PEANO AXIOMS FOR MINIMAL RINGS

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Dedicated to Professor B. Petkantchin

Was b weisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.

Richard Dedekind

A ring is said to be minimal when it does not possess proper subrings. Minimal is for example the intersection of all subrings of a given ring. The structure of the minimal rings is well known: they coincide up to isomorphism with the ring of all integers or with its factorrings.

One could, however, define the notion of minimal ring considerably more economically by following conceptually the approach of Grassmann—Dedekind—Peano—Landau [1] — [4] for axiomatic description of natural numbers, whereas the order, addition, multiplication etc. are defined starting from a simple unar operation. In the present article it is shown how this can be done.

The first four Peano axioms for natural numbers arithmetics express as a matter of fact that adding of unity generates an injection in the set of all natural numbers. Therefore, in essence, Peano deals with two axioms only: an axiom for injection and an axiom for induction. The axiomatic description of minimal rings proposed here also deals with two axioms: an axiom for bijection and an axiom for bilateral induction. The first of them concerns the adding and subtracting of unity and the second is a natural minimality request, characteristic for minimal rings. In this way one arrives to the notion of indecomposable bijective system.

In comparison with the axiomatic natural numbers construction some simplifications are presented here. So, for example, after choosing arbitrarily a zero element in an indecomposable bijective system, a natural bijection between the system and the group of its isomorphisms arises. This permits to introduce the addition almost automatically. The ring of homomorphisms of the additive group so obtained is again in one-to-one correspondence with the indecomposable bijective system, which permits it to be transformed in a minimal ring with the same facility. Paragraph 1 treats the relations between minimal rings and indecomposable bijective systems.

In the technical paragraph 2 some special subsets of an indecomposable bijective system are discussed. The importance of the recurrent sets in such kind of questions has been pointed out for the first time by Tagamlitzki [5]. The nature of the subject demands a modification of the notion of recurrent sets. It is interesting to point out that using recurrent sets in the traditional axiomatic construction of natural numbers arithmetics in the sense of Peano one could [6] introduce order in natural systems independently from the arithmetical operations in them in contrast to the approach, chosen in [4]. It is needless to emphasize the gnoseological and historical reasons which make this approach preferable.

Although in papers like this one should not expect new facts to be established, the absence of natural order in an arbitrary indecomposable bijective system made it necessary to introduce and study a surrogate — the notion of natural direction, to which paragraphs 3 and 4 are dedicated. There are exactly two natural directions in every indecomposable bijective system, which are in close relation with the addition. Although the interplay between natural direction and multiplication is not studied here, it seems that the possibilities of this notion are larger and that its further exploration is desirable by abstract consideration of directed rings in analogy with the ordered rings. The speculations in paragraphs 3 and 4 could serve as a model in this attitude.

The last paragraph 5 contains three characteristics of integers in the class of the indecomposable bijective systems. First of all, they are indecomposable bijective systems, in which unilateral induction is not possible. On the other hand, they are indecomposable bijective systems, in which natural order exists. At last, they are the infinite indecomposable bijective systems. Indecomposable bijective systems, which possess these equivalent properties, are called integral and the rest are called cyclic.

In paragraph 1— 4 those properties of indecomposable bijective systems are studied, which are intrinsic to integral as well as to cyclic systems simultaneously. In such a sense one could speak about absolute arithmetics by analogy with geometry.

§ 1. INDECOMPOSABLE BIJECTIVE SYSTEMS AND MINIMAL RINGS

Definition 1. *Bijective system* is called any nonempty set B with a bijection

(1) $$\beta : B \to B.$$

In what follows B always denotes a bijective system.

If B is a bijective system, for the sake of brevity we put

(2) $$x' = \beta(x), \quad 'x = \beta^{-1}(x) \qquad (x \in B).$$

Obviously

(3) $$'(x') = ('x)' = x \qquad (x \in B).$$

The elements x' and $'x$ of B are called *successor* and *predecessor* of x respectively.

If in a bijective system B with the bijection (1) we consider the bijection

(4) $$\beta^{-1} : B \to B,$$

we obtain again a bijective system. It is called **s y m m e t r i c** to B and is denoted by B^{-1}. The successor (predecessor) of an element of B in the symmetric to B bijective system B^{-1} coincides with its predecessor (successor) in B.

More generally, if we consider in B^{-1} an arbitrary notion N of B, we obtain again a notion in B. It is called **s y m m e t r i c** to N. If in any predicate P in B we substitute every notion with its symmetric, we obtain again a predicate in B, which is called **s y m m e t r i c** to P.

Obviously the following assertion holds.

Principle of symmetry. The symmetric statement of every statement true in B is true in B.

In what follows the application of the principle of symmetry will not always be emphasized explicitly.

D e f i n i t i o n 2. A bijective system $\beta: S \to S$ is called **i n d e c o m p o s a b l e**, when for every nonempty subset X of S from

$$(5) \qquad \beta(X) \subset X, \ \beta^{-1}(X) \subset X$$

follows

$$(6) \qquad X = S.$$

In what follows S always denotes an indecomposable bijective system.

Obviously S is an indecomposable bijective system if and only if every nonempty subset of S, which together with every its element contains its successor and its predecessor, coincides with S. When we apply this property of S, we say that we accomplish induction in S.

The notion of indecomposable bijective system is autosymmetric.

If the indecomposable bijective system S consists of the element x only, then $x' = x$. If S consists of the different elements x and y only, then $x' = y$, $y' = x$ and therefore $'x = x'$. The following proposition shows, that these equalities characterize these most simple indecomposable bijective systems.

P r o p o s i t i o n 1. If the indecomposable bijective system S has more than one element, then

$$(7) \qquad x \neq x' \qquad (x \in S)$$

and if S has more than two elements, then

$$(8) \qquad 'x \neq x' \qquad (x \in S).$$

Proof. Induction on x.

The following proposition shows, that a bijective system S is indecomposable under somewhat weaker conditions than those in definition **2**.

P r o p o s i t i o n 2. A bijective system S is indecomposable when there exists such an element s of S, that for every subset X of S containing S from (5) follows (6).

Proof. Let Y be the set of all those elements y of S, for which from (5) follows (6) for every subset X of S containing y. By hypothesis $s \in Y$. Let $y \in Y$. If X is a subset of S with (5) and $y' \in X$, then $y = \beta^{-1}(y') \in X$ and therefore $X = S$ because of $y \in Y$. Hence $y' \in Y$. Similarly it is proved that

$j \in Y$. Consequently the set Y contains s and possesses the property (5) with Y instead of X. Therefore, from the hypothesis of the proposition follows $Y = S$. Q. E. D.

In [7] it is proved that every bijective system can be represented as an union of disjoint indecomposable bijective systems.

Every ring A is a bijective system with

$$x' = x + 1, \ 'x = x - 1 \qquad (x \in A). \tag{9}$$

Definition 3. A bijective system A with (9) is said to be generated by the ring A.

The following theorem gives a criterion for indecomposability of a bijective system, generated by a ring.

Theorem 1. The bijective system generated by a ring is indecomposable iff the ring is minimal.

Proof Let A be a minimal ring and Y be the intersection of all subsets X of A, for which

$$0 \in X, \quad X + 1 \subset X, \quad X - 1 \subset X \tag{10}$$

hold. Obviously

$$0 \in Y, \quad Y + 1 \subset Y, \quad Y - 1 \subset Y. \tag{11}$$

Let X be the set of all those elements x of Y, for which $Y - x \subset Y$ holds; obviously $0 \in X$. Let $x \in X$. Then

$$Y - (x + 1) = (Y - x) - 1 \subset Y - 1 \subset Y$$

according to (11). This shows, that $X + 1 \subset X$, because from (11) follows $x + 1 \in Y$. Similarly it is proved that also $X - 1 \subset X$. So X possesses the properties (10). Hence $X = Y$ according to the definition of Y. Therefore $Y - Y \subset Y$, i. e. Y is a subgroup of the additive group of A.

Let now X be the set of all those elements x of Y, for which $Yx \subset Y$ holds. Obviously $0 \in X$. Let $x \in X$. Then

$$Y(x + 1) \subset Yx + Y \subset Y + Y \subset Y,$$

because Y is a group with respect to the addition. This proves that $X + 1 \subset X$. By analogy it is established that also $X - 1 \subset X$. So X possesses the properties (10) and consequently $X = Y$, i. e. Y contains the product of every two of its elements.

At last, from (11) follows $1 \in Y$, which proves that Y is a subring of A. Then $Y = A$, because the ring A is minimal. But Y is contained in every subset X of A with the properties (10). Hence from (10) follows $X = A$. This proves that the bijective system generated by the ring A satisfies the hypothesis of proposition 2 and is therefore indecomposable.

Let conversely the bijective system generated by the ring A be indecomposable and X be a subring of A. Then of course $X \neq \emptyset$, $X + 1 \subset X$ and $X - 1 \subset X$. Hence $X = A$, because the bijective system A is indecomposable. The ring A is therefore minimal. Q. E. D.

Corollary 1. Every minimal ring is commutative.

Proof. Let A be a minimal ring and x be an arbitrary element of A. Let Y be the set of those elements y of A, that commute with x. Obviously $0 \in Y$. Let $y \in Y$. Then

$$y'x = (y+1)x = yx + x = xy + x = x(y+1) = xy'$$

according to (9). Hence $y' \in Y$. Similarly it is proved that also $y' \in Y$. Therefore $Y = A$, because according to theorem 1 the bijective system generated by the ring A is indecomposable. Q. E. D.

Definition 4. A mapping

$$h : B_1 \to B_2 \tag{12}$$

of the bijective system B_1 into the bijective system B_2 is called a **bijective homomorphism**, when

$$h \circ \beta = \beta \circ h. \tag{13}$$

Obviously for every bijective homomorphism (12) the equality

$$h \circ \beta^{-1} = \beta^{-1} \circ h \tag{14}$$

also holds. The equalities (13) and (14) can also be written in the form

$$h(x') = (h(x))' \qquad (x \in B_1) \tag{15}$$

and

$$h('x) = '(h(x)) \qquad (x \in B_1). \tag{16}$$

From (14) it follows that (12) remains a bijective homomorphism also when each of the systems B_1 and B_2 is replaced by its symmetric.

The following two propositions give sufficient conditions for the uniqueness and surjectiveness of bijective homomorphisms.

Proposition 3. If S is an indecomposable bijective system, B is a bijective system and $s \in S$, $b \in B$, there do not exist more than one bijective homomorphism $h : S \to B$ with $h(s) = b$.

