AN EXAMPLE OF A COMPACT HAUSDORFF SPACE WHICH IS M-CONNECTED BUT IS NOT STRONGLY M-CONNECTED

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Let **M** be an arbitrary non-empty class of Hausdorff topological spaces which satisfies the following conditions:

1) If $X \in \mathbf{M}$ and Y is homeomorphic to X, then $Y \in \mathbf{M}$.

2) If $X \in \mathbf{M}$ and A is a F_σ-subset of X, then $A \in \mathbf{M}$.

We shall say that the topological space is **M**-connected if it cannot be decomposed into the sum of two closed proper* subsets whose intersection belongs to **M** [1].

We shall say that the topological space is strongly **M**-connected if it cannot be decomposed into the sum of countably many closed proper subsets whose pairwise interesections belong to **M** [1].

The **M**-connectedness and the strong **M**-connectedness are topologically invariants. Of course, from the strong **M**-connectedness follows **M**-connectedness. It is easy to see that if the Hausdorff space R is **M**-connected and contains at least two points then $R \in \mathbf{M}$.

Obviously, the space R is $\{\varnothing\}$-connected if and only if it is connected. Furthermore, if one space R is **M**-connected and $\mathbf{M}' \subset \mathbf{M}$ then R is $\mathbf{M}'$-connected. In particular every **M**-connected space is connected.

If $\mathbf{M}_n$ is the class of normal topological spaces whose dimension dim does not exceed $n-2$ $(n \geqq 1)$, then $\mathbf{M}_n$-satisfies the conditions 1) and 2) above. The topological space is $\mathbf{M}_n$-connected if and only if it is Cantor n-manifold [2]. The space is strongly $\mathbf{M}_n$-connected if and only if it is strongly Cantor n-manifold [3], [4] and [5]. It was proved in [3] that the n-dimension cube I^n is strongly $\mathbf{M}_n$-connected.

Let τ be an infinite cardinal number and $\mathbf{M}_\tau$ is the class of topological spaces which are τ-normal and its cardinal dimension k does not exceed τ [4]. By virtue of lemma 2 [4], p. 136 follows immediately that $\mathbf{M}_\tau$ satisfies the conditions 1) and 2). One space is $\mathbf{M}_\tau$-connected if and only if it is Cantor τ-manifold and is strongly $\mathbf{M}_\tau$-connected if and only if it is strongly Cantor τ-manifold [4], [5]. It was proved in [4] that the Tychonoff cube I^τ is strongly $\mathbf{M}_\tau$-connected.

The famous theorem of Sierpinsky [6], in the terminology which we have introduced, states as follows: every compact Hausdorff $\{\varnothing\}$-connected space is strongly $\{\varnothing\}$-connected.

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* The subset M of a topological space X is called proper if $M \neq X$.

The aim of this paper is to show that in rather arbitrary assumptions for $\mathbf{M}$ there are compact and $\mathbf{M}$-connected spaces which are not strongly $\mathbf{M}$-connected.

Theorem. If a non-trivial $\mathbf{M}$-connected compact Hausdorff space R exists and $\mathbf{M}\neq\{\varnothing\}$, then there exists a non-trivially $\mathbf{M}$-connected compact Hausdorff space K, which is not strongly $\mathbf{M}$-connected.

We use the following two lemmas to prove this theorem.

Lemma 1. There exists a locally compact connected subspace X of $\mathbf{R}^3$, which is not strongly connected, i. e. $X=\bigcup_{i=1}^{\infty} F_i$, where F_i are proper closed subsets of X $(i=1, 2,\ldots)$ and $F_i\cap F_j=\varnothing$ when $i\neq j$ [7], [8], p. 183.

Proof. Let X be the following subspace of $\mathbf{R}^3$ (see fig. 1). We consider on the Ox axis the open segments $a_n=\{x:n<x<n+1\}$, $n=0, 1,\ldots$. In the plane $\alpha_1=xOz$ we construct a right angle, one of which sides begins from the midpoint of the segment a_0 and has the length 1 and the other arm is parallel to Ox and is directed towards the plane yOz. We denote by F_1 the figure which consists of the segment a_0 and the sides of the right angle.

By induction we construct the plane α_{n+1} which is obtained from α_n by turning it around the Ox axis thorough an angle of $\pi/4.2^n$ in the direction S indicated in the figure. In the obtained plane α_{n+1} we construct a right angle whose one side begins from the midpoint of the segment a_n and has the length $1/2^n$. The other side is parallel to Ox and is directed towards

