PRECOMPACT MINIMAL TOPOLOGIES ON SOME TORSION FREE MODULES

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In this paper R will denote a countable Dedekind domain which is not a field, and in which every non-trivial ideal is of finite index (a part of the definitions and statements below are valid without so restrictive conditions on R).

An R-module M is called topological R-module if M is endowed with a topology such that:

i) M is a topological group with respect to the addition;

ii) for each $r \in R$, the function rx $(x \in M)$ of x is continuous.

A Hausdorff topological R-module M is said to be minimal (and the topology of M is called minimal), if every continuous module isomorphism $h: M \to M_1$, where M_1 is a Hausdorff topological R-module, is a homeomorphism.

A topological R-module M is called precompact, if M is Hausdorff, and there exists a compact R-module which contains M as a dense topological submodule.

All known minimal R-modules are precompact. The study of this special case is facilitated by the convenient and powerful Pontrjagin's theory of characters. In [4], [5] precompact minimal abelian groups are considered from that point of view. [6] contains a general approach which among the other things permits to describe the precompact minimal topologies on R^n $(n \in \mathbf{N})$. But the method of [6] can not be used to describe the precompact minimal topologies on the direct sum of countably many copies of R, for example.

For each topological R-module M, by M^* we will denote the set of all the continuous (additive) group homomorphisms $\chi: M \to \mathbf{T}^1$ where $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ is the one-dimensional torus. Obviously M^* is an Abelian group. For each $\chi \in M^*$ and for each $r \in M$ by $rx: M \to \mathbf{T}^1$ we will denote the element of M^* defined by $(r\chi)(x) = \chi(rx)$ for each $x \in M$. In this way M^* is endowed with an R-module structure.

It is not difficult to see that an R-module M is precompact, if and only if the elements of M^* separate the points of M, and the topology of M coincides with the weakest topology on M such that the elements of M^* are continuous. In that case we will endow M^* with the discrete topology.

If a topological R-module M is precompact, and the module M^{**} is endowed with the pointwise convergence topology, M^{**} is a compact R-module, and there is a natural algebraical and topological embedding

$$i: M \to M^{**} \tag{1}$$

such that $i(M)$ is dense in M^{**}. The embedding (1) is defined by $(i(x))(\chi)=\chi(x)$ for arbitrary $x \in M$ and $\chi \in M^*$. Clearly, M^{**} is the completion of M, for every precompact R-module M.

An embedding $i: M \to M_1$ of an R-module M into a topological R-module M_1 is called essential, if $i(M)$ is a dense subset of M_1 and every closed non-trivial submodule of M_1 contains non-zero elements of $i(M)$.

[6] contains a general criterion for minimality which in the present situation specializes as follows.

Proposition 1.

A precompact R-module M is minimal if and only if the embedding (1) is essential.

The present paper is a continuation and an improvement of a part of [6]. It turns out that, if a torsion free precompact module is minimal, we have

$$M^{**}=\left(\prod_{p \in P} R_p^{a_p}\right)\times(Q(R)^*)^b, \tag{2}$$

where P is the set of all the maximal ideals of R, $R_p (p \in P)$ is the p-adic completion of R (R_p is compact, since the non trivial ideals of R are of finite index), $a_p (p \in P)$ is an arbitrary cardinal number, $Q(R)$ is the field of fractions of R, and b is an arbitrary cardinal number. The proof of (2) is given in Section 1.

In the special case

$$\operatorname{card} M < \mathfrak{c} (= \operatorname{card} \mathbf{R}),$$

Proposition 1 impose further restrictions to the product on the right side of (2). In this way a possibility of describing all the precompact minimal topologies arises. This case is considered in Section 2.

1. COMPLETION OF A TORSION FREE PRECOMPACT MINIMAL R-MODULE

In this section we prove the relation (2). If $R=\mathbf{Z}$ and M is compact, (2) is well known (see [3]). The proof of (2) is based on the structural theorem 1 which in the case $R=\mathbf{Z}$ is also well known (see [2]).

Let R be a ring. An R-module M is called divisible, if for each $x \in M$ and for each $r \in R$ with $r \neq 0$ there exists an element y of M with $ry=x$. M is called injective, if for every R-module N and for every module homomorphism $f: N_1 \to M$ of a submodule N_1 of N there exists a module homomorphism $g: N \to M$ with $g/N_1=f$. In [1] the Dedekind domains are characterised as such rings R that every divisible R-module is injective.