Proof. Induction in S by using (15) and (16).

Proposition 4. If B is a bijective system and S is an indecomposable bijective system, every homomorphism $h : B \to S$ is a surjection.

Proof. Induction in S by using the equalities (15) and (16).

Definition 5. The mapping (12) is called a **bijective antihomomorphism**, when

$$h \circ \beta = \beta^{-1} \circ h. \tag{17}$$

Obviously for every bijective antihomomorphism (12) the equality

$$h \circ \beta^{-1} = \beta \circ h \tag{18}$$

also holds. The equalities (17) and (18) can be written in the form

$$h(x') = '(h(x)) \qquad (x \in B_1) \tag{19}$$

and

$$h('x) = (h(x))' \qquad (x \in B_1). \tag{20}$$

If (12) is a bijective antihomomorphism and one of the bijective systems B_1 or B_2 is substituted by its symmetric, h becomes a bijective homomorphism. Therefore the properties of the bijective homomorphisms are transferred mutatis mutandis for bijective antihomomorphisms too. When in what follows this happens, it will not always be emphasized explicitly.

Definition 6. A bijective homomorphism is called a **bijective isomorphism** when it is a bijection.

Obviously the inverse mapping of every bijective isomorphism is a bijective isomorphism. Every bijective system, that is isomorphic with an indecomposable bijective system, is also indecomposable.

Definition 7. A bijective antihomomorphism is called a **bijective antiisomorphism**, when it is a bijection.

The following proposition gives an example of an antiisomorphism.

Proposition 5. For an arbitrary ring A and for every element a of A the mapping $\nu: A \to A$, defined by $\nu(x) = a - x$ $(x \in A)$, is an antiisomorphism in the bijective system generated by the ring A.

Proof. Clear.

The following theorem shows a class of bijective isomorphisms.

Theorem 2. For every two elements x and y of an indecomposable bijective system S there exists a unique bijective homomorphism

(21) $$h: S \to S$$

with

(22) $$h(x) = y.$$

This homomorphism is an isomorphism.

Proof. The uniqueness follows from proposition 3.

To prove the existence let x be an arbitrary fixed element of S and let Y denote the set of all those elements y of S, for which there exists an isomorphism (21) with (22). Then $x \in Y$ and hence $Y \neq \emptyset$ because the identity i of S is obviously an isomorphism with $i(x) = x$. Let $y \in Y$ and let (21) be a bijective isomorphism with (22). Let us put

(23) $$k = h \circ \beta.$$

It is immediately verified, that k is a bijective isomorphism with $k(x) = y'$. Hence $y \in Y$. Using the isomorphism

(24) $$l = h \circ \beta^{-1}$$

it is proved similarly that also $'y \in Y$. Hence $Y = S$. Q. E. D.

In what follows $h_{x,y}$ denotes the isomorphism (21) with (22). Now from from (23) and (24) the equalities

(25) $$h_{x,y'} = h_{x,y} \circ \beta \qquad (x, y \in S)$$

and

(26) $$h_{x,'y} = h_{x,y} \circ \beta^{-1} \qquad (x, y \in S)$$

follow.

In the next proposition it is proved, that the group of isomorphisms of S is a commutative one.

Proposition 6. For every two bijective homomorphisms $h: S \to S$ and $k: S \to S$ the equality

(27) $$h \circ k = k \circ h$$

holds.

Proof. Let s be an arbitrary fixed element of S. From theorem 2 it follows, that (27) will be proved, if we prove

(28) $$h_{s,x} \circ h_{s,y} = h_{s,y} \circ h_{s,x} \qquad (x, y \in S).$$

For an arbitrary fixed element x of S let Y be the set of all those elements y of S, for which (28) holds. Then $s \in Y$ and hence $Y \neq \emptyset$, because $h_{s,s}$ is the identity of S. Let $y \in Y$. From (25), (28) and (13) follows $h_{s,x} \circ h_{s,y'} = h_{s,x} \circ h_{s,y} \circ \beta = h_{s,y} \circ \beta \circ h_{s,x} = h_{s,y'} \circ h_{s,x}$, i. e. $y' \in Y$. It is proved similarly that $'y \in Y$ Q. E. D.

If A_1 and A_2 are rings and

(29) $$h: A_1 \to A_2$$

is a ring homomorphism, h is obviously a homomorphism between the bijective systems, generated by the rings A_1 and A_2. The following proposition shows that in some cases the inverse statement is also true.

Proposition 7. If the ring A_1 is minimal and (29) is a homomorphism between the bijective systems, generated by the rings A_1 and A_2, for which $h(0)=0$ holds, then h is a ring homomorphism.

Proof. From $h(1)=h(0')=(h(0))'=0'=1$ follows, that h maps the unity of A_1 in the unity of A_2.

Let x be an arbitrary element of A_1 and Y be the set of all those elements y of A_1 for which

(30) $$h(x+y)=h(x)+h(y).$$

Obviously $0 \in Y$. Let $y \in Y$. Then from $h(x+y') = h((x+y)') = (h(x+y))' = (h(x)+h(y))' = h(x)+(h(y))' = h(x)+h(y')$ follows $y' \in Y$. It is proved similarly that also $'y \in Y$ holds. Now from theorem 1 follows $Y=A_1$ and hence (30) is true for all x and y from A_1.

Let x again be an arbitrary element of A_1 and Y now be the set of all those elements y of A_1, for which

(31) $$h(xy)=h(x)h(y).$$

From $h(0)=0$ follows $0 \in Y$. Let $y \in Y$. Then from $h(xy')=h(xy+x)=h(xy)+h(x)=h(x)h(y)+h(x)=h(x)(h(y))'=h(x)h(y')$ follows $y' \in Y$. It is proved similarly that also $y' \in Y$ holds. Hence $Y=A_1$ and (31) is true for all x and y of A_1. Q. E. D.

Corollary 2. If S is an indecomposible bijective system and 0 is an arbitrary element of S, there do not exist in S more than one structure of a ring with a zero 0, which generates the bijective system S.

Proof. Theorem 1 and proposition 7 with the identity of S instead of h.

According to theorem 1 every minimal ring generates an indecomposable bijective system. The inverse procedure is also possible. One can introduce in every indecomposable bijective system S such a structure of a minimal ring, that the bijective system, which this ring generates, coincides with S. In this direction there is no uniqueness. The construction of such a minimal ring is possible in so many different ways, as many elements S contains.

Theorem 3. For every indecomposable bijective system S and for every element 0 of S there exists a unique structure of a ring in S with the following properties:

i) 0 is the zero element of the ring.

ii) The bijective system generated by the ring coincides with S. This ring is commutative and minimal.

Proof. The uniqueness is established in corollary 2.

To prove the existence we shall first define addition in S. From theorem 2 it follows that for every element x of S there exists a unique bijective isomorphism

(32) $$h_{0,x}: S \rightarrow S$$

with

(33) $$h_{0,x}(0)=x$$

and that the mapping, defined by

(34) $$x \rightarrow h_{0,x} \qquad (x \in S)$$

is a bijection between S and the set of all bijective isomorphisms in S. The last set being a group, the bijection (34) permits S to be transformed into a group, so that (34) will be a group isomorphism. Let S be supplied with this group structure. From proposition 6 it follows, that the so constructed group is commutative. For this reason additive notation is accepted for it.

As the mapping defined by (34) is a group isomorphism, we have

(35) $$x+y \rightarrow h_{0,x} \circ h_{0,y} \qquad (x, y \in S).$$

On the other hand

(36) $$x+y \rightarrow h_{0,x+y} \qquad (x, y \in S)$$

according to (34). From (35) and (36) follows $h_{0,x+y} = h_{0,x} \circ h_{0,y}$, which together with the definition of $h_{\xi,\eta}$ gives $x+y=h_{0,x+y}(0)=h_{0,x}(h_{0,y}(0))$, i. e.

(37) $$x+y=h_{0,x}(y) \qquad (x, y \in S).$$

From (37) follows $x+0=h_{0,x}(0)$, i. e.

(38) $$x+0=x \qquad (x \in S)$$

according to the definition of $h_{\xi,\eta}$. Hence 0 is the zero element of the commutative group S.

Let by definition

(39) $$1=0'.$$

From (37) and (39) follows $x+1=h_{0,x}(1)=(h_{0,x}(0))'$, i. e.

(40) $$x+1=x' \qquad (x \in S),$$

according to the definition of $h_{\xi,\eta}$.

From (40) follows

(41) $$x-1='x.$$

Before defining multiplication in S let us consider the set of the homomorphisms of the additive group S.

If for a group homomorphism

(42) $$p: S \to S$$

the equality $p(1)=0$ holds, it is the zero homomorphism. Indeed, obviously $\operatorname{Ker} p \neq \emptyset$. Let $x \in \operatorname{Ker} p$. Then $p(x+1)=p(x)+p(1)=0$, i. e. $x' \in \operatorname{Ker} p$ according to (40). It is proved similarly that also $'x \in \operatorname{Ker} p$. Hence $\operatorname{Ker} p = S$.

From this it follows, that if for two group homomorphisms (42) and $q: S \to S$ the equality $p(1)=q(1)$ holds, they coincide. Hence every group homomorphism (42) is uniquely determined by its value $p(1)$.

We shall establish now, that for every element x of S there exists a group homomorphism (42) with

(43) $$p(1)=x.$$

Let X be the set of all those elements x of S, for which the statement is true. Obviously $X \neq \emptyset$. Let $x \in X$ and let (42) be a group homomorphism with (43). Then $p+i$, where i is the identity of S, is a group homomorphism with $(p+i)\ (1)=p(1)+i(1)=x+1=x'$, hence $x' \in X$. It is proved similarly that also $'x \in X$ holds. Therefore $X=S$ and the statement is proved.

Thus for every element x of S there exists a unique group homomorphism

(44) $$p_x: S \to S$$

with

(45) $$p_x(1)=x.$$

From this definition immediately follow the equalities

(46) $$p_0=0$$

and

(47) $$p_1=i.$$

From the above it is clear, that the mapping defined by

(48) $$x \to p_x \qquad (x \in S),$$

is a bijection between S and the ring Σ of the group homomorphisms in S.

The bijection (48) is an isomorphism between the group S and the additive group of the ring S, i. e. the equality

(49) $$p_{x+y}=p_x+p_y \qquad (x, y \in S)$$

holds. Indeed $p_x+p_y: S \to S$ is a group homomorphism with $(p_x+p_y)\,(1) = p_x(1)+p_y(1)=x+y$.

