PROXIMITY ALGEBRAS AND THEIR COMPACT REPRESENTATIONS

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Let R and X be arbitrary sets. The functions of the type $f: R^m \times X^n \rightarrow X$ (m, $n=0, 1, 2, \ldots$) sometimes are called *algebraic operations on X over R*. An *algebra* on X over R is an arbitrary set of algebraic operations on X over R. The general theory of algebras has a relatively long history (see [7]).

The interplay of topological and algebraic concepts is a fruitful mathematical method. In spite of that, the general notion of topological algebra did not attract much attention.

If R and X are topological spaces, an algebra $\mathbf{A}$ on X over R is called *topological* if all the operations $f \in \mathbf{A}$ are continuous. In [2] the functional approach to compact extensions of topological spaces was modified in order to study compact representations of topological algebras.

Yu. M. Smirnov [3], [4] developed the theory of compactifications of proximity spaces which in fact presents a natural way to investigate compact extensions of spaces.

The present paper might be considered as a further development of [2]; it contains a proximity approach to compact representations of algebras, which are more natural than their compact extensions (see [8]). That is why we use proximities which are not necessarily separated.

1. A PROXIMITY MAPPING EXTENSION THEOREM

Let us remind that a binary relation δ on the power set of a set X is called a *proximity on* X (see [1]) if δ satisfies the following axioms:

1. $A\delta B$ implies $A \neq \emptyset$;
2. $A\delta B$ implies $B\delta A$;
3. $A \cap B \neq \emptyset$ implies $A\delta B$;
4. $A\delta B$ and $B \subset B_1 \subset X$ imply $A\delta B_1$;
5. $A\delta B(B \cup C)$ implies $A\delta B$ or $A\delta C$;
6. $A\bar{\delta}B$ implies there exist sets A_1, $B_1 \subset X$ with $A \subset A_1$, $B \subset B_1$, $A_1 \cup B_1 = X$, $A\bar{\delta}B_1$, $A\bar{\delta}B$.

If δ is a proximity on X, the pair (X, δ) is called a *proximity space*.

A typical example of a proximity space arises if a function $h:X\to Y$ into a compact space Y is given. The relation δ_h defined by $A\delta_h B$, if $\overline{h(A)}\cap\overline{h(B)}\neq\emptyset$ $(A, B\subset X)$ is then a proximity on X. δ_h is called a proximity on X induced by h. If $h(X)$ is dense in Y, h is called a compactification of the proximity space (X, δ_h). Yu. M. Smirnov [3] proved that every proximity space has compactifications.

The following lemma gives a useful connection between a proximity space X and the convergence of nets in any compactification of X. It was proved by Hadžiivanov in [5] and used by him to show a construction of compactifications of proximity spaces. The proof presented here is more direct.

Lemma 1.

Let X be a proximity space, $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net in X, and $h:X\to Y$ be a compactification of X. Then the following two conditions are equivalent:

i) the net $\{h(x_\gamma)\}_{\gamma\in\Gamma}$ is convergent in Y;

ii) the relation $\{x_h:\gamma\in\Gamma_1\}\delta\{x_\gamma:\gamma\in\Gamma_2\}$ hold whenever Γ_1 and Γ_2 are cofinal subsets of Γ.

Proof. Suppose i) is true. Let $\xi=\lim\limits_{\gamma\in\Gamma} h(x_\gamma)$, and Γ_1, Γ_2 be cofinal subsets of Γ. Clearly $\xi\in\overline{h(\{x_\gamma:\gamma\in\Gamma_\nu\})}$ $(\nu=1, 2)$. Since h is a compactification of X, it follows $\{x_\gamma:\gamma\in\Gamma_1\}\delta\{x_\gamma:\gamma\in\Gamma_2\}$. Hence i) implies ii).

