ON TURING COMPUTABLE OPERATORS*

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We shall consider Turing computable operators not only on total functions, but also on partially defined ones.

I. Let **N** be the set of the natural numbers and **P** be the set of all functions the domains and the ranges of which are subsets of **N**. By **F** will be denoted the set of all functions belonging to **P** the domains of which are equal to **N**. We shall always identify the elements of **P** with their graphs.

The elements of **P** the domains of which are finite can be effectively listed in a sequence

$$f_0, f_1, f_2, \ldots$$

We may, for example, set

$$f_n = \{<x, y> \quad (n+1)_x = y+1\}.$$

Let m be a positive integer and let R and F be given such that

$$R \subset \mathbf{P}^m, \quad F: R \dashrightarrow \mathbf{P}.$$

The mapping F is called a partial recursive operator if there exists a recursively enumerable subset H of $\mathbf{N}^{m+2}$ such that for all $\psi_1, \ldots, \psi_m$ belonging to **P** and satisfying the condition

$$<\psi_1, \ldots, \psi_m> \in R$$

the following equality holds

$$F(\psi_1, \ldots, \psi_m) = \{<x, y> \quad \exists n_1 \ldots \exists n_m (f_{n_1} \subset \psi_1 \& \ldots$$
$$\& f_{n_m} \subset \psi_m \& <n_1, \ldots, n_m, x, y> \in H)\}.$$

If the domain R of F coincides with the whole $\mathbf{P}^m$, then F is called a recursive operator from $\mathbf{P}^m$ into **P**.

Let $\psi_1, \ldots, \psi_m, \varphi$ belong to **P**. The function φ is *partial recursive* in $<\psi_1, \ldots, \psi_m>$ if there exists a partial recursive operator F such that

$$<\psi_1, \ldots, \psi_m> \in \operatorname{Dom} F$$

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* This is an extended version of a lecture given by the author on February 15, 1973 at the Stefan Banach International Mathematical Center for Raising Research Qualifications in Warsaw.

and

$$\varphi = F(\psi_1,\ldots, \psi_m);$$

φ is strongly partial recursive in $<\psi_1,\ldots, \psi_m>$ if the same equation holds for some recursive operator F.

If $\psi_1,\ldots, \psi_m$ are elements of **F**, then by Kleene's Normal form theorem (Theorem XIX of [2]) every function φ belonging to **P** and partial recursive in $\Psi = <\psi_1,\ldots, \psi_m>$* has a representation of the form

$$\varphi(x) \simeq U(\mu y\, T_1{}^{\Psi}(e, x, y))$$

and hence φ is strongly partial recursive in Ψ. Myhill [4] has, however, shown that there exist $\psi \in$ **P** and $\varphi \in$ **F** such that φ is partial recursive in ψ but φ is not strongly partial recursive in ψ.

II. I shall give only an intuitive definition of the notion of Turing computable operator because the notion of Turing machine is well-known and it is a routine work to formulate the precise formal definition for Turing computability of operators and to make the proofs agree with it (Turing computable operators are considered, for example, in [2], § 67, in the form of uniform computability in some given functions; it is not very essential that these functions are assumed to be total).

Let F be a mapping of $\mathbf{P}^m$ into **P**. The mapping F is called a Turing computable operator from $\mathbf{P}^m$ into **P** if there exists a deterministic procedure for computing $F(\psi_1,\ldots, \psi_m)(x)$ for any given $\psi_1,\ldots, \psi_m$, x and this procedure is fully algorithmical, except that at some stages of the computation the value of some ψ_i at some point y can be demanded and the computation cannot go on before this value is given to the computer (in the cases where some demanded value $\psi_i(y)$ happens to be undefined the value $F(\psi_1,\ldots, \psi_m)(x)$ must be undefined too). A function φ is called Turing computable in some m-tuple of functions $<\psi_1,\ldots, \psi_m>$ if $\varphi = F(\psi_1,\ldots, \psi_m)$ for some Turing computable operator F.

A mapping F of $\mathbf{P}^m$ into **P** is called μ-recursive if $F(\psi_1,\ldots, \psi_m)$ is obtainable from $\psi_1,\ldots, \psi_m$ and the initial primitive recursive functions by means of a fixed succession of substitutions, primitive recursions and applications of the μ-operator.

