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A NONREALIZATION THEOREM IN THE CONTEXT OF DESCARTES' RULE OF SIGNS

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For a real degree d polynomial P with all nonvanishing coefficients, with c sign changes and p sign preservations in the sequence of its coefficients ($c + p = d$), Descartes' rule of signs says that P has $pos \leq c$ positive and $neg \leq p$ negative roots, where $pos \equiv c(\mod 2)$ and $neg \equiv p(\mod 2)$. For $1 \leq d \leq 3$, for every possible choice of the sequence of signs of coefficients of P (called sign pattern) and for every pair (pos, neg) satisfying these conditions there exists a polynomial P with exactly pos positive and neg negative roots (all of them simple); that is, all these cases are realizable. This is not true for $d \geq 4$, yet for $4 \leq d \leq 8$ (for these degrees the exhaustive answer to the question of realizability is known) in all nonrealizable cases either $pos = 0$ or $neg = 0$. It was conjectured that this is the case for any $d \geq 4$. For $d = 9$, we show a counterexample to this conjecture: for the sign pattern $(+, -, -, -, -, +, +, +, +, -)$ and the pair $(1, 6)$ there exists no polynomial with 1 positive, 6 negative simple roots and a complex conjugate pair and, up to equivalence, this is the only case for $d = 9$.

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1. INTRODUCTION

In his work *La Géométrie* published in 1637, René Descartes (1596–1650) announces his classical rule of signs which says that for the real polynomial $P(x, a) := x^d + a_{d-1}x^{d-1} + \cdots + a_0$, the number c of sign changes in the sequence of its coefficients serves as an upper bound for the number of its positive roots. When roots are counted with multiplicity, then the number of positive roots has the same parity as c. One can apply these results to the polynomial $P(-x)$ to obtain an upper bound on the number of negative roots of P. For a given c, one can find polynomials P with c sign changes with exactly c, $c-2$, $c-4$, ... positive roots. One should observe that by doing so one does not impose any restrictions on the number of negative roots.

Remark 1. It is mentioned in [1] that 18th century authors used to count roots with multiplicity while omitting the parity conclusion; later this conclusion was attributed (see [3]) to a paper of Gauss of 1828 (see [7]), although it is absent there, but was published by Fourier in 1820 (see p. 294 in [6]).

In the present paper we consider polynomials P without zero coefficients. We denote by p the number of sign preservations in the sequence of coefficients of P, and by pos_P (resp. neg_P) the number of positive and negative roots of P. Thus the following condition must be fulfilled:

$$pos_P \leq c\,, \quad pos_P \equiv c\,(\bmod 2)\,, \quad neg_P \leq p\,, \quad neg_P \equiv p\,(\bmod 2)\,. \tag{1.1}$$

Definition 1. A *sign pattern* is a finite sequence σ of $(\pm)$-signs; we assume that the leading sign of σ is $+$. For a given sign pattern of length $d+1$ with c sign changes and p sign preservations, we call (c,p) its *Descartes pair*, $c+p=d$. For a given sign pattern σ with Descartes pair (c,p), we call (pos, neg) an *admissible pair* for σ if conditions (1.1), with $pos_P = pos$ and $neg_P = neg$, are satisfied.

It is natural to ask the following question: *Given a sign pattern σ of length $d+1$ and an admissible pair (pos, neg) can one find a degree d real monic polynomial the signs of whose coefficients define the sign pattern σ and which has exactly pos simple positive and exactly neg simple negative roots ?* When the answer to the question is positive we say that the couple $(\sigma,(pos,neg))$ is *realizable*.

For $d=1$, 2 and 3, the answer to this question is positive, but for $d=4$ D. J. Grabiner showed that this is not the case, see [8]. Namely, for the sign pattern $\sigma^* := (+,+,-,+,+)$ (with Descartes pair $(2,2)$), the pair $(2,0)$ is admissible, see (1.1), but the couple $(\sigma^*,(2,0))$ is not realizable. Indeed, for a monic polynomial $P_4 := x^4 + a_3x^3 + \cdots + a_0$ with signs of the coefficients defined by σ^* and having exactly two positive roots $u<v$ one has $a_j>0$ for $j \neq 2$, $a_2<0$ and $P_4((u+v)/2) < 0$. Hence $P_4(-(u+v)/2) < 0$ because $a_j((u+v)/2)^j = a_j(-(u+v)/2)^j$, $j=0$, 2, 4 and $0 < a_j((u+v)/2)^j = -a_j(-(u+v)/2)^j$, $j=1$, 3. As $P_4(0)=a_0>0$, there are two negative roots $\xi < -(u+v)/2 < \eta$ as well.

Definition 2. We define *the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action* on couples of the form (sign pattern, admissible pair) by its two generators. Denote by $\sigma(j)$ the jth component of the sign pattern σ. The first of the generators replaces the sign pattern σ by σ^r, where σ^r stands for the *reverted* (i.e. read from the back) sign pattern multiplied by $\sigma(1)$, and keeps the same pair (pos, neg). This generator corresponds to the fact that the polynomials $P(x)$ and $x^dP(1/x)/P(0)$ are both monic and have the same numbers of positive and negative roots. The second generator exchanges *pos* with *neg* and changes the signs of σ corresponding to the monomials of odd (resp. even) powers if d is even (resp. odd); the rest of the signs are preserved. We denote the new sign pattern by σ_m. This generator corresponds to the fact that the roots of the polynomials (both monic) $P(x)$ and $(-1)^d P(-x)$ are mutually opposite, and if σ is the sign pattern of P, then σ_m is the one of $(-1)^d P(-x)$.

Remark 2. For a given sign pattern σ and an admissible pair (pos, neg), the couples $(\sigma, (pos, neg))$, $(\sigma^r, (pos, neg))$, $(\sigma_m, (neg, pos))$ and $((\sigma_m)^r, (neg, pos))$ are simultaneously realizable or not. One has $(\sigma_m)^r = (\sigma^r)_m$.

Modulo the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action Grabiner's example is the only nonrealisable couple (sign pattern, admissible pair) for $d = 4$. All cases of couples (sign pattern, admissible pair) for $d = 5$ and 6 which are not realizable are described in [1]. For $d = 7$, this is done in [5] and for $d = 8$ in [5] and [11]. For $d = 5$, there is a single nonrealizable case (up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action). The sign pattern is $(+, +, -, +, -, -,)$ and the admissible pair is $(3, 0)$. For $n = 6$, 7 and 8 there are respectively 4, 6, and 19 nonrealizable cases. In all of them one of the numbers *pos* or *neg* is 0. In the present paper we show that for $d = 9$ this is not so.

Notation 1. For $d = 9$, we denote by σ^0 the following sign pattern (we give on the first and third lines below respectively the sign patterns σ^0 and σ^0_m while the line in the middle indicates the positions of the monomials of odd powers):

$$\begin{array}{ccccccccccccc} \sigma^0 & = & (& + & - & - & - & - & + & + & + & + & - &) \\ & & & 9 & & 7 & & 5 & & 3 & & 1 & & \\ \sigma^0_m & = & (& + & + & - & + & - & - & + & - & + & + &) \end{array}$$

In a sense σ^0 is centre-antisymmetric – it consists of one plus, four minuses, four pluses and one minus.

Theorem 1. (1) *The sign pattern σ^0 is not realizable with the admissible pair* $(1, 6)$.

(2) *Modulo the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, for $d \le 9$, this is the only nonrealizable couple (sign pattern, admissible pair) in which both components of the admissible pair are nonzero.*

Remark 3. It is shown in [10] that for $d = 11$, the admissible pair $(1, 8)$ is not realizable with the sign pattern $(+-----+++++-)$. Hence Theorem 1 shows an example of a nonrealisable couple, with both components of the admissible pair different from zero, in the least possible degree (namely, 9).

Section 2 contains comments concerning the above result and realizability of sign patterns and admissible pairs in general. Section 3 contains some technical lemmas which allow to simplify the proof of Theorem 1. The plan of the proof of part (1) of Theorem 1 is explained in Section 4. The proof results from several lemmas whose proofs can be found in Section 5. The proof of part (2) of Theorem 1 is given in Section 8.

2. COMMENTS

It seems that the problem to classify, for any degree d, all couples (sign pattern, admissible pair) which are not realizable, is quite difficult. This is confirmed by Theorem 1. For the moment, only certain sufficient conditions for realizability or nonrealizability have been formulated:

- in [5] and [13] series of nonrealizable cases were found, for $d \geq 4$, even and for $d \geq 5$, odd respectively;
- in [5] sufficient conditions are given for the nonrealizability of sign patterns with exactly two sign changes.
- in [4] sufficient conditions are given for the realizability and the nonrealizability of sign patterns with exactly two sign changes.

Remark 4. For $d \leq 8$, all couples (sign pattern, admissible pairs) with $pos \geq 1$, $neg \geq 1$, are realizable. That is, in the examples of nonrealizability given in [5] and [13] one has either $pos = 0$ or $neg = 0$, so the question to construct an example of nonrealizability with $pos \neq 0 \neq neg$ was a challenging one.

