COMPUTATION OF THE VERTEX FOLKMAN NUMBERS $F(2, 2, 2, 3; 5)$ AND $F(2, 3, 3; 5)$

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In this note we show that the exact value of the vertex Folkman numbers $F(2, 2, 2, 3; 5)$ and $F(2, 3, 3; 5)$ is 12.

**Keywords:** vertex Folkman graph, vertex Folkman number

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1. NOTATIONS

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph $G$ will be denoted by $V(G)$ and $E(G)$, respectively. We call a $p$-clique of $G$ a set of $p$ vertices, each two of which are adjacent. The biggest natural number $p$ such that the graph $G$ contains a $p$-clique is denoted by $cl(G)$ (the clique number of $G$).

If $W \subseteq V(G)$, then: $G[W]$ is the subgraph of $G$ induced by $W$ and $G - W$ is the subgraph of $G$ induced by $V(G) \setminus W$. We shall use also the following notations:

- $\overline{G}$ — the complement of the graph $G$;
- $\alpha(G)$ — the independence number of $G$;
- $N_G(v), v \in V(G)$ — the set of all vertices of $G$ adjacent to $v$;
- $K_n$ — the complete graph of $n$ vertices;
- $C_n$ — the simple cycle of $n$ vertices;
- $\chi(G)$ — the chromatic number of $G$.

Let $G_1$ and $G_2$ be two graphs without common vertices. We denote by $G_1 + G_2$ the graph $G$ for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y], x \in V(G_1), y \in V(G_2)\}$. 
The Ramsey number $R(p, q)$ is the smallest natural $n$ such that for an arbitrary $n$-vertex graph $G$ either $\alpha(G) \geq p$ or $cl(G) \geq q$. We need the equality $R(3, 3) = 6$, [3].

2. VERTEX FOLKMAN NUMBERS AND THE MAIN RESULT

Definition 2.1. Let $G$ be a graph, $a_1, \ldots, a_r$ be positive integers and let

$$V(G) = V_1 \cup \ldots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,$$

be an $r$-coloring of the vertices of $G$. This coloring is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the graph $G$ does not contain a monochromatic $a_i$-clique of color $i$. The symbol $G \rightarrow (a_1, \ldots, a_r)$ means that every $r$-coloring of $V(G)$ is not $(a_1, \ldots, a_r)$-free.

The graph $G$ such that $G \rightarrow (a_1, \ldots, a_r)$ is called a vertex Folkman graph. We put

$$F(a_1, \ldots, a_r; q) = \min \{|V(G)| : G \rightarrow (a_1, \ldots, a_r) \text{ and } cl(G) < q\}.$$

It is clear that from $G \rightarrow (a_1, \ldots, a_r)$ it follows that $cl(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman, [2], proves that there exists a graph $G$ such that $G \rightarrow (a_1, \ldots, a_r)$ and $cl(G) = \max\{a_1, \ldots, a_r\}$. Therefore, if $q > \max\{a_1, \ldots, a_r\}$, then the numbers $F(a_1, \ldots, a_r; q)$ exist. Those numbers are called vertex Folkman numbers.

Let $a_1, \ldots, a_r$ be positive integers. We put

$$m = \sum_{i=1}^{r} (a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \ldots, a_r\}.$$  \hfill (1)

Obviously, $K_m \rightarrow (a_1, \ldots, a_r)$ and $K_{m-1} \rightarrow (a_1, \ldots, a_r)$. Hence, if $q \geq m + 1$, then $F(a_1, \ldots, a_r; q) = m$. The numbers $F(a_1, \ldots, a_r; m)$ exist only if $m \geq p + 1$.

For those numbers the following is known:

Theorem A ([4]). Let $a_1, \ldots, a_r$ be positive integers and let $m$ and $p$ satisfy (1), where $m \geq p + 1$. Then $F(a_1, \ldots, a_r; m) = m + p$. If $G \rightarrow (a_1, \ldots, a_r)$, $cl(G) < m$ and $|V(G)| = m + p$, then $G = K_{m-p-1} + C_{2p+1}$.

Remark. The proof of Theorem A, given in [4], is based on [4, Lemma 1, p. 251]. But the proof of this lemma is not correct, because the sentence "If we delete both endpoints of any of its edges adjacent to $x, y$", then $\alpha(G)$ decreases again." is not true (see p.252).

