TOTALLY MINIMAL TOPOLOGICAL GROUPS

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A Hausdorff topological group G is said to be minimal (and the topology of G is called minimal group topology) if every continuous group isomorphism $h: G \rightarrow G_1$, where G_1 is a Hausdorff topological group, is a homeomorphism. If a topological group is compact, it is obviously minimal. Stephenson [8] and Doïtchinov [3] give examples of non-compact minimal topological groups. In [6] and [7] all minimal group topologies on a finitely generated free Abelian group are described. In this paper totally minimal topological groups are considered. They are nearer to the compact ones than the minimal topological groups. All the groups considered here are Hausdorff.

A Hausdorff topological group G is said to be totally minimal (and the topology of G is called totally minimal group topology) if every continuous group epimorphism $h: G \rightarrow G_1$, where G_1 is a Hausdorff topological group, is open. This means that for each closed normal subgroup N of G the quotient group G/N is minimal. Clearly every compact group is totally minimal, and every totally minimal group is minimal. The example 4 given below shows that these three classes are different.

A subgroup H of a topological group G is called totally dense if for every closed normal subgroup N of G the group $N \cap H$ is dense in N. For $N=G$ it is clear that every totally dense subgroup of G is dense in G. If the group G is simple the converse is also true. Other examples of totally dense subgroups are given below.

Theorem 1. Let G be a Hausdorff topological group, and H be a dense subgroup of G. Then H is totally minimal if and only if G is totally minimal and H is totally dense in G.

The proof of Theorem 1 is based on two lemmas. The second of them is formulated in [9] and [4], and [9] contains a proof for the Abelian case.

Lemma 1. Let G and G_1 be Hausdorff topological groups, H be a dense subroup of G, and $f: G \rightarrow G_1$ be a continuous group epimorphism. If the restriction $f|_H: H \rightarrow f(H)$ is open, then the epimorphism f is also open.

Proof. Let U be a neighbourhood of the neutral element e of G. Denote by V a symmetric neighbourhood of e with $V.V \subset U$. Then $V \cap H$ is a neighbourhood of e in H. Hence, $f(V \cap H)$ is a neighbourhood of the neutral element e_1 of $f(H)$. Therefore, there is a neighbourhood W of e_1 in G_1 such that $W \cap f(H) \subset f(V \cap H)$. Now it follows that

$$f^{-1}(W) \cap H \subset V . \ker f. \tag{1}$$

The statement will be proved if we show that $W \subset f(U)$. Let $y \in W$ and x be an element of G with $y=f(x)$. Then $Vx \cap f^{-1}(W)$ is a neighbourhood of x in G. Since H is dense in G it follows that $Vx \cap f^{-1}(W) \cap H \neq \emptyset$. Now (1) implies $Vx \cap V \ker f \neq \emptyset$. Therefore, $x \in V^2 \ker f$ and hence $y=f(x) \in f(V^2) \subset f(U)$ Q. E. D.

Lemma 2. Let G be a Hausdorff topological group, N be a closed normal subgroup of G, H be a dense subgroup of G, and $p: G \to G/N$ be the corresponding canonical homomorphism. Then the restriction $p_H: H \to p(H)$ is open if and only if $N=[N \cap H]$.*

Proof. Suppose the restriction $p_H: H \to p(H)$ is open, and U is an arbitrary neighbourhood of the neutral element e of G. We prove that

(2) $$N \subset (N \cap H)U.$$

Let V be a symmetric neighbourhood of e in G such that $V.V \subset U$. Then $p(H \cap V)$ is a neighbourhood of the neutral element e_1 of G/N in $p(H)$. Hence, there is an open neighbourhood W of e_1 in G/N such that

(3) $$p(H) \cap W \subset p(H \cap V).$$

First of all we prove that

(4) $$(p^{-1}(W)N) \cap H \subset (N \cap H)V.$$

Let for this purpose $h \in (p^{-1}(W)N) \cap H$. Then there are elements v and n such that $h=vn$, $n \in N$, $v \in G$, $p(v) \in W$. Therefore,

$$p(h) \in p(H) \cap W$$

and (3) implies $p(h) \in p(H \cap V)$. That is, there are $h_1 \in H \cap V$ and $n_1 \in N$ such that $h=n_1h_1$. Clearly $n_1 \in N \cap H$. Hence,

$$h \in (N \cap H)(H \cap V) \subset (N \cap H)V$$

and (4) is proved. Since $p^{-1}(W)N$ is an open set in G, the density of H implies $p^{-1}(W)N \subset [(p^{-1}(W)N) \cap H]$. Applying (4) we obtain

$$N \subset p^{-1}(W)N \subset [(N \cap H)V] \subset (N \cap H)V^2 \subset (N \cap H)U$$

which proves (2). Therefore, $N=[N \cap H]$ and the necessity is proved.

