

ON MULTIDIMENSIONAL POINT KINEMATICS IN CURVILINEAR COORDINATES

Ivanka Hristova, Ivan Tchobanov

In the present article R denotes the set of all real numbers and E_n the n -dimensional real Euclidean space; $G(\bar{a}_\nu)_{\nu=1}^m$ denotes the Gram's determinant for the vectors

$$(1) \quad \bar{a}_\nu \in E_n \quad (\nu = 1, 2, \dots, m).$$

When $G(\bar{a}_\nu)_{\nu=1}^m \neq 0$ the Gibbs' vectors for the reper (1) are denoted by \bar{a}_ν^{-1} ($\nu = 1, 2, \dots, m$).

1. Let

$$(2) \quad \bar{e}_\nu \in E_n \quad (\nu = 1, 2, \dots, n)$$

be an orthonormal basis of E_n :

$$(3) \quad \bar{e}_\mu \bar{e}_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \quad (\mu, \nu = 1, 2, \dots, n).$$

For

$$(4) \quad p_\nu \in R \quad (\nu = 1, 2, \dots, n)$$

let

$$(5) \quad \bar{r} = \bar{r}(p_1, p_2, \dots, p_n) \in E_n,$$

where

$$(6) \quad (p_1, p_2, \dots, p_n) \in P \subset R^n$$

with

$$(7) \quad G(\bar{\pi}_\nu)_{\nu=1}^n \neq 0$$

and

$$(8) \quad \bar{\pi}_\nu = \frac{\partial \bar{r}}{\partial p_\nu} \quad (\nu = 1, 2, \dots, n).$$

Then the ordered n -tuple (6) of real numbers (4) is called the *curvilinear coordinates* of the point \bar{r} of E_n .

From (5)

$$(9) \quad \bar{r} = \sum_{\nu=1}^n x_\nu \bar{e}_\nu$$

and

$$(10) \quad \frac{\partial \bar{e}_v}{\partial p_\mu} = 0 \quad (\mu, v = 1, 2, \dots, n)$$

follows

$$(11) \quad x_v = x_v(p_1, p_2, \dots, p_n) \quad (v = 1, 2, \dots, n)$$

for (6).

The surface, defined by (5), when the v -th curvilinear coordinate (4) is fixed and the rest $n-1$ coordinates (4) vary independently is called the v -th coordinate surface for the system (6) of curvilinear coordinates.

The curve, defined by (5), when the v -th curvilinear coordinate (4) varies and the rest $n-1$ coordinate (4) are fixed is called the v -th coordinate curve for the system (6) of curvilinear coordinates.

From (7), (8) follows the existence of the vectors

$$(12) \quad \pi_v^0 = \frac{1}{\pi_v} \bar{x}_v \quad (v = 1, 2, \dots, n),$$

$$(13) \quad x_v = \pi_v^{-1} \quad (v = 1, 2, \dots, n).$$

The vector π_v^0 ($v = 1, 2, \dots, n$) is called the v -th tangent unit vector for the system (6) of curvilinear coordinates. From

$$(14) \quad \pi_\mu \bar{x}_v = \begin{cases} 1 & (\mu = v) \\ 0 & (\mu \neq v) \end{cases} \quad (\mu, v = 1, 2, \dots, n)$$

follows that the vector \bar{x}_v is orthogonal to the v -th coordinate surface for the system (6) of curvilinear coordinates.

The system (6) of curvilinear coordinates is said to be *orthogonal* when

$$(15) \quad \pi_\mu \bar{\pi}_v = 0 \quad (\mu, v = 1, 2, \dots, n; \mu \neq v)$$

and *nonorthogonal*, when at least one of the conditions (15) fails. In case of (15)

$$(16) \quad \bar{\pi}_v = \pi_v^2 \bar{x}_v \quad (v = 1, 2, \dots, n)$$

holds, i. e.

$$(17) \quad \bar{x}_v = \frac{1}{\pi_v} \bar{\pi}_v^0 \quad (v = 1, 2, \dots, n).$$

The system (6) of curvilinear coordinates is called *orthonormal*, when it is orthogonal and

$$(18) \quad \pi_v = 1 \quad (v = 1, 2, \dots, n).$$

The conditions

$$(19) \quad \bar{\pi}_v = \bar{x}_v \quad (v = 1, 2, \dots, n)$$

are necessary and sufficient for the orthogonality of the systems (6) of curvilinear coordinates.

From (8)–(11) follows

$$(20) \quad \bar{\pi}_\mu = \sum_{v=1}^n \frac{\partial x_v}{\partial p_\mu} \bar{e}_v \quad (\mu = 1, 2, \dots, n).$$

From (20), (3) follows

$$(21) \quad V G(\bar{\pi}_v)_{v=1}^n = \begin{vmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \cdots & \frac{\partial x_1}{\partial p_n} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \cdots & \frac{\partial x_2}{\partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial p_1} & \frac{\partial x_n}{\partial p_2} & \cdots & \frac{\partial x_n}{\partial p_n} \end{vmatrix}.$$

From (21), (7) follows

$$(22) \quad \frac{D(x_1, x_2, \dots, x_n)}{D(p_1, p_2, \dots, p_n)} \neq 0.$$

2. From (8) and

$$(23) \quad \bar{r} = \sum_{v=1}^n (\bar{r} \bar{\pi}_v) \bar{x}_v$$

follows

$$(24) \quad \bar{r} = \frac{1}{2} \sum_{v=1}^n \frac{\partial r^2}{\partial p_v} \bar{x}_v.$$

When the system (6) of curvilinear coordinates is orthogonal the equality (24) takes the form

$$(25) \quad \bar{r} = \frac{1}{2} \sum_{v=1}^n \frac{1}{\pi_v} \frac{\partial r^2}{\partial p_v} \bar{\pi}_v$$

according to (17).

For

$$(26) \quad p_v = p_v(t) \quad (v = 1, 2, \dots, n; t \in T \subset R)$$

from (5) follows

$$(27) \quad \bar{v} = \frac{d \bar{r}}{dt} = \sum_{v=1}^n \dot{p}_v \bar{\pi}_v.$$

From (27) and

$$(28) \quad \bar{v} = \sum_{v=1}^n (\bar{v} \bar{x}_v) \bar{\pi}_v$$

follows

$$(29) \quad \dot{p}_v = \bar{v} \bar{x}_v \quad (v = 1, 2, \dots, n).$$

The system (29) of differential equations determines the functions (26) when the system (6) of curvilinear coordinates is given, i. e. when (13) and the velocity \bar{v} are given.

From (27) follows also

$$(30) \quad \pi_v = \frac{\partial \bar{v}}{\partial \dot{p}_v} \quad (v = 1, 2, \dots, n).$$

From (30) follows

$$(31) \quad \bar{v} \pi_v = \bar{v} \frac{\partial \bar{v}}{\partial \dot{p}_v} = \frac{1}{2} \frac{\partial \bar{v}^2}{\partial \dot{p}_v} \quad (v = 1, 2, \dots, n).$$

From (31) and

$$(32) \quad \bar{v} = \sum_{v=1}^n (\bar{v} \pi_v) \bar{x}_v$$

follows

$$(33) \quad \bar{v} = \frac{1}{2} \sum_{v=1}^n \frac{\partial \bar{v}^2}{\partial \dot{p}_v} \bar{x}_v;$$

this equality is an analogue of (24). In case of an orthogonal system (6) of curvilinear coordinates the equality (33) takes the form

$$(34) \quad \bar{v} = \frac{1}{2} \sum_{v=1}^n \frac{1}{\pi_v} \frac{\partial \bar{v}^2}{\partial \dot{p}_v} \bar{x}_v$$

according to (17), which is an analogue of (25).

From (8) follows

$$(35) \quad \frac{d \bar{\pi}_v}{dt} = \sum_{\mu=1}^n \dot{p}_\mu \frac{\partial^2 \bar{r}}{\partial p_v \partial p_\mu} \quad (v = 1, 2, \dots, n).$$

From (27), (8) follows

$$(36) \quad \frac{\partial \bar{v}}{\partial \dot{p}_v} = \sum_{\mu=1}^n \dot{p}_\mu \frac{\partial^2 \bar{r}}{\partial p_\mu \partial p_v} \quad (v = 1, 2, \dots, n).$$

From (35), (36) follows

$$(37) \quad \frac{d \bar{\pi}_v}{dt} = \frac{\partial \bar{v}}{\partial \dot{p}_v} \quad (v = 1, 2, \dots, n)$$

when

$$(38) \quad \frac{\partial^2 \bar{r}}{\partial p_\mu \partial p_v} = \frac{\partial^2 \bar{r}}{\partial p_v \partial p_\mu} \quad (\mu, v = 1, 2, \dots, n).$$

In what follows the equalities (38) and analogous for the derivatives of higher order shall be tacitly assumed.

From (24) follows

$$(39) \quad v = \frac{1}{2} \sum_{v=1}^n \frac{d}{dt} \frac{\partial r^v}{\partial p_v} x_v + \frac{1}{2} \sum_{v=1}^n \frac{\partial r^v}{\partial p_v} \frac{dx_v}{dt}.$$

From (27) follows

$$(40) \quad \bar{w} = \frac{d\bar{v}}{dt} = \sum_{v=1}^n \dot{p}_v \pi_v + \sum_{v=1}^n \dot{p}_v \frac{d\pi_v}{dt}.$$

The equality (40) is also obtained by differentiating

$$(41) \quad v = v(p_1, p_2, \dots, p_n; \dot{p}_1, \dot{p}_2, \dots, \dot{p}_n),$$

i. e. from

$$(42) \quad \bar{w} = \sum_{v=1}^n \dot{p}_v \frac{\partial \bar{v}}{\partial p_v} + \sum_{v=1}^n \dot{p}_v \frac{\partial \bar{v}}{\partial \dot{p}_v}$$

according to (37), (30).

From (40) follows

$$(43) \quad \pi_v = \frac{\partial \bar{w}}{\partial p_v} \quad (v = 1, 2, \dots, n),$$

which is an analogue of (30). From (43) follows

$$(44) \quad \bar{w} \pi_v = \bar{w} \frac{\partial \bar{w}}{\partial p_v} = \frac{1}{2} \frac{\partial w^2}{\partial \dot{p}_v} \quad (v = 1, 2, \dots, n).$$

From (44) and

$$(45) \quad \bar{w} = \sum_{v=1}^n (\bar{w} \pi_v) x_v$$

follows

$$(46) \quad \bar{w} = \frac{1}{2} \sum_{v=1}^n \frac{\partial w^2}{\partial \dot{p}_v} x_v;$$

this equality is an analogue of (33). In case of an orthogonal system (6) of curvilinear coordinates the equality (46) takes the form

$$(47) \quad \bar{w} = \frac{1}{2} \sum_{v=1}^n \frac{1}{\pi_v} \frac{\partial w^2}{\partial \dot{p}_v} \pi_v^0$$

according to (17), which is an analogue of (34).

From (43), (31), (37) follows

$$(48) \quad \bar{w} \pi_v = \frac{d\bar{v}}{dt} \pi_v = \frac{d}{dt} (\bar{v} \pi_v) - \bar{v} \frac{d\pi_v}{dt} = \frac{1}{2} \frac{d}{dt} \frac{\partial v^2}{\partial p_v} - \frac{1}{2} \frac{\partial v^2}{\partial p_v} \quad (v=1, 2, \dots, n).$$

From (45), (48) follows

$$(49) \quad w = \frac{1}{2} \sum_{v=1}^n \left(\frac{d}{dt} \frac{\partial v^2}{\partial p_v} - \frac{\partial v^2}{\partial p_v} \right) \dot{x}_v.$$

In case of an orthogonal system (6) of curvilinear coordinates the equality (49) takes the form

$$(50) \quad \bar{w} = \frac{1}{2} \sum_{v=1}^n \frac{1}{\pi_v} \left(\frac{d}{dt} \frac{\partial v^2}{\partial p_v} - \frac{\partial v^2}{\partial p_v} \right) \pi_v^0$$

according to (17).

From (40), (35) follows

$$(51) \quad \bar{w} = \sum_{v=1}^n \ddot{p}_v \pi_v + \sum_{\mu=1}^n \sum_{v=1}^n \dot{p}_\mu \dot{p}_v \frac{\partial^2 \bar{r}}{\partial p_\mu \partial p_v}.$$

From (51), (38) follows

$$(52) \quad \frac{\partial \bar{w}}{\partial \dot{p}_v} = 2 \sum_{\mu=1}^n \dot{p}_\mu \frac{\partial^2 \bar{r}}{\partial p_\mu \partial p_v} \quad (v=1, 2, \dots, n);$$

this equality is an analogue of (36).

From (33) follows

$$(53) \quad \bar{w} = \frac{1}{2} \sum_{v=1}^n \frac{d}{dt} \frac{\partial v^2}{\partial \dot{p}_v} \dot{x}_v + \frac{1}{2} \sum_{v=1}^n \frac{\partial v^2}{\partial \dot{p}_v} \frac{d\dot{x}_v}{dt};$$

this equality is an analogue of (39). From (49), (53) follows

$$(54) \quad \sum_{v=1}^n \frac{\partial v^2}{\partial \dot{p}_v} \frac{d\dot{x}_v}{dt} = - \sum_{v=1}^n \frac{\partial v^2}{\partial \dot{p}_v} \dot{x}_v.$$

From (42) follows

$$(55) \quad \frac{\partial \bar{w}}{\partial \dot{p}_v} = \frac{\partial \bar{v}}{\partial p_v} \quad (v=1, 2, \dots, n).$$

3. Let

$$(56) \quad \rho_m = \frac{d^m \bar{r}}{dt^m} \quad (m=0, 1, 2, \dots),$$

i. e.

$$(57) \quad \bar{\rho}_0 = \bar{r}, \quad \bar{\rho}_1 = \bar{v}, \quad \bar{\rho}_2 = \bar{w}$$

and

$$(58) \quad \overset{(m)}{p}_v = \frac{d^m p_v}{dt^m} \quad (v=1, 2, \dots, n; m=0, 1, 2, \dots).$$

We shall prove that

$$(59) \quad \bar{\rho}_m = \sum_{v=1}^n \overset{(m)}{p}_v \frac{\partial \bar{r}}{\partial \overset{(m)}{p}_v} + m \sum_{\mu=1}^n \sum_{v=1}^n \overset{(m-1)}{p}_{\mu} \overset{(m-1)}{p}_v \frac{\partial^2 \bar{r}}{\partial \overset{(m-1)}{p}_{\mu} \partial \overset{(m-1)}{p}_v} + R_{m-2}$$

$$(m=3, 4, \dots),$$

where R_{m-2} is an expression containing (58) of order not higher than $m-2$. For $m=1, 2$ the equality (59) fails, as follows from (8), (27) and (51). Indeed from (8), (35), (51), (56), (57) follows

$$(60) \quad \bar{\rho}_3 = \sum_{v=1}^n \overset{(m)}{p}_v \frac{\partial \bar{r}}{\partial \overset{(m)}{p}_v} + 3 \sum_{\mu=1}^n \sum_{v=1}^n \overset{(m)}{p}_{\mu} \overset{(m)}{p}_v \frac{\partial^2 \bar{r}}{\partial \overset{(m)}{p}_{\mu} \partial \overset{(m)}{p}_v} + R_1$$

with

$$(61) \quad R_1 = \sum_{\lambda=1}^n \sum_{\mu=1}^n \sum_{v=1}^n \overset{(m)}{p}_{\lambda} \overset{(m)}{p}_{\mu} \overset{(m)}{p}_v \frac{\partial^3 \bar{r}}{\partial \overset{(m)}{p}_{\lambda} \partial \overset{(m)}{p}_{\mu} \partial \overset{(m)}{p}_v},$$

i. e. (59) is true for $m=3$. From (59) follows

$$(62) \quad \bar{\rho}_{m+1} = \sum_{v=1}^n \overset{(m+1)}{p}_v \frac{\partial \bar{r}}{\partial \overset{(m+1)}{p}_v} + (m+1) \sum_{\mu=1}^n \sum_{v=1}^n \overset{(m)}{p}_{\mu} \overset{(m)}{p}_v \frac{\partial^2 \bar{r}}{\partial \overset{(m)}{p}_{\mu} \partial \overset{(m)}{p}_v} + R_{m-1}$$

with

$$(63) \quad R_{m-1} = m \sum_{\mu=1}^n \sum_{v=1}^n \overset{(m-1)}{p}_{\mu} \overset{(m-1)}{p}_v \frac{\partial^2 \bar{r}}{\partial \overset{(m-1)}{p}_{\mu} \partial \overset{(m-1)}{p}_v}$$

$$+ m \sum_{\lambda=1}^n \sum_{\mu=1}^n \sum_{v=1}^n \overset{(m-1)}{p}_{\lambda} \overset{(m-1)}{p}_{\mu} \overset{(m-1)}{p}_v \frac{\partial^3 \bar{r}}{\partial \overset{(m-1)}{p}_{\lambda} \partial \overset{(m-1)}{p}_{\mu} \partial \overset{(m-1)}{p}_v} + \frac{dR_{m-2}}{dt},$$

i. e. the validity of (59) for $m+1$, that is (59) is proved by induction. From (8), (30), (43), (59) follows

$$(64) \quad \frac{\partial \bar{\rho}_m}{\partial \overset{(m)}{p}_v} = \frac{\partial \bar{r}}{\partial \overset{(m)}{p}_v} \quad (v=1, 2, \dots, n; m=1, 2, \dots).$$

From (8), (64) follows

$$(65) \quad \bar{\rho}_m \bar{\pi}_v = \bar{\rho}_m \frac{\partial \bar{r}}{\partial \overset{(m)}{p}_v} = \bar{\rho}_m \frac{\partial \bar{\rho}_m}{\partial \overset{(m)}{p}_v} = \frac{1}{2} \frac{\partial \bar{\rho}_m^2}{\partial \overset{(m)}{p}_v} \quad (v=1, 2, \dots, n; m=1, 2, \dots).$$

From (65) and

$$(66) \quad \rho_m = \sum_{v=1}^n (\bar{\rho}_m \bar{\pi}_v) \bar{x}_v \quad (m=1, 2, \dots)$$

follows

$$(67) \quad \dot{\rho}_m = \frac{1}{2} \sum_{v=1}^n \frac{\partial \rho_m^2}{\partial p_v} \bar{x}_v \quad (m=1, 2, \dots).$$

In case of an orthogonal system (6) of curvilinear coordinates the equalities (67) take the form

$$(68) \quad \dot{\rho}_m = \frac{1}{2} \sum_{v=1}^n \frac{1}{\pi_v} \frac{\partial \rho_m^2}{\partial p_v} \bar{\pi}_v^0 \quad (m=1, 2, \dots)$$

according to (17).