The result in [5] about sign patterns with exactly two sign changes, consisting of m pluses followed by n minuses followed by q pluses, with $m + n + q = d + 1$, is formulated in terms of the following quantity:

$$\kappa := \frac{d - m - 1}{m} \cdot \frac{d - q - 1}{q} \ .$$

Lemma 1. *For $\kappa \geq 4$, such a sign pattern is not realizable with the admissible pair $(0, d - 2)$. The sign pattern is realizable with any admissible pair of the form $(2, v)$.*

Lemma 1 coincides with Proposition 6 of [5]. One can construct new realizable cases with the help of the following concatenation lemma (see its proof in [5]):

Lemma 2. *Suppose that the monic polynomials P_j of degrees d_j and with sign patterns of the form $(+, \sigma_j)$, $j = 1$, 2 (where σ_j contains the last d_j components of the corresponding sign pattern) realize the pairs (pos_j, neg_j). Then:*

(1) *if the last position of σ_1 is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ realizes the sign pattern $(+, \sigma_1, \sigma_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$;*

(2) *if the last position of* σ_1 *is* $-$, *then for any* $\varepsilon > 0$ *small enough, the polynomial* $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ *realizes the sign pattern* $(+, \sigma_1, -\sigma_2)$ *and the pair* $(pos_1 + pos_2, neg_1 + neg_2)$ *(here* $-\sigma_2$ *is obtained from* σ_2 *by changing each* $+$ *by* $-$ *and vice versa).*

Remark 5. If Lemma 2 were applicable to the case treated in Theorem 1, then this case would be realizable and Theorem 1 would be false. We show here that Lemma 2 is indeed inapplicable. It suffices to check the cases $\deg P_1 \geq 5$, $\deg P_2 \leq 4$ due to the centre-antisymmetry of σ^0 and the possibility to use the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action. In all these cases the sign pattern of the polynomial P_1 has exactly two sign changes (including the first sign $+$, the four minuses that follow and the next between one and four pluses). With the notation from Lemma 1, these cases are $m = 1$, $n = 4$, $q = 1, \ldots, 4$. The respective values of κ are 9, 6, 5 and $9/2$. All of them are > 4. By Descartes' rule the polynomial P_1 can have either 0 or 2 positive roots. In the case of 2 positive roots, Lemma 2 implies that its concatenation with P_2 has at least 2 positive roots which is a contradiction. Hence P_1 has no positive roots. The polynomials P_1 and P_2 define sign patterns with $3 + q - 1$ and $4 - q$ sign preservations respectively. The polynomial P_1 has $\leq 1 + (q - 1)$ negative roots (see Lemma 1) and P_2 has $\leq 4 - q$ ones. Therefore the concatenation of P_1 and P_2 has ≤ 6 negative roots and a polynomial realizing the couple $(\sigma^0, (1, 6))$ (if any) could not be represented as a concatenation of P_1 and P_2. This, of course, does not a priori mean that such a polynomial does not exist.

3. PRELIMINARIES

Notation 2. By S we denote the set of tuples $a \in \mathbb{R}^9$ for which the polynomial $P(x, a) = x^9 + a_8 x^8 + \cdots + a_0$ realizes the pair $(1, 6)$ and the signs of its coefficients define the sign pattern σ^0. We denote by T the subset of S for which $a_8 = -1$. The notation $\bar{S}$ and $\bar{T}$ stands for the closures of the sets S and T.

By writing $a \in S$ (resp. $a \in T$) we mean that the coefficient vector a of the polynomial $P(x, a)$ (excluding the coefficient of x^9) is in S (resp. in T).

For a polynomial $P \in S$, the conditions $a_9 = 1$, $a_8 = -1$ can be obtained by rescaling the variable x and by multiplying P by a nonzero constant (a_9 is the leading coefficient of P).

Lemma 3. *For* $a \in \bar{S}$, *one has* $a_j \neq 0$ *for* $j = 7, 6, 3, 2$, *and one does not have* $a_4 = 0$ *and* $a_5 = 0$ *simultaneously.*

Proof of Lemma 3. For $a_j = 0$ (where j is one of the indices $7, 6, 3, 2$) there are less than 6 sign changes in the sign pattern σ^0_m. Descartes' rule of signs implies that the polynomial $P(., a)$ has less than 6 negative roots counted with multiplicity. The same is true for $a_5 = a_4 = 0$. □

Lemma 4. *For $a \in \bar{S}$, one has $a_0 \neq 0$.*

Remark 6. A priori the set $\bar{S}$ can contain polynomials with all roots real and nonzero. The positive ones can be either a triple root or a double and a simple roots (but not three simple roots). If $a \in S$, then $P(x, a)$ has the maximal possible number of negative roots (equal to the number of sign preservations in the sign pattern). If $a^{'} \in \bar{S}$, then the polynomial $Q(x, a^{'})$ is the limit of polynomials $Q(x, a)$ with $a \in S$. In the limit as a $\to a^{'}$, the complex conjugate pair can become a double positive, but not a double negative root, because there are no 8 sign preservations in the sign pattern.

Proof of Lemma 4. In the proof we consider the two cases $a_0 = 0 \neq a_1$ and $a_0 = a_1 = 0$, and for each of them the three possibilities $a_4 \neq 0 \neq a_5$, $a_4 = 0 \neq a_5$ and $a_4 \neq 0 = a_5$, see Lemma 3.

Suppose that for $P \in \bar{S}$, one has $a_0 = 0$ and for $j \neq 0$, $a_j \neq 0$. Hence the polynomial $P_1 := P/x$ has 6 negative roots and either 0 or 2 positive roots. We show that 0 positive roots is impossible. Indeed, the polynomial P_1 defines a sign pattern with exactly 2 sign changes. Suppose that all negative roots are distinct. If P_1 has no positive roots, then one can apply Lemma 1, according to which, as one has $\kappa = 9/2 > 4$, such a polynomial does not exist. If P_1 has a negative root $-b$ of multiplicity $m > 1$, then its perturbation

$$P_{1,\epsilon} := (x + b + \epsilon)P_1/(x + b) \ , \ 0 < \epsilon \ll 1 \, ,$$

defines the same sign pattern and instead of the root $-b$ of multiplicity m has a root $-b$ of multiplicity $m - 1$ and a simple root $-b - \epsilon$. After finitely many such perturbations, one is in the case when all negative roots are distinct, which leads to a contradiction as above.

If P_1 has 2 positive roots, then this is a double positive root g, see Remark 6. In this case, we add to P_1 a linear term $\pm\epsilon x$ (with ϵ small enough in order not to change the sign pattern) to make the double root bifurcate into a complex conjugate pair. The sign is chosen depending on whether P_1 has a minimum or a maximum at g. After this, if there are multiple negative roots, we apply perturbations of the form $P_{1,\epsilon}$ to arrive again at a contradiction.

Suppose that $a_1 = a_0 = 0$, and that for $j \geq 2$, $a_j \neq 0$. Then one considers the polynomial $P_2 := P/x^2$. It defines a sign pattern with two sign changes and one has $\kappa = 5 > 4$. Hence it has 2 positive roots, otherwise one obtains a contradiction with Lemma 1.

Suppose now that exactly one of the coefficients a_4 or a_5 is 0. We assume this to be a_4, for a_5 the reasoning is similar. Suppose also that either $a_1 \neq 0$, $a_0 = 0$ or $a_1 = a_0 = 0$, and that for $j \geq 2$, $j \neq 4$, one has $a_j \neq 0$. We treat in detail the case $a_1 \neq 0$, $a_0 = 0$, the case $a_1 = a_0 = 0$ is treated by analogy. We first make the double positive root if any bifurcate into a complex conjugate pair as above. This does not change the coefficient a_4. After this instead of perturbations

$P_{1,\epsilon}$ we use perturbations preserving the condition $a_4 = 0$. Suppose that $P_1 = (x+b)^m Q_1 Q_2$, where Q_1 and Q_2 are monic polynomials, deg $Q_2 = 2$, Q_2 having a complex conjugate pair of roots, Q_1 having $6-m$ negative roots counted with multiplicity. Then we set:

$$P_1 \mapsto P_1 + \epsilon(x+b)^{m-1}(x+h_1)(x+h_2)Q_1\,,$$

where the real numbers h_i are distinct, different from any of the roots of P and chosen in such a way that the coefficient δ of x^3 of P_1 is 0. Such a choice is possible, because all coefficients of the polynomial $(x+b)^{m-1}Q_1$ are positive, hence δ is of the form $A+(h_1+h_2)B+Ch_1h_2$, where $A>0$, $B>0$ and $C>0$. The result of the perturbation is a polynomial P_1 having six negative distinct roots and a complex conjugate pair; its coefficient of x^3 is 0. By adding a small positive number to this coefficient, one obtains a polynomial P_1 with roots as before and defining the sign pattern $(+----++++)$. For this polynomial one has $\kappa = 9/2 > 4$ which contradicts Lemma 1.

In the case $a_1 = a_0 = 0$, the polynomial P_1 thus obtained has five negative distinct roots, a complex conjugate pair of roots and a root at 0. One adds small positive numbers to its constant term and to its coefficient of x^3 and one proves in the same way that such a polynomial does not exist. □

Remark 7. One deduces from Lemmas 3 and 4 that for a polynomial in $\bar{T}$ exactly one of the following conditions holds true:

(1) all its coefficients are nonvanishing;

(2) exactly one of them is vanishing and this coefficient is either a_1 or a_4 or a_5;

(3) exactly two of them are vanishing, and these are either a_1 and a_4 or a_1 and a_5.

Lemma 5. *There exists no real degree 9 polynomial satisfying the following conditions:*

- *the signs of its coefficients define the sign pattern* σ^0*,*
- *it has a complex conjugate pair of roots with nonpositive real part,*
- *it has a single positive root,*
- *it has negative roots of total multiplicity 6.*

Proof. Suppose that such a monic polynomial P exists. We can write P in the form $P = P_1P_2P_3$, where deg $P_1 = 6$.

All roots of P_1 are negative hence $P_1 = \sum_{j=0}^{6} \alpha_j x^j$, $\alpha_j > 0$, $\alpha_6 = 1$; $P_2 = x - w$, $w > 0$; $P_3 = x^2 + \beta_1 x + \beta_0$, $\beta_j \geq 0$, $\beta_1^2 - 4\beta_0 < 0$.

By Descartes' rule of signs, the polynomial $P_1P_2 = \sum_{j=0}^{7} \gamma_j x^j$, $\gamma_7 = 1$, has exactly one sign change in the sequence of its coefficients. It is clear that as

$0 > a_8 = \gamma_6 + \beta_1$, and as $\beta_1 \geq 0$, one must have $\gamma_6 < 0$. But then $\gamma_j < 0$ for $j = 0, \ldots, 6$. For $j = 2$, 3 and 4, one has $a_j = \gamma_{j-2} + \beta_1\gamma_{j-1} + \beta_0\gamma_j < 0$ which means that the signs of a_j do not define the sign pattern σ^0. □

Remark 8. It follows from Lemma 5 that polynomials of $\bar{T}$ can only have negative roots of total multiplicity 6 and positive roots of total multiplicity 1 or 3 (i.e., either one simple, or one simple and one double or one triple positive root); these polynomials have no root at 0 (Lemma 4). Indeed, when approaching the boundary of T, the complex conjugate pair can coalesce to form a double positive (but never nonpositive) root; the latter might eventually coincide with the simple positive root.

