

ON THE DEFINITION OF VECTOR MULTIPLICATION

Ivanka Christova, Ivan Chobanov

Praemissis praemittendis

In the present paper the symbols Sgn, sgn, Ax, Df, Pr and Dm stand, for the words notation, denote, axiom, definition, proposition and proof respectively; R , F and C stand for the set of all real numbers, for an arbitrary non-trivial ordered field and for the set of all complex numbers respectively.

Let V_F be an arbitrary non-trivial Euclidean space over F , i. e. a set, consisting of more than one element, for which mappings

$$(1) \quad M_1 : V_F^2 \rightarrow V_F$$

(addition in V_F),

$$(2) \quad M_s : F \times V_F \rightarrow V_F$$

(multiplication of the elements of F with the elements of V_F),

$$(3) \quad M_b : V_F^2 \rightarrow F$$

(scalar multiplication of the elements of V_F) are defined, so that, provided

$$(4) \quad a+b \text{ sgn: } M_1((a, b))$$

(sum of a, b),

$$(5) \quad \lambda a \text{ sgn: } M_2((\lambda, a))$$

(product of λ, a),

$$(6) \quad ab \text{ sgn: } M_b((a, b))$$

(scalar product of a, b), the following conditions are satisfied:

Ax 1. $a, b, c \in V_F$ imply $(a+b)+c=a+(b+c)$.

Ax 2. There exists $o \in F$ with: $a \in V_F$ implies $a+o=a$.

Ax 3. $a \in V_F$ implies: there exists $-a \in V_F$ with $a+(-a)=o$.

Ax 4. $a \in V_F$ implies $1a=a$.

Ax 5. $\lambda, \mu \in F, a \in V_F$ imply $(\lambda\mu)a=\lambda(\mu a)$.

Ax 6. $\lambda, \mu \in F, a \in V_F$ imply $(\lambda+\mu)a=\lambda a+\mu a$.

Ax 7. $\lambda \in F; a, b \in V_F$ imply $\lambda(a+b)=\lambda a+\lambda b$.

Ax 8. $a, b \in V_F$ imply $ab=ba$.

Ax 9. $\lambda \in F; a, b \in V_F$ imply $(\lambda a)b=\lambda(ab)$.

Ax 10. $a, b, c \in V_F$ imply $(a+b)c = ac + bc$.

Ax 11. $a \in V_F$ implies $0 \leq a^2$.

Ax 12. $a \in V_F$, $a^3 = 0$ imply $a = 0$.

0 in Ax 11, Ax 12 and 1 in Ax 4 denote the zero- and unit-element of F respectively; $a - b$ sgn: $a + (-b)$ for all $a, b \in V_F$ (difference of a, b); a^2 sgn: aa for every $a \in V_F$ (scalar square of a).

By analogy with the case of the real three-dimensional vector space V , a three-dimensional vector space over an ordered field F is called any Euclidean space V_F over F , for which a mapping

$$(7) \quad M_4 : V_F^2 \rightarrow V_F$$

(vector multiplication in V_F) is defined, so that, provided

$$(8) \quad a \times b \text{ sgn: } M_4((a, b))$$

(vector product of a, b), the following conditions are satisfied:

Ax 13. $a, b, c \in V_F$ imply $a \times b \cdot c = a \cdot b \times c$.

Ax 14. $a, b, c \in V_F$ imply $(a \times b) \times c = (ac)b - (bc)a$.

Ax 15. There exist $a, b \in V_F$ with $a \times b \neq 0$.

Ax 16. $a, b \in V_F$ imply $a \times b = -(b \times a)$.

Ax 17. $\lambda \in F$; $a, b \in V_F$ imply $(\lambda a) \times b = \lambda(a \times b)$.

Ax 18. $a, b, c \in V_F$ imply $(a+b) \times c = a \times c + b \times c$.

The order, in which Ax 13—Ax 18 are usually enumerated, is another one, cf. for instance [1]. The order given here is adopted, since it is proved in the present paper, that Ax 16—Ax 18 are actually corollaries of Ax 1—Ax 15. At that, when trivial corollaries of Ax 1—Ax 12 are used, this fact will not be emphasized explicitly.

