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ON THE GRAY-HERVELLA CLASSES OF *AH*-STRUCTURES ON SIX-DIMENSIONAL SUBMANIFOLDS OF CAYLEY ALGEBRA

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It is proved that only eight Gray-Hervella classes of almost Hermitian structures can be represented on six-dimensional general-type submanifolds of the octave algebra.

Keywords: almost Hermitian manifold, Gray-Hervella classification, semi-Kählerian structure

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1. PRELIMINARIES

In this paper we consider the Gray-Hervella classes of almost Hermitian (AH) structures [11]. As it is well-known [1, 11], a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J is called almost Hermitian if the following condition holds:

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞. We recall that the fundamental form of an almost Hermitian manifold is determined by

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [1]. Its elements are the frames adapted to the structure (A-frames). They look as follows:

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\widehat{1}}, \ldots, \varepsilon_{\widehat{n}}),$$

where $p \in M^{2n}$, ε_a are the eigenvectors corresponding to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\widehat{a}}$ are the eigenvectors corresponding to the eigenvalue $-i$. Here $a = 1, \ldots, n$; $\widehat{a} = a + n$. Therefore, the matrix of the almost complex structure in an A-frame at the point p looks as follows:

$$(J_j^k) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right), \tag{1.1}$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$. By direct computing, it is easy to obtain that in A-frame the matrices of the Riemannian metric g and of the fundamental form F look as follows, respectively:

$$(g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right), \quad (F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right). \tag{1.2}$$

2. CARTAN-KIRICHENKO STRUCTURAL EQUATIONS OF AN AH-STRUCTURE

The form of the Levi-Civita connection ∇ is determined by the forms system $\{\omega_j^k\}$ on the space of the complex frames stratification over an almost Hermitian manifold. Similarly, the deplacement form ω is determined by the forms system $\{\omega^k\}$. The Cartan structural equations of the stratification space over almost Hermitian manifold look as follows:

$$\begin{aligned} &1)\ d\omega^k = \omega_j^k \wedge \omega^j; \\ &2)\ d\omega_j^k = \omega_l^k \wedge \omega_j^l + \frac{1}{2} R_{jml}^k \omega^m \wedge \omega^l, \end{aligned} \tag{2.1}$$

where $\{R_{jml}^k\}$ are the components of the Riemannian curvature tensor (or of the Riemann-Christoffel tensor [13]). Here and further $k, j, m, l = 1, \ldots, 2n$.
As J and g are the tensors of (1,1)- and (2,0)-type, respectively, and as $\nabla g = 0$, then the components of these tensors must satisfy the following system of differential equations:

$$\begin{aligned} &1)\ dJ_j^k + J_l^k \omega_j^l - J_j^l \omega_l^k = J_{j,l}^k \omega^l; \\ &2)\ dg_{kj} + g_{lj}\omega_k^l + g_{kl}\omega_j^l = 0, \end{aligned} \tag{2.2}$$

where $\{J^k_{j,l}\}$ are the components of ∇J. Taking into account (1.1) and (1.2), we can rewrite $(2.2)_1$ as follows:

$$1)\ \omega^a_{\hat{b}} = -\frac{i}{2}J^a_{\hat{b},k}\omega^k; \quad 2)\ \omega^{\hat{a}}_b = -\frac{i}{2}J^{\hat{a}}_{b,k}\omega^k; \quad 3)\ J^a_{b,k} = 0; \quad 4)\ J^{\hat{a}}_{\hat{b},k} = 0. \qquad (2.3)$$

Similarly, from $(2.2)_2$ we obtain:

$$1)\ \omega^a_{\hat{b}} + \omega^b_{\hat{a}} = 0; \quad 2)\ \omega^a_b + \omega^{\hat{b}}_{\hat{a}} = 0; \quad 3)\ \omega^{\hat{a}}_b + \omega^{\hat{b}}_a = 0. \qquad (2.4)$$

