Proximity-type Relations on Boolean Algebras and their Connections with Topological Spaces

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0.0 Preface

This thesis is in the field of General Topology. In its title, however, some algebraical notions are used. I will try to explain why this happens.

I will first say some words about the proximity structures.

In [86], Som Naimpally wrote: "The idea of nearness is one of those rare items in Mathematics – a concept which is simultaneously intuitive and rigorous. It is so intuitive that to quote Lagrange it is possible to "make it clear to the first person one meets on the street" ... Its simplicity and depth provide a powerful tool in research in Topology and Functional Analysis." I completely agree with this opinion.

F. Riesz was the first mathematician who formulated (in 1908) a set of axioms to describe the notion of *nearness* (or, *proximity*) of pair of sets (in fact, he used the term *linkage* (or, *chaining*) (in German, *Verkettung*)) (see [98]). The paper [98] is a continuation of the paper [97] (published in Hungarian in 1906) which, in turn, was provoked by the M. Fréchet paper [57]. In [57], M. Fréchet proposed the fundamental notion of a *metric space* (the importance of which for the whole mathematics is unnecessary to explain), and the more general notion of an *L-space* (where "L" comes from "Limit") (which lies in the ground of the modern theory of *sequential convergence spaces* (see [58])). With the introducing of these notions, M. Fréchet made the first attempts for describing in an abstract form that structure of Euclidean spaces and their subspaces which makes possible the defining of continuous functions between them. (Recall that in that time the notion of a topological space was still not defined, as well as the notion of a net, the Cantor Naive Theory of Sets had not still completed thirty years and the same was valid for the fundamental notions of neighborhood of a point, open and closed set, accumulation point, as well as closure and interior of a set, which were introduced by G. Cantor in the realm of subsets of Euclidean spaces.) In [97], Riesz showed, with the help of an important example, that Fréchet's "countable" approach does not suffice. Since, as I have already mentioned, the notion of a net was still not created, Riesz proposed the concept of accumulation points of subsets to be defined satisfying several axioms instead of the axiomatically described by Fréchet concept of limits of sequences. The relation of linkage, introduced in [98], was an instrument for determining in a natural way the accumulation points of subsets. The axioms of the relation of linkage, proposed by Riesz, are the following (we will write "$A\#B$" for "A is linked to B", where A and B are subsets of some set X):

1) ($A\#B$, $A \subseteq C$ and $B \subseteq D$) implies ($C\#D$),

2) $(A \cup B)\#C$ implies that $A\#C$ or $B\#C$,

3) $\forall x \in X$, $(A\#\{x\}, B\#\{x\})$ implies that $A\#B$,

4) $\forall x, y \in X$, $(x \neq y)$ implies $(\{x\}(-\#)\{y\})$ (where $(-\#)$ is the negation of $\#$).

It is not explicitly written in [98] that the relation $\#$ is symmetric (i.e., $(A\#B) \to (B\#A)$) but it seems that it is taken for granted. If one considers a point $x \in X$ as an *accumulation point of a subset* A of X iff $x\#A$, then the axioms 1)-4) correspond to those for accumulation points given in [97]. The basic example of a set with a relation of linkage comes from metric spaces: A is linked to B iff the distance between A and B is equal to zero. The paper [98] is interesting not only with the introducing of the relation of linkage but also with the Riesz idea of adding to the original set X some *ideal points* which are systems of subsets of X now known as *clusters*. This is very close to what was done by P. S. Alexandroff, H. Freudenthal, P. Samuel, S. Leader, W. Thron (and others) many years later. A notion of a cluster, analogous to that of Riesz, plays a central role in the present thesis as well. (For excellent expositions of the ideas of Riesz mentioned above, see [12, 76, 111].) So, as it is written in [12], "F. Riesz anticipated what was studied in detail much later". He didn't succeed in the creating of an expedient notion like that of the present notion of a topological space but he showed that Fréchet's notion of an L-space is not enough general and introduced in mathematics the first example of a proximity-type relation. Moreover, as it is now well known, his approach could lead to the creating of a notion equivalent to the modern notion of a topological space. I'm now going to show how this could be done because, doing this, I will have the possibility to demonstrate that the axioms of the modern proximity structures arise in a very natural way.

Let us first reason in the realm of Euclidean spaces. Let's say that a point x of an Euclidean space X *is near to* a subset A of X (and write $x\delta A$) if x is a limit of a sequence of points of A. Then, having in mind the properties of the convergent sequences, it is natural to introduce the following axioms:

(PS1) $(x\delta A) \to (A \neq \emptyset)$,

(PS2) $(\{x\} \cap A \neq \emptyset) \to (x\delta A)$,

(PS3) $(x\delta(A \cup B)) \leftrightarrow [(x\delta A) \text{ or } (x\delta B)]$,

(PS4) $[x\delta A \text{ and } (\forall a \in A)(a\delta B)] \to (x\delta B)$.

(The last axiom is a translation of the following theorem valid in Euclidean spaces: if $x = \lim_{i\to\infty} a_i$ and $a_i = \lim_{j\to\infty} b^i_j$ then there exist sequences of positive integers

$i_1, i_2, \ldots$ and $j_1, j_2, \ldots$ such that $x = \lim_{k\to\infty} b^{i_k}_{j_k}$.) Now, if we set, for every $A \subseteq X$,

$$\mathrm{cl}(A) = \{x \in X \mid x\delta A\},$$

then the above axioms can be rewritten in the following form (where we will write $\mathrm{cl}_X(A)$ instead of $\mathrm{cl}(A)$):

(PS1′) $\mathrm{cl}_X(\emptyset) = \emptyset$,

(PS2′) $A \subseteq \mathrm{cl}_X(A)$,

(PS3′) $\mathrm{cl}_X(A \cup B) = \mathrm{cl}_X(A) \cup \mathrm{cl}_X(B)$,

(PS4′) $\mathrm{cl}_X(\mathrm{cl}_X(A)) = \mathrm{cl}_X(A)$.

So, we obtained a function $\mathrm{cl}_X : P(X) \longrightarrow P(X)$, where $P(X)$ is the power set of X, which has the properties (PS1′)-(PS4′) (i.e., a *Kuratowski closure operator*). Note that the definition of Heine of a continuous function $f : X \longrightarrow Y$ can be expressed in the following form: f is continuous iff

(1) $\quad \forall A \subseteq X,\ f(\mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A))$.

(Indeed, the definition of Heine says that $f : X \longrightarrow Y$ is continuous iff for every convergent sequence (x_n) of points of X, $x = \lim_{n\to\infty} x_n$ implies $f(x) = \lim_{n\to\infty} f(x_n)$, i.e., for every $A \subseteq X$, the points of $\mathrm{cl}_X(A)$ are mapped by f in $\mathrm{cl}_Y(f(A))$.)

Let's now forget that X is an Euclidean space. Let X be an arbitrary abstract set and $\mathrm{cl}_X : P(X) \longrightarrow P(X)$ be a function satisfying the axioms (PS1′)-(PS4′). Then the pair (X, cl_X) is exactly the Kuratowski notion of a topological space. Also, it is natural to introduce the following definition: a function $f : (X, \mathrm{cl}_X) \longrightarrow (Y, \mathrm{cl}_Y)$ is said to be continuous iff it satisfies condition (1).

Note that if, in the above definition of the relation δ, the sequences were replaced by nets, then $\mathrm{cl}(A)$ would coincide with $\mathrm{cl}_X(A)$ (which is just the closure of A in the topological space (X, cl_X)) (otherwise, $\mathrm{cl}(A)$ is, in general, only a subset of $\mathrm{cl}_X(A)$). Hence, when δ is defined by means of nets, the given above definition of a continuous function $f : (X, \mathrm{cl}_X) \longrightarrow (Y, \mathrm{cl}_Y)$ can be rewritten as follows: f is continuous iff for every $x \in X$ and each $A \subseteq X$,

$$(x\delta A) \to (f(x)\delta' f(A))$$

(where δ' places the role of δ in Y). This form of the definition of a continuous function reflects in a best way our intuitive idea of what had to be a continuous function.

So, we have shown that the idea of introducing the relation "nearness between points and subsets of a set" leads to the definition of a topological space. In fact, this relation can be also regarded as a topological structure. Indeed, let (X, τ) be a topological space and cl be the generated by it closure operator. Then cl satisfies axioms (PS1′)-(PS4′). Define, for each $x \in X$ and each $A \subseteq X$, $x\delta_{cl}A$ iff $x \in \mathrm{cl}(A)$. Then it is easy to see that δ_{cl} satisfies axioms (PS1)-(PS4). Conversely, let δ be a relation between points and subsets of a set X satisfying axioms (PS1)-(PS4). Define, for each $A \subseteq X$,

$$\mathrm{cl}_\delta(A) = \{x \in X \mid x\delta A\}.$$

Then it is obvious that cl_δ satisfies axioms (PS1′)-(PS4′). Moreover, using the above notation, we have that $\delta = \delta_{cl_\delta}$ and $\mathrm{cl} = \mathrm{cl}_{\delta_{cl}}$. Thus, we can define topological spaces as pairs (X, δ) consisting of an abstract set X and a relation δ between the points and subsets of X which satisfies axioms (PS1)-(PS4). Hence, we have shown that the topological structure on a set X is in fact a kind of a "nearness structure" on X. This nearness structure is, however, not symmetrical (in the sense that it defines a relation between points and sets and not between sets and sets). If we symmetrize it (i.e., if we replace x and $\{x\}$ with C in the axioms (PS1)-(PS4) and add the symmetry axiom "$A\delta B \leftrightarrow B\delta A$"), then we will come to the notion of a *Lodato proximity* (see [79, 80]).

The ideas of F. Riesz were for a long time almost forgotten. It was V. A. Efremovič who defined proximity spaces in his lecture "Geometry of infinite proximity" at a mathematical conference in Moscow in 1936. The motivation of Efremovič was geometrical: he noted that, e.g., the Euclidean plane and the Lobačevsky plane are homeomorphic but no bijective map preserves "infinitesimality" of subsets. He published his results more than 10 years later (see [49, 50, 51, 52]). In these papers he introduced the axioms of what we now call *Efremovič proximity spaces* (or *EF-proximity spaces*) and obtained some significant results: 1) the topologies induced by EF-proximities are completely regular topologies, 2) any two far sets in a proximity space X are functionally separated by a proximally continuous function, and 3) a mapping between metric spaces is uniformly continuous iff it is proximally continuous. Many other beautiful results about proximities were also obtained by N. S. Ramm and A. S. Švarc jointly with V. A. Efremovič or separately. It can be said, however, that the theory of proximity spaces became an important part of the General Topology after the works of Ju. M. Smirnov on proximity spaces (see [103, 104, 105]) and especially after his famous Compactification Theorem which revealed completely the connection between proximities and compact extensions (see [103]). After these basic results of the Moscow proximity school, the theory of proximity spaces was developed very rapidly. A good introduction in this theory is the book [87] of S. Naimpally and B. Warrack. We will mention only that E. Čech [118] defined a more general notion of a proximity than that of EF-proximity and that the Smirnov Compactification Theorem was generalized by S. Leader [78], who introduced the notion of a *local proximity* (which plays an important role in this thesis) and proved his Local Compactification Theorem.

