DEGREE OF A MAPPING IN AXIOMATIC HOMOLOGY AND COHOMOLOGY THEORIES

Ianko Kintishev, Ivan Prodanov, Georg I. Tchobanov

The concept of degree of a mapping plays a considerable role in the axiomatic introduction to homology and cohomology. So for instance it is essential in the deriving of the well known rules for computation of the homology and cohomology groups of a finite cellular polytope. It is of course desirable the notion of the degree of a mapping to be introduced in an abstract homology and cohomology theory. In the present paper this is done for an arbitrary generalized homology theory; as usually everything is analogical in cohomology. Furthermore the exposition does not make use of the homotopy classification of the mappings of the n-dimensional sphere into itself. A consequence of this approach is a simple proof of the well known rule for computation of the degrees of smooth mappings by means of the signs of the Jacobians of the inverse images of a certain point.

In what follows mapping means a continuous mapping.

§ 1. *U*-HOMEOMORPHISMS

As usual S^n denotes the n-dimensional sphere, i. e. the set of all $x \in R^{n+1}$ with $\sum_{\nu=1}^{n+1} x_\nu^2 = 1$; S_+^n denotes the upper hemisphere, i. e. the set of all $x \in S^n$ with $x_{n+1} \geqq 0$, and S_-^n the lower hemisphere, i. e. the set of all $x \in S^n$ with $x_{n+1} \leqq 0$.

Definition 1. A subset U of S^n is called simple, whenever there exists a homeomorphism

$$\varphi : S^n \to S^n \tag{1}$$

with

$$\varphi(U) = S_+^n. \tag{2}$$

Thus for example the upper and the lower hemispheres are simple subsets. Similarly a simple subset is every n-dimensional simplex in an arbitrary triangulation of S^n.

Lemma 1. Let X be a topological space. Suppose $f_\nu : X \to S^n$ ($\nu = 0, 1$) are continuous mappings, U is a simple subset of S^n, $A \subset X$, $f_0|_A = f_1|_A$ and from $x \in X \backslash A$ follows $f_\nu(x) \in U$. Then f_0 and f_1 are homotopic.

Proof. By hypothesis there exists a homeomorphism (1) with (2). It is easily seen that $t\varphi f_1(x)+(1-t)\varphi f_0(x)\neq 0$ for $t\in[0,1]$ and $x\in X$. Thus the equality

$$F(x,t)=\varphi^{-1}\left(\frac{t\varphi f_1(x)+(1-t)\varphi f_0(x)}{\|t\varphi f_1(x)+(1-t)\varphi f_0(x)\|}\right)$$

defines a continuous mapping $F:X\times[0,1]\to S^n$. This mapping is a homotopy that connects f_0 and f_1. Q. E. D.

Corollary 1. If two homeomorphisms

$$\varphi_\nu : S^n \to S^n \qquad (\nu=0,1) \tag{3}$$

coincide on a simple subset A of S^n, then they are homotopic.

Proof. Lemma 1 with $X=S^n$ and $U=\varphi_0(A)=\varphi_1(A)$.

Definition 2. Let U be a simple subset of S^n. A mapping

$$f: S^n \to S^n \tag{4}$$

is called an U-homeomorphism, whenever there exists a homeomorphism (1) with

$$f|_U=\varphi|_U. \tag{5}$$

Thus for example if the mapping (4) is a simplicial mapping and if Δ_1 and Δ_2 are n-dimensional simplexes in S^n with

$$f(\Delta_1)=\Delta_2, \tag{6}$$

then f is a Δ_1-homeomorphism.

Let by

$$h_\nu : S^n \to S^n \qquad (\nu=1,2,\ldots,n+1) \tag{7}$$

be denoted the mapping that preserves all coordinates with exception of the ν-th, whose sign is changed to the opposite.

If f is an U-homeomorphism and if (3) are homeomorphisms with $f|_U=\varphi_\nu|_U$ $(\nu=0,1)$, then corollary 1 implies that φ_0 and φ_1 are homotopic. This justifies the following definition.

