{n+i} \\
&\qquad \le \frac{C}{n^2} + \frac{C}{n} \Big| \ln(1-x) + \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n}\, |\ln(1-x)| \\
&\qquad \le \frac{C}{n^2} + \frac{C}{n^2} + \frac{C}{n}\, |\ln(1-x)|.
\end{aligned}$$

Therefore,

$$\Big| \int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x)\big)\, dx \Big| \le \frac{C}{n} \int_0^1 (2 - \ln(1-x))\, dx \le \frac{C}{n}. \tag{2.27}$$

3. Estimation of $\left|\int_u^1 \big(\tilde M_{n,\alpha}(\phi_3;x)-\phi_3(x)\big)\,dx\right|$. The last estimation we need concerns the function $\phi_3(x)=\frac{1}{1-x}$. We have

$$\int_{\Delta_{n,k}} \phi_3(t)\,dt = \ln\Big(1+\frac{1}{n+k+1}\Big)=\frac{1}{n+k+1}+\mathcal{O}\Big(\frac{1}{(n+k)^2}\Big),$$

$$\gamma_{n,k}\int_{\Delta_{n,k}} \phi_3(t)\,dt = \frac{n+k+2}{n+1}+\mathcal{O}\Big(\frac{1}{n}\Big).$$

By Lemma 1,

$$\phi_3(x)=\frac{1}{n+\ell+1}\sum_{k=0}^{\infty}(n+\ell+k+1)m_{n+\ell,k}(x),$$

hence

$$\begin{aligned}\big|\tilde M_{n,\alpha}(\phi_3;x)-\phi_3(x)\big| &\le \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\frac{n+k+\ell+1}{n+\ell+1}\Big(\frac{n+k+\ell+2}{n+k+1}-1\Big)+\mathcal{O}\Big(\frac{1}{n}\Big)\\ &= \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\frac{n+k+\ell+1}{n+\ell+1}\cdot\frac{\ell+1}{n+k+1}+\mathcal{O}\Big(\frac{1}{n}\Big)=\mathcal{O}\Big(\frac{1}{n}\Big).\end{aligned}$$

Then

$$\Big|\int_u^1 \big(\tilde M_{n,\alpha}(\phi_3;x)-\phi_3(x)\big)\,dx\Big| \le \frac{C}{n}\int_u^1 dx \le \frac{C}{n}. \tag{2.28}$$

Now, from inequalities (2.22)–(2.28) it follows that

$$\Big\|\tilde M_{n,\alpha}\int_x^{(\cdot)}[\phi(\cdot)-\phi(u)]|(w\tilde D f)(u)|\,du\Big\|_1 \le \frac{C}{n}. \tag{2.29}$$

The estimate (2.17) is a consequence of (2.18), (2.19), (2.21), and (2.29).

Finally, the estimate (2.13) for the case $p=1$ follows from (2.14), (2.15) and (2.17).

The case $p=\infty$.

We proceed similarly to the case $p=1$: applying Holder's inequality for the smallest integer $\ge\alpha$, considering again the operator $\tilde M_{n,\alpha}$ and using the following estimation

$$\begin{aligned}&\tilde M_{n,\alpha}\Big(\int_x^{(\cdot)}[\phi(\cdot)-\phi(u)]|(w\tilde D f)(u)|\,du;x\Big)\\ &\quad\le \|w\tilde D f\|_\infty\,\tilde M_{n,\alpha}\Big(\int_x^{(\cdot)}[\phi(\cdot)-\phi(u)]\,du;x\Big)\\ &\quad\le x\big|\tilde M_{n,\alpha}(\ln t;x)-\ln x\big|\|w\tilde D f\|_\infty+(1-x)\Big|\tilde M_{n,\alpha}\Big(\frac{1}{1-t};x\Big)-\frac{1}{1-x}\Big|\|w\tilde D f\|_\infty\\ &\qquad + x\big|\tilde M_{n,\alpha}(\ln(1-t);x)-\ln(1-x)\big|\|w\tilde D f\|_\infty. \qquad\square\end{aligned}$$

For the proof of Theorem 2 we need a weighted variant of (2.8).

