

# КЪМ КИНЕМАТИКАТА НА ТОЧКА В КРИВОЛИНЕЙНИ КООРДИНАТИ

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1. Нека радиус-векторът  $\bar{r}$  на подвижна точка е зададен като функция

$$(1) \quad \bar{r} = \bar{r}(q_1, q_2, q_3)$$

на криволинейните координати  $q_\nu$  ( $\nu = 1, 2, 3$ ), които от своя страна са функции

$$(2) \quad q_\nu = q_\nu(t)$$

на времето  $t$ . Единичните тангенциални вектори  $\bar{q}_\nu^0$  ( $\nu = 1, 2, 3$ ) на координатните линии се дефинират с

$$(3) \quad \bar{q}_\nu^0 = \frac{\frac{\partial \bar{r}}{\partial q_\nu}}{\left| \frac{\partial \bar{r}}{\partial q_\nu} \right|} \quad (\nu = 1, 2, 3).$$

При

$$(4) \quad \frac{\partial \bar{r}}{\partial q_1} \times \frac{\partial \bar{r}}{\partial q_2} \cdot \frac{\partial \bar{r}}{\partial q_3} \neq 0$$

реципрочният репер  $(\bar{q}_\nu^0)^{-1}$  ( $\nu = 1, 2, 3$ ) на репера (3) е

$$(5) \quad (\bar{q}_\nu^0)^{-1} = \left| \frac{\partial \bar{r}}{\partial q_\nu} \right| \frac{1}{\frac{\partial \bar{r}}{\partial q_1} \times \frac{\partial \bar{r}}{\partial q_2} \cdot \frac{\partial \bar{r}}{\partial q_3}} \frac{\partial \bar{r}}{\partial q_{\nu+1}} \times \frac{\partial \bar{r}}{\partial q_{\nu+2}} \quad (\nu = 1, 2, 3)$$

при

$$(6) \quad \frac{\partial \bar{r}}{\partial q_{\nu+3}} = \frac{\partial \bar{r}}{\partial q_\nu} \quad (\nu = 1, 2, 3).$$

Необходимо и достатъчно условие за

$$(7) \quad (\bar{q}_\nu^0)^{-1} = \bar{q}_\nu^0 \quad (\nu = 1, 2, 3)$$

е

$$(8) \quad \frac{\partial \bar{r}}{\partial q_\mu} \cdot \frac{\partial \bar{r}}{\partial q_\nu} = 0 \quad (\mu, \nu = 1, 2, 3; \mu \neq \nu),$$

т. е. системата (2) да бъде ортогонална.

От

$$(9) \quad \bar{r} = x(q_1, q_2, q_3) \bar{i} + y(q_1, q_2, q_3) \bar{j} + z(q_1, q_2, q_3) \bar{k}$$

следва

$$(10) \quad \frac{\partial \bar{r}}{\partial q_1} \times \frac{\partial \bar{r}}{\partial q_2} \cdot \frac{\partial \bar{r}}{\partial q_3} = \frac{D(x, y, z)}{D(q_1, q_2, q_3)}.$$

От (5), (10) следва

$$(11) \quad (\bar{q}_v^0)^{-1} = \left| \frac{\partial \bar{r}}{\partial q_v} \right| \frac{D(x, y, z)}{D(q_1, q_2, q_3)} \frac{\partial \bar{r}}{\partial q_{v+1}} \times \frac{\partial \bar{r}}{\partial q_{v+2}} \quad (v = 1, 2, 3)$$

при (6).

От

$$(12) \quad r = \sum_{v=1}^3 \left[ \bar{r} (\bar{q}_v^0)^{-1} \right] \bar{q}_v^0$$

и (11) следва

$$(13) \quad \bar{r} = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{v=1}^3 \left[ \frac{\partial \bar{r}}{\partial q_v} \right] \left( \bar{r} \cdot \frac{\partial \bar{r}}{\partial q_{v+1}} \times \frac{\partial \bar{r}}{\partial q_{v+2}} \right) \bar{q}_v^0.$$

При (8) от (12), (7), (3) следва, че равенството (13) приема вида

$$(14) \quad r = \sum_{v=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \left( \bar{r} \cdot \frac{\partial \bar{r}}{\partial q_{v+1}} \times \frac{\partial \bar{r}}{\partial q_{v+2}} \right) \bar{q}_v^0.$$

## 2. Цилиндричната координатна система $(\rho, \varphi, z)$ се дефинира с

$$(15) \quad q_1 = \rho, \quad q_2 = \varphi, \quad q_3 = z$$

при

$$(16) \quad \bar{r} = \rho \cos \varphi \bar{i} + \rho \sin \varphi \bar{j} + z \bar{k}.$$

От (16) следва

$$(17) \quad \frac{\partial \bar{r}}{\partial \rho} = \cos \varphi \bar{i} + \sin \varphi \bar{j},$$

$$(18) \quad \frac{\partial \bar{r}}{\partial \varphi} = -\rho \sin \varphi \bar{i} + \rho \cos \varphi \bar{j},$$

$$(19) \quad \frac{\partial r}{\partial z} = k.$$

От (17)–(19) следва

$$(20) \quad \frac{\partial r}{\partial \rho} = 1, \quad \frac{\partial r}{\partial \varphi} = \rho, \quad \frac{\partial r}{\partial z} = 1.$$

От (17)–(20), (15), (3) следва

$$(21) \quad \rho^0 = \cos \varphi i + \sin \varphi j,$$

$$(22) \quad \varphi^0 = -\sin \varphi i + \cos \varphi j,$$

$$(23) \quad z^0 = k.$$

От (21)–(23) следва

$$(24) \quad \rho^0 \varphi^0 - \varphi^0 z^0 - z^0 \rho^0 = 0.$$

От (15)–(20), (24), (14) следва

$$(25) \quad r = \rho \rho^0 - z k,$$

което, разбира се, следва съвсем непосредствено и от (16), (21).

Сферичната координатна система  $(r, \varphi, \psi)$  се дефинира с

$$(26) \quad q_1 = r, \quad q_2 = \varphi, \quad q_3 = \psi$$

при

$$(27) \quad r = r \cos \psi \cos \varphi i + r \cos \psi \sin \varphi j + r \sin \psi k.$$

От (27) следва

$$(28) \quad \frac{\partial r}{\partial r} = \cos \psi \cos \varphi i + \cos \psi \sin \varphi j + \sin \psi k.$$

$$(29) \quad \frac{\partial r}{\partial \varphi} = -r \cos \psi \sin \varphi i + r \cos \psi \cos \varphi j,$$

$$(30) \quad \frac{\partial r}{\partial \psi} = -r \sin \psi \cos \varphi i - r \sin \psi \sin \varphi j + r \cos \psi k.$$

От (28)–(30) следва

$$(31) \quad \left| \frac{\partial r}{\partial r} \right| = 1, \quad \left| \frac{\partial r}{\partial \varphi} \right| = r \cos \psi, \quad \left| \frac{\partial r}{\partial \psi} \right| = r.$$

От (28)–(31), (26), (3) следва

$$(32) \quad \bar{r}^0 = \cos \psi \cos \varphi i + \cos \psi \sin \varphi j + \sin \psi k,$$

$$(33) \quad \varphi^0 = -\sin \varphi \bar{i} + \cos \varphi \bar{j},$$

$$(34) \quad \bar{\psi}^0 = -\sin \psi \cos \varphi \bar{i} - \sin \psi \sin \varphi \bar{j} + \cos \psi \bar{k}.$$

От (32)–(34) следва

$$(35) \quad \bar{r}^0 \varphi^0 = \varphi^0 \bar{\psi}^0 = \bar{\psi}^0 \bar{r}^0 = 0.$$

От (26)–(31), (35), (14) следва

$$(36) \quad \bar{r} = r \bar{r}^0,$$

което, разбира се, следва съвсем непосредствено и от (27), (32).

Не така тривиален е примерът с елиптичната координатна система  $(\lambda, \mu, \nu)$ , която, както е известно, се дефинира по следния начин. При

$$(37) \quad c^2 < b^2 < a^2$$

алгебричното уравнение от трета степен спрямо  $q$

$$(38) \quad \frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} + \frac{z^2}{c^2+q} - 1 = 0$$

има три различни реални корена:

$$(39) \quad q_1 = \lambda, \quad q_2 = \mu, \quad q_3 = \nu$$

с

$$(40) \quad -a^2 < \lambda < -b^2 < \mu < -c^2 < \nu.$$

При (9), (39)  $x, y, z$  се определят от системата уравнения

$$(41) \quad \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

$$(42) \quad \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1,$$

$$(43) \quad \frac{x^2}{a^2+\nu} + \frac{y^2}{b^2+\nu} + \frac{z^2}{c^2+\nu} = 1$$

при допълнителна уговорка за знаците. От (41)–(43) следва

$$(44) \quad x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)},$$

$$(45) \quad y^2 = \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-a^2)(b^2-c^2)},$$

$$(46) \quad z^2 = \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}.$$

От (44)–(46) следва

$$(47) \quad \frac{\partial x}{\partial \lambda} = \frac{x}{2(a^2 + \lambda)}, \quad \frac{\partial x}{\partial \mu} = \frac{x}{2(a^2 + \mu)}, \quad \frac{\partial x}{\partial \nu} = \frac{x}{2(a^2 + \nu)};$$

$$(48) \quad \frac{\partial y}{\partial \lambda} = \frac{y}{2(b^2 + \lambda)}, \quad \frac{\partial y}{\partial \mu} = \frac{y}{2(b^2 + \mu)}, \quad \frac{\partial y}{\partial \nu} = \frac{y}{2(b^2 + \nu)};$$

$$(49) \quad \frac{\partial z}{\partial \lambda} = \frac{z}{2(c^2 + \lambda)}, \quad \frac{\partial z}{\partial \mu} = \frac{z}{2(c^2 + \mu)}, \quad \frac{\partial z}{\partial \nu} = \frac{z}{2(c^2 + \nu)}.$$

От (9), (39), (47)–(49) следва

$$(50) \quad \frac{\partial \bar{r}}{\partial \lambda} = \frac{1}{2} \left( \frac{x}{a^2 + \lambda} \bar{i} + \frac{y}{b^2 + \lambda} \bar{j} + \frac{z}{c^2 + \lambda} \bar{k} \right),$$

$$(51) \quad \frac{\partial \bar{r}}{\partial \mu} = \frac{1}{2} \left( \frac{x}{a^2 + \mu} \bar{i} + \frac{y}{b^2 + \mu} \bar{j} + \frac{z}{c^2 + \mu} \bar{k} \right),$$

$$(52) \quad \frac{\partial \bar{r}}{\partial \nu} = \frac{1}{2} \left( \frac{x}{a^2 + \nu} \bar{i} + \frac{y}{b^2 + \nu} \bar{j} + \frac{z}{c^2 + \nu} \bar{k} \right).$$

От (47)–(49) следва

$$(53) \quad \frac{D(x, y, z)}{D(\lambda, \mu, \nu)} = \frac{1}{8xyz} \frac{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)}{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)}.$$

Нека

$$(54) \quad Q(q) = \frac{x^2}{a^2 + q} + \frac{y^2}{b^2 + q} + \frac{z^2}{c^2 + q} - 1.$$

От (54) и дефиницията на  $\lambda, \mu, \nu$  следва

$$(55) \quad Q(q) = - \frac{(q - \lambda)(q - \mu)(q - \nu)}{(a^2 + q)(b^2 + q)(c^2 + q)}.$$

От (55) следва

$$(56) \quad \left( \frac{dQ}{dq} \right)_{q=\lambda} = - \frac{(\lambda - \mu)(\lambda - \nu)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}.$$

От (54) следва

$$(57) \quad \left( \frac{dQ}{dq} \right)_{q=\lambda} = - \left[ \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right].$$

От (56), (57) следва

$$(58) \quad \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} = \frac{(\lambda - \mu)(\lambda + \nu)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}.$$

Аналогично се получава

$$(59) \quad \frac{x^2}{(a^2 + \mu)^2} + \frac{y^2}{(b^2 + \mu)^2} + \frac{z^2}{(c^2 + \mu)^2} = \frac{(\mu - \lambda)(\mu - \nu)}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)},$$

$$(60) \quad \frac{x^2}{(a^2 + \nu)^2} + \frac{y^2}{(b^2 + \nu)^2} + \frac{z^2}{(c^2 + \nu)^2} = \frac{(\nu - \lambda)(\nu - \mu)}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}.$$

От (50)–(52), (58)–(60) следва

$$(61) \quad \frac{\partial r}{\partial \lambda} = \frac{1}{2} \sqrt{\frac{(\lambda - \mu)(\lambda - \nu)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}},$$

$$(62) \quad \frac{\partial r}{\partial \mu} = \frac{1}{2} \sqrt{\frac{(\mu - \lambda)(\mu - \nu)}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}},$$

$$(63) \quad \frac{\partial r}{\partial \nu} = \frac{1}{2} \sqrt{\frac{(\nu - \lambda)(\nu - \mu)}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}}.$$

От (3), (39), (50)–(52), (61)–(63) следва

$$(64) \quad \lambda^0 = \sqrt{\frac{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(\lambda - \nu)}} \left( \frac{x}{a^2 + \lambda} i + \frac{y}{b^2 + \lambda} j + \frac{z}{c^2 + \lambda} k \right),$$

$$(65) \quad \mu^0 = \sqrt{\frac{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(\mu - \nu)}} \left( \frac{x}{a^2 + \mu} i + \frac{y}{b^2 + \mu} j + \frac{z}{c^2 + \mu} k \right),$$

$$(66) \quad \nu^0 = \sqrt{\frac{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(\nu - \mu)}} \left( \frac{x}{a^2 + \nu} i + \frac{y}{b^2 + \nu} j + \frac{z}{c^2 + \nu} k \right).$$

Чрез изваждане на двойките равенства (41), (42); (42), (43); (43), (41) се получава съответно

$$(67) \quad \frac{x^2}{(a^2 + \lambda)(a^2 + \mu)} + \frac{y^2}{(b^2 + \lambda)(b^2 + \mu)} + \frac{z^2}{(c^2 + \lambda)(c^2 + \mu)} = 0,$$

$$(68) \quad \frac{x^2}{(a^2 + \mu)(a^2 + \nu)} + \frac{y^2}{(b^2 + \mu)(b^2 + \nu)} + \frac{z^2}{(c^2 + \mu)(c^2 + \nu)} = 0,$$

$$(69) \quad \frac{x^2}{(a^2 + \nu)(a^2 + \lambda)} + \frac{y^2}{(b^2 + \nu)(b^2 + \lambda)} + \frac{z^2}{(c^2 + \nu)(c^2 + \lambda)} = 0,$$

след съкращаване съответно на  $\lambda - \mu$ ,  $\mu - \nu$ ,  $\nu - \lambda$  съгласно (40). От (64)–(69) следва

$$(70) \quad \bar{\lambda}^0 \bar{\mu}^0 = \bar{\mu}^0 \bar{\nu}^0 = \bar{\nu}^0 \bar{\lambda}^0 = 0,$$

От (39), (9), (70), (14), (3), (64)–(66), (41)–(43) следва

$$(71) \quad \bar{r} = \sqrt{\frac{(\alpha^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(\lambda - \nu)}} \bar{\lambda}^0 + \sqrt{\frac{(\alpha^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(\mu - \nu)}} \bar{\mu}^0 \\ + \sqrt{\frac{(\alpha^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(\nu - \mu)}} \bar{\nu}^0$$

— резултат, който съвсем не е така очевиден, както (25) и (36), макар и тук да се касае за ортогонална система криволинейни координати. В случая на неортогонални системи ползата от израза (13), даващ разлагане на радиус-вектора (1) на подвижната точка по единичните тангенциални вектори (3) на координатните линии, при което коефициентите на разлагането са функции на криволинейните координати, е безсъмнена. Тази полза изпъква още повече при търсенето на такива кинематични величини като например скоростта и ускорението на подвижната точка, което се прави по-долу.