The proximity spaces are pairs of a set and a proximity relation on the power set of this set. Clearly, the power set of any set can be regarded as a Boolean algebra with respect to the natural set-theoretical operations – this is the power set algebra. By the Lindenbaum-Tarski Theorem, a Boolean algebra is isomorphic to a power set algebra iff it is complete and atomic. Hence, there exist many Boolean algebras which are not isomorphic to any power set algebra. Thus, it seems very natural to study the pairs of a Boolean algebra and a proximity-type relation on it, and to regard these objects as some generalizations or variations of proximity spaces. Could such investigations lead to some interesting results about topological spaces? The answer to this question was given by M. Stone [108] by means of his Duality Theorem which is regarded as one of the most significant theorems in mathematics (see, e.g., the excellent book [75] of P. Johnstone where the influence of Stone's Duality Theorem on almost all areas of the modern mathematics is revealed). The Stone Duality Theorem was created before the birth of EF-proximity spaces and it doesn't need the notion of a proximity because the proximity-type relation used in Stone's Duality Theorem is hidden; it is the following one: two elements a and b of a Boolean algebra are *near* iff $a \wedge b \neq 0$. This nearness relation satisfies almost all of the Efremovič axioms of EF-proximity. The Stone Duality Theorem shows that all information about a zero-dimensional compact Hausdorff space X is contained in the Boolean algebra of its *clopen* (= closed and open) subsets, i.e., having the Boolean algebra $CO(X)$ of all clopen subsets of X, one can reconstruct, up to homeomorphism, the space X; also, having a Boolean algebra B, one can construct a unique, up to homeomorphism, zero-dimensional compact Hausdorff space X such that the Boolean algebras $CO(X)$ and B are isomorphic. In this way, the theory of zero-dimensional compact Hausdorff spaces transforms into the theory of Boolean algebras, and conversely. That's why, the fact that many important theorems in the theory of Boolean algebras are created by topologists, and conversely, is not a surprise. Such things happen always when there is a duality between two categories:

the existence of such a duality is a powerful instrument for the investigation of the both categories. The Stone Duality Theorem was extended to the category **HC** of compact Hausdorff spaces and continuous maps by H. de Vries [24]. The objects of the dual category **DHC** constructed by de Vries are pairs consisting of a Boolean algebra and a proximity-type relation on it; the axioms which this relation satisfies almost coincide with the axioms of EF-proximities. The de Vries Duality Theorem shows that the theory of compact Hausdorff spaces and all continuous maps between them is in fact the theory of the Boolean algebras endowed with a special kind of proximity-type relations. By the de Vries Duality, the dual object of a compact Hausdorff space X is the Boolean algebra $RC(X)$ of its *regular closed subsets* together with the relation ρ_X on $RC(X)$ defined by

$$F\rho_X G \iff F \cap G \neq \emptyset.$$

The whole information about the space X (up to homeomorphism) is contained in the pair $(RC(X), \rho_X)$. This permits to study compact Hausdorff spaces by means of algebraical methods, and conversely. So, the investigation of proximity-type relations on Boolean algebras transforms through the technique of duality or equivalence (or isomorphism) functors into an investigation of different categories of topological spaces. Let us also recall that in the case of the Stone Duality we were able to restore the points of a zero-dimensional compact Hausdorff space knowing only the Boolean algebra of its clopen sets, and, analogously, in the case of de Vries' Duality we were able to restore the points of a compact Hausdorff space X knowing only the Boolean algebra of its regular closed sets and the relation ρ_X. The philosophical significance of these facts can be understand with the help of the ideas of A. N. Whitehead [123] and T. de Laguna [23]. Briefly speaking, their theory (known as *region-based theory of space* (or *theory of events*, or *geometry of solids*) (see [120, 121, 122, 123, 23])) is based on a certain criticism of the Euclidean approach to the geometry, where the points (as well as the straight lines and planes) are taken as the basic primitive notions. A. N. Whitehead and T. de Laguna noticed that points, lines and planes are quite abstract entities which have not a separate existence in reality and proposed to put the theory of space on the base of some more realistic spatial entities. This new approach to the theory of space (and time) was influenced by the Einstein Theory of Relativity. In [120], A. N. Whitehead (who co-authored, with B. Russell, the famous book "Principia Mathematica") wrote:

"...It follows from the relativity theory that a point should be definable in terms of the

relations between material things. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning. Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. Similar explanations apply to time. Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared. This is an unwritten chapter of mathematics..."
In Whitehead [123], the notion of a *region* is taken as a primitive notion: it is an abstract analog of a spatial body; also some natural relations between regions are regarded. In [121], Whitehead considered some mereological relations like "part-of", "overlap" and some others, while in [123] he adopted from de Laguna [23] the relation of "*contact*" ("connectedness" in Whitehead's original terminology) as the only primitive relation between regions except the relation "part-of".

Let us note that neither Whitehead nor de Laguna presented their ideas in a detailed mathematical form.

The ideas of de Laguna and Whitehead lead naturally to the following general programme (or *general region-based theory of space*):

- for every topological space X belonging to some class $\mathcal{C}$ of topological spaces, define in topological terms:

  (a) a family $\mathcal{R}(X)$ of subsets of X that will serve as models of Whitehead's "regions" (and call the elements of the family $\mathcal{R}(X)$ *regions of* X);

  (b) a relation ρ_X on $\mathcal{R}(X)$ that will serve as a model of Whitehead's relation of "contact" (and call the relation ρ_X a *contact relation on* $\mathcal{R}(X)$);

- choose some (algebraic) structure which is inherent to the families $\mathcal{R}(X)$ and contact relations ρ_X, for $X \in \mathcal{C}$, fix some kind of morphisms between the obtained (algebraic) objects and build in this way a category $\mathbf{A}$;

- find a subcategory $\mathbf{T}$ of the category $\mathbf{Top}$ of topological spaces and continuous maps, with objects belonging to the class $\mathcal{C}$, which is equivalent (or isomorphic) or dually equivalent to the category $\mathbf{A}$ trough a (contravariant) functor that assigns to each object X of $\mathbf{T}$ the chosen (algebraic) structure of the family of all regions of X.

If all of this is done then, in particular, the chosen (algebraic) structure of the regions of any object X of $\mathbf{T}$ is sufficient for recovering completely (of course, up to homeomorphism) the whole space X. Hence, in this way, a "region-based theory" of the objects and morphisms of the category $\mathbf{T}$ will be obtained.

Of course, during the realization of this programme, one can find the category $\mathbf{A}$ starting with the category $\mathbf{T}$, if the latter is the desired one.

The Stone Duality [108] between the category of all Boolean algebras and their homomorphisms and the category of all compact zero-dimensional Hausdorff spaces and all continuous maps between them, which we mentioned above, is in fact the first realization of this programme, although M. Stone came to his theory guided by ideas which are completely different from those of Whitehead and de Laguna. Clearly, in Stone's Duality, the clopen subsets of compact zero-dimensional Hausdorff spaces serve as models of the regions and the algebraic structure on the regions is that of a Boolean algebra; the contact relation ρ_X here is hidden, as we have already mentioned, because it can be defined by the Boolean operations. Another celebrated example is the *localic duality* (see, e.g., [75, Corollary II.1.7]) between the category of all spatial frames and all functions between them which preserve finite meets and arbitrary joins and the category of all sober spaces and all continuous maps between them: in it the open subsets of sober spaces serve as models of the regions, the algebraic structure on the regions is that of a spatial frame, and, as above, the contact relation ρ_X between the regions is hidden because it can be recovered by the lattice operations. The de Vries duality [24] for the category $\mathbf{HC}$ of all compact Hausdorff spaces and all continuous maps between them, which was also mentioned above, is the first realization of the ideas of the general region-based theory of space in their full generality and strength (and again, as it seems, de Vries was unaware of the papers [23] and [123]): the models of the regions in de Vries' theory are the regular closed sets, the algebraic structure on the regions is that of a Boolean algebra, and, in contrast to the case of the Stone duality and localic duality, the contact relation between the regions, which is in the basis of de Vries' duality theorem, cannot be derived from the Boolean structure on the regions. Note that in [24], instead of the Boolean algebra $RC(X)$ of regular closed sets, the Boolean algebra $RO(X)$ of regular open sets was regarded ($RO(X)$ and $RC(X)$ are isomorphic Boolean algebras); also, instead of the relation ρ_X on the set $RC(X)$ (see the description of ρ_X above), de Vries used in [24] the so-called "*compingent relation*" between regular open sets whose counterpart for $RC(X)$ is the relation $\ll_X$, defined by

$$F \ll_X G \iff F \subseteq \mathrm{int}(G),$$

for $F, G \in RC(X)$ (this relation is similar to the topogenous order in the sense of Á. Császár [22]); the relations ρ_X and $\ll_X$ are inter-definable. Two more realizations of this general programme were accomplished by V. V. Fedorchuk [54] (it seems that he was also not aware of the ideas of Whitehead and de Laguna). He obtained a Duality Theorem and an Equivalence Theorem concerning the category of compact Hausdorff spaces and quasi-open maps. The regions and the chosen algebraic structure on the regions are the same as in the case of de Vries' Duality, but the morphism are different.

In this thesis, I present my contributions to the general region-based theory of space and to the region-based theory of space, and their applications in General Topology. These contributions are some duality, equivalence or isomorphism theorems concerning two kinds of categories: categories whose objects are topological spaces, and categories having as objects pairs of a Boolean algebra and a proximity-type relation on it. The thesis is based mainly on my papers [27, 28, 29, 30, 31, 32, 33, 34, 39, 40, 117]. My joint papers with D. Vakarelov [41, 42, 43] also fit very well to the topic of the thesis but they are not included in it because, otherwise, some more 100 pages should be added. Only some definitions from [41] are used here. However, the papers [41, 42, 43] have to be regarded as an irreversible part of my contributions to the topic of the thesis. The same is valid for the paper [44] because a generalization of a theorem of Iv. Prodanov [95], presented in it, is used in [39].

In the thesis, I present my generalizations of the famous Stone Duality Theorem, of the Duality Theorems of de Vries and Fedorchuk, as well as of the Equivalence Theorem of Fedorchuk. Some isomorphism theorems for Scott and Tarski consequence systems (which are pairs of a set X and a proximity-type relation on the power set $P(X)$, and arise naturally in Logic and Theoretical Computer Science) are proved as well. Some applications of the results mentioned above are presented. These applications are in the field of General Topology. In particular, a mathematical realization of the original philosophical ideas of A. N. Whitehead for Euclidean spaces is obtained.