Teorem 1. An operator is Turing computable iff it is μ-recursive.

The proof is essentially contained in § 68 and § 69 of [2].

Theorem 2. Every μ-recursive operator is recursive.

This is Theorem XVIII of [2].

III. By Theorem 1 and Theorem 2, Turing computable operators and μ-recursive operators are the same thing and they all are recursive operators. It is natural to ask whether all this three classes of operators coincide.

From Kleene's Normal form theorem it follows that on functions belonging to **F** every recursive operator coincides with some μ-recursive one. The situation is, however, different in the case where we consider

* The definition of relative partial recursiveness given above is equivalent to the definition given in [2].

the operators also on partially defined functions. This was shown by A. V Kuznecov and me in 1961* as an answer to a question of V. A. Uspenskii. I shall give now some examples.

Example 1. Let F be the mapping of $\mathbf{P}^2$ into $\mathbf{P}$ defined by the equation

$$F(\psi_1, \psi_2) = (\mathrm{Dom}\,\psi_1 \cup \mathrm{Dom}\,\psi_2) \times \{0\}.$$

It is not difficult to prove that F is a recursive operator but this operator is not Turing computable (Hint: Suppose that F is Turing computable and consider the first demanded value $\psi_i(y)$ in the computation of $F(\psi_1, \psi_2)(0)$. It is obvious that i and y will not depend from the choice of ψ_1 and ψ_2. Making use of that, choose such ψ_1 and ψ_2 that $F(\psi_1, \psi_2)(0)$ is defined but $\psi_i(y)$ is undefined).

Example 2. Consider the mapping G of $\mathbf{P}$ into $\mathbf{P}$ defined by the equation

$$G(\psi) = (\mathrm{Range}\,\psi) \times \{0\}.$$

We can prove in a quite similar way as above that G is also a recursive operator which is not Turing computable.

IV. We shall give now a necessary and sufficient condition for the Turing computability of a recursive operator (this condition is a modification of a condition given in [11]).

Theorem 3 Let m be a positive integer. Then for every recursive operator F from $\mathbf{P}^m$ into $\mathbf{P}$, the following two conditions are equivalent:

(i) F is Turing computable.

(ii) There exists a partial recursive function χ of $m+1$ variables such that for all $n_1, \ldots, n_m, x$

a) $\exists \psi_1 \ldots \exists \psi_m (f_{n_1} \subset \psi_1 \,\&\, \ldots \,\&\, f_{n_m} \subset \psi_m \,\&\, x \in \mathrm{Dom}\,F(\psi_1, \ldots, \psi_m)$

$\Rightarrow (n_1, \ldots, n_m, x) \in \mathrm{Dom}\,\chi$;

b) $\chi(n_1, \ldots, n_m, x) = 0 \Rightarrow x \in \mathrm{Dom}\,F(f_{n_1}, \ldots, f_{n_m})$;

c) $\forall y \forall i [1 \leq i \leq m \,\&\, \chi(n_1, \ldots, n_m, x) = my + i \Rightarrow y \in \mathrm{Dom}\,f_{n_i}$

$\&\, \forall \psi_1 \ldots \forall \psi_m (f_{n_1} \subset \psi_1 \,\&\, \ldots \,\&\, f_{n_m} \subset \psi_m \,\&\, x \in \mathrm{Dom}\,F(\psi_1, \ldots, \psi_m) \Rightarrow y \in$

$\mathrm{Dom}\,\psi_{n_i})]$.

Proof. Let F be a recursive operator from $\mathbf{P}^m$ into $\mathbf{P}$. Then the functions ρ defined by the equation

$$\rho(n_1, \ldots, n_m, x) \simeq F(f_{n_1}, \ldots, f_{n_m})(x)$$

is a partial recursive function. If condition (i) is satisfied, then we define the function χ in the following way:

* It seems that the fact has been known earlier to Orlovskii and Lacombe (see [10] and [5], 8.4).

$$\chi(n_1,\ldots, n_m, x) \simeq \begin{cases} 0, & \text{if } x \notin \operatorname{Dom} F(f_{n_1},\ldots, f_{n_m});\\ my+i, & \text{if } f_{n_i}(y) \text{ is the first demanded but not defined value in the Turing computation of } F(f_{n_1},\ldots, f_{n_m})(x). \end{cases}$$

The function χ is partial recursive and χ satisfies the conditions a), b), c). In order to prove the implication (ii) $\Rightarrow$ (i), suppose that χ is a function of $m+1$ variables such that conditions a), b), c) are satisfied. Then we can use the following procedure for computing $z=F(\psi_1,\ldots, \psi_m)(x)$ for any given $\psi_1,\ldots, \psi_m, x$:

1) $n_1:=a$, $n_2:=a,\ldots, n_m:=a$, where a is a natural number such that $f_a=\varnothing$.