3. Нека

$$(72) \quad \bar{a} = \bar{a}(p_1, p_2, \dots, p_n)$$

и

$$(73) \quad \bar{a} \neq 0.$$

Тогава

$$(74) \quad \bar{a}^0 = \frac{\bar{a}}{a},$$

$$(75) \quad \frac{\partial}{\partial p_\nu} \left( \frac{\bar{a}}{a} \right) = \frac{a \frac{\partial \bar{a}}{\partial p_\nu} - \bar{a} \frac{\partial a}{\partial p_\nu}}{a^2} \quad (\nu = 1, 2, \dots, n),$$

$$(76) \quad a \frac{\partial a}{\partial p_\nu} = \bar{a} \frac{\partial \bar{a}}{\partial p_\nu}. \quad (\nu = 1, 2, \dots, n).$$

От (75), (76) следва

$$(77) \quad \frac{\partial}{\partial p_\nu} \left( \frac{\bar{a}}{a} \right) = \frac{a^2 \frac{\partial \bar{a}}{\partial p_\nu} - \left( \bar{a} \frac{\partial \bar{a}}{\partial p_\nu} \right) \bar{a}}{a^3} = \frac{\left( \bar{a} \times \frac{\partial \bar{a}}{\partial p_\nu} \right) \times \bar{a}}{a^3} \quad (\nu = 1, 2, \dots, n),$$

От (77), (74) следва

$$(78) \quad \frac{\partial \bar{a}^0}{\partial p_\nu} = \frac{1}{a} \left( \bar{a}^0 \times \frac{\partial \bar{a}}{\partial p_\nu} \right) \times \bar{a}^0 \quad (\nu = 1, 2, \dots, n).$$

От (3), (78) следва

$$(79) \quad \frac{\partial \bar{q}_v^0}{\partial q_\mu} = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \left( \bar{q}_v^0 \times \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \right) \times \bar{q}_v^0 \quad (\mu, v = 1, 2, 3).$$

От (79) и

$$(80) \quad \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} = \sum_{\lambda=1}^3 \left[ \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} (\bar{q}_\lambda^0)^{-1} \right] \bar{q}_\lambda^0 \quad (\mu, v = 1, 2, 3)$$

следва

$$(81) \quad \begin{aligned} \frac{\partial \bar{q}_v^0}{\partial q_\mu} &= \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \sum_{\lambda=1}^n \left[ \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} (\bar{q}_\lambda^0)^{-1} \right] \bar{q}_\lambda^0 \\ &\quad - \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_v^0 \right) \bar{q}_v^0 \end{aligned} \quad (\mu, v = 1, 2, 3).$$

Поради (11) на (81) може да се даде още видът

$$(82) \quad \begin{aligned} \frac{\partial \bar{q}_v^0}{\partial q_\mu} &= \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{\lambda=1}^3 \left| \frac{\partial \bar{r}}{\partial q_\lambda} \right| \left( \frac{\partial \bar{r}}{\partial q_{\lambda+1}} \times \frac{\partial \bar{r}}{\partial q_{\lambda+2}} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \right) \bar{q}_\lambda^0 \\ &\quad - \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_v^0 \right) \bar{q}_v^0 \end{aligned} \quad (\mu, v = 1, 2, 3)$$

при (6).

При ортогонална криволинейна координатна система от (81), (7) следва

$$(83) \quad \frac{\partial \bar{q}_v^0}{\partial q_\mu} = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \left[ \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_{v+1}^0 \right) \bar{q}_{v+1}^0 + \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_{v+2}^0 \right) \bar{q}_{v+2}^0 \right]$$

( $\mu, v = 1, 2, 3$ ) при

$$(84) \quad \bar{q}_{v+3}^0 = \bar{q}_v^0 \quad (v = 1, 2).$$

От

$$(85) \quad \frac{d \bar{q}_v^0}{dt} = \sum_{\mu=1}^3 \dot{q}_\mu \frac{\partial \bar{q}_v^0}{\partial q_\mu} \quad (v = 1, 2, 3),$$

където, както обикновено, с точка е означено диференциране спрямо времето  $t$ , и (81) следва

$$(86) \quad \frac{dq_v^0}{dt} = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \sum_{\mu=1}^3 \dot{q}_\mu \left\{ \sum_{i=1}^3 \left[ \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} (\bar{q}_i^0)^{-1} \right] \bar{q}_i^0 - \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_v^0 \right) \bar{q}_v^0 \right\} \quad (v=1, 2, 3).$$

Поради (11) на (86) може да се даде още видът

$$(87) \quad \frac{dq_v^0}{dt} = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \sum_{\mu=1}^3 \dot{q}_\mu \left\{ \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{i=1}^3 \left| \frac{\partial \bar{r}}{\partial q_i} \right| \left( \frac{\partial \bar{r}}{\partial q_{i+1}} \right. \right. \\ \times \left. \left. \frac{\partial \bar{r}}{\partial q_{i+2}} - \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \right) \bar{q}_i^0 - \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_v^0 \right) \bar{q}_v^0 \right\} \quad (v=1, 2, 3)$$

при (6).

При ортогонална криволинейна координатна система от (87), (7) следва

$$(88) \quad \frac{dq_v^0}{dt} = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \sum_{\mu=1}^3 \dot{q}_\mu \left[ \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_{\mu+1}^0 \right) \bar{q}_{\mu+1}^0 + \left( \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \bar{q}_{\mu+2}^0 \right) \bar{q}_{\mu+2}^0 \right]$$

( $v=1, 2, 3$ ) при (84).

4. При цилиндрични координати от (17)–(19) следва

$$(89) \quad \frac{\partial^2 \bar{r}}{\partial \rho^2} = 0, \quad \frac{\partial^2 \bar{r}}{\partial \varphi \partial \varphi} = -\sin \varphi \bar{i} + \cos \varphi \bar{j}, \quad \frac{\partial^2 \bar{r}}{\partial \rho \partial z} = \bar{0};$$

$$(90) \quad \frac{\partial^2 \bar{r}}{\partial \varphi \partial \rho} = -\sin \varphi \bar{i} + \cos \varphi \bar{j}, \quad \frac{\partial^2 \bar{r}}{\partial \varphi^2} = -\rho \cos \varphi \bar{i} - \rho \sin \varphi \bar{j}, \quad \frac{\partial^2 \bar{r}}{\partial \varphi \partial z} = \bar{0};$$

$$(91) \quad \frac{\partial^2 \bar{r}}{\partial z \partial \rho} = \bar{0}, \quad \frac{\partial^2 \bar{r}}{\partial z \partial \varphi} = \bar{0}, \quad \frac{\partial^2 \bar{r}}{\partial z^2} = \bar{0}.$$

От (89)–(91), (24), (15), (83), (21)–(23), (20) следва

$$(92) \quad \frac{\partial \bar{\rho}^0}{\partial \rho} = \bar{0}, \quad \frac{\partial \bar{\rho}^0}{\partial \varphi} = \bar{\varphi}^0, \quad \frac{\partial \bar{\rho}^0}{\partial z} = \bar{0};$$

$$(93) \quad \frac{\partial \bar{\varphi}^0}{\partial \rho} = \bar{0}, \quad \frac{\partial \bar{\varphi}^0}{\partial \varphi} = -\bar{\rho}^0, \quad \frac{\partial \bar{\varphi}^0}{\partial z} = \bar{0};$$

$$(94) \quad \frac{\partial \bar{z}^0}{\partial \rho} = \bar{0}, \quad \frac{\partial \bar{z}^0}{\partial \varphi} = \bar{0}, \quad \frac{\partial \bar{z}^0}{\partial z} = \bar{0}.$$

От (92)–(94), (15), (85) следва

$$(95) \quad \frac{d\bar{\rho}^0}{dt} = \dot{\varphi} \bar{\varphi}^0, \quad \frac{d\bar{\varphi}^0}{dt} = -\dot{\rho} \bar{\rho}^0, \quad \frac{d\bar{z}^0}{dt} = \bar{0}.$$

Разбира се, резултатите (92)–(95) могат да се получат и директно от (21)–(23) поради простотата на тези изрази.

При сферични координати от (28)–(31) следва

$$(96) \quad \frac{\partial^2 \bar{r}}{\partial r^2} = \bar{0},$$

$$(97) \quad \frac{\partial^2 \bar{r}}{\partial r \partial \varphi} = \frac{\partial^2 \bar{r}}{\partial \varphi \partial r} = -\cos \psi \sin \varphi \bar{i} + \cos \psi \cos \varphi \bar{j},$$

$$(98) \quad \frac{\partial^2 \bar{r}}{\partial r \partial \psi} = \frac{\partial^2 \bar{r}}{\partial \psi \partial r} = -\sin \psi \cos \varphi \bar{i} - \sin \psi \sin \varphi \bar{j} + \cos \psi \bar{k},$$

$$(99) \quad \frac{\partial^2 \bar{r}}{\partial \varphi^2} = -r \cos \psi \cos \varphi \bar{i} - r \cos \psi \sin \varphi \bar{j},$$

$$(100) \quad \frac{\partial^2 \bar{r}}{\partial \varphi \partial \psi} = \frac{\partial^2 \bar{r}}{\partial \psi \partial \varphi} = r \sin \psi \sin \varphi \bar{i} - r \sin \psi \cos \varphi \bar{j},$$

$$(101) \quad \frac{\partial^2 \bar{r}}{\partial \psi^2} = -r \cos \psi \cos \varphi \bar{i} - r \cos \psi \sin \varphi \bar{j} - r \sin \psi \bar{k}.$$

От (96)–(101), (35), (26), (83), (32)–(34), (31) следва

$$(102) \quad \frac{\partial \bar{r}^0}{\partial r} = \bar{0}, \quad \frac{\partial \bar{r}^0}{\partial \varphi} = \cos \psi \bar{\varphi}^0, \quad \frac{\partial \bar{r}^0}{\partial \psi} = \bar{\psi}^0;$$

$$(103) \quad \frac{\partial \bar{\varphi}^0}{\partial r} = \bar{0}, \quad \frac{\partial \bar{\varphi}^0}{\partial \varphi} = -\cos \psi \bar{r}^0 + \sin \psi \bar{\psi}^0, \quad \frac{\partial \bar{\varphi}^0}{\partial \psi} = \bar{0};$$

$$(104) \quad \frac{\partial \bar{\psi}^0}{\partial r} = \bar{0}, \quad \frac{\partial \bar{\psi}^0}{\partial \varphi} = -\sin \psi \bar{\varphi}^0, \quad \frac{\partial \bar{\psi}^0}{\partial \psi} = -\bar{r}^0.$$

От (102)–(104), (26), (85) следва

$$(105) \quad \frac{d\bar{r}^0}{dt} = \cos \psi \dot{\varphi} \bar{\varphi}^0 + \dot{\psi} \bar{\psi}^0, \quad \frac{d\bar{\varphi}^0}{dt} = -\cos \psi \dot{\varphi} \bar{r}^0 + \sin \psi \dot{\varphi} \bar{\psi}^0,$$

$$\frac{d\bar{\psi}^0}{dt} = -\sin \psi \dot{\varphi} \bar{\varphi}^0 - \dot{\psi} \bar{r}^0.$$

Разбира се, резултатите (102)–(105) могат да се получат и директно от (32)–(34) поради простотата на тези изрази.