From (36), (52), (57), (59) follows

$$(69) \quad \frac{\partial \bar{\rho}_m}{\partial p_v} = m \sum_{\mu=1}^n \dot{p}_{\mu} \frac{\partial^2 \bar{r}}{\partial p_{\mu} \partial p_v} \quad (v=1, 2, \dots, n; m=1, 2, \dots).$$

From (69), (35) follows

$$(70) \quad \frac{\partial \bar{\rho}_m}{\partial p_v} = m \frac{d \bar{\pi}_v}{dt} \quad (v=1, 2, \dots, n; m=1, 2, \dots).$$

From (56), (65), (70) follows

$$(71) \quad \begin{aligned} \bar{\rho}_{m+1} \bar{\pi}_v &= \frac{d \bar{\rho}_m}{dt} \bar{\pi}_v = \frac{d}{dt} (\bar{\rho}_m \bar{\pi}_v) - \bar{\rho}_m \frac{d \bar{\pi}_v}{dt} = \frac{1}{2} \frac{d}{dt} \frac{\partial \rho_m^2}{\partial p_v} - \frac{1}{m} \bar{\rho}_m \frac{\partial \bar{\rho}_m}{\partial p_v} \\ &= \frac{1}{2} \frac{d}{dt} \frac{\partial \rho_m^2}{\partial p_v} - \frac{1}{2m} \frac{\partial \rho_m^2}{\partial p_v} \quad (v=1, 2, \dots, n; m=1, 2, \dots). \end{aligned}$$

From (71) and

$$(72) \quad \bar{\rho}_{m+1} = \sum_{v=1}^n (\bar{\rho}_{m+1} \bar{\pi}_v) \bar{x}_v \quad (m=0, 1, 2, \dots)$$

follows

$$(73) \quad \bar{\rho}_{m+1} = \frac{1}{2} \sum_{v=1}^n \left(\frac{d}{dt} \frac{\partial \rho_m^2}{\partial p_v} - \frac{1}{m} \frac{\partial \rho_m^2}{\partial p_v} \right) \bar{x}_v \quad (m=1, 2, \dots).$$

In case of an orthogonal system (6) of curvilinear coordinates the equality (73) takes the form

$$(74) \quad \bar{\rho}_{m+1} = \frac{1}{2} \sum_{v=1}^n \frac{1}{\pi_v} \left(\frac{d}{dt} \frac{\partial \rho_m^2}{\partial p_v} - \frac{1}{m} \frac{\partial \rho_m^2}{\partial p_v} \right) \bar{\pi}_v^0 \quad (m=1, 2, \dots)$$

according to (17).

From (67) follows

$$(75) \quad \bar{\rho}_{m+1} = \frac{1}{2} \sum_{v=1}^n \frac{d}{dt} \frac{\partial \rho_m^2}{\partial p_v} \bar{x}_v + \frac{1}{2} \sum_{v=1}^n \frac{\partial \rho_m^2}{\partial p_v} \frac{d \bar{x}_v}{dt} \quad (m=1, 2, \dots).$$

From (73), (75) follows

$$(76) \quad \sum_{v=1}^n \frac{\partial \rho_m^2}{\partial p_v} \frac{d \bar{x}_v}{dt} = -\frac{1}{m} \sum_{v=1}^n \frac{\partial \rho_m^2}{\partial p_v} \bar{x}_v \quad (m=1, 2, \dots).$$

4. From (35) and

$$(77) \quad \frac{d \bar{\pi}_v}{dt} = \sum_{\lambda=1}^n \left(\frac{d \bar{\pi}_v}{dt} \bar{x}_{\lambda} \right) \bar{\pi}_{\lambda} \quad (v=1, 2, \dots, n)$$

follows

$$(78) \quad \frac{d \bar{\pi}_v}{dt} = \sum_{\lambda=1}^n \sum_{\mu=1}^n \bar{p}_{\mu} \left(\frac{\partial^2 \bar{r}}{\partial p_v \partial p_{\mu}} \bar{x}_{\lambda} \right) \bar{\pi}_{\lambda} \quad (v=1, 2, \dots, n).$$

From (40), (78) follows

$$(79) \quad w = \sum_{\lambda=1}^n \left(\ddot{p}_{\lambda} + \sum_{\mu=1}^n \sum_{v=1}^n \dot{p}_{\mu} \dot{p}_v \frac{\partial^2 \bar{r}}{\partial p_v \partial p_{\mu}} \bar{x}_{\lambda} \right) \bar{\pi}_{\lambda}.$$

From (79) and

$$(80) \quad \bar{w} = \sum_{\lambda=1}^n (\bar{w} \bar{x}_{\lambda}) \bar{\pi}_{\lambda}$$

follows

$$(81) \quad \bar{w} \bar{x}_{\lambda} = \ddot{p}_{\lambda} + \sum_{\mu=1}^n \sum_{v=1}^n \dot{p}_{\mu} \dot{p}_v \frac{\partial^2 \bar{r}}{\partial p_v \partial p_{\mu}} \bar{x}_{\lambda} \quad (\lambda=1, 2, \dots, n).$$

The result (81) is also obtained from (51) by scalar multiplication by $\bar{x}_{\lambda} (\lambda=1, 2, \dots, n)$ because of (14), as well as from (29):

$$(82) \quad \bar{w} \cdot \bar{x}_{\lambda} = \ddot{p}_{\lambda} - \bar{v} \frac{d \bar{x}_{\lambda}}{dt} \quad (\lambda=1, 2, \dots, n).$$

From (27) follows

$$(83) \quad \bar{v} \frac{d\bar{x}_\lambda}{dt} = \sum_{\nu=1}^n \bar{p}_\nu \bar{\pi}_\nu \frac{d\bar{x}_\lambda}{dt} \quad (\lambda = 1, 2, \dots, n).$$

From (14), (35) follows

$$(84) \quad \bar{\pi}_\nu \frac{d\bar{x}_\lambda}{dt} = -\bar{x}_\lambda \frac{d\bar{\pi}_\nu}{dt} = -\sum_{\mu=1}^n \bar{p}_\mu \frac{\partial^2 \bar{r}}{\partial \bar{p}_\nu \partial \bar{p}_\mu} \bar{x}_\lambda \quad (\lambda = 1, 2, \dots, n).$$

From (82)–(84) again follows (81).

The knowing of the expressions (78) for the derivatives of $\bar{\pi}_\nu$ ($\nu = 1, 2, \dots, n$) is of interest for the following reason. Let $\bar{a} = \bar{a}(t)$ be a given function decomposed in the reper $\bar{\pi}_\nu$ ($\nu = 1, 2, \dots, n$):

$$(85) \quad \bar{a} = \sum_{\nu=1}^n a_\nu \bar{\pi}_\nu.$$

From (85) follows

$$(86) \quad \frac{da}{dt} = \sum_{\nu=1}^n \dot{a}_\nu \bar{\pi}_\nu + \sum_{\nu=1}^n a_\nu \frac{d\bar{\pi}_\nu}{dt}.$$

By substituting the expressions (78) in (86) one gets

$$(87) \quad \frac{da}{dt} = \sum_{\lambda=1}^n b_\lambda \bar{\pi}_\lambda$$

with

$$(88) \quad b_\lambda = \dot{a}_\lambda + \sum_{\mu=1}^n \sum_{\nu=1}^n a_\nu \bar{p}_\mu \frac{\partial^2 \bar{r}}{\partial \bar{p}_\nu \partial \bar{p}_\mu} \bar{x}_\lambda \quad (\lambda = 1, 2, \dots, n).$$

We now find the higher order derivatives of $\bar{a}(t)$ by repeating the above procedure: differentiate (87) and substitute the derivatives of $\bar{\pi}_\lambda$ ($\lambda = 1, 2, \dots, n$) with the expressions (78) once more. The difference between the results so obtained and the results from the previous paragraph is that while the derivatives of $\bar{a}(t)$ given here are decomposed in the reper $\bar{\pi}_\nu$ ($\nu = 1, 2, \dots, n$), in the preceding paragraph they are decomposed in the reper \bar{x}_ν , reciprocal to $\bar{\pi}_\nu$ ($\nu = 1, 2, \dots, n$). By the way, one could proceed in the following manner: to obtain the decompositions in the reper \bar{x}_ν ($\nu = 1, 2, \dots, n$) from the previous paragraph, then to pass from the reper \bar{x}_ν to the reper $\bar{\pi}_\nu$ ($\nu = 1, 2, \dots, n$) using the equalities

$$(89) \quad \bar{x}_\nu = \sum_{\mu=1}^n (\bar{x}_\nu \bar{x}_\mu) \bar{\pi}_\mu \quad (\nu = 1, 2, \dots, n).$$

In case of an orthogonal system (6) of curvilinear coordinates the application of the formula (50) is preceded by calculating of v^2 from the equality

$$(90) \quad v^2 = \sum_{\nu=1}^n p_\nu^2 \pi_\nu^2,$$

which is an immediate corollary from (15) and (27).

5. In the threedimensional case, which for apparent reasons is the most interesting, the calculations for nonorthogonal system (6) of curvilinear coordinates are facilitated by the fact that now one disposes of convenient expressions for the reciprocal vectors

$$(91) \quad \bar{a}_\nu^{-1} = \frac{\bar{a}_{\nu+1} \times \bar{a}_{\nu+2}}{\bar{a}_1 \times \bar{a}_2 \cdot \bar{a}_3} \quad (\nu = 1, 2, 3)$$

with

$$(92) \quad \bar{a}_{\nu+3} = \bar{a}_\nu \quad (\nu = 1, 2).$$

In this paragraph we shall consider some traditionally used orthogonal systems (6) of curvilinear coordinates in the threedimensional vector space V . At that i, j, k denote the unit vectors of the axes of a rectangular right-hand orientated Cartesian coordinate system $Oxyz$.

For the coordinate system

$$(93) \quad r = p_1 (\cos p_2 \bar{i} + \sin p_2 \bar{j}) + p_3 \bar{k}$$

with

$$(94) \quad 0 \leq p_1, 0 \leq p_2 < 2\pi, -\infty < p_3 < \infty$$

the equalities

$$(95) \quad \bar{\pi}_1 = \cos p_2 \bar{i} + \sin p_2 \bar{j},$$

$$(96) \quad \bar{\pi}_2 = p_1 (-\sin p_2 \bar{i} + \cos p_2 \bar{j}),$$

$$(97) \quad \bar{\pi}_3 = \bar{k}$$

hold. From (95)–(97) follows (15) and

$$(98) \quad \pi_1 = 1, \pi_2 = p_1, \pi_3 = 1.$$

From (93) follows

$$(99) \quad r^2 = p_1^2 + p_3^2.$$

From (25), (98), (99) follows

$$(100) \quad \bar{r} = p_1 \bar{\pi}_1^0 + p_3 \bar{\pi}_3^0.$$

From (90), (98) follows

$$(101) \quad v^2 = p_1^2 + p_1^2 p_2^2 + p_3^2.$$

From (50), (98), (101) follows

$$(102) \quad \bar{w} = (\ddot{p}_1 - p_1 \dot{p}_1^2) \bar{\pi}_1^0 + \frac{1}{p_1} \frac{d}{dt} (p_1^2 \dot{p}_2) \bar{\pi}_2^0 + \ddot{p}_3 \bar{\pi}_3^0.$$

For the coordinate system

$$(103) \quad \bar{r} = \operatorname{ch} p_1 \cos p_2 \bar{i} + \operatorname{sh} p_1 \sin p_2 \bar{j} + p_3 \bar{k}$$

with

$$(104) \quad -\infty < p_1 < \infty, \quad 0 \leq p_2 < 2\pi, \quad -\infty < p_3 < \infty$$

the equalities

$$(105) \quad \bar{\pi}_1 = \operatorname{sh} p_1 \cos p_2 \bar{i} + \operatorname{ch} p_1 \sin p_2 \bar{j},$$

$$(106) \quad \bar{\pi}_2 = -\operatorname{ch} p_1 \sin p_2 \bar{i} + \operatorname{sh} p_1 \cos p_2 \bar{j}$$

and (97) hold. From (105), (106), (97) follows (15) and

$$(107) \quad \pi_1 = \pi_2 = \sqrt{\operatorname{sh}^2 p_1 + \sin^2 p_2}, \quad \pi_3 = 1.$$

From (103) follows

$$(108) \quad r^2 = \operatorname{sh}^2 p_1 + \cos^2 p_2 + p_3^2.$$

From (25), (107), (108) follows

$$(109) \quad r = \frac{\operatorname{sh} 2p_1 \bar{\pi}_1^0 - \sin 2p_2 \bar{\pi}_2^0}{2 \sqrt{\operatorname{sh}^2 p_1 + \sin^2 p_2}} + p_3 \bar{\pi}_3^0.$$

From (90), (107) follows

$$(110) \quad v^2 = (\operatorname{sh}^2 p_1 + \sin^2 p_2) (\dot{p}_1^2 + \dot{p}_2^2) + \dot{p}_3^2.$$

From (50), (107), (110) follows

$$(111) \quad \begin{aligned} & \frac{1}{\sqrt{\operatorname{sh}^2 p_1 + \sin^2 p_2}} \left\{ \sum_{v=1}^2 \frac{d}{dt} [(\operatorname{sh}^2 p_1 + \sin^2 p_2) \dot{p}_v] \bar{\pi}_v^0 \right. \\ & \left. - \frac{1}{2} (\dot{p}_1^2 + \dot{p}_2^2) (\operatorname{sh} 2p_1 \bar{\pi}_1^0 + \sin 2p_2 \bar{\pi}_2^0) + \ddot{p}_3 \bar{\pi}_3^0 \right\}. \end{aligned}$$

For the coordinate system

$$(112) \quad r = p_1 p_2 \bar{i} + \frac{1}{2} (p_1^2 - p_2^2) \bar{j} + p_3 \bar{k}$$

with

$$(113) \quad -\infty < p_v < \infty \quad (v = 1, 2, 3)$$

the equalities

$$(114) \quad \bar{\pi}_1 = p_2 \bar{i} + p_1 \bar{j},$$

$$(115) \quad \bar{\pi}_2 = p_1 \bar{i} - p_2 \bar{j}$$

and (97) hold. From (114), (115), (97) follows (15) and

$$(116) \quad \pi_1 = \pi_2 = \sqrt{p_1^2 + p_2^2}, \quad \pi_3 = 1.$$

From (112) follows

$$(117) \quad r^2 = \frac{1}{4} (p_1^2 + p_2^2) + p_3^2.$$

From (25), (116), (117) follows

$$(118) \quad r = \frac{1}{2} \sqrt{p_1^2 + p_2^2} (p_1 \bar{\pi}_1^0 + p_2 \bar{\pi}_2^0) + p_3 \bar{\pi}_3^0.$$

From (90), (116) follows

$$(119) \quad v^2 = (p_1^2 + p_2^2) (p_1^2 + p_2^2) + p_3^2.$$

From (50), (116), (119) follows

$$(120) \quad \bar{w} = \frac{1}{\sqrt{p_1^2 + p_2^2}} \sum_{r=1}^3 \left\{ \frac{d}{dt} [(p_1^2 + p_2^2) \dot{p}_r] - (p_1^2 + p_2^2) \ddot{p}_r \right\} \bar{\pi}_r^0 + p_3 \bar{\pi}_3^0.$$