2) $h:=\chi(n_1,\ldots, n_m, x)$.

3) If $h=0$, then go to instruction 4), else go to instruction 6).

4) $z:=\rho(n_1,\ldots n_m, x)$.

5) Stop.

6) $y:=\left[\dfrac{h-1}{m}\right]$.

7) $i:=h-my$.

8) $w:=\psi_i(y)$.

9) Find a natural number n such that $f_n=f_{n_i}\cup\{<y, w>\}$.

10) $n_i:=n$.

11) Go to instruction 2).

Corollary. Let m be a positive integer and let F and G be recursive operators from $\mathbf{P}^m$ into $\mathbf{P}$. If

$$\forall \psi_1 \ldots \forall \psi_m (\operatorname{Dom} F(\psi_1,\ldots, \psi_m)=\operatorname{Dom} G(\psi_1,\ldots, \psi_m))$$

and the operator F is Turing computable, then the operator G is Turing computable too.

V. The above mentioned question of Uspenskii had a second, more difficult, part. It was the following: is it possible to construct a pair of functions φ and ψ such that φ is strongly partial recursive in ψ, but φ is not Turing computable in ψ? The question was answered positively and a proof of the result can be found in [11]. However, I prefer now another proof which gives us a simpler example of such pair. This proof* is based on two lemmas.

Lemma 1. Let φ and ψ belong to $\mathbf{P}$ and let the following conditions be satisfied:

a) ψ is potentially partial recursive;

b) φ is Turing computable in ψ;

c) $\operatorname{Dom}\varphi$ is a productive set.

Then $\operatorname{Dom}\psi$ has an infinite recursively enumerable subset.

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* I am describing the proof in a form such that classical logic is used. A variant of the proof using intuitionistic logic can be obtained by slight modifications.

Proof. Let F be a Turing computable operator such that $\varphi = F(\psi)$. By Theorem 3, there exists a partial recursive function χ of two variables such that for all n, x

a) $f_n \subset \psi \,\&\, x \in \mathrm{Dom}\, F(\psi) \Rightarrow \langle n, x \rangle \in \mathrm{Dom}\, \chi$;

b) $\chi(n, x) = 0 \Rightarrow x \in \mathrm{Dom}\, F(f_n)$;

c) $\forall y (\chi(n, x) = y + 1 \,\&\, f_n \subset \psi \,\&\, x \in \mathrm{Dom}\, F(\psi) \Rightarrow y \bar{\in} \mathrm{Dom}\, f_n \,\&$
$y \in \mathrm{Dom}\, \psi)$.

Then, given any finite subset E of $\mathrm{Dom}\, \psi$, we can effectively construct a point y of $(\mathrm{Dom}\, \psi) - E$ in the following way: we find an n such that $f_n = \psi \upharpoonright E$ and consider $\mathrm{Dom}\, F(f_n)$ which will be a recursively enumerable subset of the productive set $\mathrm{Dom}\, \varphi$; then we find a point $x \in (\mathrm{Dom}\, \varphi) - (\mathrm{Dom}\, F(f_n))$ and set $y = \chi(n, x) - 1$.

Lemma 2. There exists a $\psi \in \mathbf{P}$ such that

a) ψ is potentially primitive recursive;

b) $\mathrm{Dom}\, \psi$ has no infinite recursively enumerable subset;

c) $\mathrm{Range}\, \psi$ is a productive set;

d) $\mathrm{Dom}\, \psi$ and $\mathrm{Range}\, \psi$ are complements of some recursively enumerable subsets of $\mathbf{N}$.