As Σ is a ring, the bijection (48) permits S to be transformed into a ring, so that (48) will be a ring isomorphism. Let S be supplied with this ring structure.

From (49) it follows, that the additive group structure in S, defined by means of the bijection (34), and the additive ring group structure in S, defined by means of the bijection (48), coincide. Hence from (38) it follows, that the element 0 of the bijective system S coincides with the zero element of the ring S. In this manner the property i) from the formulation of the theorem is established.

As (48) is a ring homomorphism,

$$xy \rightarrow p_x \circ p_y \qquad (x, y \in S) \tag{50}$$

holds. From (50) follows $x1 \rightarrow p_x \circ p_1 = p_x$ according to (47), which together with the bijectivity of (48) gives $x1 = x$ $(x \in S)$. It is proved similarly that $1x = x$ $(x \in S)$ also holds. Hence the element 1 defined by (39) of the bijective system S is the unity of the ring S. Now (40) shows that property ii) from the formulation of the theorem is also fulfilled.

From theorem 1 it follows, that the so constructed ring S is minimal Its commutativity follows from corollary 1. Q. E. D.

In what follows, when an indecomposable bijective system S is given and we speak about a ring structure in S, we shall understand, that an element 0 of S is fixed and we consider the unique structure of a ring in S with the properties i) and ii) from theorem 3.

§ 2. RECURRENT SETS

Let B be a bijective system, $y \in B$ and $M \subset B$.

Definition 1. The set M is called right y-recurrent, when from $x \in M$ and $x \neq y$ follows $x' \in M$.

Definition 2. The set M is called left y-recurrent, when from $x \in M$ and $x \neq y$ follows $'x \in M$.

The notions right and left y-recurrent set are obviously symmetric. The empty set, as well as the whole set B, are right and left y-reccurent for every y from B; the one-element set $\{y\}$ is right as well as left y-recurrent. The following lemma gives another, not so trivial example.

Lemma 1. For every element y of B the set $B \backslash \{y'\}$ is right y-recurrent and the set $B \backslash \{'y\}$ is left y-reccurent.

Proof. Immediate application of definitions 1 and 2.

The following proposition permits to construct reccurent sets, in which induction is possible.

Proposition 1. The intersection of right y-reccurent sets is a right y-recurrent set and the intersection of left y-recurrent sets is a left y-recurrent set.

Proof. Immediate application of definitions 1 and 2.

Let B be a bijective system and x, y be elements of B.

Definition 3. The intersection of all right y-recurrent subsets of B, which contain x, is called the **right y-reccurent closure** of x.

Definition 4. The intersection of all left y-recurrent subsets of B, which contain x, is called the left y-recurrent closure of x.

The right y-recurrent closure of x is denoted by $R[x, y]$ and the left y-recurrent closure of x is denoted by $L[x, y]$.

The notions right and left y-recurrent closure of x are obviously symmetric. From proposition 1 it follows that $R[x, y]$ is a right and $L[x, y]$ is a left y-recurrent set. Obviously $R[y, y]=\{y\}$ and $L[y, y]=y$.

The following two propositions express some properties of $R[x, y]$ and $L[x, y]$.

Proposition 2. If x is an element of B, then

$$\beta(R[x, 'x])\subset R[x, 'x] \tag{1}$$

and

$$\beta^{-1}(L[x, x'])\subset L[x, x']. \tag{2}$$

Proof. Let $\xi\in R[x, 'x]$. Then $\xi'\in R[x, 'x]$ for $\xi\neq 'x$, since $R[x, 'x]$ is a right $'x$-recurrent set. If $\xi='x$, then $\xi'=x$, hence again $\xi'\in R[x, 'x]$. This proves (1). Now (2) follows from the principle of symmetry. Q. E. D.

Proposition 3. If x is an element of B, then

$$R[x, x']=\{x, x'\} \tag{3}$$

and

$$L[x, 'x]=\{x, 'x\}. \tag{4}$$

Proof. The set $\{x, x'\}$ contains x and is right x'-recurrent. Hence $R[x, x']\subset\{x, x'\}$. The opposite inclusion is obvious and so (3) is proved. Now (4) follows from the principle of symmetry. Q. E. D.

The next theorem studies the behavior of $R[x, y]$ and $L[x, y]$ when one of the arguments x or y is replaced by its successor or predecessor.

Theorem 1. If x and y are different elements of a bijective system B, then

$$R[x, 'y]=R[x, y]\setminus\{y\}, \tag{5}$$

$$R[x', y]\cup\{x\}=R[x, y] \tag{6}$$

and

$$L[x, y']=L[x, y]\setminus\{y\}, \tag{7}$$

$$L['x, y]\cup\{x\}=L[x, y]. \tag{8}$$

If B is an indecomposable bijective system, (6) and (8) can be written in the form

$$R[x', y]=R[x, y]\setminus\{x\} \tag{9}$$

and

$$L['x, y]=L[x, y]\setminus\{x\}. \tag{10}$$

Proof. First of all we shall establish the validity of (5). From $x\neq y$ follows

$$x\in R[x, y]\setminus\{y\}. \tag{11}$$

On the other hand, the set $R[x, y]\setminus\{y\}$ is right $'y$-reccurent. Indeed, let $\xi \in R[x, y]\setminus\{y\}$ and $\xi \neq 'y$. Then $\xi \neq y$ and $\xi \in R[x, y]$. Hence $\xi' \in R[x, y]$, because $R[x, y]$ is right y-recurrent set. But $\xi' \neq y$, therefore $\xi' \in R[x, y]\setminus\{y\}$. Now from (11) follows

(12) $$R[x, 'y] \subset R[x, y]\setminus\{y\}.$$

Obviously

(13) $$x \in R[x, 'y] \cup \{y\}.$$

On the other hand, the set $R[x, 'y] \cup \{y\}$ is right y-recurrent. Indeed, let $\xi \in R[x, 'y] \cup \{y\}$ and $\xi \neq y$. Then $\xi \in R[x, 'y]$ and therefore $\xi' \in R[x, 'y] \subset R[x, 'y] \cup \{y\}$ for $\xi \neq 'y$. For $\xi = 'y$ we have $\xi' = y$, hence again $\xi' \in R[x, 'y] \cup \{y\}$. Now from (11) follows

(14) $$R[x, y] \subset R[x, 'y] \cup \{y\}.$$

From lemma 1 it follows, that the set $B\setminus\{y\}$ is right $'y$-recurrent. On the other hand, $x \in B\setminus\{y\}$, hence $R[x, 'y] \subset B\setminus\{y\}$, consequently

(15) $$y \bar{\in} R[x, 'y] \qquad (x \neq y)$$

Now (5) follows from (12), (14) and (15) and (7) follows from (5) and from the principle of symmetry.

We shall establish now the validity of (6). Obviously

(16) $$x \in R[x', y] \cup \{x\}.$$

On the other hand, the set $R[x', y] \cup \{x\}$ is right y-recurrent. Indeed, let $\xi \in R[x', y] \cup \{x\}$ and $\xi \neq y$. If $\xi \in R[x', y]$, then $\xi' \in R[x', y] \cup \{x\}$, because $R[x', y]$ is a right y-recurrent set. If $\xi = x$, then $\xi' \in R[x', y] \subset R[x', y] \cup \{x\}$. Now from (16) follows

(17) $$R[x, y] \subset R[x', y] \cup \{x\}.$$

From $x \neq y$ follows

(18) $$x' \in R[x, y].$$

On the other hand, $R[x, y]$ is a right y-recurrent set, hence $R[x', y] \subset R[x, y]$ and consequently

(19) $$R[x', y] \cup \{x\} \subset R[x, y].$$

Now (6) follows from (17) and (19) and (8) follows from (6) and from the principle of symmetry.

The equality (9) will be proved, if we show, that

(20) $$x \bar{\in} R[x', y].$$

For an arbitraty fixed y let X denote the set, formed by y and by those elements x of B different from y, for which (20) holds. Obviously $X \neq \varnothing$. From proposition 1, § 1 follows $y' \neq y''$, because B by assumption has more than one element. Now (15) with y' instead of y and with y'' instead of x shows that $y' \bar{\in} R[y'', y]$, which is (20) with $x = y'$. Hence $y' \in X$. On the other hand $'y \bar{\in} \{y\} = R[y, y]$, hence $'y \in X$ too. Let now $x \in X$ and $x \neq y$.

First we shall establish, that $x' \in X$. If $x' = y$, this is trivial, therefore let $x' \neq y$. From (20) follows $x' \bar{\in} R[x'', y]$, because the mapping $\xi \to \xi'$ ($\xi \in B$) is a bijective isomorphism. But $R[x'', y] \subset R[x', y']$ according to (3). Therefore $x' \in R[x'', y]$, i. e. $x' \in X$. Finally we shall show, that $'x \in X$. If $'x = y$, this is trivial, therefore let $'x \neq y$. As the mapping $\xi \to {}'\xi$ ($\xi \in B$) is a bijective isomorphism from (20) follows $'x \bar{\in} R[x, 'y]$, which together with (5) gives $'x \bar{\in} R[x, y]$ because of $'x \neq y$. But this shows, that $'x \in X$ and proves (20).

In this way (9) is proved. Now (10) follows from the principle of symmetry. Q. E. D.

Corollary 1. If x and y are elements of B, then $R[x, y]$ is a left x-recurrent set and $L[x, y]$ is a right x-recurrent set.

Proof. Let $\xi \in R[x, y]$ and $\xi \neq x$. Then $'\xi \in R['x, 'y]$. But $R['x, 'y] = R[x, 'y] \cup \{'x\}$ according to (6). As $'\xi \neq {}'x$, then $'\xi \in R[x, 'y]$. Now (5) gives $'\xi \in R[x, y]$ and the statement for $R[x, y]$ is proved. Now the statement for $L[x, y]$ follows from the principle of symmetry. Q. E. D.

Corollary 2. For every element x of S the equality

$$R[x, 'x] \cup L[x, x'] = S \tag{21}$$

holds.

Proof. Let X be the set in the left side of (21). Obviously $x \in X$, hence $X \neq \emptyset$. Let $\xi \in X$. If $\xi \in R[x, 'x]$ and $\xi \neq {}'x$, then $\xi' \in X$, because $R[x, 'x]$ is a right $'x$-recurrent set. If $\xi = {}'x$, then $\xi' = x \in X$. Now let $\xi \in L[x, x']$. If $\xi \neq x$, then $\xi' \in L[x, x'] \subset X$ according to corollary 1. If $\xi = x$, then $\xi \neq {}'x$ according to proposition 1, § 1. Therefore $\xi' = x' \in R[x', x] \subset X$ holds. This proves $\xi' \in X$. From the principle of symmetry now follows $'\xi \in X$. Hence $X = S$. Q. E. D.

