THE MEASURE OF THE IMAGE OF A DIFFERENTIABLE MAPPING

Ivan Prodanov

The idea to consider the measure of the set of critical values of maps is due to Marston Morse. A. P. Morse [1] proved that the critical values of a function of m variables of class C^m constitute a set of linear measure zero. This result was generalized by A Sard [2]. He proved that if $m>n$, the set of critical values of the mapping

$$f: B \to R^n \qquad (B \subset R^m) \tag{1}$$

is of n-dimensional Hausdorff-Saks measure zero provided that $f \in C^q$ with $q \geqq m-n+1$. Using an example of Hassler Whitney [3], he showed that the hypothesis on q cannot be weakened in this case. In the case $m \leqq n$ Sard proved that the set of critical values of the mapping (1) is of m-dimensional Hausdorff-Saks measure zero if $f \in C^1$. The purpose of this paper is to establish that if $m \leqq n$, the conclusion of Sard's theorem is true *without any hypothesis on the mapping* (1) (corollary **2**). This statement is a special case of a natural estimation of the measure of the image of the map (1) provided that it is differentiable in B (theorem 1).

The conception of differentiability is often associated with the ones of open set in R^m and continuous partial derivative. It seems that this is an unnecessary restriction of this notion [4]. In the present paper we shall use the following definition.

Let B be an arbitraty subset of R^m and $x \in B$. The mapping (1) is said to be d i f f e r e n t i a b l e at the point x if there are a linear mapping

$$D_x: R^m \to R^n \tag{2}$$

and a mapping

$$\varphi: B-x \to R^n \tag{3}$$

such that

$$f(x+h) = f(x) + D_x(h) + \| h \| \varphi(h) \qquad (h \in B-x) \tag{4}$$

and

$$\lim_{h \to 0} \varphi(h) = 0^*. \tag{5}$$

Each of the linear maps (2) is called a t o t a l d e r i v a t i v e of f at the point x. The total derivative is unique if the point x is interior to B but in general this is not true.

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* We denote by $B-x$ the set of all differences $b-x$ $(b \in B)$, and by $\| \;\; \|$ the Euclidean norm.

A critical point of the map (1) is a point x in B, such that f is differentiable at x and there is a total derivative D_x of f at x with $\dim D_x(R^m) < \min(m, n)$. A critical value of f is the image under f of a critical point of f.

Let E be a metric space and $A \subset E$. Denote the diameter of A by $d(A)$. Given positive quantities m and α, let $L_m(A, \alpha)$ be the greatest lower bound of the sums $c_m \sum_{k=1}^{\infty} (d(A_k))^m$, where $A_1, A_2, \ldots, A_k, \ldots$ is an arbitrary countable covering of A by sets with $d(A_k) < \alpha$ $(k = 1, 2, \ldots)$, and

$$c_m = \frac{\pi^{\frac{m}{2}}}{2^m \Gamma\left(\frac{m}{2}+1\right)}.$$

The m-dimensional outer Hausdorff-Saks measure of A is by definition

$$L_m(A) = \sup_{\alpha>0} L_m(A, \alpha). \tag{6}$$

In R^m the outer Hausdorff-Saks measure $L_m(A)$ coincides with the m-dimensional outer Lebesgue measure of A.

Given a linear mapping

$$D: R^m \to R^n \quad (m \leq n), \tag{7}$$

an m-dimensional linear subspace V of R^n with $D(R^m) \subset V$, and an orthonormal basis of V, let $\det D$ be the determinant of D. It is clear that if C is an m-dimensional cube in R^m, then

$$L_m(D(C)) = |\det D| \, L_m(C). \tag{8}$$

Lemma 1. Given positive quantities α and ε, let (7) be a linear mapping and C_0 be the unit cube in R^m. Then there is an open covering

$$U_1, U_2, \ldots, U_k, \ldots \tag{9}$$

of $D(C_0)$ in R^n, such that

$$d(U_k) < \alpha \quad (k = 1, 2, \ldots) \tag{10}$$

and

$$c_m \sum_{k=1}^{\infty} d(U_k))^m < |\det D| + \varepsilon. \tag{11}$$

Proof. The definition of the Hausdorff-Saks measure implies that there is a covering $A_1, A_2, \ldots, A_k, \ldots$ of $D(C_0)$ in $D(R^m)$ with

$$d(A_k) < \alpha \quad (k = 1, 2, \ldots) \tag{12}$$

and

$$(13) \qquad c_m \sum_{k=1}^{\infty} (d(A_k))^m < L_m(D(C_0)) + \varepsilon.$$

It follows from (8) and (13) that

$$(14) \qquad c_m \sum_{k=1}^{\infty} (d(A_k))^m < |\det D| + \varepsilon.$$

Now we put $U_k = O(A_k, \eta_k)$, where η_k are sufficiently small positive numbers $(k = 1, 2, \ldots)$*. Q. E. D.