For the coordinate system

$$(121) \quad \bar{r} = \frac{\operatorname{sh} p_1 \bar{i} + \sin p_2 \bar{j}}{\operatorname{ch} p_1 - \cos p_2} + p_3 \bar{k}$$

with (104) the equalities

$$(122) \quad \bar{\pi}_1 = \frac{(1 - \operatorname{ch} p_1 \cos p_2) \bar{i} - \operatorname{sh} p_1 \sin p_2 \bar{j}}{(\operatorname{ch} p_1 - \cos p_2)^2},$$

$$(123) \quad \bar{\pi}_2 = -\frac{\operatorname{sh} p_1 \sin p_2 \bar{i} + (1 - \operatorname{ch} p_1 \cos p_2) \bar{j}}{(\operatorname{ch} p_1 - \cos p_2)^2},$$

and (97) hold. From (122), (123), (97) follows (15) and

$$(124) \quad \pi_1 = \pi_2 = \frac{1}{\operatorname{ch} p_1 - \cos p_2}, \quad \pi_3 = 1.$$

From (121) follows

$$(125) \quad r^2 = \frac{\operatorname{ch} p_1 + \cos p_2}{\operatorname{ch} p_1 - \cos p_2} + p_3^2.$$

From (25), (124), (125) follows

$$(126) \quad \bar{r} = -\frac{\operatorname{sh} p_1 \cos p_2 \bar{\pi}_1^0 + \operatorname{ch} p_1 \sin p_2 \bar{\pi}_2^0}{\operatorname{ch} p_1 - \cos p_2} + p_3 \bar{\pi}_3^0.$$

From (90), (124) follows

$$(127) \quad v^2 = \frac{\dot{p}_1^2 + \dot{p}_2^2}{(\operatorname{ch} p_1 - \cos p_2)^2} + \dot{p}_3^2.$$

From (50), (124), (127) follows

$$(128) \quad \bar{w} = (\operatorname{ch} p_1 - \cos p_2) \sum_{\nu=1}^2 \frac{d}{dt} \frac{\dot{p}_\nu}{(\operatorname{ch} p_1 - \cos p_2)^2} \pi_\nu^0 + \frac{(\dot{p}_1^2 + \dot{p}_2^2)(\operatorname{sh} p_1 \pi_1^0 + \sin p_2 \pi_2^0)}{(\operatorname{ch} p_1 - \cos p_2)^2} + \dot{p}_3 \pi_3^0.$$

For the coordinate system

$$(129) \quad \bar{r} = p_1 (\cos p_2 \cos p_3 \bar{i} + \cos p_2 \sin p_3 \bar{j} + \sin p_2 \bar{k})$$

with

$$(130) \quad 0 \leq p_1, \quad -\frac{\pi}{2} \leq p_2 \leq \frac{\pi}{2}, \quad 0 \leq p_3 < 2\pi$$

the equalities

$$(131) \quad \pi_1 = \cos p_2 \cos p_3 \bar{i} + \cos p_2 \sin p_3 \bar{j} + \sin p_2 \bar{k},$$

$$(132) \quad \pi_2 = p_1 (-\sin p_2 \cos p_3 \bar{i} - \sin p_2 \sin p_3 \bar{j} + \cos p_2 \bar{k}),$$

$$(133) \quad \pi_3 = p_1 (-\cos p_2 \sin p_3 \bar{i} + \cos p_2 \cos p_3 \bar{j})$$

hold. From (131)–(133) follows (15) and

$$(134) \quad \pi_1 = 1, \quad \pi_2 = p_1, \quad \pi_3 = p_1 \cos p_2.$$

From (129) follows

$$(135) \quad \bar{r}^2 = p_1^2.$$

From (25), (134), (135) follows

$$(136) \quad \bar{r} = p_1 \pi_1^0.$$

From (90), (134) follows

$$(137) \quad v^2 = \dot{p}_1^2 + p_1^2 \dot{p}_2^2 + p_1^2 \cos^2 p_2 \dot{p}_3^2.$$

From (50), (134), (137) follows

$$(138) \quad \bar{w} = (\ddot{p}_1 - p_1 \cos^2 p_2 \dot{p}_3^2 + p_1 \dot{p}_2^2) \pi_1^0 + \frac{1}{p_1 \cos p_2} \frac{d}{dt} (p_1^2 \cos^2 p_2 \dot{p}_3) \pi_2^0 + \frac{1}{p_1} \left[\frac{d}{dt} (p_1^2 \dot{p}_2) + \frac{1}{2} p_1^2 \sin 2p_2 \dot{p}_3^2 \right] \pi_3^0.$$

For the coordinate system

$$(139) \quad \bar{r} = \operatorname{sh} p_1 \cos p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \operatorname{ch} p_1 \sin p_2 \bar{k}$$

with (130) the equalities

$$(140) \quad \pi_1 = \operatorname{ch} p_1 \cos p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \operatorname{sh} p_1 \sin p_2 \bar{k},$$

$$(141) \quad \pi_2 = -\operatorname{sh} p_1 \sin p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \operatorname{ch} p_1 \cos p_2 \bar{k},$$

$$(142) \quad \pi_3 = \operatorname{ch} p_1 \cos p_2 (-\sin p_3 \bar{i} + \cos p_3 \bar{j})$$

hold. From (140)–(142) follows (15) and

$$(143) \quad \pi_1 = \pi_2 = \sqrt{\operatorname{sh}^2 p_1 + \cos^2 p_2}, \quad \pi_3 = \operatorname{sh} p_1 \cos p_2.$$

From (139) follows

$$(144) \quad r^2 = \operatorname{sh}^2 p_1 + \sin^2 p_2.$$

From (25), (143), (144) follows

$$(145) \quad r = \frac{\operatorname{sh} 2p_1 \pi_1^0 + \sin 2p_2 \pi_2^0}{2 \sqrt{\operatorname{sh}^2 p_1 + \cos^2 p_2}}.$$

From (90), (143) follows

$$(146) \quad v^2 = (\operatorname{sh}^2 p_1 + \cos^2 p_2)(\dot{p}_1^2 + \dot{p}_2^2) + \operatorname{sh}^2 p_1 \cos^2 p_2 \dot{p}_3^2.$$

From (50), (143), (146) follows

$$(147) \quad w = \frac{1}{\sqrt{\operatorname{sh}^2 p_1 + \cos^2 p_2}} \left\{ \sum_{v=1}^2 \frac{d}{dt} [(\operatorname{sh}^2 p_1 + \cos^2 p_2) \dot{p}_v] \pi_v^0 \right. \\ \left. - \frac{1}{2} (\dot{p}_1^2 + \dot{p}_2^2) (\operatorname{sh} 2p_1 \pi_1^0 - \sin 2p_2 \pi_2^0) \right\} \\ + \frac{1}{\operatorname{sh} p_1 \cos p_2} \frac{d}{dt} (\operatorname{sh}^2 p_1 \cos^2 p_2 \dot{p}_3) \pi_3^0.$$

For the coordinate system

$$(148) \quad \bar{r} = \operatorname{ch} p_1 \cos p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \operatorname{sh} p_1 \sin p_2 \bar{k}$$

with

$$(149) \quad -\infty < p_1 < \infty, \quad -\frac{\pi}{2} \leq p_2 \leq \frac{\pi}{2}, \quad 0 \leq p_3 < 2\pi$$

the equalities

$$(150) \quad \bar{\pi}_1 = \operatorname{sh} p_1 \cos p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \operatorname{ch} p_1 \sin p_2 \bar{k},$$

$$(151) \quad \bar{\pi}_2 = -\operatorname{ch} p_1 \sin p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \operatorname{sh} p_1 \cos p_2 \bar{k}$$

$$(152) \quad \bar{\pi}_3 = \operatorname{ch} p_1 \cos p_2 (-\sin p_3 \bar{i} + \cos p_3 \bar{j})$$

hold. From (150)–(152) follows (15) and

$$(153) \quad \pi_1 = \pi_2 = \operatorname{ch}^2 p_1 - \cos^2 p_2, \quad \pi_3 = \operatorname{ch} p_1 \cos p_2.$$

From (148) follows

$$(154) \quad r^2 = \operatorname{sh}^2 p_1 + \cos^2 p_2.$$

From (25), (153), (154) follows

$$(155) \quad \bar{r} = \frac{\operatorname{sh} 2p_1 \bar{\pi}_1^0 - \sin 2p_2 \bar{\pi}_2^0}{2 \sqrt{\operatorname{ch}^2 p_1 - \cos^2 p_2}}.$$

From (90), (153) follows

$$(156) \quad v^2 = (\operatorname{ch}^2 p_1 - \cos^2 p_2)(\dot{p}_1^2 + \dot{p}_2^2) + \operatorname{ch}^2 p_1 \cos^2 p_2 \dot{p}_3^2.$$

From (50), (153), (156) follows

$$(157) \quad \bar{w} = \frac{1}{\sqrt{\operatorname{ch}^2 p_1 - \cos^2 p_2}} \left\{ \sum_{v=1}^2 \frac{d}{dt} [(\operatorname{ch}^2 p_1 - \cos^2 p_2) \dot{p}_v] \bar{\pi}_v^0 - \frac{1}{2} (\dot{p}_1^2 + \dot{p}_2^2) (\operatorname{sh} 2p_1 \bar{\pi}_1^0 + \sin 2p_2 \bar{\pi}_2^0) + \frac{1}{\operatorname{ch} p_1 \cos p_2} \frac{d}{dt} (\operatorname{ch}^2 p_1 \cos^2 p_2 \dot{p}_3) \bar{\pi}_3^0. \right.$$

For the coordinate system

$$(158) \quad \bar{r} = p_1 p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \frac{1}{2} (p_1^2 - p_2^2) \bar{k}$$

with

$$(159) \quad 0 \leq p_1, 0 \leq p_2, 0 \leq p_3 < 2\pi$$

the equalities

$$(160) \quad \bar{\pi}_1 = p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + p_1 \bar{k},$$

$$(161) \quad \bar{\pi}_2 = p_1 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) - p_2 \bar{k},$$

$$(162) \quad \bar{\pi}_3 = p_1 p_2 (-\sin p_3 \bar{i} + \cos p_3 \bar{j})$$

hold. From (160)–(162) follows (15) and

$$(163) \quad \pi_1 = \pi_2 = \sqrt{p_1^2 + p_2^2}, \quad \pi_3 = p_1 p_2.$$

From (158) follows

$$(164) \quad r^2 = \frac{1}{4} (p_1^2 + p_2^2)^2.$$

From (125), (163), (164) follows

$$(165) \quad \bar{r} = \frac{1}{2} \sqrt{p_1^2 + p_2^2} (p_1 \bar{\pi}_1^0 + p_2 \bar{\pi}_2^0).$$

From (90), (163) follows

$$(166) \quad v^2 = (p_1^2 + p_2^2)(p_1^2 + p_2^2) + p_1^2 p_2^2 p_3^2.$$

From (50), (163), (166) follows

$$(167) \quad \bar{w} = \frac{1}{\sqrt{p_1^2 + p_2^2}} \left\{ \sum_{v=1}^2 \frac{d}{dt} [(p_1^2 + p_2^2) p_v] \bar{\pi}_v^0 - (p_1^2 + p_2^2) p_v \bar{\pi}_v^0 - \frac{1}{p_1 p_2} \frac{d}{dt} (p_1^2 p_2^2 p_3) \bar{\pi}_3^0. \right.$$

For the coordinate system

$$(168) \quad \bar{r} = \frac{\operatorname{sh} p_1 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + \sin p_2 \bar{k}}{\operatorname{ch} p_1 - \cos p_2}$$

with (130) the equalities

$$(169) \quad \bar{\pi}_1 = \frac{(1 - \operatorname{ch} p_1 \cos p_2)(\cos p_3 \bar{i} + \sin p_3 \bar{j}) - \operatorname{sh} p_1 \sin p_2 \bar{k}}{(\operatorname{ch} p_1 - \cos p_2)^2},$$

$$(170) \quad \bar{\pi}_2 = \frac{-\operatorname{sh} p_1 \sin p_2 (\cos p_3 \bar{i} + \sin p_3 \bar{j}) + (\operatorname{ch} p_1 \cos p_2 - 1) \bar{k}}{(\operatorname{ch} p_1 - \cos p_2)^2},$$

$$(171) \quad \bar{\pi}_3 = \frac{\operatorname{sh} p_1 (-\sin p_3 \bar{i} + \cos p_3 \bar{j})}{\operatorname{ch} p_1 - \cos p_2}.$$

From (169)–(171) follows (15) and

$$(172) \quad \pi_1 = \pi_2 = \frac{1}{\operatorname{ch} p_1 - \cos p_2}, \quad \pi_3 = \frac{\operatorname{sh} p_1}{\operatorname{ch} p_1 - \cos p_2}.$$

From (168) follows

$$(173) \quad r^2 = \frac{\operatorname{ch} p_1 + \cos p_2}{\operatorname{ch} p_1 - \cos p_2}.$$

From (25), (172), (173) follows

$$(174) \quad r = - \frac{\operatorname{sh} p_1 \cos p_2 \bar{\pi}_1^0 + \operatorname{ch} p_1 \sin p_2 \bar{\pi}_2^0}{\operatorname{ch} p_1 - \cos p_2}.$$

From (90), (172) follows

$$(175) \quad v^2 = \frac{p_1^2 + p_2^2 + \operatorname{sh}^2 p_1 p_3^2}{(\operatorname{ch} p_1 - \cos p_2)^2}.$$

From (50), (172), (175) follows

$$(176) \quad \bar{w} = (\operatorname{ch} p_1 - \cos p_2) \left[\sum_{\nu=1}^2 \frac{d}{dt} \frac{\dot{p}_\nu}{(\operatorname{ch} p_1 - \cos p_2)^2} \bar{\pi}_\nu^0 + \frac{\dot{p}_1^2 + \dot{p}_2^2 + (\operatorname{ch} p_1 \cos p_2 - 1) \dot{p}_3^2}{(\operatorname{ch} p_1 - \cos p_2)^3} \sin p_2 \bar{\pi}_1^0 + \frac{\dot{p}_1^2 + \dot{p}_2^2 + \operatorname{sh}^2 p_1 \dot{p}_3^2}{(\operatorname{ch} p_1 - \cos p_2)^3} \sin p_2 \bar{\pi}_2^0 + \frac{1}{\operatorname{sh} p_1} \frac{d}{dt} \frac{\operatorname{sh}^2 p_1 \dot{p}_3}{(\operatorname{ch} p_1 - \cos p_2)} \bar{\pi}_3^0 \right].$$

For the coordinate system

$$(177) \quad r = \frac{\sin p_2 (\cos p_3 i + \sin p_3 j) + \operatorname{sh} p_1 k}{\operatorname{ch} p_1 - \cos p_2}$$

with

$$(178) \quad -\infty < p_1 < \infty, \quad 0 \leq p_2 \leq \pi, \quad 0 \leq p_3 < 2\pi$$

the equalities

$$(179) \quad \pi_1 = -\frac{\operatorname{sh} p_1 \sin p_2 (\cos p_3 i + \sin p_3 j) + (\operatorname{ch} p_1 \cos p_2 - 1) k}{(\operatorname{ch} p_1 - \cos p_2)^2},$$

$$(180) \quad \pi_2 = \frac{(\operatorname{ch} p_1 \cos p_2 - 1) (\cos p_3 i + \sin p_3 j) - \operatorname{sh} p_1 \sin p_2 k}{(\operatorname{ch} p_1 - \cos p_2)^2},$$

$$(181) \quad \pi_3 = \frac{\sin p_2 (-\sin p_3 i + \cos p_3 j)}{\operatorname{ch} p_1 - \cos p_2}$$

hold.

From (179)–(181) follows (15) and

$$(182) \quad \pi_1 = \pi_2 = \frac{1}{\operatorname{ch} p_1 - \cos p_2}, \quad \pi_3 = \frac{\sin p_2}{\operatorname{ch} p_1 - \cos p_2}.$$

From (177) follows (173). From (25), (182), (173) follows (174). From (90), (182) follows

$$(183) \quad v^2 = \frac{\dot{p}_1^2 + \dot{p}_2^2 + \sin p_2 \dot{p}_3^2}{(\operatorname{ch} p_1 - \cos p_2)^2}.$$

From (50), (182), (183) follows

$$(184) \quad \bar{w} = (\operatorname{ch} p_1 - \cos p_2) \left[\sum_{\nu=1}^2 \frac{d}{dt} \frac{\dot{p}_\nu}{(\operatorname{ch} p_1 - \cos p_2)^2} \bar{\pi}_\nu^0 + \frac{\dot{p}_1^2 + \dot{p}_2^2 + \sin^2 p_2 \dot{p}_3^2}{(\operatorname{ch} p_1 - \cos p_2)^3} \operatorname{sh} p_1 \bar{\pi}_1^0 + \frac{\dot{p}_1^2 + \dot{p}_2^2 + (1 - \operatorname{ch} p_1 \cos p_2) \dot{p}_3^2}{(\operatorname{ch} p_1 - \cos p_2)^3} \sin p_2 \bar{\pi}_2^0 \right]$$

$$+ \frac{1}{\sin p_2} \frac{d}{dt} \left[\frac{\sin^2 p_2 p_3}{(\operatorname{ch} p_1 - \cos p_2)^2} \pi_3^0 \right].$$

6. Let E be an arbitrary nontrivial real Euclidean space. We shall prove some proposition that are frequently used [Th=theorem, Dm =proof].