Corollary 3. If for some element x of S the relation

$$'x \in R[x, 'x] \tag{22}$$

holds, then

$$R[x, 'x] = S \tag{23}$$

and if the relation

$$x' \in L[x, x'] \tag{24}$$

holds, then

$$L[x, x'] = S. \tag{25}$$

Proof. Let $\xi \in R[x, 'x]$. Then $'\xi \in R[x, 'x]$ according to corollary 1 for $\xi \neq x$ and according to (22) for $\xi = x$. Hence $\beta^{-1}(R[x, 'x]) \subset R[x, 'x]$. On the other hand, from proposition 2 follows $\beta(R[x, 'x]) \subset R[x, 'x]$. Now (23) is proved by induction. The second part of the statement follows from the first and from the principle of symmetry. Q. E. D.

The next proposition deals with a property of the intervals in S.

Proposition 4. For every two elements x and y of S the inclusions

$$R[x, y] \cap R[y, x] \subset \{x, y\} \tag{26}$$

and

(27) $$L[x,\ y] \cap L[y,\ x] \subset \{x,\ y\}$$

hold.

Proof. Let x be an arbitrary fixed element of S and Y be the set of the elements y of S, for which (26) holds. As $R[x,\ x]=\{x\}$, then $x \in Y$ and hence $Y \neq \emptyset$. From proposition 3 follows $x' \in Y$, therefore let $y \in Y$ and $y \neq x$. If $y'=x$, obviously $y' \in Y$, therefore let $y' \neq x$. Then from (5) and (9) follow the inclusions

$$R[x,\ y'] \subset R[x,\ y] \cup \{y'\} \text{ and } R[y',\ x] \subset R[y,\ x] \setminus \{y\}$$

respectively. Hence

$$R[x,\ y'] \cap R[y',\ x] \subset \{x, y'\}$$

according to (26). Consequently from $y \in Y$ follows $y' \in Y$. The relation $'y \in Y$ is established by analogy and so (26) is proved. Now (27) follows from (26) and from the principle of symmetry. Q. E. D.

The next two propositions show, that if one of the arguments is fixed, the sets $R[x,\ y]$ and $L[x, y]$ determine the other one uniquely.

Proposition 5. If x, y and z are elements of S, from $y \in R[x,\ z]$ and $z \in R\,[x,\ y]$, as well as from $y \in L\,[x,\ z]$ and $z \in L\,[x,\ y]$ follows $y=z$.

Proof. Let y and z be arbitrary fixed elements of S. For $y=z$ there is nothing to prove, therefore let

(28) $$y \neq z$$

and let X be the set of the elements x of S, for which

(29) $$y \overline{\in} R[x,\ z]$$

or

(30) $$z \overline{\in} R[x,\ y]$$

holds.

From (9) and (28) follows $z \overline{\in} R[z',\ y]$, hence

(31) $$z' \in X.$$

From (28) and $R(y,\ y]=\{y\}$ follows $z \overline{\in} R[y,\ y]$, hence

(32) $$y \in X.$$

It is proved similarly that also

(33) $$z \in X.$$

Let now $x \in X$ and let for example (29) hold. We shall prove

(34) $$x' \in X.$$

Because of (31) we can suppose $x \neq z$. But then from (6) follows $R[x',\ z] \subset R[x,\ z]$, which together with (29) gives $y \overline{\in} R[x',\ z]$ and hence (34). Now we shall prove

(35) $$'x \in X.$$

Because of (33) we can suppose $'x \neq z$. But then (9) gives $R['x, z] \subset R[x, z] \cup \{'x\}$. Hence either $y \bar{\in} R['x, z]$ and then (35) holds, or $y \in R['x, z]$ and then because of (29) $'x = y$ holds, in which case (35) is true because of (32).

Consequently for (29) the relations (34) and (35) hold. It is proved similarly that these relations are also true for (30). Hence $X = S$ and the first part of the statement is proved. Now the second follows from the principle of symmetry. Q. E. D.

Proposition 6. If x, y and z are elements of S, from $y \in R[z, x]$ and $z \in R[y, x]$, as well as from $y \in L[z, x]$ and $z \in L[y, x]$ follows $y = z$.

Proof. For $z = x$ the equality $R[z, x] = \{x\}$ holds and the statement is trivial, therefore let $z \neq x$. Then

$$R[z, x] \setminus \{z\} = R[z', x] \tag{36}$$

according to (9). If we suppose $y \neq z$, from $y \in R[z, x]$ and (36) would follow $y \in R[z', x]$ and therefore $R[y, x] \subset R[z', x]$, which contradicts (36) and $z \in R[y, x]$). Hence $y = z$ and the first part of the proposition is proved. Now the second follows from the principle of symmetry. Q. E. D.

§ 3. EQUIDIRECTED TRIPLES

In the present paragraph we discuss bijective systems B and indecomposable bijective systems S with three elements at least.

For an arbitrary set M the set of all ordered triples (x, y, z) of elements of M with $x \neq y \neq z \neq x$ is denoted by $\tilde{M}$.

Definition 1. Two elements of $\tilde{B}$ are called elementary equidirected, when one of them is obtained from the other by replacing its first component with its successor or by even permutation of its components.

In this way a binary relation $\triangle$ is defined, which is reflexive and symmetric, but is not transitive in general. Let $\uparrow$ denote the least equivalence relation in $\tilde{B}$, which contains as a subset the relation $\triangle$ in $\tilde{B}$.

Definition 2. Two elements of $\tilde{B}$ are called equidirected, when they are in the relation $\uparrow$.

If λ and μ are equidirected elements of $\tilde{B}$, we shall write $\lambda \uparrow \mu$. The relations $\triangle$ and $\uparrow$ in $\tilde{B}$ are obviously autosymmetric.

The following proposition, which enumerates some triples, equidirected with (x, y, z), is often used at inductive studying of the relation $\uparrow$.

Proposition 1. If (x, y, z) and (ξ, η, ζ) are elements of $\tilde{B}$ and $\xi \in \{'x, x, x'\}$, $\eta \in \{'y, y, y'\}$, $\zeta \in \{'z, z, z'\}$, then $(x, y, z) \uparrow (\xi, \eta, \zeta)$.

Proof. Immediate verification.

The next proposition gives a connection between the relation $\uparrow$ and intervals in B.

Proposition 2. If the triple (x, y, z) is equidirected with the triple (ξ, η, ζ), from

(1) $$y \in R[x, z]$$

follows at least one of the relations

(2) $$\eta \in R[\xi, \zeta], \quad \zeta \in R[\eta, \xi], \quad \xi \in R[\zeta, \eta]$$

and from

(3) $$y \in L[x, z]$$

follows at least one of the relations

(4) $$\eta \in L[\xi, \zeta], \zeta \in L[\eta, \xi], \xi \in L[\zeta, \eta].$$

Proof. First we shall prove that from (1) follows at least one of the relations (2), when the triple (x, y, z) is elementary equidirected with the triple (ξ, η, ζ). This is obviously true, if the triple (ξ, η, ζ) is obtained from the triple (x, y, z) by an even permutation of its components. Hence according to definition 1 we have to consider only the cases $(\xi, \eta, \zeta)=(x', y, z)$ and $(\xi, \eta, \zeta)=('x, y, z)$, in either of which from (1) follows the first of the relations (2) according to (9), § 2.

In order to prove, that from (1) follows at least one of the relations (2) in the general case, let T denote the set of all triples (u, v, w) of $\widetilde{B}$, for which at least one of the relations $v \in R[u, w]$, $w \in R[v, u]$ or $u \in R[w, v]$ holds. Let E be the epuivalence relation in B, whose equivalence classes are T and $\widetilde{B} \setminus T$. From the just proved part of the statement it follows, that the relation $\uparrow$ in $\widetilde{B}$ is a subset of the relation E. Hence $\uparrow \subset E$. Therefore the first part of the statement is proved. Now the second part follows from the principle of symmetry. Q. E. D.

C o r o l l a r y 1. If x is an arbitrary element of B, none of the triples $('x, x, x')$ and $(x', x, 'x)$ is equidirected with the other.

Proof. If some two of the elements $'x$, x and x' coincide, the statement is trivially true, therefore let all three of them be different from one another. Obviously $x \in R['x, x']$. If the statement were not true, from proposition 2 it would follow, that at least one of the relations

(5) $$x \in R[x', 'x], \ 'x \in R[x, x'], \ x' \in R['x, x]$$

would hold. But the first and third of them are impossible according to (5), § 2 and (9), § 2 respectively, and the second relation (5) is impossible according to proposition 3, § 2. Q. E. D.

The following proposition shows that the directions are invariant with respect to bijective isomorphisms.

P r o p o s i t i o n 3. If S is an indecomposable bijective system, $h: S \to S$ is a bijective homomorphism and $(\xi, \eta, \zeta) \in \widetilde{S}$, then

(6) $$(h(\xi), h(\eta), h(\zeta)) \in \widetilde{S}$$

and

(7) $$(\xi, \eta, \zeta) \uparrow (h(\xi), h(\eta), h(\zeta)).$$

Proof. The validity of (6) follows from theorem 2, § 1.

Let x be an arbitrary fixed element of S. There exists an element y of S with $h=h_{x,y}$. Therefore (7) will be proved, if we show, that

(8) $$(\xi, \eta, \zeta) \uparrow (h_{x,y}(\xi), h_{x,y}(\eta), h_{x,y}(\zeta))$$

for every y of S. Let Y be the set of the elements y of S, for which (8) is true. Obviously $x \in Y$, because $h_{x,x}$ is the identity of S. Hence $Y \neq \emptyset$. Let $y \in Y$. From (13), § 1 and (25), § 1 follows $h_{x,y'}=\beta \circ h_{x,y}$. Therefore

(9) $$(h_{x,y'}(\xi), h_{x,y'}(\eta), h_{x,y'}(\zeta))=((h_{x,y}(\xi))', (h_{x,y}(\eta))', h_{x,y}(\zeta))').$$

From (8), (9) and proposition 1 follows $y' \in Y$. It is proved similarly that also $'y \in Y$. Hence $Y=S$. Q. E. D.