A correct proof of Theorem A is given in [13] (see also p.66, Theorem 7.4 in this volume).

The numbers $F(a_1, \ldots, a_r; m - 1)$ exist only if $m \geq p + 2$. For those numbers the following is known:

Theorem B ([13]). Let $a_1, \ldots, a_r$ be positive integers. Let $m$ and $p$ satisfy (1), where $m \geq p + 2$. Then $F(a_1, \ldots, a_r; m - 1) \geq m + p + 2$.  

72
Theorem C ([14]). Let \( a_1, \ldots, a_r \) be positive integers and let \( m \) and \( p \) satisfy (1). Let \( m \geq p + 2 \), \( G \to (a_1, \ldots, a_r) \) and \( \text{cl}(G) < m - 1 \). Then:

(a) \(|V(G)| \geq m + p + \alpha(G) - 1|; \\
(b) \text{if } |V(G)| = m + p + \alpha(G) - 1, \text{ then } |V(G)| \geq m + 3p. \\

It is clear that for each permutation \( \varphi \) of the symmetric group \( S_r \)

\[ G \to (a_1, \ldots, a_r) \iff G \to (a_{\varphi(1)}, \ldots, a_{\varphi(r)}). \]

Note that if \( a_1 = 1 \), then \( F(a_1, \ldots, a_r; q) = F(a_2, \ldots, a_r; q) \). Therefore, we can assume that \( 2 \leq a_1 \leq \cdots \leq a_r \).

The next theorem implies that, in the special situation \( a_1 = \cdots = a_r = 2 \), \( r \geq 5 \), the inequality from Theorem B is exact.

Theorem D.

\[ F(2, \ldots, 2, r) = \begin{cases} 11, & r = 3 \text{ or } r = 4; \\ r + 5, & r \geq 5. \end{cases} \]

It is clear that \( G \to (2, \ldots, 2) \iff \chi(G) \geq r + 1. \)

Mycielski in [5] presents an 11-vertex graph \( G \) such that \( G \to (2, 2, 2) \) and \( \text{cl}(G) = 2 \), proving that \( F(2, 2, 2; 3) \leq 11 \). Chvátal, [1], proves that Mycielski graph is the smallest such graph and hence \( F(2, 2, 2; 3) = 11 \). The inequality \( F(2, 2, 2, 2; 4) \geq 11 \) is proved in [8] and inequality \( F(2, 2, 2, 2; 4) \leq 11 \) is proved in [7] and [12] (see also [9]). The equality \( F(2, \ldots, 2; r) = r + 5, r \geq 5 \), is proved in [7], [12] and later in [4]. Only few other numbers of the type \( F(a_1, \ldots, a_r; m - 1) \) are known, namely: \( F(3, 3; 4) = 14 \) (the inequality \( F(3, 3; 4) \leq 14 \) is proved in [6] and the opposite inequality \( F(3, 3; 4) \geq 14 \) is verified by means of computers in [18]); \( F(3, 4; 5) = 13, [10]; F(2, 2, 4; 5) = 13, [11]; F(4, 4; 6) = 14, [15]; F(2, 2, 2, 4; 6) = F(2, 3, 4; 6) = 14, [16]. \)

In this paper we will calculate another two numbers of this type.

Theorem 2.1. \( F(2, 2, 2, 3; 5) = F(2, 3, 3; 5) = 12. \)

3. THE LEMMAS

We consider the graph \( P \), whose complementary graph \( \overline{P} \) is given in Fig. 1. For this graph we put

\[ A = \{a_1, \ldots, a_8\}, \quad B = \{b_1, b_2, b_3, b_4\}. \]
Lemma 3.1 (Main Lemma). $P \rightarrow (2, 3, 3)$.

To prove the main Lemma, we make use of the next lemmas.

Lemma 3.2. Let $W \subseteq V(\overline{P})$ and $\overline{P}[W] = C_5$.

(a) If $W \cap B = \{b_1\}$, then $W = \{b_1, a_1, a_2, a_7, a_8\}$.
(b) If $W \cap B = \{b_2\}$, then $W = \{b_2, a_1, a_2, a_3, a_4\}$.
(c) If $W \cap B = \{b_3\}$, then $W = \{b_3, a_3, a_4, a_5, a_6\}$.
(d) If $W \cap B = \{b_4\}$, then $W = \{b_4, a_5, a_6, a_7, a_8\}$.