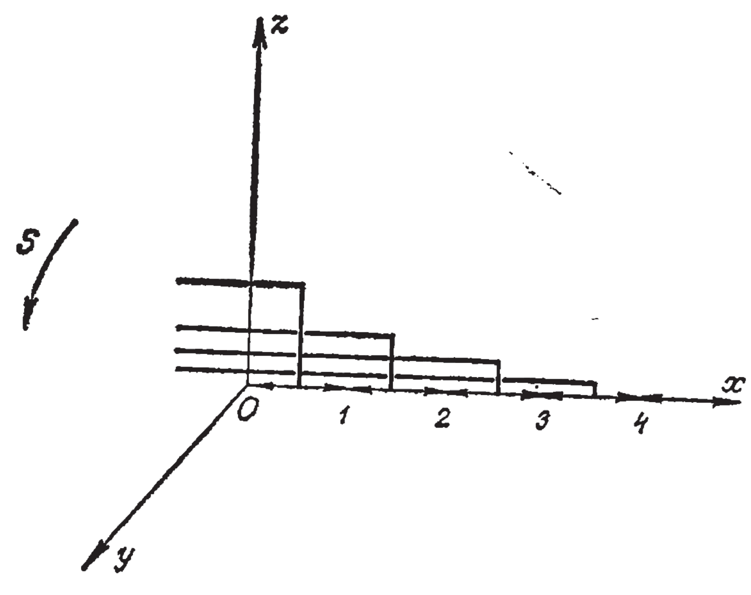

Fig. 1

the plane yOz. We denote by F_{n+1} the figure which consists of the segment a_n and the sides of the right angle. We define $X=\bigcup_{n=1}^{\infty} F_n$ (see fig. 1).

It is easy to verify that this space satisfies the above necessary properties.

Lemma 2. If X is a connected space and R is a non-trivial Hausdorff **M**-connected space, then $X\times R$ is a non-trivial **M**-connected space.*

Proof. Let us suppose the contrary. Then there exist two proper closed subsets F_1 and F_2 of $X\times R$, so that

(1) $$X\times R=F_1\cup F_2 \text{ and } F_1\cap F_2\in \mathbf{M}.$$

Let x be an arbitrary point of X. We will now show that only one of the next two inclusions is true.

(2) $$\{x\}\times R\subset F_1 \quad \{x\}\times R\subset F_2.$$

First of all, let us assume that for some $x_0\in X$ the above inclusions are true. Then we have $\{x_0\}\times R\subset F_1\cap F_2$. Obviously the set $\{x_0\}\times R$ is closed in $X\times R$, thus it is closed in $F_1\cap F_2$. Hence, by virtue of condition 2) we have $\{x_0\}\times R\in \mathbf{M}$. Furthermore, by virtue of condition 1) we assume that $R\in \mathbf{M}$ (because $\{x_0\}\times R$ and R are homeomorphic). This contradicts to the assumption that R is a non-trivial Hausdorff **M**-connected space. Thus we proved that the both inclusions (2) cannot hold at the same time. Now we shall show that at least one of them holds.

By virtue of the first formula of (1) we obtain.

(3) $$\{x\}\times R=((\{x\}\times R)\cap F_1)\cup((\{x\}\times R)\cap F_2).$$

The addends in the right side of this equality are closed subsets of $\{x\}\times R$ and its intersection is $(\{x\}\times R)\cap F_1\cap F_2=(\{x\}\times R)\cap F_1)\cap((\{x\}\times R)\cap F_2)$. This set is a closed subset of $F_1\cap F_2$ and with respect to the second formula of (1) we obtain that $(\{x\}\times R)\cap F_1\cap F_2\in \mathbf{M}$. As we note above $\{x\}\times R$ is **M**-connected and therefore the formula (3) implies that at least one of the sets $(\{x\}\times R)\cap F_1$ and $(\{x\}\times R)\cap F_2$ is not proper, i. e. $(\{x\}\times R)\cap F_1=\{x\}\times R$, or $(\{x\}\times R)\cap F_2=\{x\}\times R$. Thus we proved (2).

Let us put $X_i=\{x\in X:\{x\}\times R\subset F_i\}$, $i=1, 2$.
As we have already noted above we have

(4) $$X=X_1\cup X_2,\quad X_1\cap X_2=\varnothing.$$

It is easy to verify that the sets X_i, $i=1, 2$, are closed. By (4) and our assumption of connectedness of X obtain $X=X_{i_0}$ for some $i_0=1, 2$. Therefore, $X\times R\subset F_{i_0}$ which is contrary to our assumption that F_i, $i=1, 2$ are proper subsets of $X\times R$. Thus, the assumption that the space $X\times R$ is not **M**-connected leads to a contradiction, hence lemma 2 is proved.

Let us note (the proof is analogical) that if X is strong connected** and R is a non-trivial Hausdorff strongly **M**-connected space then $X\times R$ is a non-trivial strongly **M**-connected space.

We shall now state the theorem formulated above.

Let us denote

(5) $$L=X\times R,\quad K=L\cup\{\omega\},$$

* For the proof of this lemma we use condition 1) and the following weaker condition of condition 2) if $X\in \mathbf{M}$ and F is closed subset of X, then $F\in \mathbf{M}$.

** The space is called strong connected if it cannot be decomposed into the snm of finitely or countably many disjoint proper closed subsets, i. e. it is strong $\{\varnothing\}$-connected.

where X is the space in lemma 1, R is the space in the formulated theorem and K is Alexandroff's compactification of L, where ω is the point of infinity.

We will show that:

a) K is not strong **M**-connected;

b) K is **M**-connected.