Suppose now $N=[N \cap H]$, and prove that the restriction $p_H: H \to p(H)$ is open. Let U be an arbitrary neighbourhood of e in H, and V be a symmetric neighbourhood of e in H with $V^2 \subset U$. Then $[V]$ is a neighbourhood of e in G, and therefore, $p([V])$ is a neighbourhood of e_1 in G/N. First of all we prove that

(5) $$p([V]) \cap p(h) \subset p(U).$$

Let for this purpose $p(h) \in p([V])$ $(h \in H)$. Clearly $h=\bar{v}n$ where $v \in [V]$ and $n \in N$. But $N=[N \cap H]$ and there is a net $\{n_\alpha\} \subset N \cap H$ with $\lim n_\alpha = n$. At the same time there is a net $\{v_\alpha\} \subset V$ with $\lim v_\alpha = v$. Therefore, $\lim hn_\alpha^{-1}v_\alpha^{-1} = e$,

* $[X]$ denotes the closure of a subset X of G in G.

and hence for each sufficiently large α we have $hn_\alpha^{-1}v_\alpha^{-1}\in V$ because $n_\alpha\in H$, $v_\alpha\in H$. From this it follows that $h\in Vv_\alpha n_\alpha\subset V^2N$, and $p(h)\in p(V^2)\subset p(U)$ which proves (5). Since $p([V])$ is a neighbourhood of e_1 in G/N (5) implies that the restriction $p|_H$ is open. Q. E. D.

Proof of theorem 1. Let H be a totally minimal subgroup of G. First we prove that G is also totally minimal. Let for this purpose G_1 be a Hausdorff topological group and $f:G\to G_1$ be a continuous epimorphism. The restriction $f|_H:H\to f(H)$ is open by the definition of a totally minimal group. Now lemma 1 implies that the epimorphism f is open, and therefore, G is totally minimal. Now we prove that H is totally dense in G. Let N be a closed normal subgroup of G, and $f:G\to G/N$ be the corresponding canonical epimorphism. Since the group H is totally minimal, the restriction $f|_H$ is open. Applying Lemma 2, we find $N=[N\cap H]$ and the necessity is proved.

Let now G be totally minimal, and H be a totally dense subgroup of G. Consider an arbitrary continuous epimorphism $f:H\to H_1$, where H_1 is a Hausdorff topological group. Then the intersection $N=\bigcap[f^{-1}(W)]$ where W runs over all neighbourhoods of the neutral element of H_1 is a closed normal subgroup of G. Since the group H_1 is Hausdorff we have $N\cap H=\ker f$. Now H being totally dense implies $N=[\ker f]$. Denote by $p:G\to G/N$ the corresponding canonical epimorphism. Endow G/N with the topology τ having as a base of neighbourhoods of the neutral element the set $\{p([f^{-1}(W)])\}$ where W runs over all the neighbourhoods of the neutral element of H_1. Since $[f^{-1}(W)]$ are neighbourhoods of e in G the topology τ is weaker than the quotient topology N on G/N. On the other hand the definition of N implies that τ is Hausdorff and the minimality of T implies $T=\tau$. Consider the monomorphism $g:H_1\to G/N$ defined by $g(f(x))=p(x)$ for each $x\in H$. We find a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{i} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ H_1 & \xrightarrow{g} & G/N \end{array}$$

where i is the canonical embedding of H in G. It follows from Lemma 2 that the restriction $p/_H:H\to g(H_1)$ is open. On the other hand the definition of τ implies that $g:H_1\to g(H_1)$ is a homeomorphism, and since the same is true for $i:H\to H$, we find that the epimorphism f is open. Q. E. D.

The above argument gives more. Let Q be a class of continuous epimorphisms of Hausdorff topological groups, with the properties:

1. If in the diagram $F\xrightarrow{f}G\xrightarrow{g}H$ g is a topological group isomorphism and $f\in Q$ then $g\circ f\in Q$;

2. If in the diagram $F\xrightarrow{g}G\xrightarrow{f}H$ g is a topological group isomorphism and $f\in Q$ then $f\circ g\in Q$;

3. Every open group epimorphism belongs to Q;

4. If G and G_1 are Hausdorff topological groups, $f: G \to G_1$ is a continuous epimorphism, and H is a dense subgroup of G, the restriction $f|_H$: $H \to f(H)$ belongs to Q if and only if f belongs to Q.

Obviously the class of all continuous epimorphisms has the properties 1—4. An epimorphism $f: G \to G_1$ of Hausdorff topological groups G and G_1 is called almost-open if for each neighbourhood U of the neutral element of G the set $[f(U)]$ is a neighbourhood of the neutral element of G_1. It is clear that the class of all continuous almost-open epimorphisms also has the properties 1—4.