Th. 1. From

$$(185) \quad \alpha_{\mu\nu} \in R \quad (\mu, \nu = 1, 2, \dots, n),$$

$$(186) \quad \operatorname{Det} \alpha_{\mu\nu} \neq 0 \quad (\mu, \nu = 1, 2, \dots, n),$$

$$(187) \quad a_\nu \in E \quad (\nu = 1, 2, \dots, n),$$

$$(188) \quad \sum_{\nu=1}^n \alpha_{\mu\nu} a_\nu = 0 \quad (\mu = 1, 2, \dots, n)$$

follows

$$(189) \quad a_\nu = 0 \quad (\nu = 1, 2, \dots, n).$$

Dm. From (185), (186) follows

$$(190) \quad \sum_{\nu=1}^n \alpha_{\mu\nu}^2 \neq 0 \quad (\mu = 1, 2, \dots, n).$$

From (190), (188) follows

$$(191) \quad G(a_\nu)_{\nu=1}^n = 0.$$

From (191) and

$$(192) \quad \sum_{\nu=1}^n a_\nu^2 \neq 0$$

without loss of generality [because of the possibility of a new numbering of the vectors (187)] would follow the existence of $l < n$ with

$$(193) \quad G(\bar{a}_\nu)_{\nu=1}^n \neq 0$$

and

$$(194) \quad \bar{a}_\nu = \sum_{\lambda=1}^l \beta_{\nu\lambda} \bar{a}_\lambda \quad (\nu = 1, 2, \dots, n)$$

for appropriate

$$(195) \quad \beta_{\nu\lambda} \in R \quad (\lambda = 1, 2, \dots, l; \nu = 1, 2, \dots, n).$$

From (188), (194) follows

$$(196) \quad \sum_{\lambda=1}^l \left(\sum_{\nu=1}^n \alpha_{\mu\nu} \beta_{\nu\lambda} \right) \bar{a}_\lambda = \bar{0} \quad (\mu = 1, 2, \dots, n).$$

From (196), (193) follows

$$(197) \quad \sum_{\nu=1}^n \alpha_{\mu\nu} \beta_{\nu\lambda} = 0 \quad (\lambda = 1, 2, \dots, l; \mu = 1, 2, \dots, n).$$

From (194) follows

$$(198) \quad \beta_{\nu\lambda} = \bar{a}_\nu \bar{a}_\lambda^{-1} \quad (\lambda = 1, 2, \dots, l; \nu = 1, 2, \dots, n),$$

where

$$(199) \quad \bar{a}_\lambda^{-1} \in E \quad (\lambda = 1, 2, \dots, l)$$

is the reciprocal reper to the reper

$$(200) \quad \bar{a}_\lambda \in E \quad (\lambda = 1, 2, \dots, l).$$

From (198) follows

$$(201) \quad \beta_{\nu\nu} = 1 \quad (\nu = 1, 2, \dots, n).$$

From (197), (186) however follows

$$(202) \quad \beta_{\nu\lambda} = 0 \quad (\lambda = 1, 2, \dots, l; \nu = 1, 2, \dots, n).$$

The contradiction (201), (202) refute (192) and proves (189).

Th 2. From (185), (187), (188), (192) follows

$$(203) \quad \text{Det } \alpha_{\mu\nu} = 0 \quad (\mu, \nu = 1, 2, \dots, n).$$

Dm. Th 1.

Th 3. From (185)–(187)

$$(204) \quad \bar{b}_\nu \in E \quad (\nu = 1, 2, \dots, n),$$

$$(205) \quad \sum_{\nu=1}^n \alpha_{\mu\nu} \bar{a}_\nu = \sum_{\nu=1}^n \alpha_{\mu\nu} \bar{b}_\nu \quad (\mu = 1, 2, \dots, n)$$

follows

$$(206) \quad \bar{a}_\nu = \bar{b}_\nu \quad (\nu = 1, 2, \dots, n).$$

Dm. From (205) follows

$$(207) \quad \sum_{\nu=1}^n \alpha_{\mu\nu} (\bar{a}_\nu - \bar{b}_\nu) = \bar{0} \quad (\mu = 1, 2, \dots, n).$$

From (207), (186), Th 1 follows (206).

Th 4. From (185), (187)

$$(208) \quad \bar{b}_\mu = \sum_{\nu=1}^n \alpha_{\mu\nu} \bar{a}_\nu \quad (\mu = 1, 2, \dots, n),$$

$$(209) \quad G(\bar{b}_\nu)_{\nu=1}^n \neq 0$$

follows (186).

Dm. From (203) follows the existence of

$$(210) \quad \alpha_v \in R \quad (v=1, 2, \dots, n)$$

with

$$(211) \quad \sum_{v=1}^n \alpha_v^2 \neq 0$$

and

$$(212) \quad \sum_{v=1}^n \alpha_v \alpha_{\mu v} = 0 \quad (v=1, 2, \dots, n).$$

From (208), (212) follows

$$(213) \quad \sum_{v=1}^n \alpha_v \bar{b}_v = \sum_{v=1}^n \left(\sum_{\mu=1}^n \alpha_{\mu} \alpha_{\mu v} \right) \bar{a}_v = 0$$

for (210), (211) contrary to (209), which refutes (203) and proves (196).

Th 5. From (185), (187), (208), (209) follows

$$(214) \quad G(\bar{a}_v)_{v=1}^n \neq 0.$$

Dm. From (185), (187), (208), (209) follows (186) (Th 4). Let there exists (210) with (211) and

$$(215) \quad \sum_{v=1}^n \alpha_v \bar{a}_v = \bar{0}.$$

From (186) follows that the system of equations

$$(216) \quad \sum_{v=1}^n \beta_{\mu} \alpha_{\mu v} = 0 \quad (v=1, 2, \dots, n)$$

has a solution

$$(217) \quad \beta_{\mu} \in R \quad (\mu=1, 2, \dots, n)$$

with

$$(218) \quad \sum_{\mu=1}^n \beta_{\mu}^2 \neq 0,$$

otherwise we would have

$$(219) \quad \alpha_v = 0 \quad (v=1, 2, \dots, n),$$

contrary to (211). Now from (208), (216), (215) follows

$$(220) \quad \sum_{v=1}^n \beta_{\mu} \bar{b}_{\mu} = \sum_{v=1}^n \left(\sum_{\mu=1}^n \beta_{\mu} \alpha_{\mu v} \right) \bar{a}_v = 0$$

for (217), (218), contrary to (219), which refutes (215) for (210), (211) and proves (214).

Th 6. From (185), (187), (208), (203) follows

$$(221) \quad G(b)_{\nu=1}^n = 0.$$

Dm. Th 4.

Th 7. From (185), (187), (208), (191) follows (221).

Dm. Th 5.

Th 8. From (185)–(187), (208), (214) follows (209).

Dm. From (208), (214) follows

$$(222) \quad \alpha_{\mu\nu} = \bar{a}_\nu^{-1} \bar{b}_\mu \quad (\mu, \nu = 1, 2, \dots, n).$$

From (222), (186) follows

$$(223) \quad G(\bar{a}_\nu^{-1}, \bar{b}_\nu)_{\nu=1}^n \neq 0.$$

We have

$$(224) \quad (G(\bar{a}_\nu^{-1}, \bar{b}_\nu)_{\nu=1}^n)^2 = G(\bar{a}_\nu^{-1})_{\nu=1}^n G(b_\nu)_{\nu=1}^n$$

according to [1], p. 229. From (214) and

$$(225) \quad G(a_\nu)_{\nu=1}^n G(\bar{a}_\nu^{-1})_{\nu=1}^n = 1$$

follows

$$(226) \quad G(a_\nu^{-1})_{\nu=1}^n \neq 0.$$

From (223), (224), (226) follows (209).

Th 9. From (185), (187), (208) follows (209) iff (186), (214).

Dm. Th 4, Th 5, Th 8.

Th 10. From (185), (187), (208), (209) follows

$$(227) \quad \bar{a}_\nu^{-1} = \sum_{\mu=1}^n \alpha_{\mu\nu} b_\mu^{-1} \quad (\nu = 1, 2, \dots, n).$$

Dm. From (185), (187), (208), (209) follows (214) (Th 5). From (208), (214) follows (222). From (208), (209), (214) follows

$$(228) \quad \bar{a}_\nu^{-1} = \sum_{\mu=1}^n (\bar{a}_\nu^{-1} b_\mu) \bar{b}_\mu^{-1} \quad (\nu = 1, 2, \dots, n).$$

From (222), (228) follows (227).

Th 11. From (185), (186), (204), (209),

$$(229) \quad \bar{c}_\nu = \sum_{\mu=1}^n \alpha_{\mu\nu} \bar{b}_\mu^{-1} \quad (\nu = 1, 2, \dots, n)$$

follows

$$(230) \quad \bar{b}_\mu = \sum_{\nu=1}^n \alpha_{\mu\nu} \bar{c}_\nu^{-1} \quad (\mu = 1, 2, \dots, n).$$

Dm. From (204), (209) follows

$$(231) \quad G(\bar{b}_v^{-1})_{v=1}^n \neq 0.$$

From (185), (187), (213), (229) follows

$$(232) \quad G(\bar{c}_v)_{v=1}^n \neq 0$$

(Th 8). From (185), (229), (232) follows (230) (Th 11) because of

$$(233) \quad (\bar{b}_v^{-1})^{-1} = \bar{b}_v \quad (v=1, 2, \dots, n).$$

Th 12. From (158), (204), (209) follows: the only solution (207) of the system of equations (208) is defined by (227).

Dm. Th 10, Th 11, Th 3.

Th 13. From (185), (187), (208), (209),

$$(234) \quad a_\mu a_v = \begin{cases} 1 & (\mu = v) \\ 0 & (\mu \neq v) \end{cases} \quad (\mu, v = 1, 2, \dots, n)$$

follows

$$(235) \quad a_v = \sum_{\mu=1}^n \alpha_{\mu v} \bar{b}_\mu^{-1} \quad (v = 1, 2, \dots, n).$$

Dm. From (234), (226) (Th 8) follows

$$(236) \quad a_v^{-1} = \bar{a}_v \quad (v = 1, 2, \dots, n).$$

From (227), (236) follows (235).

Th 14. From (185), (187), (208), (209),

$$(237) \quad \bar{b}_\mu \bar{b}_v = \begin{cases} 1 & (\mu = v) \\ 0 & (\mu \neq v) \end{cases} \quad (\mu, v = 1, 2, \dots, n)$$

follows

$$(238) \quad \bar{a}_v^{-1} = \sum_{\mu=1}^n \alpha_{\mu v} \bar{b}_\mu \quad (v = 1, 2, \dots, n).$$

Dm. From (237), (209) follows

$$(239) \quad \bar{b}_v^{-1} = \bar{b}_v \quad (v = 1, 2, \dots, n).$$

From (227), (239) follows (238).

Th 15. From (185), (187), (208), (209), (234), (237) follows

$$(240) \quad \bar{a}_v = \sum_{\mu=1}^n \alpha_{\mu v} \bar{b}_\mu \quad (v = 1, 2, \dots, n).$$

Dm Th 13, Th 14.

Th 16. From (185), (187), (208), (209), (235) follows (234).

Dm. From (227) (Th 10) and (235) follows (236). From (236) follows (234).

Th 17. From (185), (187), (208), (209), (238) follows (237).

Dm. From (227) (Th 10) and (238) follows

$$(241) \quad \sum_{\mu=1}^n \alpha_{\mu\nu} (\bar{b}_{\mu}^{-1} - \bar{b}_{\mu}) = \bar{0} \quad (\nu = 1, 2, \dots, n).$$

From (241), (186) (Th 4) follows (239) (Th 3). From (239) follows (237).

7. Let

$$(242) \quad q_{\nu} \in R \quad (\nu = 1, 2, \dots, n)$$

and

$$(243) \quad p_{\nu} = p_{\nu}(q_1, q_2, \dots, q_n) \quad (\nu = 1, 2, \dots, n)$$

for

$$(244) \quad (q_1, q_2, \dots, q_n) \in Q \subset R^n$$

with

$$(245) \quad \frac{D(p_1, p_2, \dots, p_n)}{D(q_1, q_2, \dots, q_n)} \neq 0.$$

From

$$(246) \quad \rho = \bar{\rho}(q_1, q_2, \dots, q_n) = \bar{r}(p_1(q_1, q_2, \dots, q_n),$$

$$p_2(q_1, q_2, \dots, q_n), \dots, p_n(q_1, q_2, \dots, q_n))$$

follows

$$(247) \quad G\left(\frac{\partial \bar{\rho}}{\partial q_{\nu}}\right)_{\nu=1}^n \neq 0.$$

Indeed for

$$(248) \quad \rho_{\nu} = \frac{\partial \bar{\rho}}{\partial q_{\nu}} \quad (\nu = 1, 2, \dots, n)$$

from (246), (8) follows

$$(249) \quad \rho_{\mu} = \sum_{\nu=1}^n \frac{\partial p_{\nu}}{\partial q_{\mu}} \pi_{\nu} \quad (\mu = 1, 2, \dots, n).$$

From (249) and

$$(250) \quad \sum_{\mu=1}^n r_{\mu} \bar{\rho}_{\mu} = \bar{0} \quad (r_{\mu} \in R; \mu = 1, 2, \dots, n)$$

follows

$$(251) \quad \sum_{\nu=1}^n \left(\sum_{\mu=1}^n r_{\mu} \frac{\partial p_{\nu}}{\partial q_{\mu}} \right) \pi_{\nu} = \bar{0}.$$

From (251), (7) follows

$$(252) \quad \sum_{\mu=1}^n r_\mu \frac{\partial p_\nu}{\partial q_\mu} = 0 \quad (\nu = 1, 2, \dots, n).$$

From (252) and

$$(253) \quad \sum_{\nu=1}^n r_\nu^2 \neq 0$$

would follow

$$(254) \quad \frac{D(p_1, p_2, \dots, p_n)}{D(q_1, q_2, \dots, q_n)} = 0,$$

contrary to (245). Hence

$$(255) \quad r_\nu = 0 \quad (\nu = 1, 2, \dots, n)$$

which together with (250), (248) proves (247).

From (249) follows

$$(256) \quad \frac{\partial p_\lambda}{\partial q_\mu} = \bar{\pi}_\lambda^{-1} \bar{\rho}_\mu \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (256) follows

$$(257) \quad \frac{D(p_1, p_2, \dots, p_n)}{D(q_1, q_2, \dots, q_n)} = G(\bar{\pi}_1^{-1}, \bar{\rho}_1, \dots, \bar{\pi}_n^{-1}, \bar{\rho}_n) \neq 0$$

for (7), (248), (247) according to [1], p. 294, 299. Thus conversely from (247) follows (245).

