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EVOLUTION OF PERIODIC WAVES IN DISPERSIVE MEDIUM

OGNIAN KAMENOV

Огнян Каменов. ЭВОЛЮЦИЯ ПЕРИОДИЧЕСКИХ ВОЛН В ДИСПЕРСНОЙ СРЕДЕ

В настоящей работе найдены пять классов решений обобщеного уравнения Кортевега – де Фриза: бипериодические, солитонные, типа уединеной волны, рациональные и волны Стокса. Анализировано условие их реализаций, а также эффекты влияния дисперсной среды на вольновые параметры.

Ognian Kamenov. EVOLUTION OF PERIODIC WAVES IN DISPERSIVE MEDIUM

In the present paper five classes of wave solutions of a generalized Kortveg – de Vries equation are found analytically: biperiodical, solitary, solitone, rational and Stokes' waves. The conditions for their realization are analyzed as well as the impact of the dispersive medium on the wave parameters.

1. INTRODUCTION

The propagation of one-dimensional waves with finite amplitude in dispersive medium can be modelled by the generalized equation of Kortveg – de Vries (K–dV):

$$u_t + \frac{3}{2}uu_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0, \tag{1}$$

as it is shown by Kakutani and Ono in [6]. Here the parameters α and β characterize the dispersion of the medium and can take both positive and negative values. Actually, (1) describes the evolution of waves for which the angle between their front and the gradient of the external field tends to the critical angle

$\varphi_c = \operatorname{arc\,tg}\left(\sqrt{m_1/m_0} - \sqrt{m_0/m_1}\right)$, where m_1 and m_0 are the masses of ion and electron, respectively. In this case the coefficient in front of the third derivative in the classic K–dV equation decreases, causing debalance between the dispersive effects and the non-linear ones. The fifth derivative βu_{xxxxx} compensates for this debalance.

Kawahara has shown numerically in [7] the existence of oscillatory and monotonous-solitary wave solutions of (1). He has found out the existence of two types of solitary waves — compressive and rarefactive ones, corresponding to the negative dispersion ($\alpha > 0$, $\beta > 0$) and to the positive one ($\alpha < 0$, $\beta < 0$). Kano and Nakayama [6] have found an unbounded biperiodical solution, depending on the wave phase velocity.

In the present paper analytical and bounded biperiodic solutions have been found. By them in the boundary case of degeneration of the elliptical Jacobbi's function another two classes of wave solutions can be determined — solitary and rational solutions. This fact confirms the unique property of the non-linear periodic waves, called by many authors ([8, 5, 15]) non-linear principle of superposition. Under specified conditions of the dispersive medium solitone waves could be also realized, but not as a consequence of the solitary ones.

2. PERIODIC WAVE SOLUTIONS

Without a loss of generality we shall suppose that because otherwise after the transformations $u \to -u$, $x \to -x$, $t \to t$ we shall obtain an equation analogous to the eq. (1) in which $\alpha \to -\alpha$, $\beta \to -\beta$. Before reducing (1), let introduce the following transformations:

$$x \to x/\sqrt[4]{\beta}, \quad t \to t/\sqrt[4]{\beta}.$$

We look for a stationary solution in the form

$$u(x,t) = \zeta(z), \tag{2}$$

where

$$z = k(x - vt) + C_1, \tag{3}$$

as a result of which (1) is reduced to the non-linear equation

$$\frac{3}{4}\zeta^2(z) - v\zeta(z) + C = k^4\frac{d^4\zeta}{dz^4} - \left(\alpha k^2/\sqrt{\beta}\right)\frac{d^2\zeta}{dz^2}. \tag{4}$$

The constant C (in the general case, it is possible C to be complex) characterizes the phase shift, k is the wave number ($k > 0$) and v is the phase velocity of the stationary waves with phase z given by (3). Although the integration constant C in (4) has no dynamic meaning, it will be shown further that $C \neq 0$ for periodic wave solutions, that is opposite to the boundary conditions of [7] imposing $C = 0$. We look for a particular solution of (4) in the form

$$\zeta(z) = 6A\wp^2(z, G_2G_3) + B\wp(z, G_2, G_3) - AG_2/2, \tag{5}$$

where $\wp(z, G_2, G_3)$ is the biperiodic elliptical function of Weierstrasse with real invariants G_2 and G_3, and the constants A and B are unknown for the present. If we replace (5) in (2) and make use of the fact that

$$\wp^{(2n)}(z, G_2, G_3) = \sum_{j=0}^{n+1} b_{n-j}(G_2, G_3)\wp^j(z, G_2, G_3), \quad n = 1, 2, \ldots$$

