

ЗА УСТОЙЧИВОСТТА НА ЕДНА СИСТЕМА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ СЪС ЗАКЪСНЯВАЩ АРГУМЕНТ

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Нека е дадена системата диференциални уравнения със закъсняващ аргумент:

$$(1) \quad \begin{aligned} \ddot{x}(t) + Q(x) &= \varepsilon q(t, x, \dot{x}, \ddot{x}, z, \dot{z}, \ddot{z}, \ddot{\dot{z}}, \varepsilon), \\ \ddot{z}(t) &= \varepsilon F(t, x, \dot{x}, \ddot{x}, z, \dot{z}, \ddot{z}, \ddot{\dot{z}}, \varepsilon), \end{aligned}$$

където $z = \{z_1, z_2, \dots, z_N\}$, $F = \{F_1, F_2, \dots, F_N\}$ са N -мерни вектори, x — едномерна координата, $x = x(t-h)$, $z = z(t-h)$, $t \in (0, \infty)$, $h = \text{const} > 0$ — закъснение, ε — малък параметър.

В настоящата работа се изследва въпросът за асимптотическата устойчивост на решенията на тази система при достатъчно малки стойности на ε . За системи от вида (1) без закъснения този въпрос е разгледан в (1).

Ще направим следните предположения:

1) Пораждащата система ($\varepsilon = 0$) има решения [1, 2] от вида:

$$(2) \quad \begin{aligned} x^{(0)} &= \varphi[\omega(t - t_0 - \tau^*), \omega], \\ z_i^{(0)} &= \tau_i^*, \end{aligned}$$

където φ е периодична функция относно първия си аргумент с период T_0 , $\omega = \frac{2\pi}{T_0}$ — собствена честота, а τ^* и τ_i^* са константи.

2) Функциите q и F_i са непрекъснати и периодични по t , а по останалите аргументи допускат частни производни, които удовлетворяват условието на Липшиц с независещи от t константи в някоя околност на пораждащото решение.

3) Функцията $Q(x)$ допуска втора производна, която удовлетворява условието на Липшиц в някоя околност на пораждащото решение.

4) Системата от $N+1$ нелинейни уравнения относно $\tau, \tau_1, \tau_2, \dots, \tau_N$

$$(3) \quad \begin{aligned} P(\tau, \tau_1, \dots, \tau_N) &= \int_0^T q(t, x_0, \dot{x}_0, \ddot{x}_0, z_0, \dot{z}_0, \ddot{z}_0, \ddot{\dot{z}}_0, 0) \dot{x}_0 dt = 0, \\ P_i(\tau, \tau_1, \dots, \tau_N) &= \int_0^T F_i(t, x_0, \dot{x}_0, \ddot{x}_0, z_0, \dot{z}_0, \ddot{z}_0, \ddot{\dot{z}}_0, 0) dt = 0 \end{aligned}$$

допуска реално решение $\tau^*, \tau_1^*, \tau_2^*, \dots, \tau_N^*$, т. е.

$$\frac{\partial (P, P_i)}{\partial (\tau^*, \tau_i^*)} \neq 0.$$

5) При достатъчно малки стойности на ϵ основната начална задача [3] за системата (1) има решения от вида

$$(4) \quad \begin{aligned} x(t, \epsilon) &= x^{(0)}[\omega(t - t_0 + \tau^*), \omega] + \epsilon y(t, \epsilon), \\ z_i(t, \epsilon) &= \tau_i^* + \epsilon y_i(t, \epsilon), \end{aligned}$$

където функциите y, y_i са периодични с период T ($T = n T_0$, n — цяло положително число). За да изследваме устойчивостта на това решение, да положим в (1)

$$x = x(t, \epsilon) + s,$$

$$z_i = z_i(t, \epsilon) + r_i.$$

За вариациите s и r_i получаваме системата

$$(5) \quad \begin{aligned} \ddot{s} + \frac{dQ}{dx} s &= \epsilon \left[\frac{\partial q}{\partial x} s + \frac{\partial q}{\partial x} \dot{s} + \frac{\partial q}{\partial x} \ddot{s} + \frac{\partial q}{\partial x} \dot{s} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} r_j \right. \\ &\quad \left. + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \dot{r}_j + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \ddot{r}_j + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \dot{r}_j \right], \\ \ddot{r}_i &= \epsilon \left[\frac{\partial F_i}{\partial x} s + \frac{\partial F_i}{\partial x} \dot{s} + \frac{\partial F_i}{\partial x} \ddot{s} + \frac{\partial F_i}{\partial x} \dot{s} + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} r_j \right. \\ &\quad \left. + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} \dot{r}_j + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} \ddot{r}_j + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} \dot{r}_j \right], \end{aligned}$$

където производните на функциите Q, q и F_i се пресмятат за несмутеното движение (4).