Proof. It is sufficient to prove the proposition (a).

Let $W \cap B = \{b_1\}$. From $b_2, b_4 \notin W$ and $\overline{P}[W] = C_5$ it follows that $a_2, a_7 \in W$.

From $a_7 \in W$ it follows that $a_8 \in W$ or $a_6 \in W$. From $a_2 \in W$ it follows that $a_1 \in W$ or $a_3 \in W$. Since in $\{a_1, a_3, a_6, a_8\}$ only $a_1$ and $a_8$ are adjacent in $\overline{P}$, we have $W = \{b_1, a_1, a_2, a_7, a_8\}$.

Lemma 3.3. Let $W \subseteq V(\overline{P})$, $\overline{P}[W] = C_5$ and $|W \cap B| = 2$. Then the two vertices of $W \cap B$ are adjacent in $\overline{P}$.

Proof. Assume the contrary and let for example $W = \{b_1, b_3\}$. From $\overline{P}[W] = C_5$ it follows that there exists $u \in W$ such that $u \in N_{\overline{P}}(b_1) \cap N_{\overline{P}}(b_3)$. Since $N_{\overline{P}}(b_1) \cap N_{\overline{P}}(b_3) = \{b_2, b_4\}$, this contradicts equality $W = \{b_1, b_3\}$.

Lemma 3.4. Let $W \subseteq V(\overline{P})$ and $\overline{P}[W] = C_5$.

(a) If $W \cap B = \{b_1, b_2\}$, then $W = \{b_1, b_2, a_1, a_7, a_8\}$ or $W = \{b_1, b_2, a_2, a_3, a_4\}$.
(b) If $W \cap B = \{b_2, b_3\}$, then $W = \{b_2, b_3, a_1, a_2, a_3\}$ or $W = \{b_2, b_3, a_4, a_5, a_6\}$.
(c) If $W \cap B = \{b_3, b_4\}$, then $W = \{b_3, b_4, a_3, a_4, a_5\}$ or $W = \{b_3, b_4, a_6, a_7, a_8\}$. 

74
Proof. It is sufficient to prove the proposition (a).

Let \( W \cap B = \{b_1, b_2\} \). From \( b_1 \in W \) and \( b_4 \not\in W \) it follows that \( a_2 \in W \) or \( a_7 \in W \). Let \( a_2 \in W \). Since \( \overline{P}(a_1, a_2, b_1, b_2) = C_4 \), we have \( a_1 \not\in W \). Hence, \( a_3 \in W \). Therefore, from \( \overline{P}[W] = C_5 \) it follows that \( W = \{b_1, b_2, a_2, a_3, a_4\} \). Let \( a_7 \in W \). From \( \overline{P}[W] = C_5 \) it follows that \( a_6 \in W \) or \( a_8 \in W \). Since \( N_{\overline{P}}(a_6) \cap N_{\overline{P}}(b_2) = \{b_3\} \) and \( b_3 \not\in W \), we have \( a_8 \in W \). From \( N_{\overline{P}}(a_8) \cap N_{\overline{P}}(b_2) = \{a_1\} \) it follows that \( W = \{b_1, b_2, a_1, a_7, a_8\} \). \( \square \)

Lemma 3.5. Let \( W \subseteq V(\overline{P}) \) and \( \overline{P}[W] = C_5 \).

(a) If \( W \cap B = \{b_1, b_2, b_3\} \), then
\[
W = \{b_1, b_2, b_3, a_2, a_3\} \quad \text{or} \quad W = \{b_1, b_2, b_3, a_6, a_7\}.
\]
(b) If \( W \cap B = \{b_2, b_3, b_4\} \), then
\[
W = \{b_2, b_3, b_4, a_4, a_5\} \quad \text{or} \quad W = \{b_2, b_3, b_4, a_1, a_8\}.
\]
(c) If \( W \cap B = \{b_1, b_3, b_4\} \), then
\[
W = \{b_1, b_3, b_4, a_6, a_7\} \quad \text{or} \quad W = \{b_1, b_3, b_4, a_2, a_3\}.
\]
(d) If \( W \cap B = \{b_1, b_2, b_4\} \), then
\[
W = \{b_1, b_2, b_4, a_1, a_8\} \quad \text{or} \quad W = \{b_1, b_2, b_4, a_4, a_5\}.
\]