(the concrete derivatives are given in Appendix A), then the left-hand and the right-hand sides of (2) become forth-degree polynomials of $\wp$. The condition for identity of these polynomials, i.e. the equality of the coefficients in front of the corresponding powers, is actually a condition for the unknown constants A, B and also for the unknown invariants G_2 and G_3. The system becomes:

$$A(3A - 560k^4) = 0,$$

$$3A\left(3B + 40\alpha k^2/\sqrt{\beta}\right) = 0, \tag{6}$$

$$v = 720AG_3k^4/B + 6\alpha k^2\beta^{-1/2}(1 - 3AG_2/B) - 3AG_2/4, \tag{7}$$

$$v = 6k^4(28G_2 - B/A) + B^2/(8A) - 3AG_2/4, \tag{8}$$

$$C + 3A^2G_2/16 - AvG_2/2 = 3k^4G_2(3AG_2 - B) + \alpha k^2\beta^{-1/2}(12AG_3 + BG_2/2). \tag{9}$$

It is easy to obtain a non-trivial ($A \neq 0$, $B \neq 0$) family of solutions, depending on the parameters

$$A = 560k^4/3, \quad B = -40\alpha k^2/(3\sqrt{\beta}), \tag{10}$$

$$G_2 = 40\varepsilon, \quad G_3 = \frac{\gamma}{k^2}(\varepsilon + \mu_1\gamma - \mu_0\gamma^2), \tag{11}$$

$$v = 160k^4\varepsilon + 3k^2\gamma/7 + 5\gamma^2/42, \tag{12}$$

$$\begin{aligned} C = {} & 40\varepsilon[20k^4\gamma(4k^2 - 1/3) + 50k^4\gamma^2/3 - 582400k^8\varepsilon/3] \\ & + \gamma^2(2240k^2\varepsilon + 26k^2\gamma/7 - 5\gamma^2/63) \end{aligned} \tag{13}$$

(we have denoted for short $\gamma = \alpha/\sqrt{\beta}$, $\mu_0 = 1/28224$, $\mu_1 = 13/7840$), by means of which we obtain according to (5) the solution of (1) in the form

$$\begin{aligned} u(x,t) = {} & 1120k^4\wp^2(k(x - vt) + C_1, G_2, G_3) \\ & - 40k^2\gamma/3\wp(k(x - vt) + C_1, G_2, G_3) - 11200k^4\varepsilon/3. \end{aligned} \tag{14}$$

The solution obtained is a biperiodical elliptical function that is not suitable for use and practical analysis in this form because of the fact that the Weierstrasse's function has multiple poles on the real axis in the complex space $z + C_1$. For these peculiar points to be isolated from the solution, we have to choose the complex constant C_1 so that to have the poles translated half a period in the positive direction of the imaginary axis. More precisely, this procedure can he done as follows. We use the designations

$$\omega_1 = \frac{2}{\sqrt{e_1 - e_3}} \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - M^2 \sin^2 \xi}}, \quad \omega_2 = \frac{2i}{\sqrt{e_1 - e_3}} \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - M^{-2} \sin^2 \xi}} \tag{15}$$

for the primitive periods (for which is fulfilled $\mathrm{Im}(\omega_1/\omega_2) > 0$) of the function $\wp$, M is the modulo of the standard elliptical integral of first kind, i.e.

$$0 \leq \frac{e_2 - e_3}{e_1 - e_3} = M^2 \leq 1, \tag{16}$$

and the real numbers $e_1 > e_2 > e_3$ are the roots of the cubic equation

$$4Y^3 - G_2 Y - G_3 = 0. \tag{17}$$

A sufficient condition for the fulfilment of the above condition is

$$G_2^3 > 27G_3^2. \tag{18}$$

Now let the phase shift be half a period in the positive direction of the imaginary axis, i.e.