The structure of the thesis is the following. In the preliminary Chapter 0, I introduce some notation and list some of the definitions, which are necessary for the exposition; they are from the following fields: general topology, category theory and the theories of proximity spaces and Boolean algebras. Of course, the given list of definitions is far from completeness and in the text I either refer to some textbooks or recall explicitly some of the necessary definitions. I list, as well, very few facts from the areas mentioned above, mainly in order to introduce some notation which I use throughout the text. In Chapter 1, based on the paper [117], I present the basic notion of a *contact algebra* (introduced in [41]) and some of its specializations, the notion of an *MVD-algebra*, and a new proof of the Roeper Representation Theorem [99] which is of great importance for this thesis. This new proof is based on the lattice-theoretical generalizations (presented in [117]) of some well-known theorems from the theory of proximity spaces. The new proof of Roeper's Representation Theorem, given here, is at the root of the proofs of my generalizations of de Vries' ([24]) and Fedorchuk's ([54]) Duality Theorems and Fedorchuk's Equivalence Theorem ([54]). These generalizations are presented in Chapter 2, which is based on the papers [27, 28, 29, 31]. I extend the de Vries Duality to the category of locally compact Hausdorff spaces and continuous maps, and Fedorchuk's Duality (as well as Fedorchuk's Equivalence) to the category of locally compact Hausdorff spaces and continuous skeletal maps; I also prove many other duality theorems about some cofull subcategories of the categories mentioned above. In Chapter 3, whose exposition follows that of the paper [33], using the ideas developed in Chapter 2, I extend the Stone Duality to the category of zero-dimensional locally compact Hausdorff spaces and continuous maps and obtain, as well, some duality theorems for certain cofull subcategories of this category. The next Chapter 4 of the thesis contains many applications in the field of general topology of the results obtained in the previous chapters. In its first section, based on the paper [40], I introduce, using some ideas connected with the MVD-algebras, a class of non-symmetric proximities and construct by means of them all Hausdorff locally compact extensions of a Tychonoff space. In this way I obtain a new generalization of the famous Smirnov Compactification Theorem [103]. The second section of Chapter 4 is based on the results of the paper [32]. In it, using the duality theorems proved in Chapter 2, I characterize the functions between Tychonoff spaces which have continuous extensions of special kind (namely, I regard the following kinds of map extensions: open, quasi-open, perfect, skeletal, injective, surjective) over arbitrary, but fixed, Hausdorff local compactifications of these spaces; in particular, I generalize many results of Smirnov [103], Leader [78], Poljakov [89], Ponomarev [90] and Taĭmanov [6]. In section 3 of Chapter 4, based on the paper [30], I regard an analogous problem to that discussed in section 2, but now I'm interested only in map extensions over Hausdorff zero-dimensional local compactifications. In the proofs, presented in this section, I use the duality theorems from Chapter 3. I obtain, in particular, a generalization of Dwinger's Compactification Theorem [48] and of many results of Banaschewski [8] and Bezhanishvili [13]. In the last fourth section of Chapter 4, based on the paper [34], I obtain a Whiteheadian-type description of Euclidean spaces, spheres, tori and Tychonoff cubes, presenting in this way a mathematical realization of the original philosophical ideas of Whitehead [123, 121, 122] for Euclidean spaces. In the last Chapter 5, based on the paper [39], an isomorphism theory for Scott consequence systems (introduced by D. Vakarelov in [113] in an analogy to a similar notion given by D. Scott in [100]) and for Tarski consequence systems (see [59, 113]) is developed. In this way, it is shown that there exist a very strong connection between the theory of Scott and Tarski consequence systems (which lies in the field of logic) and the theory of some topological objects, namely, some class of hyperspaces. I introduce the category **SSyst** of Scott consequence systems and some natural morphisms between them and show, in particular, that the category of distributive lattices and lattice homomorphisms is isomorphic to a reflective full subcategory of the category **SSyst**.

Let me add that in the beginning of each chapter and of some of the sections, a detailed description of the problems, regarded in the corresponding chapter or section, is given; also, the history of these problems, their motivation, the main results obtained previously in connection with them, as well as the aims of my investigations and a brief description of the main obtained results are presented.

Sections and displayed formulae are numbered consecutively within each chapter, with the chapter number included. When it is necessary, the sections are divided in subsections which are numbered in the same manner as sections. All theorems, propositions, lemmas, definitions, examples, remarks etc. shared only one numbering sequence, i.e., they are not numbered independently. Thus, when I cite a text-unit, it is not necessary to mention its kind, i.e., to explain whether it is a theorem, definition, etc. For example, by 4.3.2.1 (or by Proposition 4.3.2.1) I mean the first text-unit in Subsection 2 of Section 3 of Chapter 4 (which in this thesis happens to be a proposition). So, the text-units have four coordinates. The displayed formulas, however, have only two coordinates: the first of them is the number of the chapter in which the corresponding formula appears and the second is its serial number in the numbering sequence of the displayed formulas in the respective chapter. For example, by (2.11) I mean the eleventh displayed formula in Chapter 2. Of course, not each of the displayed formulae is supplied with a number. The above explanation concerns only those of them which are numbered.

Let me explain how I form the names of the categories in this thesis. The letter "C" stays for "compact spaces", "L" stays for "locally", "H" – for "Hausdorff", "P" – for "perfect maps", "Q" – for "quasi-open maps", "O" – for "open maps", "S" – for "skeletal maps". For example, "**HLC**" means "the category of locally compact Hausdorff spaces and continuous maps", and "**POHLC**" means "the cofull subcategory of the category **HLC** determined by the open perfect maps" (i.e., **POHLC** is the category of locally compact Hausdorff spaces and open perfect maps). When I define a dual (resp., equivalent) category of a subcategory **K** of the category **Top** of all topological spaces and continuous maps, I denote it by **DK** (resp., by **EK**).

Finally, I want to express my gratitude to Professor D. Vakarelov who introduced me to the ideas of Whitehead and de Laguna and called my attention to the paper [99] of P. Roeper, which is of great importance for this thesis. I'm very thankful to him for the wonderful collaboration as well.

Chapter 0

Foreword

0.1 Notation

0.1.1 Concrete objects

0.1.1.1. We denote by:

- $\mathbb{N}^+$ the positive natural numbers,
- $\mathbb{R}$ the real line (with its natural topology),
- $\mathbb{S}^n$ (where $n \in \mathbb{N}^+$) the n-dimensional sphere (with its natural topology),
- $\mathbb{D}$ the set of all dyadic numbers of the interval $(0, 1)$,
- $\mathbb{Q}$ the topological space of all rational numbers with their natural topology,
- $\mathbb{I}$ the subspace $[0, 1]$ $(= \{x \in \mathbb{R} \mid 0 \leq x \leq 1\})$ of $\mathbb{R}$,
- $\mathbf{2}$ the two-point set $\{0, 1\}$ endowed with the discrete topology and the Boolean algebra $\{0, 1\}$ with $0 \neq 1$.

0.1.2 General notation

0.1.2.1. If X is a set then we will denote by

$$P(X)$$

the power set of X, by

$$|X|$$

the cardinality of X, and by

$$Fin(X)$$

the set of all non-empty finite subsets of X.

0.1.2.2. Let X and Y be sets. If $f : X \longrightarrow Y$ is a function, $M \subseteq X$ and $Z \subseteq Y$ then:

- $f|M$ is the restriction of f having M as a domain and Y as a codomain, i.e.,

$$f|M : M \longrightarrow Y,$$

- $f_{\restriction M}$ is the restriction of f having M as a domain and $f(M)$ as a codomain, i.e.,

$$f_{\restriction M} : M \longrightarrow f(M),$$

- f_Z is the restriction of f with domain $f^{-1}(Z)$ and codomain Z, i.e.,

$$f_Z : f^{-1}(Z) \longrightarrow Z,$$

- if $X \subseteq Y$ then we will denote by

$$i_{X,Y} : X \longrightarrow Y$$

  (or, simply, by i_X) the function defined by $i_X(x) = x$, for every $x \in X$,

- we will set

$$f^{\#}(M) = \{y \in Y \mid f^{-1}(y) \subseteq M\},$$

- if $\mathcal{A} \subseteq P(X)$ (resp., $\mathcal{B} \subseteq P(Y)$) then we will write

$$f(\mathcal{A}) \text{ (resp., } f^{-1}(\mathcal{B}))$$

  for the set $\{f(A) \mid A \in \mathcal{A}\}$ (resp., $\{f^{-1}(B) \mid B \in \mathcal{B}\}$).

0.1.2.3. If $(A, \leq)$ is a poset and $a \in A$, we set

$$\downarrow_A (a) = \{b \in A \mid b \leq a\}$$

(we will write even "$\downarrow (a)$" instead of "$\downarrow_A (a)$" when there is no ambiguity); if $B \subseteq A$ then we set

$$\downarrow (B) = \bigcup \{\downarrow (b) \mid b \in B\}.$$

0.1.2.4. If $\mathbf{C}$ denotes a category, we write $X \in |\mathbf{C}|$ if X is an object of $\mathbf{C}$, and $f \in \mathbf{C}(X,Y)$ if f is a morphism of $\mathbf{C}$ with domain X and codomain Y.

0.1.2.5. If (X,τ) is a topological space and M is a subset of X, we denote by $\mathrm{cl}_{(X,\tau)}(M)$ (or simply by $\mathrm{cl}(M)$ or $\mathrm{cl}_X(M)$) the closure of M in (X,τ) and by $\mathrm{int}_{(X,\tau)}(M)$ (or briefly by $\mathrm{int}(M)$ or $\mathrm{int}_X(M)$) the interior of M in (X,τ).

The set of all clopen (= closed and open) subsets of a topological space X will be denoted by

$$CO(X)$$

and the set of all compact open subsets of X by

$$KO(X).$$

0.1.2.6. We denote by:

- **Set** the category of sets and functions,
- **Top** the category of topological spaces and continuous maps,
- **HC** the category of compact Hausdorff spaces and continuous maps,
- **HLC** the category of locally compact Hausdorff spaces and continuous maps,
- **PHLC** the category of locally compact Hausdorff spaces and perfect maps,
- **QHC** the category of compact Hausdorff spaces and quasi-open maps,
- **SHLC** the category of locally compact Hausdorff spaces and continuous skeletal maps,
- **PSHLC** the category of locally compact Hausdorff spaces and skeletal perfect maps,
- **OHLC** the category of locally compact Hausdorff spaces and open maps,
- **OHC** the category of compact Hausdorff spaces and open maps,
- **POHLC** the category of locally compact Hausdorff spaces and open perfect maps,
- **Stone** the category of all compact Hausdorff zero-dimensional spaces (= *Stone spaces*) and their continuous maps,

- **BoolSp** the category of zero-dimensional locally compact Hausdorff spaces (= *Boolean spaces*) and continuous maps,
- **PBoolSp** the category of zero-dimensional locally compact Hausdorff spaces and perfect maps,
- **SBoolSp** the category of zero-dimensional locally compact Hausdorff spaces and continuous skeletal maps,
- **OBoolSp** the category of zero-dimensional locally compact Hausdorff spaces and open maps,
- **QPBoolSp** the category of zero-dimensional locally compact Hausdorff spaces and quasi-open perfect maps,
- **POBoolSp** the category of zero-dimensional locally compact Hausdorff spaces and open perfect maps,
- **QStone** the category of compact zero-dimensional Hausdorff spaces and quasi-open maps,
- **OStone** the category of compact zero-dimensional Hausdorff spaces and open maps,
- **BoolAlg** the category of Boolean algebras and Boolean homomorphisms,
- **CBool** the category of Boolean algebras and complete Boolean homomorphisms,
- **OBool** the category of Boolean algebras and Boolean homomorphisms φ having lower adjoint ψ (i.e., the pair (ψ, φ) forms a Galois connection),
- **DLat** the category of distributive lattices and lattice homomorphisms,
- **Frm** the category of all frames and frame homomorphisms.