При елиптични координати от (47)–(52) следва

$$(106) \quad \frac{\partial^2 \bar{r}}{\partial \lambda^2} = -\frac{1}{4} \left[ \frac{x}{(a^2 + \lambda)^2} \bar{i} + \frac{y}{(b^2 + \lambda)^2} \bar{j} + \frac{z}{(c^2 + \lambda)^2} \bar{k} \right],$$

$$(107) \quad \frac{\partial^2 \bar{r}}{\partial \mu^2} = -\frac{1}{4} \left[ \frac{x}{(a^2 + \mu)^2} \bar{i} + \frac{y}{(b^2 + \mu)^2} \bar{j} + \frac{z}{(c^2 + \mu)^2} \bar{k} \right],$$

$$(108) \quad \frac{\partial^2 \bar{r}}{\partial \nu^2} = -\frac{1}{4} \left[ \frac{x}{(a^2 + \nu)^2} \bar{i} + \frac{y}{(b^2 + \nu)^2} \bar{j} + \frac{z}{(c^2 + \nu)^2} \bar{k} \right],$$

$$(109) \quad \frac{\partial^2 \bar{r}}{\partial \lambda \partial \mu} = \frac{\partial^2 \bar{r}}{\partial \mu \partial \lambda} = \frac{1}{4} \left[ \frac{x}{(a^2 + \lambda)(a^2 + \mu)} \bar{i} \right. \\ \left. + \frac{y}{(b^2 + \lambda)(b^2 + \mu)} \bar{j} + \frac{z}{(c^2 + \lambda)(c^2 + \mu)} \bar{k} \right],$$

$$(110) \quad \frac{\partial^2 \bar{r}}{\partial \mu \partial \nu} = \frac{\partial^2 \bar{r}}{\partial \nu \partial \mu} = \frac{1}{4} \left[ \frac{x}{(a^2 + \mu)(a^2 + \nu)} \bar{i} \right. \\ \left. + \frac{y}{(b^2 + \mu)(b^2 + \nu)} \bar{j} + \frac{z}{(c^2 + \mu)(c^2 + \nu)} \bar{k} \right],$$

$$(111) \quad \frac{\partial^2 \bar{r}}{\partial \nu \partial \lambda} = \frac{\partial^2 \bar{r}}{\partial \lambda \partial \nu} = \frac{1}{4} \left[ \frac{x}{(a^2 + \nu)(a^2 + \lambda)} \bar{i} \right. \\ \left. + \frac{y}{(b^2 + \nu)(b^2 + \lambda)} \bar{j} + \frac{z}{(c^2 + \nu)(c^2 + \lambda)} \bar{k} \right].$$

От (106), (65) следва

$$(112) \quad \frac{\partial^2 \bar{r}}{\partial \lambda^2} \bar{\mu}^0 = -\frac{1}{4} \sqrt{\frac{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(\mu - \nu)}} s$$

при

$$(113) \quad s = \frac{x^2}{(a^2 + \lambda)^2(a^2 + \mu)} + \frac{y^2}{(b^2 + \lambda)^2(b^2 + \mu)} + \frac{z^2}{(c^2 + \lambda)^2(c^2 + \mu)}.$$

Чрез диференциране на (58) спрямо  $\mu$  поради (47)–(49) от (113) следва

$$(114) \quad s = \frac{\gamma - \lambda}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}.$$

От (112), (114) следва

$$(115) \quad \frac{\partial^2 \bar{r}}{\partial \lambda^2} \bar{\mu}^0 = -\frac{1}{4} \sqrt{\frac{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(\mu - \gamma)}} \frac{\gamma - \lambda}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}.$$

Аналогично се получава

$$(116) \quad \frac{\partial^2 \bar{r}}{\partial \lambda^2} \bar{\gamma}^0 = -\frac{1}{4} \sqrt{\frac{(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)}{(\gamma - \lambda)(\gamma - \mu)}} \frac{\mu - \lambda}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}.$$

От (115), (116), (70), (39), (83), (61) следва

$$(117) \quad \begin{aligned} \frac{\partial \bar{\lambda}^0}{\partial \lambda} &= \frac{1}{2(\lambda - \mu)} \sqrt{\frac{(\gamma - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \gamma)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \bar{\mu}^0 \\ &+ \frac{1}{2(\lambda - \gamma)} \sqrt{\frac{(\mu - \lambda)(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)}{(\gamma - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \bar{\gamma}^0. \end{aligned}$$

Аналогично се получава

$$(118) \quad \begin{aligned} \frac{\partial \bar{\mu}^0}{\partial \mu} &= \frac{1}{2(\mu - \lambda)} \sqrt{\frac{(\gamma - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \gamma)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \bar{\lambda}^0 \\ &+ \frac{1}{2(\mu - \gamma)} \sqrt{\frac{(\lambda - \mu)(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)}{(\gamma - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \bar{\gamma}^0, \end{aligned}$$

$$(119) \quad \begin{aligned} \frac{\partial \bar{\gamma}^0}{\partial \gamma} &= \frac{1}{2(\gamma - \lambda)} \sqrt{\frac{(\mu - \gamma)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)}} \bar{\lambda}^0 \\ &+ \frac{1}{2(\gamma - \mu)} \sqrt{\frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)}} \bar{\mu}^0. \end{aligned}$$

От (109), (65) следва

$$(120) \quad \frac{\partial^2 \bar{r}}{\partial \lambda \partial \mu} \bar{\mu}^0 = \frac{1}{4} \sqrt{\frac{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(\mu - \gamma)}} t$$

при

$$(121) \quad t = \frac{x^2}{(a^2 + \lambda)(a^2 + \mu)^2} + \frac{y^2}{(b^2 + \lambda)(b^2 + \mu)^2} + \frac{z^2}{(c^2 + \lambda)(c^2 + \mu)^2}.$$

Чрез диференциране на (59) спрямо  $\lambda$  поради (47)–(49) се получава

$$(122) \quad t = \frac{\gamma - \mu}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}.$$

От (120), (122) следва

$$(123) \quad \frac{\partial^2 \bar{r}}{\partial \lambda \partial \mu} \bar{\mu}^0 = -\frac{1}{4} \sqrt{\frac{\nu - \mu}{(\lambda - \mu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}}.$$

От (109), (66) следва

$$(124) \quad \frac{\partial^2 \bar{r}}{\partial \lambda \partial \mu} \bar{\nu}^0 = \frac{1}{4} \sqrt{\frac{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(\nu - \mu)}} u$$

при

$$(125) \quad u = \frac{x^2}{(a^2 + \lambda)(a^2 + \mu)(a^2 + \nu)} + \frac{y^2}{(b^2 + \lambda)(b^2 + \mu)(b^2 + \nu)} + \frac{z^2}{(c^2 + \lambda)(c^2 + \mu)(c^2 + \nu)}.$$

Чрез диференциране на (67) спрямо  $\nu$  поради (47)–(49) се получава

$$(126) \quad u = 0.$$

От (124), (126) следва

$$(127) \quad \frac{\partial^2 \bar{r}}{\partial \lambda \partial \mu} \bar{\nu}^0 = \bar{0}.$$

От (123), (127), (70), (39), (83), (61) следва

$$(128) \quad \frac{\partial \bar{\lambda}^0}{\partial \mu} = \frac{1}{2(\lambda - \mu)} \sqrt{\frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \bar{\mu}^0.$$

Аналогично се получава

$$(129) \quad \frac{\partial \bar{\lambda}^0}{\partial \nu} = \frac{1}{2(\lambda - \nu)} \sqrt{\frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \bar{\nu}^0,$$

$$(130) \quad \frac{\partial \bar{\mu}^0}{\partial \lambda} = \frac{1}{2(\mu - \lambda)} \sqrt{\frac{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \bar{\lambda}^0,$$

$$(131) \quad \frac{\partial \bar{\mu}^0}{\partial \nu} = \frac{1}{2(\mu - \nu)} \sqrt{\frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \bar{\nu}^0,$$

$$(132) \quad \frac{\partial \bar{\nu}^0}{\partial \lambda} = \frac{1}{2(\nu - \lambda)} \sqrt{\frac{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \bar{\lambda}^0,$$

$$(133) \quad \frac{\partial \bar{\nu}^0}{\partial \mu} = \frac{1}{2(\nu - \mu)} \sqrt{\frac{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \bar{\mu}^0.$$

От (117)–(119), (128)–(133), (39), (85) следва

$$(134) \quad \frac{d\bar{\lambda}^0}{dt} = \frac{1}{2(\lambda - \mu)} \left[ \sqrt{\frac{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \dot{\lambda} \right. \\ \left. + \sqrt{\frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \dot{\mu} \right] \bar{\mu}^0 \\ + \frac{1}{2(\lambda - \nu)} \left[ \sqrt{\frac{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \dot{\lambda} \right. \\ \left. + \sqrt{\frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \dot{\nu} \right] \bar{\nu}^0,$$

$$(135) \quad \frac{d\bar{\mu}^0}{dt} = \frac{1}{2(\mu - \lambda)} \left[ \sqrt{\frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \dot{\lambda} \right. \\ \left. + \sqrt{\frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \dot{\mu} \right] \bar{\lambda}^0 \\ + \frac{1}{2(\mu - \lambda)} \left[ \sqrt{\frac{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \dot{\mu} \right. \\ \left. + \sqrt{\frac{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \dot{\nu} \right] \bar{\nu}^0,$$

$$(136) \quad \frac{d\bar{\nu}^0}{dt} = \frac{1}{2(\nu - \lambda)} \left[ \sqrt{\frac{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \dot{\lambda} \right. \\ \left. + \sqrt{\frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \dot{\nu} \right] \bar{\lambda}^0 \\ + \frac{1}{2(\nu - \mu)} \left[ \sqrt{\frac{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \dot{\mu} \right. \\ \left. + \sqrt{\frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \dot{\nu} \right] \bar{\mu}^0.$$

За разлика от (92)–(95), респ. (102)–(105), резултатите (117)–(119), (128)–(133), (134)–(136) съвсем не са очевидно следствие от (64)–(66) поради сложността на тези изрази, а са планомерна последица от прилагането на релациите (83), (85) в случая на елиптични координати. При клиногонална криволинейна координатна система значението на тези релации нараства чувствително.

5. При

$$(137) \quad \bar{v} = \frac{d\bar{r}}{dt}$$

от (1) следва

$$(138) \quad \bar{v} = \sum_{v=1}^3 \dot{q}_v \frac{\partial \bar{r}}{\partial q_v}.$$

От (138), (3) следва

$$(139) \quad \bar{v} = \sum_{v=1}^3 \dot{q}_v \left| \frac{\partial \bar{r}}{\partial q_v} \right| \bar{q}_v^0.$$

От (139) и

$$(140) \quad \bar{v} = \sum_{v=1}^3 \left[ \bar{v}(\bar{q}_v^0)^{-1} \right] \bar{q}_v^0$$

следва

$$(141) \quad \dot{q}_v \left| \frac{\partial \bar{r}}{\partial q_v} \right| = \frac{d\bar{r}}{dt} (\bar{q}_v^0)^{-1} \quad (v=1, 2, 3)$$

съгласно (137).

От (138) следва

$$(142) \quad \frac{\partial \bar{v}}{\partial \dot{q}_v} = \frac{\partial \bar{r}}{\partial q_v} \quad (v=1, 2, 3).$$

От (142), (3)-следва

$$(143) \quad \bar{v} \bar{q}_v^0 = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) \quad (v=1, 2, 3).$$

От (143) и

$$(144) \quad \bar{v} = \sum_{v=1}^3 (\bar{v} \bar{q}_v^0) (\bar{q}_v^0)^{-1}$$

следва

$$(145) \quad \bar{v} = \sum_{v=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) (\bar{q}_v^0)^{-1}.$$

От (145), (11) следва

$$(146) \quad v = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{v=1}^3 \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) \frac{\partial \bar{r}}{\partial \dot{q}_{v+1}} \times \frac{\partial \bar{r}}{\partial \dot{q}_{v+2}}$$

при (6).

В случая на ортогонална криволинейна координатна система от (45), (7) следва

$$(147) \quad \bar{v} = \sum_{v=1}^3 \left| \frac{\partial \bar{r}}{\partial \dot{q}_v} \right| \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) \bar{q}_v^0.$$

Поради (3) на (147) може да се даде още видът

$$(148) \quad \bar{v} = \sum_{v=1}^3 \frac{1}{\left( \frac{\partial \bar{r}}{\partial \dot{q}_v} \right)^2} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) \frac{\partial \bar{r}}{\partial \dot{q}_v}.$$

6. При цилиндрични координати от (15), (20), (139) следва

$$(149) \quad \bar{v} = \dot{\rho} \bar{\rho}^0 + \rho \dot{\varphi} \bar{\varphi}^0 + \dot{z} \bar{k}.$$

При сферични координати от (26), (31), (139) следва

$$(150) \quad \bar{v} = \dot{r} \bar{r}^0 + r \cos \psi \dot{\varphi} \bar{\varphi}^0 + r \dot{\psi} \bar{\psi}^0.$$

При елиптични координати от (39), (61)–(63), (139) следва

$$(151) \quad \begin{aligned} \bar{v} = & \frac{1}{2} \left[ \sqrt{\frac{(\lambda - \mu)(\lambda - \nu)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \dot{\lambda} \bar{\lambda}^0 \right. \\ & \left. + \sqrt{\frac{(\mu - \lambda)(\mu - \nu)}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \dot{\mu} \bar{\mu}^0 + \sqrt{\frac{(\nu - \lambda)(\nu - \mu)}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \dot{\nu} \bar{\nu}^0 \right]. \end{aligned}$$

От (149), (24) следва

$$(152) \quad v^2 = \dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2.$$

От (150), (35) следва

$$(153) \quad v^2 = \dot{r}^2 + r^2 \cos^2 \psi \dot{\varphi}^2 + r^2 \dot{\psi}^2.$$

От (151), (70) следва

$$(154) \quad \begin{aligned} v^2 = & \frac{1}{4} \left[ \frac{(\lambda - \mu)(\lambda - \nu)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \dot{\lambda}^2 \right. \\ & \left. + \frac{(\mu - \lambda)(\mu - \nu)}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \dot{\mu}^2 + \frac{(\nu - \lambda)(\nu - \mu)}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \dot{\nu}^2 \right]. \end{aligned}$$

7. При

$$(155) \quad \omega = \frac{dv}{dt} - \frac{d^2r}{dt^2}$$

съгласно (137) от (138) следва

$$(156) \quad \bar{\omega} = \sum_{\nu=1}^3 \left( \ddot{q}_\nu \frac{\partial \bar{r}}{\partial q_\nu} + \sum_{\mu=1}^3 \dot{q}_\mu \dot{q}_\nu \frac{\partial^2 \bar{r}}{\partial q_\nu \partial q_\mu} \right).$$

От (156), (3), (80) следва

$$(157) \quad \bar{\omega} = \sum_{\nu=1}^3 \left[ \ddot{q}_\nu \frac{\partial \bar{r}}{\partial q_\nu} + \sum_{\lambda=1}^3 \sum_{\mu=1}^3 \dot{q}_\lambda \dot{q}_\mu \frac{\partial^2 \bar{r}}{\partial q_\lambda \partial q_\mu} (\bar{q}_\nu^0)^{-1} \right] \bar{q}_\nu^0.$$

Поради (11) на (157) може да се даде още видът

$$(158) \quad \bar{\omega} = \sum_{\nu=1}^3 \frac{\partial \bar{r}}{\partial q_\nu} \left[ \ddot{q}_\nu + \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{\lambda=1}^3 \sum_{\mu=1}^3 \dot{q}_\lambda \dot{q}_\mu \frac{\partial \bar{r}}{\partial q_{\nu+1}} \right. \\ \left. \times \frac{\partial \bar{r}}{\partial q_{\nu+2}} \cdot \frac{\partial^2 \bar{r}}{\partial q_\lambda \partial q_\mu} \right] \bar{q}_\nu^0$$

при (6).