Proof. We take an immune set A and a productive set B such that A and B are complements of some recursively enumerable subsets of $\mathbf{N}$ and then we set

$$\psi = \{\langle 2^u . 3^v, v \rangle \mid u \in A \,\&\, v \in B \,\&\, u > v\}.$$

The main result is the following:

Theorem 4. There exist φ and ψ belonging to $\mathbf{P}$ such that φ is strongly partial recursive in ψ, φ is not Turing computable in ψ and the following additional conditions are satisfied:

a) ψ is potentially primitive recursive;

b) $\mathrm{Dom}\, \psi$ and $\mathrm{Range}\, \psi$ are complements of some recursively enumerable subsets of $\mathbf{N}$.

Proof. Take ψ as in Lemma 2 and set

$$\varphi = (\mathrm{Range}\, \psi) \times \{0\}.$$

Remark 1. From this proof it follows once more that the operator G considered in Example 2 is not Turing computable.

VI. We shall now consider a kind of reducibility which will be called partial c-reducibility.

Let all finite subsets of $\mathbf{N}$ be effectively listed in the sequence

$$s_0, s_1, s_2, \ldots$$

We can, for example, set

$$s_n = \{x_1, x_2, \ldots, x_k\},$$

where $x_1 < x_2 < \ldots < x_k$ and $n = 2^{x_1} + 2^{x_2} + \ldots + 2^{x_k}$.

Let $A_0, A_1,\ldots, A_m$ be some subsets of $\mathbf{N}$. The set A_0 is partially c-reducible to $<A_1,\ldots, A_m>$ if there exist partial recursive functions $\theta_1,\ldots, \theta_m$ such that

$$A_0=\{x\ \ \exists n_1\ldots\exists n_m(\theta_1(x)=n_1\ \&\ldots\&\ \theta_m(x)=n_m$$

$$\&\ s_{n_1}\subset A_1\ \&\ldots\&\ s_{n_m}\subset A_m)\}.$$

If we omit the words "partially" and "partial" in this definition, then we shall have a notion studied by Jockusch in his dissertation (see [5], Exercise 8—27). But the partial c-reducibility is essentially different from the c-reducibility, because the c-reducibility is a truth-table reducibility and the partial c-reducibility is not (the last can be concluded from the fact that a non-recursive recursively enumerable set is always partially c-reducible to a recursive set).

Theorem 5. Let $A_0, A_1,\ldots, A_m$ be subsets of $\mathbf{N}$ and let $\alpha_0, \alpha_1,\ldots, \alpha_m$ be constant functions such that $\mathrm{Dom}\,\alpha_i=A_i$ for $i=0, 1,\ldots, m$. Then the following two conditions are equivalent:

(i) A_0 is partially c-reducible to $<A_1,\ldots, A_m>$.

(ii) α_0 is Turing computable in $<\alpha_1,\ldots, \alpha_m>$.

Proof. The verification of the implication ((i) $\Rightarrow$ (ii)) is straightforward. In order to prove the converse implication, suppose that

$$\alpha_0=F(\alpha_1,\ldots, \alpha_m)$$

for some Turing computable operator F and take recursive functions $\beta_1,\ldots, \beta_m$ such that

$$\alpha_1\subset\beta_1,\ldots, \alpha_m\subset\beta_m.$$

Then for $i=1,\ldots, m$ define θ_i as the set of the pairs $<x, n>$ satisfying the following condition: $x\in\mathrm{Dom}\,F(\beta_1,\ldots, \beta_m)$ and s_n is the set of the natural numbers y for which the value $\beta_i(y)$ has been demanded in the Turing computation of $F(\beta_1,\ldots, \beta_m)(x)$.

The partial c-reducibility implies enumeration reducibility. But we shall show that the partial c-reducibility is stronger than enumeration reducibility. Namely, the following theorem will be proved.

Theorem 6. There exist subsets A_1, A_1, A of $\mathbf{N}$ such that

a) $A_1\cup A_2$ is not partially c-reducible to $<A_1, A_2>$;

b) $A_1\cap A_2=\emptyset$;

c) $A_1=\{x\ \ 2x\in A\}$, $A_2=\{x\ \ 2x+1\in A\}$;

d) $A_1\cup A_2$ is not partially c-reducible to A;

e) each of the sets A_1, A_2, A belongs to the intersection of the classes Σ_2 and Π_2 of the Kleene-Mostowski hierarchy.

*Proof.** Take immune sets B_1 and B_2 belonging to $\Sigma_2\cap\Pi_2$ such that

$$B_1\cap B_2=\emptyset,\ B_1\cup B_2=\mathbf{N}$$

* Using classical logic.