The next proposition reduces examining of precompact minimal topologies on torsion free R-modules to an algebraic problem.

Proposition 2. If M is a precompact minimal torsion free R-module, M^* is a divisible module.

Proof. We first prove that M^{**} is torsion free. Assume the contrary. Let $\xi \in M^{**}$, $r \in R$ be elements with $\xi \neq 0$, $r \neq 0$ and $r\xi=0$. Clearly, the set I

of all the $x \in R$ with $x\xi=0$ is a non-trivial ideal of R, and we have $R\xi=R/I$. Since I is of finite index, the submodule $R\xi$ of M^{**} is finite and therefore closed. Now proposition 1 implies that $i(M)$ contains non-zero elements of $R\xi$ in contradiction with the fact that M is torsion free. Hence M^{**} is torsion free.

The proposition will be proved, if we show that for each $r \in R$ with $r \neq 0$ the equality

$$rM^* = M^* \tag{3}$$

holds. For this purpose we first prove that the submodule rM^* of M^* separates the points of M^{**}. Let $\xi \in M^{**}$ and $\xi \neq 0$. Then $r\xi \neq 0$, and hence there exists $\chi \in M^*$ with $(r\xi)(\chi) \neq 0$, i. e. $\xi(r\chi) \neq 0$. The canonical isomorphism between M^* and $(M^{**})^*$ now shows that $(r\chi)(\xi) \neq 0$, and hence rM^{**} separates the points of M^{**}. But the compactness of M^{**} implies that M^* is a minimal group of continuous characters of M^{**} which separates the points of M^{**} (see [3]). Thus (3) is proved. Q. E. D.

The following lemma is a preparation for the proof of theorem 1 below.

Lemma 1.

Let R be a Dedekind domain, M be a divisible R-module and A be a divisible submodule of M. Then there exists a divisible submodule B of M with $A \cap B = \{0\}$ and $A+B=M$.

Proof. Since A is a divisible R-module, A is injective. Therefore there exists a module homomorphism $h: M \to A$ with $h(x)=x$ for each $x \in A$. Let $B=\ker h$. We only need to prove that the R-module B is divisible. Let $x \in B$ and $r \in R$, $r \neq 0$. By conditions, there is an element $y_1 \in M$ with $ry_1=x$. Let $y=y_1-h(y_1)$. Then $y \in B$, and $ry=ry_1-f(ry_1)=x$. Q. E. D.

For an arbitrary maximal ideal p of a Dedekind domain R we denote by $R_{(p)}$ the localization of R at p, and by $R(p)$ the R-module defined by

$$R(p)=Q(R)/R_{(p)}. \tag{4}$$

It is not difficult to see that every non-trivial submodule of $R(p)$ is generated by an element of $R(p)$ of the type

$$\frac{1}{\pi^n}+R_{(p)} \qquad (n \in \mathbf{N},\ \pi \in p \setminus p^2). \tag{5}$$

Actually, the R-module generated by the element (5) depends on n but does not depend on the special choice of $\pi \in p \setminus p^2$.

The following theorem is an algebraic background of Section 2.

Theorem 1.

Let R be a Dedekind domain, and M be a divisible R-module. Then M is a direct sum of copies of $R(p)$ (where p runs over a suitable set of maximal ideals of R) and of copies of $Q(R)$.

Proof. Let τ be the torsion part of M. Clearly, τ is a divisible R-module. Hence lemma 1 implies that $M=\tau \oplus M_1$ where M_1 is a divisible submodule of M. Clearly M_1 is torsion free. Therefore M_1 is a linear space over $Q(R)$. Hence M_1 is a direct sum of copies of $Q(R)$.

For an arbitrary maximal ideal p of R, denote by τ_p the set of all the elements x of τ with $p^n x=0$ for a suitable positive integer n. It is well known that

$$\tau=\bigoplus_{p\in P}\tau_p.$$

Therefore the theorem will be proved, if we show that for each $p\in P$, the R-module τ_p is a direct sum of copies of $R(p)$.