As it is well known, the following two propositions are proved on the basis of Ax 1—Ax 12.

Pr 1. The elements a_v ($v=1, 2, \dots, n$) of an Euclidean space are linearly independent iff their Gram determinant is different from zero.

Pr 2. If a_v ($v=1, 2, \dots, n$) are linearly independent elements of an Euclidean space and r_v ($v=1, 2$) are such elements of the n -dimensional linear space, generated by a_v ($v=1, 2, \dots, n$), that

$$(9) \quad r_1 a_v = r_2 a_v \quad (v=1, 2, \dots, n),$$

then

$$(10) \quad r_1 = r_2.$$

We shall now prove some auxiliary propositions.

Pr 3. $a, b, c, d \in V_F$ imply

$$(11) \quad a \times b \cdot c \times d = \begin{vmatrix} ac & ad \\ bc & bd \end{vmatrix}.$$

Dm. (11) follows from

$$(12) \quad \begin{aligned} a \times b \cdot c \times d &= (a \times b) \times c \cdot d \\ &= ((ac)b - (bc)a)d = (ac)(bd) - (bc)(ad) \end{aligned}$$

(Ax 13, Ax 14).

Pr 4. $a, b \in V_F$ imply

$$(13) \quad (a \times b)^3 = \begin{vmatrix} a^2 & ab \\ ba & b^2 \end{vmatrix}.$$

Dm. Pr 3.

Pr 5. $a, b \in V_F$ imply: a, b are linearly independent elements of V_F iff

$$(14) \quad a \times b \neq o.$$

Dm. Pr 1, Pr 4.

Pr 6. $a \in V_F$ imply $a \times a = o$.

Dm. Pr 5.

Pr 7. $a \in V_F$ imply $o \times a = o$.

Dm. Pr 5.

Pr 8. $a, b \in V_F$ imply $a \cdot a \times b = 0$.

Dm. Ax 13, Pr 6.

Pr 9. $a, b \in V_F$ imply $a \cdot b \times a = 0$.

Dm. Ax 8, Ax 13, Pr 6.

Pr 10. $a, b \in V_F$ imply

$$(15) \quad (a \times b)^3 = (b \times a)^3.$$

Dm. Pr 4.

Pr 11. $a, b \in V_F$ imply

$$(16) \quad (a \times b) \times (b \times a) = 0.$$

Dm. (16) follows from

$$(17) \quad (a \times b) \times (b \times a) = (a \cdot b \times a) b - (b \cdot b \times a) a = 0b - 0a = o$$

(Ax 14, Pr 9, Pr 8).

Pr 12. $a, r \in V_F$,

$$(18) \quad a \neq o,$$

$$(19) \quad r \times a = o$$

imply

$$(20) \quad r = \alpha a \quad (\alpha \in F).$$

Dm. (18) implies

$$(21) \quad a^3 \neq 0;$$

(19) implies

$$(22) \quad o = o \times a = (r \times a) \times a = (ra) a - a^2 r$$

(Pr 7, Ax 14); (21), (22) imply (20) with $\alpha = ra/a^2$.

Pr 13. $a, b \in V_F$ imply

$$(23) \quad a \times b = \lambda (b \times a) \quad (\lambda \in F).$$

Dm. If

$$(24) \quad b \times a = o,$$

then

$$(25) \quad a \times b = o.$$

(Pr 10); (24), (25) imply (23) for every $\lambda \in F$. If

$$(26) \quad b \times a \neq o,$$

then Pr 11, Pr 12 imply (23).

Pr 14. $a, b \in V_F$ imply

$$(27) \quad a \times b = \epsilon (b \times a) \quad (\epsilon = \pm 1).$$

Dm. Pr 13, Pr 10.

Pr 15. $a, b \in V_F$,

$$(28) \quad a \times b = b \times a$$

imply (25).

Dm. (28) implies

$$(29) \quad (a \times b)^2 = a \times b \cdot b \times a = \begin{vmatrix} a & b & a^2 \\ b^2 & ba \end{vmatrix}$$

(Pr 3); (29) implies

$$(30) \quad (a \times b)^2 = -(a \times b)^2$$

(Pr 4), i. e. (25).