Corollary 2. For arbitrary elements x and ξ of S the relations

(10) $$('x, x, x') \uparrow ('\xi, \xi, \xi')$$

and

(11) $$(x', x, 'x) \uparrow (\xi', \xi, '\xi)$$

hold.

Proof. Proposition 3 with $h=h_{x,\xi}$.

The next proposition strengthens corollary 1.

Proposition 4. If x, z and ξ are elements of S, then

(12) $$('x, x, z) \uparrow ('\xi, \xi, \xi') \qquad ('x \neq z \neq x)$$

and

(13) $$(x', x, z) \uparrow (\xi', \xi, '\xi) \qquad (x' \neq z \neq x).$$

Proof. We shall establish first, that

(14) $$('x, x, z) \uparrow ('x, x, x') \qquad ('x \neq z \neq x).$$

Let Z be the set, consisting of $'x$, x and of those elements z of S, for which (14) is true. Obviously $Z \neq \emptyset$. Let $z \in Z$.

If $z='x$, then $z'=x$, hence $z' \in Z$. If $z=x$, then $z'=x'$ and obviously $('x, x, z) \uparrow ('x, x, x')$, therefore again $z' \in Z$. Let $'x \neq z \neq x$. If z' coincides with $'x$ or with x, obviously $z' \in Z$, therefore let $'x \neq z' \neq x$. From proposition 1 now follows $('x, x, z') \uparrow ('x, x, z)$, which together with (14) gives $('x, x, z') \uparrow ('x, x, x')$ and this shows, that $z' \in Z$. Consequently from $z \in Z$ follows $z' \in Z$.

It is proved similarly that also $'z \in Z$. Hence $Z=S$ and (14) is proved. Now (12) follows from (10) and (14), and (13) follows from (12) and from the principle of symmetry. Q. E. D.

In the following teorem the basic property of the relation $\uparrow$ is formulated.

Theorem 1. The relation $\uparrow$ divides $\mathfrak{S}$ into two classes. All triples of the type

(15) $$('\xi, \xi, \xi') \qquad (\xi \in S)$$

belong to one of them, and all triples of the type

$$(16) \qquad (\xi', \xi, '\xi) \qquad (\xi \in S)$$

belong to the other.

Proof. From corollaries 1 and 2 it follows, that all the triples of the type (15) are equivalent between one another, that all the triples of the type (16) are equivalent between one another and that no triple of the type (15) is equivalent to a triple of the type (16). Therefore the theorem will be proved, if we show, that for every triple (x, y, z) from $\widetilde{S}$ either

$$(17) \qquad (x, y, z) \uparrow ('\xi, \xi, \xi')$$

or

$$(18) \qquad (x, y, z) \uparrow (\xi', \xi, '\xi)$$

is true for some ξ from S.

Let x be an arbitrary fixed element of S and let Y be the set, formed by x and by all elements y of S with $y \neq x$ for which some of the relations (17) or (18) hold for an arbitrary element (x, y, z) of $\widetilde{S}$. From proposition 4 follows $Y \neq \emptyset$. Let $y \in Y$.

If $y = x$, then $(x, y', z) = (x, x', x) \uparrow ('\xi, \xi, \xi')$ according to (12) with x' instead of x. Hence $y' \in Y$. If $y \neq x$ and $y' = x$, obviously $y' \in Y$, therefore let $y \neq x \neq y'$. Now let $(x, y', z) \in \widetilde{S}$, i. e. $x \neq z \neq y'$. If $z \neq y$, then $(x, y, z) \uparrow (x, y', z)$ according to proposition 1 and as at least one of the relations (17) or (18) is true, at least one of the relations

$$(19) \qquad (x, y', z) \uparrow ('\xi, \xi, \xi')$$

or

$$(20) \qquad (x, y', z) \uparrow (\xi', \xi, '\xi)$$

will be also true. If $z = y$, then $(x, y', z) = (x, y', y) \uparrow (y', y, x) \uparrow (\xi', \xi, '\xi)$ according to proposition 4, i. e. the relation (19) will be true. Consequently from $y \in Y$ follows $y' \in Y$. Now $'y \in Y$ follows from the principle of symmetry. Q. E. D.

§ 4. NATURAL DIRECTIONS

Definition 1. For an arbitrary set M a subset D of $\widetilde{M}$ is called a direction in M, when it possesses the following properties:

i) From $(x, y, z) \in D$ follows $(y, z, x) \in D$.

ii) From $(x, y, z) \in D$ and $(x, z, t) \in D$ follows $(x, y, t) \in D$.

When in a set M a direction is given, M is called a directed set.

The following proposition strengthens the conditions i) and ii).

Proposition 1. If D is a direction in M, then:

j) From $(x, y, z) \in D$ follows $(z, x, y) \in D$.

jj) From $(x, y, z) \in D$ and $(x, z, t) \in D$ follows $(y, z, t) \in D$.

Proof. j) Clear.

jj) From the hypothesis of the proposition, from j) and i) follow $(z, x, y) \in D$ and $(z, t, x) \in D$, which together with ii) gives $(z, t, y) \in D$. Now the statement follows from j). Q. E. D.

For a given direction D in M we shall denote by $\bar{D}$ the set of all elements (x, y, z) of $\tilde{M}$, for which $(z, y, x) \in D$ holds.

Proposition 2. $\bar{D}$ is a direction in M.

Proof. Let $(x, y, z) \in \bar{D}$, i. e. $(z, y, x) \in D$. From j) follows $(x, z, y) \in D$, i. e. $(y, z, x) \in D$, whence the property i) for $\bar{D}$ is verified.

Let $(x, y, z) \in \bar{D}$ and $(x, z, t) \in \bar{D}$, i. e. $(z, y, x) \in D$ and $(t, z, x) \in D$. From j) follows $(x, z, y) \in D$ and $(x, t, z) \in D$. Then from ii) and i) we obtain consecutively $(x, t, y) \in D$ and $(t, y, x) \in D$, i. e. $(x, y, t) \in \bar{D}$ whence the property ii) for $\bar{D}$ is also verified. Q. E. D.

Definition 2. The direction $\bar{D}$ in M is called opposite to the direction D in M.

Obviously $\bar{\bar{D}} = D$.

Proposition 3. For every direction D in M the equality $D \cap \bar{D} = \emptyset$ holds.

Proof. Otherwise there would exist a triple (x, y, z) with $(x, y, z) \in D$ and $(x, y, z) \in \bar{D}$. From the last relation follows $(z, y, x) \in D$, which together with j) gives $(x, z, y) \in D$. Now from $(x, y, z) \in D$ and ii) follows $(x, z, z) \in D$, which is impossible. Q. E. D.

For a given partial order $<$ in M we shall denote by $D(<)$ the set of all those elements (x, y, z) of $\tilde{M}$, for which one of the conditions

$$x<y<z, \tag{1}$$

$$y<z<x \tag{2}$$

or

$$z<x<y \tag{3}$$

holds.

Proposition 4. For every partial order $<$ in M the set $D(<)$ is a direction in M.

Proof. Immediate verification of conditions i) and ii) of the definition 1.

For the direction $D(<)$ in M we shall say, that it is generated by the order $<$ in M. The opposite to $<$ order generates the opposite to $D(<)$ direction.

If in M a direction D is given and m is an arbitrary element of M, let by definition

$$x<y \qquad (x, y \in M), \tag{4}$$

when

$$x=m \neq y \tag{5}$$

or

$$(m, x, y) \in D \tag{6}$$

holds. It is verified immediately, that this relation is a partial order in M. It is called a (m, D)-order in M. Similarly it is immediately verified, that for it $D(<) \subset D$ holds. In the general case this inclusion is a strict one.

Definition 3. A direction D in M is said to be a full one, when $D \cup \bar{D} = \tilde{M}$.

If the order $<$ in M is a full one, the direction $D(<)$ is also full. Conversely, if an order $<$ in an at least three-element set M generates a full direction in M, the order is also full.

Proposition 5. If D is a full direction in M and m is an element of M, the direction in M, generated by the (m, D)-order in M, coincides with D.

Proof. Because of $D(<) \subset D$ it is sufficient to prove only the inclusion $D \subset D(<)$. Let $(x, y, t) \in D$. If m coincides with some of the elements x, y or z of M, because of i) and j) without loss of generality we can suppose, that $m = x$. Then $m \neq y$, hence $m < y$ according to the definition of the (m, D)-order in M. From the same definition follows $y < z$. Therefore (1) holds and consequently $(x, y, z) \in D(<)$. Let now m be different from all three of x, y and z. As the direction D is a full one, one of the relations

(6) $$(m, x, y) \in D$$

or

(7) $$(m, y, x) \in D$$

and one of the relations

(8) $$(m, y, z) \in D$$

or

(9) $$(m, z, y) \in D$$

will hold.

In case of (6) and (8) from the definition of the (m, D)-order in M follows (1), hence

(10) $$(x, y, z) \in D(<).$$

In case of (7) from i) follows $(y, x, m) \in D$. On the other hand, again from i) and $(x, y, z) \in D$ follows $(y, z, x) \in D$. Now ii) and jj) give $(y, z, m) \in D$ and $(z, x, m) \in D$, i. e. $(m, y, z) \in D$ and $(m, z, x) \in D$ according to j). But then (2) holds, i. e. again (10) is true. It is proved similarly that in case of (9) (10) holds again. Q. E. D.

Definition 4. A nonempty direction D in a bijective system B is called a natural direction in B, when from $(x, y, z) \in D$ follows $(x', y, z) \in D$ for $(x'; y, z) \in \tilde{B}$ and $('x, y, z) \in D$ for $('x, y, z) \in \tilde{B}$.

Obviously if there exists a natural direction in B, B is at least a three-element set. If D is a natural direction in B, the opposite direction $\bar{D}$ of D is also a natural direction in B. The notion natural direction is autosymmetric.

The next theorem guarantees the existence and uniqueness of the natural directions in the indecomposable bijective systems.

Theorem 1. **In every indecomposable bijective system S with three elements at least there exist just two natural directions. They coincide with the equivalence classes, generated by the relation $\uparrow$, are mutually opposite and each of them is full.**

Proof. Let D be a natural direction in S. We shall prove first, that from $\lambda \in D$ and $\lambda \uparrow \mu$ follows $\mu \in D$. Indeed, let E be the equivalence relation in $\tilde{S}$, whose equivalence classes are D and $\tilde{S} \setminus D$. From definition 4 follows, that the relation Δ in $\tilde{S}$ is a subset of the relation E. Hence $\uparrow \subset E$ and the statement is proved.