The introduction of the Cartesian coordinates is based on the following circumstance: for

$$(258) \quad \bar{r} = \sum_{\nu=1}^n p_\nu \bar{\pi}_\nu = \sum_{\nu=1}^n q_\nu \bar{\rho}_\nu$$

with

$$(259) \quad \frac{\partial \bar{\pi}_\nu}{\partial p_\mu} = \frac{\partial \bar{\rho}_\nu}{\partial q_\mu} = 0 \quad (\mu, \nu = 1, 2, \dots, n)$$

there exists a linear dependence between the curvilinear coordinates (6) and (244). Indeed, from (258) follows

$$(260) \quad p_\mu = \sum_{\nu=1}^n q_\nu \bar{\pi}_\mu^{-1} \bar{\rho}_\nu \quad (\mu = 1, 2, \dots, n).$$

From (259) follows

$$(261) \quad \frac{\partial \bar{\pi}_\mu^{-1}}{\partial p_\lambda} = 0 \quad (\lambda, \mu = 1, 2, \dots, n)$$

and

$$(262) \quad \frac{\partial \bar{\rho}_\nu}{\partial p_\lambda} = \sum_{\sigma=1}^n \frac{\partial q_\sigma}{\partial p_\lambda} \frac{\partial \bar{\rho}_\nu}{\partial q_\sigma} = 0 \quad (\lambda, \nu = 1, 2, \dots, n).$$

From (261), (262) follows

$$(263) \quad \frac{\partial (\bar{\pi}_\mu^{-1} \bar{\rho}_\nu)}{\partial p_\lambda} = 0 \quad (\lambda, \mu, \nu = 1, 2, \dots, n).$$

It is proved analogically that

$$(264) \quad \frac{\partial (\bar{\pi}_\mu^{-1} \bar{\rho}_\nu)}{\partial q_\lambda} = 0 \quad (\lambda, \mu, \nu = 1, 2, \dots, n).$$

For (2), (3), (6), (9)–(11) let by definition

$$(265) \quad \text{grad } p_\mu = \sum_{\nu=1}^n \frac{\partial p_\mu}{\partial x_\nu} e_\nu \quad (\mu = 1, 2, \dots, n).$$

It is not difficult to see that

$$(266) \quad \bar{\pi}_\nu^{-1} = \text{grad } p_\nu \quad (\nu = 1, 2, \dots, n).$$

Indeed from (9)–(11), (8) follows

$$(267) \quad \frac{\partial r}{\partial x_\mu} = e_\mu = \sum_{\nu=1}^n \frac{\partial p_\nu}{\partial x_\mu} \bar{\pi}_\nu \quad (\mu = 1, 2, \dots, n).$$

From (3) follows

$$(268) \quad \bar{e}_\mu^{-1} = \bar{e}_\nu \quad (\nu = 1, 2, \dots, n).$$

From (267), (268), (22), Th 14 follows

$$(269) \quad \bar{\pi}_\mu^{-1} = \sum_{\nu=1}^n \frac{\partial p_\mu}{\partial x_\nu} \bar{e}_\nu \quad (\mu = 1, 2, \dots, n),$$

i. e. (266) according to (265).

From (266) follows, that the system (6) of curvilinear coordinates is orthogonal iff

$$(270) \quad \text{grad } p_\lambda \text{ grad } p_\mu = 0 \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu).$$

From (266) and

$$(271) \quad \bar{r} = \sum_{\nu=1}^n (\bar{r} \bar{x}_\nu) \bar{\pi}_\nu$$

follows

$$(272) \quad \bar{r} = \sum_{\nu=1}^n (\bar{r} \operatorname{grad} p_\nu) \bar{\pi}_\nu.$$

From (266) still follows

$$(273) \quad \bar{x}_\nu^0 = -\frac{\operatorname{grad} p_\nu}{\operatorname{grad} p_\nu} \quad (\nu = 1, 2, \dots, n).$$

In the case (270) from (16), (266) follows

$$(274) \quad \bar{\pi}_\nu = \pi_\nu^2 \operatorname{grad} p_\nu \quad (\nu = 1, 2, \dots, n)$$

and

$$(275) \quad \operatorname{grad} p_\nu = \frac{1}{\pi_\nu} \bar{\pi}_\nu^0 \quad (\nu = 1, 2, \dots, n).$$

8. As a nontrivial application let us consider the elliptical coordinates (6), which are defined as the n different real roots of the equation

$$(276) \quad \sum_{\nu=1}^n \frac{x_\nu^2}{a_\nu^2 + p} - 1 = 0$$

of n -th degree with respect to p for

$$(277) \quad a_1^2 > a_2^2 > \dots > a_n^2.$$

At that for definiteness let

$$(278) \quad -a_1^2 < p_1 < -a_2^2 < p_2 < \dots < -a_n^2 < p_n.$$

Below for brevity we shall use the notations

$$(279) \quad S_\lambda = \sum_{\sigma=1}^n \frac{x_\sigma^2}{a_\sigma^2 + p_\lambda},$$

$$(280) \quad S_{\lambda\mu} = \sum_{\sigma=1}^n \frac{x_\sigma^2}{(a_\sigma^2 + p_\lambda)(a_\sigma^2 + p_\mu)},$$

$$(281) \quad S_{\lambda\mu\nu} = \sum_{\sigma=1}^n \frac{x_\sigma^2}{(a_\sigma^2 + p_\lambda)(a_\sigma^2 + p_\mu)(a_\sigma^2 + p_\nu)} \quad (\lambda, \mu, \nu = 1, 2, \dots, n)$$

for (278).

From the definition of elliptical coordinates and from (279) follows

$$(282) \quad S_\lambda = 1 \quad (\lambda = 1, 2, \dots, n).$$

From (282), (280) follows

$$(283) \quad (p_\lambda - p_\mu) S_{\lambda\mu} = S_\lambda - S_\mu = 0 \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (283), (278) follows

$$(284) \quad S_{\lambda\mu} = 0 \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu).$$

Let

$$(285) \quad P(p) = \sum_{\nu=1}^n \frac{x_\nu^2}{a_\nu^2 + p} - 1.$$

If we reduce to a common denominator the sum in the right side of (285) the numerator becomes a polynomial of n -th degree in p , whose zeros are the elliptical coordinates (6) with (278), (279), (282) and the coefficient of p^n is equal to -1 . Hence

$$(286) \quad P(p) = - \prod_{\nu=1}^n \frac{p - p_\nu}{a_\nu^2 + p}.$$

From (286) follows

$$(287) \quad \left(\frac{dP(p)}{dp} \right)_{p=p_\lambda} = -R_\lambda \quad (\lambda = 1, 2, \dots, n),$$

where

$$(288) \quad R_\lambda = \prod_{\substack{\nu=1 \\ (\nu \neq \lambda)}}^n (p_\lambda - p_\nu) \left(\prod_{\nu=1}^n (a_\nu^2 + p_\lambda) \right)^{-1} \quad (\lambda = 1, 2, \dots, n).$$

From (285), (280) follows

$$(289) \quad \left(\frac{dP(p)}{dp} \right)_{p=p_\lambda} = -S_{\lambda\lambda} \quad (\lambda = 1, 2, \dots, n).$$

From (287), (289) follows

$$(290) \quad S_{\lambda\lambda} = R_\lambda \quad (\lambda = 1, 2, \dots, n).$$

From (279), (282), (280) follows

$$(291) \quad \frac{\partial S_\lambda}{\partial x_\mu} = \frac{2x_\mu}{a_\mu^2 + p_\lambda} - S_{\lambda\lambda} \frac{\partial p_\lambda}{\partial x_\mu} = 0 \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (291) follows

$$(292) \quad \frac{\partial p_\lambda}{\partial x_\mu} = \frac{2x_\mu}{a_\mu^2 + p_\lambda} - S_{\lambda\lambda}^{-1} \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (292), (290) follows

$$(293) \quad \frac{\partial p_\lambda}{\partial x_\mu} = \frac{2x_\mu}{a_\mu^2 + p_\lambda} R_\lambda^{-1} \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (293), (265) follows

$$(294) \quad \text{grad } p_\lambda = 2R_\lambda^{-1} \sum_{v=1}^n \frac{x_v}{a_v^2 + p_\lambda} \bar{e}_v \quad (\lambda = 1, 2, \dots, n).$$

From (294) follows

$$(295) \quad \text{grad } p_\lambda \text{ grad } p_\mu = 4R_\lambda^{-1} R_\mu^{-1} S_{\lambda\mu} \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (295), (284) follows (270).

From (294), (3), (280) follows

$$(296) \quad (\text{grad } p_\lambda)^2 = 4R_\lambda^{-2} S_{\lambda\lambda} \quad (\lambda = 1, 2, \dots, n).$$

From (296), (290) follows

$$(297) \quad |\text{grad } p_\lambda| = 2R_\lambda^{-1/2} \quad (\lambda = 1, 2, \dots, n).$$

From (270) follows

$$(298) \quad \pi_v^0 = x_v^0 \quad (v = 1, 2, \dots, n).$$

From (298), (273), (294), (297) follows

$$(299) \quad \bar{\pi}_\lambda^0 = R_\lambda^{-1/2} \sum_{v=1}^n \frac{x_v}{a_v^2 + p_\lambda} \bar{e}_v \quad (\lambda = 1, 2, \dots, n).$$

From

$$(300) \quad \bar{r} = \sum_{\lambda=1}^n (r \bar{\pi}_\lambda^0) \pi_\lambda^0$$

according to (270) and from (9), (3), (279), (282) follows

$$(301) \quad \bar{r} = \sum_{\lambda=1}^n R_\lambda^{-1/2} \bar{\pi}_\lambda^0.$$

So in order to find the expression (301) for the radius vector \bar{r} of an arbitrary point in function of the elliptical coordinates and the elliptical tangential unit vectors (299) it is not necessary to solve the system of equations (282) with (271) in respect to x_v ($v = 1, 2, \dots, n$). From (299) however it is seen, that in order to express the elliptical tangential unit vectors in function of the elliptical coordinates only it is necessary to solve these equations.

In order to find the expression (301) one could use the equality (24), which in the case of an orthogonal system (6) of curvilinear coordinates takes the form (25). From (9), (3) follows

$$(302) \quad r^2 = \sum_{v=1}^n x_v^2,$$

and then

$$(303) \quad \frac{\partial r^2}{\partial p_\lambda} = 2 \sum_{v=1}^n x_v \frac{\partial x_v}{\partial p_\lambda} \quad (\lambda = 1, 2, \dots, n).$$

From (279), (282), (280), (290) follows

$$(304) \quad \sum_{v=1}^n \frac{x_v}{a_v^2 + p_\mu} \frac{\partial x_v}{\partial p_\lambda} = \begin{cases} R_\lambda & (\lambda = \mu), \\ 0 & (\lambda \neq \mu) \end{cases}$$

($\lambda, \mu = 1, 2, \dots, n$). From the system of equations (303) one could determine the quantities $x_v \frac{\partial x_v}{\partial p_\lambda}$ ($\lambda, v = 1, 2, \dots, n$), which are necessary for (303). This determination also does not require to know x_v ($v = 1, 2, \dots, n$) in function of the elliptical coordinates.

Let

$$(305) \quad \Delta_n = \begin{vmatrix} \frac{1}{a_1^2 + p_1} & \frac{1}{a_2^2 + p_1} & \cdots & \frac{1}{a_n^2 + p_1} \\ \frac{1}{a_1^2 + p_2} & \frac{1}{a_2^2 + p_2} & \cdots & \frac{1}{a_n^2 + p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a_1^2 + p_n} & \frac{1}{a_2^2 + p_n} & \cdots & \frac{1}{a_n^2 + p_n} \end{vmatrix}.$$

From (305) follows

$$(306) \quad \Delta_n = \begin{vmatrix} \frac{a_n^2 - a_1^2}{(a_1^2 + p_1)(a_n^2 + p_1)} & \frac{a_n^2 - a_2^2}{(a_2^2 + p_1)(a_n^2 + p_1)} & \cdots & \frac{1}{a_n^2 + p_1} \\ \frac{a_n^2 - a_1^2}{(a_1^2 + p_2)(a_n^2 + p_2)} & \frac{a_n^2 - a_2^2}{(a_2^2 + p_2)(a_n^2 + p_2)} & \cdots & \frac{1}{a_n^2 + p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_n^2 - a_1^2}{(a_1^2 + p_n)(a_n^2 + p_n)} & \frac{a_n^2 - a_2^2}{(a_2^2 + p_n)(a_n^2 + p_n)} & \cdots & \frac{1}{a_n^2 + p_n} \end{vmatrix} = P_n \delta_n,$$

where

$$(307) \quad P_\lambda = \prod_{\substack{v=1 \\ (v \neq \lambda)}}^n (a_\lambda^2 - a_v^2) \left(\prod_{v=1}^n (a_\lambda^2 + p_v) \right)^{-1} \quad (\lambda = 1, 2, \dots, n)$$

and

$$(308) \quad \delta_n = \begin{vmatrix} \frac{1}{a_1^2 + p_1} & \frac{1}{a_2^2 + p_1} & \cdots & \frac{1}{a_{n-1}^2 + p_1} & 1 \\ \frac{1}{a_1^2 + p_2} & \frac{1}{a_2^2 + p_2} & \cdots & \frac{1}{a_{n-1}^2 + p_2} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a_1^2 + p_n} & \frac{1}{a_2^2 + p_n} & \cdots & \frac{1}{a_{n-1}^2 + p_n} & 1 \end{vmatrix}.$$

From (308) follows

$$(309) \quad \delta_n = \begin{vmatrix} \frac{p_n - p_1}{(a_1^2 + p_1)(a_2^2 + p_n)} & \frac{p_n - p_1}{(a_2^2 + p_1)(a_2^2 + p_n)} & \cdots & 0 \\ \frac{p_n - p_2}{(a_1^2 + p_2)(a_2^2 + p_n)} & \frac{p_n - p_2}{(a_2^2 + p_2)(a_2^2 + p_n)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{p_n - p_{n-1}}{(a_1^2 + p_{n-1})(a_2^2 + p_n)} & \frac{p_n - p_{n-1}}{(a_2^2 + p_{n-1})(a_2^2 + p_n)} & \cdots & 0 \\ \frac{1}{a_1^2 + p_n} & \frac{1}{a_2^2 + p_n} & \cdots & 1 \end{vmatrix} = R_n \Delta_{n-1},$$

where

$$(310) \quad \Delta_{n-1} = \begin{vmatrix} \frac{1}{a_1^2 + p_1} & \frac{1}{a_2^2 + p_1} & \cdots & \frac{1}{a_{n-1}^2 + p_1} \\ \frac{1}{a_1^2 + p_2} & \frac{1}{a_2^2 + p_2} & \cdots & \frac{1}{a_{n-1}^2 + p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a_1^2 + p_{n-1}} & \frac{1}{a_2^2 + p_{n-1}} & \cdots & \frac{1}{a_{n-1}^2 + p_{n-1}} \end{vmatrix}.$$

From (306)–(310) by induction follows

$$(311) \quad \Delta_n = \frac{\prod_{\mu=2}^n \prod_{v=1}^{\mu-1} (a_\mu^2 - a_v^2) (p_\mu - p_v)}{\prod_{\mu=1}^n \prod_{v=1}^n (a_\mu^2 + p_v)} \quad (n=2, 3, \dots).$$

From (279), (282), (306), (308) follows

$$(312) \quad x_n^2 = \frac{\delta_n}{\Delta_n} = P_n^{-1}.$$

Analogically to (312) one receives in general

$$(313) \quad x_\lambda^2 = P_\lambda^{-1} \quad (\lambda = 1, 2, \dots, n),$$

i. e.

$$(314) \quad x_\lambda = \varepsilon_\lambda P_\lambda^{-1/2} \quad (\lambda = 1, 2, \dots, n),$$

where $\varepsilon_\lambda (\lambda = 1, 2, \dots, n)$ take the values +1 and -1 independently from one another.