Suppose now that ii) is true, and that the net $\{h(x_\gamma)\}_{\gamma\in\Gamma}$ does not converge. Since Y is compact, the net $\{x_\gamma\}_{\gamma\in\Gamma}$ has two different cluster points ξ_1 and ξ_2. Let U_ν be a closed neighbourhood of ξ_ν $(\nu=1, 2)$ with $U_2\cap U_2=\emptyset$. Since h is a compactification of X, the relation $h^{-1}(U_1)\bar{\delta}h^{-1}(U_2)$ holds. Let $\Gamma_\nu=\{\gamma\in\Gamma:h(x_\gamma)\in U_\nu\}$ $(\nu=1, 2)$. It is clear that Γ_ν is a cofinal subset of Γ $(\nu=1, 2)$. On the other hand, $\{x_\gamma:\gamma\in\Gamma_\nu\}\subset h^{-1}(U_\nu)(\nu=1, 2)$. Hence

$$\{x_\gamma:\gamma\in\Gamma_1\}\bar{\delta}\{x_\gamma:\gamma\in\Gamma_2\}$$

which contradicts ii). Q. E. D.

Let X be a proximity space, R be a topological space, $\rho\in R$, and $\pi:R\times X\to X$ be the corresponding projection. Two subsets A_1, A_2 of $R\times X$ are said to be close near ρ, if for each neighbourhood U of ρ in R, the relation

$$(1) \qquad (\pi[(U\times X)\cap A_1])\delta(\pi[(U\times X)\cap A_2])$$

in X holds. In this case we shall write $A_1\delta A_2$ (mod ρ).

Let R be a topological space, and X, X_1 be two proximity spaces. A function $f:R\times X\to X_1$ is said to be a proximity mapping, if $A\delta B$ (mod ρ) implies

$$f(A)\delta f(B) \qquad (A,\ B\subset R\times X,\ \rho\in R).$$

It is clear that, if R is a one point space, the above concept of proximity mapping turns into the standard one.

The following theorem is the main result of the present section. It is a generalisation of the standard theorem of extension of proximity mappings [3].

Theorem 1.

Let R be a topological space, X, X_1 be proximity spaces, $h:X\to Y$ and $h_1:X_1\to Y_1$ be compactifications of X and Y, respectively, and $f:R\times X\to X_1$ be a function. Then the following two conditions are equivalent:

i) f is a proximity mapping;

ii) there exists a continuous mapping

(2) $$\hat{f}: R\times Y\to Y_1$$

such that the diagram

(3) $$\begin{array}{ccc} R\times X & \xrightarrow{f} & X_1 \\ \downarrow{\scriptstyle i_R\times h} & & \downarrow{\scriptstyle h_1} \\ R\times Y & \xrightarrow{\hat{f}} & Y_1 \end{array}$$

is commutative.

Proof. Suppose i) is true. First of all, we prove that if the nets $\{r_\gamma\}_{\gamma\in\Gamma}$ $(r_\gamma\in R)$ and $\{h(x_\gamma)\}_{\gamma\in\Gamma}$ $(x_\gamma\in X)$ are convergent, the net $\{h_1(f(x_\gamma, r_\gamma))\}_{\gamma\in\Gamma}$ is also convergent. Let Γ_1, Γ_2 be cofinal subsets of Γ, and $\rho=\lim\limits_{\gamma\in\Gamma} r_\gamma$. Let U be an arbitrary neighbourhood of f, and

(4) $$A_\nu=\{(r_\gamma,\ x_\gamma):\gamma\in\Gamma_\nu\} \qquad (\nu=1,\ 2).$$

Clearly,

(5) $$\pi[(U\times X)\cap A_\nu]\supset\{x_\gamma:\gamma\in\Gamma_\nu'\} \qquad (\nu=1,\ 2)$$

where Γ_ν' is a cofinal subset of Γ_ν $(\nu = 1,\ 2)$. It follows from lemma 1 that

(6) $$\{x_\gamma:\gamma\in\Gamma_1'\}\delta\{x_\gamma:\gamma\in\Gamma_2'\}.$$

From (5) and (6) it follows that the relation (1) takes place. Hence $A_1\delta A_2$ (mod ρ). Since f is a proximity mapping, it follows $f(A_1)\delta f(A_2)$, and (4) implies

(7) $$\{f(r_\gamma,\ x_\gamma):\gamma\in\Gamma_1\}\delta\{f(r_\gamma,\ x_\gamma):\gamma\in\Gamma_2\}.$$

Now, from lemma 1 it follows that the net $\{h_1(f(r_\gamma,\ x_\gamma)\}_{\gamma\in\Gamma}$ is convergent.