При ортогонална криволинейна координатна система от (157), (7) следва

$$(159) \quad \bar{\omega} = \sum_{\nu=1}^3 \left( \ddot{q}_\nu \frac{\partial \bar{r}}{\partial q_\nu} + \sum_{\lambda=1}^3 \sum_{\mu=1}^3 \dot{q}_\lambda \dot{q}_\mu \frac{\partial^2 \bar{r}}{\partial q_\lambda \partial q_\mu} \bar{q}_\nu^0 \right) \bar{q}_\nu^0.$$

От (138) следва

$$(160) \quad \frac{\partial \bar{v}}{\partial q_\mu} = \sum_{\nu=1}^3 \dot{q}_\nu \frac{\partial^2 \bar{r}}{\partial q_\nu \partial q_\mu} \quad (\mu = 1, 2, 3).$$

От (160) и

$$(161) \quad \frac{d}{dt} \frac{\partial r}{\partial q_\mu} = \sum_{\nu=1}^3 \dot{q}_\nu \frac{\partial^2 \bar{r}}{\partial q_\mu \partial q_\nu} \quad (\mu = 1, 2, 3)$$

следва

$$(162) \quad \frac{\partial v}{\partial q_\nu} = \frac{d}{dt} \frac{\partial r}{\partial q_\nu} \quad (\nu = 1, 2, 3).$$

От (155), (3), (142), (162) следва

$$(163) \quad \begin{aligned} \left| \frac{\partial \bar{r}}{\partial q_v} \right| \bar{w} \bar{q}_v^0 &= \frac{d\bar{v}}{dt} \frac{\partial \bar{r}}{\partial q_v} = \frac{d}{dt} \left( \bar{v} \frac{\partial \bar{r}}{\partial q_v} \right) - \bar{v} \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \\ &= \frac{d}{dt} \left( \bar{v} \frac{\partial \bar{v}}{\partial \dot{q}_v} \right) - \bar{v} \frac{\partial \bar{v}}{\partial q_v} = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) - \frac{\partial}{\partial q_v} \left( \frac{v^2}{2} \right) \end{aligned}$$

( $v = 1, 2, 3$ ). От (163) следва

$$(164) \quad \bar{w} = \sum_{v=1}^3 \left| \frac{\partial \bar{r}}{\partial q_v} \right| \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) - \frac{\partial}{\partial q_v} \left( \frac{v^2}{2} \right) \right] (\bar{q}_v^0)^{-1}.$$

Поради (11) на (164) може да се даде още видът

$$(165) \quad \bar{w} = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{v=1}^3 \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) - \frac{\partial}{\partial q_v} \left( \frac{v^2}{2} \right) \right] \frac{\partial \bar{r}}{\partial q_{v+1}} \times \frac{\partial \bar{r}}{\partial q_{v+2}}$$

при (6).

В случая на ортогонална криволинейна координатна система от (164), (7) следва

$$(166) \quad \bar{w} = \sum_{v=1}^3 \left| \frac{1}{\frac{\partial \bar{r}}{\partial q_v}} \right| \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) - \frac{\partial}{\partial q_v} \left( \frac{v^2}{2} \right) \right] \bar{q}_v^0.$$

От (166) следва

$$(167) \quad \frac{\partial \bar{w}}{\partial \dot{q}_v} = \frac{\partial \bar{r}}{\partial q_v} \quad (v = 1, 2, 3).$$

От (167), (3) следва

$$(168) \quad \bar{w} \bar{q}_v^0 = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial}{\partial \dot{q}_v} \left( \frac{w^2}{2} \right) \quad (v = 1, 2, 3).$$

От (168) следва

$$(169) \quad \bar{w} = \sum_{v=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial}{\partial \dot{q}_v} \left( \frac{w^2}{2} \right) (\bar{q}_v^0)^{-1}.$$

Поради (11) на (169) може да се даде още видът

$$(170) \quad \bar{w} = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{v=1}^3 \frac{\partial}{\partial \dot{q}_v} \left( \frac{w^2}{2} \right) \frac{\partial \bar{r}}{\partial q_{v+1}} \times \frac{\partial \bar{r}}{\partial q_{v+2}}$$

при (6).

При ортогонална криволинейна координатна система от (169), (7) следва

$$(171) \quad \bar{w} = \sum_{\nu=1}^3 \left[ \frac{1}{\frac{\partial r}{\partial q_\nu}} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{w^2}{2} \right) \bar{q}_\nu^0 \right].$$

От (169), (164) следва

$$(172) \quad \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{w^2}{2} \right) = \frac{d}{dt} \frac{\partial}{\partial q_\nu} \left( \frac{v^2}{2} \right) - \frac{\partial}{\partial q_\nu} \left( \frac{v^2}{2} \right) \quad (\nu = 1, 2, 3).$$

8. При цилиндрични координати от (15), (20), (152), (166) следва

$$(173) \quad \bar{w} = (\ddot{\rho} - \rho \dot{\varphi}^2) \bar{\rho}^0 + \frac{1}{\rho} \frac{d}{dt} (\rho^2 \dot{\varphi}) \bar{\varphi}^0 + \ddot{z} \bar{k}.$$

При сферични координати от (26), (31), (153), (166) следва

$$(174) \quad \bar{w} = (\ddot{r} - r \cos^2 \psi \dot{\varphi}^2 + r \dot{\psi}^2) \bar{r}^0 + \frac{1}{r \cos \psi} \frac{d}{dt} (r^2 \cos^2 \psi \dot{\varphi}) \bar{\varphi}^0 \\ + \frac{1}{r} \left[ \frac{d}{dt} (r^2 \dot{\psi}) + r^2 \sin \psi \cos \psi \dot{\varphi}^2 \right] \bar{\psi}^0.$$

При елиптични координати от (29), (61)–(63), (154), (166) следва

$$(175) \quad \bar{w} = \frac{1}{2} \sqrt{\frac{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(\lambda - \nu)}} \left[ \frac{1}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \times \right. \\ \left\{ (\lambda - \mu)(\lambda - \nu) \ddot{\lambda} + \frac{1}{2} \left[ (2\lambda - \mu - \nu) - (\lambda - \mu)(\lambda - \nu) \left( \frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) \right] \dot{\lambda}^2 \right. \\ \left. + (\nu - \lambda) \dot{\lambda} \dot{\mu} + (\mu - \lambda) \dot{\lambda} \dot{\nu} \right\} + \frac{1}{2} \frac{\mu - \nu}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \dot{\mu}^2 \\ + \frac{1}{2} \frac{\nu - \mu}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \dot{\nu}^2 \right] \bar{\lambda}^0 + \frac{1}{2} \sqrt{\frac{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(\mu - \nu)}} \\ \times \left[ \frac{1}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \left\{ (\mu - \lambda)(\mu - \nu) \dot{\mu}^2 + \frac{1}{2} \left[ (2\mu - \lambda - \nu) \right. \right. \right. \\ \left. \left. \left. - (\mu - \lambda)(\mu - \nu) \left( \frac{1}{a^2 + \mu} + \frac{1}{b^2 + \mu} + \frac{1}{c^2 + \mu} \right) \right] \dot{\mu}^2 + (\nu - \mu) \dot{\lambda} \dot{\mu} + (\lambda - \mu) \dot{\lambda} \dot{\nu} \right\} \right]$$

$$\begin{aligned}
 & + \frac{1}{2} \left[ \frac{\lambda - \gamma}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \dot{\lambda}^2 - \frac{1}{2} \left( \frac{\gamma - \lambda}{(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)} \dot{\gamma}^2 \right) \mu^0 \right] \dot{\mu}^0 \\
 & + \frac{1}{2} \sqrt{\frac{(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)}{(\gamma - \lambda)(\gamma - \mu)}} \left[ \frac{1}{(a^2 + \gamma)(b^2 + \gamma)(c^2 + \gamma)} \left\{ (\gamma - \lambda)(\gamma - \mu) \dot{\gamma}^2 \right. \right. \\
 & \quad \left. \left. + \frac{1}{2} \left[ (2\gamma - \lambda - \mu) - (\gamma - \lambda)(\gamma - \mu) \left( \frac{1}{a^2 + \gamma} + \frac{1}{b^2 + \gamma} + \frac{1}{c^2 + \gamma} \right) \right] \dot{\gamma}^2 \right. \right. \\
 & \quad \left. \left. + (\mu - \gamma)\dot{\lambda}\dot{\gamma} + (\lambda - \gamma)\dot{\mu}\dot{\gamma} \right\} + \frac{1}{2} \frac{\lambda - \mu}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \dot{\lambda}^2 \right. \\
 & \quad \left. + \frac{1}{2} \frac{\mu - \lambda}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \dot{\mu}^2 \right] \dot{\gamma}^0.
 \end{aligned}$$

При ортогонална криволинейна координатна система от (138), (8) следва

$$(176) \quad v^2 = \sum_{v=1}^3 \left( \frac{\partial \bar{r}}{\partial q_v} \right)^2 \dot{q}_v^2.$$

От (176) следва

$$(177) \quad \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) = \left( \frac{\partial \bar{r}}{\partial q_v} \right)^2 \ddot{q}_v \quad (v = 1, 2, 3).$$

От (177) следва

$$(178) \quad \frac{d}{dt} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) = \left( \frac{\partial \bar{r}}{\partial q_v} \right)^2 \ddot{q}_v + 2 \sum_{\mu=1}^3 \frac{\partial \bar{r}}{\partial q_v} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \dot{q}_\mu \dot{q}_v$$

( $v = 1, 2, 3$ ). От (176), записано във вида

$$(179) \quad v^2 = \sum_{\mu=1}^3 \left( \frac{\partial \bar{r}}{\partial q_\mu} \right)^2 \dot{q}_\mu^2,$$

следва

$$(180) \quad \frac{\partial}{\partial q_v} \left( \frac{v^2}{2} \right) = \sum_{\mu=1}^3 \frac{\partial \bar{r}}{\partial q_\mu} \frac{\partial^2 \bar{r}}{\partial q_\mu \partial q_v} \dot{q}_\mu^2 \quad (v = 1, 2, 3).$$

От (178), (180) следва

$$(181) \quad \frac{d}{dt} \frac{\partial}{\partial \dot{q}_v} \left( \frac{v^2}{2} \right) = \frac{\partial}{\partial q_v} \left( \frac{v^2}{2} \right) = \left( \frac{\partial \bar{r}}{\partial q_v} \right)^2 \ddot{q}_v$$

$$\begin{aligned}
 & + \frac{\partial \bar{r}}{\partial q_v} \frac{\partial^2 \bar{r}}{\partial q_v^2} \dot{q}_v^2 + 2 \frac{\partial \bar{r}}{\partial q_v} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{v+1}} \dot{q}_v \dot{q}_{v+1} + 2 \frac{\partial \bar{r}}{\partial q_v} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{v+2}} \dot{q}_v \dot{q}_{v+2} \\
 & - \frac{\partial \bar{r}}{\partial q_{v+1}} \frac{\partial^2 \bar{r}}{\partial q_{v+1} \partial q_v} \dot{q}_{v+1}^2 - \frac{\partial \bar{r}}{\partial q_{v+2}} \frac{\partial^2 \bar{r}}{\partial q_{v+2} \partial q_v} \dot{q}_{v+2}^2 \quad (v=1, 2, 3)
 \end{aligned}$$

при

$$(182) \quad q_{v+3} = q_v \quad (v=1, 2).$$

От (39), (61)–(63), (50)–(52), (106)–(111), (70), (181) отново следва (175), като се има пред вид (113), (114), (121), (122) и т. н., както и следващото от (58) чрез диференциране тъждество

$$\begin{aligned}
 (183) \quad & \frac{x^2}{(a^2+\lambda)^3} + \frac{y^2}{(b^2+\lambda)^3} + \frac{z^2}{(c^2+\lambda)^3} = -\frac{1}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \left[ 2\lambda - \mu - v \right. \\
 & \left. - (\lambda - \mu)(\lambda - v) \left( \frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda} \right) \right]
 \end{aligned}$$

съгласно (47)–(49), и т. н.

9. Резултатите (167)–(172) могат да се обобщят. Нека

$$(184) \quad \bar{\rho}_v = \frac{d^v \bar{r}}{dt^v} \quad (v=0, 1, 2, \dots),$$

т. е. например

$$(185) \quad \bar{\rho}_0 = \bar{r}, \quad \bar{\rho}_1 = \bar{v}, \quad \bar{\rho}_2 = \bar{w}$$

съгласно (137), (155) и т. н. От (156), (184), (185) следва

$$\begin{aligned}
 (186) \quad & \bar{\rho}_3 = \sum_{v=1}^3 \dot{q}_v \frac{\partial \bar{r}}{\partial q_v} + 3 \sum_{v=1}^3 \sum_{\mu=1}^3 \ddot{q}_v \dot{q}_\mu \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \\
 & + \sum_{i=1}^3 \sum_{\mu=1}^3 \sum_{v=1}^3 \dot{q}_i \dot{q}_\mu \dot{q}_v \frac{\partial^3 \bar{r}}{\partial q_i \partial q_\mu \partial q_v}.
 \end{aligned}$$

От (186) следва

$$(187) \quad \frac{\partial \bar{\rho}_3}{\partial q_v} = \frac{\partial \bar{r}}{\partial q_v} \quad (v=1, 2, 3).$$

От (187), (3) следва

$$(188) \quad \bar{\rho}_3 \bar{q}_v^0 = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial}{\partial q_v} \left( \frac{\bar{\rho}_3^2}{2} \right) \quad (v=1, 2, 3).$$

От (188) и

$$(189) \quad \bar{\rho}_3 = \sum_{v=1}^3 (\bar{\rho}_3 \bar{q}_v^0) (\bar{q}_v^0)^{-1}$$

следва

$$(190) \quad \bar{\rho}_3 = \sum_{v=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial \left( \frac{\bar{\rho}_3^2}{2} \right)}{\partial q_v} (\bar{q}_v^0)^{-1}.$$

Поради (11) на (190) може да се даде още видът

$$(191) \quad \bar{\rho}_3 = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{v=1}^3 \frac{\partial \left( \frac{\bar{\rho}_3^2}{2} \right)}{\partial q_v} \frac{\partial \bar{r}}{\partial q_{v+1}} \times \frac{\partial \bar{r}}{\partial q_{v+2}}$$

при (6).