Definition 3. An U-homeomorphism f is called positive on U, whenever the homeomorphism (1) with (5) is homotopic to the identity mapping of S^n; whenever it is homotopic to h_1, f is called negative on U.

Thus for example the Δ_1-homeomorphism f with (6) is positive on Δ_1 if it preserves the orientation of Δ_1 in mapping it onto Δ_2; it is negative on Δ_1, if this orientation is changed.

§ 2. BASIC THEOREM

Let H be an arbitrary generalized homology functor: the Steenrod-Eilenberg axioms hold with a possible exception of the Dimension axiom. Let us recall that for an arbitrary topological space X, $\widetilde{H}_k(X)$ denotes the reduced k-dimensional homology group of X, i. e. the kernel of the homomorphism

$$p_*: H_k(X) \to H_k(P_t),$$

where P_t is an arbitrary one point space and $p: X \to P_t$ is the only possible mapping.

Lemma 2. For every integer k and for every $\xi \in \tilde{H}_k(S^n)$

$$\tilde{h}_{1*}(\xi) = -\xi \tag{8}$$

holds.

Proof. Induction on n.

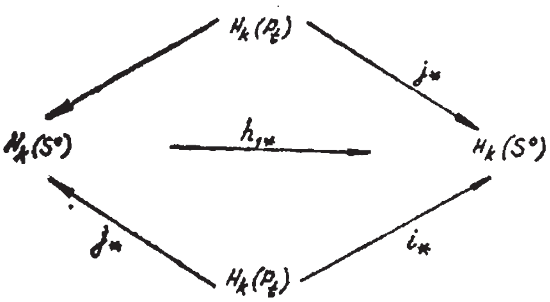
Let $i: P_t \to S^0$ and $j: P_t \to S^0$ are mappings defined by $i(P_t) = -1$ and $j(P_t) = 1$. From the direct sum theorem [1] it follows that for every $\xi \in H_k(S^0)$ there exist unique $\eta \in H_k(P_t)$ and $\zeta \in H_k(P_t)$ with

$$\xi = i_*(\eta) + j_*(\zeta). \tag{9}$$

Since $\xi \in \tilde{H}_k(S^0)$ precisely when $p_*(\xi) = 0$ and since $p_* i_*(\eta) = \eta$ and $p_* j_*(\zeta) = \zeta$, from (9) it follows, that $\xi \in \tilde{H}_k(S^0)$ exactly when η and ζ from (9) satisfy the condition

$$\eta + \zeta = 0. \tag{10}$$

From (9), (10) and from the commutativity of the diagram



follows

$$h_{1*}(\xi) = h_{1*} i_*(\eta) + h_{1*} j_*(\zeta)$$

$$= j_*(\eta) + i_*(\zeta) = -j_*(\zeta) - i_*(\eta) = -\xi,$$

which proves (8) for $n=0$.

Let the statement be true for $n-1$. Then its validity for n follows from the commutativity of the diagram

$$\begin{array}{ccc}
\widetilde{H}_k(S^n) & \xrightarrow{\widetilde{h}_{1*}} & \widetilde{H}_k(S^n) \\
\downarrow & & \downarrow \\
H_k(S^n, S_-^n) & \xrightarrow{h_{1*}} & H_k(S^n, S_-^n) \\
\uparrow & & \uparrow \\
H_k(S^n_+, S^{n-1}) & \xrightarrow{h_{1*}} & H_k(S^n_+, S^{n-1}) \\
\widetilde{\delta}\downarrow & & \downarrow\widetilde{\delta} \\
\widetilde{H}_{k-1}(S^{n-1}) & \xrightarrow{\widetilde{h}_{1*}} & \widetilde{H}_{k-1}(S^{n-1})
\end{array}$$

and from the fact that the vertical homomorphisms are isomorphisms. Q. E. D.