Pr 16. $a, b \in V_F$ imply

$$(31) \quad a \times b = -(b \times a).$$

Dm. Trivial if (25); Pr 14, Pr 15 if (14).

Pr 17. Ax 1—Ax 14 imply Ax 16.

Dm. Cf. the proof of Pr 16.

Pr 18. $a, b, c \in V_F$ imply

$$(32) \quad a \times (b \times c) = (ac) b - (ab) c.$$

Dm. (32) follows from

$$(33) \quad a \times (b \times c) = -((b \times c) \times a) = -((ba)c - (ca)b) = (ac)b - (ab)c$$

(Pr 16, Ax 14).

Pr 19. $a, b, c, d \in V_F$ imply

$$(34) \quad (a \times b) \times (c \times d) = (a \cdot c \times d) b - (b \cdot c \times d) a.$$

Dm. Ax 14.

Pr 20. $a, b, c, d \in V_F$ imply

$$(35) \quad (a \times b) \times (c \times d) = (a \times b \cdot d) c - (a \times b \cdot c) d.$$

Dm. Pr 18.

Pr 21. $a, b, c, d \in V_F$ imply

$$(36) \quad (a \times b \cdot c)d = (b \times c \cdot d)a + (c \times a \cdot d)b + (a \times b \cdot d)c.$$

Dm. Pr 19, Pr 20, Ax 13, Pr 16.

Pr 22. $a, b, c; p, q, r \in V_F$ imply

$$(37) \quad (a \times b \cdot c)(p \times q \cdot r) = \begin{vmatrix} ap & aq & ar \\ bp & bq & br \\ cp & cq & cr \end{vmatrix}.$$

Dm. Pr 21 with $d = p \times q$ implies

$$(38) \quad (a \times b \cdot c)p \times q = (b \times c \cdot p \times q)a + (c \times a \cdot p \times q)b$$

$$+ (a \times b \cdot p \times q)c = \begin{vmatrix} bp & bq \\ cp & cq \end{vmatrix} a + \begin{vmatrix} cp & cq \\ ap & aq \end{vmatrix} b + \begin{vmatrix} ap & aq \\ bp & bq \end{vmatrix} c;$$

(38) implies

$$(39) \quad (a \times b \cdot c)(p \times q \cdot r) = \begin{vmatrix} bp & bq \\ cp & cq \end{vmatrix} ar + \begin{vmatrix} cp & cq \\ ap & aq \end{vmatrix} br$$

$$+ \begin{vmatrix} ap & aq \\ bp & bq \end{vmatrix} cr = \begin{vmatrix} ap & aq & ar \\ bp & bq & br \\ cp & cq & cr \end{vmatrix}.$$

Pr 23. $a, b, c \in V_F$ imply

$$(40) \quad (a \times b \cdot c)^2 = \begin{vmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{vmatrix}.$$

Dm. Pr 22.

Pr 24. $a, b, c \in V_F$ imply: a, b, c are linearly independent elements of V_F iff

$$(41) \quad a \times b \cdot c \neq 0.$$

Dm. Pr 1, Pr 23.

Pr 25. $a, b \in V_F$, (14),

$$(42) \quad c = a \times b.$$

imply: a, b, c are linearly independent elements of V_F .

Dm. (42), (14) imply

$$(43) \quad a \times b \cdot c = (a \times b)^2 \neq 0.$$

Then Pr 24.

Pr 26. $a, b, c, d \in V_F$ imply: a, b, c, d are linearly dependent elements of V_F .

Dm. Trivial if $a \times b \cdot c = 0$ because of Pr 24. If (41), then

$$(44) \quad d = \frac{b \times c \cdot d}{a \times b \cdot c} a + \frac{c \times a \cdot d}{a \times b \cdot c} b + \frac{a \times b \cdot d}{a \times b \cdot c} c$$

according to Pr 21.

Pr 27. If V_F is a non-trivial Euclidean space and a mapping (7) is defined, so that, provided (8), the conditions Ax 13 and Ax 14 are satisfied, then V_F is an one-dimensional or a three-dimensional Euclidean space.