From this follows, that D is an union of equivalence classes in $\tilde{S}$ generated by the relation $\uparrow$. As by definition $D \neq \emptyset$, the direction D contains at least one class. If we suppose, that D contains both equivalence classes in $\tilde{S}$, generated by the relation $\uparrow$ (theorem 1, § 3), we get a contradiction with proposition 3. Hence D coincides with one of them. As the opposite direction $\bar{D}$ is also a natural one, it coincides with the other equivalence class, generated by $\uparrow$. Therefore the theorem will be proved, if we establish, that each of these equivalence classes is a natural direction in S.

So let D be one of the equivalence classes, in which the relation $\uparrow$ decomposes the set $\tilde{S}$. From definitions 1 and 2 of § 3 follows, that D possesses the property i).

In order to prove, that D possesses the property ii) also, let

(11) $$(x, y, z) \in D$$

and

(12) $$(x, z, t) \in D.$$

In order to prove $(x, y, t) \in \tilde{S}$ we must establish, that $t \neq y$. Otherwise we would have (11) and

(13) $$(x, z, y) \in D.$$

Let for example D be the equivalence class, generated by the relation $\uparrow$, which contains all the triples (15), § 3. As $\xi \in R['\xi, \xi']$, from (11) and proposition 2, § 3 follows, that at least one of the relations

(14) $$y \in R[x, z],\ x \in R[z, y],\ z \in R[y, x]$$

holds. Analogically, from (13) follows at least one of the relations

(15) $$z \in R[x, y],\ x \in R[y, z],\ y \in R[z, x].$$

But there is no relation (14), consistent with any relation (15). In order to prove that it is sufficient, with a view to the symmetry, to show that the first of the relations (14) is not consistent with the relations (15). Indeed, from $y \in R[x, z]$ and $z \in R[x, y]$ follows $y = z$ according to proposition 5, § 2, which is in contradiction with $(x, y, z) \in \tilde{S}$. Analogically from $y \in R[x, z]$ and $x \in R[y, z]$ follows $x = y$ according to proposition 6, § 2, which leads to the same contradiction. Finally, from $y \in R[x, z]$ and $y \in R[z, x]$ follows $y = x$ or $y = z$ according to proposition 3, § 2, which again is a contradiction. Hence $(x, y, t) \in \tilde{S}$.

Now we shall prove that from (11) and (12) follows

(16) $$(x, y, t) \in D.$$

To this end let x, y and t be arbitrary elements of S with $(x, y, t) \in \tilde{S}$ and Z be the set, composed of x, y, t and of those elements z of S, which are different from x, y, t and for which from (11) and (12) follows (16).

We shall prove first the relations

(17) $$x' \in Z,\ 'x \in Z,\ y' \in Z,\ 'y \in Z,\ t' \in Z,\ 't \in Z.$$

For $z=x'$ the relation (11) and (12) take the form $(x, y, x') \in D$ and $(x, x', t) \in D$ respectively and cannot be simultaneously fulfilled, because the first is equivalent with $(x', x, y) \in D$, which together with the second gives $(x', x, y) \uparrow (x, x', t)$ in contradiction to proposition 4, § 3 and corollary 1, § 3. Hence $x' \in Z$. It is proved similarly that $'x \in Z$.

For $z=y'$ the relations (11) and (12) take the form $(x, y, y') \in D$ and $(x, y', t) \in D$ respectively. From proposition 1, § 3 follows $(x, y', t) \uparrow (x, y, t)$, hence the second of these relations implies (16). Hence $y' \in Z$. It is proved similarly that $'y \in Z$.

For $z=t'$ the relations (11) and (12) take the form $(x, y, t') \in D$ and $(x, t', t) \in D$ respectively. From proposition 1, § 3 follows $(x, y, t') \uparrow (x, y, t)$, hence the first of these relations implies (16). Hence $t' \in Z$. It is proved similarly that $t' \in Z$.

Therefore the relations (17) are proved.

Let $z \in Z$, $z \neq x$, $z \neq y$, $z \neq t$ and

(18) $$(x, y, z') \in D$$
and
(19) $$(x, z', t) \in D.$$

From (18), (19) and proposition 1, § 3 follow (11) and (12). As $z \in Z$, this implies (16). Hence $z' \in Z$. It is proved analogically, that also $'z \in Z$. Therefore $Z=S$ and ii) is proved.

Therefore it is proved, that D is a direction in S. From proposition 1, § 3 immediately follows, that D is a natural direction. Q. E. D.

Consequently in every indecomposable bijective system S there exist just two natural directions. Every two elements of each of them are equidirected. According to theorem 1 and theorem 1, § 3 one of these directions consists of all those elements of $\tilde{S}$, which are equidirected with any triple of the type $('x, x, x')$ and the other — of all the elements of $\tilde{S}$, equidirected with any triple of the type $(x', x, 'x)$ $(x \in S)$.

Definition 5. The natural direction in S, that contains all triples $('x, x, x')$ $(x \in S)$, is called positive direction in S and the natural direction in S, that contains all triples $(x', x, 'x)$ $(x \in S)$, is called negative direction in S.

From proposition 4, § 3 follows, that the positive direction in S contains also all elements of $\tilde{S}$ of the type $('x, x, z)$ and the negative direction in S contains all elements of $\tilde{S}$ of the type (x', x, z). The notions positive and negative direction are obviously symmetric. It is clear, that positive directions are transformed in positive directions and negative direction in negative directions by isomorphisms between indecomposable bijective systems.

For the elements of the positive direction in S is said, that they are positive directed, and for the elements of the negative direction in S — that are negative directed. When two elements of $\tilde{S}$ belong to the same natural direction in S it is said, that they are equally directed, and otherwise — that are opposite directed.

The next proposition sets up a connection between the arithmetical operations addition and subtraction and the natural directions in S.

Proposition 6 For every element (x, y, z) of $\tilde{S}$ and for every element a of S the triples (x, y, z) and $(x+a, y+a, z+a)$ are equally directed, and the triples (x, y, z) and $(a-x, a-y, a-z)$ are opposite directed.

Proof. From (37), § 1 follows, that the mapping $s \rightarrow s+a$ $(s \in S)$ is a bijective homomorphism and the first part of the proposition follows from proposition 3, § 3.

In order to prove the second part, let us consider together with S also its symmetric bijective system S^{-1}. The mapping $\sigma: S \rightarrow S^{-1}$, defined by $\sigma(s)=a-s$ $(s \in S)$, is a bijective isomorphism according to proposition 5, § 1. Therefore it transforms for example the positive direction in S in the positive direction in S^{-1}. However, the last coincides with the negative direction in S. Hence σ changes the directions of the triples in $\tilde{S}$. Q. E. D.

§ 5. CYCLIC AND INTEGRAL SYSTEMS

Definition 1. A bijective system

$$\beta: C \rightarrow C \tag{1}$$

is called cyclic, when for every nonempty subset X of C from

$$\beta(X) \subset X \tag{2}$$

follows

$$X=C. \tag{3}$$

In what follows C denotes a cyclic system.

Obviously every cyclic system is an indecomposable bijective system.

The definition property of the cyclic systems is, so to say, an axiom for a right induction. The next proposition shows, that in cyclic systems left induction can be performed also, i. e. that the notion cyclic system is autosymmetric.

Proposition 1. A necessary and sufficient condition for the bijective system (1) to be cyclic is for every nonempty subset Y of C from

$$\beta^{-1}(Y) \subset Y \tag{4}$$

to follow

$$Y=C. \tag{5}$$

Proof. Necessity. Let, contrary to (5), $Y \neq C$ and let $X = C \setminus Y$. Then $\emptyset \neq X \subset C$ and (2) holds, whence (3) follows in contradiction with $Y \neq \emptyset$. The sufficiency is proved analogically. Q. E. D.

The next two propositions strengthen the axiom for the right induction and proposition 1.

Proposition 2. A necessary and sufficient condition for the bijective system (1) to be cyclic is such an element c of C to exist, that for every subset X of C from

$$c \in X \tag{6}$$

and (2) to follow (3).

Proof. Let T be the set of all those elements t of C, for which from

$$t \in X \subset C \tag{7}$$

and (2) to follow (3). Obviously

$$c \in T. \tag{8}$$

Let

$$t \in T, \tag{9}$$

$$t' \in X \subset C \tag{10}$$

and (2) holds. If $X_1 = X \cup \{t\}$, then $t \in X_1 \subset C$ and $\beta(X_1) \subset X_1$ because of (2) and (10). Now from (9) follows $X_1 = C$. Hence $'t \in X$ and consequently $t \in X$ according to (2). Then from (2) and (9) follows $X = Z$. Consequently $t' \in T$, which shows, that

$$\beta(T) \subset T. \tag{11}$$

From (8) and (11) follows

$$T = C. \tag{12}$$

according to the condition of the proposition. The equality (12) expresses, that for every nonempty subset X of C from (2) follows (3). Therefore C is a cyclic system. Q. E. D.

Proposition 3. A necessary and sufficient condition for the bijective system (1) to be cyclic is such an element c of C to exist, that for every subset Y of C from

$$c \in Y \tag{13}$$

and (4) to follow (5).

Proof. Analogical to that of proposition 2.

Definition 2. A bijective system

$$\beta : I \to I \tag{14}$$

is called integral, when it is not cyclic.

This means, that either there exists a subset X of I with

$$\emptyset \neq X \neq I \tag{15}$$

and (2) or there exists a subset Y of I with

(16) $$\emptyset \neq Y \neq I$$
and (4) (proposition 1).

In what follows I denotes an integral system.

The notion integral system is autosymmetric. No integral system is isomorphic with a cyclic system.

The next two propositions follow directly from the definition of an integral system and from propositions 2 and 3.

Proposition 4. A necessary and sufficient condition for the bijective system (14) to be integral is for every element i of I a subset X of I to exist with

(17) $$i \in X \neq I$$
and (2).

Proposition 5. A necessary and sufficient condition for the bijective system (14) to be integral is for every element i of I a subset Y of I to exist with

(18) $$i \in Y \neq I$$
and (4).

From the definition of a cyclic system, from proposition 1 and from proposition 2, § 2 follows, that for every element c of an arbitrary cyclic system C the equalities

(19) $$R[c, {}'c] = C$$
and

(20) $$L[c, c'] = C$$

hold, and from propositions 4 and 5 follows, that for every two elements x and y of an arbitrary integral system I the relations

(21) $$R[x, y] \neq I$$
and

(22) $$L[x, y] \neq I$$
hold.