Theorem 1. Let

(11) $$U_1, U_2, \ldots, U_m$$

be mutually disjoint simple subsets of S^n and U be a simple subset of S^n Suppose (4) is an U_μ-homeomorphism ($\mu=1, 2, \ldots, m$),

(12) $$f(U_\mu)=U \qquad (\mu=1, 2, \ldots, m)$$

and

(13) $$f^{-1}(U)=\bigcup_{\mu=1}^{m} U_\mu.$$

If the mapping f is positive on a of the sets (11) and negative on b of them $(a+b=m)$, then

(14) $$\widetilde{f}_*(\xi)=(a-b)\,\xi$$

holds.

Proof. Induction on m.

For $m=1$ and for f positive or negative the statement follows from definition 3 and lemma 1 or lemma 2 respectively.

Let the statement be true for $m-1$. Suppose $\varphi_\mu: S^n \to S^n$ are homeomorphisms with $\varphi_\mu(U_\mu)=S^n_+$ ($\mu=1, 2, \ldots, m$), $X_1=S^n\setminus\bigcup_{\mu=1}^{m-1} U_\mu$, $X_2=S^n\setminus U_m$,

(15) $$f_1(x)=\begin{cases} f(x) & (x\in X_1) \\ \varphi_\mu^{-1} h_{n+1}\varphi_\mu f(x) & (x\in U_\mu,\ \mu=1, 2, \ldots, m-1) \end{cases}$$

and

(16) $$f_2(x)=\begin{cases} f(x) & (x\in X_2), \\ \varphi_m^{-1} h_{n+1}\varphi_m f(x) & (x\in U_m). \end{cases}$$

Let for definiteness f be for example positive on U_m. From the inductive assumption and from the validity of the statement for $m=1$ it follows that

(17) $$\widetilde{f}_{1*}(\xi)=\xi$$

and

(18) $$\tilde{f}_{2*}(\xi) = (a-b-1)\,\xi.$$

The diagram

(19)
$$\begin{array}{ccc} \tilde{H}_k(S^n) & \xrightarrow{\tilde{i}_*} & H_k(S^n,\ S^n \setminus \bigcup_{\mu=1}^{m} U_\mu) \\ \tilde{f}_*,\ \tilde{f}_{1*}, \tilde{f}_{2*} \downarrow & & \downarrow f_*,\ f_{1*},\ f_{2*} \\ \tilde{H}_k(S^n) & \xrightarrow{\tilde{i}_*} & H_k(S^n,\ S^n \setminus U) \end{array}$$

(where at the vertical arrows $\tilde{f}_*$ and f_*, $\tilde{f}_{1*}$ and f_{1*}, $\tilde{f}_{2*}$ and f_{2*} are taken simultaneously) is commutative, whence

(20) $$\tilde{i}_*(\tilde{f}_* - \tilde{f}_{1*} - \tilde{f}_{2*}) = (f_* - f_{1*} - f_{2*})\tilde{i}_*.$$

On the other hand the additivity theorem [1] shows that

(21) $$f_* = f_{1*} + f_{2*}.$$

From (20) and (21) follows

(22) $$\tilde{f}_* = \tilde{f}_{1*} + \tilde{f}_{2*},$$

since the homomorphism $\tilde{i}_*$ in the lower row of the diagram (19) obviously is an isomorphism. Now (14) follows from (17), (18) and (22). Q. E. D.

Corollary 2. For every continuous mapping (4) there exists an integer π, such that for any generalized homology theory H and for any integer k for the homomorphism

(23) $$\tilde{f}_* : \tilde{H}_k(S^n) \to \tilde{H}_k(S^n)$$

the equality

(24) $$\tilde{f}_*\xi = \pi\xi \qquad (\xi \in \tilde{H}_k(S^n))$$

holds.

Proof. For any simplicial approximation g of f let U be an arbitrary simplex in S^n and (11) be its inverse images by g. Then theorem 1 is applicable. Q. E. D.

