ON A PROPERTY OF A CLASS OF LINEAR POSITIVE OPERATORS

Blagovest Sendov and Borislav D. Boyanov

In [1] is given the following definition of left, respectively right, slant convergence.

Definition 1. Suppose $f_1, f_2, \dots$ is a sequence each term of which is a real-valued function on [0, 1] and x is in [0, 1]. The statement that $f_1, f_2, \dots$ left slant converges at x means that there is a number L_x so that if $\varepsilon > 0$ there exists a number Δ, $0 \leqq \Delta < x$, so that if $\Delta < d < x$, there is a positive number N such that if n is an integer greater than N and $\Delta \leqq q \leqq d$, then $|f_n(q) - L_x| < \varepsilon$.

The definition for right slant convergence is entirely analogous.

Definition 2. We say that the operators $L_n(f; x)$ $(n = 1, 2, \dots)$ are operators of Korovkin type if

a) L_n is a linear positive operator,

b) $L_n(1; x) = 1 + \alpha_n(x)$,
$L_n(t; x) = x + \beta_n(x)$,
$L_n(t^2; x) = x^2 + \gamma_n(x)$,

where $\alpha_n(x)$, $\beta_n(x)$, $\gamma_n(x)$ converge uniformly to zero on the finite interval $[a, b]$.

We shall now prove

Theorem 1. Let $\{L_n(f; x)\}_1^\infty$ be a sequence of operators of Korovkin type and let the functions

$$(*) \qquad h_z = \begin{cases} 1, & 0 \leqq x < z \\ 0, & z \leqq x \leqq 1 \end{cases}, \quad 0 < z < 1,$$

are contained in the domain of the operators L_n $(n = 1, 2, \dots)$. Suppose $f(x)$ is a bounded real-valued function on [0, 1] and lies in the domain of the given operators. If $f(t-0)$ exists for some $t \in (0, 1)$, then the sequence $\{L_n(f; x)\}_1^\infty$ left slant converges at t to $f(t-0)$.

The particular case where L_n is the Bernstein polynomial operator is discussed in [1].

In the proof of Theorem 1. we shall make use of a result of one of the authors [2] concerning the approximation of a function relative to the Hausdorff metric. We now recall some definitions. We refer the reader to [2] and [3] for details and proofs of the matter summarized in this section.

Definition 3. Let $f(x)$ be a bounded function defined on an interval $[a, b]$. By the completed graph $\bar{f}$ of the function $f(x)$ we mean the intersection of all closed and convex (relative to the Y axis) point sets in the plane which contain the graph of the function $f(x)$. Notice that the completed graph of a continuous function coincides with its graph.

Definition 4. For each pair of bounded functions $f(x)$ and $g(x)$ we define the Hausdorff distance $r(f, g)$ as

$$r(f, g) = \max\{\max_{X \in \bar{f}} \min_{Y \in \bar{g}} \|X-Y\|_0,\ \max_{X \in \bar{g}} \min_{Y \in \bar{f}} \|X-Y\|_0\},$$

where

$$\|X-Y\|_0 = \|X(x_1, y_1) - Y(x_2, y_2)\|_0 = \max[|x_1-x_2|, |y_1-y_2|].$$

Definition 5. Denote by $\mu(f; \delta)$ the modulus of non-monotonicity of the function $f(x)$:

$$\mu(f; \delta) = \sup_{|x_1-x_2| \leqq \delta} \{\sup_{x_1 \leqq x \leqq x_2} [|f(x_1)-f(x)| + |f(x_2)-f(x)|] - |f(x_1)-f(x_2)|\}.$$

We say that the function $f(x)$ is locally monotonic if

$$\lim_{\delta \to 0} \mu(f; \delta) = 0.$$

Theorem 2. Let $L_n(f; x)$ $(n=1, 2\ldots)$ be a sequence of operators of Korovkin type. If the function $f(x)$ defined on the interval $[a, b]$ is locally monotonic, continuous at the points a and b and lies in the domain of definition of the operators L_n $(n=1, 2, \ldots)$, then the sequence $\{L_n(f; x)\}_1^\infty$ converges relative to the Hausdorff distance to the function $f(x)$.

The proof is given in [2].

We are able now to prove Theorem 1.

Proof. Let ε be an arbitrary positive number. Denote

$$M = \sup_{x \in [0, 1]} |f(x)|, \quad T = f(t-0), \quad \varepsilon_0 = \varepsilon/4.$$

Since $f(t-0)$ exists there is a number $y \in (0, t)$, such that $|f(x)-T| < \varepsilon_0$ for $y \leqq x < t$.

It is seen that the functions

$$\theta_1(x) = \begin{cases} M & 0 \leqq x < y \\ T+\varepsilon_0 & y \leqq x < t \\ M & t \leqq x \leqq 1 \end{cases} \qquad \theta_2(x) = \begin{cases} -M & 0 \leqq x < y \\ T-\varepsilon_0 & y \leqq x < t \\ -M & t \leqq x \leqq 1 \end{cases}$$

are locally monotonic, lie in the domain of definition of L_n and satisfy the inequality

$$\theta_1(x) \leqq f(x) \leqq \theta_2(x) \qquad x \in (0, 1]$$

It follows from the above inequality and the positivity of L_n that

(1) $$L_n(\theta_1;\, x) \leqq L_n(f;\, x) \leqq L_n(\theta_2;\, x).$$

Chose $\Delta = y+(t-y)/4$. Let $\Delta < d < t$. By Theorem 2 there exists a positive integer N, so that if $n > N$ then

$$r(L_n(\theta_i),\ \theta_i) < \varepsilon_1 \qquad (i=1, 2),$$

where $\varepsilon_1 = \min\,[\varepsilon_0,\ (t-d)/4]$. Since $\theta_1(x)$ and $\theta_2(x)$ are constants on $[\Delta,\ d]$ we get

$$T-\varepsilon_0-\varepsilon_1 \leqq L_n(\theta_1;\, x), \quad L_n(\theta_2;\, x) \leqq T+\varepsilon_0+\varepsilon_1$$

for $x \in [\Delta,\ d]$. This with (1) gives

$$|L_n(f;\, x)-T| \leqq \varepsilon_0+\varepsilon_1 < \varepsilon \qquad x \in [\Delta,\ d],$$

which is what we wanted to show.

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ОБ ОДНОМ СВОЙСТВЕ ОДНОГО КЛАССА ЛИНЕЙНЫХ ПОЛОЖИТЕЛЬНЫХ ОПЕРАТОРОВ

Бл. Сендов и Б. Д. Боянов

(РЕЗЮМЕ)

Исследовано свойство косой (slant) сходимости одной последовательности функции в данной точке. Доказана следующая

Теорема. Пусть $\{L_n(f;x)\}_1^\infty$ последовательность линейных положительных операторов типа Коровкина и пусть функции (*) $h_z(x)$ принадлежат области определения D операторов L_n, $n=1, 2, \ldots$. Пусть $f(x)$ вещественная функция, определенная и ограниченная на (0, 1) и $f(x) \in D$. Если $f(t-0)$ существует для некоторого $t \in (0, 1)$, то последовательность $\{L_n(f;\ x)\}_1^\infty$ стремится косо слева к $f(t-0)$ в точке t.