0.2 Category theory

0.2.1 Some definitions

We will now remind some of the basic definitions in category theory.

Definition 0.2.1.1. A *category* is a quadruple $\mathbf{A} = (\mathcal{O}, hom, id, \circ)$ consisting of:

(1) a class $\mathcal{O}$, whose members are called $\mathbf{A}$-*objects*,

(2) for each pair (A, B) of $\mathbf{A}$-objects, a set $hom(A, B)$, whose members are called $\mathbf{A}$-*morphisms from* A *to* B (or, simply, *morphisms*) (the statement "$f \in hom(A, B)$" is usually expressed by "$f : A \longrightarrow B$"),

(3) for each $\mathbf{A}$-object A, a morphism $id_A \in hom(A, A)$, called the $\mathbf{A}$-*identity on* A,

(4) a *composition law* associating with each $f \in hom(A, B)$ and each $g \in hom(B, C)$ an $\mathbf{A}$-morphism $g \circ f \in hom(A, C)$, called the *composite of* f *and* g, subject to the following conditions:

(a) composition is associative,

(b) $\mathbf{A}$-identities act as identities with respect to the composition,

(c) the sets $hom(A, B)$ are pairwise disjoint.

For simplicity, when $\mathbf{A} = (\mathcal{O}, hom, id, \circ)$ is a category, we write

$$|\mathbf{A}|$$

instead of $\mathcal{O}$, and

$$\mathbf{A}(A, B)$$

instead of $hom(A, B)$.

Definition 0.2.1.2. Let $\mathbf{A}$ be a category and $A, B \in |\mathbf{A}|$. A morphism $f \in \mathbf{A}(A, B)$ is called an *isomorphism* provided that there exists a morphism $g \in \mathbf{A}(B, A)$ with $g \circ f = id_A$ and $f \circ g = id_B$.

Definition 0.2.1.3. A category $\mathbf{A}$ is said to be a *subcategory* of a category $\mathbf{B}$ provided that the following conditions are satisfied:

(i) $|\mathbf{A}| \subseteq |\mathbf{B}|$,

(ii) for each $A, B \in |\mathbf{A}|$, $\mathbf{A}(A, B) \subseteq \mathbf{B}(A, B)$,

(iii) for each $A \in |\mathbf{A}|$, the $\mathbf{B}$-identity on A is the $\mathbf{A}$-identity on A,

(iv) the composition law in $\mathbf{A}$ is the restriction of the composition law in $\mathbf{B}$ to the morphisms of $\mathbf{A}$;

$\mathbf{A}$ is called a *full subcategory of* $\mathbf{B}$ if, in addition to the above, for each $A, B \in |\mathbf{A}|$, $\mathbf{A}(A, B) = \mathbf{B}(A, B)$.

We say that a subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is a *cofull subcategory* if $|\mathbf{A}| = |\mathbf{B}|$.

Definition 0.2.1.4. The *opposite category* $\mathbf{C}^{op}$ of a category $\mathbf{C}$ has the same objects as $\mathbf{C}$, and a morphism $f : C \longrightarrow D$ in $\mathbf{C}^{op}$ is a morphism $f : D \longrightarrow C$ in $\mathbf{C}$. That is $\mathbf{C}^{op}$ is just $\mathbf{C}$ with all of the morphisms formally turned around. It is convenient to have a notation to distinguish an object (resp. morphism, composition) in $\mathbf{C}$ from the same one in $\mathbf{C}^{op}$. Thus, let us write (in this definition only) $\bar{f} : \bar{D} \longrightarrow \bar{C}$ in $\mathbf{C}^{op}$ for $f : C \longrightarrow D$ in $\mathbf{C}$ and analogously for the composition in $\mathbf{C}^{op}$. With this notation we can define the composition and identities in $\mathbf{C}^{op}$ in terms of the corresponding operations in $\mathbf{C}$, namely,

$$id_{\bar{C}} = \overline{id_C}, \text{ and}$$

$$\bar{f}\bar{\circ}\bar{g} = \overline{g \circ f}.$$

Definition 0.2.1.5. If $\mathbf{A}$ and $\mathbf{B}$ are categories, then a *covariant functor* (resp., *contravariant functor*) F *from* $\mathbf{A}$ *to* $\mathbf{B}$ is a function that assigns to each $A \in |\mathbf{A}|$ a $\mathbf{B}$-object $F(A)$, and to each $f \in \mathbf{A}(A, B)$ an $F(f) \in \mathbf{B}(F(A), F(B))$ (resp., $F(f) \in \mathbf{B}(F(B), F(A))$), in such a way that

(1) F preserves identity morphisms, and

(2) $F(f \circ g) = F(f) \circ F(g)$ (resp., $F(f \circ g) = F(g) \circ F(f)$) whenever $f \circ g$ is defined.

The covariant and contravariant functors F from $\mathbf{A}$ to $\mathbf{B}$ will be denoted by $F : \mathbf{A} \longrightarrow \mathbf{B}$.

In the sequel, by a "functor", we will mean a "covariant functor".

Note that any contravariant functor from $\mathbf{A}$ to $\mathbf{B}$ is a covariant functor from $\mathbf{A}$ to $\mathbf{B}^{op}$ (or from $\mathbf{A}^{op}$ to $\mathbf{B}$) so that we can even avoid the use of the notion of a contravariant functor. However, in many cases it is very convenient to have such a notion.

For any category $\mathbf{A}$, there is *the identity functor* $Id_{\mathbf{A}} : \mathbf{A} \longrightarrow \mathbf{A}$ defined by $Id_{\mathbf{A}}(A) = A$ for every $A \in |\mathbf{A}|$, and $Id_{\mathbf{A}}(f) = f$ for every $f \in \mathbf{A}(A, B)$.

If $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $G : \mathbf{B} \longrightarrow \mathbf{C}$ are two functors then the *composite functor* $G \circ F : \mathbf{A} \longrightarrow \mathbf{C}$ is defined by $(G \circ F)(f) = G(F(f))$ and $(G \circ F)(A) = G(F(A))$, for any $\mathbf{A}$-morphism f and any $\mathbf{A}$-object A.

Definition 0.2.1.6. Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a functor or a contravariant functor; then:

- F is called *faithful* (resp., *full*) provided that all the hom-set restrictions are injective (resp., surjective),

- F is called *isomorphism-dense* if for any $B \in |\mathbf{B}|$ there exists $A \in |\mathbf{A}|$ such that $F(A)$ is isomorphic to B.

A functor (resp., a contravariant functor) F is called an *equivalence* (resp., *duality*) if it is full, faithful, and isomorphism-dense.

Two categories $\mathbf{A}$ and $\mathbf{B}$ are *equivalent* (resp., *dually equivalent*) if there exists an equivalence (resp., duality) functor from $\mathbf{A}$ to $\mathbf{B}$.

A functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ is called an *isomorphism* provided that there exists a functor $G : \mathbf{B} \longrightarrow \mathbf{A}$ such that $G \circ F = Id_{\mathbf{A}}$ and $F \circ G = Id_{\mathbf{B}}$. If there exists an isomorphism $F : \mathbf{A} \longrightarrow \mathbf{B}$, then the category $\mathbf{A}$ is said to be *isomorphic to* the category $\mathbf{B}$.

Definition 0.2.1.7. Let $\mathbf{A}$ and $\mathbf{B}$ be categories and $F, G : \mathbf{A} \longrightarrow \mathbf{B}$ be functors. A *natural transformation τ from F to G* (denoted by $\tau : F \longrightarrow G$) is a function that assigns to each $\mathbf{A}$-object A a $\mathbf{B}$-morphism $\tau_A : F(A) \longrightarrow G(A)$ in such a way that the following *naturality condition* holds:

$$G(f) \circ \tau_A = \tau_{A'} \circ F(f),$$

for each $\mathbf{A}$-morphism $f : A \longrightarrow A'$. A natural transformation $\tau : F \longrightarrow G$ whose components τ_A are isomorphisms is called a *natural isomorphism from F to G*; in this case we write

$$F \cong G.$$

Definition 0.2.1.8. A *contravariant adjunction*

$$(T, S, \eta, \varepsilon) : \mathbf{A} \longrightarrow \mathbf{B}$$

between two categories $\mathbf{A}$ and $\mathbf{B}$ consists of two contravariant functors

$$T : \mathbf{A} \longrightarrow \mathbf{B}$$

and

$$S : \mathbf{B} \longrightarrow \mathbf{A}$$

and two natural transformations

$$\eta : Id_{\mathbf{B}} \longrightarrow T \circ S$$

and

$$\varepsilon : Id_{\mathbf{A}} \longrightarrow S \circ T$$

(called, respectively, *unit* and *co-unit*) such that

$$T(\varepsilon_A) \circ \eta_{TA} = id_{TA}$$

and

$$S(\eta_B) \circ \varepsilon_{SB} = id_{SB},$$

for all $A \in |\mathbf{A}|$ and $B \in |\mathbf{B}|$.

It is well known that a contravariant adjunction $(T, S, \eta, \varepsilon)$ *is a duality iff* η *and* ε *are natural isomorphisms.*

Definition 0.2.1.9. Let $\mathbf{A}$ be a subcategory of $\mathbf{B}$, and let B be a $\mathbf{B}$-object.

(1) An $\mathbf{A}$-*reflection* (or $\mathbf{A}$-*reflection arrow*) for B is a $\mathbf{B}$-morphism $r : B \longrightarrow A$ from B to an $\mathbf{A}$-object A with the following universal property:

for any $\mathbf{B}$-morphism $f : B \longrightarrow A'$ from B into some $\mathbf{A}$-object A', there exists a unique $\mathbf{A}$-morphism $f' : A \longrightarrow A'$ such that $f = f' \circ r$.

By an "abuse of language", an $\mathbf{A}$-object A is called an $\mathbf{A}$-*reflection for* $B \in |\mathbf{B}|$ provided that there exists an $\mathbf{A}$-reflection $r : B \longrightarrow A$ for B with codomain A.