From (299), (314) follows

$$(315) \quad \bar{\pi}_\lambda^0 = R_\lambda^{-1/2} \sum_{\nu=1}^n \frac{\varepsilon_\nu}{P_\nu^{1/2}(a_\nu^2 + p_\lambda)} \bar{e}_\nu, \quad (\lambda = 1, 2, \dots, n).$$

From (315), (301) follows

$$(316) \quad \bar{r} = \sum_{\nu=1}^n \sum_{\lambda=1}^n \frac{\varepsilon_\nu}{R_\lambda P_\nu^{1/2}(a_\nu^2 + p_\lambda)} \bar{e}_\nu.$$

On the other hand, from (9), (314) follows

$$(317) \quad \bar{r} = \sum_{\lambda=1}^n \varepsilon_\lambda P_\lambda^{-1/2} \bar{e}_\lambda.$$

From (316), (317) follows the equality

$$(318) \quad \sum_{\nu=1}^n \frac{1}{R_\lambda(a_\nu^2 + p_\lambda)} = 1 \quad (\lambda = 1, 2, \dots, n).$$

9. Let H be a nontrivial real Hilbert space,

$$(319) \quad s_\nu \in R \quad (\nu = 1, 2, \dots, n),$$

$$(320) \quad a = \bar{a}(s_1, s_2, \dots, s_m)$$

for

$$(321) \quad (s_1, s_2, \dots, s_m) \in S \subset R^m.$$

When

$$(322) \quad \bar{a} \neq 0$$

from

$$(323) \quad \bar{a}^0 = \frac{\bar{a}}{a}$$

follows

$$(324) \quad \frac{\partial \bar{a}^0}{\partial s_\nu} = \frac{1}{a^2} \left(a \frac{\partial \bar{a}}{\partial s_\nu} - \frac{\partial a}{\partial s_\nu} \bar{a} \right) \quad (\nu = 1, 2, \dots, m).$$

From (323), (324) and

$$(325) \quad a \frac{\partial a}{\partial s_v} = \bar{a} \frac{\partial \bar{a}}{\partial \bar{s}_v} \quad (v=1, 2, \dots, m)$$

follows

$$(326) \quad \frac{\partial \bar{a}^0}{\partial s_v} = \frac{1}{a} \left(\frac{\partial a}{\partial s_v} - \left(a^0 \frac{\partial \bar{a}}{\partial \bar{s}_v} \right) \bar{a}^0 \right) \quad (v=1, 2, \dots, m).$$

From (8), (12), (326) follows

$$(327) \quad \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} = \frac{1}{\left| \frac{\partial r}{\partial p_\lambda} \right|} \left(\frac{\partial^2 r}{\partial p_\lambda \partial p_\mu} - \left(\bar{\pi}_\lambda^0 \frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \right) \bar{\pi}_\lambda^0 \right) \quad (\lambda, \mu=1, 2, \dots, n).$$

From the obvious equality

$$(328) \quad \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} = \sum_{v=1}^n \left(\frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} (\bar{\pi}_v^0)^{-1} \right) \bar{\pi}_v^0 \quad (\lambda, \mu=1, 2, \dots, n)$$

and from (327) follows

$$(329) \quad \begin{aligned} \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} &= \frac{1}{\left| \frac{\partial r}{\partial p_\lambda} \right|} \sum_{v=1}^n \left(\frac{\partial^2 r}{\partial p_\lambda \partial p_\mu} (\bar{\pi}_v^0)^{-1} \right) \bar{\pi}_v^0 \\ &\quad - \frac{1}{\left| \frac{\partial r}{\partial p_\lambda} \right|} \sum_{v=1}^n \left(\bar{\pi}_\lambda^0 \frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \right) (\bar{\pi}_\lambda^0 (\bar{\pi}_v^0)^{-1}) \bar{\pi}_v^0, \end{aligned}$$

i. e.

$$(330) \quad \begin{aligned} \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} &= \frac{1}{\left| \frac{\partial r}{\partial p_\lambda} \right|} \sum_{v=1}^n \left(\frac{\partial^2 r}{\partial p_\lambda \partial p_\mu} (\bar{\pi}_v^0)^{-1} \right) \bar{\pi}_v^0 \\ &\quad - \frac{1}{\left| \frac{\partial r}{\partial p_\lambda} \right|} \left(\frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \bar{\pi}_\lambda^0 \right) \bar{\pi}_\lambda^0 \quad (\lambda, \mu=1, 2, \dots, n). \end{aligned}$$

In the case (15) of an orthogonal system of curvilinear coordinates the equality (330) takes the form

$$(331) \quad \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} = \frac{1}{\left| \frac{\partial r}{\partial p_\lambda} \right|} \sum_{\substack{v=1 \\ (v \neq \lambda)}}^n \left(\frac{\partial^2 r}{\partial p_\lambda \partial p_\mu} \bar{\pi}_v^0 \right) \bar{\pi}_v^0 \quad (\lambda, \mu=1, 2, \dots, n).$$

From (26) follows

$$(332) \quad \frac{d\bar{\pi}_\lambda^0}{dt} = \sum_{\mu=1}^n p_\mu \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} \quad (\lambda = 1, 2, \dots, n).$$

From (332), (330) follows

$$(333) \quad \begin{aligned} \frac{d\bar{\pi}_\lambda^0}{dt} &= \frac{1}{\left| \frac{\partial \bar{r}}{\partial p_\lambda} \right|} \sum_{\mu=1}^n p_\mu \sum_{\nu=1}^n \left(\frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} (\bar{\pi}_\nu^0)^{-1} \right) \bar{\pi}_\nu^0 \\ &- \frac{1}{\left| \frac{\partial \bar{r}}{\partial p_\lambda} \right|} \sum_{\mu=1}^n p_\mu \left(\frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \bar{\pi}_\lambda^0 \right) \bar{\pi}_\lambda^0 \quad (\lambda = 1, 2, \dots, n). \end{aligned}$$

In the case (15) of an orthogonal system of curvilinear coordinates the equality (333) takes the form

$$(334) \quad \frac{d\bar{\pi}_\lambda^0}{dt} = \frac{1}{\left| \frac{\partial \bar{r}}{\partial p_\lambda} \right|} \sum_{\nu=1}^n \sum_{\mu=1}^n p_\mu \left(\frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \bar{\pi}_\nu^0 \right) \bar{\pi}_\nu^0 \quad (\lambda = 1, 2, \dots, n).$$

10. From (284), (290) follows

$$(335) \quad S_{\lambda\mu} = \begin{cases} R_\lambda & (\lambda = \mu) \\ 0 & (\lambda \neq \mu) \end{cases} \quad (\lambda, \mu = 1, 2, \dots, n).$$

From

$$(336) \quad \frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

for

$$(337) \quad a \neq b \neq c \neq a$$

follows

$$(338) \quad A = \frac{1}{(a-b)(a-c)}, \quad B = \frac{1}{(b-a)(b-c)}, \quad C = \frac{1}{(c-a)(c-b)}.$$

From (338) follows

$$(339) \quad A + B + C = 0.$$

Let

$$(340) \quad \lambda \neq \mu \neq \nu \neq \lambda \quad (\lambda, \mu, \nu = 1, 2, \dots, n).$$

From (340), (278) follows

$$(341) \quad p_\lambda \neq p_\mu \neq p_\nu \neq p_\lambda \quad (\lambda, \mu, \nu = 1, 2, \dots, n).$$

From (278) follows

$$(342) \quad S_{\lambda\mu\nu} = \sum_{\sigma=1}^n x_{\sigma}^2 \left(\frac{A_{\lambda}}{a_{\sigma}^2 + p_{\lambda}} + \frac{A_{\mu}}{a_{\sigma}^2 + p_{\mu}} + \frac{A_{\nu}}{a_{\sigma}^2 + p_{\nu}} \right)$$

with

$$(343) \quad A_{\lambda} + A_{\mu} + A_{\nu} = 0.$$

From (342), (279) follows

$$(344) \quad S_{\lambda\mu\nu} = A_{\lambda}S_{\lambda} + A_{\mu}S_{\mu} + A_{\nu}S_{\nu}.$$

From (344), (343), (282) follows

$$(345) \quad S_{\lambda\mu\nu} = 0 \quad (\lambda, \mu, \nu = 1, 2, \dots, n; \lambda \neq \mu \neq \nu \neq \lambda).$$

From (313), (307) follows

$$(346) \quad 2x_{\lambda} \frac{\partial x_{\lambda}}{\partial p_{\mu}} = \frac{\partial P_{\lambda}^{-1}}{\partial p_{\mu}} = \frac{P_{\lambda}^{-1}}{a_{\lambda}^2 + p_{\mu}} = \frac{x_{\lambda}^2}{a_{\lambda}^2 + p_{\mu}} \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (346) follows

$$(347) \quad \frac{\partial x_{\lambda}}{\partial p_{\mu}} = \frac{x_{\lambda}}{2(a_{\lambda}^2 + p_{\mu})} \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (290) follows

$$(348) \quad \frac{\partial S_{\lambda\mu}}{\partial p_{\mu}} = \frac{\partial R_{\lambda}}{\partial p_{\mu}} \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (280), (346) follows

$$(349) \quad \frac{\partial S_{\lambda\mu}}{\partial p_{\mu}} = \sum_{\sigma=1}^n \frac{1}{(a_{\sigma}^2 + p_{\lambda})^2} 2x_{\sigma} \frac{\partial x_{\sigma}}{\partial p_{\mu}} = S_{\lambda\lambda\mu} \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu).$$

From (288) follows

$$(350) \quad \frac{\partial R_{\lambda}}{\partial p_{\mu}} = -\frac{R_{\lambda}}{p_{\lambda} - p_{\mu}} \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu).$$

From (348)–(350) follows

$$(351) \quad S_{\lambda\lambda\mu} = \frac{R_{\lambda}}{p_{\mu} - p_{\lambda}} \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu).$$

From (9), (10), (347) follows

$$(352) \quad \frac{\partial \bar{r}}{\partial p_{\lambda}} = \frac{1}{2} \sum_{\nu=1}^n \frac{x_{\nu}}{a_{\nu}^2 + p_{\lambda}} \bar{e}_{\nu} \quad (\lambda = 1, 2, \dots, n).$$

From (352), (347) follows

$$(353) \quad \frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} = \frac{1}{4} \sum_{\nu=1}^n \frac{x_\nu}{(a_\nu^2 + p_\lambda)(a_\nu^2 + p_\mu)} \bar{e}_\nu \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu)$$

and

$$(354) \quad \frac{\partial^2 \bar{r}}{\partial p_\lambda^2} = -\frac{1}{4} \sum_{\nu=1}^n \frac{x_\nu}{(a_\nu^2 + p_\lambda)^2} \bar{e}_\nu \quad (\lambda = 1, 2, \dots, n).$$

The equalities (353), (354) can be written together in the form

$$(355) \quad \frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} = \frac{2 \operatorname{sgn}^2(\lambda - \mu) - 1}{4} \sum_{\nu=1}^n \frac{x_\nu}{(a_\nu^2 + p_\lambda)(a_\nu^2 + p_\mu)} \bar{e}_\nu \quad (\lambda, \mu = 1, 2, \dots, n).$$

From (355), (299), (3), (281) follows

$$(356) \quad \frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \bar{\pi}_\nu^0 = \frac{2 \operatorname{sgn}^2(\lambda - \mu) - 1}{4} R_\nu^{-1/2} S_{\lambda \mu \nu}.$$

From (331) follows, that in order to find $\frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu}$ ($\lambda, \mu = 1, 2, \dots, n$) we can restrict ourselves to the case

$$(357) \quad \lambda \neq \nu$$

In (356). From (356), (345), (351), (357) follows

$$(358) \quad \frac{\partial^2 \bar{r}}{\partial p_\lambda^2} \bar{\pi}_\nu^0 = -\frac{1}{4(p_\nu - p_\lambda)} R_\nu^{-1/2} R_\lambda \quad (\lambda, \nu = 1, 2, \dots, n; \lambda \neq \nu),$$

$$(359) \quad \frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \bar{\pi}_\nu^0 = 0 \quad (\lambda, \mu, \nu = 1, 2, \dots, n; \lambda \neq \mu \neq \nu \neq \lambda),$$

$$(360) \quad \frac{\partial^2 \bar{r}}{\partial p_\lambda \partial p_\mu} \bar{\pi}_\mu^0 = \frac{1}{4(p_\lambda - p_\mu)} R_\mu^{1/2} \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu).$$

In accordance to the cases (358)–(360) from (331) follows

$$(361) \quad \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\lambda} = \frac{R_\lambda}{4 \left| \frac{\partial \bar{r}}{\partial p_\lambda} \right|} \sum_{\substack{\nu=1 \\ (\nu \neq \lambda)}}^n \frac{1}{(p_\nu - p_\lambda) R_\nu^{1/2}} \bar{\pi}_\nu^0,$$

$$(362) \quad \frac{\partial \bar{\pi}_\lambda^0}{\partial p_\mu} = -\frac{R_\mu^{1/2} \bar{\pi}_\mu^0}{4 \left| \frac{\partial \bar{r}}{\partial p_\lambda} \right| (p_\lambda - p_\mu)} \quad (\lambda, \mu = 1, 2, \dots, n; \lambda \neq \mu).$$

From (334) and (361), (362) the expression for $\frac{d\pi_\lambda^0}{dt}$ ($\lambda=1, 2, \dots, n$) in elliptical coordinates is obtained.

11. The results from § 6 and especially Th 9, Th 10 suggest a manner for an explicit construction of an orthogonal basis of a finite-dimensional linear space, defined by some basis.

Let

$$(187) \quad a_v \in E \quad (v=1, 2, \dots, n),$$

$$(214) \quad G(a_v)_{v=1}^n \neq 0$$

and let

$$(363) \quad L(a_v)_{v=1}^m = \left\{ a : a = \sum_{v=1}^m \alpha_v a_v (\alpha_v \in R) \right\} \quad (m=1, 2, \dots, n)$$

be a m -dimensional linear space, spanned on the reper

$$(364) \quad \bar{a}_\mu \in E \quad (\mu=1, 2, \dots, m).$$

Let us seek vectors

$$(204) \quad \bar{b}_v \in E \quad (v=1, 2, \dots, n)$$

with

$$(209) \quad G(\bar{b}_v)_{v=1}^n \neq 0$$

and

$$(365) \quad \bar{b}_\mu \in L(\bar{a}_v)_{v=1}^m, \quad (\mu=1, 2, \dots, m),$$

$$(366) \quad \bar{b}_\mu \bar{b}_v = 0 \quad (\mu, v=1, 2, \dots, n; \mu \neq v).$$

The condition (365) is equivalent to

$$(367) \quad \bar{b}_\mu = \sum_{v=1}^n \alpha_{\mu v} \bar{a}_v \quad (\mu=1, 2, \dots, n),$$

where

$$(368) \quad \alpha_{\mu v} \in R \quad (\mu=1, 2, \dots, n; v=1, 2, \dots, n).$$

Let

$$(369) \quad \alpha_{\mu v} = 0 \quad (\mu=1, 2, \dots, n-1; v=\mu+1, \mu+2, \dots, n).$$

From (368), (369) follows, that

$$(370) \quad \text{Det } \alpha_{\mu v} \neq 0 \quad (\mu, v=1, 2, \dots, n)$$

iff

$$(371) \quad \alpha_{vv} \neq 0 \quad (v=1, 2, \dots, n);$$

these conditions are supposed satisfied.

From (370), (214), (367), Th 9 follows (209).

From $m=1, 2, \dots, n$ let us consider successively the systems of equations

$$(372) \quad b_\mu = \sum_{\nu=1}^m \alpha_{\mu\nu} a_\nu \quad (\mu=1, 2, \dots, m).$$

Let

$$(373) \quad \bar{a}_{m\mu}^{-1} = \frac{(-1)^{\mu+1}}{G(a_\nu)_{\nu=1}^n} \begin{vmatrix} a_1 & a_2 & \dots & a_m \\ a_1^2 & a_1 a_2 & \dots & a_1 a_m \\ \bar{a}_2 \bar{a}_1 & a_2^2 & \dots & \bar{a}_2 a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_{\mu-1} a_1 & \bar{a}_{\mu-1} \bar{a}_2 & \dots & a_{\mu-1} a_n \\ \bar{a}_{\mu+1} \bar{a}_1 & a_{\mu+1} a_2 & \dots & \bar{a}_{\mu+1} \bar{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_m a_1 & a_m a_2 & \dots & a_m^2 \end{vmatrix}$$

$(\mu=1, 2, \dots, m)$ be the Gibbs' vectors of the reper (364). Then from the system of equations (372), Th 10 and (369) follows especially

$$(374) \quad \bar{a}_{mm}^{-1} = \sum_{\mu=1}^m \alpha_{\mu m} \bar{b}_{m\mu}^{-1} = \alpha_{mm} \bar{b}_{mm}^{-1},$$

where

$$(375) \quad \bar{b}_{m\mu}^{-1} \in E \quad (\mu=1, 2, \dots, m)$$

are the Gibbs' vectors of the reper

$$(376) \quad \bar{b}_\mu \in E \quad (\mu=1, 2, \dots, m).$$

The conditions

$$(377) \quad \bar{b}_\mu \bar{b}_\nu = 0 \quad (\mu, \nu=1, 2, \dots, m; \mu \neq \nu)$$

are equivalent to

$$(378) \quad \bar{b}_\mu = b_\mu^2 \bar{b}_{m\mu}^{-1} \quad (\mu=1, 2, \dots, m).$$

From (374), (378), (371), (209) follows

$$(379) \quad \bar{b}_m = \frac{1}{\alpha_{mm} b_m^2} \bar{a}_{mm}^{-1} \quad (m=1, 2, \dots, n).$$

Since the conditions (214), (209), (365), (366) are independent from the norming of the vectors (204), because of (373), one may put

$$(380) \quad \bar{b}_\nu = \hat{a}_\nu, \quad (\nu=1, 2, \dots, n)$$

instead of (379), where

$$(381) \quad \hat{a}_v = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_v \\ a_1^2 & \bar{a}_1 \bar{a}_2 & \dots & \bar{a}_1 \bar{a}_v \\ \bar{a}_2 \bar{a}_1 & a_2^2 & \dots & \bar{a}_2 \bar{a}_v \\ \dots & \dots & \dots & \dots \\ \bar{a}_{v-1} \bar{a}_1 & \bar{a}_{v-1} \bar{a}_2 & \dots & \bar{a}_{v-1} \bar{a}_v \end{vmatrix} \quad (v=1, 2, \dots, n).$$

Naturally the above considerations have only heuristic and not demonstrative character. It is established in the following paragraph, that the vectors (381) form an orthogonal basis of (363) and some of their fundamental properties are studied. The norming of the basis (381) leads to simple explicit expressions. Some of the propositions of § 12 are well known and are given here only for completeness.