4. PLAN OF THE PROOF OF PART (1) OF THEOREM 1

Suppose that there exists a monic polynomial $P(x, a^*)|_{a_8^*=-1}$ with signs of its coefficients defined by the sign pattern σ^0, with 6 distinct negative, a simple positive and two complex conjugate roots.

Then for a close to $a^* \in \mathbb{R}^8$, all polynomials $P(x, a)$ share with $P(x, a^*)$ these properties. Therefore the interior of the set T is nonempty. In what follows we denote by Γ the connected component of T to which a^* belongs. Denote by $-\delta$ the value of a_7 for $a = a^*$ (recall that this value is negative).

Lemma 6. *There exists a compact set $K \subset \bar{\Gamma}$ containing all points of $\bar{\Gamma}$ with $a_7 \in [-\delta, 0)$. Hence there exists $\delta_0 > 0$ such that for every point of $\bar{\Gamma}$, one has $a_7 \leq -\delta_0$, and for at least one point of K and for no point of $\bar{\Gamma}\backslash K$, the equality $a_7 = -\delta_0$ holds.*

Proof. Suppose that there exists an unbounded sequence $\{a^n\}$ of values $a \in \bar{\Gamma}$ with $a_7^n \in [-\delta, 0)$. Hence one can perform rescalings $x \mapsto \beta_n x$, $\beta_n > 0$, such that the largest of the moduli of the coefficients of the monic polynomials $Q_n := (\beta_n)^{-9}P(\beta_n x, a^n)$ equals 1. These polynomials belong to $\bar{S}$, not necessarily to $\bar{T}$ because a_8 after the rescalings, in general, is not equal to -1. The coefficient of x^7 in Q_n equals $a_7^n/(\beta_n)^2$. The sequence $\{a^n\}$ is unbounded, so there exists a subsequence β_{n_k} tending to ∞. This means that the sequence of monic polynomials $Q_{n_k} \in \bar{S}$ with bounded coefficients has a polynomial in $\bar{S}$ with $a_7 = 0$ as one of its limit points which contradicts Lemma 3.

Hence the moduli of the roots and the tuple of coefficients a_j of $P(x, a) \in \bar{\Gamma}$ with $a_7 \in [-\delta, 0)$ remain bounded from which the existence of K and δ_0 follows. □

The above lemma implies the existence of a polynomial $P_0 \in \bar{\Gamma}$ with $a_7 = -\delta_0$. We say that P_0 is a_7-*maximal*. Our aim is to show that no polynomial of $\bar{\Gamma}$ is a_7-maximal which contradiction will be the proof of Theorem 1.

Definition 3. A real univariate polynomial is *hyperbolic* if it has only real (not necessarily simple) roots. We denote by $H \subset \bar{\Gamma}$ the set of hyperbolic polynomials in $\bar{\Gamma}$. Hence these are monic degree 9 polynomials having positive and negative roots of respective total multiplicities 3 and 6 (roots at the origin are impossible by Lemma 4). By $U \subset \bar{\Gamma}$ we denote the set of polynomials in $\bar{\Gamma}$ having a complex conjugate pair, a simple positive root and negative roots of total multiplicity 6. Thus $\bar{\Gamma} = H \cup U$ and $H \cap U = \emptyset$. We denote by U_0, U_2, $U_{2,2}$, U_3 and U_4 the subsets of U for which the polynomial $P \in U$ has respectively 6 simple negative roots, one double and 4 simple negative roots, at least two negative roots of multiplicity ≥ 2, one triple and 3 simple negative roots and a negative root of multiplicity ≥ 4.

The following lemma on hyperbolic polynomials is proved in [10]. It is used in the proofs of the other lemmas.

Lemma 7. *Suppose that V is a hyperbolic polynomial of degree $d \geq 2$ with no root at 0. Then:*

(1) *V does not have two or more consecutive vanishing coefficients.*

(2) *If V has a vanishing coefficient, then the signs of its surrounding two coefficients are opposite.*

(3) *The number of positive (of negative) roots of V is equal to the number of sign changes in the sequence of its coefficients (in the one of $V(-x)$).*

By a sequence of lemmas we consecutively decrease the set of possible a_7-*maximal* polynomials until in the end it turns out that this set must be empty. The proofs of the lemmas of this section except Lemma 6 are given in Sections 5 (Lemmas 7 – 12), 6 (Lemma 13) and 7 (Lemmas 14 –16).

Lemma 8. (1) *No polynomial of $U_{2,2} \cup U_4$ is a_7-maximal.*

(2) *For each polynomial of U_3, there exists a polynomial of U_0 with the same values of a_7, a_5, a_4 and a_1.*

(3) *For each polynomial of $U_0 \cup U_2$, there exists a polynomial of $H \cup U_{2,2}$ with the same values of a_7, a_5, a_4 and a_1.*

Lemma 8 implies that if there exists an a_7-maximal polynomial in $\bar{\Gamma}$, then there exists such a polynomial in H. So from now on, we aim at proving that H contains no such polynomial hence H and $\bar{\Gamma}$ are empty.

Lemma 9. *There exists no polynomial in H having exactly two distinct real roots.*

Lemma 10. *The set H contains no polynomial having one triple positive root and negative roots of total multiplicity* 6.

Lemma 10 and Remark 6 imply that a polynomial in H (if any) satisfies the following condition:

Condition A. Any polynomial $P \in H$ has a double and a simple positive roots and negative roots of total multiplicity 6.

Lemma 11. *There exists no polynomial $P \in H$ having exactly three distinct real roots and satisfying the conditions $\{a_1 = 0, a_4 = 0\}$ or $\{a_1 = 0, a_5 = 0\}$.*

It follows from Lemma 11 and Lemma 3 that a polynomial $P \in H$ having exactly three distinct real roots (hence a double and a simple positive and an 6-fold negative one) can satisfy at most one of the conditions $a_1 = 0$, $a_4 = 0$ and $a_5 = 0$.

Lemma 12. *No polynomial in H having exactly three distinct real roots is a_7-maximal.*

Thus an a_7-maximal polynomial in H (if any) must satisfy Condition A and have at least four distinct real roots.

Lemma 13. *The set H contains no polynomial having a double and a simple positive roots and exactly two distinct negative roots of total multiplicity 6, and which satisfies either the conditions $\{a_1 = a_4 = 0\}$ or $\{a_1 = a_5 = 0\}$.*

At this point we know that an a_7-maximal polynomial of H satisfies Condition A and one of the two following conditions:

Condition B. It has exactly four distinct real roots and satisfies exactly one or none of the equalities $a_1 = 0$, $a_4 = 0$ or $a_5 = 0$.

Condition C. It has at least five distinct real roots.

Lemma 14. *The set H contains no a_7-maximal polynomial satisfying Conditions A and B.*

Therefore an a_7-maximal polynomial in H (if any) must satisfy Conditions A and C.

Lemma 15. *The set H contains no a_7-maximal polynomial having exactly five distinct real roots.*

Lemma 16. *The set H contains no a_7-maximal polynomial having at least six distinct real roots.*

Hence the set H contains no a_7-maximal polynomial at all. It follows from Lemma 8 that there is no such polynomial in $\bar{\Gamma}$. Hence $\bar{\Gamma} = \emptyset$.

5. PROOFS OF LEMMAS 7, 8, 9, 10, 11 AND 12

Proof of Lemma 7. Part (1). Suppose that a hyperbolic polynomial V with two or more vanishing coefficients exists. If V is degree d hyperbolic, then $V^{(k)}$ is also hyperbolic for $1 \leq k < d$. Therefore we can assume that V is of the form $x^\ell L + c$, where $\deg L = d - \ell$, $\ell \geq 3$, $L(0) \neq 0$ and $c = V(0) \neq 0$. If V is hyperbolic and $V(0) \neq 0$, then such is also $W := x^d V(1/x) = cx^d + x^{d-\ell} L(1/x)$ and also $W^{(d-\ell)}$

which is of the form $ax^\ell + b$, $a \neq 0 \neq b$. However given that $\ell \geq 3$, this polynomial is not hyperbolic.

For the proof of part (2) we use exactly the same reasoning, but with $\ell = 2$. The polynomial $ax^2 + b$, $a \neq 0 \neq b$, is hyperbolic if and only if $ab < 0$.

To prove part (3) we consider the sequence of coefficients of $V := \sum_{j=0}^{d} v_j x^j$, $v_0 \neq 0 \neq v_d$. Set $\Phi := \sharp\{k | v_k \neq 0 \neq v_{k-1}, v_k v_{k-1} < 0\}$, $\Psi := \sharp\{k | v_k \neq 0 \neq v_{k-1}, v_k v_{k-1} > 0\}$ and $\Lambda := \sharp\{k | v_k = 0\}$. Then $\Phi + \Psi + 2\Lambda = d$. By Descartes' rule of signs the number of positive (of negative) roots of V is $pos_V \leq \Phi + \Lambda$ (resp. $neg_V \leq \Psi + \Lambda$). As $pos_V + neg_V = d$, one must have $pos_V = \Phi + \Lambda$ and $neg_V = \Psi + \Lambda$. It remains to notice that $\Phi + \Lambda$ is the number of sign changes in the sequence of coefficients of V (and $\Psi + \Lambda$ equals the number of sign changes in the sequence of coefficients of $V(-x)$), see part (2) of the lemma. □

Proof of Lemma 8. Part (1). A polynomial of $U_{2,2}$ or U_4 respectively is representable in the form:

$$P^\dagger := (x+u)^2(x+v)^2 S\Delta \quad \text{and} \quad P^* := (x+u)^4 S\Delta\,,$$

where $\Delta := (x^2 - \xi x + \eta)(x - w)$ and $S := x^2 + Ax + B$. All coefficients u, v, w, ξ, η, A, B are positive and $\xi^2 - 4\eta < 0$ (see Lemma 5); for A and B this follows from the fact that all roots of $P^\dagger/\Delta$ and P^*/Δ are negative. (The roots of $x^2 + Ax + B$ are not necessarily different from $-u$ and $-v$.) We consider the two Jacobian matrices

$$J_1 := (\partial(a_8, a_7, a_1, a_4)/\partial(\xi, \eta, w, u)) \quad \text{and} \quad J_2 := (\partial(a_8, a_7, a_1, a_5)/\partial(\xi, \eta, w, u))\,.$$

In the case of $P^\dagger$ their determinants equal

$$\begin{aligned} \det J_1 &= (A^2u^2v + 2A^2uv^2 + 2Au^2v^2 + Auv^3 + 2ABu^2 + 5ABuv \\ &\quad + 2ABv^2 + 3Bu^2v + 2Buv^2 + Bv^3 + 2B^2u + B^2v)\Pi\,, \end{aligned}$$

$$\begin{aligned} \det J_2 &= (A^2uv + Au^2v + 2Auv^2 + 2ABu \\ &\quad + ABv + 2Bu^2 + 4Buv + 2Bv^2)\Pi\,, \end{aligned}$$

where $\Pi := -2v(w+u)(-\eta - w^2 + w\xi)(\xi u + \eta + u^2)$.