Let t be an arbitrary non-zero element of τ_p with $pt=0$. Clearly, Rt is a one-dimensional linear space over the field R/p. Since the submodule N of $R(p)$ generated by $\frac{1}{\pi}+R_{(p)}$ is also a one-dimensional linear space over R/p there is a homomorphism $k:N\to Rt$ with

$$k\left(\frac{1}{\pi}+R_{(p)}\right)=t.$$

Since the module τ_p is clearly divisible, and R is a Dedekind domain, there is an extension

$$h:R(p)\to\tau_p \tag{6}$$

of k. Therefore there exists a module homomorphism (6) with

$$h\left(\frac{1}{\pi}+R_{(p)}\right)=t. \tag{7}$$

On the other hand, every non-zero submodule of $R(p)$ contains $\frac{1}{\pi}+R_{(p)}$. Hence (7) and $t\neq 0$ imply $\ker h=0$. Therefore τ_p contains a copy A of $R(p)$ with $t\in A$.

For each element $t\in\tau_p$ with $t\neq 0$, $pt\neq 0$ choose a copy M_t of $R(p)$ with $t\in M_t$. Denote by Γ a maximal among the sets $\Delta\subset M$ with the following two properties: i) $t\in\Delta$ implies $t\neq 0$, $pt=0$; ii) the sum of the modules $\{M_t:t\in\Delta\}$ is direct. The theorem will be proved, if we show that

$$\tau_p=\sum_{t\in\Gamma}M_t. \tag{8}$$

Assume the contrary. Since the module on the right side of (8) is clearly divisible, it follows from lemma 1 that there is a non-zero submodule H of τ_p which is divisible and

$$\left(\sum_{t\in\Gamma}M_t\right)\cap H=\{0\}.$$

Choosing $t\in H$ with $t\neq 0$ and $pt=0$, we obtain a possibility to extend Γ which is a contradiction. Q. E. D.

Corollary 1. if M is a torsion free precompact minimal R-module, the equality (2) holds.

Indeed, it follows from Proposition 2 that M^* is a divisible R-module. Therefore the statement follows from Theorem 1, from the well known equality

$$(\bigoplus_{\alpha \in A} M_\alpha)^* = \prod_{\alpha \in A} M_\alpha^*,$$

and from the equalities $R(p)^* = R_p$ $(p \in P)$ which are fulfilled, since the non-trivial ideals of R are of finite index (see [6]). Q. E. D.

2. PRECOMPACT MINIMAL TOPOLOGIES ON TORSION FREE R-MODULES M WITH CARD $M < \mathfrak{c}$

This section contains a full description of the precompact minimal topologies on torsion free R-modules M with card $M < \mathfrak{c}$. In particular, we have a more transparent solution of a part of the problems in [6].

Let us denote by $H(R)$ the set of all the characters $\chi : Q(R) \to \mathbf{T}$ with $\chi | R = 0$. Clearly, $H(R)$ is a closed submodule of $Q(R)^*$, and $H(R) = (Q(R)/R)^*$. But for every Dedekind domain R we have

$$Q(R)/R = \bigoplus_{p \in P} R(p).$$

Hence

$$H(R) = \prod_{p \in P} R(p)^* = \prod_{p \in P} R_p$$

for every Dedekind domain R in which every non-trivial ideal is of finite index. Thus $Q(R)^*$ contains a copy of $\prod_{p \in P} R_p$.

The following theorem shows that there are only two origins of precompact minimal topologies on the torsion free R-modules M with card $M < \mathfrak{c}$.

Theorem 2. *Let R be a countable Dedekind domain which is not a field, and in which every non-trivial ideal is of finite index. Let M be a torsion free precompact minimal R-module with card $M < \mathfrak{c}$. Then only the following two possibilities may arise:*

i) there is a set $\pi \subset P$ with $M^{**} = \prod_{p \in \pi} R_p$;

ii) $M^{**} = Q(R)^*$.

Proof. Let $p \in P$. For an arbitrary $\xi \in R_p$ with $\xi \neq 0$, consider the submodule

$$R_p(1, \xi) = \{(\lambda, \lambda\xi) : \lambda \in R_p\}$$

of $R_p \times R_p$. It is clear that if ξ runs over R_p, the submodule $R_p(1, \xi)$ runs over a continuum of closed submodules of $R_p \times R_p$ with

$$(R_p(1, \xi_1)) \cap (R_p(1, \xi_2)) = \{0\},$$

whenever $\xi_1 \neq \xi_2$. Since card $M < \mathfrak{c}$, the embedding (1) cannot be essential, if M^{**} contains a copy of $R_p \times R_p$ $(p \in P)$. Thus, it follows from Proposition 1 that M^{**} does not contain a product of the type $R_p \times R_p$ $(p \in P)$. On the other hand, we have already seen that $Q(R)^*$ contains a copy of each R_p $(p \in P)$. Therefore the Theorem follows from Corollary 1. Q. E. D.