При ортогонална криволинейна координатна система от (190), (7) следва

$$(192) \quad \bar{\rho}_3 = \sum_{v=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial \left( \frac{\bar{\rho}_3^2}{2} \right)}{\partial q_v} \bar{q}_v^0.$$

Поради (3) на (192) може да се даде още видът

$$(193) \quad \bar{\rho}_3 = \sum_{v=1}^3 \frac{1}{\left( \frac{\partial \bar{r}}{\partial q_v} \right)^2} \frac{\partial \left( \frac{\bar{\rho}_3^2}{2} \right)}{\partial q_v} \frac{\partial \bar{r}}{\partial q_v},$$

От (156) следва

$$(194) \quad \frac{\partial \bar{\omega}}{\partial \dot{q}_\mu} = 2 \sum_{v=1}^3 \dot{q}_v \frac{\partial^2 \bar{r}}{\partial q_v \partial q_\mu} \quad (\mu = 1, 2, 3).$$

От (194), (161) следва

$$(195) \quad \frac{\partial \bar{\omega}}{\partial \dot{q}_v} = 2 \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \quad (\nu = 1, 2, 3).$$

От (184), (185), (3), (167), (195) следва

$$(196) \quad \begin{aligned} \left| \frac{\partial \bar{r}}{\partial q_v} \right| \bar{\rho}_3 \bar{q}_v^0 &= \frac{d \bar{\omega}}{dt} \frac{\partial \bar{r}}{\partial q_v} = \frac{d}{dt} \left( \bar{w} \frac{\partial \bar{r}}{\partial q_v} \right) - \bar{w} \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v}, \\ &= \frac{d}{dt} \left( \bar{w} \frac{\partial \bar{w}}{\partial \dot{q}_v} \right) - \frac{1}{2} \bar{w} \frac{\partial \bar{\omega}}{\partial \dot{q}_v} = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_v} \left( \frac{\bar{w}^2}{2} \right) - \frac{1}{2} \frac{\partial}{\partial \dot{q}_v} \left( \frac{\bar{w}^2}{2} \right) \end{aligned}$$

( $\gamma = 1, 2, 3$ ). От (196), (189) следва

$$(197) \quad \bar{\rho}_3 = \sum_{\nu=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_\nu} \right|} \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{w^2}{2} \right) - \frac{1}{2} \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{w^2}{2} \right) \right] (\bar{q}_\nu^0)^{-1}.$$

Поради (11) на (197) може да се даде още видът

$$(198) \quad \rho_3 = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{\nu=1}^3 \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{w^2}{2} \right) - \frac{1}{2} \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{w^2}{2} \right) \right] \frac{\partial \bar{r}}{\partial q_{\nu+1}} \times \frac{\partial r}{\partial q_{\nu+2}}$$

при (6).

При ортогонална криволинейна координатна система от (197), (7) следва:

$$(199) \quad \bar{\rho}_3 = \sum_{\nu=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_\nu} \right|} \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{w^2}{2} \right) - \frac{1}{2} \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{w^2}{2} \right) \right] \bar{q}_\nu^0.$$

Поради (11) на (199) може да се даде още видът

$$(200) \quad \bar{\rho}_3 = \sum_{\nu=1}^3 \frac{1}{\left( \frac{\partial \bar{r}}{\partial q_\nu} \right)^2} \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{w^2}{2} \right) - \frac{1}{2} \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{w^2}{2} \right) \right] \frac{\partial \bar{r}}{\partial q_\nu}.$$

От (190), (197), (4) следва

$$(201) \quad \frac{\partial}{\partial q_\nu} \left( \frac{\rho_3^2}{2} \right) = \frac{d}{dt} \frac{d}{\partial \dot{q}_\nu} \left( \frac{w^2}{2} \right) - \frac{1}{2} \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{w^2}{2} \right) \quad (\nu = 1, 2, 3).$$

10. От начина, по който бяха изведени тъждествата (190), (197), (201), е ясно как те могат да се обобщят за произволна натурална стойност на  $\nu$  в (184), т. е. как могат да се изведат тъждествата

$$(202) \quad \bar{\rho}_n = \sum_{\nu=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_\nu} \right|} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{\rho_n^2}{2} \right) (\bar{q}_\nu^0)^{-1},$$

$$(203) \quad \bar{\rho}_{n+1} = \sum_{\nu=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_\nu} \right|} \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{\rho_n^2}{2} \right) - \frac{1}{n} \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{\rho_n^2}{2} \right) \right] (\bar{q}_\nu^0)^{-1},$$

$$(204) \quad \frac{\partial}{\partial q_\nu} \left( \frac{\rho_{n+1}^2}{2} \right) = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\nu} \left( \frac{\rho_n^2}{2} \right) - \frac{1}{n} \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{\rho_n^2}{2} \right)$$

( $v = 1, 2, 3$ ) за всяко  $n = 1, 2, \dots$ , където е положено

$$(205) \quad \overset{(n)}{q}_v = \frac{d^n q_v}{dt^n} \quad (v = 1, 2, 3; n = 1, 2, \dots).$$

Верността на (202) в случаите  $n = 1, 2, 3$  се потвърждава от (145), (169), (190); тази на (203) в случаите  $n = 1, 2$  — от (164), (197); най-после от (172), (201) следва верността на (204) за  $n = 1, 2$ .

Най-напред ще установим верността на тъждеството

$$(206) \quad \rho_n = \sum_{v=1}^3 \overset{(n)}{q}_v \frac{\partial \bar{r}}{\partial q_v} + n \sum_{v=1}^3 \sum_{\mu=1}^3 \overset{(v-1)}{q}_v \cdot \overset{(v-1)}{q}_{\mu} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{\mu}} + R_n$$

( $n = 3, 4, \dots$ ), където  $R_n$  означава израз, в който производните (205) фигурират от ред, не по-висок от  $n - 2$ . Верността на (206) при  $n = 3$  се вижда от (186). От (206), (184) следва

$$(207) \quad \begin{aligned} \bar{\rho}_{n+1} &= \sum_{v=1}^3 \overset{(n+1)}{q}_v \frac{\partial \bar{r}}{\partial q_v} + \sum_{v=1}^3 \sum_{\mu=1}^3 \overset{(n)}{q}_v \overset{(n)}{q}_{\mu} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{\mu}} \\ &\quad + n \sum_{v=1}^3 \sum_{\mu=1}^3 \overset{(v)}{q}_v \cdot \overset{(v)}{q}_{\mu} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{\mu}} + R_{n+1} \\ &= \sum_{v=1}^3 \overset{(n+1)}{q}_v \frac{\partial \bar{r}}{\partial q_v} + (n+1) \sum_{v=1}^3 \sum_{\mu=1}^3 \overset{(v)}{q}_v \cdot \overset{(v)}{q}_{\mu} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{\mu}} + R_{n+1}, \end{aligned}$$

където е положено

$$(208) \quad \begin{aligned} R_{n+1} &= n \sum_{v=1}^3 \sum_{\mu=1}^3 \left[ \overset{(n-1)}{q}_v \cdot \overset{(n-1)}{q}_{\mu} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{\mu}} \right. \\ &\quad \left. + \overset{(n-1)}{q}_v \cdot \overset{(n-1)}{q}_{\mu} \sum_{\lambda=1}^3 \overset{(n-1)}{q}_{\lambda} \frac{\partial^3 \bar{r}}{\partial q_v \partial q_{\mu} \partial q_{\lambda}} \right] + \frac{\partial \lambda_n}{\partial t}. \end{aligned}$$

От (208) и дефиницията  $R_n$  следва, че  $R_{n+1}$  означава израз, в който производните (205) фигурират от ред, не по-висок от  $n - 1$ . Равенството (207) доказва (206) за всяко  $n = 3, 4, \dots$

От (206) следва

$$(209) \quad \frac{\partial \bar{\rho}_n}{\partial q_v} = \frac{\partial \bar{r}}{\partial q_v} \quad (v = 1, 2, 3).$$

От (209), (3) следва

$$(210) \quad \bar{\rho}_n q_v^0 = \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) \quad (v = 1, 2, 3).$$

От (210) и

$$(211) \quad \bar{\rho}_n = \sum_{v=1}^3 (\bar{\rho}_n \bar{q}_v^0) (\bar{q}_v^0)^{-1}$$

следва (202). Поради (11) на (202) може да се даде още видът

$$(212) \quad \bar{\rho}_n = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{v=1}^3 \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) \frac{\partial r}{\partial q_{v+1}} \times \frac{\partial r}{\partial q_{v+2}}$$

при (6).

При ортогонална криволинейна координатна система от (202), (7) следва

$$(213) \quad \bar{\rho}_n = \sum_{v=1}^3 \frac{1}{\left| \frac{\partial r}{\partial q_v} \right|} \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) \bar{q}_v^0.$$

Поради (3) на (213) може да се даде още видът

$$(214) \quad \bar{\rho}_n = \sum_{v=1}^3 \frac{1}{\left( \frac{\partial r}{\partial q_v} \right)^2} \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) \frac{\partial r}{\partial q_v}.$$

От (206) следва

$$(215) \quad \frac{\partial \bar{\rho}_n}{\partial q_v} = n \sum_{\mu=1}^3 \bar{q}_{\mu} \frac{\partial^2 \bar{r}}{\partial q_v \partial q_{\mu}} \quad (v=1, 2, 3).$$

От (215), (191) следва

$$(216) \quad \frac{\partial \bar{\rho}_n}{\partial q_v} = n \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \quad (v=1, 2, 3).$$

От (184), (3), (209), (216) следва

$$(217) \quad \begin{aligned} \frac{\partial \bar{r}}{\partial q_v} \bar{\rho}_{n+1} \bar{q}_v^0 &= \frac{d \bar{\rho}_n}{dt} \frac{\partial \bar{r}}{\partial q_v} = \frac{d}{dt} \left( \bar{\rho}_n \frac{\partial \bar{r}}{\partial q_v} \right) - \bar{\rho}_n \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \\ &= \frac{d}{dt} \left[ \bar{\rho}_n \frac{\partial \bar{r}}{\partial q_v} \right] - \frac{1}{n} \bar{\rho}_n \frac{\partial \bar{\rho}_n}{\partial q_v} = \frac{d}{dt} \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) - \frac{1}{n} \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) \end{aligned}$$

(v=1, 2, 3). От (217) и (211) с  $n+1$  вместо  $n$  следва (203). Поради (11) на (203) може да се даде още видът

$$(218) \quad \bar{\rho}_{n+1} = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{v=1}^3 \left[ \frac{d}{dt} \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) - \frac{1}{n} \frac{\partial}{\partial q_v} \left( \frac{\rho_n^2}{2} \right) \right] \frac{\partial \bar{r}}{\partial q_{v+1}} \times \frac{\partial \bar{r}}{\partial q_{v+2}}$$

при (6).

При ортогонална криволинейна координатна система от (203), (7) следва

$$(219) \quad \bar{\rho}_{n+1} = \sum_{\nu=1}^3 \left| \frac{d\bar{r}}{dq_\nu} \right| \left[ \frac{d}{dt} \frac{\partial}{\partial q_\nu} \left( \frac{\rho_n^2}{2} \right) - \frac{1}{n} \frac{\partial}{\partial q_\nu} \left( \frac{\rho_n^2}{2} \right) \right] \bar{q}_\nu^0.$$

Поради (3) на (219) може да се даде още видът

$$(220) \quad \bar{\rho}_{n+1} = \sum_{\nu=1}^3 \frac{1}{\left( \frac{d\bar{r}}{dq_\nu} \right)^2} \left[ \frac{d}{dt} \frac{\partial}{\partial q_\nu} \left( \frac{\rho_n^2}{2} \right) - \frac{1}{n} \frac{\partial}{\partial q_\nu} \left( \frac{\rho_n^2}{2} \right) \right] \frac{d\bar{r}}{dq_\nu}.$$

Тъждеството (204) е директно следствие от (202) с  $n+1$  вместо  $n$  и от (203).

11. Нека реперът (3) на изобщо клиногоналната криволинейна координатна система  $q_\nu$  ( $\nu=1, 2, 3$ ) е „твърд“ или „нееластичен“, т. е. нека

$$(221) \quad \frac{d}{dt} (\bar{q}_\mu^0 \bar{q}_\nu^0) = 0 \quad (\mu, \nu = 1, 2, 3).$$

(в същност при  $\mu=\nu$  равенствата (221) са тривиални, тъй като  $\bar{q}_\nu^0$  ( $\nu=1, 2, 3$ ) са единични вектори, т. е. постоянни). Тогава реперът (3) притежава моментална ъглова скорост  $\bar{\omega}$ , която се дефинира с

$$(222) \quad \bar{\omega} = \frac{1}{2} \sum_{\nu=1}^3 \bar{q}_\nu^0 \times \frac{d(\bar{q}_\nu^0)^{-1}}{dt}$$

или с

$$(223) \quad \bar{\omega} = \frac{1}{2} \sum_{\nu=1}^3 (\bar{q}_\nu^0)^{-1} \times \frac{d\bar{q}_\nu^0}{dt}.$$

От (11) следва, че на (223) може да се даде още видът

$$(224) \quad \bar{\omega} = \frac{1}{2} \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{\nu=1}^3 \left| \frac{\partial \bar{r}}{\partial p_\nu} \right| \left( \frac{\partial \bar{r}}{\partial q_{\nu+1}} \times \frac{\partial \bar{r}}{\partial q_{\nu+2}} \right) \times \frac{d\bar{q}_\nu^0}{dt}$$

при (6). В случая на ортогонална криволинейна координатна система  $q_\nu$  ( $\nu=1, 2, 3$ ) с (221) от (222), (223), (7) следва

$$(225) \quad \bar{\omega} = \frac{1}{2} \sum_{\nu=1}^3 \bar{q}_\nu^0 \times \frac{d\bar{q}_\nu^0}{dt}.$$

Векторът  $\bar{\omega}$  има следното свойство: за произволен вектор

$$(226) \quad \bar{a} = \bar{a}(t),$$

за който

$$(227) \quad \frac{d}{dt} (\bar{a} \bar{q}_v^0) = 0 \quad (v=1, 2, 3),$$

или, което е същото,

$$(228) \quad \frac{d}{dt} [\bar{a}(\bar{q}_v^0)^{-1}] = 0 \quad (v=1, 2, 3),$$

е в сила

$$(229) \quad \frac{d\bar{\omega}}{dt} = \bar{\omega} \times \bar{a}.$$

От (78) с  $n=1$ ,  $p_1=t$  и (3) следва

$$(230) \quad \frac{d\bar{q}_v^0}{dt} = \frac{1}{\left| \frac{\partial \bar{r}}{\partial \bar{q}_v} \right|} \left( \bar{q}_v^0 \times \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right) \times \bar{q}_v^0 \quad (v=1, 2, 3).$$