Следвайки метода на Ляпунов [2, 4], решенията на тази система търсим във вида

$$s = \xi e^{\alpha t}, \quad r_i = \eta_i e^{\alpha t},$$

където α е характеристичен показател на системата (1), а ξ, η_i са периодични функции. Относно ξ и η_i получаваме

$$\begin{aligned}
 \ddot{\xi} + \frac{dQ}{dx} \dot{\xi} &= \varepsilon \left\{ \frac{\partial q}{\partial x} \dot{\xi} + \frac{\partial q}{\partial x} (\dot{\xi} + \alpha \xi) + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \eta_j + \sum_{j=1}^N \frac{\partial q}{\partial z_j} (\dot{\eta}_j + \alpha \eta_j) \right. \\
 &\quad \left. + e^{-\alpha h} \left[\frac{\partial q}{\partial x} \dot{\xi} + \frac{\partial q}{\partial x} (\dot{\xi} + \alpha \xi) + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \eta_j + \sum_{j=1}^N \frac{\partial q}{\partial z_j} (\dot{\eta}_j + \alpha \eta_j) \right] \right\} - 2\alpha \dot{\xi} - \alpha^2 \xi, \\
 (6) \quad \ddot{\eta}_i &= \varepsilon \left\{ \frac{\partial F_i}{\partial x} \dot{\xi} + \frac{\partial F_i}{\partial x} (\dot{\xi} + \alpha \xi) + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} \eta_j + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} (\dot{\eta}_j + \alpha \eta_j) \right. \\
 &\quad \left. + e^{-\alpha h} \left[\frac{\partial F_i}{\partial x} \dot{\xi} + \frac{\partial F_i}{\partial x} (\dot{\xi} + \alpha \xi) + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} \eta_j + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} (\dot{\eta}_j + \alpha \eta_j) \right] \right\} - 2\alpha \dot{\eta}_i - \alpha^2 \eta_i.
 \end{aligned}$$

Тъй като $\frac{\partial(P, P_i)}{\partial(\tau^*, \tau_j^*)} \neq 0$, то [1, 2]

$$\begin{aligned}
 \alpha &= \alpha_1 \sqrt{\varepsilon} + \alpha_2 \varepsilon + \alpha_3 \varepsilon^{3/2} + \dots, \\
 (7) \quad \xi &= \xi^{(0)} + \xi^{(1)} \sqrt{\varepsilon} + \xi^{(2)} \varepsilon + \xi^{(3)} \varepsilon^{3/2} + \dots, \\
 \eta_i &= \eta_i^{(0)} + \eta_i^{(1)} \sqrt{\varepsilon} + \eta_i^{(2)} \varepsilon + \eta_i^{(3)} \varepsilon^{3/2} + \dots
 \end{aligned}$$

Като заместим α , ξ и η_i от (7) в (6), за нулевото и първото приближение на ξ и η_i получаваме съответно

$$\begin{aligned}
 (8) \quad \ddot{\xi}^{(0)} + \left(\frac{dQ}{dx} \right)_0 \dot{\xi}^{(0)} &= 0, \\
 \ddot{\eta}_i^{(0)} &= 0
 \end{aligned}$$

и

$$\begin{aligned}
 (9) \quad \ddot{\xi}^{(1)} + \left(\frac{dQ}{dx} \right)_0 \dot{\xi}^{(1)} &= -2\alpha_1 \dot{\xi}^{(0)}, \\
 \ddot{\eta}_i^{(1)} &= -2\alpha_1 \dot{\eta}_i^{(0)}.
 \end{aligned}$$

Уравненията (8) и (9) имат периодични решения

$$\xi^{(0)} = a_0 \dot{x}_0 = a_0 u,$$

$$\eta_i^{(0)} = a_i^{(0)}$$

и

$$\xi^{(1)} = a_1 u + \alpha_1 a_0 v,$$

$$\eta_i^{(1)} = a_i^{(1)},$$

където u, v са периодични решения съответно на уравненията

$$(9_1) \quad \ddot{u} + Q'(x_0) u = 0,$$

$$\ddot{v} + Q'(x_0) v = -2 \alpha_1 \dot{u},$$

а $a_0, a_1, a_i^{(0)}, a_i^{(1)}$ са константи. От (6) и (7) следва, че за функциите $\xi^{(2)}, \eta_i^{(2)}$ имаме