Here and further $a, b, c, d = 1, \ldots, n$; $\hat{a} = a + n$. Substituting (2.3) and (2.4) in Cartan structural equations (2.1), we get

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + \omega^a_{\hat{b}} \wedge \omega^{\hat{b}} = \omega^a_b \wedge \omega^b - \frac{i}{2}J^a_{\hat{b},c}\omega^c \wedge \omega^{\hat{b}} + \frac{i}{2}J^a_{[\hat{b},\hat{c}]}\omega^{\hat{c}} \wedge \omega^{\hat{b}}; \\ d\omega^{\hat{a}} &= \omega^{\hat{a}}_b \wedge \omega^b + \omega^{\hat{a}}_{\hat{b}} \wedge \omega^{\hat{b}} = \omega^{\hat{a}}_{\hat{b}} \wedge \omega^{\hat{b}} + \frac{i}{2}J^{\hat{a}}_{b,\hat{c}}\omega^{\hat{c}} \wedge \omega^b - \frac{i}{2}J^{\hat{a}}_{[b,c]}\omega^c \wedge \omega^b. \end{aligned} \qquad (2.5)$$

We denote $\omega_k = g_{kj}\omega^j$. In particular, $\omega_a = \omega^{\hat{a}} = \overline{\omega^a}$. Taking into account this fact as well as (2.4), we can rewrite (2.5) as follows:

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c; \\ d\omega_a &= -\omega^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c, \end{aligned} \qquad (2.6)$$

where

$$B^{ab}{}_c = -\frac{i}{2}J^a_{\hat{b},c}; \quad B_{ab}{}^c = \frac{i}{2}J^{\hat{a}}_{b,\hat{c}}; \quad B^{abc} = \frac{i}{2}J^a_{[\hat{b},\hat{c}]}; \quad B_{abc} = -\frac{i}{2}J^{\hat{a}}_{[b,c]}. \qquad (2.7)$$

The functions $\{B^{ab}{}_c\}$, $\{B_{ab}{}^c\}$, $\{B^{abc}\}$, $\{B_{abc}\}$ serve as components of complex tensors of an almost Hermitian manifold (M^{2n}, J, g) [2], because, considering the differential continuations of (2.7), it is not difficult to see that

$$\begin{aligned} &dB^{ab}{}_c + B^{ab}{}_d\omega^d_c - B^{db}{}_c\omega^a_d - B^{ad}{}_c\omega^b_d = B^{ab}{}_{c,k}\omega^k; \\ &dB_{ab}{}^c - B_{ab}{}^d\omega^c_d + B_{db}{}^c\omega^d_a + B_{ad}{}^c\omega^d_b = B_{ab}{}^c{}_{,k}\omega^k; \\ &dB^{abc} - B^{dbc}\omega^a_d - B^{adc}\omega^b_d - B^{abd}\omega^c_d = B^{abc}{}_{,k}\omega^k; \\ &dB_{abc} + B_{dbc}\omega^d_a + B_{adc}\omega^d_b + B_{abd}\omega^d_c = B_{abc,k}\omega^k. \end{aligned}$$

Definition 2.1 ([4]). The tensors with the components $\{B^{ab}{}_c\}$ and $\{B_{ab}{}^c\}$ are called virtual Kirichenko tensors of first and second order, respectively.

Definition 2.2 ([4]). The tensors with the components $\{B^{abc}\}$ and $\{B_{abc}\}$ are called structural Kirichenko tensors of first and second order, respectively.

Definition 2.3. The equations (2.6) are called Cartan-Kirichenko structural equations of an almost Hermitian structure on the manifold M^{2n}.