Lemma 2. Let the mapping (1) be differentiable at a point $x \in B$, and D_x be a total derivative of f at x. Then for each $\alpha > 0$ and any $\varepsilon > 0$ there is a $\delta(x) > 0$, such that for every m-dimensional cube C in R^m the conditions

$$(15) \qquad d(C) < \delta(x),$$

and

$$(16) \qquad x \in C$$

imply

$$(17) \qquad L_m(f(C \cap B), \alpha) \leq (|\det D_x| + \varepsilon) L_m(C).$$

Proof. Let (9) be an open covering of $D_x(C_0)$ in R^n possessing the properties (10) and (11) (with D_x instead of D), and $\Delta_x > 0$ be such that

$$(18) \qquad O(D_x(C_0), \Delta_x) \subset \bigcup_{k=1}^{\infty} U_k.$$

It follows from (5) that there is a positive number $\delta(x)$, such that the relations $h \in B - x$, $\|h\| < \delta(x)$ imply $\|\varphi(h)\| < \dfrac{\Delta_x}{d(C_0)}$.

Let now a m-dimensional cube C in R^m satisfies (15) and (16). From (4) and the choice of $\delta(x)$ it follows that

$$f(C \cap B) \subset f(x) + O\left(D_x(C - x), \frac{d(C)\Delta_x}{d(C_0)}\right).$$

Therefore

$$f(C \cap B) \subset f(x) + D_x(c - x) + O\left(D_x(C - c), \frac{d(C)}{d(C_0)} \Delta_x\right),$$

where c is the center of the cube C. Hence

$$(19) \qquad f(C \cap B) \subset f(x) + D_x(c - x) + \bigcup_{k=1}^{\infty} \frac{d(C)}{d(C_0)} U_k$$

by (18) and linearity of D_x. Let

* We denote by $O(A_k, \eta_k)$ the η_k-neighbourhood of A_k in R^n.

$$V_k = f(x) + D_x(c-x) + \frac{d(C)}{d(C_0)}\ U_k \qquad (k=1,\ 2,\ldots).$$

It follows from (19) that the sets V_k $(k=1,\ 2,\ldots)$ form an open covering of $f(C\cap B)$. If $\delta(x)$ is chosen with $\delta(x)\leqq d(C_0)$, (10) and (15) imply that $d(V_k)<\alpha$ $(k=1,2,\ldots)$. Moreover

$$d(V_k)=\frac{d(C)}{d(C_0)}\ d(U_k) \qquad (k=1,2,\ldots). \tag{20}$$

Hence

$$L_m(f(C\cap B),\ \alpha)\leqq c_m \sum_{k=1}^{\infty}\left(\frac{d(C)}{d(C_0)}\right)^m (d(U_k))^m$$

$$\leqq(|\det D_x|+\varepsilon)\left(\frac{d(C)}{d(C_0)}\right)^m=(|\det D_x|+\varepsilon)L_m(C)$$

by (11). Q. E. D.

Lemma 3. For each map

$$\delta: B\to(0,\infty) \quad (B\subset R^m) \tag{21}$$

there is a sequence

$$C_1,\ C_2,\ldots,\ C_k,\ldots \tag{22}$$

of closed m-dimensional cubes in R^m with the following properties:

i) C_k each C_l have no common interior points $(k\neq l)$;

ii) $B\subset\bigcup_{k=1}^{\infty}C_k$;

iii) for each $k=1,\ 2,\ldots$ there is a x in B, such that

$$x\in C_k \tag{23}$$

and

$$d(C_k)<\delta(x). \tag{24}$$

Proof. Without loss of generality we may suppose that there is a cube C in R^m, such that $B\subset C$. For every $k=1,\ 2,\ldots$ we cut each of the edges of C into 2^k equal parts, and denote by Γ_k the set of the so found 2^{mk} closed subcubes of C. Let $\Gamma=\bigcup_{k=1}^{\infty}\Gamma_k$. It is clear that if two cubes of Γ have common interior points, one of them contains the other. We shall say that a cube $\gamma\in\Gamma$ is normal if there is a point x in $B\cap\gamma$ with $d(\gamma)<\delta(x)$. Let Γ' be the set of all maximal (with respect to the inclusion) normal elements of Γ. It is obvious that the cubes of Γ' have the desired properties. Q. E. D.