The essential embeddings of R-modules into products of the type $\prod_{p\in\pi} R_p$ ($\pi\subset P$), are examined in [6]. Among other things, the following proposition was proved there.

Proposition 3. A dense submodule M of $\prod_{p\in\pi} R_p$ ($\pi\subset P$) is essentially embedded in $\prod_{p\in\pi} R_p$, if and only if M contains non-zero elements of each of the factors R_p ($p\in\pi$).

Corollary 2. A free R-module M may be essentially embedded into $\prod_{p\in\pi} R_p$ ($\pi\subset P$), if and only if

$$\operatorname{card}\pi\leqq\dim M\leqq\mathfrak{c}.$$

The essential embeddings of R-modules into $Q(R)^*$ were also examined in [6]. Ammong other things, the following proposition was proved there.

Proposition 4. Every non-trivial closed submodule of $Q(R)^*$ contains non-zero elements of $H(R)$.

Corollary 3. A dense submodule M of $Q(R)^*$ is essentially embedded into $Q(R)^*$, if and only if M contains non-zero elements of each of the submodules $R(p)^*=R_p$ ($p\in P$) of $Q(R)^*$.

The following theorem is a consequence of Theorem 2, Proposition 1, Proposition 3, Proposition 4, Corollary 2, and Corollary 3.

Theorem 3. Let R be a countable Dedekind domain which is not a field, and in which every non-trivial ideal is of finite index. Let M be a torsion free precompact minimal R-module with card $M<\mathfrak{c}$. Then only the following two possibilities may arise:

i) the module M^* is periodic;

ii) the module M^* is torsion free.

If i) takes place, there is a set $\pi\subset P$ with card $\pi\leqq\dim M$ and a homeomorphic embedding

$$i: M\to\prod_{p\in\pi} R_p \tag{9}$$

such that $i(M)$ is dense in $\prod_{p\in\pi} R_p$ and contains a non-zero element of each of the factors R_p ($p\in\pi$). Conversely, every embedding (9) of an R-module M with just enumerated properties induces a precompact minimal topology on M.

If ii) takes place, the inequality card $P\leqq\dim M$ holds and there is a homeomorphic embedding

$$i: M\to Q(R)^* \tag{10}$$

such that $i(M)$ is dense in $Q(R)^*$ and contains non-zero elements of each of the submodules $R(p)^*=R_p$ of $Q(R)^*$. Conversely, every embedding (10) of an R-module M with just enumerated properties induces a precompact minimal topology on M.

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ПРЕДКОМПАКТНИ МИНИМАЛНИ ТОПОЛОГИИ В НЯКОИ МОДУЛИ БЕЗ ТОРЗИЯ

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Нека R е изброим дедекиндов пръстен, в който нетривиалните идеали имат краен индекс. В предлаганата работа се изучават минималните топологии в R модулите без торзия и с мощност по-малка от мощността на континуума. Ето главният резултат на работата в случая $R=\mathbf{Z}$.

Теорема 3. Нека M е предкомпактна минимална група без торзия и с мощност по-малка от мощността на континуума. Тогава групата M^* на непрекъснатите характери на M или е периодична, или няма торзия. В първия случай съществува множество π от прости числа и хомеоморфно групово влагане

$$i: M \to \prod_{p \in \pi} \mathbf{Z}_p \tag{9}$$

(където $\mathbf{Z}_p$ е компактната група на целите p-адични числа) със свойствата: групата $i(M)$ е гъста в $\prod_{p \in \pi} \mathbf{Z}_p$; групата $i(M)$ съдържа различни от нулата елементи от всеки от множителите $\mathbf{Z}_p$ ($p \in \pi$). Обратно, всяко мономорфно влагане (9) с тези свойства индуцира минимална групова топология в M. Във втория случай M е група с безкраен ранг и съществува хомеоморфно групово влагане

$$i: M \to \mathbf{Q}^* \tag{10}$$

със свойствата: групата $i(M)$ е гъста в $\mathbf{Q}^*$; за всяко просто p групата $i(M)$ съдържа различни от нулата елементи на каноничния екземпляр на групата $\mathbf{Z}_p$ в $\mathbf{Q}^*$. Обратно, всяко мономорфно влагане (10) с тези свойства индуцира минимална групова топология в M.