We remark that according to (2.4)

$$J^{a}_{\hat{b},k} + J^{b}_{\hat{a},k} = 0.$$

So, we have

$$B^{ab}{}_{c} + B^{ba}{}_{c} = 0.$$

Similarly,

$$B_{ab}{}^{c} + B_{ba}{}^{c} = 0.$$

Thus, we have proved

Proposition 2.1. *The virtual Kirichenko tensors of an almost Hermitian manifold are skew-symmetric relative to the first pair of indices.*

From (2.7) we obtain the following result:

Proposition 2.2. *The structural Kirichenko tensors of an almost Hermitian manifold are skew-symmetric relative to the second pair of indices.*

Owing to the reality of ∇J, from the given definitions we have

Proposition 2.3. $B^{ab}{}_{c} = \overline{B_{ab}{}^{c}}$, $B^{abc} = \overline{B_{abc}}$.

3. THE MAIN RESULT

Let $\mathbf{O} \equiv \mathbf{R}^8$ be the Cayley algebra. As it is well-known [10], two non-isomorphic three-fold vector cross products are defined on it by means of the relations

$$P_1(X,Y,Z) = -X(\overline{Y}Z) + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

$$P_2(X,Y,Z) = -(X\overline{Y})Z + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

where $X,Y,Z \in \mathbf{O}$, $\langle \cdot,\cdot \rangle$ is the scalar product in $\mathbf{O}$, and $X \to \overline{X}$ is the conjugation operator. Moreover, any other three-fold vector cross product in the octave algebra is isomorphic to one of the two above-mentioned.

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle \cdot,\cdot \rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1,2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [10]. The point $p \in M^6$ is called general [12] if

$$e_0 \notin T_p(M^6) \not\subset L(e_0)^{\perp},$$

where e_0 is the unit of Cayley algebra and $L(e_0)^{\perp}$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general-type submanifold [12, 13].

Naturally, we can consider the following question:

Which of the sixteen Gray-Hervella classes of almost Hermitian structures can be represented on six-dimensional submanifolds of Cayley algebra?

A partial answer is known just for a six-dimensional special $M^6 \subset \mathbf{O}$, or the so-called Calabi submanifolds [13]. We recall that an almost Hermitian submanifold $M^6 \subset \mathbf{O}$ is called special [13] if

$$e_0 \notin T_p(M^6) \subset L(e_0)^{\perp}.$$

Such six-dimensional almost Hermitian submanifolds of the octave algebra were studied by A. Gray [8 – 10], E. Calabi [6], K. Yano and T. Sumitomo [17].
We answer this question in the case of general-type $M^6 \subset \mathbf{O}$, i.e. we shall prove the following

Main Theorem. *Just the eight Gray-Hervella classes of almost Hermitian structures can be represented on six-dimensional general-type submanifolds of Cayley algebra, namely:*

$$K,\ AK,\ NK,\ SH,\ QK,\ W_1 \oplus W_3,\ W_2 \oplus W_3,\ SK.$$

Proof. Let us use the characterization of Gray-Hervella classes of AH-structures in terms of Kirichenko tensors [2], as listed in Table 1.

Table 1

Characterization of Gray-Hervella classes of AH-structures in terms of Kirichenko tensors [2]

Class	Condition
K	$B^{abc} = 0, \quad B^{ab}{}_c = 0$
$NK = W_1$	$B^{abc} = -B^{bac}, \quad B^{ab}{}_c = 0$
$AK = W_2$	$B^{(abc)} = 0, \quad B^{ab}{}_c = 0$
$SH = W_3$	$B^{abc} = 0, \quad B^{ab}{}_b = 0$
W_4	$B^{abc} = 0, \quad B^{ab}{}_c = \alpha^{[a}\delta_c^{b]}$
$QK = W_1 \oplus W_2$	$B^{ab}{}_c = 0$
$H = W_3 \oplus W_4$	$B^{abc} = 0$
$VG = W_1 \oplus W_4$	$B^{abc} = -B^{bac}, \quad B^{ab}{}_c = \alpha^{[a}\delta_c^{b]}$
$W_1 \oplus W_3$	$B^{abc} = -B^{bac}, \quad B^{ab}{}_b = 0$
$W_2 \oplus W_3$	$B^{(abc)} = 0, \quad B^{ab}{}_b = 0$
$W_2 \oplus W_4$	$B^{(abc)} = 0, \quad B^{ab}{}_c = \alpha^{[a}\delta_c^{b]}$
$SK = W_1 \oplus W_2 \oplus W_3$	$B^{ab}{}_b = 0$
$G1 = W_1 \oplus W_3 \oplus W_4$	$B^{abc} = -B^{bac}$
$G2 = W_2 \oplus W_3 \oplus W_4$	$B^{(abc)} = 0$
$W_1 \oplus W_2 \oplus W_4$	$B^{ab}{}_c = \alpha^{[a}\delta_c^{b]}$
W	no condition