(such pair of immune sets can actually be constructed). Then take a productive set C which is the complement of some recursively enumerable set and set

$$A_1 = B_1 \cap C, \ A_2 = B_2 \cap C,$$

$$A = A_1 \text{ join } A_2.$$

The conditions b), c) and e) are obviously satisfied. In order to prove the statement d), consider the functions

$$\varphi = (A_1 \cup A_2) \times \{0\},$$

$$\psi = A \times \{0\}$$

and suppose that φ is Turing computable in ψ. Since Dom φ is a productive set, by Lemma 1 we can conclude that the set A has an infinite recursively enumerable subset and this is impossible, because none of the sets A_1 and A_2 has any infinite recursively enumerable subset. Hence φ is not Turing computable in ψ. Thus, by Theorem 5, $A_1 \cup A_2$ is not partially c-reducible to A. For the proof of the statement a), consider the functions ψ_1, ψ_2 where

$$\psi_i = A_i \times \{0\} \qquad (i = 1, 2),$$

From the equations

$$\psi_1(x) \simeq \psi(2x),$$

$$\psi_2(x) \simeq \psi(2x+1)$$

it follows immediately that ψ_1 and ψ_2 are Turing computable in ψ and therefore (by the transitivity of relative Turing computability) φ cannot be Turing computable in $<\psi_1, \psi_2>$. Hence, by Theorem 5, $A_1 \cup A_2$ cannot be partial c-reducible to $<A_1, A_2>$.

R e m a r k 2. The functions φ and ψ considered in the proof above have the main property formulated in Theorem 4: φ is strongly partial recursive in ψ but φ is not Turing computable in ψ.

R e m a r k 3. By considering the functions φ, ψ_1, ψ_2 from the proof above, we can once more see that the operator F considered in Example 1 is not Turing computable.

R e m a r k 4. It is possible to prove Theorem 6 without making use of Theorem 5, but in such a direct proof we have to repeat in essence the proof of Lemma 1.

VII. At the end, I wish to give an account on a relatively simple effective enumeration of the Turing computable operators. This enumeration, in essence, is contained in [12].

Let $\psi_1, \ldots, \psi_m$ be elements of **P** and let $\Psi = <\psi_1, \ldots, \psi_m>$. For e and x belonging to **N**, we define the symbol $\{e\}^{\Psi}(x)$ by the following inductive definition:

1) $$\{(a-1)^2-1\}^{\Psi}(x) = (a-x)^2+2a$$

for all natural a and x.

2) $$\{(a+1)^2\}^{\Psi}(x)=a$$
for all natural a and x.

3) $$\{i^2-1\}^{\Psi}(x)=\psi_i(x)$$
for $i=1, 2,\ldots, m$ and $x \in \mathrm{Dom}\,\psi_i$.

4) $$\{(a+b-2)^2+2b+2\}^{\Psi}(x)=\{\{a\}^{\Psi}(x)\}^{\Psi}(\{b\}^{\Psi}(x))$$
for all natural a, b, x such that the right side of the equation is defined.

It is not difficult to see that this definition is consistent, i. e. for any e and x there exists at most one z such that $\{e\}^{\Psi}(x)=z$ (obviously, there exist infinitely many pairs $<e, x>$ for which $\{e\}^{\Psi}(x)$ will be undefined).

Theorem 7. Let F be a mapping of $\mathbf{P}^m$ into $\mathbf{P}$. Then F is a Turing computable operator iff

$$\exists e\,\forall\psi_1\ldots\forall\psi_m\,\forall x\,[F(\psi_1,\ldots,\psi_m)(x)\simeq\{e\}^{<\psi_1,\ldots,\psi_m>}(x)].$$

The proof that every operator F having this representation is Turing computable is based on the following: from the definition of the symbol $\{e\}^{\Psi}(x)$ it is easy to obtain a deterministic procedure for computing $\{e\}^{\Psi}(x)$ using finitely many values $\psi_i(y)$ demanded in the course of the computation. The statement that every Turing computable operator from $\mathbf{P}^m$ into $\mathbf{P}$ has the representation above can be proved by using some techniques from the combinatory logic approximately so as it is done in [6] and also in Wagner's papers [8], [9]* and in Strong's paper [7] (Theorem 1 must be used in the proof too). The possibility of applying these techniques can be observed from the following lemma.