The integer π from (24) is called **degree** of f and is denoted by $\deg f$.

§ 3. DEGREES OF SMOOTH MAPPINGS

It is well known that the n-dimensional sphere S^n is an orientable smooth manifold. In what follows we shall suppose fixed the orientation, generated by the stereographic projection

(25) $$p : S^n \setminus \{\sigma\} \to R^n \qquad (\sigma = (0, 0, \ldots, 0, -1)).$$

Let ξ and η be arbitrary points of S^n, T_ξ and T_η be the tangent spaces in ξ and η respectively and $D : T_\xi \to T_\eta$ be a linear isomorphism. Then D

is called **positive**, when it preserves the orientation, and **negative**, when the orientation is changed. In the first case we write $\operatorname{sign} D=1$ and in the second $\operatorname{sign} D=-1$. Obviously this definition does not depend on the choice of the orientation.

Lemma 3. Let the mapping $f: S^n \to S^n$ be differentiable at a point $\xi \in S^n$ and its total differential be nondegenerate. Then there exist closed neighbourhoods V and W of ξ in S^n with $V \subset \mathring{W}$ and a continuous mapping

(26) $$g: S^n \to S^n$$

with the following properties:

i) $g|_{S^n \setminus W} = f|_{S^n \setminus W}$,

ii) $g \sim f$,

iii) $g(x) \neq g(\xi)$ for $x \in W \setminus V$,

iv) g is a V-homeomorphism, which is positive for $\operatorname{sign} df_\xi = 1$ and negative for $\operatorname{sign} df_\xi = -1$.

Proof. Thus the lemma states that the mapping f can be corrected in a certain sense in vicinity of ξ. In order to establish this we shall first transfer f to R^n with the help of the stereographic projection (25).

Without loss of generality we can assume that $\xi \neq \sigma$ and $f(\xi) \neq \sigma$. Let

(27) $$\eta = p(\xi)$$

and

(28) $$M = p(f^{-1}(\sigma)).$$

Since $f(\xi) \neq \sigma$, then

(29) $$\eta \in R^n \setminus M.$$

The set M of course is a closed subset of R^n, hence the set $R^n \setminus M$ is open.

Let us transfer f to R^n, i. e. let us consider the mapping

(30) $$F: R^n \setminus M \to R^n,$$

defined by

(31) $$F = p \circ f \circ p^{-1}.$$

Since by hypothesis df_ξ exists and is nondegenerate, the same is true for $dF_\eta: R^n \to R^n$. According to the definition of the total differential the equality

(32) $$F(y) - F(\eta) - dF_\eta(y-\eta) = r(y)\,\|y-\eta\|$$

holds, where the mapping $r: R^n \setminus M \to R^n$ satisfies $\lim_{y\to\eta} r(y) = 0$.

Let c be a positive constant, such that

(33) $$\|dF_\eta(y-\eta)\| \geqq 2c\|y-\eta\|$$

for every $y \in R^n$ and let the positive real number ρ be such that $B(\eta, \rho) \subset R^n \setminus M$ and the inequality

(34) $$\|r(y)\| < c$$

holds for $\|y-\eta\|<\rho$. Furthermore let $0<\rho'<\rho$ and $\varphi: R^n \to [0, 1]$ be a continuous mapping such that

(35) $$\varphi(y)=0$$

for $y \in R^n \setminus B(\eta, \rho)$ and

(36) $$\varphi(y)=1$$

for $y \in B(\eta, \rho')$.

Now we correct F in vicinity of η. To this end let us consider the mapping $G: R^n \setminus M \to R^n$, defined by

(37) $$G(y)=(1-\varphi(y))F(y)+\varphi(y)(F(\eta)+dF_\eta(y-\eta)).$$

From (35) and (37) follows

(38) $$G(y)=F(y)$$

for $y \in B(\eta, \rho) \cup M$.