Lemma 8. *Let $1 < p < \infty$. Then, for all functions $f \in L_p(w)$ such that $\varphi D^2 f \in L_p(w)$, there exists a constant C such that the next inequality is true*

$$\|wD\varphi Df\|_p \le C\big(\|wf\|_p + \|w\varphi D^2 f\|_p\big).$$

Proof. The proof is analogous to the proof of [16, Lemma 3], using the obvious

$$|D\varphi(x)| = |(1-x)(1-3x)| < 2(1-x), \qquad 0 \le x < 1,$$

and $w(x) \sim w(1-2^{-k})$ for $x \in (1-2^{-k}, 1-2^{-k-1})$. □

3. PROOFS OF THEOREM 1 AND THEOREM 2

Proof of Theorem 1. We establish the direct inequality by means of a standard argument.

Let $1 \le p \le \infty$. For any $g \in W_p(w)$ such that $f - g \in L_p(w)$ we have, by virtue of (2.11) and Lemma 7,

$$\begin{aligned} \|w(f-\tilde{M}_n f)\|_p &\le \|w(f-g)\|_p + \|w(g-\tilde{M}_n g)\|_p + \|w\tilde{M}_n(f-g)\|_p \\ &\le 2\|w(f-g)\|_p + \frac{C}{n}\,\|w\tilde{D}g\|_p \\ &\le C\Big(\|w(f-g)\|_p + \frac{1}{n}\,\|w\tilde{D}g\|_p\Big). \end{aligned}$$

Taking the infimum on g we obtain the inequality (1.7) in the theorem. □

Proof of Theorem 2. For every $c \in \mathbb{R}$, by virtue of Lemma 8, we have

$$\begin{aligned} \|wD\varphi Dg\|_p &= \|wD\varphi D(g-c)\|_p \\ &\le C\big(\|w\varphi D^2(g-c)\|_p + \|w(g-c)\|_p\big) \\ &= C\big(\|w\varphi D^2 g\|_p + \|w(g-c)\|_p\big). \end{aligned}$$

Using the latter inequality and the obvious

$$\|w\tilde{D}g\|_p \le \|wD\varphi Dg\|_p + \|w\varphi D^2 g\|_p$$

we have for $t > 0$

$$\begin{aligned} &\|w(f-g)\|_p + t\|w\tilde{D}g\|_p \\ &\quad \le \|w(f-g)\|_p + t\|wD\varphi Dg\|_p + t\|w\varphi D^2 g\|_p \\ &\quad = \|w(f-g)\|_p + Ct\big(\|w\varphi D^2 g\|_p + \|w(g-c)\|_p\big) + t\|w\varphi D^2 g\|_p \\ &\quad = C\big(\|w(f-g)\|_p + t\|w\varphi D^2 g\|_p\big) + Ct\|w(g-f+f-c)\|_p \\ &\quad \le C\big(\|w(f-g)\|_p + t\|w\varphi D^2 g\|_p\big) + Ct\|w(g-f)\|_p + Ct\|w(f-c)\|_p \\ &\quad \le C\big(\|w(f-g)\|_p + t\|w\varphi D^2 g\|_p + t\|w(f-c)\|_p\big). \end{aligned}$$

By taking infimum over all functions $g \in W_p(w)$ and all real constants c we obtain the inequality

$$\tilde{K}_w(f,t)_p \le C \inf \left\{ \|w(f-g)\|_p + t\|w\varphi D^2 g\|_p \, : \, f-g \in L_p(w),\, g \in W_p(w) \right\} + CtE_0(f).$$

To complete the proof in the case $\alpha \ge 0$, it remains to take into consideration that in the definition of $K_w(f,t)_p$ we can, equivalently, assume that g is in C^2 in a neighbourhood of 0 if $f \in L_p(w)$ (see [3, p. 110]).

To complete the proof for $\alpha < 0$, we will show that if $g, Dg \in AC_{loc}(0,1)$ and $wg, w\varphi D^2 g \in L_p[0,1)$, then

$$\lim_{x \to 1^-} \varphi(x) Dg(x) = 0.$$

To this end, we first apply [5, Lemma 1] to get $(1-x)^{\alpha+1} Dg(x) \in L_p[1/2,1)$.

Next, we use [8, Lemma 3.1(a)], transformed for a singularity at $x = 1$, with $G = \varphi Dg$ and $\gamma = \alpha - 1 < -1$ to derive

$$\lim_{x \to 1^-} G(x) = \lim_{x \to 1^-} \varphi(x) Dg(x) = 0. \qquad \square$$

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