Dm. Pr 5, Pr 25, Pr 26 and the proofs of these propositions.

Pr 28. If V_F is a non-trivial Euclidean space and a mapping (7) is defined, so that, provided (8), the conditions Ax 13 and Ax 14 are satisfied, the V_F is a three-dimensional Euclidean space iff Ax 15 holds.

Dm. Pr 27, Pr 25.

Pr 29. $\lambda \in F; a, b \in V_F$ imply

$$(45) \quad (\lambda a) \times b = \lambda(a \times b).$$

Dm. (42) is trivial if Ax 15 does not hold. If Ax 15 holds, let

$$(46) \quad a_v \in V_F \quad (v=1, 2, 3)$$

be a basis of V_F (Pr 28) and

$$(47) \quad r_1 = (\lambda a) \times b,$$

$$(48) \quad r_2 = \lambda(a \times b).$$

Then

$$(49) \quad r_1 a_v = (\lambda a) \times b \cdot a_v = (\lambda a) \cdot b \times a_v,$$

$$= \lambda(a \cdot b \times a_v) = \lambda(a \times b \cdot a_v) \quad (v=1, 2, 3)$$

(Ax 13, Ax 9) and

$$(50) \quad r_1 a_v = (\lambda(a \times b)) a_v = \lambda(a \times b \cdot a_v) \quad (v=1, 2, 3)$$

(Ax 9); (49), (50) imply (9) (with $n=3$), i. e. (10) (Pr 2). Now (10), (47), (48) imply (45).

Pr 30. Ax 1—Ax 14 imply Ax 17.

Dm. Cf. the proof of Pr 29.

Pr 31. $a, b, c \in V_F$ imply

$$(51) \quad (a+b) \times c = a \times c + b \times c.$$

Dm. (51) is trivial if Ax 15 does not hold. If Ax 15 holds, let (46) be a basis of V_F and

$$(52) \quad r_1 = (a+b) \times c,$$

$$(53) \quad r_2 = a \times c + b \times c.$$

Then

$$(54) \quad r_1 a_v = (a+b) \times c \cdot a_v = (a+b) \cdot c \times a_v,$$

$$= a \cdot c \times a_v + b \cdot c \times a_v = a \times c \cdot a_v + b \times c \cdot a_v, \quad (v=1, 2, 3)$$

(Ax 13, Ax 10) and

$$(55) \quad r_2 a_v = (a \times c + b \times c) a_v = a \times c \cdot a_v + b \times c \cdot a_v, \quad (v=1, 2, 3)$$

(Ax 10); (54), (55) imply (9) (with $n=3$), i. e. (10) (Pr 2). Now (10), (52), (53) imply (51).

Pr 32. Ax 1 — Ax 14 imply Ax 18.

Dm. Cf. the proof of Pr 31.

Df 1. F being a non-trivial ordered field, a set V_F , for which mappings (1)–(3), (7) are defined, such that, provided (4)–(6), (8), the conditions Ax 1—Ax 15 are satisfied, is called a three-dimensional vector space over F .

Supplement. As it is well known, if L_C is a linear space over the field C of the complex numbers with two at least linearly independent elements, it is impossible to define scalar multiplication of the elements of L_C in such a manner, that L_C turns to be an Euclidean space with respect to this scalar multiplication [1]. If however Ax 8 is subjected to a slight modification, a definition of scalar multiplication, which preserves Ax 9—Ax 12 could be given. The pseudo-Euclidean spaces so obtained are called Hermitean spaces over C .

More precisely, a set H_C is called a Hermitean space over the field F of the complex numbers, iff mappings

$$(56) \quad H_1 : H_C^2 \rightarrow H_C,$$

$$(57) \quad H_2 : C \times H_C \rightarrow H_C,$$

$$(58) \quad H_3 : H_C^2 \rightarrow C$$

are defined, such that, provided

$$(59) \quad a+b \text{ sgn: } H_1((a, b)),$$

$$(60) \quad \lambda a \text{ sgn: } H_2((\lambda, a)),$$

$$(61) \quad ab \text{ sgn: } H_3((a, b)),$$

the following conditions are satisfied:

Ax 1H. $a, b, c \in H_C$ imply $(a+b)+c=a+(b+c)$.