Let $B(\eta, \rho)$ be a closed ball and let $W=p^{-1}(B(\eta, \rho))$. Now we return again G on S^n, i. e. we consider the mapping $g: S^n \to S^n$, defined by

(39) $$g|_{S^n \setminus W}=f|_{S^n \setminus W}$$

and

(40) $$g|_W=p^{-1} \circ G \circ p|_W.$$

From (31), (38) follows that $p^{-1} \circ G \circ p$ coincides with g in a neighbourhood of W. Therefore (39) and (40) imply that the mapping $g: S^n \to S^n$ so defined is continuous. This proves i).

From (32) and (37) follows

$$\|G(y)-F(y)\|=\varphi(y)\|r(y)\| . \|y-\eta\|,$$

which together with (33) — (36) gives

(41) $$\|G(y)-F(y)\|<c\rho \qquad (y \in R^n \setminus M).$$

From (39), (41) and from the obvious inequality $\|p(x)-p(-x)\| \geqq \sqrt{2} (x \in S^n \setminus \{\sigma\})$ follows

(42) $$f(x)+g(x) \neq 0$$

for every $x \in S^n$, provided ρ is so small, that $c\rho<\sqrt{2}$. From (42) follows that ii) holds.

In order to establish iii) let $y \in B(\eta, \rho) \setminus B(\eta, \rho')$. From (32) and (37) follows

$$G(y)-G(\eta)=dF_\eta(y-\eta)+(1-\varphi(y))r(y)\|y-\eta\|,$$

whence

$$\|G(y)-G(\eta)\| \geqq \|dF_\eta(y-\eta)\|-|1-\varphi(y)| . \|r(y)\| . \|y-\eta\|$$
$$\geqq 2c\|y-\eta\|-\|r(y)\| . \|y-\eta\| \geqq c\|y-\eta\| \geqq c\rho'>0,$$

i. e.

(43) $$G(y) \neq G(\eta) \quad (y \in B(\eta, \rho) \setminus B(\eta, \rho')).$$

Let $V=p^{-1}(B(\eta, \rho'))$. Now from (43) and (40) follows iii).

Let at last the mapping $\delta: S^n \to S^n$ be defined by

$$\delta(x) = \begin{cases} p^{-1}(F(\eta) + dF\eta_n\ (p(x) - \eta)) & \text{for } x \neq \sigma, \\ \sigma & \text{for } x = \sigma. \end{cases} \tag{44}$$

Clearly δ is a homeomorphism with $\delta|_V = g|_V$, i. e. g is a V homeomorphism.

From sign $df_\xi = 1$ follows sign $dF_\eta = 1$. Hence the linear mappings $F(\eta) + dF_\eta(y - \eta)$ and y with argument $y \in R^n$ are homotopic, all intermediate mappings being linear and nondegenerate. Therefore this homotopy can be transferred on the sphere by means of the stereographic projection (25) and we conclude that in this case

$$\delta \sim \mathrm{id}_{S^n}. \tag{45}$$

Analogically from sign $df_\xi = -1$ follows sign $dF_\eta = -1$ and the mapping $F(\eta) + dF_\eta(y - \eta)$ is homotopic to the linear mapping $H_1: R^n \to R^n$, which changes the sign of the first coordinate only, leaving all other coordinates unchanged. This homotopy can be transferred on the sphere too and we conclude that in this case

$$\delta \sim h_1. \tag{46}$$

This proves iv). Q. E. D.