These determinants are nonzero. Indeed, each of the factors is either a sum of positive terms or equals $-\eta - w^2 + w\xi < -\xi^2/4 - w^2 + w\xi = -(\xi/2 - w)^2 \leq 0$. Thus one can choose values of (ξ, η, w, v) close to the initial one (u, A and B remain fixed) to obtain any values of (a_8, a_7, a_1, a_4) or (a_8, a_7, a_1, a_5) close to the initial one. In particular, with $a_8 = -1$, $a_1 = a_4 = 0$ or $a_8 = -1$, $a_1 = a_5 = 0$ while a_7 can have values larger than the initial one. Hence this is not an a_7-maximal polynomial. (If the change of the value of (ξ, η, w, v) is small enough, the values of the coefficients a_j, $j = 0$, 2, 3, 5 or 4 and 6 can change, but their signs remain the same.) The same reasoning is valid for P^* as well in which case one has

$$\begin{aligned} \det J_1 &= (3A^2u^2 + 3Au^3 + 9ABu + 6Bu^2 + 3B^2)M\,, \\ \det J_2 &= (A^2u + 3Au^2 + 3AB + 8Bu)M\,, \end{aligned}$$

with $M := -4u^2(w+u)(-\eta - w^2 + w\xi)(\xi u + \eta + u^2)$.

To prove part (2), we observe that if the triple root of $P \in U_3$ is at $-u < 0$, then in case when P is increasing (resp. decreasing) in a neighbourhood of $-u$ the polynomial $P - \varepsilon x^2(x+u)$ (resp. $P + \varepsilon x^2(x+u)$), where $\varepsilon > 0$ is small enough, has three simple roots close to $-u$; it belongs to $\bar{\Gamma}$, its coefficients a_j, $2 \neq j \neq 3$, are the same as the ones of P, the signs of a_2 and a_3 are also the same.

For the proof of part (3), we observe first that 1) for $x < 0$ the polynomial P has three maxima and three minima and 2) for $x > 0$ one of the following three things holds true: either $P' > 0$, or there is a double positive root γ of P', or P' has two positive roots $\gamma_1 < \gamma_2$ (they are both either smaller than or greater than the positive root of P). Suppose first that $P \in U_0$. Consider the family of polynomials $P - t$, $t \geq 0$. Denote by t_0 the smallest value of t for which one of the three things happens: either $P - t$ has a double negative root v (hence a local maximum), or $P - t$ has a triple positive root γ or $P - t$ has a double and a simple positive roots (the double one is at γ_1 or γ_2). In the second and third cases one has $P - t_0 \in H$. In the first case, if $P - t_0$ has another double negative root, then $P - t_0 \in U_{2,2}$ and we are done. If not, then consider the family of polynomials

$$P_s := P - t_0 - s(x^2 - v^2)^2(x^2 + v^2) = P - t_0 - s(x^6 - v^2x^4 - x^2v^4 + v^6)\,, \quad s \geq 0\,.$$

The polynomial $-(x^6 - v^2x^4 - x^2v^4 + v^6)$ has double real roots at $\pm v$ and a complex conjugate pair. It has the same signs of the coefficients of x^6, x^4 and 1 as $P - t_0$ and P. The rest of the coefficients of $P - t_0$ and P_s are the same. As s increases, the value of P_s for every $x \neq \pm v$ decreases. So for some $s = s_0 > 0$ for the first time one has either $P_s \in U_{2,2}$ (another local maximum of P_s becomes a double negative root) or $P_s \in H$ (P_s has positive roots of total multiplicity 3, but not three simple ones). This proves part (3) for $P \in U_0$.

If $P \in U_2$ and the double negative root is a local minimum, then the proof of part (3) is just the same. If this is a local maximum, then one skips the construction of the family $P - t$ and starts constructing the family P_s directly. □

Proof of Lemma 9. Suppose that such a polynomial exists. Then it must be of the form $P := (x+u)^6(x-w)^3$, $u > 0$, $w > 0$. The conditions $a_8 = -1$ and $a_1 > 0$ read:

$$6u - 3w = -1 \quad \text{and} \quad 3u^5w^2(u - 2w) > 0\,.$$

In the plane of the variables (u, w) the domain $\{u > 0, w > 0, u - 2w > 0\}$ does not intersect the line $6u - 3w = -1$ which proves the lemma. □

Proof of Lemma 10. Represent the polynomial in the form $P = (x+u_1)\cdots(x+u_6)(x-\xi)^3$, where $u_j > 0$ and $\xi > 0$. The numbers u_j are not necessarily distinct. The coefficient a_8 then equals $u_1 + \cdots + u_6 - 3\xi$. The condition $a_8 = -1$ implies $\xi = \xi_* := (u_1 + \cdots + u_6 + 1)/3$. Thus

$$P(x) = (x+u_1)\cdots(x+u_6)\Big(x - \frac{u_1 + \cdots + u_6 + 1}{3}\Big)^3$$

and for the coefficient a_1 we have

$$27a_1 = (u_1 + \cdots + u_6 + 1)^2\, u_1u_2\cdots u_6 \Big(3 - (u_1 + \cdots + u_6 + 1) \sum_{j=1}^{6} \frac{1}{u_j} \Big) .$$

The last factor in this representation is negative, hence $a_1 < 0$, a contradiction. □

Proof of Lemma 11. Suppose that such a polynomial exists. Then it must be of the form $(x+u)^6(x-w)^2(x-\xi)$, where $u > 0$, $w > 0$, $\xi > 0$, $w \neq \xi$. One checks numerically (say, using MAPLE), for each of the two systems of algebraic equations $a_8 = -1$, $a_1 = 0$, $a_4 = 0$ and $a_8 = -1$, $a_1 = 0$, $a_5 = 0$, that each real solution (u, w, ξ) or (u, v, w) contains a nonpositive component. □

Proof of Lemma 12. Making use of Condition A formulated after Lemma 10, we consider only polynomials of the form $(x+u)^6(x-w)^2(x-\xi)$, where u, w, ξ are positive and $w \neq \xi$. Consider the Jacobian matrix

$$J_1^* := (\partial(a_8, a_7, a_1)/\partial(u, w, \xi)) .$$

Its determinant equals $-12u^4(u+w)(u-5w)(w-\xi)(u+\xi)$. All factors except $u - 5w$ are nonzero. Thus for $u \neq 5w$, one has $\det J_1 \neq 0$, so one can fix the values of a_8 and a_1 and vary the one of a_7 arbitrarily close to the initial one by choosing suitable values of u, w and ξ. Hence the polynomial is not a_7-maximal. For $u = 5w$, one has $a_3 = -2500w^5(\xi + 5w) < 0$ which is impossible. Hence there exist no a_7-maximal polynomials which satisfy only the condition $a_1 = 0$ or none of the conditions $a_1 = 0$, $a_4 = 0$ or $a_5 = 0$. To see that there exist no such polynomials satisfying only the condition $a_4 = 0$ or $a_5 = 0$ one can consider the matrices $J_4^* := (\partial(a_8, a_7, a_4)/\partial(u, w, \xi))$ and $J_5^* := (\partial(a_8, a_7, a_5)/\partial(u, w, \xi))$. Their determinants equal respectively

$$-60u(u+w)(2u-w)(\xi-w)(\xi+u) \quad \text{and} \quad -12u(u+w)(5u-w)(\xi-w)(\xi+u) .$$

They are nonzero respectively for $2u \neq w$ and $5u \neq w$, in which cases in the same way we conclude that the polynomial is not w_7-maximal. If $u = w/2$, then $a_1 = -(1/64)w^7(10\xi - w)$ and $a_8 = w - \xi$. As $a_1 > 0$ and $a_8 < 0$, one has $w > 10\xi$ and $\xi > w > 10\xi$ which is a contradiction. If $w = 5u$, then $a_6 = 20u^2(u+\xi) > 0$ which is again a contradiction. □

6. PROOF OF LEMMA 13

The multiplicities of the negative roots of P define the following a priori possible cases:

$$\text{A)} \quad (5,1)\,, \quad \text{B)} \quad (4,2) \quad \text{and} \quad \text{C)} \quad (3,3)\,.$$

In all of them the proof is carried out simultaneously for the two possibilities $\{a_1 = a_4 = 0\}$ and $\{a_1 = a_5 = 0\}$. In order to simplify the proof we fix one of the roots to be equal to -1 (this can be achieved by a change $x \mapsto \beta x$, $\beta > 0$, followed by $P \mapsto \beta^{-9} P$). This allows to deal with one less parameter. By doing so we may no longer require that $a_8 = -1$, but only that $a_8 < 0$.

Case A) We use the following parametrization:

$$P = (x+1)^5 (s\,x+1)(t\,x-1)^2(w\,x-1)\,,\ s>0\,,\ t>0\,,\ w>0\,,\ t\neq w\,,$$

i.e. the negative roots of P are at -1 and $-1/s$ and the positive ones at $1/t$ and $1/w$.