Now, we write out the Cartan-Kirichenko structural equations for six-dimensional general-type almost Hermitian submanifolds of Cayley algebra [13]:

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + \frac{1}{\sqrt{2}}\varepsilon^{ah[b}D_h{}^{c]}\omega_b \wedge \omega_c + \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hc}\omega^c \wedge \omega_b; \\ d\omega_a &= -\,\omega^b_a \wedge \omega_b + \frac{1}{\sqrt{2}}\varepsilon_{ah[b}D^h{}_{c]}\omega_b \wedge \omega_c + \frac{1}{\sqrt{2}}\varepsilon_{abh}D^{hc}\omega_c \wedge \omega^b. \end{aligned} \tag{3.1}$$

Here $\varepsilon^{abc} = \varepsilon^{abc}_{123}$, $\varepsilon_{abc} = \varepsilon^{123}_{abc}$ are the components of Kronecker tensor of third order [16]:

$$\begin{aligned} D_{cj} &= \mp T^8_{cj} + iT^7_{cj}, \quad D_{\widehat{cj}} = \mp T^8_{\widehat{cj}} - iT^7_{\widehat{cj}}; \\ D_h{}^c &= D_{h\widehat{c}}, \quad D^h{}_c = D_{\widehat{h}c}, \quad D^{hc} = D_{\widehat{h}\widehat{c}}, \end{aligned} \tag{3.2}$$

where $\{T^{\varphi}_{kj}\}$, $\varphi = 7, 8$, are components of the configuration tensor (using Gray's terminology [8] or of the Euler curvature tensor [7]). Now, we assume that the indices a, b, c, d, h range from 1 to 3; the indices k, j range from 1 to 6, and we set $\widehat{a} = a + 3$.

Comparing (2.6) and (3.1), we get the following relations for virtual Kirichenko tensors of an almost Hermitian $M^6 \subset \mathbf{O}$ [5]:

$$B_{ab}{}^c = \frac{1}{\sqrt{2}}\varepsilon_{abh}D^{hc}, \quad B^{ab}{}_c = \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hc}.$$

From (3.2) it follows that the tensor D_{hb} is symmetric relative to the indices h and b; ε^{abh} is skew-symmetric relative to these indices. Hence, $\varepsilon^{abh}D_{hb} = 0$, and therefore

$$B^{ab}{}_b = \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hb} = 0.$$

So, for an arbitrary almost Hermitian structure induced by means of a three fold vector cross product on six-dimensional submanifolds of Cayley algebra, the following identity is fulfilled:

$$B^{ab}{}_b \equiv 0.$$

Consequently, as it is clear from the given table, all these almost Hermitian structures must be semi-Kählerian (SK). The class of SK-structures containes only eight classes of AH-structures, namely:

$$K,\ NK,\ AK,\ SH,\ QK,\ W_1 \oplus W_3,\ W_2 \oplus W_3,\ SK. \quad \Box$$

We remark that this result is similar to A. Gray's conclusion that a six-dimensional special submanifold $M^6 \subset \mathbf{O}$ is semi-Kählerian [9, 11]. So, all six-dimesional almost Hermitian submanifolds of Cayley algebra are semi-Kählerian, i.e. they belong to one of the eight above-mentioned Gray-Hervella classes of AH-manifolds.

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