$$C_1 = C_0(e_1 - e_3)^{-1/2} + \omega_2/2, \tag{19}$$

where $C_0 = \text{const}$, $C_0 \in \mathbb{R}$. After taking into account the transformations

$$\begin{aligned}
&\wp(k(x - vt) + C_1, G_2, G_3) \\
&\qquad = e_3 + (e_1 - e_3)\mathrm{sn}^{-2}\left(k(x - vt)\sqrt{e_1 - e_3} + C_1\sqrt{e_1 - e_3}, G_2, G_3\right) \\
&\qquad = e_3 + (e_2 - e_3)\mathrm{sn}^2\left(C_0 + k(x - vt)\sqrt{e_1 - e_3}, M\right),
\end{aligned}$$

we obtain from (14) a boundary periodic solution, having no poles on the real axis, of the form

$$\begin{aligned}
u(x,t) = {} & 1120k^4(e_2 - e_3)^2\mathrm{cn}^4\left(k(x - vt)\sqrt{e_1 - e_3} + C_0, M\right) \\
& - 40k^2(e_2 - e_3)(56k^2 e_2 - \gamma/3)\mathrm{cn}^2\left(k(x - vt)\sqrt{e_1 - e_3} + C_0; M\right) \\
& + 40k^2(28k^2 e_2^2 - e_2\gamma/3 - 280k^2\varepsilon/3.
\end{aligned} \tag{20}$$

The period of the cnoidal waves (20) is $2K/\left(k\sqrt{e_1 - e_3}\right)$ with x, because the arguments of cn are real and K is the standard elliptical integral of Legendre with modulo M. Actually, $u(x,t)$ is a superposition of two impulses with different amplitudes proportional to e_2 and e_3. A typical peculiarity of the periodic solution (20) obtained here is the dependence of the amplitude and the phase shift not only on the wave number, but also on the dispersion parameters α and β. It is easy to realize from (12) that these periodic waves can propagate in two directions, i.e. on fixed α and $\beta > 0$, the freedom of choosing ε allows to obtain $v > 0$ — a propagation in the positive direction of Ox or $v < 0$ — a propagation in the negative direction of $-Ox$.

3. SOLITARY WAVES

Toda [8] has shown first that the cnoidal wave in the K–dV equation could be presented as a superposition of repeated solitary-wave profiles. Later on, a number of authors ([5, 8, 15]) have ascertained this unique property, called a *non-linear principle of superposition*, for a considerable part of the non-linear evolutionary equations. Parker [15] has made the most detailed analysis of the indicated non-linear principle for the equation ILW, revealing a more profound essence of the phenomenon — that really we do not have one superposition (in the common meaning of the linear theory) of solitary waves of the type sech^2, but we have a superposition of their forms, due to the different velocities of propagation of the periodic and the solitary waves. In the present section we try to find a confirmation of the non-linear principle of superposition for the periodic wave packs in dispersive medium too. For this purpose we investigate in details the solution (20) for the right boundary value of the modulo M, i.e. when $M = 1$. In this case the function $\mathrm{cn}(z, 1)$ becomes degenerated according to the relation

$$\mathrm{cn}(z,1) = \mathrm{sech}(z). \tag{21}$$

The condition $M = 1$, according to (16) and (17), means that

$$e_1 = e_2 \neq e_3. \tag{22}$$

The last condition is possible only if the following three relations are satisfied at the same time:

$$G_2^3 = 27G_3^2, \quad G_2 > 0, \quad G_3 < 0, \tag{23}$$

or if we have in mind (11), (15) and the condition $\beta > 0$:

$$\varepsilon^3 - (\mu_2\gamma^2/k^4)\varepsilon^2 - 2(\mu_2\gamma^3/k^5)(\mu_1 k - \mu_0\gamma)\varepsilon - (\mu_2\gamma^4/k^6)(\mu_1 k - \mu_0\gamma)^2 = 0, \tag{24}$$