The condition $a_1 = w+2t-s-5 = 0$ yields $s = w+2t-5$. With this s one has

$$a_3 = a_{32}w^2 + a_{31}w + a_{30}\,, \quad a_4 = a_{42}w^2 + a_{41}w + a_{40}\,, \quad \text{where}$$

$$a_{32} = -2t+5\,, \quad a_{31} = -(2t-5)^2\,, \quad a_{30} = -2t^3 + 20t^2 - 50t + 40\,,$$

$$a_{42} = t^2 - 10t + 10\,, \quad a_{41} = 2t^3 - 25t^2 + 70t - 50\,, \quad a_{40} = -10t^3 + 55t^2 - 100t + 45\,.$$

The coefficient a_{30} has a single real root $6.7245\ldots$ hence $a_{30} < 0$ for $t > 6.7245\ldots$. On the other hand, for $t > 6.7245\ldots$,

$$a_{32}w^2 + a_{31}w = w(-2t+5)(w+2t-5) = w(-2t+5)s < 0\,.$$

Thus the inequality $a_3 > 0$ fails for $t > 6.7245\ldots$. Observing that $a_{41} = (2t-5)a_{42}$ one can write

$$a_4 = (w+2t-5)w\,a_{42} + a_{40} = s\,w\,a_{42} + a_{40}\,.$$

The real roots of a_{42} (resp. a_{40}) equal $1.127\ldots$ and $8.872\ldots$ (resp. $0.662\ldots$). Hence for $t \in [1.127\ldots, 8.872\ldots]$, the inequality $a_4 > 0$ fails. There remains to consider the possibility $t \in (0, 1.127\ldots)$.

It is to be checked directly that for $s = w+2t-5$, one has

$$a_8/t = 10t^2w + 5t\,w^2 - 2t^2 - 29t\,w - 2w^2 + 5t + 10w = (5t-2)w\,s + t(5-2t)\,,$$

which is nonnegative (hence $a_8 < 0$ fails) for $t \in [2/5, 5/2]$. Similarly

$$a_6 = a_6^* w(w+2t-5) + a_6^\dagger = a_6^* w\,s + a_6^\dagger\,, \quad \text{where}$$

$$a_6^* = 10t^2 - 20t + 5\,, \quad a_6^\dagger = -5(t-1)(4t^2 - 9t + 1)\,.$$

The real roots of a_6^* (resp. $a_6^\dagger$) equal $1.707\ldots > 2/5 = 0.4$ and $0.293\ldots$ (resp. $1 > 2/5$, $0.117\ldots$ and $2.133\ldots$) hence for $t \in (0, 2/5)$ one has $a_6^* > 0$ and $a_6^\dagger > 0$, i.e. $a_6 > 0$ and the equality $a_6 = 0$ or the inequality $a_6 < 0$ is impossible. □

Case B) We parametrize P as follows:

$$P = (x+1)^4 (Tx^2 + Sx - 1)^2 (w\,x-1)\,,\ T>0\,,\ w>0\,.$$

Here we presume S to be real, not necessarily positive. The factor $(Tx^2 + Sx - 1)^2$ contains the double positive and negative roots of P.

From $a_1 = w + 2S - 4 = 0$ one finds $S = (4 - w)/2$. With this S one has

$$a_8/T = (4w - 1)T + 4w - w^2\,, \quad a_5 = a_{52}T^2 + a_{51}T + a_{50}\,, \quad \text{where}$$

$$a_{52} = w - 4\,, \quad a_{51} = -4w^2 + 10w - 16\,, \quad a_{50} = (3/2)w^3 - 9w^2 + 16w - 12\,.$$

Suppose first that $w > 1/4$. The inequality $a_8 < 0$ is equivalent to

$$T < T_0 := (w^2 - 4w)/(4w - 1)\,.$$

As $T > 0$, this implies $w > 4$.

For $T = T_0$, one obtains $a_5 = 3C/2(4w - 1)^2$, where the numerator $C := 6w^5 - 40w^4 + 85w^3 - 54w^2 + 32w - 8$ has a single real root $0.368\ldots$. Hence for $w > 4$, one has $C > 0$ and $a_5|_{T=T_0} > 0$. On the other hand, $a_{50} = a_5|_{T=0}$ has a single real root $3.703\ldots$, so for $w > 4$ one has $a_5|_{T=0} > 0$. For $w > 4$ fixed, and for $T \in [0, T_0]$, the value of the derivative

$$\partial a_5/\partial T = (2w - 8)T - 4w^2 + 10w - 16$$

is maximal for $T = T_0$; this value equals

$$-2(7w^3 - 14w^2 + 21w - 8)/(4w - 1)\,,$$

which is negative because the only real root of the numerator is $0.510\ldots$. Thus $\partial a_5/\partial T < 0$ and a_5 is minimal for $T = T_0$. Hence the inequality $a_5 < 0$ fails for $w > 1/4$. For $w = 1/4$ one has $a_8 = 15/16 > 0$.

So suppose that $w \in (0, 1/4)$. In this case the condition $a_8 < 0$ implies $T > T_0$. For $T = T_0$ one gets

$$a_4 = 3D/2(4w - 1)^2\,, \quad \text{where} \quad D := 8w^5 - 32w^4 + 54w^3 - 85w^2 + 40w - 6$$

has a single real root $2.719\ldots$. Therefore for $w \in (0, 1/4)$ one has $D < 0$ and $a_4|_{T=T_0} < 0$. The derivative $\partial a_4/\partial T = -w^2 - 2T - 4$ being negative one has $a_4 < 0$ for $w \in (0, 1/4)$, i.e. the inequality $a_4 > 0$ fails. □

Case C) We set

$$P := (x + 1)^3(sx + 1)^3(tx - 1)^2(wx - 1)\,,\ s > 0\,,\ t > 0\,,\ w > 0\,,\ t \neq w\,.$$

The condition $a_1 = w + 2t - 3s - 3 = 0$ implies $s = s_0 := (w + 2t - 3)/3$. For $s = s_0$, one has $27a_8 = t(w + 2t - 3)^2 H^*$, where

$$H^* := 6wt^2 - 2t^2 + 3w^2t - 5wt + 3t + 6w - 2w^2\,. \tag{6.1}$$

We show first that for $s = s_0$, the case $a_1 = a_5 = 0$ is impossible. To fix the ideas, we represent in Figure 1 the sets $\{H^* = 0\}$ (solid curve) and $\{a_5^* = 0\}$ (dashed curve), where $a_5^* := a_5|_{s=s_0}$. Although we need only the nonnegative values of t and w, we show these curves also for the negative values of the variables to make things more clear. (The lines $t = 2/3$ and $w = 1/3$ are asymptotic lines for the set $\{H^* = 0\}$). For $t \geq 0$ and $w \geq 0$, the only point, where $H^* = a_5^* = 0$, is the point $(0; 3)$. However, at this point one has $a_8 = 0$, i.e. this does not correspond to the required sign pattern.

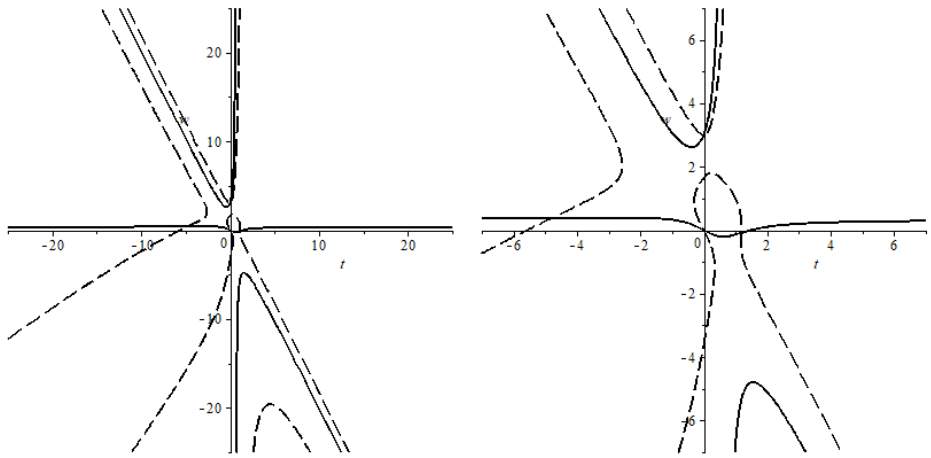


Figure 1: The sets $\{H^* = 0\}$ (solid curve) and $\{a_5^* = 0\}$ (dashed curve), with 3 and 4 connected components respectively.

Lemma 17. (1) *For* $(t, w) \in \Omega_1 \cup \Omega_2$, *where* $\Omega_1 = [3/2, \infty) \times [1/3, \infty)$ *and* $\Omega_2 = [0, 3/2] \times [0, 3]$, *one has* $H^* \geq 0$.

(2) *For* $(t, w) \in \Omega_3 := [3/2, \infty) \times [0, 1/3]$, *one has* $a_5^* < 0$.

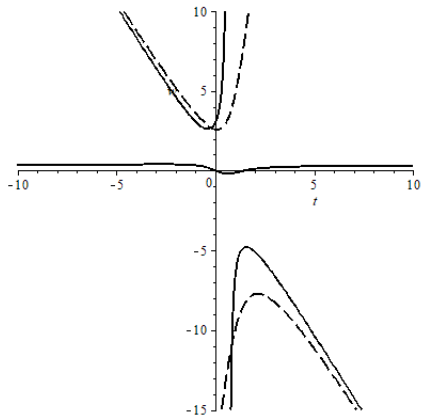
(3) *For* $(t, w) \in \Omega_4 := [0, 3/2] \times [3, \infty)$, *the two conditions* $H^* < 0$ *and* $a_5^* = 0$ *do not hold simultaneously.*

Lemma 17 (which is proved after the proof of Lemma 12) implies that in each of the sets Ω_j, $1 \leq j \leq 4$, at least one of the two conditions $H^* < 0$ (i. e. $a_8 < 0$) and $a_5^* = 0$ fails. There remains to notice that $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 = \{t \geq 0, w \geq 0\}$.