Proof. It is sufficient to prove the proposition (a). Let \( W \cap B = \{b_1, b_2, b_3\} \). From \( b_1 \in W \) and \( \overline{P}[W] = C_5 \) it follows that \( a_2 \in W \) or \( a_7 \in W \). Let \( a_2 \in W \). Since \( N_{\overline{P}}(a_2) \cap N_{\overline{P}}(b_3) = \{a_3\} \), we have \( W = \{b_1, b_2, b_3, a_2, a_3\} \). If \( a_7 \in W \), then from \( N_{\overline{P}}(a_7) \cap N_{\overline{P}}(b_3) = \{a_6\} \) it follows that \( W = \{b_1, b_2, b_3, a_6, a_7\} \). \( \square \)

4. A PROOF OF THE MAIN LEMMA

Assume that \( P \Rightarrow (2, 3, 3) \) and let \( V_1 \cup V_2 \cup V_3 \) be a \((2, 3, 3)\)-free 3-coloring of \( V(\overline{P}) \). From \( \alpha(P) = 2 \) it follows that
\[
|V_1| \leq 2. \tag{2}
\]
Since \( V_i, i = 2, 3, \) contains no 3-clique, from \( \alpha(P) = 2 \) and \( R(3, 3) = 6 \) it follows that
\[
|V_i| \leq 5, \quad i = 2, 3. \tag{3}
\]
The equality \( |V(P)| = 12 \) together with (2) and (3) imply that \( |V_1| = 2, |V_2| = |V_3| = 5 \). We put \( G_i = \overline{P}[V_i], i = 2, 3 \). Since \( \alpha(G_i) = cl(G_i) = 2 \), from \( |V_i| = 5, i = 2, 3 \), it follows that \( G_2 = G_3 = C_5 \). Obviously, \( \overline{P}[A] = C_8 \). Hence \( V_i \cap B \neq \emptyset, i = 2, 3 \). Assume that \( |V_2 \cap B| \leq |V_3 \cap B| \). From \( |B| = 4 \) it follows that \( 1 \leq |V_2 \cap B| \leq 2 \).

Case 1. \( |V_2 \cap B| = 1 \). Without a loss of generality we can assume that \( V_2 \cap B = \{b_1\} \). According to Lemma 3.2(a), \( V_2 = \{b_1, a_1, a_2, a_7, a_8\} \).

Subcase 1a. \( |V_3 \cap B| = 1 \). Suppose that \( V_3 \cap B = \{b_2\} \) or \( V_3 \cap B = \{b_4\} \). Then, according to Lemma 3.2, \( V_2 \cap V_3 \neq \emptyset \), which is a contradiction. Let \( V_3 \cap B = \{b_3\} \).
Then \( V_3 = \{b_3, a_3, a_4, a_5, a_6\} \) (see Lemma 3.2(c)). Hence \( V_1 = \{b_2, b_4\} \). This contradicts the assumption that \( V_1 \) is independent in \( P \).

**Subcase 1b.** \( |V_3 \cap B| = 2 \). According to Lemma 3.3, \( V_3 \cap B = \{b_2, b_3\} \) or \( V_3 \cap B = \{b_3, b_4\} \). Without a loss of generality we can assume that \( V_3 \cap B = \{b_2, b_3\} \). From \( V_2 \cap V_3 = \emptyset \) and Lemma 3.4(b) it follows that \( V_3 = \{b_2, b_3, a_4, a_5, a_6\} \). Hence \( V_1 = \{a_3, b_4\} \). This contradicts the assumption that \( V_1 \) is independent in \( P \).

**Subcase 1c.** \( |V_3 \cap B| = 3 \). It is clear that \( V_3 \cap B = \{b_2, b_3, b_4\} \). From \( V_2 \cap V_3 = \emptyset \) and Lemma 3.5(b) it follows that \( V_3 = \{b_2, b_3, b_4, a_4, a_5\} \). Hence \( V_1 = \{a_3, a_6\} \). This contradicts the assumption that \( V_1 \) is an independent set in \( P \).