12. Let C be the set of all complex numbers, E_C be a complex Euclidean space (pre-Hilbert space over C) and

$$(382) \quad \bar{a}_v \in E_C \quad (v=1, 2, \dots, n),$$

$$(383) \quad b_v \in E_C \quad (v=1, 2, \dots, n).$$

Then by definition

$$(384) \quad G(\bar{a}_v, \bar{b}_v)_{v=1}^n = \begin{vmatrix} \bar{a}_1 \bar{b}_1 & \bar{a}_2 \bar{b}_1 & \dots & \bar{a}_n \bar{b}_1 \\ \bar{a}_1 \bar{b}_2 & \bar{a}_2 \bar{b}_2 & \dots & \bar{a}_n \bar{b}_2 \\ \dots & \dots & \dots & \dots \\ \bar{a}_1 \bar{b}_n & \bar{a}_2 \bar{b}_n & \dots & \bar{a}_n \bar{b}_n \end{vmatrix}$$

and

$$(385) \quad G(\bar{a}_v)_{v=1}^n = G(\bar{a}_v, \bar{a}_v)_{v=1}^n.$$

Obviously (385) is the determinant of Gram for the vectors (382).

From (384), (385) follows

$$(386) \quad G(\bar{a}_v)_{v=1}^n = \begin{vmatrix} a_1^2 & \bar{a}_1 \bar{a}_2 & \dots & \bar{a}_1 \bar{a}_n \\ \bar{a}_2 \bar{a}_1 & a_2^2 & \dots & \bar{a}_2 \bar{a}_n \\ \dots & \dots & \dots & \dots \\ \bar{a}_n \bar{a}_1 & \bar{a}_n \bar{a}_2 & \dots & a_n^2 \end{vmatrix}.$$

For $z \in C$ let \bar{z} denote the conjugate number of z . According to the definition of E_C the equality

$$(387) \quad \bar{a} \bar{b} = \underline{\bar{b} a} \quad (\bar{a} \bar{b} \in E_C)$$

holds; particularly

$$(388) \quad a^2 = \bar{a}^2 = \underline{a} \bar{a} = \bar{a} \underline{a} = \underline{a}^2 = \bar{a}^2 \quad (\bar{a} \in E_C).$$

Th 18. From (382) follows

$$(389) \quad G(\underline{a}_v)_{v=1}^n = G(\bar{a}_v)_{v=1}^n.$$

Dm. The assertion is an immediate consequence from the definition (385), (384), (386) and from (387), (388).

Th. 19. From (382) follows (191) iff (382) are linearly dependent elements of E_C .

Dm. Let there exists

$$(390) \quad \alpha_v \in C \quad (v=1, 2, \dots, n)$$

with

$$(391) \quad \sum_{v=1}^n \alpha_v \underline{a}_v \neq 0,$$

$$(392) \quad \sum_{v=1}^n \alpha_v \bar{a}_v = \bar{0}.$$

From (392) follows

$$(393) \quad \sum_{v=1}^n \alpha_v \bar{a}_v \bar{a}_\mu = \bar{0} \bar{a}_\mu = 0 \quad (\mu=1, 2, \dots, n).$$

Since the determinant of the linear with respect to (391) system (393) is (385), from (391) follows (191).

Let now conversely (191) holds. Then there exists (390) with (391) and

$$(394) \quad \sum_{v=1}^n \alpha_v \bar{a}_v \bar{a}_\mu = 0 \quad (\mu=1, 2, \dots, n).$$

From (394) follows

$$(395) \quad \sum_{v=1}^n (\alpha_v \underline{a}_v) (\bar{a}_v \bar{a}_\mu) = 0 \quad (\mu=1, 2, \dots, n).$$

Since

$$(396) \quad (\alpha \bar{a}) (\beta \bar{b}) = (\alpha \beta) (\bar{a} \bar{b})$$

$(\alpha, \beta \in C, \bar{a}, \bar{b} \in E_C)$, from (395) follows

$$(397) \quad \sum_{v=1}^n (\alpha_v \bar{a}_v) (\alpha_\mu \bar{a}_\mu) = 0 \quad (\mu=1, 2, \dots, n).$$

From (397) follows

$$(398) \quad \sum_{\mu=1}^n \sum_{v=1}^n (\alpha_v \bar{a}_v) (\alpha_\mu \bar{a}_\mu) = \left(\sum_{v=1}^n \alpha_v \bar{a}_v \right)^2 = 0,$$

i. e. (392) for (390), (391), since according to the definition of E_C , from $\bar{a} \in E_C$ and $a^2=0$ follows $\bar{a}=\bar{0}$.

Let (382) hold and let

$$(399) \quad \alpha_{\mu v} \in C \quad (\mu, v = 1, 2, \dots, n).$$

Then by definition

$$(400) \quad \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \dots & \alpha_{3n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} = \sum_{v=1}^n \alpha_v \bar{a}_v,$$

where $\alpha_v (v=1, 2, \dots, n)$ is the cofactor of $\alpha_{1v} (v=1, 2, \dots, n)$ in the determinant

$$(401) \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \dots & \alpha_{3n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}.$$

For (382) let by definition

$$(402) \quad \hat{a}_\mu = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_\mu \\ \bar{a}_1^2 & \bar{a}_2 \bar{a}_1 & \dots & \bar{a}_\mu \bar{a}_1 \\ \bar{a}_1 \bar{a}_2 & \bar{a}_2^2 & \dots & \bar{a}_\mu \bar{a}_2 \\ \dots & \dots & \dots & \dots \\ \bar{a}_1 \bar{a}_{\mu-1} & \bar{a}_2 \bar{a}_{\mu-1} & \dots & \bar{a}_\mu \bar{a}_{\mu-1} \end{vmatrix} \quad (\mu = 1, 2, \dots, n).$$

The 20. From (382) follows

$$(403) \quad \hat{a}_\mu \bar{a}_v = \begin{cases} (-1)^{\mu+1} G(\bar{a}_\lambda)_{\lambda=1}^\mu (v=\mu), \\ 0 \quad (v=1, 2, \dots, \mu-1) \end{cases} \quad (\mu = 1, 2, \dots, n).$$

Dm. Immediate consequence from the definitions (402), (400), (384), (385).

Th 21. From (382) follows

$$(404) \quad \bar{a}_v a_\mu = \begin{cases} (-1)^{\mu+1} G(\bar{a}_1)_{i=1}^\mu (v=\mu), \\ 0 \quad (v=1, 2, \dots, \mu-1) \end{cases} \quad (\mu=1, 2, \dots, n).$$

Dm. Th 20, (387), Th 18.

In the following, when there is no danger of misunderstanding, we shall put for brevity

$$(405) \quad G_0 = 1, \quad G_\mu = G(\bar{a}_v)_{v=1}^\mu \quad (\mu=0, 1, 2, \dots, n).$$

Th 22. From (382) follows

$$(406) \quad \hat{a}_\mu \hat{a}_v = \begin{cases} G_{\mu-1} G_\mu & (\mu=v) \\ 0 & (\mu \neq v) \end{cases} \quad (\mu, v=1, 2, \dots, n).$$

Dm. Let

$$(407) \quad v \leq \mu.$$

From the definitions (400), (402) follows

$$(408) \quad \hat{a}_v = \sum_{i=1}^n \alpha_i \bar{a}_i \quad (v=1, 2, \dots, \mu)$$

for appropriate

$$(409) \quad \alpha_\lambda \in C \quad (\lambda=1, 2, \dots, v).$$

From (408), (409) follows

$$(410) \quad \begin{aligned} \hat{a}_\mu \hat{a}_v &= \sum_{\lambda=1}^v \hat{a}_\mu (\alpha_\lambda \bar{a}_\lambda) = \sum_{\lambda=1}^v \alpha_\lambda (\hat{a}_\mu \bar{a}_v) \\ &= \begin{cases} \alpha_\mu (\hat{a}_\mu \bar{a}_\mu) & (v=\mu) \\ 0 & (v=1, 2, \dots, \mu-1) \end{cases} \quad (\mu=1, 2, \dots, n) \end{aligned}$$

according to (396) with $\alpha=1$, (407), Th 21. From (402), (408), Th 18 however follows

$$(411) \quad \alpha_\mu = (-1)^{\mu+1} G(\bar{a}_\lambda)_{\lambda=1}^{\mu-1} = \underline{\alpha}_\mu \quad (\mu=1, 2, \dots, n)$$

and from Th 20 follows

$$(412) \quad \hat{a}_\mu \bar{a}_\mu = (-1)^{\mu+1} G(\bar{a}_\lambda)_{\lambda=1}^\mu \quad (\mu=1, 2, \dots, n).$$

From (410)–(412), (405) follows (406) for (407). It is clear, that since the products in the left side of (406) are real, the condition (407) is no restriction of the generality.

Th 23. From (382) follows

$$(413) \quad 0 \leq G(\bar{a}_v)_{v=1}^{\mu} \quad (\mu = 0, 1, 2, \dots, n).$$

Dm. (413) follows by induction from (405) and

$$(414) \quad G_{\mu-1} G_{\mu} = \hat{a}_{\mu}^2 \geq 0 \quad (\mu = 1, 2, \dots, n)$$

according to Th 22.

Th 24. From (382), (214) follows

$$(415) \quad 0 < G(\bar{a}_v)_{v=1}^{\mu} \quad (\mu = 0, 1, 2, \dots, n).$$

Dm. Th 23, (405), Th 19.

Th 25. From (382), (214) follows

$$(416) \quad G(\hat{a}_v)_{v=1}^n \neq 0.$$

Dm. From (390),

$$(417) \quad \sum_{v=1}^n \alpha_v \hat{a}_v = \bar{0},$$

Th 22 follows

$$(418) \quad \sum_{v=1}^n \alpha_v \hat{a}_v \hat{a}_{\mu} = \alpha_{\mu} \hat{a}_{\mu}^2 = 0 \quad (\mu = 1, 2, \dots, n).$$

From Th 22, Th 24 follows

$$(419) \quad 0 < \hat{a}_{\mu}^2 \quad (\mu = 1, 2, \dots, n).$$

From (418), (419) follows

$$(420) \quad \alpha_{\mu} = 0 \quad (\mu = 1, 2, \dots, n),$$

which proves (416).

Th 26. From (382), (214) follows

$$(421) \quad L(\hat{a}_v)_{v=1}^{\mu} = L(\bar{a}_v)_{v=1}^{\mu} \quad (\mu = 1, 2, \dots, n).$$

Dm. Th. 25, (402), (363) with *C* instead of *R*.

For (382), (214) let by definition

$$(422) \quad \tilde{a}_{\mu} = -\frac{(-1)^{\mu+1}}{\sqrt{G(\bar{a}_v)_{v=1}^{\mu-1} G(\bar{a}_v)_{v=1}^{\mu}}} \hat{a}_{\mu} \quad (\mu = 1, 2, \dots, n).$$

Then:

Th 27. From (382), (214) follows

$$(423) \quad \tilde{a}_{\mu} \bar{a}_v = \begin{cases} \sqrt{G_{\mu-1}^{-1} G_{\mu}} & (v = \mu) \\ 0 & (v = 1, 2, \dots, \mu-1) \end{cases} \quad (\mu = 1, 2, \dots, n).$$

Dm. From (422), Th 20, (405) follows

$$(424) \quad \tilde{a}_\mu \bar{a}_\nu = \frac{(-1)^{\mu+1}}{\sqrt[n]{G_{\mu-1} G_\mu}} \partial_\mu \bar{a}_\nu$$

$$= \begin{cases} \frac{(-1)^{\mu+1}}{\sqrt[n]{G_{\mu-1} G_\mu}} (-1)^\mu = \sqrt[n]{G_{\mu-1}^{-1} G_\mu} & (\nu = \mu) \\ 0 & (\nu = 1, 2, \dots, \mu-1) \end{cases} \quad (\mu = 1, 2, \dots, n).$$

Th 28. From (382), (214) follows

$$(425) \quad \bar{a}_\nu \tilde{a}_\mu = \begin{cases} \sqrt[n]{G_{\mu-1}^{-1} G_\mu} & (\nu = \mu) \\ 0 & (\nu = 1, 2, \dots, \mu-1) \end{cases} \quad (\mu = 1, 2, \dots, n).$$

Dm. Th 37, (387).

Th 29. From (382), (214) follows

$$(426) \quad G(\bar{a}_\nu, \tilde{a}_\nu)_{\nu=1}^n = \sqrt[n]{G_n}.$$

Dm. From (384), (425), (405) follows

$$(427) \quad G(\bar{a}_\nu, \tilde{a}_\nu)_{\nu=1}^n = \begin{vmatrix} \bar{a}_1 \tilde{a}_1 & \bar{a}_2 \tilde{a}_1 & \dots & \bar{a}_n \tilde{a}_1 \\ \bar{a}_1 \tilde{a}_2 & \bar{a}_2 \tilde{a}_2 & \dots & \bar{a}_n \tilde{a}_2 \\ \dots & \dots & \dots & \dots \\ \bar{a}_1 \tilde{a}_n & \bar{a}_2 \tilde{a}_n & \dots & \bar{a}_n \tilde{a}_n \end{vmatrix}$$

$$= \prod_{\nu=1}^n \sqrt[n]{G_{\nu-1}^{-1} G_\nu} = \sqrt[n]{G_n}.$$

Th 30. From (382), (214) follows

$$(428) \quad \tilde{a}_\mu \tilde{a}_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \quad (\mu, \nu = 1, 2, \dots, n).$$

Dm. (422), Th 22, (405).

Th 31. From (382), (214) follows

$$(429) \quad G(\tilde{a}_\nu)_{\nu=1}^\mu = 1 \quad (\mu = 1, 2, \dots, n).$$

Dm. (386), Th (30).

Th 32. From (382), (214) follows

$$(430) \quad L(\tilde{a}_\nu)_{\nu=1}^\mu = L(\bar{a}_\nu)_{\nu=1}^\mu \quad (\mu = 1, 2, \dots, n).$$

Dm. Th 31, (422). Th 26.

In the light of Th 31, Th 32 the meaning of Th 29 is that the repers (382) and

$$(431) \quad \tilde{a}_v \in E \quad (v=1, 2, \dots, n)$$

are equally orientated.

It is well known that for (382), (214) the reper

$$(432) \quad \bar{a}_v^{-1} \in E \quad (v=1, 2, \dots, n)$$

of Gibbs of the reper (382) is defined by

$$(433) \quad a_v^{-1} = \frac{(-1)^{v+1}}{G(a_\lambda)_{\lambda=1}^n} \begin{vmatrix} \bar{a}_1 & a_2 & \dots & \bar{a}_n \\ a_1^2 & \bar{a}_2 \bar{a}_1 & \dots & \bar{a}_n \bar{a}_1 \\ a_1 a_2 & a_2^2 & \dots & a_n \bar{a}_2 \\ \dots & \dots & \dots & \dots \\ a_1 a_{v-1} & a_2 a_{v-1} & \dots & a_n a_{v-1} \\ a_1 a_{v+1} & a_2 a_{v+1} & \dots & \bar{a}_n \bar{a}_{v+1} \\ \dots & \dots & \dots & \dots \\ a_1 a_n & a_2 a_n & \dots & a_n^2 \end{vmatrix}$$

($v=1, 2, \dots, n$). Then:

Th 33. From (382), (214) follows

$$(434) \quad \tilde{a}_v^{-1} = \tilde{a}_v \quad (v=1, 2, \dots, n).$$

Dm. Th 30.

Th 34. From (382), (214) follows

$$(435) \quad a_\mu \bar{a}_v^{-1} = \begin{cases} \sqrt{G(a_\lambda)_{\lambda=1}^{\mu-1} G^{-1}(a_\lambda)_{\lambda=1}^\mu} & (\mu=v), \\ 0 & (\mu=1, 2, \dots, v-1), \quad (v=1, 2, \dots, n). \end{cases}$$

Dm. (422), (402), (384), (385).

Th 35. From (382), (214) follows

$$(436) \quad \bar{a}_v^{-1} \tilde{a}_\mu = \begin{cases} \sqrt{G(\bar{a}_\lambda)_{\lambda=1}^{\mu-1} G^{-1}(a_\lambda)_{\lambda=1}^\mu} & (\mu=v), \\ 0 & (\mu=1, 2, \dots, v-1), \quad (v=1, 2, \dots, n). \end{cases}$$

Dm. Th 34, (387).