(2) $\mathbf{A}$ is called a *reflective subcategory of* $\mathbf{B}$ provided that each $\mathbf{B}$-object has an $\mathbf{A}$-reflection.

Definition 0.2.1.10. A category $\mathbf{C}$ is called a *construct* (or a *concrete category over* $\mathbf{Set}$) if there exists a faithful functor $U : \mathbf{C} \longrightarrow \mathbf{Set}$. U is called a *forgetful functor* (or *underlying functor*). For each $\mathbf{C}$-object A, $U(A)$ is called the *underlying set of* A and for each $\mathbf{C}$-morphism f, $U(f)$ is called the *underlying map of the morphism* f.

Definition 0.2.1.11. An object H is called a *coseparator* (or a *cogenerator*) *in a category* $\mathbf{C}$ provided that, for each pair of distinct $\mathbf{C}$-morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B$, there is a $\mathbf{C}$-morphism $k : B \longrightarrow H$ such that $k \circ f \neq k \circ g$.

Definition 0.2.1.12. For any category $\mathbf{A}$ and any $\mathbf{A}$-object A, there is the *covariant hom-functor*

$$\mathbf{A}(A, -) : \mathbf{A} \longrightarrow \mathbf{Set},$$

defined by

$$\mathbf{A}(A, -)(B) = \mathbf{A}(A, B),$$

for any $\mathbf{A}$-object B, and, for each $f \in \mathbf{A}(B, C)$,

$$\mathbf{A}(A, -)(f) : \mathbf{A}(A, -)(B) \longrightarrow \mathbf{A}(A, -)(C)$$

is determined by the formula

$$\mathbf{A}(A,-)(f)(g) = f \circ g,$$

for any $g \in \mathbf{A}(A,B)$. Usually, one writes $\mathbf{A}(A,B)$ instead of $\mathbf{A}(A,-)(B)$, and $\mathbf{A}(A,f)$ instead of $\mathbf{A}(A,-)(f)$. Also, instead of $\mathbf{A}(A,-)$, the notation $hom_{\mathbf{A}}(A,-)$ is often used.

Analogously, one defines the *contravariant hom-functor*

$$\mathbf{A}(-,A) : \mathbf{A} \longrightarrow \mathbf{Set}$$

by

$$\mathbf{A}(-,A)(B) = \mathbf{A}(B,A)$$

for any $\mathbf{A}$-object B, and, for each $f \in \mathbf{A}(B,C)$,

$$\mathbf{A}(-,A)(f) : \mathbf{A}(-,A)(C) \longrightarrow \mathbf{A}(-,A)(B)$$

is determined by the formula

$$\mathbf{A}(-,A)(f)(g) = g \circ f,$$

for any $g \in \mathbf{A}(C,A)$. Usually, one writes $\mathbf{A}(B,A)$ instead of $\mathbf{A}(-,A)(B)$, and $\mathbf{A}(f,A)$ instead of $\mathbf{A}(-,A)(f)$.

A functor $F : \mathbf{A} \longrightarrow \mathbf{Set}$ is called *representable (by an* $\mathbf{A}$*-object* A*)* provided that F is naturally isomorphic to the hom-functor $\mathbf{A}(A,-) : \mathbf{A} \longrightarrow \mathbf{Set}$.

A contravariant functor $F : \mathbf{A} \longrightarrow \mathbf{Set}$ is called *representable (by an* $\mathbf{A}$*-object* A*)* provided that F is naturally isomorphic to the contravariant hom-functor $\mathbf{A}(-,A) : \mathbf{A} \longrightarrow \mathbf{Set}$.

Definition 0.2.1.13 ([37, 38])**.** Let $U : \mathbf{A} \longrightarrow \mathbf{Set}$ and $V : \mathbf{B} \longrightarrow \mathbf{Set}$ be faithful (covariant) functors (in [37, 38] they are even arbitrary functors, but here we will regard only the situation with faithful functors (as it is done in [93])). A contravariant adjunction $(T,S,\eta,\varepsilon) : \mathbf{A} \longrightarrow \mathbf{B}$ is called *strictly* $(\tilde{A},\tilde{B})$*-represented*, with $\tilde{A} \in |\mathbf{A}|$ and $\tilde{B} \in |\mathbf{B}|$, if

$$V \circ T = \mathbf{A}(-,\tilde{A}) \text{ and } U \circ S = \mathbf{B}(-,\tilde{B});$$

(T,S,η,ε) is *strictly represented* if it is strictly $(\tilde{A},\tilde{B})$-represented for suitable $\tilde{A},\tilde{B}$. For such adjunctions, the units and co-units are essentially evaluation maps; more precisely, for $A \in |\mathbf{A}|$, $x \in UA$, and $B \in \mathbf{B}$, $y \in VB$, consider

$$\varphi_{A,x} : \mathbf{A}(A,\tilde{A}) \longrightarrow U\tilde{A},\ \ s \mapsto (Us)(x),$$

$$\psi_{B,y} : \mathbf{B}(B, \tilde{B}) \longrightarrow V\tilde{B},\ t \mapsto (Vt)(y),$$

$$\tau = \tau_{T,S} : U\tilde{A} \longrightarrow V\tilde{B},\ \tilde{x} \mapsto (V[(U\varepsilon_{\tilde{A}})(\tilde{x})])(1_{\tilde{A}}),$$

$$\sigma = \sigma_{T,S} : V\tilde{B} \longrightarrow U\tilde{A},\ \tilde{y} \mapsto (U[(V\eta_{\tilde{B}})(\tilde{y})])(1_{\tilde{B}});$$

then τ and σ are bijective, with $\sigma = \tau^{-1}$, and

$$V[(U\varepsilon_A)(x)] = \tau \circ \varphi_{A,x}, \quad U[(V\eta_B)(y)] = \sigma \circ \psi_{B,y}.$$

We call a strictly $(\tilde{A}, \tilde{B})$-represented adjunction $(T, S, \eta, \varepsilon)$ *natural* if, for every $A \in |\mathbf{A}|$ and $B \in |\mathbf{B}|$,

$$((U\varepsilon_A)(x) : TA \longrightarrow \tilde{B})_{x \in UA} \text{ is a } V\text{-initial family, and}$$

$$((V\eta_B)(y) : SB \longrightarrow \tilde{A})_{y \in VB} \text{ is a } U\text{-initial family.}$$

(Recall that if $W : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor, then a family $(f_i : A \longrightarrow A_i)_{i \in I}$ of morphisms in $\mathbf{C}$ (such families are called $\mathbf{C}$-*sources*) is said to be W-*initial* if, for any $\mathbf{C}$-source $(g_i : B \longrightarrow A_i)_{i \in I}$ and any $\mathbf{D}$-morphism $h : WB \longrightarrow WA$ with $Wf_i \circ h = Wg_i$ $(i \in I)$, there exists a unique $\mathbf{C}$-morphism $t : B \longrightarrow A$ with $Wt = h$ and $f_i \circ t = g_i$ $(i \in I)$.)

For the notions and notation not defined here see [1, 75].

0.2.2 Some theorems

Theorem 0.2.2.1. *A (contravariant) functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ is an equivalence (resp., a duality) iff there exists a (contravariant) functor $G : \mathbf{B} \longrightarrow \mathbf{A}$ such that $Id_{\mathbf{A}} \cong G \circ F$ and $F \circ G \cong Id_{\mathbf{B}}$.*

Proposition 0.2.2.2. *An object C of a category $\mathbf{A}$ is a coseparator in $\mathbf{A}$ if and only if the contravariant functor $\mathbf{A}(-, C) : \mathbf{A} \longrightarrow \mathbf{Set}$ is faithful.*

0.3 Boolean algebras

0.3.1 Some definitions

Definition 0.3.1.1. A binary relation "$\leq$" in a set X is called a *partial order* (or, simply, an *order*) if it is reflexive, transitive and antisymmetric; if "$\leq$" is an order in X then the pair $(X, \leq)$ is called a *partially ordered set* (or, simply, an *ordered set*, and even a *poset*). In a poset $(X, \leq)$, for any $a, b \in X$, the symbol $a \vee b$ denotes $\sup\{a, b\}$, i.e., the smallest element $c \in X$ – if one exists - such that $a \leq c$ and $b \leq c$; it is called the *join of* a and b; further, $a \wedge b$ stands for $\inf\{a, b\}$, which is defined dually, and is called the *meet of* a and b. The definitions of the join and meet of a subset of a poset are analogous. The meet and join of a subset E of a poset, if they exist, are denoted by $\bigwedge E$ and $\bigvee E$ respectively.

Definition 0.3.1.2. A *pseudolattice* is a poset having all finite non-empty meets and joins; the pseudolattices with a *bottom element* (= *zero* = *the least element*) (denoted by 0) will be called *0-pseudolattices*; the pseudolattices with a bottom element and *top element* (= *unit* = *the greatest element*) (denoted by 1) will be called *lattices*. We do not require the elements 0 and 1 to be distinct. The *lattice homomorphisms* are the functions between lattices which preserve the distinguished elements 0 and 1 and the operations join and meet.

A lattice in which every subset has a meet (i.e., an infimum) and a join (i.e., a supremum) is said to be *complete*.

Let L be a 0-pseudolattice and $a \in L$. An element $\neg a$ of L is called a *pseudo-complement of* a provided that $\neg a$ is the largest element of L whose meet with a is 0; that is $(\forall x \in L)(a \wedge x = 0 \text{ iff } x \leq \neg a)$.

A pseudolattice is called *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all a, b, c.

In any lattice, an element x satisfying $x \wedge a = 0$ and $x \vee a = 1$ is called a *complement of* a. In a distributive lattice complements are unique when they exist.

A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element a will be denoted by a^*.

The *Boolean algebra homomorphisms* are just the lattice homomorphisms.

Definition 0.3.1.3. If $(A, \leq)$ is a poset and $B \subseteq A$ then B is said to be a *dense subset of* A if for any $a \in A \setminus \{0\}$ there exists $b \in B \setminus \{0\}$ such that $b \leq a$; when $(B, \leq_1)$ is a poset and $f : A \longrightarrow B$ is a map, then we will say that f is a *dense map* if $f(A)$ is a dense subset of B.

Definition 0.3.1.4. A subset F of a lattice B is called a *filter in* B if it satisfies the following conditions:

(F1) $1 \in F$,

(F2) $a, b \in F$ implies that $a \wedge b \in F$, and

(F3) $a \leq b$ and $a \in F$ imply that $b \in F$.

A filter F is a *proper filter* if $0 \notin F$.

In the sequel, by a "filter", we will always understand a "proper filter".

An *ultrafilter in* B is a maximal (with respect to the inclusion) filter in B. The set of all ultrafilters in B will be denoted by

$$Ult(B).$$

Definition 0.3.1.5. A subset I of a 0-pseudolattice A is an *ideal of* A if the following three conditions are satisfied:

(I1) $0 \in I$,

(I2) if $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$,

(I3) if $x \in I$ and $y \in I$, then $x \vee y \in I$.