$$\begin{aligned} \ddot{\xi}^{(2)} + \left(\frac{dQ}{dx} \right)_0 \xi^{(2)} &= -x_1 \left(\frac{d^2Q}{dx^2} \right)_0 \xi^{(0)} + \left(\frac{\partial q}{\partial x} \right)_0 \xi^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)}, \\ &+ \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \eta_j^{(0)} + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \eta_j^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 \bar{\xi}^{(0)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)} \\ &+ \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \dot{\eta}_j^{(0)} + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \dot{\eta}_j^{(1)} - 2 \alpha_1 \dot{\xi}^{(1)} - 2 \alpha_2 \xi^{(0)} - \alpha_1^2 \xi^{(0)}, \\ (10) \quad \ddot{\eta}_i &= \left(\frac{\partial F_i}{\partial x} \right)_0 \xi^{(0)} + \left(\frac{\partial F_i}{\partial x} \right)_0 \dot{\xi}^{(0)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 \eta_j^{(0)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 \dot{\eta}_j^{(0)} \\ &+ \left(\frac{\partial F_i}{\partial x} \right)_0 \bar{\xi}^{(0)} + \left(\frac{\partial F_i}{\partial x} \right)_0 \dot{\bar{\xi}}^{(0)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 \bar{\eta}_j^{(0)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 \dot{\bar{\eta}}_j^{(0)} \\ &- 2 \alpha_1 \dot{\eta}_i^{(1)} - 2 \alpha_2 \dot{\eta}_i^{(0)} - \alpha_1^2 \eta_i^{(0)}, \end{aligned}$$

където знакът 0 при скобите означава, че в производните на функциите Q, q и F_i аргументите са взети за пораждащото решение. Уравненията (10) имат периодични решения с период T , когато са изпълнени условията

$$\begin{aligned} \int_0^T \left\{ - \left(\frac{d^2Q}{dx^2} \right)_0 x_1 a_0 u^2 + \left(\frac{\partial q}{\partial x} \right)_0 a_0 u^2 + \left(\frac{\partial q}{\partial x} \right)_0 a_0 u \dot{u} + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 a_j^{(0)} u \right. \\ \left. + \left(\frac{\partial q}{\partial x} \right)_0 a_0 u \bar{u} + \left(\frac{\partial q}{\partial x} \right)_0 a_0 \bar{u} \dot{u} + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 a_j^{(0)} u - 2 \alpha_1^2 a_0 u \dot{v} - \alpha_1^2 a_0 u^2 \right\} dt = 0, \\ (11) \quad \int_0^T \left\{ \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 u + \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 \dot{u} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 a_j^{(0)} + \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 \bar{u} + \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 \dot{\bar{u}} \right. \\ \left. + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 a_j^{(0)} - \alpha_1^2 a_i^{(0)} \right\} dt = 0. \end{aligned}$$

От (3) получаваме

$$\frac{\partial P}{\partial \tau^*} = \int_0^T \left\{ \left(\frac{\partial q}{\partial x_0} \right) u + \left(\frac{\partial q}{\partial x_0} \right) u \bar{u} + \left(\frac{\partial q}{\partial x_0} \right) u \dot{u} + \left(\frac{\partial q}{\partial x_0} \right) u \bar{u} + q \dot{u} \right\} dt,$$

$$\frac{\partial P}{\partial \tau_j^*} = \int_0^T \left\{ \left(\frac{\partial q}{\partial z_j} \right)_0 u + \left(\frac{\partial q}{\partial z_j} \right)_0 u \bar{u} \right\} dt,$$

(12)

$$\frac{\partial P_i}{\partial \tau_*^*} = \int_0^T \left\{ \left(\frac{\partial F_i}{\partial x_0} \right) u + \left(\frac{\partial F_i}{\partial x_0} \right) \bar{u} + \left(\frac{\partial F_i}{\partial x_0} \right) \dot{u} + \left(\frac{\partial F_i}{\partial x_0} \right) \bar{u} \right\} dt,$$

$$\frac{\partial P_i}{\partial \tau_j^*} = \int_0^T \left\{ \left(\frac{\partial F_i}{\partial z_j} \right)_0 + \left(\frac{\partial F_i}{\partial z_j} \right)_0 \right\} dt.$$

Като вземем пред вид, че

$$-\int_0^T \left(\frac{d^2 Q}{dx^2} \right)_0 x_1 u^2 dt = \int_0^T q \dot{u} dt$$

(вж. Малкин [2], 408), от (11) и (12) следва

$$a_0 \left(\frac{\partial P}{\partial \tau^*} - \alpha_1^2 K T \right) + \sum_{j=1}^N a_j^{(0)} \frac{\partial P}{\partial \tau_j^*} = 0,$$

(13)

$$a_0 \frac{\partial P_i}{\partial \tau^*} + \sum_{j=1}^N a_j^{(0)} \left(\frac{\partial P_i}{\partial \tau_j^*} - \delta_{ij} \alpha_1^2 T \right) = 0,$$

където $K = u^2 + u \dot{v} - \dot{u} v$ е детерминантата на Вронски за решенията $u_1 = u$, $u_2 = u t + v$ на първото уравнение на (9₁).