$$\varepsilon > 0, \tag{25}$$

$$(\varepsilon\gamma + \mu_1\gamma^2)k^2 < \mu_0\gamma^3, \tag{26}$$

where we have denoted $\mu_2 = (3/40)^3$ and $\gamma = \alpha/\sqrt{\beta}$. It is clear that the cubic equation (24) with respect to ε has for every $k > 0$ and every $\gamma \in \mathbb{R}$ at least one positive root that can be determined using the Cardano's formula

$$\begin{aligned}\varepsilon_0(k,\gamma) = &\sqrt[3]{-Q(k,\gamma)/2 + \sqrt{D(k,\gamma)}} \\ &+ \sqrt[3]{-Q(k,\gamma)/2 - \sqrt{D(k,\gamma)}} + \mu_2\gamma^2/3k^4,\end{aligned} \tag{27}$$

where $Q(k,\gamma)$ and $D(k,\gamma)$ are given in Appendix B. On specified dispersion parameters α and β (i.e. γ is given) those real values of k, satisfying at the same time the inequalities $k > 0$ and $[\varepsilon_0(k,\gamma)\gamma + \mu_1\gamma^2]k^2 < \mu_0\gamma^3$, determine the permissible wave numbers $k_0 > 0$ for the type of waves being considered here, i.e. the solitary ones. The intervals will depend on the introduced dispersion parameter γ. In other words, if we determine $\varepsilon(k,\gamma) > 0$ through (27) and if $k_0 > 0$ satisfies (26), then the roots e_j, $j = 1, 2, 3$, of (18) could be determined analytically and they have specific values

$$e_1 = e_2 = (-G_3)^{1/3}/2 = \sqrt{\mu_3\varepsilon(k,\gamma)}, \quad e_3 = -2\sqrt{\mu_3\varepsilon_0(k,\gamma)}, \quad \mu_3 = 10/3. \tag{28}$$

The solitary solution of (1) takes the following form:

$$\begin{aligned}u(x,t) = &\,33600k_0^4\varepsilon_0\,\mathrm{sech}^4\left(k_0\sqrt[4]{30\varepsilon_0}(x - v_0t) + C_0\right) \\ &- 40k_0^2\sqrt{30\varepsilon_0}\left(56k_0^2\sqrt{\mu_3\varepsilon_0} - \gamma/3\right)\mathrm{sech}^2\left(k_0\sqrt{30\varepsilon_0}(x - v_0t) + C_0\right) \\ &- 40k_0^2\gamma\sqrt{\mu_3\varepsilon_0}/3.\end{aligned} \tag{29}$$

We have denoted by v_0 the phase velocity that in the case of solitary waves takes the value

$$v_0 = 160k_0^4\varepsilon_0 + 3k_0^2\gamma/7 + 5\gamma^2/42. \tag{30}$$

The solitary solution obtained in (29) confirms the non-linear principle of superposition for the generalized dispersion equation (1) too, i.e. the periodic waves from (20) are a superposition of solitary forms , propagating in general with different phase velocity v_0 from the one of the periodic waves.

4. SOLITONE WAVES

The solitary-wave solution (29) allows another interesting class of non-periodic waves to be analyzed, i.e. the solitone ones. A necessary condition for this is $u(x,t) \to 0$ when $k_0(x - v_0t) \to \pm\infty$, where $u(x,t)$ is given by (29).

We shall show that the generalized dispersion equation (1) does not have a solitone solution when $\beta > 0$ (which means that when $\beta < 0$ the same is valid too). In fact, the free term in (29) becomes zero when $k_0 > 0$ in two cases: when $\gamma = 0$ (i.e. when $\alpha = 0$) or when $\varepsilon_0 = 0$. In the first case, according to (11), $G_3 = 0$, that makes the fulfilment of (23) impossible and this is the reason for the impossibility the solitary waves (29) to be realized. If we suppose $\varepsilon_0 = 0$, that is permissible according to (24), the solution (29) is identically zero.

The only case when solitone waves can be generated in (1) is the one when $\alpha \neq 0$, $\beta = 0$, i.e. when (1) takes the form

$$u_t + \frac{3}{4}uu_x + \alpha u_{xxx} = 0, \tag{31}$$

which differs from the classic K–dV equation only in the coefficients in front of the non-linear and the dispersion terms. After integrating twice and making the integration constants equal to zero, we obtain the one-solitone solution in the form

$$u(x,t) = C_0 \operatorname{sech}^2\left[\sqrt{C_0/16\alpha}(x - C_0t/4)\right], \tag{32}$$

where C_0 is a real constant for which $\operatorname{sign} C_0 = \operatorname{sign}\alpha$.