**Case 2.** \( |V_2 \cap B| = 2 \). It is clear that \( |V_3 \cap B| = 2 \). According to Lemma 3.3, we can assume that \( V_2 \cap B = \{b_1, b_2\} \) and \( V_3 \cap B = \{b_3, b_4\} \). Because of the Lemma 3.4(a) we have the following two subcases:

**Subcase 2a.** \( V_2 = \{b_1, b_2, a_2, a_3, a_4\} \). From Lemma 3.4(c) and \( V_2 \cap V_3 = \emptyset \) it follows that \( V_3 = \{b_3, b_4, a_6, a_7, a_8\} \). Hence \( V_1 = \{a_1, a_5\} \). This contradicts the assumption that \( V_1 \) is independent in \( P \).

**Subcase 2b.** \( V_2 = \{b_1, b_2, a_1, a_7, a_8\} \). From Lemma 3.4(c) and \( V_2 \cap V_3 = \emptyset \) it follows that \( V_3 = \{b_3, b_4, a_3, a_4, a_5\} \). Hence \( V_2 = \{a_2, a_6\} \). This contradicts the assumption that \( V_1 \) is independent in \( P \).

5. A PROOF OF THEOREM 2.1

It is obvious that from \( G \to (2, 3, 3) \) it follows that \( G \to (2, 2, 2, 3) \). Therefore

\[
F(2, 2, 2, 3; 5) \leq F(2, 3, 3; 5).
\]

From the above inequality it becomes clear that it is sufficient to prove that \( F(2, 3, 3; 5) \leq 12 \) and \( F(2, 2, 2, 3; 5) \geq 12 \).

1. **Proof of the inequality** \( F(2, 3, 3; 5) \leq 12 \). According to the main Lemma, \( P \to (2, 3, 3) \). Since \( \text{cl}(P) = 4 \) and \( |V(P)| = 12 \), we have \( F(2, 3, 3; 5) \leq 12 \).

2. **Proof of the inequality** \( F(2, 2, 2, 3; 5) \geq 12 \). According to Theorem B, \( F(2, 2, 2, 3; 5) \geq 11 \). Assume that \( F(2, 2, 2, 3; 5) = 11 \) and let \( G \) be a graph such that \( |V(G)| = 11 \), \( \text{cl}(G) < 5 \) and \( G \to (2, 2, 3) \). From Theorem C(a) it follows that \( \alpha(G) \leq 3 \). According to Theorem C(b), \( \alpha(G) \neq 3 \). Hence

\[
\alpha(G) = 2. \tag{4}
\]

Assume that there exist \( u, v \in V(G) \) such that \( N_G(u) \supseteq N_G(v) \). It is clear that \( \{u, v\} \not\in E(G) \). From \( F(2, 2, 2, 3; 5) \geq 11 \) it follows that \( G - v \not\to (2, 2, 2, 3) \). Consider an arbitrary \( (2, 2, 2, 3) \)-free 4-coloring of \( G - v \). If we color the vertex \( v \) with the same color as the vertex \( u \), we will obtain \( (2, 2, 2, 3) \)-free 4-coloring of \( G \), which is a contradiction. Therefore:

\[
N_G(v) \subsetneq N_G(u) \text{ for all } u, v \in V(G) \tag{5}
\]
If \( |N_G(v)| = |V(G)| - 1 \) for some \( v \in V(G) \), then \( cl(G - v) < 4 \) and \( G - v \to (2, 2, 2, 2) \). This contradicts Theorem D. Hence, \( |N_G(v)| \neq |V(G)| - 1 \), \( \forall v \in V(G) \). This, together with (5) imply that

\[
|N_G(v)| \leq |V(G)| - 3 \quad \text{for all } v \in V(G). \tag{6}
\]

From \( F(2, 2, 4; 5) = 13 \), [11], it follows that \( G \not\to (2, 2, 4) \). Let \( V_1 \cup V_2 \cup V_3 \) be \( (2, 2, 4) \)-free 3-coloring of \( V(G) \). It follows from (4) that \( |V_1| \leq 2 \), \( |V_2| \leq 2 \). According to (6) and (4), we may assume that \( |V_1| = |V_2| = 2 \). We put \( G_1 = G[V_3] \). It is clear that from \( G \to (2, 2, 2, 3) \) it follows that \( G_1 \to (2, 3) \). According to Theorem A, \( G_1 = \overline{C_7} \) (Fig. 2). Let \( V_1 = \{a, b\} \), \( V_2 = \{c, d\} \) and \( G_2 = G[a, b, c, d] \). From (4) it follows that \( E(G_2) \) contains two independent edges. Without a loss of generality we can assume that \([a, c], [b, d] \in E(G_2)\). It is sufficient to consider the next two cases.