Theorem 1. Let the mapping (1) be differentiable at every point $x\in B$, $\lambda\geqq 0$ and

$$|\det D_x|\leqq\lambda \tag{25}$$

for every x in B. Then

$$L_m(f(B)) \leqq \lambda L_m(B). \tag{26}$$

Proof. Let α and ε be arbitrary positive numbers. Since L_m coincides with the m-dimensional outer Lebesgue measure in R^m, there is an open set U in R^m, such that $B \subset U$ and

$$L_m(U) < L_m(B) + \varepsilon. \tag{27}$$

For any point x in B we choose a positive number $\delta(x)$, such that

$$O(x, \delta(x)) \subset U, \tag{28}$$

and (15) and (16) imply (17). Consider the sequence (22) with the properties i), ii) and iii). Obviously

$$L_m(f(B), \alpha) \leqq \sum_{k=1}^{\infty} L_m(f(C_k \cap B), \alpha), \tag{29}$$

since $B \subset \bigcup_{k=1}^{\infty} C_k$. Since the sequence (22) has the properties (23) and (24) we have

$$L_m(f(C_k \cap B), \alpha) \leqq (\lambda + \varepsilon) L_m(C_k) \tag{30}$$

by the choice of $\delta(x)$ and (25). From (29) and (30) it follows that

$$L_m(f(B), \alpha) \leqq (\lambda + \varepsilon) \sum_{k=1}^{\infty} L_m(C_k). \tag{31}$$

But the cubes C_k have no common interior points and $C_k \subset U$ by $B \subset U$, iii) and (28). Hence

$$\sum_{k=1}^{\infty} L_m(C_k) < L_m(B) + \varepsilon. \tag{32}$$

Now by (27) from (31) and (32) it follows

$$L_m(f(B), \alpha) \leqq (\lambda + \varepsilon)(L_m(B) + \varepsilon). \tag{33}$$

Since α and ε are arbitrary positive numbers, (26) follows from (33). Q. E. D.

Corollary 1. If $\lambda \geqq 0$ and A_λ is the set of all points x in B, such that the map (1) is differentiable at x and satisfies (25), then

$$L_m(f(A_\lambda)) \leqq \lambda L_m(A_\lambda). \tag{34}$$

Corollary 2. The set of the critical values of the map (1) is of m-dimensional Hausdorff-Saks measure zero.

Corollary 3. Let

$$f: [a, b] \to R \qquad (a < b;\ a, b \in R) \tag{35}$$

be a continuous function, and K be the set of all the points x in $B=[a, b]$, such that f is differentiable and $f'(x)=0$. If $f(B\setminus K)$ is of Lebesgue measure zero, the function (33) is a constant.

Proof. Indeed, corollary 2 implies that $f(K)$ is of Lebesgue measure zero. Since $f(B)\subset f(K)\cup f(B\setminus K)$, the set $f(B)$ has Lebesgue measure zero. On the other hand $f(B)$ is obviously an interval. Q. E. D.

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МЯРКА НА ОБРАЗА НА ДИФЕРЕНЦИРУЕМО ИЗОБРАЖЕНИЕ

И. Проданов

(РЕЗЮМЕ)

Идеята да се разгледа мярката на множеството на критичните точки на едно изображение произхожда от Мастън Морз. А. П. Морз [1] доказа, че критичните стойности на една функция на m променливи от клас C^m образуват множество с линейна мярка нула. Този резултат бе обобщен от А. Сард [2]. Той доказа, че ако $m>n$, множеството на критичните стойности на изображението (1) има n-мерна мярка на Хаусдорф — Сакс нула, стига да е изпълнено условието $f\in C^q$ с $q\geqq m-n+1$. Като използува един пример на Х. Уитни [3], той показа, че този резултат не може да бъде подобрен. В случая $m\leqq n$ Сард установи, че множеството на критичните стойности на изображението (1) има m-мерна мярка на Хаусдорф — Сакс нула, стига да е налице условието $f\in C^1$. В предлаганата работа се доказва, че при $m\leqq n$ заключението на теоремата на Сард запазва валидността си без никакви предположения за изображението (1) (следствие 2). Това твърдение е специален случай от една естествена оценка на мярката на образа на изображението (1), стига то да е диференцируемо в B (теорема 1).