A Hausdorff topological group G is said to be Q-totally minimal if every epimorphism $f: G \to G_1$ with $f \in$ Q is open. If Q coincides with the class of all continuous epimorphisms this definition gives totally minimal topological groups. If Q is the class of all almost-open continuous epimorphisms we obtain B — complete groups, which are studied by various authors ([1], [4], [5], [9]). Each totally minimal topological group is obviously Q-totally minimal for each class Q with the properties 1—4.

Of course, the notion of Q-minimal topological group may be introduced. A Hausdorff topological group G is called Q-minimal if every continuous group isomorphism $h: G \to G_1$ with $h \in$ Q is a homeomorphism. If Q coincides with the class of all continuous epimorphisms this definition gives minimal topological groups. If Q is the class of all almost-open epimorphisms we obtain B_r-complete groups studied in [1], [4], [5], [9]. Obviously G is Q-totally minimal if and only if for each closed normal subgroup N of G the quotient group G/N is Q minimal.

Theorem 2. Let G be a Hausdorff topological group and H be a dense subgroup of G. Then:

i) H is Q-totally minimal if and only if G is Q totally minimal and H is totally dense in G;

ii) H is Q-minimal if and only if G is Q-minimal and for each closed normal subgroup N of G the group $N \cap H$ does not reduce to the neutral element of G.

The proof of i) is a Q-version of the proof of Theorem 1, and the proof of ii) is analogous; we will omit them. Let us note that Theorem 2 generalise also Theorem 1.4 of [4] (see also [8]).

Example 1. Let H be a dense subgroup of the group $SO(3)$. Then H is totally minimal. Indeed, the group $SO(3)$ is compact and so is totally minimal, and H is totally dense because $SO(3)$ is simple.

Example 2. Let G be a compact Lie group, and H be the subgroup of G generated by the all the periodic elements of G. Then H is totally minimal.

Indeed, let N be a closed subgroup of G. Then N is a compact Lie group and the set of all the periodic elements of G is dense in N. Therefore, $N = [N \cap H]$.

In particular, if $\mathbf{Q}$ is the additive group of rationals, and $\mathbf{Z}$ is the subgroup of integers, the group $(\mathbf{Q}/\mathbf{Z})^n$ is totally minimal for each natural number n.

Indeed, $(\mathbf{Q}/\mathbf{Z})^n$ is the subgroup of the periodic elements of the n-dimensional torus T^n.

Example 3. A group topology on $\mathbf{Z}^n$ is totally minimal if and only if it is induced by a dense embedding

$$\varphi: \mathbf{Z}^n \to \mathbf{Z}_{p_1} \times \mathbf{Z}_{p_2} \times \ldots \times \mathbf{Z}_{p_k}, \tag{6}$$

where $k \leqq n$, $p_1, p_2, \ldots, p_k$ are different prime numbers, and $\mathbf{Z}_{p_i}$ are the corresponding compact groups of p_i-adic integers, with

$$\varphi(\mathbf{Z}^n) \cap (0 \times \ldots \times 0 \times (\mathbf{Z}_{p_i} \setminus p_i \mathbf{Z}_{p_i}) \times 0 \times \ldots \times 0) \neq \emptyset \tag{7}$$

for each $i = 1, 2, \ldots, k$.

Indeed, [6] contains a description of all the precompact minimal topologies on $\mathbf{Z}^n$. From [7] we know that every totally minimal group topology on $\mathbf{Z}^n$ is precompact. From Theorem 7.3 of [6] it follows that the desired topologies are induced by dense embeddings (6) with

$$\varphi(\mathbf{Z}^n) \cap (0 \times \ldots \times 0 \times (\mathbf{Z}_{p_i} \setminus 0) \times 0 \times \ldots \times 0) \neq \emptyset \tag{8}$$

for each $i = 1, 2, \ldots, k$.

Now we prove the necessity. Let $\mathbf{Z}^n$ be endowed by a totally minimal group topology. Then there is a dense embedding (6) with (8). Let $N = 0 \times \ldots \times 0 \times \mathbf{Z}_{p_i} \times 0 \times \ldots \times 0$ $(i = 1, 2, \ldots, k)$. Clearly N is a closed subgroup of $\prod_{i=1}^{k} \mathbf{Z}_{p_i}$. Now Theorem 1 implies that

$$N = [N \cap \varphi(\mathbf{Z}^n)]. \tag{9}$$

Since $0 \times \ldots \times 0 \times (\mathbf{Z}_{p_i} / p_i \mathbf{Z}_{p_i}) \times 0 \times \ldots \times 0$ is a non-empty open subset of N, (9) implies (7), and the necessity is proved.