![Graph \( \overline{C_7} \)](image)

Fig. 2. Graph \( \overline{C_7} \)

**Case 1.** \( E(G_2) = \{[a, c], [b, d]\} \). From \( cl(G) < 5 \) it follows that one of the vertices \( a, c \) is not adjacent to some of the vertices \( v_1, \ldots, v_7 \) (see Fig. 2). Without a loss of generality we may assume that \( v_1 \) and \( a \) are not adjacent. Consider the 4-coloring

\[
\{v_4, v_5\} \cup \{v_6, v_7\} \cup \{c, d\} \cup \{v_1, v_2, v_3, a, b\}.
\]

Since \( G \to (2, 2, 2, 3) \), we have that \( \{v_1, v_2, v_3, a, b\} \) contains a 3-clique. Hence \( v_1, v_3 \in N_G(b) \). Similarly, \( v_1, v_6 \in N_G(b) \). So, \( v_1, v_3, v_6 \in N_G(b) \). Similarly, \( v_1, v_3, v_6 \in N_G(d) \). Hence \( \{v_1, v_3, v_6, b, d\} \) is a 5-clique, which is a contradiction.

**Case 2.** \( E(G_2) \supseteq \{[a, c], [b, d], [a, d]\} \). As in case 1, we may assume that \( a \) and \( v_1 \) are not adjacent. Then from (4) it follows that \( v_2, v_7 \notin N_G(a) \). From (4) it follows also that \( a \) is adjacent to some of the vertices \( v_4, v_5 \). Without a loss of generality we may assume that \( v_4 \) and \( a \) are adjacent. So,

\[
v_2, v_4, v_7 \in N_G(a). \tag{7}
\]
From (7) and $\text{cl}(G) < 5$ it follows that $d$ is not adjacent to any of the vertices $v_2$, $v_4$, $v_7$. Hence, it is sufficient to consider the next three subcases.

**Subcase 2a.** The vertex $d$ is not adjacent to $v_2$. Consider the 4-coloring

$$\{v_5, v_6\} \cup \{v_1, v_7\} \cup \{a, b\} \cup \{v_2, v_3, v_4, c, d\}$$

of $V(G)$. From $G \rightarrow (2, 2, 2, 3)$ it follows that $\{v_2, v_3, v_4, c, d\}$ contains a 3-clique. Hence, $v_2, v_4 \in N_G(c)$. Similarly, $v_2, v_7 \in N_G(c)$. From (7) it follows that $\{v_2, v_4, v_7, a, c\}$ is a 5-clique, which contradicts $\text{cl}(G) < 5$.

**Subcase 2b.** The vertex $d$ is not adjacent to $v_4$. Consider the 4-coloring (8). As in the subcase 2a it follows that $v_2, v_4 \in N_G(c)$. Similarly, from the 4-coloring

$$\{v_1, v_7\} \cup \{v_2, v_3\} \cup \{a, b\} \cup \{v_4, v_5, v_6, c, d\}$$

it follows that $v_4, v_6 \in N_G(c)$. So,

$$v_2, v_4, v_6 \in N_G(c).$$

(9)

According to (7), (9) and $\text{cl}(G) < 5$, the vertex $c$ is not adjacent to $v_7$. Consider the 4-coloring

$$\{v_3, v_4\} \cup \{v_5, v_6\} \cup \{a, b\} \cup \{v_1, v_2, v_7, c, d\}.$$

Since $G \rightarrow (2, 2, 2, 3)$, then $\{v_1, v_2, v_7, c, d\}$ contains a 3-clique. Hence, $v_2, v_7 \in N_G(d)$. Similarly, from $G \rightarrow (2, 2, 2, 3)$ and the 4-coloring

$$\{v_1, v_2\} \cup \{v_3, v_4\} \cup \{a, b\} \cup \{v_5, v_6, v_7, c, d\}$$

it follows that $v_5, v_7 \in N_G(d)$. Then

$$v_2, v_5, v_7 \in N_G(d).$$

(10)

From (7), (9) and $\text{cl}(G) < 5$ it follows that $a$ and $v_6$ are not adjacent. From (7), (10) and $\text{cl}(G) < 5$ it follows that $a$ and $v_5$ are not adjacent. So, the vertex $a$ is not adjacent to $v_5$ and $v_6$, which contradicts (4).