An ideal I of a lattice A is called a *proper ideal* if $1 \notin I$.

In the sequel, by an "ideal", we will always understand a "proper ideal".

An ideal I is a *complete ideal* if $\bigvee M \in I$ for each subset M of I such that $\bigvee M$ exists. For every filter F of a Boolean algebra A,

$$F^* = \{x^* \mid x \in F\}$$

is an ideal of A, the *ideal dual to* F. For every ideal I of A,

$$I^* = \{x^* \mid x \in I\}$$

is a filter in A, the *filter dual to* I.

Definition 0.3.1.6. A homomorphism φ between two Boolean algebras is called *complete* if it preserves all joins (and, consequently, all meets) that happen to exist; this means that if $\{a_i\}$ is a family of elements in the domain of φ with join a, then the family $\{\varphi(a_i)\}$ has a join and that join is equal to $\varphi(a)$.

For the notions and notation not defined here see [48, 75, 77, 102].

0.3.2 Some theorems

Proposition 0.3.2.1. *A filter u in a Boolean algebra B is an ultrafilter iff $\forall b \in B$, either $b \in u$ or $b^* \in u$. Also, a filter u in a Boolean algebra B is an ultrafilter iff it satisfies the following axiom:*

(G) If $x \vee y \in u$ then $x \in u$ or $y \in u$.

The following well-known variant of the famous "Grill Lemma", which can be proved exactly as Lemma 5.7 of [87], is valid for Boolean algebras:

Theorem 0.3.2.2. *Let $(B, 0, 1, \vee, \wedge, ^*)$ be a Boolean algebra and Γ be a subset of B such that $0 \notin \Gamma$ and $a \vee b \in \Gamma$ iff $a \in \Gamma$ or $b \in \Gamma$. If $a_0 \in \Gamma$ then there exists an ultrafilter u in B such that $a_0 \in u$ and $u \subseteq \Gamma$.*

We will need the following well-known fact (see, e.g., [102]):

Proposition 0.3.2.3. *Let A and B be complete Boolean algebras and $\varphi : A \longrightarrow B$ be a Boolean monomorphism (i.e., φ is a Boolean homomorphism and an injection). If $\varphi(A)$ is dense in B then $\varphi(A) = B$.*

Definition and Proposition 0.3.2.4. Let us recall the notion of *lower adjoint* for posets. Let $\varphi : A \longrightarrow B$ be an order-preserving map between posets. If

$$\varphi_\Lambda : B \longrightarrow A$$

is an order-preserving map satisfying the following condition
(Λ) for all $a \in A$ and all $b \in B$, $b \leq \varphi(a)$ iff $\varphi_\Lambda(b) \leq a$
(i.e., the pair $(\varphi_\Lambda, \varphi)$ forms a *Galois connection between posets B and A*) then we will say that φ_Λ is a *lower adjoint* of φ. It is easy to see that condition (Λ) is equivalent to the following two conditions:
(Λ1) $\forall b \in B$, $\varphi(\varphi_\Lambda(b)) \geq b$;
(Λ2) $\forall a \in A$, $\varphi_\Lambda(\varphi(a)) \leq a$.
It is well known that $\varphi \circ \varphi_\Lambda \circ \varphi = \varphi$, $\varphi_\Lambda \circ \varphi \circ \varphi_\Lambda = \varphi_\Lambda$, and

(1) $\quad \varphi_\Lambda$ preserves all joins which exist in B.

Further, φ is an injection iff

(2) $\quad \varphi_\Lambda(\varphi(a)) = a, \forall a \in A$;

φ is a surjection iff

(3) $\quad \varphi(\varphi_\Lambda(b)) = b, \forall b \in B$.

Note that if $\varphi(0) = 0$ then:
(a) $\varphi_\Lambda(0) = 0$ (use (Λ2)), and
(b) $\varphi_\Lambda(b) \neq 0$, for every $b \in B \setminus \{0\}$ (use (Λ1)).

Note that if $\varphi : A \longrightarrow B$ is an (order-preserving) map between posets, A has all meets and φ preserves them then, by the Adjoint Functor Theorem (see, e.g., [75]), φ has a lower (or *left*) adjoint φ_Λ and, for all $b \in B$,

(4) $\quad \varphi_\Lambda(b) = \bigwedge\{a \in A \mid \varphi(a) \geq b\}.$

Finally, if $\psi : A \longrightarrow B$ is an (order-preserving) map between posets, A has all joins and ψ preserves them, then, by the Adjoint Functor Theorem, ψ has a *right adjoint*

$$\psi_P : B \longrightarrow A,$$

i.e., setting $\varphi = \psi_P$, we have that $\psi = \varphi_\Lambda$; ψ_P preserves all meets which exist in B.

Recall that if $\varphi' : B \longrightarrow C$ is a map between posets, B has all meets and φ' preserves them, then $(\varphi' \circ \varphi)_\Lambda = \varphi_\Lambda \circ \varphi'_\Lambda$.

Definition and Proposition 0.3.2.5. Let us fix the notation for the Stone Duality. The Stone contravariant functors which define the Stone duality will be denoted by

$$S^a : \mathbf{BoolAlg} \longrightarrow \mathbf{Stone}$$

and

$$S^t : \mathbf{Stone} \longrightarrow \mathbf{BoolAlg}.$$

For every $A \in |\mathbf{BoolAlg}|$,

$$S^a(A) \text{ is the set } Ult(A)$$

of all ultrafilters in A endowed with a topology having as an open base the family $\{\lambda^S_A(a) \mid a \in A\}$, where

$$\lambda^S_A(a) = \{u \in Ult(A) \mid a \in u\}$$

for every $a \in A$.

For every $X \in |\mathbf{Stone}|$,

$$S^t(X) = CO(X).$$

If $f \in \mathbf{Stone}(X, Y)$ then

$$\varphi = S^t(f) \in \mathbf{BoolAlg}(S^t(Y), S^t(X))$$

is defined by the formula

$$\varphi(F) = f^{-1}(F),$$

for every $F \in CO(Y)$.

If $\varphi \in \mathbf{BoolAlg}(B_1, B_2)$ then

$$f = S^a(\varphi) \in \mathbf{Stone}(S^a(B_2), S^a(B_1))$$

is defined by the formula

$$f(u) = \varphi^{-1}(u),$$

for every $u \in Ult(B_2)$.

For every Boolean algebra A, the map

$$\lambda_A^S : A \longrightarrow S^t(S^a(A)),\ a \mapsto \lambda_A^S(a),$$

is a Boolean isomorphism.

Definition and Proposition 0.3.2.6. For every subset E of a Boolean algebra A, the set

$$\{x \in A \mid x \leq e_1 \vee \ldots \vee e_n,\ \text{where } n \in \mathbb{N}^+ \text{ and } e_1, \ldots, e_n \in E\}$$

is an ideal of A, called *the ideal generated by E*; it is the least ideal of A including E.

Let I be an ideal of A and F its dual filter. I is *trivial* if $I = \{0\}$, i.e., if F is the *trivial filter*. I is *principal* if I is the ideal $\{x \in A \mid x \leq a\}$ generated by some $a \in A$, i.e., if F is the *principal filter generated by* a^*. I is *prime* if it is proper and $x \wedge y \in I$ implies that $x \in I$ or $y \in I$; i.e., if F is a *prime filter*.

Definition and Proposition 0.3.2.7. Recall that a *frame* is a complete lattice L satisfying the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\},$$

for every $a \in L$ and every $S \subseteq L$. A lattice homomorphism between two frames is called a *frame homomorphism* if it preserves arbitrary joins.

If A is a distributive 0-pseudolattice, we denote by

$$Idl(A)$$

the frame of all ideals of A, where the meet of an arbitrary family of ideals is the intersection of the family, and the join is the ideal generated by the union of the family.

If $J \in Idl(A)$ then we will write $\neg_A J$ (or simply $\neg J$) for the *pseudocomplement of J* in $Idl(A)$, i.e.,

$$\neg J = \bigvee \{I \in Idl(A) \mid I \wedge J = \{0\}\}.$$

Note that

$$\neg J = \{a \in A \mid (\forall b \in J)(a \wedge b = 0)\}$$

(see Stone [107]).

Recall that an ideal J of A is called *simple* (Stone [107]) if

$$J \vee \neg J = A.$$

The set of all simple ideals of A will be denoted by

$$Si(A).$$

As it is proved in [107], the set $Si(A)$ is a Boolean algebra with respect to the lattice operations in $Idl(A)$.

Recall also that the *regular elements of the frame* $Idl(A)$ (i.e., those $J \in Idl(A)$ for which $\neg\neg J = J$) are called *normal ideals* (Stone [107]).

0.4 General topology

0.4.1 Some definitions and notation

We don't assume that completely regular topological spaces and normal topological spaces are T_1-spaces. When they are supposed to be T_1-spaces, we use the terms, respectively, Tychonoff spaces and T_4-spaces.

Definition 0.4.1.1. An *extension* of a space X is a pair (Y, e), where Y is a space and $e : X \longrightarrow Y$ is a dense embedding of X into Y.

Two extensions (Y_i, e_i), $i = 1, 2$, of X are called *isomorphic* (or *equivalent*) if there exists a homeomorphism $\varphi : Y_1 \longrightarrow Y_2$ such that $\varphi \circ e_1 = e_2$. Clearly, the relation of isomorphism is an equivalence in the class of all extensions of X; the equivalence class of an extension (Y, e) of X will be denoted by

$$[(Y, e)].$$

We write

$$(Y_1, e_1) \leq (Y_2, e_2) \text{ (respectively, } (Y_1, e_1) \leq_{sur} (Y_2, e_2))$$

and say that the extension (Y_2, e_2) is *projectively larger* than the extension (Y_1, e_1) if there exists a continuous mapping (resp., a continuous surjection) $\varphi : Y_2 \longrightarrow Y_1$ such that $\varphi \circ e_2 = e_1$. These relations are *preorders* (i.e., they are reflexive and transitive). We write

$$(Y_1, e_1) \leq_{in} (Y_2, e_2)$$

and say that the extension (Y_2, e_2) is *injectively larger* than the extension (Y_1, e_1) if there exists a continuous mapping $\varphi : Y_1 \longrightarrow Y_2$ such that $\varphi \circ e_1 = e_2$ and φ is a homeomorphism from Y_1 to the subspace $\varphi(Y_1)$ of Y_2. This relation is also a preorder. The equivalence relations associated with these three preorders (i.e., (Y_1, e_1) projectively (injectively) larger than (Y_2, e_2) and conversely) coincide with the relation of isomorphism (defined above) on the class of all Hausdorff extensions of X (see [9]).

Setting for every two Hausdorff extensions $(Y_i, e_i),\ i = 1, 2$, of a Hausdorff space X,

$$[(Y_1, e_1)] \leq [(Y_2, e_2)] \text{ iff } (Y_1, e_1) \leq (Y_2, e_2),$$

we obtain a well-defined relation on the set of all, up to equivalence, Hausdorff extensions of X; it is already an order. The same can be done for the preorders $\leq_{sur}$ and $\leq_{in}$.