Now, we show that for $s = s_0$, the case $a_1 = a_4 = 0$ is impossible. In Figure 2 we show the sets $\{H^* = 0\}$ (solid curve) and $\{a_4^* = 0\}$ (dashed curve), where $a_4^* := a_4|_{s=s_0}$. We use the notation introduced in Lemma 17. By part (1) of Lemma 17 the case $a_1 = a_4 = 0$ is impossible for $(t, w) \in \Omega_1 \cup \Omega_2$.

Lemma 18. (1) *For* $(t, w) \in \Omega_3$, *one has* $a_4^* > 0$.

(2) *For* $(t, w) \in \Omega_4$, *the two conditions* $H^* < 0$ *and* $a_4^* = 0$ *do not hold simultaneously.*



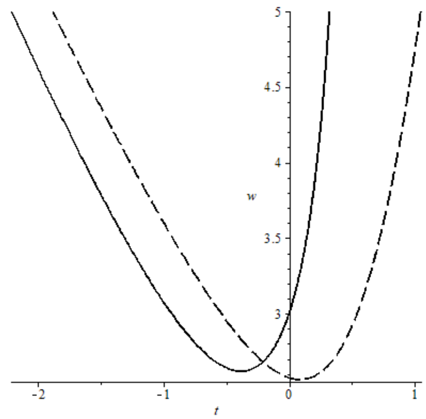


Figure 2: The sets $\{H^* = 0\}$ (solid curve) and $\{a_4^* = 0\}$ (dashed curve), with 3 and 2 connected components respectively.

Thus the couple of conditions $H^* < 0$, $a_4^* = 0$ fails for $t \geq 0$, $w \geq 0$. This proves Lemma 13. Lemma 18 is proved after Lemma 17 . □

Proof of Lemma 17. Part (1). Consider the quantity H^* as a polynomial in the variable w:

$$H^* = b_2 w^2 + b_1 w + b_0 ,$$

where

$$b_2 = 3t - 2 , \quad b_1 = 6t^2 - 5t + 6 , \quad b_0 = -2t(t - 3/2) .$$

Its discriminant $\Delta_w := b_1^2 - 4b_0 b_2 = 9(2t^2 - 3t + 2)(2t^2 + t + 2)$ is positive for any real t. This is why for $t \neq 2/3$, the polynomial H^* has 2 real roots; for $t = 2/3$, it is a linear polynomial in w and has a single real root $-5/24$. When H^* is considered as a polynomial in the variable t, one sets

$$\begin{aligned} &H^* := c_2 t^2 + c_1 t + c_0 , \quad \text{where} \\ &c_2 = 6w - 2 , \quad c_1 = 3w^2 - 5w + 3 , \quad c_0 = -2w(w - 3) . \end{aligned} \tag{6.2}$$

Its discriminant

$$\Delta_t := c_1^2 - 4c_0 c_2 = 9(w^2 + 5w + 1)(w^2 - 3w + 1)$$

is negative if and only if $w \in (-4.79\ldots, 0.20\ldots) \cup (-0.38\ldots, 2,61\ldots)$. One checks directly that $H^*|_{w=1/3} = (5/3)t + 16/9$ which is positive for $t \geq 0$. Next, one has $H^*|_{w=0} = b_0$ which is negative for $t > 3/2$. Finally, for $t > 3/2$, the ratio b_0/b_2 is negative which means that for $t > 3/2$ fixed, the polynomial H^* has one positive and one negative root, so the positive root belongs to the interval $(0, 1/3)$ (because

$H^*|_{w=1/3} > 0$). Hence $H^* \geq 0$ for $(t,w) \in \Omega_1$ and $H^* > 0$ for (t,w) in the interior of Ω_1.

Suppose now that $(t,w) \in [0,3/2] \times [0,3]$. For $t \in (2/3,3/2]$ fixed, one has $b_2 > 0$, $b_1/b_2 > 0$ and $b_0/b_2 > 0$ which implies that H^* has two negative roots, and for $(t,w) \in (2/3,3/2] \times [0,3]$, one has $H^* > 0$. For $t \in [0,2/3)$ fixed, one has $b_2 < 0$, $b_1/b_2 < 0$, $b_0/b_2 < 0$ and H^* has a positive and a negative root; given that $b_2 < 0$, H^* is positive between them. For $w = 3$ and $t \geq 0$, one has $H^* = t(16t+15) \geq 0$, with equality only for $t = 0$. Therefore $H^* > 0$ for $(t,w) \in [0,2/3) \times [0,3]$. And for $t = 2/3$, one obtains $H^* = (16/3)w + 10/9$ which is positive for $w \geq 0$.

Part (2). One has

$$\begin{aligned} a_5^* &= -8t^5 + 8t^4w + 6t^3w^2 - 4t^2w^3 - 2tw^4 - 24t^4 \\ &\quad -66t^3w - 63t^2w^2 - 12tw^3 + 3w^4 + 84t^3 + 153t^2w \\ &\quad +90tw^2 - 3w^3 - 144t^2 - 144tw - 36w^2 + 108t + 54w\,. \end{aligned}$$

Consider a_5^* as a polynomial in w. Set $R_w :=\mathrm{Res}(a_5^*, \partial a_5^*/\partial w, w)/2125764$. Then $R_w = (2t-3)R_w^1 R_w^2$, where

$$\begin{aligned} R_w^1 &= 32t^5 + 16t^4 - 80t^3 + 184t^2 - 142t - 63\,, \\ R_w^2 &= 10t^{10} - 80t^9 + 365t^8 - 928t^7 + 1564t^6 - 1788t^5 \\ &\quad +1345t^4 - 668t^3 + 208t^2 - 40t + 4\,. \end{aligned}$$

The real roots of R_w^1 (resp. R_w^2) equal $-2.56\ldots$, $-0.30\ldots$ and $1.18\ldots$ (resp. $0.34\ldots$ and $1.16\ldots$). That is, the largest real root of R_w is $3/2$. One has

$$a_5^*|_{w=0} = -4t(2t^4 + 6t^3 - 21t^2 + 36t - 27)\,,$$

with real roots equal to $-5.55\ldots$, 0 and $1.18\ldots$. This means that for $t > 3/2$, the signs of the real roots of a_5^* do not change and their number (counted with multiplicity) remains the same. For $t = 3/2$ and $t = 2$, one has

$$a_5^* = -30w^3 - (45/2)w^2 - (243/4) \quad \text{and} \quad a_5^* = -w^4 - 43w^3 - 60w^2 - 22w - 328$$

respectively, which quantities are negative. Hence $a_5^* < 0$ for $t \geq 3/2$ from which Part (2) follows.

Part (3). Consider the resultant

$$\begin{aligned} R^\flat &:= \mathrm{Res}(H^*, a_5^*, t) = -52488w(w-3)R^\sharp(w^2 - w + 1)^2\,, \\ R^\sharp &:= 5w^6 - 16w^5 + 40w^4 - 23w^3 + 61w^2 - 16w - 2\,. \end{aligned}$$

The real roots of $R^\sharp$ equal $-0.09\ldots$ and $0.37\ldots$; the factor $w^2 - w + 1$ has no real roots. Thus the largest real root of $R^\flat$ equals 3. For $w = 3$, one has

$$a_5^* = -4t^2(2t^3 + 15t + 90) \leq 0\,,$$

with equality if and only if $t = 0$. For $w > 3$ and $t \geq 0$, the sets $\{H^* = 0\}$ and $\{a_5^* = 0\}$ do not intersect (because $R^\flat < 0$). We showed in the proof of part (1) of the lemma that the discriminant Δ_t is positive for $w \geq 3$. Hence each horizontal line $w = w_0 > 3$ intersects the set $\{H^* = 0\}$ for two values of t; one of them is positive and one of them is negative (because $c_0/c_2 < 0$); we denote them by t_+ and t_-.

The discriminant R_t :=Res$(a_5^*, \partial a_5^*/\partial t, t)$ equals $2176782336(w-3)R_t^1 R_t^2$, where

$$\begin{aligned} R_t^1 &:= 5w^{12} + 50w^{11} + 100w^{10} - 2513w^9 + 10781w^8 - 25932w^7 + 46604w^6 \\ &\quad -70411w^5 + 86678w^4 - 82706w^3 + 65264w^2 - 43104w + 16896\,, \\ R_t^2 &:= 8w^4 + 154w^3 - 68w^2 - 239w - 352\,. \end{aligned}$$

The factor R_t^1 is without real roots. The real roots of R_t^2 (both simple) equal $-19.61\ldots$ and $1.81\ldots$. Hence for each $w = w_0 > 3$, the polynomial a_5^* has one and the same number of real roots. Their signs do not change with t. Indeed, a_5^* is a degree 5 polynomial in t, with leading coefficient and constant term equal to -8 and $3w(w-3)(w^2+2w-6)$ respectively; the real roots of the quadratic factor equal $-3.64\ldots$ and $1.64\ldots$.

For $w_0 > 3$, the polynomial a_5^* has exactly 3 real roots $t_1 < t_2 < t_3$. For any $w_0 > 3$, the signs of these roots and of the roots $t_\pm$ of H^* and the order of these 5 numbers on the real line are the same. For $w = 4$, one has

$$t_1 = -3.3\ldots < t_- = -1.6\ldots < t_2 = -0.8\ldots < t_+ = 0.2\ldots < t_3 = 0.3\ldots$$

Hence the only positive root t_3 of a_5^* belongs to the domain where $H^* > 0$. Hence one cannot have $a_5^* = 0$ and $H^* < 0$ at the same time. Lemma 17 is proved. □

Proof of Lemma 18. Part (1). One has

$$\begin{aligned} a_4^* &:= -20t^4 - 22t^3w - 30t^2w^2 - 10tw^3 + w^4 + 66t^3 + 45t^2w + 36tw^2 \\ &\quad +15w^3 - 135t^2 - 54tw - 54w^2 + 108t + 54w - 81\,. \end{aligned}$$

Consider a_4^* as a polynomial in t. Its discriminant $\Delta_t^\bullet$:=Res$(a_4^*, \partial a_4^*/\partial t, t)$ is of the form $170061120\,\Delta^\flat\Delta^\sharp(w^2-w+1)^2$, where

$$\begin{aligned} \Delta^\flat &:= 9w^4 + 48w^3 + 82w^2 + 56w + 205\,, \\ \Delta^\sharp &:= 3w^4 + 14w^3 - 63w^2 + 51w - 82\,. \end{aligned}$$

Only the factor $\Delta^\sharp$ has real roots, and these are $w_- := -7.72\ldots$ and $w_+ := 2.56\ldots$; they are simple. For $w \in (w_-, w_+)$, the quantity a_4^* is negative. Indeed, $a_4^*|_{w=0} = -20t^4 + 66t^3 - 135t^2 + 108t - 81$ which polynomial has no real roots; hence this is the case of $a_4^*|_{w=w_0}$ for any $w_0 \in (w_-, w_+)$. This proves Part (1), because the set Ω_3 belongs to the strip $\{w_- < w < w_+\}$.