Let $(\Gamma,\ \xi)$ be an arbitrary element of $R\times Y$. Since $h(X)$ is dense in Y there exists a net

(8) $$\{r_\gamma,\ x_\gamma)\}_{\gamma\in\Gamma} \qquad (r_\gamma\in R,\ x_\gamma\in X)$$

with

(9) $$\lim_{\gamma\in\Gamma} r_\gamma=r,\quad \lim_{\gamma\in\Gamma} h(x_\gamma)=\xi.$$

We have already proved that the limit $\lim\limits_{\gamma\in\Gamma} h_1(f(r_\gamma,\ x_\gamma))$ exists. It is not difficult to see that this limit does not depend on the special choice of the net (8) with (9). We define the function (2) by

$$\hat{f}(r,\xi)=\lim_{\gamma\in\Gamma} h_1(f(r_\gamma, x_\gamma)). \tag{10}$$

It is obvious that the diagram (3) is commutative. Since the space Y_1 is regular, and $h_1(X_1)$ is dense in Y_1, it follows from a well known theorem [6] that the continuity of $\hat{f}$ will be proved, if we show that the condition

$$\lim_{\gamma\in\Gamma}(r_\gamma, \xi_\gamma)=(r,\xi) \quad (r_\gamma,\ r\in R;\quad \xi_\gamma\in h(X),\ \xi\in Y)$$

implies

$$\lim_{\gamma\in\Gamma}\hat{f}(r_\gamma, \xi_\gamma)=\hat{f}(r,\xi). \tag{11}$$

Let $\xi_\gamma = h(x_\gamma)\ (x_\gamma\in X)$. Since the diagram (3) is commutative, we have $\hat{f}(r_\gamma, \xi_\gamma)=h_1(f(r_\gamma, x_\gamma))$. Now, the definition (10) implies (11), and the implication i) $\Longrightarrow$ ii) is proved.

Suppose now that ii) is true. Let $A_\nu\subset R\times R$ $(\nu=1, 2)$, $\rho\in R$, and $A_1\delta A_2$ (mod ρ). That is, for each neighbourhood U of ρ in R, the relation (1) holds. Since h is a compactification of X, from (1) it follows

$$\overline{h(\pi[(U\times X)\cap A_1])}\cap\overline{h(\pi[U\times X)\cap A_2])}.$$

Let ξ_U be an arbitrary element of this intersection. Since the neighbourhoods U of ρ form a directed set, we have a net $\{\xi_U\}$ of elements of the compact space Y. Without loss of generality we may suppose that the net $\{\xi_U\}$ is convergent. Let $\xi_0=\lim_{U\ni\rho}\xi_U$. From the definition of ξ_U it follows that

$$\xi_0\in\overline{h(\pi[(U\times X)\cap A_\nu)} \qquad (\nu=1,2) \tag{12}$$

for each neighbourhood U of ρ in R. (12) implies that for each neighbourhood U of ρ in R and for each neighbourhood V of ξ_0 in Y there exists a point

$$(r_{UV},\ \alpha_{UV})\in(U\cap X)\cap A_1 \tag{13}$$

with

$$h(\alpha_{UV})\in V. \tag{14}$$

From (13) it follows

$$(r_{UV},\ h(\alpha_{UV}))\in(i_R\times h)(A_1) \tag{15}$$

and $r_{U\nu}\in U$. Now, (14) implies

$$(r_{UV},\ h(\alpha_{UV}))\in U\times V. \tag{16}$$

Since $U\times V$ is an arbitrary neighbourhood of (ρ,ξ_0) in $R\times Y$, (15) and (16) imply

$$(\rho,\ \xi_0)\in\overline{(i_R\times h)(A_1)}. \tag{17}$$

Similarly,

$$(\rho,\ \xi_0)\in\overline{(i_R\times h)(A_2)}. \tag{18}$$

Since $\hat{f}$ is continuous, (17) and (18) imply

$$\overline{\hat{f}((i_R\times h)(A_1))} \cap \overline{\hat{f}((i_R\times h)(A_2))} \neq \emptyset .$$