**Subcase 2c.** The vertex $d$ is not adjacent to $v_7$. This subcase is analogous with subcase 2b.

6. THE EXTREMA L GRAPHS

By $G - e, e \in E(G)$, we denote the subgraph of $G$ such that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

Consider the graph $\overline{P}$ from Fig. 1. For this graph we set: $P_0 = P$, $P_1 = P - [a_1, a_6]$, $P_2 = P - [a_1, a_6]$, $P_3 = P - [a_2, a_6]$, $P_4 = P - [a_4, a_7]$,$P_5 = P - [a_3, a_7]$, $P_6 = P_2 - [a_4, a_8]$, $P_7 = P_2 - [a_2, a_7]$, $P_8 = P_3 - [a_4, a_7]$, $P_9 = P_7 - [a_2, a_6]$, $P_{10} = P_8 - [a_3, a_8]$, $P_{11} = P_9 - [a_4, a_8]$.

We need the next theorem.
Theorem E, [17]. Let the graph \( G \) be such that \(|V(G)| = 12\), \(c_l(G) = 4\) and \(\alpha(G) = 2\). Then \( G \) is isomorphic to one of the graphs \( P_i\), \(i = 0, \ldots, 11\).

Definition 6.1. We say that the graph \( G \) is extremal if \(|V(G)| = 12\), \(c_l(G) < 5\), \(G \to (2, 3, 3)\) or \(G \to (2, 2, 2, 3)\).

According to Theorem C(a), if \( G \) is extremal, then \(\alpha(G) \leq 4\). From Theorem C(b) it follows that \(\alpha(G) \neq 4\). Hence \(\alpha(G) = 2\) or \(\alpha(G) = 3\). In this section we describe all critical graphs \( G \) with \(\alpha(G) = 2\).

Theorem 6.1. Let \( G \) be extremal graph such that \(G \to (2, 3, 3)\) and \(\alpha(G) = 2\). Then \( G \) is isomorphic to the graph \( P \).

Proof. According to Theorem E, the graph \( G \) is isomorphic to one of the graphs \( P_i\), \(i = 0, \ldots, 11\). The 3-coloring

\[
\{a_7, a_8\} \cup \{b_2, a_1, a_4, a_5, a_6\} \cup \{b_1, b_3, b_4, a_2, a_3\}
\]

of \( P_1 \) is \((2, 3, 3)\)-free and the 3-coloring

\[
\{a_1, a_5\} \cup \{b_1, b_2, a_2, a_3, a_4\} \cup \{b_3, b_4, a_6, a_7, a_8\}
\]

of \( P_2 \) is \((2, 3, 3)\)-free. Hence \( G \) is not a subgraph of \( P_1 \) and \( P_2 \). Thus \( G = P \). \( \Box \)

Theorem 6.2. \( P_i \to (2, 2, 2, 3)\), \(i = 0, \ldots, 11\). If an extremal graph \( G \) is such that \(G \to (2, 2, 2, 3)\) and \(\alpha(G) = 2\), then \( G \) is isomorphic to one of the graphs \( P_i\), \(i = 0, \ldots, 11\).