От (230), (223) следва

$$(231) \quad \bar{\omega} = \frac{1}{2} \sum_{v=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \left\{ \bar{q}_v^0 \times \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} + \left[ \bar{q}_v^0 \times (\bar{q}_v^0)^{-1} \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right] \bar{q}_v^0 \right\}.$$

При ортогонална криволинейна координатна система от (225), (230) следва

$$(232) \quad \bar{\omega} = \frac{1}{2} \sum_{v=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_v} \right|} \bar{q}_v^0 \times \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v}.$$

12. При цилиндрични координати от (21)–(23) следва

$$(233) \quad \bar{\rho}^0 \times \bar{\varphi}^0 = \bar{k}, \quad \bar{\varphi}^0 \times \bar{z}^0 = \bar{\rho}^0, \quad \bar{z}^0 \times \bar{\rho}^0 = \bar{\varphi}^0,$$

а от (15), (24), (95), (225), (233) следва

$$(234) \quad \bar{\omega} = \dot{\varphi} \bar{k}.$$

При сферични координати от (32)–(34) следва

$$(235) \quad \bar{r}^0 \times \bar{\varphi}^0 = \bar{\psi}^0, \quad \bar{\varphi}^0 \times \bar{\psi}^0 = \bar{r}^0, \quad \bar{\psi}^0 \times \bar{r}^0 = \bar{\varphi}^0,$$

а от (26), (35), (105), (225), (235) следва

$$(236) \quad \bar{\omega} = \sin \psi \dot{\varphi} \bar{r}^0 - \dot{\psi} \bar{\varphi}^0 + \cos \psi \dot{\varphi} \bar{\psi}^0.$$

При елиптични координати от (44)–(46) следва

$$(237) \quad x = \xi \sqrt{\frac{(a^2 + \lambda)(a^2 + \mu)(a^2 + \nu)}{(a^2 - b^2)(a^2 - c^2)}} \quad (\xi = \pm 1),$$

$$(238) \quad y = \eta \sqrt{\frac{(b^2 + \lambda)(b^2 + \mu)(b^2 + \nu)}{(b^2 - a^2)(b^2 - c^2)}} \quad (\eta = \pm 1),$$

$$(239) \quad z = \zeta \sqrt{\frac{(c^2 + \lambda)(c^2 + \mu)(c^2 + \nu)}{(c^2 - a^2)(c^2 - b^2)}} \quad (\zeta = \pm 1).$$

От

$$(240) \quad \epsilon = \xi \eta \zeta$$

и (237)–(239) следва

$$(241) \quad \epsilon = \pm 1.$$

От (64), (65) следва

$$(242) \quad \lambda^0 \times \bar{\mu}^0 = \sqrt{\frac{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(\lambda - \nu)}} \sqrt{\frac{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(\mu - \nu)}} \Delta$$

при

$$(243) \quad \Delta = \begin{vmatrix} i & j & k \\ \frac{x}{a^2 + \lambda} & \frac{y}{b^2 + \lambda} & \frac{z}{c^2 + \lambda} \\ \frac{x}{a^2 + \mu} & \frac{y}{b^2 + \mu} & \frac{z}{c^2 + \mu} \end{vmatrix}$$

От (237)–(243), (66) следва

$$(244) \quad \lambda^0 \times \bar{\mu}^0 = \epsilon \bar{\nu}^0.$$

От (244), (241), (70) следва

$$(245) \quad \bar{\mu}^0 \times \bar{\nu}^0 = \epsilon \lambda^0, \quad \bar{\nu}^0 \times \bar{\lambda}^0 = \epsilon \bar{\mu}^0.$$

От (39), (70), (134)–(136), (225), (244), (245) следва

$$(246) \quad \bar{\omega} = \frac{\epsilon}{2} \left\{ \frac{1}{\mu - \nu} \left[ \sqrt{\frac{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \right] \mu \right. \\ \left. + \sqrt{\frac{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \nu \right] \bar{\lambda}^0$$

$$\begin{aligned}
 & + \frac{1}{\nu - \lambda} \left[ \sqrt{\frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}} \right. ; \\
 & + \sqrt{\frac{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \dot{\lambda} \left. \right] \bar{\mu}^0 \\
 & + \frac{1}{\lambda - \mu} \left[ \sqrt{\frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}} \dot{\lambda} \right. \\
 & \left. + \sqrt{\frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}} \dot{\mu} \right] \bar{\nu}^0 \}.
 \end{aligned}$$

13. Моменталното ъглово ускорение  $\bar{\epsilon}$  се дефинира с

$$(247) \quad \bar{\epsilon} = \frac{d\bar{\omega}}{dt}.$$

От (247) и (222), (223) следва съответно

$$(248) \quad \bar{\epsilon} = \frac{1}{2} \sum_{\nu=1}^3 \bar{q}_{\nu}^0 \times \frac{d^2(\bar{q}_{\nu}^0)^{-1}}{dt^2},$$

$$(249) \quad \bar{\epsilon} = \frac{1}{2} \sum_{\nu=1}^3 (\bar{q}_{\nu}^0)^{-1} \times \frac{d^2\bar{q}_{\nu}^0}{dt^2}.$$

От (11) следва, че на (249) може да се даде още видът

$$(250) \quad \bar{\epsilon} = \frac{1}{2} \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{\nu=1}^3 \left| \frac{\partial \bar{r}}{\partial q_{\nu}} \right| \left( \frac{\partial \bar{r}}{\partial q_{\nu+1}} \times \frac{\partial \bar{r}}{\partial q_{\nu+2}} \right) \times \frac{d^2\bar{q}_{\nu}^0}{dt^2}$$

при (6). В случая на ортогонална криволинейна координатна система  $q_{\nu}$  ( $\nu = 1, 2, 3$ ) с (221) от (248), (249), (7) следва

$$(251) \quad \bar{\epsilon} = \frac{1}{2} \sum_{\nu=1}^3 \bar{q}_{\nu}^0 \times \frac{d^2\bar{q}_{\nu}^0}{dt^2}.$$

Векторът  $\bar{\epsilon}$  има свойството, че за произволен вектор (226) с (227) или (228)  $\bar{\epsilon}$  е в сила

$$(252) \quad \frac{d^2\bar{a}}{dt^2} = \bar{\epsilon} \times \bar{a} + \bar{\omega} \times (\bar{\omega} \times \bar{a})$$

при (222) или (223).

От (85) следва

$$(253) \quad \frac{d^2\bar{q}_v^0}{dt^2} = \sum_{\mu=1}^3 \ddot{q}_\mu \frac{\partial \bar{q}_v^0}{\partial q_\mu} + \sum_{\lambda=1}^3 \sum_{\mu=1}^3 \dot{q}_\lambda \dot{q}_\mu - \frac{\partial^2 \bar{q}_v^0}{\partial q_\lambda \partial q_\mu}.$$

От (78) следва

$$(254) \quad \begin{aligned} \frac{\partial^2 \bar{a}^0}{\partial p_v \partial p_\mu} &= -\frac{1}{a^2} \frac{\partial a}{\partial p_\mu} \left( \bar{a}^0 \times \frac{\partial \bar{a}}{\partial p_v} \right) \times \bar{a}^0 + \frac{1}{a} \left( \frac{\partial \bar{a}^0}{\partial p_\mu} \times \frac{\partial \bar{a}}{\partial p_v} \right) \times \bar{a}^0 \\ &\quad + \frac{1}{a} \left( \bar{a}^0 \times \frac{\partial^2 \bar{a}}{\partial p_v \partial p_\mu} \right) \times \bar{a}^0 + \frac{1}{a} \left( \bar{a}^0 \times \frac{\partial \bar{a}}{\partial p_v} \right) \times \frac{\partial \bar{a}^0}{\partial p_\mu}. \end{aligned}$$

От (254), (78), (76) следва

$$(255) \quad \begin{aligned} \frac{\partial^2 \bar{a}^0}{\partial p_v \partial p_\mu} &= \frac{1}{a^2} \left[ 3 \left( \bar{a}^0 \frac{\partial \bar{a}}{\partial p_\mu} \right) \left( \bar{a}^0 \frac{\partial \bar{a}}{\partial p_v} \right) \bar{a}^0 + \left( \bar{a}^0 \times \frac{\partial^2 \bar{a}}{\partial p_v \partial p_\mu} \right) \times \bar{a}^0 \right. \\ &\quad \left. - \left( \bar{a}^0 \frac{\partial \bar{a}}{\partial p_\mu} \right) \frac{\partial \bar{a}}{\partial p_v} - \left( \bar{a}^0 \frac{\partial \bar{a}}{\partial p_v} \right) \frac{\partial \bar{a}}{\partial p_\mu} - \left( \frac{\partial \bar{a}}{\partial p_\mu} \frac{\partial \bar{a}}{\partial p_v} \right) \bar{a}^0 \right]. \end{aligned}$$

От (255) с  $n=1$  и  $p_1=t$  следва

$$(256) \quad \begin{aligned} \frac{d^2 \bar{a}^0}{dt^2} &= \frac{1}{a^2} \left[ 3 \left( \bar{a}^0 \frac{d \bar{a}}{dt} \right)^2 \bar{a}^0 + \left( \bar{a}^0 \times \frac{d^2 \bar{a}}{dt^2} \right) \times \bar{a}^0 \right. \\ &\quad \left. - 2 \left( \bar{a}^0 \frac{d \bar{a}}{dt} \right) \frac{d \bar{a}}{dt} - \left( \frac{d \bar{a}}{dt} \right)^2 \bar{a}^0 \right]. \end{aligned}$$

От (256), (3) следва

$$(257) \quad \begin{aligned} \frac{d^2 \bar{q}_v^0}{dt^2} &= \frac{1}{\left( \frac{\partial \bar{r}}{\partial q_v} \right)^2} \left[ 3 \left( \bar{q}_v^0 \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right)^2 \bar{q}_v^0 + \left( \frac{\partial \bar{r}}{\partial q_v} \times \frac{d^2}{dt^2} \frac{\partial \bar{r}}{\partial q_v} \right) \times \bar{q}_v^0 \right. \\ &\quad \left. - 2 \left( \bar{q}_v^0 \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right) \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} - \left( \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right)^2 \bar{q}_v^0 \right] \end{aligned}$$

( $v=1, 2, 3$ ). От (257), (249) следва

$$(258) \quad \begin{aligned} \bar{\epsilon} &= \frac{1}{2} \sum_{v=1}^3 \frac{1}{\left( \frac{\partial \bar{r}}{\partial q_v} \right)^2} \left\{ 3 \left( \bar{q}_v^0 \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right)^2 (\bar{q}_v^0)^{-1} \times \bar{q}_v^0 \right. \\ &\quad + \frac{\partial \bar{r}}{\partial q_v} \times \frac{d^2}{dt^2} \frac{\partial \bar{r}}{\partial q_v} - \left[ (\bar{q}_v^0)^{-1} \frac{\partial \bar{r}}{\partial q_v} \times \frac{d^2}{dt^2} \frac{\partial \bar{r}}{\partial q_v} \right] \bar{q}_v^0 \\ &\quad \left. - 2 \left( \bar{q}_v^0 \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right) (\bar{q}_v^0)^{-1} \times \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} - \left( \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_v} \right)^2 (\bar{q}_v^0)^{-1} \times \bar{q}_v^0 \right\}. \end{aligned}$$

При ортогонална криволинейна координатна система от (258), (7) следва

$$(259) \quad \bar{\varepsilon} = \frac{1}{2} \sum_{\nu=1}^3 \frac{1}{\left| \frac{\partial \bar{r}}{\partial q_\nu} \right|^2} \left[ \frac{\partial \bar{r}}{\partial q_\nu} \times \frac{d^2}{dt^2} \frac{\partial \bar{r}}{\partial q_\nu} - \left( \bar{q}_\nu^0 \frac{\partial \bar{r}}{\partial q_\nu} \times \frac{d^2}{dt^2} \frac{\partial \bar{r}}{\partial q_\nu} \right) \bar{q}_\nu^0 \right. \\ \left. - 2 \left( \bar{q}_\nu^0 \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_\nu} \right) \bar{q}_\nu^0 \times \frac{d}{dt} \frac{\partial \bar{r}}{\partial q_\nu} \right].$$

От (164) с

$$(260) \quad \bar{r} = \bar{q}_\nu^0 \quad (\nu = 1, 2, 3)$$

следва

$$(261) \quad \frac{d^2 \bar{q}_\nu^0}{dt^2} = \sum_{\mu=1}^3 \frac{1}{\left| \frac{\partial \bar{q}_\nu^0}{\partial q_\mu} \right|} \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] - \frac{\partial}{\partial q_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] \right\} (\bar{q}_\mu^0)^{-1}$$

( $\nu = 1, 2, 3$ ) при

$$(262) \quad \frac{\partial \bar{q}_\nu^0}{\partial q_\mu} \neq 0 \quad (\mu, \nu = 1, 2, 3).$$

Поради (11) на (261) може да се даде още видът

$$(263) \quad \frac{d^2 \bar{q}_\nu^0}{dt^2} = \frac{D(q_1, q_2, q_3)}{D(x, y, z)} \sum_{\mu=1}^3 \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] \right. \\ \left. - \frac{\partial}{\partial q_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] \right\} \frac{\partial \bar{q}_\nu^0}{\partial q_{\mu+1}} \times \frac{\partial \bar{q}_\nu^0}{\partial q_{\mu+2}} \quad (\nu = 1, 2, 3)$$

при (6). При ортогонална криволинейна координатна система от (261), (7) следва

$$(264) \quad \frac{d^2 \bar{q}_\nu^0}{dt^2} = \sum_{\mu=1}^3 \frac{1}{\left| \frac{\partial \bar{q}_\nu^0}{\partial q_\mu} \right|} \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] - \frac{\partial}{\partial q_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] \right\} \bar{q}_\mu^0$$

( $\nu = 1, 2, 3$ ).