Lemma 3. There exist natural numbers p, q, k, s with the properties a) — e) below (where Ψ must be considered as an abbreviation for $<\psi_1,\ldots,\psi_m>$).

a) $\forall\psi_1\ldots\forall\psi_m\,\forall x(\{p\}^{\Psi}(x)=x\dot{-}1)$;

b) $\forall\psi_1\ldots\forall\psi_m\forall x(\{q\}^{\Psi}(x)=x+1)$;

c) $\forall a\,\exists b\,\forall\psi_1\ldots\forall\psi_m[\{k\}^{\Psi}(a)=b\,\&\,\forall x(\{b\}^{\Psi}(x)=a)]$;

d) $\forall a\,\exists b\,\forall\psi_1\ldots\forall\psi_m[\{b\}^{\Psi}(0)=k\,\&\,\forall x(\{b\}^{\Psi}(x+1)=a)]$;

e) $\forall a\,\forall b\,\exists c\,\exists d\,\forall\psi_1\ldots\forall\psi_m[\{s\}^{\Psi}(a)=c\,\&\,\{c\}^{\Psi}(b)=d\,\&$
$\forall x[\{d\}^{\Psi}(x)\simeq\{\{a\}^{\Psi}(x)\}^{\Psi}(\{b\}^{\Psi}(x))]]$.

For the proof of this lemma, see [13] (the natural numbers p, q, k, s can be explicitly found; we can set $q=6$, but the values of p, k, s

* There exist Wagner's publications on the same subject in IBM Res. Rep., earlier than [8] and [9]. The first of them is in 1963, as well as my paper [6]. Unfortunately, these publications are inaccessible for me.

Added July 13, 1974: At the present time, copies of those papers are at my disposal by the kindness of Dr. Wagner.

which I can calculate are so great that it is not convenient to write them here).

It is worth notice that for recursive operators from $\mathbf{P}^m$ into $\mathbf{P}$ there is an enumeration theorem too (see [1], Chapter 10, Theorem 3.1), but it is not known whether an enumeration of them is possible which is not far more complicated than the enumeration of Turing computable operators considered here.

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ОБ ОПЕРАТОРАХ, ВЫЧИСЛИМЫХ В СМЫСЛЕ ТЬЮРИНГА

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(РЕЗЮМЕ)

В статье излагается в несколько расширенном виде содержание лекции, прочитанной автором 15 февраля 1973 г. в Международном математическом центре им. Банаха в Варшаве. Операторы, вычислимые в смысле Тьюринга (μ-рекурсивные операторы), в этой статье, как и в более ранней работе автора [11] (возникшей в связи с некоторыми вопросами В. А. Успенского), рассматриваются не только на всюду

определенных функциях, но и на частично определенных. Указываются некоторые примеры рекурсивных операторов, которые не вычислимы в смысле Тьюринга, и дается одно условие, которое необходимо и достаточно для Тьюринговой вычислимости рекурсивного оператора (это условие является модификацией одного условия из [11]). Строится более простой, чем в [11], пример такой пары функций φ и ψ, что φ является результатом применения некоторого рекурсивного оператора к ψ, но φ не вычислима в смысле Тьюринга относительно ψ. Вводится понятие частичной с-сводимости множеств натуральных чисел, а именно: множество A_0 называется частично с-сводящимся к системе множеств $<A_1,\ldots, A_m>$, если существуют такие частично рекурсивные функции $\theta_1,\ldots, \theta_m$, что множеству A_0 принадлежат в точности те натуральные числа x, для которых определены все функции $\theta_1, \ldots, \theta_m$ и конечные множества натуральных чисел со стандартными номерами $\theta_1(x),\ldots, \theta_m(x)$ содержатся соответственно в множествах $A_1,\ldots, A_m$. Исследуется связь этого понятия с относительной тьюринговой вычислимостью и строятся непересекающиеся множества натуральных чисел A_1 A_2, для которых не верно, что $A_1 \bigcup A_2$ частично с-сводится к $<A_1, A_2>$. В конце статьи приводится один относительно простой способ эффективного перечисления совокупности операторов, вычислимых в смысле Тьюринга (этот способ основан на применении методов комбинаторной логики в духе работы автора [6]).