A common feature of the non-linear waves is revealed in this solitone solution, namely a dependence between the amplitude and the phase velocity. For the case being considered the phase velocity does not depend on the dispersion parameter α, but the wave number $k = \sqrt{C_0/16\alpha}$ depends considerably on it, that emphasizes the rôle of the dispersive medium for the solitone waves propagating in it. Another special feature of the one-solitone waves is also evident — the impulses having a greater amplitude propagate faster, but they are narrower.

5. RATIONAL SOLUTIONS

The rational solutions of the non-linear evolutional equations belong to the particular solutions, having poles on the real axis. In the general case these are complex or singular solutions (or both). For the classic K–dV equation the rational solution can be obtained in the form of a long-wave approximation to the one-solitone solution. In this section we shall show that the rational solution of (1) can be obtained as a boundary case of the solitary solution (20), namely the boundary case $M = 0$.

The last is realized when

$$e_2 = e_3 \neq e_1 \tag{33}$$

or when the following relations are satisfied at the same time:

$$G_2^3 = 27G_3^2, \quad G_2 > 0, \quad G_3 > 0. \tag{34}$$

The first two conditions coincide with (24) and (25), and the inequality $G_3 > 0$ is satisfied for those positive values $k_1 > 0$, for which the following relation is fulfilled: $[\varepsilon_0(k_1,\gamma)\gamma + \mu_1\gamma^2]k_1^2 > \mu_0\gamma^3$. For these values of $\varepsilon_0(k_1,\gamma) = \varepsilon_1$ and $k_1 > 0$ we obtain from (34) and (17) (under the condition of (33))

$$e_2 = e_3 = -m, \quad e_1 = 2m, \quad m = (G_2/12)^{1/2} = (10\varepsilon_1)^{1/2}/3. \tag{35}$$

In this case the function $\wp(z + C, G_2, G_3)$ degenerates according to the formula

$$\wp(z + C, 12m^2, 8m^3) = -m + 3m\sin^{-2}\left[(3m)^{1/2}(z + C)\right]$$

and then we get from (14) the irregular solution for $u(x,t)$

$$\begin{aligned}(36)\ u(x,t) = &\sqrt{10\varepsilon_1}[\sin^{-2}(k_1(x - v_1t) + C_1) - 1/3] \\ &\times \left\{1120k_1^4\sqrt{10\varepsilon_1}\sin^{-2}(k_1(x - v_1t) + C_1) - k_1^2(1120k_1^2\sqrt{10\varepsilon_1}/3 + 40\gamma/3)\right\} \\ &- 11200k_1^4\varepsilon_1/3.\end{aligned}$$

The usual form of the rational solution can be obtained from (36) by means of the series

$$\sin^{-2}\theta = \theta^{-2} + \sum_{n=-\infty}^{\infty}(\theta - n\pi)^{-2}$$

and then

$$\begin{aligned}(37)\quad u(x,t) = &\sqrt{10\varepsilon_1}\left\{\sum_{n=-\infty}^{\infty}(k_1x - k_1v_1t + C_1 - n\pi)^{-2}\right. \\ &\left.+ [k_1(x - v_1t) + C_1]^{-2} - 1/3\right\}1120k_1^4\sqrt{10\varepsilon_1}\left\{\sum_{n=-\infty}^{\infty}[k_1(x - v_1t) + C_1 - n\pi]^{-2}\right. \\ &\left.+ [k_1(x - v_1t) + C_1]^{-2} - k_1^2\left(1120k_1^2\sqrt{10\varepsilon_1}/3 + 40\gamma/3\right)\right\} - 11200k_1^4\varepsilon_1/3.\end{aligned}$$

The rational solution obtained has infinite number two-fold poles that can not be isolated, because of which they have to be excluded from the definite area, namely

$$(n\pi - C_1)/k_1 < (x - v_1t < [(n + 1)\pi - C_1]/k_1,$$

where $C_1 \in \mathbb{R}$ characterizes the phase shift. The presence of rational solutions is an unusual property of some evolutionary equations, especialy when they have infinite number of poles (as it is in (37)). In these cases we have a transformation of the time-dependent evolution into evolution of a dynamic system with finite number degrees of freedom.