Системата (13) има ненулево решение относно величините a_0 , $a_j^{(0)}$, когато

$$\det \begin{vmatrix} \frac{\partial P}{\partial \tau^*} - \alpha_1^2 K T & \frac{\partial P}{\partial \tau_j^*} \\ \frac{\partial P_i}{\partial \tau^*} & \frac{\partial P_i}{\partial \tau_j^*} - \delta_{ij} \alpha_1^2 T \end{vmatrix} = 0.$$

Корените на това уравнение относно $p = \alpha_1^2 T$ са различни от нула. Необходимо условие за устойчивост на решенията на системата е всички p_v ($v = 0, 1, 2, \dots, N$) да бъдат строго отрицателни. Тогава всички $\alpha_1^{(v)} = \sqrt[p]{p_v}$ ще бъдат чисто имагинерни и изследването на устойчивостта ще се определя от α_2 .

От уравненията (6) и (7) получаваме, че функциите $\xi^{(3)}$ и $\eta_i^{(3)}$ удовлетворяват уравненията

$$\begin{aligned}
 & \dot{\xi}^{(3)} + \left(\frac{d}{dx} Q \right)_0 \xi^{(3)} = - \left(\frac{d^2 Q}{dx^2} \right)_0 x_1 \xi^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 \xi^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 (\dot{\xi}^{(1)} + \alpha_1 \xi^{(0)}) \\
 & + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \eta_j^{(1)} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} (\dot{\eta}_j^{(1)} + \alpha_1 \eta_j^{(0)}) + \left(\frac{\partial q}{\partial x} \right)_0 \bar{\xi}^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 (\dot{\bar{\xi}}^{(1)} + \alpha_1 \bar{\xi}^{(0)}) \\
 & + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \bar{\eta}_j^{(1)} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} (\dot{\bar{\eta}}_j^{(1)} + \alpha_1 \bar{\eta}_j^{(0)}) - \alpha_1 h \left[\left(\frac{\partial q}{\partial x} \right)_0 \bar{\xi}^{(0)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)} \right. \\
 & \left. + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \bar{\eta}_j^{(0)} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \dot{\eta}_j^{(0)} \right] - 2 \alpha_3 \dot{\xi}^{(0)} - 2 \alpha_2 \dot{\bar{\xi}}^{(1)} - 2 \alpha_1 \dot{\bar{\xi}}^{(2)} - \alpha_1^2 \bar{\xi}^{(1)} - 2 \alpha_1 \alpha_2 \bar{\xi}^{(0)}, \\
 (14) \quad & \eta_i = \left(\frac{\partial F_i}{\partial x} \right)_0 \xi^{(1)} + \left(\frac{\partial F_i}{\partial x} \right)_0 (\dot{\xi}^{(1)} + \alpha_1 \xi^{(0)}) + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 \eta_j^{(1)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 (\dot{\eta}_j^{(1)} + \alpha_1 \eta_j^{(0)}) \\
 & + \left(\frac{\partial F_i}{\partial x} \right)_0 \bar{\xi}^{(1)} + \left(\frac{\partial F_i}{\partial x} \right)_0 (\dot{\bar{\xi}}^{(1)} + \alpha_1 \bar{\xi}^{(0)}) + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} \bar{\eta}_j^{(1)} + \sum_{j=1}^N \frac{\partial F_i}{\partial z_j} (\dot{\bar{\eta}}_j^{(1)} + \alpha_1 \bar{\eta}_j^{(0)}) \\
 & - \alpha_1 h \left[\left(\frac{\partial F_i}{\partial x} \right)_0 \bar{\xi}^{(0)} + \left(\frac{\partial F_i}{\partial x} \right)_0 \dot{\xi}^{(0)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 \bar{\eta}_j^{(0)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 \dot{\eta}_j^{(0)} \right] \\
 & - 2 \alpha_3 \dot{\eta}_i^{(0)} - 2 \alpha_2 \eta_i^{(1)} - 2 \alpha_1 \eta_i^{(2)} - \alpha_1^2 \eta_i^{(1)} - 2 \alpha_1 \alpha_2 \eta_i^{(0)}.
 \end{aligned}$$

Условието за периодичност на $\xi^{(3)}$ се записва така:

$$\begin{aligned}
 (15) \quad & \int_0^T \left\{ \frac{d^2 Q}{dx^2} x_1 \xi^{(1)} \xi^{(0)} - \left(\frac{\partial q}{\partial x} \right)_0 \xi^{(1)} \xi^{(0)} - \left(\frac{\partial q}{\partial x} \right)_0 (\xi^{(0)} \dot{\xi}^{(1)} + \alpha_1 \xi^{(0)2}) \right. \\
 & - \sum_{j=1}^N \frac{\partial q}{\partial z_j} \eta_j^{(1)} \xi^{(0)} - \sum_{j=1}^N \frac{\partial q}{\partial z_j} (\dot{\eta}_j^{(1)} \xi^{(0)} + \alpha_1 \eta_j^{(0)} \xi^{(0)}) - \left(\frac{\partial q}{\partial x} \right)_0 \bar{\xi}^{(1)} \xi^{(0)} \\
 & - \left. \left(\frac{\partial q}{\partial x} \right)_0 (\dot{\bar{\xi}}^{(1)} \xi^{(0)} + \alpha_1 \bar{\xi}^{(0)} \xi^{(0)}) - \sum_{j=1}^N \frac{\partial q}{\partial z_j} \bar{\eta}_j^{(1)} \xi^{(0)} - \sum_{j=1}^N \frac{\partial q}{\partial z_j} (\dot{\bar{\eta}}_j^{(1)} \xi^{(0)} \right. \\
 & \left. + \alpha_1 \bar{\eta}_j^{(0)} \xi^{(0)}) + \alpha_1 h \left[\left(\frac{\partial q}{\partial x} \right)_0 \bar{\xi}^{(0)} \xi^{(0)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)} \xi^{(0)} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \bar{\eta}_j^{(0)} \xi^{(0)} \right. \right. \\
 & \left. \left. + \alpha_1 \bar{\eta}_j^{(0)} \xi^{(0)} \right] \right\} dt = 0
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \dot{\eta}_j^{(0)} \dot{\xi}^{(0)} \Big] + 2 \alpha_1 \dot{\xi}^{(2)} \dot{\xi}^{(0)} + 2 \alpha_2 \dot{\xi}^{(1)} \dot{\xi}^{(0)} + 2 \alpha_3 \dot{\xi}^{(0)} \dot{\xi}^{(0)} + \alpha_1^2 \dot{\xi}^{(1)} \dot{\xi}^{(0)} \\
 & + 2 \alpha_1 \alpha_2 \dot{\xi}^{(0)2} \Big\} dt = 0.
 \end{aligned}$$

За да елиминираме $\dot{\xi}^{(2)}$, умножаваме първите уравнения на (9) и (10) съответно с $\dot{\xi}^{(2)}$ и $\dot{\xi}^{(1)}$ и ги изваждаме. Получаваме

$$\begin{aligned}
 & \int_0^T \left\{ - \left(\frac{d^2 Q}{dx^2} \right)_0 x_1 \dot{\xi}^{(0)} \dot{\xi}^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)} \dot{\xi}^{(1)} + \left(\frac{\partial q}{\partial x} \right) \dot{\xi}^{(0)} \dot{\xi}^{(1)} + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \dot{\eta}_j^{(0)} \dot{\xi}^{(1)} \right. \\
 & + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \dot{\eta}_j^{(0)} \dot{\xi}^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)} \dot{\xi}^{(1)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)} \dot{\xi}^{(1)} + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \dot{\eta}_j^{(0)} \dot{\xi}^{(1)} \\
 & \left. + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \dot{\eta}_j^{(0)} \dot{\xi}^{(1)} - 2 \alpha_1 \dot{\xi}^{(1)} \dot{\xi}^{(1)} - 2 \alpha_2 \dot{\xi}^{(0)} \dot{\xi}^{(1)} - \alpha_1^2 \dot{\xi}^{(0)} \dot{\xi}^{(1)} + 2 \alpha_1 \dot{\xi}^{(0)} \dot{\xi}^{(2)} \right\} dt = 0.
 \end{aligned} \tag{16}$$

По-нататък събираме (15) и (16) и групирате

$$\begin{aligned}
 & \int_0^T \left\{ \left[\left(\frac{\partial q}{\partial x} \right)_0 - 2 \alpha_2 \right] [\alpha_1 \dot{\xi}^{(0)2} + \dot{\xi}^{(1)} \dot{\xi}^{(0)} - \dot{\xi}^{(0)} \dot{\xi}^{(1)}] + \sum_{j=1}^N \frac{\partial q}{\partial z_j} (\eta_j^{(1)} \dot{\xi}^{(0)} - \eta_j^{(0)} \dot{\xi}^{(1)}) \right. \\
 & + \alpha_1 \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \eta_j^{(0)} \dot{\xi}^{(0)} + \left(\frac{\partial q}{\partial x} \right)_0 (\dot{\xi}^{(1)} \dot{\xi}^{(0)} - \dot{\xi}^{(0)} \dot{\xi}^{(1)}) \\
 & + \left(\frac{\partial q}{\partial x} \right)_0 [\dot{\xi}^{(1)} \dot{\xi}^{(0)} + \alpha_1 \dot{\xi}^{(0)} \dot{\xi}^{(0)} - \dot{\xi}^{(1)} \dot{\xi}^{(0)}] + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 (\bar{\eta}_j^{(1)} \dot{\xi}^{(0)} - \bar{\eta}_j^{(0)} \dot{\xi}^{(1)}) \\
 & + \alpha_1 \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 \bar{\eta}_j^{(0)} \dot{\xi}^{(0)} - \alpha_1 h \left[\left(\frac{\partial q}{\partial x} \right)_0 \bar{\xi}^{(0)} \dot{\xi}^{(0)} + \left(\frac{\partial q}{\partial x} \right)_0 \dot{\xi}^{(0)} \bar{\xi}^{(0)} \right. \\
 & \left. + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \bar{\eta}_j^{(0)} \dot{\xi}^{(0)} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} \bar{\eta}_j^{(0)} \dot{\xi}^{(0)} \right] \Big\} = 0
 \end{aligned} \tag{17}$$