Ax 2H. There exists $o \in H$ with: $a \in H_C$ imply $a+o=a$.

Ax 3H. $a \in H_C$ implies: there exists $-a \in H_C$ with $a+(-a)=o$.

Ax 4H. $a \in H_C$ implies $1a=a$.

Ax 5H. $\lambda, \mu \in C, a \in H_C$ imply $(\lambda \mu)a=\lambda(\mu a)$.

Ax 6H. $\lambda, \mu \in C, a \in H_C$ imply $(\lambda+\mu)a=\lambda a+\mu a$.

Ax 7H. $\lambda \in C; a, b \in H_C$ imply $\lambda(a+b)=\lambda a+\lambda b$.

Ax 8H. $a, b \in H_C$ imply $ab=ba$.

Ax 9H. $\lambda \in C; a, b \in H_C$ imply $(\lambda a)b=\lambda(ab)$.

Ax 10H. $a, b, c \in H_C$ imply $(a+b)c=ac+bc$.

Ax 11H. $a \in H_C$ implies $0 \leq a^2$.

Ax 12H. $a \in H_C, a^2=0$ imply $a=o$.

0 in Ax 11H, Ax 12H and 1 in Ax 4H denote the zero- and unit-element of C respectively; $a-b \text{ sgn: } a+(-b)$ for all $a, b \in H_C$; \overline{ab} denotes the conjugate number of the complex number ab ; $a^2 \text{ sgn: } aa$ for every $a \in H_C$ (Ax 8H implies $aa=\overline{aa}$, i. e. $a^2 \in R$ for every $a \in H_C$ and this fact justifies the use of inequality sign in Ax 11H).

The question now quite naturally arises: given a Hermitean space H_C , is it possible to define a mapping

$$(62) \quad H_4 : H_C^2 \rightarrow H_C,$$

so that, provided

$$(63) \quad a \times b \text{ sgn} : H_4((a, b)),$$

the following conditions to be satisfied:

Ax 13H. $a, b, c \in H_C$ imply $a \times b \cdot c = a \cdot b \times c$.

Ax 14H. $a, b, c \in H_C$ imply $(a \times b) \times c = (ac) b - (bc) a$.

Ax 15H. There exist $a, b \in H_C$ with $a \times b \neq 0$.

As we shall immediately show (Pr 37H), the answer of this question is negative. Since we shall prove this statement by reductio ad absurdum, let us suppose that there exists a set H_C with the mappings (56)–(58), (62), satisfying the conditions Ax 1H — Ax 15H, provided (59)–(61), (63). Then for H_C will hold mutatis mutandis, i. e. obtained by formal change of F and V_F with C and H_C respectively, propositions, analogous to Pr 1—Pr 32; we shall denote these analogous propositions by Pr 1H—Pr 32H respectively. Indeed, as it is well known, Pr 1H and Pr 2H hold for Hermitean spaces as well as Pr 1 and Pr 2 for Euclidean. As regards Pr 3H—Pr 32H, they are corollaries of Ax 1H—Ax 14H, as it is easily seen by reviewing the corresponding proofs: Ax 8H occurs in the proof of Pr 9H only, and this last statement is obviously true.

Before proving Pr 37H we shall give some auxiliary propositions.

Pr 33H. $\lambda \in C; a, b \in H_C$ imply $a(\lambda b) = \bar{\lambda}(ab)$.

Dm. $a(\lambda b) = (\lambda b)a = \bar{\lambda}(\bar{b}a) = \bar{\lambda}(\bar{b}a) = \bar{\lambda}(ab)$ (Ax 8H, Ax 9H).

Pr 34H. $a, b \in H_C$ imply $a \times (\lambda b) = \bar{\lambda}(a \times b)$.

Dm. $a \times (\lambda b) = -(\lambda b) \times a = -(\lambda(b \times a)) = \bar{\lambda}(-(b \times a)) = \bar{\lambda}(a \times b)$ (Pr 16H,

Pr 29H).