6. ANOTHER PERIODIC SOLUTIONS

The Stokes' investigations [12] in 1847 laid the begining of the non-linear dispergating waves. The fundamental results of his analyses are two — there exist periodic wave packs in the non-linear systems and the amplitude is present in the dispersion relations. Actually, the simplicity of the procedure for determining Stokes' waves in non-linear evolutionary equations is due to its initial purpose, namely, the determination of a better approximation to the linear wave pack. In his fundamental work Boyd [3] has used the Stokes' procedure for determining an approximative solution of the FKVD equation (of order $O(\eta^6)$).

In this section we analyze the possibility for evolution of namely such Stokes' waves in the generalized dispersion eq. (1). Let η be a positive number, specifying the small wave amplitudes in a dispersive medium. We suppose that $0 < \eta \ll 1$, and if we ignore the non-linear term of (1), it is easy to get

$$u_1(kx - vt) = \eta \cos[kx - (\alpha k^3 + \beta k^5)t] \tag{38}$$

that represents a solution of the linear problem. In the case of small amplitudes η this "linear" solution could be improved by the asymptotic development (to a given in advance order $s \geq 2$) of $u(x,t)$ and the phase velocity with the small amplitude

$$u(x,t) = \eta u_1(\theta) + \eta^2 u_2(\theta) + \cdots + \eta^s u_s(\theta), \quad \theta = kx - vt, \tag{39}$$

$$v(\eta) = v_0 + \eta v_1 + \eta^2 v_2 + \cdots + \eta^{s-1} v_{s-1}. \tag{40}$$

Substituting (39) and (40) in (1) and making equal the functions and expressions in front of the identical powers of η, we obtain a recurent system of s ordinary differential equations for determining $u_n(\theta)$, $n = 1, 2, \ldots, s$; $v_j(\eta)$, $j = 0, 1, \ldots, s-1$:

$$\beta k^5 u_n^{(5)} - \alpha k^3 u_n^{(3)} + v_0 u_n' = \varphi_n(\theta), \quad n = 1, 2, \ldots, s, \tag{41}$$

where $\varphi_n(\theta)$ are known for every $n = 1, 2, \ldots, s$:

$$\varphi_n(\theta) = \sum_{j=1}^{n-1} \left\{ -v_{n-j} u_j'(\theta) + (3k/8)[u_j(\theta) u_{s-j}(\theta)]' \right\}. \tag{42}$$

It is not difficult to prove by induction that the solution of the system (41) represents a finite Fourier series

$$u_n(\theta) = \sum_{j=1}^{n} A_{nj} \cos(j\theta), \quad n = 1, 2, \ldots, s, \tag{43}$$

where A_{nj} are constants. The periodicity of the solution (39) with period 2π is evident. The elimination of the resonant terms for each separate equation is a delicate moment in solving the system (41). This can be realized thanks to the fact that for the j-th series the unknown function is $u_j(\theta)$, v_{j-1}. The last have to be defined properly, so to have the coefficient in front of the secular term equal to zero for every $j = 2, 3, \ldots, s$. Here are the first several terms of the asymptotic solution of (1) according to (39) and (43):

$$(44) \quad u(x,t,\eta) = \eta\cos\theta + \eta^2\left[\frac{3k\cos(2\theta)}{4f(2)}\right] + \eta^3\left[\frac{27k^2\cos(3\theta)}{32f(3)f(2)}\right] + \eta^4\frac{27k^3}{64f(2)}\left[\frac{\cos(4\theta)}{f(4)}\left(1+\frac{3}{f(3)}\right)\frac{\cos(2\theta)}{f(2)}\left(\frac{2}{f(2)}-\frac{3}{f(3)}\right)\right] + \cdots,$$

and the phase velocity v is of the form

$$(45) \qquad v(\eta) = (\beta k^5 + \alpha k^3) - \eta^2\left[\frac{9k^2}{16f(2)}\right] + O.\eta^3 + \cdots,$$

where $f(n) = \beta k^5(n^5 - n) - \alpha k^3(n^3 - n)$, $n = 1, 2, \ldots, s$.