Notation 0.4.1.2. The Alexandroff (one-point) compactification of a locally compact Hausdorff non-compact space X will be denoted by

$$\alpha X;$$

the added point will be usually denoted by ∞_X, so that we can think that

$$\alpha X = X \cup \{\infty_X\}.$$

The Stone-Čech compactification of a Tychonoff space X will be denoted by βX.

The ordered set of all, up to equivalence, Hausdorff locally compact (resp., compact) extensions of a space (X, τ) will be denoted by

$$(\mathcal{LC}(X, \tau), \leq) \text{ (resp., } (\mathcal{C}(X, \tau), \leq)),$$

where " $\leq$" is the projective order defined in 0.4.1.1. We will also write briefly $(\mathcal{LC}(X), \leq)$ and $(\mathcal{C}(X), \leq)$.

Let X be a space and (Y, f) be an extension of X. If $A \subseteq X$ then we set

$$Ex_Y(A) = Y \setminus cl_Y(f(X \setminus A)).$$

We will often write $Ex(A)$ instead of $Ex_Y(A)$ when this does not lead to ambiguity.

Definitions 0.4.1.3. Recall that:

- a continuous map is *closed* if the image of each closed set is closed;
- a continuous map is *open* if the image of each open set is open;
- a map is *perfect* if it is compact (i.e., point inverses are compact sets) and closed;
- a continuous map $f : X \longrightarrow Y$ is called *quasi-open* ([82]) if for every non-empty open subset U of X, $\mathrm{int}(f(U)) \neq \emptyset$ holds;
- a function $f : X \longrightarrow Y$ is called *skeletal* ([83]) if

  (5) $\quad \mathrm{int}(f^{-1}(\mathrm{cl}(V))) \subseteq \mathrm{cl}(f^{-1}(V))$

  for every open subset V of Y.
- A continuous map $f : X \longrightarrow Y$ is *irreducible* if $f(X) = Y$ and if, for each proper closed subset A of X, $f(A) \neq Y$.

Definition 0.4.1.4. Recall that two subsets A, B of a topological space X are *completely separated* if there is a continuous real-valued function $f : X \longrightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Definition 0.4.1.5. Recall that a subset F of a topological space (X, τ) is called *regular closed* if $F = \mathrm{cl}(\mathrm{int}(F))$. Clearly, F is regular closed iff it is the closure of an open set. The collection of all regular closed subsets of (X, τ) will be denoted by

$$RC(X, \tau)$$

(we will often write simply $RC(X)$).

A subset U of topological space (X, τ) such that $U = \mathrm{int}(\mathrm{cl}(U))$ is said to be *regular open*. The set of all regular open subsets of (X, τ) will be denoted by

$$RO(X, \tau)$$

(or briefly, by $RO(X)$).

Notation 0.4.1.6. Let (X, τ) be a topological space. We will denote by

$$CR(X, \tau)$$

the family of all compact regular closed subsets of (X, τ). We will often write $CR(X)$ instead of $CR(X, \tau)$.

For all undefined here notions and notation see [53].

0.4.2 Some theorems

Proposition 0.4.2.1. (a) *For any topological space* (X,τ)*, the collection* $RC(X,\tau)$ *becomes a complete Boolean algebra* $(RC(X,\tau),0,1,\wedge,\vee,^*)$ *under the following operations:*

$$1=X, 0=\emptyset, F^*=\mathrm{cl}(X\setminus F), F\vee G=F\cup G, F\wedge G=\mathrm{cl}(\mathrm{int}(F\cap G)).$$

The infinite operations are given by the formulas:

$$\bigvee_{\gamma\in\Gamma} F_\gamma=\mathrm{cl}(\bigcup_{\gamma\in\Gamma} F_\gamma)\ (=\mathrm{cl}(\bigcup_{\gamma\in\Gamma}\mathrm{int}(F_\gamma))=\mathrm{cl}(\mathrm{int}(\bigcup_{\gamma\in\Gamma} F_\gamma))),$$

and

$$\bigwedge\{F_\gamma\mid\gamma\in\Gamma\}=\mathrm{cl}(\mathrm{int}(\bigcap\{F_\gamma\mid\gamma\in\Gamma\})).$$

(b) *For any topological space* (X,τ)*, the collection* $RO(X,\tau)$ *becomes a complete Boolean algebra* $(RO(X,\tau),0,1,\wedge,\vee,^*)$ *under the following operations:*

$$U\vee V=\mathrm{int}(\mathrm{cl}(U\cup V)),\ U\wedge V=U\cap V,\ U^*=\mathrm{int}(X\setminus U),\ 0=\emptyset,\ 1=X.$$

The infinite operations are given by the formulas:

$$\bigwedge_{i\in I} U_i=\mathrm{int}(\mathrm{cl}(\bigcap_{i\in I} U_i))\ (=\mathrm{int}(\bigcap_{i\in I} U_i))$$

and

$$\bigvee_{i\in I} U_i=\mathrm{int}(\mathrm{cl}(\bigcup_{i\in I} U_i)).$$

The following statement is well-known (see, e.g., [21], p.271).

Lemma 0.4.2.2. *Let* X *be a dense subspace of a topological space* Y*. Then the functions*

$$r: RC(Y)\longrightarrow RC(X),\ F\mapsto F\cap X,$$

and

$$e: RC(X)\longrightarrow RC(Y),\ G\mapsto \mathrm{cl}_Y(G),$$

are Boolean isomorphisms between Boolean algebras $RC(X)$ *and* $RC(Y)$*, and* $e\circ r=id_{RC(Y)}$*,* $r\circ e=id_{RC(X)}$*. (We will sometimes write* $r_{X,Y}$ *(resp.,* $e_{X,Y}$*) instead of* r *(resp.,* e*).)*

We will often use (even without citing it explicitly) the following well-known assertion (see, e.g., [53, Theorem 3.3.2]):

Proposition 0.4.2.3. *For every compact subspace K of a locally compact space X and every open set $V \subseteq X$ that contains K there exists an open set $U \subseteq X$ such that $K \subseteq U \subseteq \mathrm{cl}(U) \subseteq V$ and $\mathrm{cl}(U)$ is compact.*

The next proposition is well known (see, e.g., [5]):

Proposition 0.4.2.4. *Let $f : X \longrightarrow Y$ be a perfect map between two locally compact Hausdorff non-compact spaces. Then the map f has a continuous extension*

$$\alpha(f) : \alpha X \longrightarrow \alpha Y;$$

moreover, $\alpha(f)(\infty_X) = \infty_Y$.

0.5 Proximity spaces

0.5.1 Some definitions and notation

Definitions 0.5.1.1. Let X be a non-empty set. A symmetric binary relation δ on the power set $P(X)$ of X is called a *basic proximity on X* or simply *proximity on X*, if it satisfies the following conditions:

(P1) $\emptyset(-\delta)A$ for every $A \subseteq X$ ("$-\delta$" means "not δ");

(P2) $A\delta A$ for every $A \neq \emptyset$;

(P3) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;

The pair (X, δ) is called a *basic proximity space* or simply *proximity space*. When x is a point of X, we write $x\delta A$ in place of $\{x\}\delta A$. A basic proximity is called *separated* if it satisfies the axiom

(SP) $\forall x, y \in X,\ x\delta y$ implies $x = y$.

In such a case the pair (X, δ) is called a *separated basic proximity space.*

A function $f : (X_1, \delta_1) \longrightarrow (X_2, \delta_2)$ between two basic proximity spaces (X_i, δ_i), $i = 1, 2$, is called *proximally continuous* (or, a *proximity mapping*) if $A\delta_1 B$ implies $f(A)\delta_2 f(B)$ $(A, B \subseteq X_1)$.

If $\delta_i,\ i = 1, 2$, are two basic proximities on a set X then we write

$$\delta_1 \geq \delta_2$$

if the identity function $id : (X, \delta_1) \longrightarrow (X, \delta_2)$ is proximally continuous (i.e., if, for $A, B \subseteq X,\ A\delta_1 B$ implies $A\delta_2 B$). This relation is an order in the set of all basic proximities on a set X.

If M is a subset of X then the *restriction* δ_M of δ to M is defined as follows: for $A, B \subseteq M$, $A\delta_M B$ iff $A\delta B$. It is easy to see that (M, δ_M) is a basic proximity space.

We write

$$A \ll_\delta B$$

(or simply $A \ll B$) if $A(-\delta)(X \setminus B)$. When x is a point of X, we write $x \ll A$ in place of $\{x\} \ll A$.

A separated basic proximity δ on X which satisfies the condition

(EF) If $A \ll B$, then there exists a $C \subseteq X$ such that $A \ll C \ll B$

is called an *Efremovič proximity* (or an *EF-proximity*); the pair (X, δ) is called an *Efremovič proximity space* if δ is an Efremovič proximity.

Definitions 0.5.1.2. Let (X, δ) be a basic proximity. Then the operator cl_δ on $P(X)$ defined by

$$\mathrm{cl}_\delta(A) = \{x \in X : x\delta A\}.$$

is a Čech closure operator (see [118]). Hence

$$\tau_\delta = \{X \setminus A : A = \mathrm{cl}_\delta(A)\}$$

is a topology on X.

A basic proximity δ on a set X which satisfies the condition

(LO) $\mathrm{cl}_\delta(A)\ \delta\ \mathrm{cl}_\delta(B)$ implies $A\delta B$

is called a *Lodato proximity*; a pair (X, δ) is called a *Lodato proximity space* if δ is a Lodato proximity.

It is well known that if (X, δ) is a basic proximity then the closure operator cl_{τ_δ} generated by τ_δ could not coincide with cl_δ. If, however, δ is a Lodato proximity, then cl_δ coincides with cl_{τ_δ}.

If δ satisfies the axiom (EF) then (X, τ_δ) is a completely regular space; if, moreover, δ is separated (i.e., δ is an Efremovič proximity) then (X, τ_δ) is a Tychonoff space ([103],[87]).

Every Efremovič proximity is a Lodato proximity.

Definition 0.5.1.3. If (X, τ) is a topological space, we say that (X, τ) *admits a proximity*, if there is a basic proximity δ on X such that $\tau = \tau_\delta$; in this case we also say that δ is a *proximity on the space* (X, τ).

We will denote by $\mathcal{EP}(X)$ the set of all EF-proximities on a Tychonoff space X.

Examples 0.5.1.4. Here are a few examples of proximity spaces:

1. Let X be a set having at least two points. For $A, B \subseteq X$, set

$$A\delta B \Longleftrightarrow A \neq \emptyset \text{ and } B \neq \emptyset.$$

   This is the *trivial basic proximity in* X. It is not separated but it satisfies the axiom (EF).