Proof. Let \( V_1 \cup V_2 \cup V_3 \cup V_4 \) be a 4-coloring of \( V(P_i) \) and \( V_i\), \(i = 1, 2, 3\), be independent. From \(\alpha(P_i) = 2\) it follows that \(|V_i| \leq 2\), \(i = 1, 2, 3\). Hence \(|V_4| \geq 6\). From \(\alpha(P_i) = 2\) and \(R(3, 3) = 6\) it follows that \( V_4 \) contains a 3-clique. Thus \( P_i \) does not have a \((2, 2, 2, 3)\)-free 4-coloring and hence \( P_i \to (2, 2, 2, 3)\). According to Theorem E, the graph \( G \) is isomorphic to one of the graphs \( P_i\), \(i = 0, \ldots, 11\). \( \Box \)

7. THE VERTEX FOLKMAN NUMBERS \( F(2, \ldots, 2, p; q) \)

AND THE RAMSEY NUMBERS \( R(3, q) \)

Theorem 7.1. Let \( p \geq 2\), \( r \) and \( q \) be positive integers such that

\[
R(3, p) + 2r < R(3, q).\tag{11}
\]

Then \( F(2, \ldots, 2, p; q) \leq R(3, p) + 2r \).

Proof. Let \( G \) be a graph such that \(|V(G)| = R(3, p) + 2r\), \(c_l(G) < q\) and

\[
\alpha(G) = 2.\tag{12}
\]
According to (11), the graph $G$ exists. Let $V_1 \cup \ldots \cup V_{r+1}$ be an $(r+1)$-coloring of $V(G)$. Suppose that $V_i, i = 1, \ldots, r$, are independent. From (12) it follows that $|V_i| \leq 2, i = 1, \ldots, r$. Hence $|V_{r+1}| \geq R(3, p)$. According to the definition of $R(3, p)$ and (12), $V_{r+1}$ contains a $p$-clique. Thus $G$ does not have a $(2, \ldots, 2, p)$-free coloring and hence $G \to (2, \ldots, 2, p)$. From $cl(G) < q$ and $|V(G)| = R(3, p) + 2r$ it follows that $F(2, \ldots, 2, p; q) \leq R(3, p) + 2r$. □

Consider the table of the known Ramsey numbers $R(3, p)$, [19]:

<table>
<thead>
<tr>
<th>$p$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(3, p)$</td>
<td>6</td>
<td>9</td>
<td>14</td>
<td>18</td>
<td>23</td>
<td>28</td>
<td>36</td>
<td>40–43</td>
</tr>
</tbody>
</table>

From this table and Theorem 7.1 it follows:

$F(2, 2, 4; 5) \leq 13$ (in [11] it is proved $F(2, 2, 4; 5) = 13$);
$F(2, 2, 6; 7) \leq 22$ (in [11] it is proved $F(2, 2, 6; 7) \leq 26$);
$F(2, 2, 7; 8) \leq 27$ (in [11] it is proved $F(2, 2, 7; 8) \leq 30$);
$F(2, 2, 8; 9) \leq 32$ (in [11] it is proved $F(2, 2, 8; 9) \leq 34$);
$F(2, 2, 9; 10) \leq 40$ if $R(3, 10) \neq 40$ (in [11] it is proved $F(2, 2, 9; 10) \leq 38$);
$F(2, 2, 2, 3; 5) \leq 12$ (according to Theorem 2.1, $F(2, 2, 2, 3; 5) = 12$);
$F(2, 2, 2, 5; 7) \leq 20$ (in [11] it is proved $F(2, 2, 2, 5; 7) \leq 23$).

8. ON THE NUMBERS $F(2, \ldots, 2, p; p + r - 1)$

We put $F(2, \ldots, 2, p; p + r - 1) = F_r(2, p)$.

The proof of Theorem 5 from [13] establishes the following statement:

**Theorem F.** Let $G \to (2, \ldots, 2, p)$. Then $K_r + G \to (2, \ldots, 2, p)$ for any $r$.

From Theorem 2.1, Theorem F and Theorem B it follows that

$r + 8 \leq F_r(2, 3) \leq r + 9$, $r \geq 3$.

The exact value of $F_2(2, 3) = F(2, 2, 3; 4)$ is unknown.

From Theorem B, Theorem F and the inequalities $F_2(2, 6) \leq 22, F_2(2, 7) \leq 27, F_2(2, 8) \leq 32$ and $F(2, 2, 2, 5; 7) \leq 20$ it follows that

$r + 14 \leq F_r(2, 6) \leq r + 20$, $r \geq 2$;

$r + 16 \leq F_r(2, 7) \leq r + 25$, $r \geq 2$;
\[ r + 18 \leq F_r(2, 8) \leq r + 30, \quad r \geq 2; \]
\[ r + 12 \leq F_r(2, 5) \leq r + 17, \quad r \geq 3. \]

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