или

$$\begin{aligned}
 & \int_0^T \left\{ \left(\left(\frac{\partial q}{\partial x} \right)_0 - 2\alpha_2 \right) a_1 a_0^2 K + \sum_{j=1}^N \frac{\partial q}{\partial z_j} [a_j^{(1)} a_0 u - a_j^{(0)} (a_1 u + \alpha_1 a_0 v)] \right. \\
 & + \alpha_1 a_0 \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 a_j^{(0)} u + \frac{\partial q}{\partial x} [a_0 u (a_1 \dot{u} + \alpha_1 a_0 \dot{v}) - a_0 \dot{u} (a_1 u + \alpha_1 a_0 v)] \\
 (18) \quad & + \frac{\partial q}{\partial x} [a_0 u (a_1 \dot{u} + \alpha_1 a_0 \dot{v}) + \alpha_1 a_0^2 u \dot{u} - a_0 \dot{u} (a_1 u + \alpha_1 a_0 v)] \\
 & + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 [a_0 u a_j^{(1)} - a_j^{(0)} (a_1 u + \alpha_1 a_0 v)] + \alpha_1 \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 a_0 u a_j^{(0)} \\
 & - \alpha_1 h \left[\left(\frac{\partial q}{\partial x} \right)_0 a_0^2 u \dot{u} + \left(\frac{\partial q}{\partial x} \right)_0 a_0^2 u \dot{u} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} a_0 u a_j^{(0)} \right] \\
 & \left. + \left(\frac{\partial q}{\partial x} \right)_0 a_0 u (a_1 u + \alpha_1 a_0 v) - \left(\frac{\partial q}{\partial x} \right)_0 a_0 u (a_1 u + \alpha_1 a_0 v) \right\} dt = 0.
 \end{aligned}$$

Като съкратим на a_0 и въведем $\frac{\partial P}{\partial \tau^*}$ и $\frac{\partial P}{\partial \tau_j^*}$, получаваме

$$(19) \quad a_1 \frac{\partial P}{\partial \tau^*} + \sum_{j=1}^N \frac{\partial P}{\partial \tau_j^*} a_j^{(1)} - a_1 \left[\frac{\partial P}{\partial \tau} + \frac{1}{a_0} \sum_{j=1}^N \frac{\partial P}{\partial \tau_j^*} a_j^{(0)} \right] + \alpha_1 \gamma = 0,$$

където

$$\begin{aligned}
 \gamma = & \int_0^T \left\{ \left(\frac{\partial q}{\partial x} - 2\alpha_2 \right) a_0 K - \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 v a_j^{(0)} + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 u a_j^{(0)} \right. \\
 & + \frac{\partial q}{\partial x} a_0 (u \dot{v} - \dot{u} v) + \frac{\partial q}{\partial x} a_0 (u \dot{v} + u \dot{u} - \dot{u} v) - \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 v a_j^{(0)} \\
 & \left. + \sum_{j=1}^N \left(\frac{\partial q}{\partial z_j} \right)_0 u a_j^{(0)} - h \left[a_0 \left(\frac{\partial q}{\partial x} \right)_0 u \dot{u} + a_0 \left(\frac{\partial q}{\partial x} \right)_0 u \dot{u} + \sum_{j=1}^N \frac{\partial q}{\partial z_j} u a_j^{(0)} \right] \right\} dt.
 \end{aligned}$$

Съгласно (13) уравнение (19) може да се запише и така:

$$(20) \quad a_1 \left(\frac{\partial P}{\partial \tau^*} - \alpha_1^2 K T \right) + \sum_{j=1}^N \frac{\partial P}{\partial \tau_j^*} a_j^{(1)} + \alpha_1 \gamma = 0.$$