Definition 3. An order $<$ in a bijective system B is called a **right natural order** in B, when from $b \in B$ follows $b < b'$. An order $<$ in a bije tive system B is called a **left natural order** in B, when from $b \in B$ follows $b < {}'b$.

The notions right and left natural order in a bijective system are symmetric. From definition 3 immediately follows, that for every right natural order $<$ in B from $b \in B$ follows $b' < b$ and for every left natural order $<$ in B from $b \in B$ follows $b' < b$. The opposite order of every right natural order in B is a left natural order in B and conversely.

For an arbitrary order $<$ in an arbitrary set M by definition

(23) $$[m) = \{x \in M : m \leqq x\}$$
and

(24) $$(m] = \{x \in M : x \leqq m\}$$
for every element m of M.

The next proposition sets up a connection between the natural orders in the indecomposable bijective systems and the operations R and L in them.

Proposition 6. For every right natural order in an indecomposable bijective system S and for every element s of S the equalities

$$[s) = R[s, 's] \tag{25}$$

and

$$(s] = L[s, s'] \tag{26}$$

hold and for every left natural order in S and for every element s of S the equalities

$$[s) = L[s, s'] \tag{27}$$

and

$$(s] = R[s, 's] \tag{28}$$

hold.

Proof. For symmetry reasons it is sufficient to prove only (25). According to the definition of a right natural order $\beta([s)) \subset [s)$ holds and as $s \in [s)$, then

$$R[s, 's] \subset [s). \tag{29}$$

Analogically it is proved, that

$$L[s, s'] \subset (s]. \tag{30}$$

We shall prove now, that also

$$[s) \subset R[s, 's] \tag{31}$$

holds. To this end let $x \in [s)$. If $x \in R[s, 's]$, from corollary 2, § 2 would follow $x \in L[s, s']$, which together with (30) would imply $x \in (s]$, wherefrom $x = s$, contrary to $x \in R[s, 's]$. Therewith it is proved (31). Now (25) follows from (29) and (31). Q. E. D.

Corollary 1. If C is a cyclic system, there does not exist a natural order in C.

Proof. Let us suppose the contrary and let $c \in C$. If for example $<$ is a right natural order in C, then $c < c'$. On the other hand, from (19) and (25) follows $c' < c$, which is a contradiction. Q. E. D.

Corollary 2. If I is an integral system, there does not exist in I more than one right and more than one left natural order.

Proof. Equalities (25) and (27).

Corollary 3. If S_1 and S_2 are indecomposable bijective systems with right (resp. left) natural orders, for every bijective isomorphism $h : S_1 \to S_2$ from

$$x < y \qquad (x, y \in S_1) \tag{32}$$

follows

$$h(x) < h(y) \tag{33}$$

and for every bijective antiisomorphism $n : S_1 \to S_2$ from (32) follows

$$n(y) < n(x). \tag{34}$$

Proof. The statement follows from

$$h(R[s,\ 's]) = R[h(s),\ '(h(s))] \text{ and } n(R[s,\ 's]) = L[n(s),\ (n(s))']$$

(resp. from $h(L[s,\ s']) = R[h(s),\ (h(s))']$ and $n\,(L[s,\ s'] = R[n(s),\ '(n(s))]$ and from proposition 6. Q. E. D.

The following two lemmas are preparatory for the proof of the existence of natural orders in the integral systems.

L e m m a 1. If x and y are arbitrary elements of I, the conditions

$$x = y, \tag{35}$$

$$R[x,\ 'x] = R[y,\ 'y] \tag{36}$$

and

$$L[x,\ x'] = L[y,\ y'] \tag{37}$$

are equivalent.

Proof. It is sufficient to prove, that from (36) follows (35). To this end it is sufficient to establish, that x is the only element of $R[x,\ 'x]$, which is not a successor of an element of $R[x,\ 'x]$. From (21) and corollary 3, § 2 follows $'x \mathrel{\overline{\in}} R[x,\ 'x]$, i. e. x indeed is not a successor of an element of $R[x,\ 'x]$. From corollary 1, § 2 follows, that every other element of $R[x,\ 'x]$ is a successor of an element of $R[x,\ 'x]$. Q. E. D.

L e m m a 2. If x and y are arbitrary elements of I, from

$$x \in R[y,\ 'y] \tag{38}$$

follows

$$y \in R[x,\ 'x] \tag{39}$$

and from

$$x \in L[y,\ y'] \tag{40}$$

follows

$$y \in L[x,\ x']. \tag{41}$$

Proof. It is sufficient to prove, that from (38) follows (39), since the second part of the statement follows from the principle of symmetry. Let us suppose, that along with (38) also

$$y \in R[x,\ 'x] \tag{42}$$

holds. From (38), (42) and corollary 2, § 2 follows

$$x \in L(y,\ y'] \tag{43}$$

and

$$y \in L[x,\ x']. \tag{44}$$

From (43), (44) and from the definition of L follows $L\,[x,\ x'] \subseteq L[y,\ y']$ and $L[y,\ y'] \subset L[x,\ x']$, i. e. $L[x,\ x'] = L[y,\ y']$, which together with lemma 1 shows, that $x = y$, contrary to (38). Q. E. D.

The following theorem guarantees the existence of natural orders in the integral systems.

Theorem 1. In every integral system there exists just one right and just one left natural order. They are mutually opposite and each of them is a full one.

Proof. The uniqueness is proved in corollary 2.

The existence will be proved, if we construct for example a right natural order in I. Let by definition $x<y$ just when $x\neq y$ and $y\in R[x, 'x]$.

From this definition follows immediately, that there does not exist an element x of I with $x<x$.

Let now $x<y$ and $y<z$. Then $x\neq y$, $y\in R[x, 'x]$ and $z\in R[y, 'y]$. From $y\in R[x, 'x]$ follows the inclusion $R[y, 'y]\subset R[x, 'x]$. Hence $z\in R[x, 'x]$. If $x=z$, we would have $R[x, 'x]=R[z, 'z]\subset R[y, 'y]\subset R[x, 'x]$, wherefrom $R[x, 'x] =R[y, 'y]$. Now from lemma 1 follows $x=y$, contrary to the condition.

From (21) follows, that I has at least two elements. From proposition 1, § 1 follows $x\neq x'$ and $x\neq 'x$ for every x of I. From $x\neq 'x$ follows $x'\in R[x, 'x]$, which together with $x\neq x'$ gives $x<x'$. Hence $<$ is a right natural order in I.

Now we shall prove, that for every two elements x and y of I at least one of the following three possibilities

$$x=y, \quad x<y, \quad y<x \tag{45}$$

really takes place. If the first two relations (45) do not hold, then $x\neq y$ and $y\in R[x, 'x]$. From the last relation and from lemma 2 follows $x\in R[y, 'y]$, which together with $x\neq y$ gives the third relation (45).

The opposite order of the so defined right natural order in I is obvious a left natural order in I. Therefore the right and the left natural orders in I are mutually opposite. Q. E. D.

Corollary 4. An indecomposable bijective system S is integral iff there exists a natural order in it.

Proof. Corollary 1 and theorem 1.

The next proposition shows a characteristic property of the natural orders in the integral systems.

Proposition 7. If $<$ is the right natural order in I, from

$$x<y \tag{46}$$

follows

$$x'\leqq y \tag{47}$$

and

$$x\leqq 'y \tag{48}$$

and if $<$ is the left natural order in I, from (46) follows

$$'x\leqq y \tag{49}$$

and

$$x\leqq y'. \tag{50}$$

Proof. It is sufficient to prove the statement for the right natural order $<$ in I. From (46) and proposition 6 follows $x\neq y$ and $x\in L[y, y']$, which together with corollary 1, § 2 gives $x'\in L[y, y']$ Now (47) follows from proposition 6. It is proved analogically, that from (46) follows (48). Q. E. D.

Corollary 5. If $<$ is the right natural order in I, for every element i of I the equalities

$$[i') = \{x \in I : i < x\} \tag{51}$$

and

$$('i] = \{x \in I : x < i\} \tag{52}$$

hold and if $<$ is the left natural order in I, for every element i of I the equalities

$$['i) = \{x \in I : i < x\} \tag{53}$$

and

$$(i'] = \{x \in I : x < i\} \tag{54}$$

hold.

Proof. Clearly.

Corollary 6. If $<$ is a natural order in I, for every two elements x and y of I with $x \leqq y$ the interval $[x, y]$ is a finite set.

Proof. Let $<$ be for example the right natural order in I and x be an arbitrary fixed element of I. Let Y be the set of those elements y of I with $x \leqq y$, for which the statement is true. Obviously $x \in Y$. Let $y \in Y$. From proposition 7 follows $[x, y'] = [x, y] \cup \{y'\}$, hence $y' \in I$. Therefore $R[x, 'x] \subset Y$ and from proposition 6 follows $[x) \subset Y$. Q. E. D.

The following theorem shows, that the natural orders in I are in a certain sense good.

Theorem 2. If $<$ is a natural order in I, every nonempty minorized subset of I has a smallest element.

Proof. Let $<$ be for example the right natural order in I, M be a nonempty minorized subset of I and m be a minorant of M. From proposition 6 follows $M \subset R\,[m, 'm]$. Let X be the set of all elements x of $R\,[m, 'm]$, which are minorants of M. Obviously $m \in X$ and $X \neq R[m, 'm]$. Therefore from the definition of $R[m, 'm]$ follows, that there exists an element ξ of X with $\xi' \bar{\in} X$. Since ξ is a minorant of M, the statement will be proved, if we establish, that $\xi \in M$. Otherwise for every y of M the inequality $\xi < y$ would hold, wherefrom would follow $\xi' \leqq y$ according to proposition 7. But this would mean, that ξ' is a minorant of M and consequently $\xi' \in X$, which is a contradiction. Therefore M has a smallest element. Q. E. D.

The next theorem guarantees the existence of bijective homomorphisms in certain cases.

Theorem 3. Let i and b be arbitrary elements of the integral system I and of the bijective system B respectively. Then there exists an unique bijective homomorphism $h : I \to B$ with

$$h(i) = b. \tag{55}$$

Proof. The uniqueness follows from proposition 3, § 1.