Th 36. From (382), (214) follows

$$(437) \quad \tilde{a}_\mu = \sum_{v=1}^n (\tilde{a}_\mu \bar{a}_v^{-1}) \bar{a}_v \quad (\mu=1, 2, \dots, n).$$

Dm. From Th 32, Th 34 follows

$$(438) \quad \tilde{a}_\mu = \sum_{v=1}^n (\tilde{a}_\mu \bar{a}_v^{-1}) \bar{a}_v = \sum_{v=1}^\mu (\tilde{a}_\mu \bar{a}_v^{-1}) \bar{a}_v \quad (\mu = 1, 2, \dots, n).$$

Th 37. From (382), (214) follows

$$(439) \quad \tilde{a}_\mu = \sum_{v=\mu}^n (\tilde{a}_\mu \bar{a}_v) \bar{a}_v^{-1} \quad (\mu = 1, 2, \dots, n).$$

Dm. From Th 32, Th 27 follows

$$(440) \quad \tilde{a}_\mu = \sum_{v=1}^n (\tilde{a}_\mu \bar{a}_v) \bar{a}_v^{-1} = \sum_{v=\mu}^n (\tilde{a}_\mu \bar{a}_v) \bar{a}_v^{-1}$$

$$(\mu = 1, 2, \dots, n).$$

Th 38. From (382), (214),

$$(441) \quad a \in L(a_v)_{v=1}^\mu \quad (\mu = 1, 2, \dots, n)$$

follows

$$(442) \quad \bar{a} = \sum_{v=1}^\mu (\bar{a} \tilde{a}_v) \tilde{a}_v.$$

Dm. From (441), Th 32 follows

$$(443) \quad a = \sum_{v=1}^\mu \alpha_v \tilde{a}_v$$

with appropriate

$$(444) \quad \alpha_v \in C \quad (v = 1, 2, \dots, \mu).$$

From (443), Th 30 follows

$$(445) \quad \bar{a} \tilde{a}_\lambda - \sum_{v=1}^\mu \alpha_v (\bar{a}_v \tilde{a}_\lambda) = \alpha_\lambda \quad (\lambda = 1, 2, \dots, \mu).$$

From (443), (445) follows (442).

Th 39. From (382), (214), (444) follows: the only solution (441) of the system of equations

$$(446) \quad \bar{a} \tilde{a}_v = \alpha_v \quad (v = 1, 2, \dots, \mu)$$

is (443)

Dm. From (441), (446), Th 38 follows (443), i. e. the system (446) does not have more than one solution. From (443), Th 38 follows (446), i. e. (443) is indeed a solution of the system (446).

Th 40. From (382), (214) follows: the only solution $\bar{a} \in E_C$ of the system of equations

$$(447) \quad \bar{a} \tilde{a} = 0 \quad (v = 1, 2, \dots, n)$$

is $\bar{a} = \bar{0}$.

Dm. Th 39.

Th 41. From (382), (214) follows

$$(448) \quad a_\nu = \sum_{\mu=1}^n (\bar{a}_\nu \tilde{a}_\mu) \tilde{a}_\mu \quad (\nu = 1, 2, \dots, n).$$

Dm. Th 32, Th 38.

Th 42. From (382), (214) follows

$$(449) \quad \bar{a}_\nu^{-1} = \sum_{\mu=\nu}^n (\bar{a}_\nu^{-1} \tilde{a}_\mu) \tilde{a}_\mu \quad (\nu = 1, 2, \dots, n).$$

Dm. From

$$(450) \quad L(\bar{a}_\nu^{-1})_{\nu=1}^n = L(\bar{a}_\nu)_{\nu=1}^n,$$

Th 38, Th 35 follows

$$(451) \quad \bar{a}_\nu^{-1} = \sum_{\mu=1}^n (\bar{a}_\nu^{-1} \tilde{a}_\mu) \tilde{a}_\mu = \sum_{\mu=\nu}^n (\bar{a}_\nu^{-1} \tilde{a}_\mu) \tilde{a}_\mu$$

($\nu = 1, 2, \dots, n$).

Th 43. From (382), (214), (441), $\bar{b} \in E_C$ follows

$$(452) \quad \bar{a} \bar{b} = \sum_{\nu=1}^{\mu} (\tilde{a} \bar{a}_\nu) (\tilde{a}_\nu \bar{b}).$$

Dm. Th 38.

Th 44. From (382), (214), (441), $\bar{b} \in E_C$ follows

$$(453) \quad \bar{b} \bar{a} = \sum_{\nu=1}^{\mu} (\tilde{a}_\nu \bar{a}) (\bar{b} \tilde{a}_\nu).$$

Dm. Th 43, (387).

N. B. The equality (453) may be written in the form

$$(454) \quad \bar{b} \bar{a} = \sum_{\nu=1}^{\mu} (\bar{b} \tilde{a}_\nu) (\tilde{a}_\nu \bar{a}),$$

but (454) and therefore (453) is not to be obtained from (452) by a formal change of \bar{a} with \bar{b} , since the roles of \bar{a} and \bar{b} in Th 43 are not equivalent.

Th 45. From (382), (214), (441) follows

$$(455) \quad a^2 = \sum_{\nu=1}^n (\bar{a} \tilde{a}_\nu) (\tilde{a}_\nu \bar{a}).$$

Dm. Th 43.

Th 46. From (382), (214),

$$(456) \quad \lambda \leq \mu \quad (\lambda, \mu = 1, 2, \dots, n)$$

follows

$$(457) \quad \bar{a}_\lambda \bar{a}_\mu = \sum_{v=1}^{\lambda} (\bar{a}_\lambda \tilde{a}_v) (\tilde{a}_v \bar{a}_\mu).$$

Dm. From

$$(458) \quad \bar{a}_\lambda \in L(\bar{a}_v)_{v=1}^\lambda,$$

and Th 43 follows (457).

Th 47. From (382), (214),

$$(459) \quad \mu \leq \lambda \quad (\lambda, \mu = 1, 2, \dots, n)$$

follows

$$(460) \quad \bar{a}_\lambda^{-1} \bar{a}_\mu^{-1} = \sum_{v=\lambda}^n (\bar{a}_\lambda^{-1} \tilde{a}_v) (\tilde{a}_v \bar{a}_\mu^{-1}).$$

Dm. From Th 42 follows

$$(461) \quad \bar{a}_\lambda^{-1} = \sum_{v=\lambda}^n (\bar{a}_\lambda^{-1} \tilde{a}_v) \tilde{a}_v \quad (\lambda = 1, 2, \dots, n).$$

From (461) follows (460).

N. B. In the proofs of Th 46 and Th 47 the conditions (456) and (459) seemingly are not used. But in reality that is not true. Indeed, if in Th 46 $\mu < \lambda$ holds, from Th 27 follows

$$(462) \quad \tilde{a}_v \bar{a}_\mu = 0 \quad (\mu = 1, 2, \dots, v-1)$$

and (457) would take the form

$$(463) \quad \bar{a}_\lambda \bar{a}_\mu = \sum_{v=1}^{\mu} (\bar{a}_\lambda \tilde{a}_v) (\tilde{a}_v \bar{a}_\mu).$$

Analogically if in Th 47 $\lambda < \mu$ holds, from Th 34 follows

$$(464) \quad \tilde{a}_v \bar{a}_\mu^{-1} = 0 \quad (v = 1, 2, \dots, \mu-1)$$

and (460) would take the form

$$(465) \quad \bar{a}_\lambda^{-1} \bar{a}_\mu^{-1} = \sum_{v=\mu}^n (\bar{a}_\lambda^{-1} \tilde{a}_v) (\tilde{a}_v \bar{a}_\mu^{-1}).$$

Th 48. From (382), (214), (390) follows: the only solution

$$(466) \quad \bar{a} \notin L(\bar{a}_v)_{v=1}^n$$

of the system of equations

$$(467) \quad \bar{a} \bar{a}_v = \alpha_v \quad (v = 1, 2, \dots, n)$$

is

$$(468) \quad \bar{a} = \sum_{v=1}^n \beta_v \tilde{a}_v,$$

where

$$(469) \quad \beta_v \in C \quad (v=1, 2, \dots, n)$$

is the solution of the system of equations

$$(470) \quad \sum_{v=1}^n \beta_v \tilde{a}_v \bar{a}_\mu = c_\mu \quad (\mu = 1, 2, \dots, n).$$

Dm. From Th 27, Th 24 follows, that the system of equations (470) has a unique solution (469). Then from (468), Th 27, (470) follows

$$(471) \quad \bar{a} \bar{a}_\mu = \sum_{v=1}^n \beta_v \tilde{a}_v \bar{a}_\mu = \sum_{v=1}^n \beta_v \tilde{a}_v a_\mu = a_\mu \quad (\mu = 1, 2, \dots, n),$$

i. e. (468) with (469), (470) is indeed a solution of (467). The uniqueness of the solution is established in [1].

Th 49. From (382), (214) follows: the only solution (466) of the system of equations

$$(472) \quad \bar{a} \bar{a}_v = \begin{cases} 1 & (v=\mu) \\ 0 & (v \neq \mu) \end{cases} \quad (\mu, v = 1, 2, \dots, n)$$

is

$$(473) \quad \bar{a}_\mu^{-1} = \sum_{v=1}^n \beta_{\mu v} \tilde{a}_v,$$

where

$$(474) \quad \beta_{\mu v} \in C \quad (\mu, v = 1, 2, \dots, n)$$

is the solution of the system of equations

$$(475) \quad \sum_{v=1}^\lambda \beta_{\mu v} \tilde{a}_v \bar{a}_\lambda = \begin{cases} 1 & (\lambda=\mu) \\ 0 & (\lambda \neq \mu) \end{cases} \quad (\lambda, \mu = 1, 2, \dots, n).$$

Dm. Th 48, (450) and

$$(476) \quad \bar{a}_\mu^{-1} \bar{a}_v = \begin{cases} 1 & (v=\mu) \\ 0 & (v \neq \mu) \end{cases} \quad (\mu, v = 1, 2, \dots, n).$$

Th 50. From (382), (214), (390) follows: the only solution (466) of the system of equations (467) is

$$(477) \quad \bar{a} = \sum_{v=1}^n \alpha_v \sum_{\mu=v}^n (\bar{a}_\mu^{-1} \tilde{a}_\mu) \tilde{a}_\mu.$$

Dm. From (382), (214), (466), (467) follows

$$(478) \quad \bar{a} = \sum_{v=1}^n \alpha_v \bar{a}_v^{-1}.$$

From (478), Th 42 follows (477).

13. We shall call the reper (422) semireciprocal to the reper (382). From (428) follows, that the semireciprocal repers to a given reper in E_C of-

fer orthonormal bases of the linear space spanned on the given reper. From (422), (402) follows, that by the manner given in the preceding paragraph one could construct $n!$ orthonormal bases of the linear space in question.

From (423), (425) follows, that the semireciprocal vectors are not only mutually orthogonal, but they are orthogonal to a part of the vectors of the given reper. Namely the vector (422) is orthogonal to all those vectors (382), whose indices are less than his.

As it is well known, a Gram-Schmidt orthogonalization is called the following inductive process [2], [3]: for a given reper (382) is put consequently

$$(479) \quad \begin{aligned} \bar{s}_1 &= \bar{a}_1, \\ \bar{s}_2 &= \bar{a}_2 - \frac{\bar{a}_2 \cdot \bar{s}_1}{\bar{s}_1^2} \bar{s}_1, \\ \bar{s}_3 &= \bar{a}_3 - \frac{\bar{a}_3 \cdot \bar{s}_1}{\bar{s}_1^2} \bar{s}_1 - \frac{\bar{a}_3 \cdot \bar{s}_2}{\bar{s}_2^2} \bar{s}_2 \end{aligned}$$

and in general

$$(480) \quad \bar{s}_\mu = \bar{a}_\mu - \sum_{\nu=1}^{\mu-1} \frac{\bar{a}_\mu \cdot \bar{s}_\nu}{\bar{s}_\nu^2} \bar{s}_\nu \quad (\mu = 1, 2, \dots, n).$$

From (402), (480) by induction follows

$$(481) \quad L(\bar{s}_\nu)_{\nu=1}^\mu = L(\hat{a}_\nu)_{\nu=1}^\mu \quad (\mu = 1, 2, \dots, n),$$

as especially

$$(482) \quad \bar{s}_1 = \hat{a}_1,$$

$$(483) \quad \bar{s}_2 = -\frac{1}{\hat{a}_1^2} \hat{a}_2.$$

The equalities (481)—(483) naturally hint, that may be in general

$$(484) \quad \bar{s}_\nu = \alpha_\nu \hat{a}_\nu, \quad (\nu = 1, 2, \dots, n)$$

hold for appropriate (390). As will be shown the answer of this question is affirmative.

First of all the equalities

$$(485) \quad \bar{s}_\mu \hat{a}_\nu = 0 \quad (\mu, \nu = 1, 2, \dots, n; \mu \neq \nu)$$

hold. Indeed, from (481) follows

$$(486) \quad \bar{s}_\mu = \sum_{\nu=1}^{\mu} \alpha_\nu \hat{a}_\nu, \quad (\mu = 1, 2, \dots, n)$$

for appropriate (444). Now (485) follows from (486) and Th 22.

On the other hand from (485), (422) follows

$$(487) \quad \bar{s}_\mu \hat{a}_\nu = 0 \quad (\mu, \nu = 1, 2, \dots, n; \mu \neq \nu)$$

and from (481), (422) follows

$$(488) \quad L(\bar{s}_\nu)_{\nu=1}^\mu = L(\hat{a}_\nu)_{\nu=1}^\mu \quad (\mu = 1, 2, \dots, n).$$

Now from (487), (488), Th 32, Th 39 follows

$$(489) \quad s_\mu = (s_\mu \tilde{\hat{a}}_\mu) \tilde{\hat{a}}_\mu \quad (\mu = 1, 2, \dots, n),$$

which together with (422) proves (484).

In this manner the orthogonal basis (480) of Gram-Schmidt almost (with accuracy to scalar factors) coincides with the orthogonal basis (402). The advantage of (402) over the inductive definition (480) consists first in the fact, that the vectors (402) are given explicitly by the vectors (382), whereas the vectors (480) are constructed step by step; moreover the vectors (402) permit the daintily norming (422).

In § 11 the heuristic motives were shown, by which we arrived to the orthogonalization (402). Our purpose was the orthogonalization of the curvilinear coordinates. Lately our colleague Iv. Dimovski drew our attention to the fact, that the equalities (422) are to be found in the course [4] of G. Stanilov, in whose preface the author writes: "Here we propose explicite formulae for the system of orthonormalized vectors, which, it seems to us, are new."

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КЪМ МНОГОМЕРНАТА КИНЕМАТИКА НА ТОЧКА В КРИВОЛИНЕЙНИ КООРДИНАТИ

И. Христова, И. Чобанов

(РЕЗЮМЕ)

В настоящата работа е изграден основният апарат на многомерната кинематика на точка в криволинейни координати (6) за произволно реално Хилбертово пространство H на базата на реципрочните вектори (433) на Гибс за произволен репер (432), където E е лежащото в основата на H реално евклидово пространство. Съществена роля в разглежданятията играят разлаганията на произволен вектор от породеното от репера (432) n -мерно линейно пространство по векторите на репера (432) и по реципрочните им вектори (433), които позволяват за произволна система (6) криволинейни координати да се получат представянията (24), (33) и (46) съответно за радиус-вектора r на произволна точка, за скоростта \dot{r} и за ускорението \ddot{r} чрез реципрочния репер (13) на репера (8) на тангенциалните вектори към координатните линии на (6). При (56), (58) са в сила тъждествата (67) за ускоренията от произволен ред, чито извод се основава на тъждеството (59), където R_{m-2} е израз, съдържащ ускоренията (58) на криволинейните координати (6) от ред, не по-висок от $m-2$. Познаването на представянията (78) за производните на тангенциалните вектори (8) към координатните линии на системата (6) криволинейни координати представлява интерес поради следната причина. За произволна функция (85) са в сила тъждествата (86)—(88), поради което при наличието на (78) производните на (85) от произволен ред могат да се получат чрез повтаряне на посочената процедура с многократно използване на (78).

В случая на тримерно векторно пространство разглежданятията придобиват особено прост вид поради удобното представяне (91) с (92) на векторите на Гибс с използване на векторно произведение. като примери са разгледани традиционно използвани в тримерния случай ортогонални координатни системи (93), (103), (112), (121), (129), (139), (148), (158), (168) и (177). като нетривиално приложение е разгледана подробно системата елиптични координати (6) в n -мерно пространство, които се дефинират като n -те различни реални корена на уравнението (276) от n -та степен спрямо r при (288). Получено е представянето (301) при (288) за радиус-вектора на произволна точка, където единичните тангенциални вектори към координатните линии се определят от (299). За частните производни на последните спрямо криволинейните координати (6) са получени изразите (330), водещи до представянията (333) на скоростите на единичните тангенциални вектори, които позволяват намирането на скоростта и ускорението на произволна точка в елиптични координати. Окончателните изрази за частните производни на единичните тангенциални вектори са (361) и (362).

Задачата за получаване на нови ортогонални системи криволинейни координати естествено води до необходимостта от експлицитна ортогонализационна схема за произволна база в едно евклидово пространство; евристичните съображения за построяване на такава схема са изложени в т. 11 от работата. В следващата част тази идея се провежда консеквентно и се стига до системата вектори (422) при (402) и (384), (385), (405) за репера (382). Изведени са основните релации, които свързват векторите (422) с изходния репер (382) и с неговия репер на Гибс и е показано, че ортогоналната база (402) съвпада с ортогоналната база (480) на Грам — Шмит с точност до скаларни множители. По-точно, доказва се тъждеството (439), което свързва векторите (480) на Грам — Шмит, дефинирани индуктивно с (479), (480), със семиreichпрочните вектори (422) за репера (382). Доколкото е известно на авторите, явни изрази за тези вектори се дават за пръв път в курса [4] на Г. Станилов.