To prove the sufficiency let (6) be a dense embedding with (7). Let N be a non-trivial closed subgroup of $\prod_{i=1}^{k} \mathbf{Z}_{p_i}$. It is known that then there are closed subgroups A_i of $\mathbf{Z}_{p_i}$ $(i = 1, 2, \ldots, k)$, such that

$$N = A_1 \times A_2 \times \ldots \times A_k. \tag{10}$$

In the same way we find closed subgroups B_i of A_i $(i = 1, 2, \ldots, k)$ with

$$[N \cap \varphi(\mathbf{Z}^n)] = B_1 \times B_2 \times \ldots \times B_k. \tag{11}$$

Let for example $B_1 \neq A_1$. Then $A_1 \neq \{0\}$, $B_1 \subset p_1 A_1$, and (10), (11) imply

$$A_1 \times A_2 \times \ldots \times A_k \cap \varphi(\mathbf{Z}^n) \subset p_1 A_1 \times A_2 \times \ldots \times A_k. \tag{12}$$

But $A_1 = p_1^l \mathbf{Z}_{p_i}$ $(l = 0, 1, \ldots)$ and (12) contradicts (7) with $i = 1$. Hence $B_1 = A_1$. In the same way we prove $B_i = A_i$ $(i = 2, 3, \ldots, k)$. Now (10) and (11) imply $N = [N \cap \varphi(\mathbf{Z}^n)]$ and the sufficiency follows from Theorem 1.

Example 4. Let p and q be different prime numbers and $\varphi:\mathbf{Z}^2\to\mathbf{Z}_p\times\mathbf{Z}_q$ be the embedding defined by $\varphi(1,0)=(1,1)$, $\varphi(0,1)=(0,q)$. Since (8) takes place the topology on $\mathbf{Z}^2$ induced by φ is minimal. Let $A=\varphi(\mathbf{Z}^2)\cap(0\times\mathbf{Z}_q)$. Clearly A is a closed subgroup of $\varphi(\mathbf{Z}^2)$. But the topological group $\varphi(\mathbf{Z}^2)/A$ may be identified with the subgroup B of $\mathbf{Z}_p\times\mathbf{Z}_q$ generated by the element $(1,1)$, and the topology of B is not minimal.

Example 5. In [6] it is shown that $\mathbf{Z}_p$ induces a minimal topology on every subgroup $G\subset\mathbf{Z}_p$, and that this property characterises the groups $\mathbf{Z}_p$ in the class of compact Abelian groups. It is not difficult to see that $\mathbf{Z}_p$ induces a totally minimal topology on every subgroup $G\subset\mathbf{Z}_p$ and that this property characterizes the groups $\mathbf{Z}_p$ in the class of compact Abelian groups.

Example 6. Let G be a compact Abelian group and T be the subgroup of the periodic elements of G. Then every dense totally minimal subgroup H of G contains T.

Indeed, for each $x\in T$ the subgroup N generated by x is finite, and hence closed in G. Now Theorem 1 implies $N=[N\cap H]$. But $N\cap H$ is also finite and $[N\cap H]=N\cap H$. Therefore, $x\in H$.

Example 7. Let n be a natural number, $\{G_\alpha\}$ be a family of finite Abelian groups with $nG_\alpha=0$ for each α. Then every totally minimal subgroup of $\prod_\alpha G_\alpha$ is compact.

Indeed, let H be a totally minimal subgroup of $G=\prod_\alpha G_\alpha$. It follows from Example 6 that H is closed in G since $[H]$ is a compact periodic Abelian group.

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ТОТАЛНО МИНИМАЛНИ ТОПОЛОГИЧНИ ГРУПИ

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(РЕЗЮМЕ)

Една хаусдорфова топологична група G се нарича **минимална**, ако всеки непрекъснат групов изоморфизъм $f: G \rightarrow G_1$, където G_1 е хаусдорфова топологична група, е хомеоморфизъм. Очевидно всяка компактна топологична група е минимална. Първите примери за некомпактни минимални топологични групи са дадени в [3] и [8]. В тази работа се разглеждат тотално минимални топологични групи, които са по-близо до компактните, отколкото минималните.

Една хаусдорфова топологична група G се нарича **тотално минимална**, ако всеки непрекъснат епиморфизъм $f: G \rightarrow G_1$, където G_1 е хаусдорфова топологична група, е отворен. Очевидно всяка компактна група е тотално минимална. От разгледаните примери се вижда, че има тотално минимални групи, които не са компактни, а също и минимални групи, които не са тотално минимални. Доказана е следната теорема.

Теорема 1. Нека G е хаусдорфова топологична група и H е нейна навсякъде гъста подгрупа. Тогава H е тотално минимална тогава и само тогава, когато G е тотално минимална и за всяка затворена инвариантна подгрупа N на G подгрупата $N \cap H$ е навсякъде гъста в N.