In order to prove the existence let $<$ be for example the right natural order in I. From theorem 2 and from the theorem for definition by induction in the well ordered sets follows, that there exists a mapping $h_1 : [i) \to B$ with $h_1(i) = b$ and

$$h_1(x') = (h_1(x))' \qquad (x \in [i)) \tag{56}$$

and a mapping $h_2 : (i] \to B$ with $h_2(i) = b$ and

(57) $$h_2('x) = '(h_2(x)) \qquad (x \in (i]).$$

Since the right natural order in I is a full one, then $(i] \cup [i) = I$. Therefore the equalities

(58) $$h(x) = \begin{cases} h_1(x) & \text{for } x \in [i), \\ h_2(x) & \text{for } x \in (i] \end{cases}$$

define a mapping $h: I \to B$ with (55). From (56) and (57) follows, that the so defined h is a bijective homomorphism Q. E. D.

Corollary 7. Every two integral systems are isomorphic.

Proof. Let I_1 and I_2 be integral systems and $i_1 \in I_1$, $i_2 \in I_2$. From theorem 3 follows, that there exist homomorphisms $h_1: I_1 \to I_2$ and $h_2: I_2 \to I_1$ with $h_1(i_1) = i_2$ and $h_2(i_2) = i_1$. From proposition 3, § 1 follows $h_2 \circ h_1 = j_2$, where j_2 is the identity of I_2. Q. E. D.

Corollary 8. If I is an integral system and S is an indecomposable bijective system, there exists a ring epimorphism $h: I \to S$.

Proof. According to theorem 3 there exists a bijective homomorphism $h: I \to S$. From proposition 7, § 1 follows, that h is a ring homomorphism. From proposition 4, § 1 follows, that h is an epimorphism. Q. E. D.

Definition 4. If $<$ is the right natural order in I, the element x of I is called *positive* when $0 < x$ and *negative* when $x < 0$.

The notions positive and negative element are symmetric. From corollary 5 and proposion 6 and from the definition inequalities follows, that the set of the positive elements of I coincides with $R[1,0]$ and the set of the negative elements of I coincides with $L[-1, 0]$.

The following theorem shows, that the right natural order $<$ in I is coordinated with the ring operations in I.

Theorem 4. The right natural order $<$ in I possesses the following properties:

i) 0 is not a positive element of I.

ii) For every element $x \neq 0$ of I at least one of the elements x and $-x$ is positive.

iii) The sum of two positive elements of I is a positive element of I.

iv) The product of two positive elements of I is a positive element of I.

Proof. i) Clearly.

ii) From proposition 5, § 1 follows, that the mapping $\nu: I \to I$, defined by $\nu(x) = -x (x \in I)$, is an antiisomorphism. Hence from corollary 3 follows, that for every negative element x of I the element $-x$ is positive. Now ii) follows from the completeness of the natural orders in I.

iii) Let x and y be positive element of I. The mapping $h: I \to I$, defined by $h(z) = x + z$ $(z \in I)$, is a bijective isomorphism. Since $0 < y$, from corollary 3 follows $h(0) < h(y)$, i. e. $x < x + y$. Now iii) follows from the inequality $0 < x$.

iv) For an arbitrary fixed positive element x of I let Y be the set of all those elements y of $R[1,0]$, for which xy belongs to $R[1,0]$. Obviously $1 \in Y$. Let $y \in Y$. Then $xy' = x(y+1) = xy + x \in R[1,0]$ according to iii). From he definition of $R[1,0]$ now follows $Y = R[1,0]$. Q. E. D.

The next theorem shows a connection between the natural orders and the natural directions in I.

Theorem 5. The right natural order in I is the only order in I, which does not possess a smallest element and generates the positive direction in I and the left natural order in I is the only order in I, which does not possess a largest element and generates the negative order in I.

Proof. It is sufficient to prove the statement for the right natural order $<$ in I. We have to prove, that the direction $D(<)$ (proposition 4, § 4) is the positive direction in I. We shall establish first, that it is a natural direction in I.

Let $(x, y, z) \in D(<)$. Then at least one of the relations

$$x<y<z, \quad y<z<x, \quad z<x<y \tag{59}$$

hold. Let $y \neq x' \neq z$. Then from proposition 7 it follows, that every of the relations (59) implies the respective relation

$$x'<y<z, \quad y<z<x', \quad z<x'<y. \tag{60}$$

Therefore $(x', y, z) \in D(<)$. It is proved analogically, that from $y \neq 'x \neq z$ follows $('x, y, z) \in D(<)$. Consequently $D(<)$ is a natural direction in I.

Since for every x of I the inequalities $'x<x<x'$ hold, then $('x, x, x') \in D(<)$. Therefore $D(<)$ is the positive direction in I.

Let now $\{$ be a direction in I without a smallest element, which generates the positive direction in I. Since the last one is full, the order $\{$ is also full. Therefore if we suppose, that it is not a right natural order in I, an element x of I would exist with $x' \{ x$. Since the order $\{$ does not have a smallest element, an element y of I would exist with $y<x'$. Hence the triple (y, x', x) would be positive directed in contradiction with the properties of the positive direction in I. Consequently $\{$ is the right natural order in I. Q. E. D.

The next theorem gives a necessary and sufficient condition for an indecomposable bijective system to be cyclic.

Theorem 6. An indecomposable bijective system is cyclic iff it is a finite set.

Proof. Les S be a finite indecomposable bijective system. We suppose that S is an integral system and denote by $<$ the right natural order in S. For an arbitrary element s of S let us consider the mapping $\sigma: S \to S$, defined by

$$\sigma(x)=\begin{cases} x & \text{for } x \leqq s, \\ x' & \text{for } s<x \end{cases}. \tag{61}$$

From corollary 5, proposition 6, corrolary 3, § 2 and (21) follows, that σ is an injection and that $\sigma(S) \neq S$, which contradicts to the finiteness of S.

Let inversely S be a cyclic system. According to corollary 8 there exists a ring epimorphism $h: I \to S$. It is not a monomorphism, since in such a case I and S would be isomorphic. Therefore there exist different from 0 elements x of I with

$$h(x)=0. \tag{62}$$

From theorem 4 follows that without loss of generality we can suppose $0<x$. We shall prove that

$$S=h([0, x]). \tag{63}$$

Indeed, obviously $0 \in h([0, x])$. Let

$$s \in h([0, x]). \tag{64}$$

Then $s = h(i)$, where $i \in [0, x]$. If $i \neq x$, then $s' = h(i') \in h([0, x])$ according to proposition 7. If $i = x$, then $h(i) = h(x) = 0 = h(0)$ by (62). Therefore $s' = h(i') = h(0') \in h([0, x])$, according to proposition 7. Consequently from (64) follows $s' \in h([0, x])$. It is proved analogically, that from (64) follows also $'s \in h([0, x])$. This proves (63). Now corollary 6 shows, that S is a finite set. Q. E. D.

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Постъпила на 17. XII. 1974 г.

АКСИОМИ НА ПЕАНО ЗА МИНИМАЛНИТЕ ПРЪСТЕНИ

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(РЕЗЮМЕ)

Един пръстен се нарича минимален, когато не притежава собствени подпръстени. Минимален пръстен е например сечението на всичките подпръстени на даден пръстен. Известно е как изглеждат минималните пръстени: с точност до изоморфизъм те съвпадат с пръстена на целите числа или с негови факторпръстени.

На понятието минимален пръстен обаче може да се даде и значително по-пестелива дефиниция, като се следва идейно пътят на Грасман — Дедекинд — Пеано — Ландау [1] — [4] за аксиоматическо описание на естествените числа, при който наредбата, събирането, умножението и пр. се дефинират, като се изхожда от проста унарна операция. В настоящата работа е показано как може да стане това.

Първите четири аксиоми на Пеано за аритметиката на естествените

числа изразяват в същност, че прибавянето на единицата поражда инекция в множеството на естествените числа. Ето защо у Пеано по същество се касае за две аксиоми: аксиома за инекция и аксиома за индукция. И в предлаганото тук аксиоматично описание на минималните пръстени също се касае за две аксиоми: аксиома за биекция и аксиома за двустранна индукция. Първата описва прибавянето и изваждането на единицата, а втората е естествено изискване за минималност, характерна за минималните пръстени. Така се стига до понятието неразложима биективна система.

В сравнение с аксиоматическото построяване на естествените числа тук са налице някои опростявания. Така например, след като в една неразложима биективна система се избере по произволен начин един нулев елемент, съвсем естествено възниква биекция между нея и групата на изоморфизмите ѝ. Това позволява почти автоматичното въвеждане на събирането. Пръстенът на хомоморфизмите на така получената адитивна група отново се намира в еднозначно обратимо съответствие с неразложимата биективна система, което позволява тя да се превърне със същата леснина в минимален пръстен. На връзката между минималните пръстени и неразложимите биективни системи е посветен § 1.

В техническия § 2 се изучават някои специални подмножества на една неразложима биективна система. Значението на рекурентните множества при въпроси от подобен род бе изтъкнато за пръв път от Тагамлицки [5]. Естеството на нещата налага тук да се използува една модификация на понятието рекурентно множество. Интересно е да се отбележи, че с помощта на рекурентни множества при традиционното аксиоматическо изграждане на аритметиката на естествените числа по Пеано наредбата може [6] да се въведе независимо от аритметичните операции за разлика от избрания в [4] поход. Излишно е да се изтъкват гносеологическите и историческите причини, поради които този начин е за предпочитане.

Въпреки че в работи с характера на настоящата не би трябвало да се очаква установяването на нови факти, отсъствието на естествена наредба в произволна неразложима биективна система наложи въвеждането и изучаването на един сурогат — понятието естествена посока, на което са посветени § 3 и 4. Във всяка неразложима биективна система има точно две естествени посоки, които са тясно свързани със събирането. Макар и тук да не се изучава връзката на естествените посоки с умножението, възможностите на тази концепция са, изглежда, по-големи и е желателно по-нататъшното ѝ използуване чрез абстрактно изучаване на насочените пръстени по аналогия с наредените. Разглежданията от § 3 и 4 биха могли да служат за модел в тази насока.

Последният § 5 съдържа три характеристики на целите числа в класа на неразложимите биективни системи. Преди всичко това са неразложими биективни системи, в които не е възможна едностранна индукция. От друга страна, това са неразложими биективни системи, в които съществуват естествени наредби. Най-после това са безкрайните неразложими биективни системи. Неразложимите биективни системи, които притежават тези еквивалентни помежду си свойства, са наречени интегрални, а останалите — циклични.

В § 1 — 4 се изучават онези свойства на неразложимите биективни системи, които са присъщи както на интегралните, така и на цикличните системи. В този смисъл по аналогия с геометрията би могло да се говори за абсолютна аритметика.