Сега ще запишем условието за периодичност на $\eta^{(3)}$:

$$\begin{aligned}
 & \int_0^T \left\{ \left(\frac{\partial F_i}{\partial x} \right)_0 (a_1 u + \alpha_1 a_0 v) + \left(\frac{\partial F_i}{\partial \dot{x}} \right)_0 [a_1 \dot{u} + \alpha_1 a_0 \dot{v} + \alpha_1 a_0 u] \right. \\
 & + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 a_j^{(1)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial \dot{z}_j} \right)_0 \alpha_1 a_j^{(0)} + \left(\frac{\partial F_i}{\partial x} \right)_0 (a_1 \bar{u} + \alpha_1 a_0 \bar{v}) \\
 (21) \quad & + \left. \left(\frac{\partial F_i}{\partial x} \right)_0 [a_1 \bar{u} + \alpha_1 a_0 \bar{v} + \alpha_1 a_0 \bar{u}] + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial \bar{z}_j} \right)_0 a_j^{(1)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial \dot{\bar{z}}_j} \right)_0 \alpha_1 a_j^{(0)} \right. \\
 & - \alpha_1 h \left[\left(\frac{\partial F_i}{\partial x} \right)_0 a_0 \bar{u} + \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 \dot{\bar{u}} + \sum_{j=1}^N \frac{\partial F_i}{\partial \bar{z}_j} a_j^{(0)} \right] - \alpha_1^2 a_i^{(1)} - 2 \alpha_1 \alpha_2 a_i^{(0)} \left. \right\} dt = 0.
 \end{aligned}$$

или

$$\begin{aligned}
 & a_1 \int_0^T \left\{ \left(\frac{\partial F_i}{\partial x} \right)_0 u + \left(\frac{\partial F_i}{\partial \dot{x}} \right)_0 \dot{u} + \left(\frac{\partial F_i}{\partial \bar{x}} \right)_0 \bar{u} + \left(\frac{\partial F_i}{\partial \dot{\bar{x}}} \right)_0 \dot{\bar{u}} \right\} dt \\
 & + \sum_{j=1}^N \left[\left(\frac{\partial F_i}{\partial z_j} \right)_0 + \left(\frac{\partial F_i}{\partial \dot{z}_j} \right)_0 \right] a_j^{(1)} - \sum_{j=1}^N \alpha_1^2 T \delta_{ij} a_j^{(1)} + \alpha_1 \int_0^T \left\{ \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 v \right. \\
 (22) \quad & + \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 (\dot{v} + u) + \left(\frac{\partial F_i}{\partial \bar{x}} \right)_0 a_0 \bar{v} + \left(\frac{\partial F_i}{\partial \dot{\bar{x}}} \right)_0 a_0 (\dot{\bar{v}} + \bar{u}) + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial \bar{z}_j} \right)_0 a_j^{(0)} \\
 & + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial \dot{\bar{z}}_j} \right)_0 a_j^{(0)} - h \left[a_0 \left(\frac{\partial F_i}{\partial x} \right)_0 \bar{u} + a_0 \left(\frac{\partial F_i}{\partial \dot{x}} \right)_0 \dot{\bar{u}} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial \bar{z}_j} \right)_0 a_j^{(0)} \right. \\
 & \left. \left. - 2 \alpha_2 a_i^{(0)} \right] \right\} dt = 0.
 \end{aligned}$$

Като въведем $\frac{\partial P_i}{\partial \tau^*}$ и $\frac{\partial P_i}{\partial \tau_j^*}$, получаваме

$$a_1 \frac{\partial P_i}{\partial \tau} + \sum_{j=1}^N \left(\frac{\partial P_i}{\partial \tau_j} - \alpha_1^2 \delta_{ij} T \right) a^{(1)} + \alpha_1 \gamma_i = 0. \quad (i=1, 2, \dots, N),$$

където

$$\gamma_i = \int_0^T \left\{ \left(\frac{\partial F_i}{\partial x} \right)_0 a_0 v + \left(\frac{\partial F_i}{\partial \dot{x}} \right)_0 a_0 (\dot{v} + u) + \left(\frac{\partial F_i}{\partial \bar{x}} \right)_0 a_0 \bar{v} + \left(\frac{\partial F_i}{\partial \dot{\bar{x}}} \right)_0 a_0 (\dot{\bar{v}} + u) \right\} dt.$$

$$\begin{aligned}
 & + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 a_j^{(0)} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial \bar{z}_j} \right)_0 a_j^{(0)} - h \left[a_0 \left(\frac{\partial F_i}{\partial \bar{x}} \right)_0 \bar{u} + a_0 \left(\frac{\partial F_i}{\partial x} \right)_0 \dot{u} \right. \\
 & \quad \left. + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial z_j} \right)_0 a_j^{(0)} \right] - 2 \alpha_2 a_i^{(0)} \} dt.
 \end{aligned}$$