Now, from the commutativity of the diagram (3), it follows

$$\overline{h_1(f(A_1))} \cap \overline{h_1(f(A_2))} \neq \emptyset.$$

Hence, $f(A_1)\delta f(A_2)$, since h_1 is by condition a compactification of X_1. Q. E. D.

2. COMPACT REPRESENTATIONS OF ALGEBRAS

At the beginning we reminded the definition of the product of a family of proximity spaces. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of proximity spaces $X = \prod_{\alpha\in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ $(\alpha \in A)$ be the corresponding projections. X usually is endowed with a proximity δ, defined in the following manner. Two subsets A, B of X are called close, if for each finite covering $\{A_\mu\}_{\mu=1}^m$ of A in X and for each finite covering $\{B_\nu\}_{\nu=1}^n$ of B in X there exists $\mu = 1, 2, \ldots, m$ and $\nu = 1, 2, \ldots, n$ such that $\Pi_\alpha(A_\mu)\delta\pi_\alpha(B_\nu)$ in X_α for each $\alpha \in A$. The so found proximity space (X, δ) is called the product of the family $\{X_\alpha\}_{\alpha\in A}$. If is well known that if $h_\alpha : X_\alpha \to Y_\alpha$ $(\alpha \in A)$ are compactifications, the function $\prod_{\alpha\in A} h_\alpha : X \to \prod_{\alpha\in A} Y_\alpha$ is a compactification of X.

Let R be a topological space, and X be a proximity space. For each non-negative integer n we have just endowed X^n with a proximity. So it makes sense to ask whether an algebraic operation

(19) $$f: R^m \times X^n \to X$$

is a proximity mapping or not. Let **A** be an algebra on X over R. We shall call **A** a proximity algebra, if each element (19) of **A** is a proximity mapping. In this case we shall also say that the proximity of X is compatible with **A**.

Let R, X, Y be sets, **A** be an algebra on X over R, and **B** be an algebra on Y over R. A function

(20) $$h : X \to Y$$

is called a homomorphism if there exists a preserving the number of arguments bijection $f \to f_h$ between **A** and **B** such that the diagram

(21) $$\begin{array}{ccc} R^m\times X^n & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i_R^m\times h^n} & & \downarrow{\scriptstyle h} \\ R^m\times Y^n & \xrightarrow{f_h} & Y \end{array}$$

is commutative for each $f \in A$.

If R and Y are topological spaces, the algebra **B** is topological and the set $h(X)$ is dense in Y, the homomorphism (20) is called a representation of X into Y. We shall consider representations (20) such that the algebra Y is compact, and will call them compact representations of X.

The next theorem studies the connection between the proximities on X compatible with **A** and the compact representations of X.

Theorem 2.

Let R be a topological space, X be a set, **A** be an algebra on X over R, and (20) be a compact representation of **A**. Then the proximity δ_h on X, induced by h is compatible with **A**.

Conversely let R be a topological space, X be a proximity space, **A** be a proximity algebra on X over R, and (20) be a compactification of X. Then there exists a unique topological algebra **B** on Y over R such that h is a compact representation of **A**.

Proof. We first prove the first part of the theorem. The function (20) is by definition a compactification of (X, δ_h). Hence $h^n: X^n \to Y^n$ is a compactification of X^n. Now from the commutativity of the diagram (21) and from theorem 1 it follows that (19) is a proximity mapping. Hence δ_h is compatible with **A**.