От (261), (249) следва

$$(265) \quad \bar{\varepsilon} = \frac{1}{2} \sum_{\mu=1}^3 \sum_{\nu=1}^3 \frac{1}{\left| \frac{\partial \bar{q}_\nu^0}{\partial q_\mu} \right|} \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] \right. \\ \left. - \frac{\partial}{\partial q_\mu} \left[ \frac{1}{2} \left( \frac{d \bar{q}_\nu^0}{dt} \right)^2 \right] \right\} (\bar{q}_\nu^0)^{-1} \times (\bar{q}_\mu^0)^{-1}.$$

При ортогонална криволинейна координатна система от (265), (7) следва

$$(266) \quad \bar{\epsilon} = \frac{1}{2} \sum_{\mu=1}^3 \sum_{\nu=1}^3 \frac{1}{\left| \frac{\partial \bar{q}_\nu^0}{\partial q_\mu} \right|} \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\mu} \left[ \frac{1}{2} \left( \frac{d\bar{q}_\nu^0}{dt} \right)^2 \right] - \frac{\partial}{\partial q_\mu} \left[ \frac{1}{2} \left( \frac{d\bar{q}_\nu^0}{dt} \right)^2 \right] \right\} \bar{q}_\nu^0 \times \bar{q}_\mu^0.$$

14. Формулите (261), (263)–(266) са неприложими за цилиндричната и сферичната координатна система поради (262) и (92)–(94), (102)–(104). Напротив, за елиптичната координатна система те са приложими, тъй като съгласно (117)–(119), (128)–(133) условията (262) са изпълнени.

При цилиндрични координати от (247), (234) следва непосредствено

$$(267) \quad \bar{\epsilon} = \dot{\varphi} \bar{k}.$$

При сферични координати от (247), (236), (105) следва

$$(268) \quad \bar{\epsilon} = (\cos \psi \dot{\varphi} \dot{\psi} + \sin \psi \dot{\varphi}) \bar{r}^0 - \dot{\psi} \bar{\varphi}^0 + (\cos \psi \ddot{\varphi} - \sin \psi \dot{\varphi} \dot{\psi}) \bar{\psi}^0.$$

При елиптични координати от (134)–(136) следва

$$(269) \quad \begin{aligned} \left( \frac{d\bar{\lambda}^0}{dt} \right)^2 &= \frac{1}{4} \left[ \frac{1}{(\lambda - \mu)^2} \frac{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \right. \\ &+ \frac{1}{(\lambda - \nu)^2} \frac{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \left. \dot{\lambda}^2 + \frac{1}{2(\lambda - \mu)^2} \dot{\lambda} \dot{\mu} \right. \\ &+ \frac{1}{2(\lambda - \nu)^2} \dot{\lambda} \dot{\nu} + \frac{1}{4(\lambda - \mu)^2} \frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \dot{\mu}^2 \\ &+ \frac{1}{4(\lambda - \nu)^2} \frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \dot{\nu}^2, \end{aligned}$$

$$(270) \quad \begin{aligned} \left( \frac{d\bar{\mu}^0}{dt} \right)^2 &= \frac{1}{4(\mu - \lambda)^2} \frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \dot{\lambda}^2 \\ &+ \frac{1}{2(\mu - \lambda)^2} \dot{\lambda} \dot{\mu} + \frac{1}{4} \left[ \frac{1}{(\mu - \lambda)^2} \frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \right. \\ &+ \frac{1}{(\mu - \nu)^2} \frac{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \left. \dot{\mu}^2 + \frac{1}{2(\mu - \nu)^2} \dot{\mu} \dot{\nu} \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4(\mu-\nu)^2} \frac{(\nu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)} \ddot{\gamma}_2, \\
 (271) \quad \left( \frac{d\bar{\lambda}^0}{dt} \right)^2 = & \frac{1}{4(\nu-\lambda)^2} \frac{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\mu-\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \dot{\lambda}^2 \\
 & + \frac{1}{2(\nu-\lambda)^2} \dot{\lambda} \dot{\nu} + \frac{1}{4(\nu-\mu)^2} \frac{(\mu-\lambda)(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\lambda-\nu)(a^2+\mu)(b^2+\mu)(c^2+\mu)} \dot{\mu}^2 \\
 & + \frac{1}{2(\nu-\mu)^2} \dot{\mu} \dot{\nu} + \frac{1}{4} \left[ \frac{1}{(\nu-\lambda)^2} \frac{(\mu-\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)} \right. \\
 & \left. + \frac{1}{(\nu-\mu)^2} \frac{(\lambda-\nu)(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\mu-\lambda)(a^2+\nu)(b^2+\nu)(c^2+\nu)} \right] \dot{\nu}^2.
 \end{aligned}$$

От (269)–(271) следва

$$\begin{aligned}
 (272) \quad \frac{\partial}{\partial \dot{\lambda}} \left[ \frac{1}{2} \left( \frac{d\bar{\lambda}^0}{dt} \right)^2 \right] = & \frac{1}{4} \left\{ \left[ \frac{1}{(\lambda-\mu)^2} \frac{(\nu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\mu-\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \right. \right. \\
 & \left. \left. + \frac{1}{(\lambda-\nu)^2} \frac{(\mu-\lambda)(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\nu-\mu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \right] \dot{\lambda} + \frac{1}{(\lambda-\mu)^2} \dot{\mu} + \frac{1}{(\lambda-\nu)^2} \dot{\nu} \right\},
 \end{aligned}$$

$$(273) \quad \frac{\partial}{\partial \dot{\mu}} \left[ \frac{1}{2} \left( \frac{d\bar{\lambda}^0}{dt} \right)^2 \right] = \frac{1}{4(\lambda-\mu)^2} \left[ \dot{\lambda} + \frac{(\mu-\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\nu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)} \dot{\mu} \right],$$

$$(274) \quad \frac{\partial}{\partial \dot{\nu}} \left[ \frac{1}{2} \left( \frac{d\bar{\lambda}^0}{dt} \right)^2 \right] = \frac{1}{4(\mu-\nu)^2} \left[ \dot{\lambda} + \frac{(\nu-\mu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\mu-\lambda)(a^2+\nu)(b^2+\nu)(c^2+\nu)} \dot{\nu} \right];$$

$$(275) \quad \frac{\partial}{\partial \dot{\lambda}} \left[ \frac{1}{2} \left( \frac{d\bar{\mu}^0}{dt} \right)^2 \right] = \frac{1}{4(\lambda-\lambda)^2} \left[ \dot{\mu} + \frac{(\lambda-\nu)(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\nu-\mu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \dot{\lambda} \right],$$

$$\begin{aligned}
 (276) \quad \frac{\partial}{\partial \dot{\mu}} \left[ \frac{1}{2} \left( \frac{d\bar{\mu}^0}{dt} \right)^2 \right] = & \frac{1}{4} \left\{ \frac{1}{(\mu-\lambda)^2} \dot{\lambda} \right. \\
 & \left. + \left[ \frac{1}{(\mu-\lambda)^2} \frac{(\nu-\mu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\mu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)} \right. \right. \\
 & \left. \left. + \frac{1}{(\mu-\nu)^2} \frac{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\nu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)} \right] \dot{\mu} + \frac{1}{(\mu-\nu)^2} \dot{\nu} \right\},
 \end{aligned}$$

$$(277) \quad \frac{\partial}{\partial \dot{\nu}} \left[ \frac{1}{2} \left( \frac{d\bar{\mu}^0}{dt} \right)^2 \right] = \frac{1}{4(\mu-\nu)^2} \left[ \dot{\mu} + \frac{(\nu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)} \dot{\nu} \right];$$

$$(278) \quad \frac{\partial}{\partial \dot{\lambda}} \left[ \frac{1}{2} \left( \frac{d\bar{\nu}^0}{dt} \right)^2 \right] = \frac{1}{4(\nu-\lambda)^2} \left[ \dot{\nu} + \frac{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\mu-\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \dot{\lambda} \right],$$

$$(279) \quad \frac{\partial}{\partial \mu} \left[ \frac{1}{2} \left( \frac{d\bar{v}^0}{dt} \right)^2 \right] = \frac{1}{4(\nu - \mu)^2} \left[ \dot{\nu} + \frac{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \dot{\mu} \right],$$

$$(280) \quad \begin{aligned} \frac{\partial}{\partial \nu} \left[ \frac{1}{2} \left( \frac{d\bar{v}^0}{dt} \right)^2 \right] &= \frac{1}{4} \left\{ \frac{1}{(\nu - \lambda)^2} \dot{\lambda} + \frac{1}{(\nu - \mu)^2} \dot{\mu} \right. \\ &+ \left[ \frac{1}{(\nu - \lambda)^2} \frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \right. \\ &\left. \left. + \frac{1}{(\nu - \mu)^2} \frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \right] \dot{\nu} \right\}. \end{aligned}$$

От (117)–(119), (128) – (133), (70) следва

$$(281) \quad \begin{aligned} \left| \frac{\partial \bar{\lambda}^0}{\partial \lambda} \right| &= \frac{1}{2(\mu - \lambda)} \frac{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \\ &+ \frac{1}{2(\nu - \lambda)} \frac{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}, \end{aligned}$$

$$(282) \quad \begin{aligned} \left| \frac{\partial \bar{\mu}^0}{\partial \mu} \right| &= \frac{1}{2(\mu - \lambda)} \frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \\ &+ \frac{1}{2(\nu - \mu)} \frac{(\lambda - \nu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}, \end{aligned}$$

$$(283) \quad \begin{aligned} \left| \frac{\partial \bar{\nu}^0}{\partial \nu} \right| &= \frac{1}{2(\nu - \lambda)} \frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \\ &+ \frac{1}{2(\nu - \mu)} \frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}, \end{aligned}$$

$$(284) \quad \left| \frac{\partial \bar{\lambda}^0}{\partial \mu} \right| = \frac{1}{2(\mu - \lambda)} \frac{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)},$$

$$(285) \quad \left| \frac{\partial \bar{\lambda}^0}{\partial \nu} \right| = \frac{1}{2(\nu - \lambda)} \frac{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)},$$

$$(286) \quad \left| \frac{\partial \bar{\mu}^0}{\partial \lambda} \right| = \frac{1}{2(\mu - \lambda)} \frac{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \nu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)},$$

$$(287) \quad \left| \frac{\partial \bar{\mu}^0}{\partial \nu} \right| = \frac{1}{2(\nu - \mu)} \frac{(\lambda - \nu)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)},$$

$$(288) \quad \left| \frac{\partial \bar{\nu}^0}{\partial \lambda} \right| = \frac{1}{2(\nu - \lambda)} \frac{(\mu - \lambda)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \mu)(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)},$$

$$(289) \quad \frac{\partial \bar{\gamma}^0}{\partial \mu} = \frac{1}{2(\nu - \mu)} \frac{(\lambda - \mu)(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}{(\nu - \lambda)(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}.$$

От (266), (70), (39), (245), (240) следва

$$(290) \quad 4\bar{\xi}\bar{\eta}\bar{\zeta}\bar{\epsilon} =$$

$$\left\{ \frac{1}{\frac{\partial \mu^0}{\partial \nu}} \left[ \frac{d}{dt} \frac{\partial}{\partial \nu} \left( \frac{d\mu^0}{dt} \right)^2 - \frac{\partial}{\partial \nu} \left( \frac{d\mu^0}{dt} \right)^2 \right] - \frac{1}{\frac{\partial \nu^0}{\partial \mu}} \left[ \frac{d}{dt} \frac{\partial}{\partial \mu} \left( \frac{d\nu^0}{dt} \right)^2 - \frac{\partial}{\partial \mu} \left( \frac{d\nu^0}{dt} \right)^2 \right] \right\} \bar{\lambda}^0$$

$$+ \left\{ \frac{1}{\frac{\partial \nu^0}{\partial \lambda}} \left[ \frac{d}{dt} \frac{\partial}{\partial \lambda} \left( \frac{d\nu^0}{dt} \right)^2 - \frac{\partial}{\partial \lambda} \left( \frac{d\nu^0}{dt} \right)^2 \right] - \frac{1}{\frac{\partial \lambda^0}{\partial \nu}} \left[ \frac{d}{dt} \frac{\partial}{\partial \nu} \left( \frac{d\lambda^0}{dt} \right)^2 - \frac{\partial}{\partial \nu} \left( \frac{d\lambda^0}{dt} \right)^2 \right] \right\} \bar{\mu}^0$$

$$+ \left\{ \frac{1}{\frac{\partial \lambda^0}{\partial \mu}} \left[ \frac{d}{dt} \frac{\partial}{\partial \mu} \left( \frac{d\lambda^0}{dt} \right)^2 - \frac{\partial}{\partial \mu} \left( \frac{d\lambda^0}{dt} \right)^2 \right] - \frac{1}{\frac{\partial \mu^0}{\partial \lambda}} \left[ \frac{d}{dt} \frac{\partial}{\partial \lambda} \left( \frac{d\mu^0}{dt} \right)^2 - \frac{\partial}{\partial \lambda} \left( \frac{d\mu^0}{dt} \right)^2 \right] \right\} \bar{\nu}^0$$

при (281)–(289), (272)–(280), (269)–(271).