Part (2). The discriminant $\mathrm{Res}(a_4^*, H^*, t)$ equals $-26244\, R^{\triangle}(w^2 - w + 1)^2$ whose factor

$$R^{\triangle} := 2w^6 + 16w^5 - 61w^4 + 23w^3 - 40w^2 + 16w - 5$$

has exactly two real (and simple) roots which equal $-10.90\ldots$ and $2.68\ldots$. Hence for $w \geq 3 > w_+$,

(1) the sets $\{H^* = 0\}$ and $\{a_4^* = 0\}$ do not intersect;

(2) the numbers of positive and negative roots of H^* and a_4^* do not change; for H^* this follows from formula (6.2); for a_4^* whose leading coefficient as a polynomial in t equals -20, this results from $a_4^*|_{t=0} = w^4 + 15w^3 - 54w^2 + 54w - 81$ whose real roots $-18.1\ldots$ and $2.5\ldots$ (both simple) are < 3.

Hence for $w = w_0 \geq 3$, one has $h_- < A_- < 0 \leq h_+ < A_+$, where h_- and h_+ (resp. A_- and A_+) are the two roots of $H^*|_{w=w_0}$ (resp. of $a_4^*|_{w=w_0}$), with equality only for $w_0 = 3$. It is sufficient to check this string of inequalities for one value of w_0, say, for $w_0 = 4$, in which case one obtains

$$h_- = -1.63\ldots < A_- = -1.26\ldots < h_+ = 0.22\ldots < A_+ = 0.85\ldots\,.$$

Hence for $w = w_0 \geq 3$, the only positive root of the polynomial $a_4^*|_{w=w_0}$ belongs to the domain $\{H^* > 0\}$. This proves Part (2) of the Lemma. □

7. PROOFS OF LEMMAS 14, 15 AND 16

Proof of Lemma 14. We are using the following:

Notation 3. If ζ_1, ζ_2, …, ζ_k are distinct roots of the polynomial P (not necessarily simple), then by P_{ζ_1}, P_{ζ_1,ζ_2}, …, $P_{\zeta_1,\zeta_2,\ldots,\zeta_k}$ we denote the polynomials

$$P/(x-\zeta_1)\,,\ P/(x-\zeta_1)(x-\zeta_2)\,,\ \ldots\,,\ P/(x-\zeta_1)(x-\zeta_2)\ldots(x-\zeta_k)\,.$$

Denote by u, v, w and t the four distinct roots of P (all nonzero). Hence

$$P = (x-u)^m(x-v)^n(x-w)^p(x-t)^q\ ,\ m+n+p+q = 9\,.$$

For $j = 1$, 4 or 5, we show that the Jacobian matrix $J := (\partial(a_8, a_7, a_j)/\partial(u, v, w, t))^{\top}$ (where a_8, a_7, a_j are the corresponding coefficients of P expressed as functions of (u, v, w, t)) is of rank 3. (The entry in position $(2,3)$ of J is $\partial a_7/\partial w$.) Hence one can vary the values of (u, v, w, t) in such a way that a_8 and a_j remain fixed (the value of a_8 being -1) and a_7 takes all possible nearby values. Hence the polynomial is not a_7-maximal.

The entries of the four columns of J are the coefficients of x^8, x^7 and x^j of the polynomials $-mP_u = \partial P/\partial u$, $-nP_v$, $-pP_w$ and $-qP_t$. By abuse of language we say that the linear space $\mathcal{F}$ spanned by the columns of J is generated by the polynomials P_u, P_v, P_w and P_t. As

$$P_{u,v} = \frac{P_u - P_v}{v-u}\,, \quad P_{u,w} = \frac{P_u - P_w}{w-u} \quad \text{and} \quad P_{u,t} = \frac{P_u - P_t}{t-u}\,,$$

one can choose as generators of $\mathcal{F}$ the quadruple $(P_u,\, P_{u,v},\, P_{u,w},\, P_{u,t})$; in the same way one can choose $(P_u,\, P_{u,v},\, P_{u,v,w},\, P_{u,v,t})$ or $(P_u,\, P_{u,v},\, P_{u,v,w},\, P_{u,v,w,t})$ (the latter polynomials are of respective degrees 8, 7, 6 and 5). As $(x-t)P_{u,v,w,t} = P_{u,v,w}$, $(x-w)P_{u,v} = P_{u,v,w}$ etc. one can choose as generators the quadruple

$$\psi := (x^3 P_{u,v,w,t}\ ,\ x^2 P_{u,v,w,t}\ ,\ x P_{u,v,w,t}\ ,\ P_{u,v,w,t})\,.$$

Set $P_{u,v,w,t} := x^5 + Ax^4 + \cdots + G$. The coefficients of x^8, x^7 and x^5 of the quadruple ψ define the matrix

$$J^* := \begin{pmatrix} 1 & 0 & 0 & 0 \\ A & 1 & 0 & 0 \\ D & C & B & A \end{pmatrix}.$$

Its columns span the space $\mathcal{F}$ hence $\operatorname{rank} J^* = \operatorname{rank} J$. As at least one of the coefficients B and A is nonzero (Lemma 7) one has $\operatorname{rank} J^* = 3$ and the lemma follows (for the case $j = 6$). In the cases $j = 5$ and $j = 1$ the last row of J^* equals respectively $(\,E\ D\ C\ B\,)$ and $(\,0\ 0\ G\ F\,)$ and in the same way $\operatorname{rank} J^* = 3$. $\square$

Proof of Lemma 15. We are using Notation 3 and the method of proof of Lemma 14. Denote by u, v, w, t, h the five distinct real roots of P (not necessarily simple). Thus using Lemma 10 one can assume that

$$\begin{gathered} P = (x+u)^\ell (x+v)^m (x+w)^n (x-t)^2 (x-h)\,, \\ u,\, v,\, w,\, t,\, h > 0\,, \quad \ell + m + n = 6\,. \end{gathered} \tag{7.1}$$

Set $J := (\partial(a_8, a_7, a_j, a_1)/\partial(u, v, w, t, h))^\top$, $j = 4$ or 5. The columns of J span a linear space $\mathcal{L}$ defined by analogy with the space $\mathcal{F}$ of the proof of Lemma 14, but spanned by 4-vector-columns.

Set $P_{u,v,w,t,h} := x^4 + ax^3 + bx^2 + cx + d$. Consider the vector-column

$$(0, 0, 0, 0, 1, a, b, c, d)^\top\,.$$

The similar vector-columns defined when using the polynomials $x^s P_{u,v,w,t,h}$, $1 \le s \le 4$, instead of $P_{u,v,w,t,h}$ are obtained from this one by successive shifts by one position upward. To obtain generators of $\mathcal{L}$ one has to restrict these vector-columns to the rows corresponding to x^8 (first), x^7 (second), x^j ($(9-j)$th) and x (eighth row).

Further we assume that $a_1 = 0$. If this is not the case, then at most one of the conditions $a_4 = 0$ and $a_5 = 0$ is fulfilled and the proof of the lemma can be finished by analogy with the proof of Lemma 14.

Consider the case $j = 5$. The rank of J is the same as the rank of the matrix

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ c & b & a & 1 & 0 \\ 0 & 0 & 0 & d & c \end{pmatrix} \quad \begin{matrix} x^8 \\ x^7 \\ x^5 \\ x \end{matrix} \,.$$

One has $\operatorname{rank} M = 2 + \operatorname{rank} N$, where $N = \begin{pmatrix} a & 1 & 0 \\ 0 & d & c \end{pmatrix}$. Given that $d \neq 0$, see Lemma 4, one can have $\operatorname{rank} N < 2$ only if $a = c = 0$. We show that the condition $a = c = 0$ leads to the contradiction that one must have $a_8 > 0$. We set $u = 1$ to reduce the number of parameters, so we require only the inequality $a_8 < 0$, but not the equality $a_8 = -1$, to hold true. We have to consider the following cases for the values of the triple (ℓ, m, n) (see (7.1)): 1) $(4,1,1)$, 2) $(3,2,1)$ and 3) $(2,2,2)$. Notice that

$$P_{u,v,w,t,h}|_{u=1} = (x+1)^{\ell-1}(x+v)^{m-1}(x+w)^{n-1}(x-t)\,.$$

In case 1) one has

$$a = 3 - t\,, \quad b = 3 - 3t\,, \quad c = 1 - 3t \quad \text{and} \quad d = -t\,, \tag{7.2}$$

so the condition $a = c = 0$ leads to the contradiction $3 = t = 1/3$.

In case 2) one obtains

$$a = 2 + v - t\,, \quad b = 1 + 2v - (2+v)t\,, \quad c = v - (1+2v)t \quad \text{and} \quad d = -vt\,. \tag{7.3}$$

Thus, the condition $a = c = 0$ yields $v = -1$, $t = 1$. This is also a contradiction because v must be positive.

In case 3) one gets

$$\begin{aligned} &a = 1 + v + w - t\,, \quad b = v + (1+v)w - (1+v+w)t\,, \\ &c = vw - (v + (1+v)w)t\,, \quad d = -vwt\,. \end{aligned} \tag{7.4}$$

Expressing v and w as functions of t from the system of equations $a = c = 0$, one obtains two possible solutions: $v = t$, $w = -1$ and $v = -1$, $w = t$. In both cases one of the variables (v, w) is negative which is a contradiction.