Theorem 2. If the mapping

$$f: S^n \to S^n \tag{47}$$

and the point $\eta \in S^n$ are such that the set $f^{-1}(\eta)$ consists of a finite number of points x_μ ($\mu = 1, 2, \ldots, m$) and if at every point x_μ the mapping (47) is differentiable and its total differential df_{x_μ} is nondegenerate then

$$\deg f = \sum_{\mu=1}^{m} \operatorname{sign} df_{x_\mu}. \tag{48}$$

Proof. Let the neighbourhoods V_μ and W_μ of x_μ ($\mu = 1, 2, \ldots, m$) and the mappings

$$g_\mu: S^n \to S^n \qquad (\mu = 1, 2, \ldots, m) \tag{49}$$

be defined by applying lemma 3 for $\xi = x_\mu$ ($\mu = 1, 2, \ldots, m$). We can assume furthermore that $W_\mu \cap W_\nu = \varnothing$ ($\mu \neq \nu$). Let $h: S^n \to S^n$ be defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \notin \bigcup_{\mu=1}^{m} W_\mu, \\ g_\mu(x) & \text{for } x \in W_\mu \qquad (\mu = 1, 2, \ldots, m). \end{cases} \tag{50}$$

Lemma 3, i) implies that h is continuous.

Let the mappings $f_\mu: S^n \to S^n$ ($\mu = 1, 2, \ldots, m$) be defined by

$$f_\mu(x)=\begin{cases} f(x) & \text{for } x \overline{\in} \bigcup\limits_{\nu=1}^{\mu} W_\nu, \\ g_\nu(x) & \text{for } x \in W_\mu \end{cases} \qquad (\nu=1,2,\ldots,\mu). \tag{51}$$

From (51) and lemma 3, i) and ii) follows that the mappings f_μ are continuous and that

$$f \sim f_1 \sim \ldots \sim f_m = h. \tag{52}$$

The set $K=S^n \setminus \bigcup\limits_{\mu=1}^{m} \overset{\circ}{V}_\mu$ is compact and consequently such is the set $h(K)$. Since the points x_μ ($\mu=1,2,\ldots,m$) are the only inverse images of η, from (50) and lemma 3, iii) follows that $\eta \overline{\in} h(K)$. Then there exists a neighbourhood U of η with $U \cap h(K)=\emptyset$ and U is a simple subset of S^n. Furthermore obviously

$$h^{-1}(U) \subset \bigcup_{\mu=1}^{m} V_\mu. \tag{53}$$

Let $U_\mu=h^{-1}(U) \cap V_\mu$. Then

$$h^{-1}(U)=\bigcup_{\mu=1}^{m} U_\mu. \tag{54}$$

From lemma 3, iv) follows that h is an U_μ-homeomorphism ($\mu=1,2,\ldots,m$), which is positive or negative depending on sign $d f_{x_\mu}$. Besides we have $h(U_\mu)=U$ ($\mu=1,2,\ldots,m$). From (54) and the remark above follows that theorem 1 is applicable for h, whence

$$\deg h=\sum_{\mu=1}^{m} \operatorname{sign} d f_{x_\mu}. \tag{55}$$

From (52) now follows $\deg f=\deg h$. Q. E. D.

REFERENCE

1. Eilenberg, S. and Steenrod, N. E.: Foundations of algebraic topology. Princeton, 1952.

Постъпила на 17. XII. 1974 г.

СТЕПЕН НА ИЗОБРАЖЕНИЕ В АКСИОМАТИЧНИТЕ ТЕОРИИ НА ХОМОЛОГИИТЕ И КОХОМОЛОГИИТЕ

Я. Кинтишев, И. Проданов, Г. Чобанов

(РЕЗЮМЕ)

Понятието *степен на изображение* играе съществена роля при аксиоматичното въвеждане на хомологиите и кохомологиите. На него се опира съществено например извеждането на известните правила за пресмятане на хомологиите и кохомологиите на краен клетъчен комплекс. Желателно е, разбира се, понятието степен на изображение да се въведе в рамките на абстрактната теория на хомологиите и кохомологиите. В предлаганата работа това се извършва за произволна екстраординарна теория на хомологиите; разбира се, при кохомологиите нещата, както обикновено, стоят аналогично. При това изложението не използува хомотопната класификация на изображенията на n-мерната сфера в себе си. Следствие от този подход е и едно просто доказателство на известното правило за пресмятане на степен на гладко изображение на сферата в себе си с помощта на знаците на якобианите в прообразите на някоя точка.