По такъв начин за константите $a_1, a_j^{(1)}$ ($j = 1, 2, \dots, N$) получаваме следната система от $N+1$ уравнения:

$$\begin{aligned}
 a_1 \left(\frac{\partial P}{\partial \tau^*} - \alpha_1^2 K T \right) + \sum_{j=1}^N \frac{\partial P}{\partial \tau_j^*} a_j^{(1)} + \alpha_1 \gamma &= 0, \\
 a_1 \frac{\partial P_i}{\partial \tau^*} + \sum_{j=1}^N \left(\frac{\partial P_i}{\partial \tau_j^*} - \alpha_1^2 \delta_{ij} T \right) a_j^{(1)} + \alpha_1 \gamma_i &= 0 \quad (i = 1, 2, \dots, N).
 \end{aligned}$$

Тъй като основната детерминанта на тази линейна система относно $a_1, a_j^{(1)}$ ($j = 1, 2, \dots, N$) е равна на нула, то за да бъде тя разрешима, съгласно теоремата на Руше трябва

$$\begin{array}{c}
 \begin{array}{cc|c}
 \frac{\partial P}{\partial \tau^*} - \alpha_1^2 K T & \frac{\partial P}{\partial \tau_j^*} & \cdots \cdots \gamma \\
 \text{rang} & & \\
 \hline
 \frac{\partial P_i}{\partial \tau^*} & \frac{\partial P_i}{\partial \tau_j^*} - \alpha_1^2 \delta_{ij} T & \cdots \cdots \gamma_i \\
 & p = p_r^* & p = p_v^*
 \end{array}
 \end{array}
 = \text{rang} \quad .$$

На всяко p_r може да се получи съответно $\alpha_2^{(r)}$. Ако $\operatorname{Re}(\alpha_2^{(r)}) < 0$ ($r = 1, 2, \dots, N+1$), възмутеното решение на системата (5) е асимптотически устойчиво. В частност, ако при някоя стойност на v $\operatorname{Re}(\alpha_2^{(v)}) = 0$, съответните им характеристически показатели трябва да бъдат пресметнати с по-голяма точност.

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ON THE STABILITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH A RETARDATING ARGUMENT

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(SUMMARY)

In this paper the authors consider the system of differential equations with a retarded argument

$$(1) \quad \begin{aligned} \ddot{x}(t) + Q(x) &= \varepsilon q(t, x, \dot{x}, \bar{x}, z, \dot{z}, \bar{z}, \ddot{z}, \varepsilon), \\ \ddot{z}(t) &= \varepsilon F(t, x, \dot{x}, \bar{x}, z, \dot{z}, \bar{z}, \ddot{z}, \varepsilon) \end{aligned}$$

where

$z = \{z_1, z_2, \dots, z_N\}$, $F = \{F_1, F_2, \dots, F_N\}$ are N -dimensional vectors, x is a one-dimensional coordinate, $\bar{x} = x(t-h)$, $\bar{z} = z(t-h)$, $t \in (0, \infty)$, $h = \text{const} > 0$: is a constant, ε is a small parameter.

The following assumptions are made:

1. The function $Q(x)$ has a second derivative, which satisfies the Lipschitz' condition in a neighbourhood of the generating solution ($\varepsilon = 0$)

$$\begin{aligned} x^{(0)} &= \varphi[\omega(t - t_0 + \tau), \omega], \\ z_i^{(0)} &= \tau_i, \end{aligned}$$

where φ is a periodic function with respect to its first argument with a period T_0 , $\omega = 2\pi/T_0$ is the proper frequency, τ, τ_i are constants.

2. The functions q and F_i are continuous and periodical with respect to t , and relatively to the other arguments they have partial derivatives, which satisfy the Lipschitz' condition.

3. The system of $N+1$ non-linear equations in $\tau, \tau_1, \dots, \tau_N$

$$\begin{aligned} P(\tau, \tau_1, \tau_2, \dots, \tau_N) &= \int_0^T q(t, x_0, \bar{x}_0, \dot{x}_0, \bar{x}_0, z_0, \bar{z}_0, \dot{z}_0, \bar{z}_0, 0) \dot{x}_0 dt = 0 \\ P(\tau, \tau_1, \tau_2, \dots, \tau_N) &= \int_0^T F_i(t, x_0, \bar{x}_0, \dot{x}_0, \bar{x}_0, z_0, \bar{z}_0, \dot{z}_0, \bar{z}_0, 0) dt = 0 \end{aligned}$$

has a real solution $\tau^*, \tau_1^*, \dots, \tau_N^*$.

4. For sufficiently small values of ε the basic initial problem for the system (1) has solution, as follows:

$$(2) \quad \begin{aligned} x(t, \varepsilon) &= x^{(0)} [\omega(t - t_0 + \tau^*), \omega] + \varepsilon y(t, \varepsilon), \\ z_i(t, \varepsilon) &= \tau_i^* + \varepsilon y_i(t, \varepsilon), \end{aligned}$$

where the functions y and y_i are periodical with a period T ($T = n T_0$, n is a whole positive number).

In the paper conditions for the asymptotic stability of the solution (2), for sufficiently small values of ε , are found.