First of all, let us note, that K is a non-trivial compact Hausdorff space.

Proof of a). We know that $X=\bigcup_{i=1}^{\infty} F_i$, where F_i $(i=1, 2, \ldots)$ are disjoint proper closed subsets of X, i. e. $F_i \cap F_j = \emptyset$, $i \neq j$ (see Lemma 1). Therefore, the sets $\Phi_n=(F_n \times R) \cup \{\omega\}$ are proper closed subsets of K and $K=\bigcup_{n=1}^{\infty} \Phi_n$. Besides that $\Phi_i \cap \Phi_j = \{\omega\}$ when $i \neq j$. Since $\mathbf{M} \neq \{\emptyset\}$ it is easy to see that $\{\omega\} \in \mathbf{M}$. Thus a) is proved.

Proof of b). By virtue of lemma 2 we obtain that $L=X \times R$ is **M**-connected.

Let us suppose that K is not **M**-connected. Then there exist two proper closed subsets K_1 and K_2 of K, so that

$$K=K_1 \cup K_2 \quad \text{and} \quad K_1 \cap K_2 \in \mathbf{M}. \tag{6}$$

Now we shall prove that $K_1 \cap K_2 \setminus \{\omega\} \in \mathbf{M}$. Applying condition 2) of **M** and the right side of formula (6) it is sufficient to show that $K_1 \cap K_2 \setminus \{\omega\}$ is a F_τ-subset of $K_1 \cap K_2$. This is true because $K_1 \cap K_2 \setminus \{\omega\}$ is a closed subset of L and L can be represented as a countable sum of compact subspaces.* Therefore, we have

$$L=(K_1 \setminus \{\omega\}) \cup (K_2 \setminus \{\omega\}), \tag{7}$$

where the sets $K_1 \setminus \{\omega\}$ and $K_2 \setminus \{\omega\}$ are closed subsets of L and their intersection $(K_1 \setminus \{\omega\}) \cap (K_2 \setminus \{\omega\}) = K_1 \cap K_2 \setminus \{\omega\}$ belongs to **M**. Since L is **M**-connected we obtain that $L=K_{i_0} \setminus \{\omega\}$ for some $i_0=1, 2$. Hence $K=K_{i_0}$, which is a contradiction to our assumption that $K_i (i=1, 2)$ are proper subsets of K. The obtained contradiction proves our theorem.

Corollary 1. For every natural number $n>1$ there exists a compact subset of Euclidean space which is Cantor n-manifold but is not strongly Cantor n-manifold, [1].

Corollary 2. For every infinite cardinal number τ there exists a compact Hausdorff space which is Cantor τ-manifold but is not strongly Cantor τ-manifold.

It can be verified that if condition 2) from the definition of **M** is replaced with the following two conditions

2') If $X \in \mathbf{M}$ and F is a closed subset of X then $F \in \mathbf{M}$;

2'') If $X \in \mathbf{M}$ and $x \in X$, then $X \setminus \{x\} \in \mathbf{M}$, then the formulated propositions above hold and the proofs are analogical.

* X can be represented as a countable sum of compact subspaces X_n and therefore $L=X \times R=\bigcup_{n=1}^{\infty} (X_n \times R)$, where $X_n \times R$, $n=1, 2, \ldots$, are compact.

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ПРИМЕР НА M-СВЪРЗАНО КОМПАКТНО ХАУСДОРФОВО ПРОСТРАНСТВО, КОЕТО НЕ Е СИЛНО M-СВЪРЗАНО

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Нека **M** е някакъв клас от топологични пространства, който удовлетворява следните условия:

1) класът **M** е топологически затворен;

2) класът **M** съдържа всяко F_σ-подмножество на всеки свой елемент.

Ще казваме, че едно топологично пространство е **M**-свързано, ако то не може да се представи във вид на обединение на две свои собствени затворени подмножества, сечението на които принадлежи на **M**. Едно топологично пространство ще наричаме силно **M**-свързано, ако то не може да се представи като обединение на изброимо много свои затворени собствени подмножества, сечението на всеки две от които е елемент на класа **M**.

Основна теорема. Ако класът **M** съдържа елемент, различен от празното множество, и съществува компактно хаусдорфово **M**-свързано пространство, което има поне две различни точки, то съществува нетривиално компактно хаусдорфово **M**-свързано пространство, което не е силно **M**-свързано.

Доказателството се основава на следните две леми:

Лема 1. Съществува локално компактно свързано подпространство на тримерното евклидово пространство, което не е силно свързано.

Лемма 2. Топологично произведение $X \times R$ на свързаното пространство X и нетривиалното М-свързано пространство R е също нетривиално М-свързано пространство.

Следствие 1. За произволно естествено число n, $n > 1$, съществува компактно подмножество на евклидово пространство, което е канторово n-многообразие, но не е силно канторово n-многообразие.

Следствие 2. За произволно кардинално число τ, по-голямо или равно на алеф нула, съществува компактно хаусдорфово канторово τ-многообразие, което не е силно канторово τ-многообразие.

В края на статията се привеждат резултати, аналогични на основната теорема.