The 2π-periodic solution obtained in (44) and (45) shows also the dependence of the waves amplitudes on the dispersion parameters α and β. These waves just like the biperiodic ones obtained in (20) could be in two directions. Really, when the amplitudes are small ($0 < \eta \ll 1$), the sign of v is determined by $v_0 = \beta k^5 + \alpha k^3$. If the dispersion of the medium is positive ($\alpha < 0$, $\beta < 0$), then $v < 0$, and hence the waves will propagate in the negative direction of the real axis. When the dispersion is negative ($\alpha > 0$, $\beta > 0$), then within $O(\eta)$ the sign of v is positive and the wave propagation is in the positive direction of the real axis. Concluding, we will mention that the solution given in the form (39) could be interpreted as a finite Fourier series for a periodic wave pack.

7. CONCLUDING REMARKS

Finally, we shall comment in brief the conditions under which the analytically obtained different classes of solutions of the dispersion eq. (1) have been realized. Apparently, the boundary periodic waves, obtained in (20), can be generated in an arbitrary chosen α (we suppose $\beta > 0$ according to Section 2), i.e. both for the negative dispersion ($\alpha > 0$, $\beta > 0$) and for the mixed type ($\alpha < 0$, $\beta < 0$), when $\varepsilon > 0$ is chosen properly, so that the inequality (18) to be fulfilled. Periodic waves with bounded amplitudes can be generated in the appropriate dispersive medium. Their phase velocity will depend both on the dispersive medium (i.e. on $\gamma = \alpha/\sqrt{\beta}$) and on the choice of $\varepsilon > 0$, and the wave number according to (12) and the amplitudes will vary proportionally to e_2 and e_3.

Obviously, the solitary waves, given by (29), are possible to exist on considerably more bounded conditions. If $k > 0$ is arbitrary, then a restriction for γ is given by the inequality (26), where ε is an whichever positive root of (24) and, vice versa, if γ is arbitrary, then the wave number is restricted by (26), i.e. in return for the lengths of the solitary waves. The phase velacity of the solitary waves differs from the one of the periodic waves, as it was shown in Section 3.

In Section 4 it was ascertained that solitone waves can develop only in dispersive medium for which $\beta = 0$ and $\alpha \neq 0$ is arbitrary.

As a second boundary case of the solitary waves is the localized rational solution determined in Section 5, having movable two-fold peculiarities.

If we suppose small amplitudes of the waves evolving according to (1), then the periodic waves could be considered as a finite Fourier cosine series that represents in fact Stokes' waves.

APPENDIX A

$$\zeta(z) = 6A\wp^2(z, G_2, G_3) + B\wp'(z, G_2, G_3) - AG_2/2,$$

$$\zeta^2(z) = 36A^2\wp^4(z, G_2, G_3) + 12AB\wp^3(z, G_2, G_3) + (B^2 - 6A^2G_2)\wp^2(z, G_2, G_3) - ABG_2\wp(z, G_2, G_3) + A^2G_2^2/4,$$

$$\frac{d^2\zeta}{dz^2} = 120A\wp^3(z, G_2, G_3) + (6B - 18AG_2)\wp(z, G_2, G_3) - (12AG_3 + BG_2/2),$$

$$\frac{d^4\zeta}{dz^4} = 5040A\wp^4(z, G_2, G_3) + 36(B - 28AG_2)\wp^2(z, G_2, G_3) - 720AG_3\wp(z, G_2, G_3) - 3G_2(B - 3AG_2).$$

APPENDIX B

$$Q(k, \gamma) = \frac{\mu_2\gamma^4}{27k^{12}}[k^3(\mu_1 k - \mu_0\gamma)(18\mu_2\gamma - \mu_1 k^4 + \mu_0\gamma k^3) - 2\mu_2^2\gamma^2],$$

$$D(k, \gamma) = Q^2/4 + (\mu_2^2\gamma^4/3k^8)^3.$$

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