We pass to the proof of the second part of the theorem. Let the function (19) belong to **A**. Hence f is a proximity mapping. Since $h^n: X^n \to Y^n$ is a compactification of X^n, it follows from theorem 1 that there exists a continuous mapping

$$f_h: R^m \times Y^n \to Y$$

such that the diagram (21) is commutative. If f runs over **A**, f_h runs over a topological algebra **B** on Y over R. It is clear that h is a representation of **A** into **B**. The uniqueness of **B** is trivial. Q. E. D.

We pass to the problem of comparing of compact representations of algebras.

Let R be a topological space, X be a set, **A** be an algebra on X over R, (20) be a representation of X, and

(22) $$k: X \to Z$$

be another representation of X. The representation h is called finer than k, if there exists a continuous representation $\chi: Y \to Z$, such that the diagram

(23) $$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "k"'] & Y \arrow[d, "\chi"] \\ & Z \end{tikzcd}$$

is commutative. h and k are said to be equivalent, if each of them is finer than the other; this means that χ in the diagram (23) is an algebraical and topological isomorphism.

Let two proximities δ_1 and δ_2 on a set X are given. They say that the proximity δ_1 is stronger than δ_2 and write $\delta_1 \geq \delta_2$, if $A \delta_1 B$ implies

$A\delta_2 B$. Clearly, this means that the identity $i_x:(X, \delta_1) \to (X, \delta_2)$ is a proximity mapping.

The following theorem shows a connection between the above two ordering relations.

Theorem 3.

Let R be a topological space, X be a set, A be an algebra on X over R, and (20), (22) be compact representations of A. Then h is finer than k if and only if $\delta_h \geqq \delta_k$.

Proof. Let h be finer than k. The diagram (23) may be written in the form

$$
\begin{tikzcd}
(X, \delta_h) \arrow[r, "i_x"] \arrow[d, "h"] & (X, \delta_k) \arrow[d, "k"] \\
Y \arrow[r, dotted, "\chi"] & Z
\end{tikzcd} \quad . \tag{24}
$$

Since h and k are compactifications, and χ is continuous, Theorem 1 implies that i_x is a proximity mapping. Hence $\delta_h \geqq \delta_k$.

Let now $\delta_h \geqq \delta_k$. Then i_x in (24) is a proximity mapping. Since h and k are compactifications, by Theorem 1 there exists a continuous mapping χ such that the diagram (24) is commutative. Clearly χ is a homomorphism of Y into Z. Hence h is finer than k. Q. E. D.

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БЛИЗОСТНИ АЛГЕБРИ И КОМПАКТНИТЕ ИМ ПРЕДСТАВЯНИЯ

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(РЕЗЮМЕ)

Нека R е топологично пространство, а X е близостно пространство. Две подмножества A_1 и A_2 на $R\times X$ се наричат близки около някоя точка $\rho\in R$, когато за всяка околност U на ρ е изпълнено условието (1), където $\pi:R\times X\to X$ е съответната проекция. Едно изображение $f:R\times X\to X_1$, където X_1 е близостно пространство, се нарича близостно непрекъснато, когато за всяка точка $\rho\in R$ и за всеки две подмножества A_1, A_2 на $R\times X$, които са близки около ρ, е в сила $f(A_1)\delta f(A_2)$.

В първия параграф е доказано следното обобщение на известната теорема на Ю. М. Смирнов [3], [4] за продължаване на близостно непрекъснати изображения.

Теорема 1. Нека R е топологично пространство, X и X_1 са близостни пространства,

$$h:X\to Y \text{ и } h_1:X_1\to Y_1$$

са компактните разширения на близостните пространства X и X_1, а $f:R\times X\to X_1$ е произволно изображение. Тогава следните две условия са еквивалентни:

а) изображението f е близостно непрекъснато;

б) съществува такова непрекъснато изображение (2), че диаграмата (3) е комутативна.

Във втория параграф на работата теорема 1 се използува за изучаване на компактните представяния на близостните алгебри.