Now consider the case $j = 4$. The matrices M and N equal respectively

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ d & c & b & a & 1 \\ 0 & 0 & 0 & d & c \end{pmatrix}\,, \quad N = \begin{pmatrix} b & a & 1 \\ 0 & d & c \end{pmatrix}\,.$$

One has $\operatorname{rank} N < 2$ only for $b = 0$, $d = ac$ (because $d \neq 0$).

In case 1) these conditions lead to the contradiction $1 = t = (3 \pm \sqrt{5})/2$, see (7.2).

In case 2) one expresses the variable t from the condition $b = 0$: $t = t^{\bullet} := (1 + 2v)/(2 + v)$. Set $a^{\bullet} := a|_{t=t^{\bullet}}$, $c^{\bullet} := c|_{t=t^{\bullet}}$ and $d^{\bullet} := d|_{t=t^{\bullet}}$. The quantity $d^{\bullet} - a^{\bullet}c^{\bullet}$ equals $3(v^2 + v + 1)^2/(2 + v)^2$ which vanishes for no $v \geq 0$. So case 2) is also impossible.

In case 3) the condition $b = 0$ implies $t = t^{\triangle} := (vw + v + w)/(1 + v + w)$. Set $a^{\triangle} := a|_{t=t^{\triangle}}$, $c^{\triangle} := c|_{t=t^{\triangle}}$ and $d^{\triangle} := d|_{t=t^{\triangle}}$. The quantity $d^{\triangle} - a^{\triangle}c^{\triangle}$ equals $(w^2 + w + 1)(v^2 + v + 1)(v^2 + vw + w^2)/(1 + v + w)^2$ which is positive for any $v \geq 0$, $w \geq 0$. Hence case 3) is impossible. The lemma is proved. □

Proof of Lemma 16. We use the same ideas and notation as in the proof of Lemma 15. Six of the six or more real roots of P are denoted by (u, v, w, t, h, q). The space $\mathcal{L}$ is defined by analogy with the one of the proof of Lemma 15. The Jacobian matrix J is of the form

$$J := (\partial(a_8, a_7, a_j, a_1)/\partial(u, v, w, t, h, q))^{\top}.$$

Set $P_{u,v,w,t,h,q} := x^3 + ax^2 + bx + c$ and consider the vector-column

$$(0, 0, 0, 0, 0, 1, a, b, c)^{\top}.$$

Its successive shifts by one position upward correspond to the polynomials $x^s P_{u,v,w,t,h,q}$, $s \leq 5$. In the case $j = 5$ the matrices M and N look like this:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ c & b & a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b \end{pmatrix}, \quad N = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & 0 & c & b \end{pmatrix}.$$

One has $\operatorname{rank} M = 2 + \operatorname{rank} N$ and $\operatorname{rank} N = 2$, because at least one of the two coefficients b and c is nonzero (Lemma 7). Hence $\operatorname{rank} M = 4$ and the lemma is proved by analogy with Lemmas 14 and 15. In the case $j = 4$ the matrices M and N look like this:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ 0 & c & b & a & 1 & 0 \\ 0 & 0 & 0 & 0 & c & b \end{pmatrix}, \quad N = \begin{pmatrix} b & a & 1 & 0 \\ 0 & 0 & c & b \end{pmatrix}.$$

The matrix N is of rank 4, because either $b \neq 0$ or $b = 0$ and both a and c are nonzero (Lemma 7). Hence rank $M = 4$. □

8. PROOF OF PART (2) OF THEOREM 1

We remind that we consider polynomials with positive leading coefficients. For $d = 9$, we denote by σ a sign pattern and by σ^* the shortened sign pattern (obtained from σ by deleting its last component).

Lemma 19. *For $d = 9$, if $pos \geq 2$ and $neg \geq 2$, then such a couple (sign pattern, admissible pair) is realizable.*

Proof. Suppose that the last two components of σ are equal (resp. different). Then the pair $(pos, neg-1)$ (resp. $(pos-1, neg)$) is admissible for the sign pattern σ^* and the couple $(\sigma^*,\ (pos\ ,\ neg-1))$ (resp. $(\sigma^*,\ (pos-1\ ,\ neg))$) is realizable by some degree 8 polynomial P, see Remark 4. Hence the couple $(\sigma,\ (pos\ ,\ neg))$ is realizable by the concatenation of the polynomials P and $x+1$ (resp. P and $x-1$). □

Lemma 19 implies that in any nonrealizable couple with $pos > 0$ and $neg > 0$, one of the numbers pos, neg equals 1. Using the the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (i.e changing if necessary $P(x)$ to $-P(-x)$) one can assume that $pos = 1$. This implies that the last component of the sign pattern is $-$.

Lemma 20. *For $d = 9$, if $pos = 1$, $neg \geq 2$ and the last two components of σ are $(-\ ,\ -)$, then such a couple $(\sigma,\ (pos\ ,\ neg))$ is realizable.*

Proof. The couple $(\sigma^*,\ (pos\ ,\ neg-1))$ is realizable by some polynomial P, see Remark 4. Hence the concatenation of P and $x+1$ realizes the couple $(\sigma,\ (pos\ ,\ neg))$. □

Hence for any nonrealizable couple $(\sigma,\ (pos\ ,\ neg))$, one has $pos = 1$, $neg \geq 2$ and the last two components of σ are $(+\ ,\ -)$. Thus, the couple $(\sigma^*,\ (0\ ,\ neg))$ is nonrealizable. The first and the last components of σ^* are $+$. There are 19 such couples modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, see [11]:

Case	Sign pattern	Admissible pair(s)
A	$(++------++)$	$(0,6)$
B	$(+-------++)$	$(0,6)$
C	$(++++-----+)$	$(0,6)$
D	$(+++------+)$	$(0,6)$
E	$(+-+---+-+)$	$(0,2)$
F	$(+-+-+---+)$	$(0,2)$
$G1-G2$	$(+-+------+)$	$(0,2)\ ,\ (0,4)$
$H1-H2$	$(+---+---+)$	$(0,2)\ ,\ (0,4)$
$I1-I3$	$(+--------+)$	$(0,2)\ ,\ (0,4)\ ,\ (0,6)$

$$J \qquad (+++----++) \qquad (0,6)$$

$$K \qquad (+----+--+) \qquad (0,4)$$

$$L \qquad (+-------++) \qquad (0,4)$$

$$M \qquad (+-++-----+) \qquad (0,4)$$

$$N \qquad (+-+------++) \qquad (0,4)$$

$$Q \qquad (+----+-++) \qquad (0,4)$$

To obtain all couples $(\sigma^*, (0,\ neg))$ giving rise to nonrealizable couples $(\sigma, (1,\ neg))$ by concatenation with $x-1$, one has to add to the above list of cases $(A-Q)$ the cases obtained from them by acting with the first generator of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, i.e. the one replacing σ by σ^r, see Definition 2. The second generator (the one replacing σ by σ^m) has to be ignored, because it exchanges the two components of the admissible pair and the condition $pos = 1$ could not be maintained. The cases that are to be added are denoted by $(A^r - Q^r)$. E.g.

$$N^r \qquad (++-----+-+) \qquad (0,4)\,.$$

One can observe that, due to the center-symmetry of certain sign patterns, one has $A = A^r$, $E = E^r$, $Hj = Hj^r$, $j = 1,\ 2$ and $Ij = Ij^r$, $j = 1, 2, 3$.

With the only exception of case C^r, we show that all cases $(A-Q)$ and $(A^r - Q^r)$, are realizable which proves part (2) of the theorem. We do this by means of Lemma 2. We explain this first for the following cases:

$$B^r, \quad C, \quad D, \quad E, \quad F, \quad F^r, \quad G1, \quad G1^r, \quad G2, \quad G2^r, \quad H1, \quad H2,$$

$$I1, \quad I2, \quad I3, \quad K, \quad K^r, \quad L^r, \quad M, \quad M^r, \quad N^r \quad \text{and} \quad Q^r\,.$$

In all of them the last three components of σ are $(-+-)$, and we set $P_2^\dagger := x^2 - x + 1$ (see part (2) of Lemma 2). The polynomial $P_2^\dagger$ has no real roots and defines the sign pattern $\sigma^\dagger := (+-+)$. Denote by $\tilde{\sigma}$ the sign pattern obtained from σ by deleting its two last components. Hence $(1, neg)$ is an admissible pair for the sign pattern $\tilde{\sigma}$, and the couple $(\tilde{\sigma}, (1, neg))$ is realizable by some degree 7 monic polynomial $\tilde{P}_1$, see Remark 4. By Lemma 2 the concatenation of $\tilde{P}_1$ and $P_2^\dagger$ realizes the couple $(\sigma, (1, neg))$.

In cases A, B, J, L, N and Q, the last four components of the sign pattern σ are $(-++-)$. We set $P_2^\triangle := (x+2)((x^2-2)+1) = x^3 - 2x^2 - 3x + 10$. Hence $P_2^\triangle$ realizes the couple $((+--+), (0,1))$. Denote by $\sigma^\triangle$ the sign pattern obtained from σ by deleting its three last components. Hence $(1, neg-1)$ is an admissible pair for the sign pattern $\sigma^\triangle$, and the couple $(\sigma^\triangle, (1, neg-1))$ is realizable by some degree 6 monic polynomial $P_1^{\triangle}$, see Remark 4. By Lemma 2 the concatenation of $P_1^{\triangle}$ and $P_2^{\triangle}$ realizes the couple $(\sigma, (1, neg))$.

In the two remaining cases D^r and J^r, the last six components of σ are $(--++-)$. The sign pattern $\sigma^{\ddagger} := (++---+)$ is realizable by some degree 5 polynomial $P_2^{\ddagger}$, see [1]. Denote by $\sigma^{\diamond}$ the sign pattern obtained from σ by deleting its five last components. Hence in cases D^r and J^r one has $\sigma^{\diamond} = (+----)$ and $\sigma^{\diamond} = (++---)$ respectively. Thus the couple $(\sigma^{\diamond}, (1,3))$ is realizable by some monic degree 4 polynomial $P_1^{\diamond}$ (see Remark 4), and the concatenation of $P_1^{\diamond}$ and $P_2^{\ddagger}$ realizes the couple $(\sigma, (1, neg))$. Part (2) of Theorem 1 is proved.

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