Pr 35H. $\lambda \in C; a, b, c \in H_C$ imply $a \times (\lambda b) \cdot c = \bar{\lambda}(a \times b \cdot c)$,

Dm. $a \times (\lambda b) \cdot c = (\lambda(a \times b)) \cdot c = \bar{\lambda}(a \times b \cdot c)$. (Pr 34H, Pr 9H).

Pr 36H. $\lambda \in C; a, b, c \in H_C$ imply $a \times (\lambda b) \cdot c = \bar{\lambda}(a \times b \cdot c)$.

Dm. $a \times (\lambda b) \cdot c = a \cdot (\lambda b) \times c = a \cdot (\lambda(b \times c)) = \bar{\lambda}(a \cdot b \times c) = \bar{\lambda}(a \times b \cdot c)$

(Ax 13H, Pr 34H, Pr 33H).

Pr 37H. There exists no set H_C with the mappings (56)–(58), (62) satisfying the conditions Ax 1H — Ax 15H, provided (59)–(61), (63).

Dm. Otherwise let $\lambda \in C; a, b, c \in H_C$ with (41) (Ax 15H, Pr 25H, Pr 24H). Pr 35H and Pr 36H imply

$$(64) \quad \lambda(a \times b \cdot c) = \bar{\lambda}(a \times b \cdot c)$$

(64) and (41) imply $\lambda = \bar{\lambda}$, i. e. $C = R$. This accomplishes the proof.

Let us note at last, that if Π_C is a linear space over the field C of the complex numbers, for which scalar multiplication is defined, satisfying the conditions Ax 8 — Ax 10 (with C and Π_C instead of F and V_F relatively), cf. [2] then vector multiplication in Π_C may be defined, satisfying the conditions Ax 13—Ax 18 (with C and Π_C instead of F and V_F relatively), as it is easily seen by constructing a model of Π_C in the set C^3 of all triples (z_1, z_2, z_3) of complex numbers. In this model by definition

$$(65) \quad (z_1, z_2, z_3) + (\zeta_1, \zeta_2, \zeta_3) \text{ sgn: } (z_1 + \zeta_1, z_2 + \zeta_2, z_3 + \zeta_3),$$

$$(66) \quad \lambda (z_1, z_2, z_3) \text{ sgn: } (\lambda z_1, \lambda z_2, \lambda z_3) \quad (\lambda \in C).$$

$$(67) \quad (z_1, z_2, z_3) (\zeta_1, \zeta_2, \zeta_3) \text{ sgn: } \sum_{v=1}^3 z_v \zeta_v,$$

$$(68) \quad (z_1, z_2, z_3) (\zeta_1, \zeta_2, \zeta_3) \text{ sgn: } (z_2 \zeta_3 - z_3 \zeta_2, z_3 \zeta_1 - z_1 \zeta_3, z_1 \zeta_2 - z_2 \zeta_1)$$

and all the axioms Ax 1—Ax 10, Ax 13—Ax 18 (with C and Π_C instead of F and V_F respectively) are satisfied. For Π_C however do not hold propositions, obtained mutatis mutandis from Pr 17, Pr 30 and Pr 32, since the analogues of Pr 1 and Pr 2 are not true for Π_C , as well as other corollaries of Ax 11 and Ax 12.

LITERATURE

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ВЪРХУ ДЕФИНИЦИЯТА НА ВЕКТОРНОТО УМНОЖЕНИЕ

Ив. Христова, Ив. Чобанов

(РЕЗЮМЕ)

Както е известно, тримерно векторно пространство над нареденото поле F се нарича множество V_F , за което са дефинирани изображения (1)—(3) и (7), такива, че при (4)—(6) и (8) са изпълнени условията Ax 1—Ax 18. В настоящата бележка се показва, че традиционните аксиоми Ax 16—Ax 18, които за векторното умножение са аналоги на съответните аксиоми Ax 8—Ax 10 за скаларното умножение, в действителност се оказват следствия от аксиомите Ax 1—Ax 14; затова условията Ax 1—Ax 15 са достатъчни за дефиниране на тримерно векторно пространство над нареденото поле.