### 15. Нека

$$(291) \quad \bar{a}_\nu = \bar{a}_\nu(t) \quad (\nu = 1, 2, 3)$$

са векторни функции, за които

$$(292) \quad \bar{a}_1 \times \bar{a}_2 \cdot \bar{a}_3 \neq 0,$$

$$(293) \quad \frac{d}{dt} (\bar{a}_\mu \bar{a}_\nu) = 0 \quad (\mu, \nu = 1, 2, 3)$$

за всяко  $t$ . Необходимо и достатъчно условие за

$$(294) \quad \bar{a}_\nu^{-1} = \bar{a}_\nu \quad (\nu = 1, 2, 3)$$

е

$$(295) \quad \bar{a}_\mu \bar{a}_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \quad (\mu, \nu = 1, 2, 3)$$

и при (295) моменталната ъглова скорост на репера (291) с (293) е

$$(296) \quad \bar{\omega} = \frac{1}{2} \sum_{\nu=1}^3 \bar{a}_\nu \times \frac{d\bar{a}_\nu}{dt}.$$

Нека специално

$$(297) \quad \bar{a}_1 = \tau^0, \quad \bar{a}_2 = \bar{\gamma}^0, \quad \bar{a}_3 = \bar{\beta}^0$$

са единичните вектори на триедъра на Frenet. От формулите на Frenet

$$(298) \quad \frac{d\bar{\tau}^0}{ds} = \kappa \bar{\gamma}^0,$$

$$(299) \quad \frac{d\bar{\nu}^0}{ds} = -\kappa \bar{\tau}^0 + \sigma \bar{\beta}^0,$$

$$(300) \quad \frac{d\bar{\beta}^0}{ds} = -\sigma \bar{\gamma}^0,$$

където  $s$  е дължина на дъга, а  $\kappa$  и  $\sigma$  са съответно кривината и торзията, следва, че равенството (296) приема вида

$$(301) \quad \omega = \frac{1}{2} (\kappa \bar{\tau}^0 \times \bar{\gamma}^0 - \kappa \bar{\gamma}^0 \times \bar{\tau}^0 + \sigma \bar{\gamma}^0 \times \bar{\beta}^0 - \sigma \bar{\beta}^0 \times \bar{\gamma}^0) \frac{ds}{dt}.$$

От (301) и

$$(302) \quad \bar{\tau}^0 \times \bar{\gamma}^0 = \bar{\beta}^0, \quad \bar{\gamma}^0 \times \bar{\beta}^0 = \bar{\tau}^0$$

следва

$$(303) \quad \omega = (\sigma \bar{\tau}^0 + \kappa \bar{\beta}^0) \frac{ds}{dt}.$$

Векторът

$$(304) \quad \delta = \sigma \bar{\tau}^0 + \kappa \bar{\beta}^0$$

се нарича вектор на Darboux. От (303), (304) се вижда, че  $\delta$  представлява ъгловата скорост на репера (297) на Frenet, когато траекторията се описва с постоянна бързина, равна на единица:

$$(305) \quad \frac{ds}{dt} = 1.$$

От (247), (298), (300), (303) следва

$$(306) \quad \bar{\varepsilon} = \left[ \frac{d\sigma}{ds} \left( \frac{ds}{dt} \right)^2 + \sigma \frac{d^2 s}{dt^2} \right] \bar{\tau}^0 + \left[ \frac{d\kappa}{ds} \left( \frac{ds}{dt} \right)^2 + \kappa \frac{d^2 s}{dt^2} \right] \bar{\beta}^0.$$

Ако движението е равномерно, т. е.

$$(307) \quad \frac{d^2 s}{dt^2} = 0,$$

от (306) следва

$$(308) \quad \bar{\varepsilon} = \left[ \frac{d\sigma}{ds} \bar{\tau}^0 + \frac{d\kappa}{ds} \bar{\beta}^0 \right] \left( \frac{ds}{dt} \right)^2.$$

При (305) равенството (308) приема вида

$$(309) \quad \bar{\varepsilon} = \frac{d\sigma}{ds} \bar{\tau}^0 + \frac{d\kappa}{ds} \bar{\beta}^0.$$

От (303) следва

$$(310) \quad \bar{\omega} \bar{\gamma}^0 = 0.$$

От (303), (299) следва също

$$(311) \quad \bar{\omega} \frac{d\bar{\gamma}^0}{ds} = 0.$$

Резултатът (306) може да се изведе и директно без помощта на (298), (300). Наистина известно е, че

$$(312) \quad \frac{d\bar{\omega}}{dt} = \frac{\delta \bar{\omega}}{\delta t},$$

където при (296)  $\frac{\delta \bar{\omega}}{\delta t}$  е локалната производна на  $\bar{\omega}$  спрямо репера (291) с (293). От (312), (303) следва

$$(313) \quad \frac{d\bar{\omega}}{dt} = \frac{d}{dt} \left( \sigma \frac{ds}{dt} \right) \bar{\gamma}^0 + \frac{d}{dt} \left( \chi \frac{ds}{dt} \right) \bar{\beta}^0.$$

Равенството (306) следва от (313) и (247).

16. Последната забележка в предната точка позволява да се избегнат сложните пресмятания, чрез които в т. 14 бе намерен изразът (290) за моменталното ъглово ускорение  $\bar{\epsilon}$  в случая на елиптична координатна система. Същото важи, разбира се, и за сферични координати и изобщо за всяка криволинейна координатна система  $q_v$  ( $v=1, 2, 3$ ) с „твърд“ репер (3), т. е. за който е в сила (221).

Наистина непосредствено се вижда, че изразът (268) за моменталното ъглово ускорение  $\bar{\epsilon}$  в сферични координати е равен на локалната производна на моменталната ъглова скорост  $\bar{\omega}$ , определена с (236) спрямо репера  $\bar{r}^0, \bar{\varphi}^0, \bar{\psi}^0$ :

$$(314) \quad \bar{\epsilon} = \frac{d}{dt} (\sin \psi \dot{\varphi}) \bar{r}^0 - \frac{d\psi}{dt} \bar{\varphi}^0 + \frac{d}{dt} (\cos \psi \dot{\varphi}) \bar{\psi}^0.$$

От (312), (246), (240) следва

$$(315) \quad 2\xi\eta\zeta\bar{\epsilon} = \frac{d}{dt} \left\{ \frac{1}{\mu-\nu} \left[ \sqrt{\frac{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\nu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)}} \cdot \right. \right. \\ \left. \left. + \sqrt{\frac{(\nu-\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}} \cdot \right] \right\} \bar{\lambda}^0 \\ + \frac{d}{dt} \left\{ \frac{1}{\nu-\lambda} \left[ \sqrt{\frac{(\mu-\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}} \cdot \right. \right. \\ \left. \left. + \sqrt{\frac{(\lambda-\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\mu-\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \cdot \right] \right\} \bar{\mu}^0$$

$$+\frac{d}{dt} \left\{ \frac{1}{\lambda-\mu} \left[ \sqrt{\frac{(\lambda-\nu)(a^2+\mu)(b^2+\lambda)(c^2+\mu)}{(\nu-\mu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} ; \right. \right.$$

$$\left. \left. + \sqrt{\frac{(\nu-\mu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\lambda-\nu)(a^2+\mu)(b^2+\mu)(c^2+\mu)}} \mu \right] \right\} \nu^0.$$

Горната забележка може да се отрази и като друга обща формула за моменталното ъглово ускорение  $\bar{\epsilon}$ . За целта е достатъчно да се даде разложение

$$(316) \quad \bar{\omega} = \sum_{r=1}^3 [\bar{\omega} (\bar{q}_r^0)^{-1}] \bar{q}_r^0$$

на моменталната ъглова скорост  $\bar{\omega}$  в репера (3); тогава от (316), (312), (247) следва

$$(317) \quad \bar{\epsilon} = \sum_{r=1}^3 \frac{d}{dt} [\bar{\omega} (\bar{q}_r^0)^{-1}] \bar{q}_r^0.$$

Тук обаче не ще привеждаме експлицитно такова разложение (316) на вектора  $\bar{\omega}$ .

16. Във връзка с горните разглеждания естествено възникват два въпроса, които тук само ще скицираме. И двата се отнасят до ортогонална система криволинейни координати  $q_r$  ( $r = 1, 2, 3$ ).

Първият въпрос засяга дефиницията на *естествена криволинейна координатна система*. Така ще наричаме ортогонална система криволинейни координати, за която единичните тангенциални вектори (3) към координатните линии съвпадат съответно с векторите  $\bar{t}^0$ ,  $\bar{n}^0$ ,  $\bar{\beta}^0$  на триедъра на Frenet.

Вторият въпрос засяга целесъобразна дефиниция на Euler'овите ъгли  $\psi$ ,  $\varphi$ ,  $\theta$  и на линията на възлите  $\bar{\gamma}^0$  на подвижния ортогонален репер (3) спрямо неподвижния ортогонален репер  $\bar{i}$ ,  $\bar{j}$ ,  $\bar{k}$ , както и свързаните с Euler'овите ъгли кинематични въпроси, като например представянето на моменталната ъглова скорост  $\bar{\omega}$  във вида

$$(318) \quad \bar{\omega} = \dot{\psi} \bar{k} + \dot{\vartheta} \bar{\gamma}^0 + \dot{\varphi} \bar{\zeta}^0$$

при съответна дефиниция на  $\bar{\zeta}^0$  като един от векторите (3).

Решението на първия въпрос позволява създаването на общ метод за построяване на различни ортогонални системи криволинейни координати. Вторият въпрос има отражение в динамиката на тела в криволинейни координати.

В настоящата работа се ограничихме до кинематични разглеждания. Специалните системи криволинейни координати позволяват развитието на обща теория на динамиката на точка, подчинена на кон-

кретни крайни връзки — постоянен цилиндър, сфера, конус, елипсоид, прост и двоен хиперболоид и пр.

Дискусионен остава въпросът, коя от двете форми на ускорението — (164) или (169) — е за предпочтение за извеждане на динамичните уравнения на материална точка. И в двета случая силата  $\bar{F}$  следва да се представи във вида

$$(319) \quad \bar{F} = \sum_{\nu=1}^3 (\bar{F} \bar{q}_\nu^0) (\bar{q}_\nu^0)^{-1}.$$

Вярно е, че при работа с уравненията (169) следва да се преесмята „енергията на ускорението“, докато в уравненията (164) фигурира само „кинетичната енергия“. За прилагането на уравненията (164) обаче се налага допълнително диференциране спрямо криволинейните координати  $q_\nu$  ( $\nu = 1, 2, 3$ ) за намиране на частните производни

$$(320) \quad \frac{\partial}{\partial q_\nu} \left( \frac{v^2}{2} \right) \quad (\nu = 1, 2, 3),$$

както и за намиране на производните спрямо времето

$$(321) \quad \frac{d}{dt} \frac{\partial}{\partial q_\nu} \left( \frac{v^2}{2} \right) \quad (\nu = 1, 2, 3),$$

докато при (169) намирането на производните

$$(322) \quad \frac{\partial}{\partial \ddot{q}_\nu} \left( \frac{w^2}{2} \right) \quad (\nu = 1, 2, 3)$$

е съвсем просто поради квадратната зависимост

$$(323) \quad w^2 = \sum_{\nu=1}^3 \sum_{\lambda=1}^3 \left( \ddot{q}_\nu \frac{\partial \bar{r}}{\partial \bar{q}_\nu} + \sum_{\mu=1}^3 \dot{q}_\nu \dot{q}_\mu \frac{\partial^2 \bar{r}}{\partial \bar{q}_\nu \partial \bar{q}_\mu} \right) \\ \times \left( \ddot{q}_\lambda \frac{\partial \bar{r}}{\partial \bar{q}_\lambda} + \sum_{\kappa=1}^3 \dot{q}_\lambda \dot{q}_\kappa \frac{\partial^2 \bar{r}}{\partial \bar{q}_\lambda \partial \bar{q}_\kappa} \right)$$

на  $w^2$  от  $\ddot{q}_\nu$  ( $\nu = 1, 2, 3$ ) съгласно (156).

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## ZUR PUNKTKINEMATIK IN KRUMMLINIGEN KOORDINATEN

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(ZUSAMMENFASSUNG)

Die vorliegende Arbeit enthält die Grundelemente der Punktkinematik in krummlinigen Koordinaten  $q_\nu$  ( $\nu = 1, 2, 3$ ), direkt ausgedrückt durch den Grundvektor  $\bar{q}_\nu^0$  ( $\nu = 1, 2, 3$ ) der Einheitstangentialvektoren zu den

Koordinatenlinien und durch den reziproken Reper  $(\bar{q}_v^0)^{-1}$  ( $v=1, 2, 3$ ), definiert durch (5), (6), ohne Zuhilfenahme des üblichen Descartesschen Orthogonalrepers  $\bar{i}, \bar{j}, \bar{k}$ . Der Radiusvektor  $r$  des beweglichen Punktes wird durch (13), (10) ausgedrückt und im Falle eines orthogonalen krummlinigen Koordinatensystems — durch (14). Die Ableitungen  $\frac{\partial \bar{q}_v^0}{\partial q_\mu}$  ( $\mu, v=1, 2, 3$ ) der Einheitstangentialvektoren zu den Koordinatenlinien in bezug auf die krummlinigen Koordinaten sind durch (81) oder (82) gegeben; diese Ausdrücke gehen bei orthogonalem krummlinigen Koordinatensystem

in (83) und (84) über. Die Ableitungen  $\frac{d\bar{q}_v^0}{dt}$  ( $v=1, 2, 3$ ) der Einheitstangentialvektoren zu den Koordinatenlinien werden durch (86) und (87) gegeben, wobei wieder diese Ausdrücke bei orthogonalem System in (88) übergehen. Die Formel (88) ermöglicht die direkte Auswertung der Ableitungen beliebiger Ordnung  $\frac{d^n \bar{r}}{dt^n}$  des Radiusvektors  $\bar{r}$  des beweglichen Punktes in bezug auf die Zeit  $t$ , zerlegt im Grundreper  $\bar{q}_v^0$  ( $v=1, 2, 3$ ), wieder ohne Zuhilfenahme des Descartesschen Repers  $\bar{i}, \bar{j}, \bar{k}$ . Unter der Bedingung (184) sind die Identitäten (202)–(204) mit (205) abgeleitet. Bei unelastischem Reper  $\bar{q}_v^0$  ( $v=1, 2, 3$ ), d. h. bei dem (221), gilt, wird für die momentane Winkelgeschwindigkeit  $\bar{\omega}$  des Repers  $\bar{q}_v^0$  ( $v=1, 2, 3$ ) der Ausdruck (224) abgeleitet, der bei orthogonalem Koordinatensystem in (225) übergeht, bzw. die Ausdrücke (231), (232); für die momentane Winkelbeschleunigung  $\bar{\epsilon}$  gilt (258), in orthogonalem System — (259). Diese Ausdrücke können auch als (265), (266) dargestellt werden. Die obigen Betrachtungen werden durch Beispiele meistens in elliptischem krummlinigem Koordinatensystem illustriert. Man zeigt, daß zur Auswertung von  $\bar{\epsilon}$  am zweckmäßigsten die Gleichung (312) zu benutzen ist, nämlich zwischen der absoluten und der lokalen Ableitung der momentanen Winkelgeschwindigkeit  $\bar{\omega}$  in bezug auf den Reper  $\bar{q}_v^0$  ( $v=1, 2, 3$ ). Es wird der Begriff des natürlichen krummlinigen Koordinatensystems eingeführt, d. h. desjenigen, für welches der Reper  $\bar{q}_v^0$  ( $v=1, 2, 3$ ) mit dem Frenetschen Reper zusammenfällt, was die Konstruktion von neuen breiten Klassen von krummlinigen Koordinatensystemen ermöglicht. Es wird auch die Frage angedeutet über die Einführung von Eulerschen Winkeln  $\varphi, \psi, \theta$  für den beweglichen Reper  $\bar{q}_v^0$  ( $v=1, 2, 3$ ) und den unbeweglichen Descartesschen Reper  $\bar{i}, \bar{j}, \bar{k}$ .