2. Let (X, τ) be a T_4-space and define

$$A\delta B \Longleftrightarrow \mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset.$$

   Then δ is an Efremovič proximity on the space X. We call it a *standard proximity.*

3. Let (X, τ) be a locally compact Hausdorff space, and, for $A, B \subseteq X$, define

$$A(-\delta)B \Longleftrightarrow (\mathrm{cl}(A) \cap \mathrm{cl}(B) = \emptyset \text{ and either } \mathrm{cl}(A) \text{ or } \mathrm{cl}(B) \text{ is compact}).$$

   Then δ is an Efremovič proximity on the space X.

4. Let (X, d) be a metric space and, for $A, B \subseteq X$, define

$$A\delta B \Longleftrightarrow d(A, B) = 0,$$

   where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

   Then δ is an Efremovič proximity on the space (X, τ_d).

5. Let (X, τ) be a completely regular space. We can define a basic proximity δ on X, satisfying the axiom (EF), by

$$A(-\delta)B \Longleftrightarrow A \text{ and } B \text{ are completely separated.}$$

Definition 0.5.1.5. Let X be a set. A *stack* in X is a family $\mathcal{S}$ of subsets of X satisfying the condition

$$B \supseteq A \in \mathcal{S} \Rightarrow B \in \mathcal{S}.$$

A *grill* ([18]) $\mathcal{G}$ in X is a stack in X satisfying $\emptyset \notin \mathcal{G}$ and

$$(A \cup B) \in \mathcal{G} \ \Rightarrow \ (A \in \mathcal{G} \text{ or } B \in \mathcal{G}).$$

Definition 0.5.1.6. Let (X, δ) be a basic proximity space. A grill $\mathcal{G}$ in X is called a *clan in* (X, δ) ([112]) iff

$$A, B \in \mathcal{G} \;\Rightarrow\; A\delta B.$$

A clan σ in (X, δ) is called a *cluster* ([78]) if it satisfies the following condition:

$$\text{if } A \subseteq X \text{ and } A\delta B \text{ for every } B \in \sigma, \text{ then } A \in \sigma.$$

For each $x \in X$, the collection

$$\sigma_x^\delta = \{A \subseteq X : A\delta x\}$$

is a cluster. Such a cluster is called a *point-cluster*.

If σ is a cluster in (X, δ) and $\{x\} \in \sigma$ for some $x \in X$, then $\sigma = \sigma_x^\delta$.

If (X, δ) is separated, then no cluster on (X, δ) can contain more than one point.

Definition 0.5.1.7. Recall that a proximity space (X, δ) is said to be *compact* iff the topological space (X, τ_δ) is compact.

Definition 0.5.1.8. ([71]) A non-empty collection $\mathcal{B}$ of subsets of a set X is called a *boundedness in* X iff

(i) $A \in \mathcal{B}$ and $B \subseteq A$ implies $B \in \mathcal{B}$, and

(ii) $A, B \in \mathcal{B}$ implies $A \cup B \in \mathcal{B}$.

The elements of $\mathcal{B}$ are called *bounded sets*.

Definitions 0.5.1.9. ([78]) A *local proximity space* $(X, \beta, \mathcal{B})$ consists of a set X, a basic proximity β on X, and a boundedness $\mathcal{B}$ in X subject to the following axioms:

(LP1) If $A \in \mathcal{B}$, $C \subseteq X$ and $A \ll_\beta C$ then there exists $B \in \mathcal{B}$ such that $A \ll_\beta B \ll_\beta C$;

(LP2) If $A\beta C$, then there is a $B \in \mathcal{B}$ such that $B \subseteq C$ and $A\beta B$.

Note that (LP2) implies that every singleton set, and hence every finite subset of X, is bounded.

Two local proximity spaces $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$ are said to be *isomorphic* if there exists a bijection between X_1 and X_2 which preserves in both directions the bounded sets and proximity relations.

A local proximity space $(X, \delta, \mathcal{B})$ is said to be *separated* if δ is the identity relation on singletons.

Recall that if $(X, \beta, \mathcal{B})$ is a separated local proximity space then β is a Lodato proximity and induces a Tychonoff topology $\tau_{(X,\beta,\mathcal{B})}$ on X by defining

$$\mathrm{cl}(M) = \{x \in X \mid x\beta M\}$$

for every $M \subseteq X$ ([78]).

If (X, τ) is a topological space then we say that $(X, \beta, \mathcal{B})$ is a *local proximity space on* (X, τ) if $\tau_{(X,\beta,\mathcal{B})} = \tau$.

A function $f : X_1 \longrightarrow X_2$ between two local proximity spaces $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$ is said to be an *equicontinuous mapping* (see [78]) (or a *bounded p-map*) if the following two conditions are fulfilled:

(EQ1) $A\beta_1 B$ implies $f(A)\beta_2 f(B)$, for $A, B \subseteq X$, and

(EQ2) $B \in \mathcal{B}_1$ implies $f(B) \in \mathcal{B}_2$.

Notation 0.5.1.10. The set of all separated local proximity spaces on a Tychonoff space (X, τ) will be denoted by

$$\mathcal{LP}(X, \tau).$$

An order in $\mathcal{LP}(X, \tau)$ is defined by

$$(X, \beta_1, \mathcal{B}_1) \preceq (X, \beta_2, \mathcal{B}_2) \Longleftrightarrow (\beta_2 \subseteq \beta_1 \text{ and } \mathcal{B}_2 \subseteq \mathcal{B}_1)$$

(see [78]).

The ordered set of all separated local proximity spaces on a Tychonoff space (X, τ) will be denoted by $(\mathcal{LP}(X, \tau), \preceq)$.

Definition and Proposition 0.5.1.11. Let $(X, \beta, \mathcal{B})$ be a separated local proximity space. Define a binary relation α_β on $P(X)$ by

$$A\alpha_\beta B \Longleftrightarrow [(A\beta B) \text{ or } A, B \not\in \mathcal{B}],$$

for $A, B \subseteq X$. Then, by [78, Theorem I] (or by [87, Theorem 9.17]), α_β is an EF-proximity on (X, τ_β). It is called *the Alexandroff extension of* β.

If $X \not\in \mathcal{B}$, the family

$$\sigma_{\alpha_\beta} = \{A \subseteq X : A \not\in \mathcal{B}\}$$

is a cluster in (X, α_β) (see [78] or [87]).

Definition 0.5.1.12. Let X be a set. A binary relation β defined on the power set of X is called a *Pervin proximity* if β satisfies axioms (P1), (P2) and (P3) of 0.5.1.1 and the following conditions:

(PP1) $A(-\beta)\emptyset$ for every $A \subseteq X$,

(PP2) $A(-\beta)B$ implies there exists an $E \subseteq X$ such that $A(-\beta)E$ and $(X \setminus E)(-\beta)B$,

(PP3) $(A \cup B)\beta C$ iff $A\beta C$ or $B\beta C$.

For all undefined here notions and notation see [53] and [87].

0.5.2 Some theorems

Fact 0.5.2.1. (see, e.g., [87]) *If (X, τ) is a compact Hausdorff space, then it admits a unique Efremovič proximity δ, namely the standard one.*

Fact 0.5.2.2. (see, e.g., [87]) *Let (X, δ) be a Lodato proximity space and $A, B \subseteq X$. Then $A \ll_\delta B$ implies $\mathrm{cl}(A) \ll_\delta B$ and $A \ll_\delta \mathrm{int}(B)$.*

Theorem 0.5.2.3. (The Smirnov Compactification Theorem ([103])) *Let (X, τ) be a Tychonoff space. Then the ordered sets $(\mathcal{C}(X, \tau), \leq)$ and $(\mathcal{EP}(X), \leq)$ are isomorphic. The isomorphism between them is defined as follows. To every Efremovič proximity space (X, δ) on (X, τ) corresponds a Hausdorff compactification*

$$(S_m(X, \delta), s_\delta)$$

of (X, τ), where $S_m(X, \delta)$ consists of all clusters in (X, δ), i.e.,

$$S_m(X, \delta) = \mathrm{Clust}(X, \delta),$$

its topology is generated by the closed base

$$\{\{\sigma \in \mathrm{Clust}(X, \delta) \mid A \in \sigma\} \mid A \subseteq X\}$$

and, for every $x \in X$,

$$s_\delta(x) = \sigma_x^\delta$$

(here σ_x^δ is the point-cluster from 0.5.1.6) (see, e.g., [87]).

$(S_m(X, \delta), s_\delta)$ *is called* the Smirnov compactification of (X, δ).

Conversely, for every Hausdorff compactification (Y, f) of (X, τ), the corresponding Efremovič proximity $\delta_{(Y,f)}$ on the space (X, τ) is defined by the formula

$$A\delta_{(Y,f)}B \Longleftrightarrow \mathrm{cl}_{(Y,f)}(f(A)) \cap \mathrm{cl}_{(Y,f)}(f(B)) \neq \emptyset,$$

for every $A, B \in P(X)$. Then the map

$$S_m : (\mathcal{EP}(X), \leq) \longrightarrow \mathcal{C}(X, \tau), \quad \delta \mapsto [(S_m(X, \delta), s_\delta)]$$

is the required isomorphism and $S_m^{-1}([(Y, f)]) = \delta_{(Y,f)}$.

We will need the following well-known result (see, e.g., [87], Theorem 7.10):

Theorem 0.5.2.4. (Ju. M. Smirnov [103]) *Let (X, δ) and (Y, δ') be two Efremovič proximity spaces and f be a mapping between them. Then there exists a continuous mapping*

$$S_m f : S_m(X, \delta) \longrightarrow S_m(Y, \delta')$$

such that

$$S_m f \circ s_\delta = s_{\delta'} \circ f$$

iff f is a proximity mapping. It is defined by the formula

$$S_m f(\sigma) = \{C \subseteq Y : (\forall A \in \sigma)(C \delta' f(A))\},$$

for every $\sigma \in S_m(X, \delta)$.

The next theorem of Leader ([78]) and its proof are of great importance for our investigations.

Theorem 0.5.2.5. ([78]) *Let (X, τ) be a Tychonoff space. Then there exists an isomorphism*

$$\gamma_{(X,\tau)} : (\mathcal{LC}(X, \tau), \leq) \longrightarrow (\mathcal{LP}(X, \tau), \preceq).$$

Namely, if (Y, l) is a locally compact Hausdorff extension of X then

$$\gamma_{(X,\tau)}([(Y, l)]) = (X, \beta_{(Y,l)}, \mathcal{B}_{(Y,l)}),$$

where

$$\mathcal{B}_{(Y,l)} = \{F \subseteq X : \mathrm{cl}_Y(l(F)) \text{ is compact}\}$$

and, for $A, B \subseteq X$,

$$A \beta_{(Y,l)} B \Longleftrightarrow \mathrm{cl}_Y(l(A)) \cap \mathrm{cl}_Y(l(B)) \neq \emptyset.$$

The description of the map $(\gamma_{(X,\tau)})^{-1}$ is the following: let $(X, \beta, \mathcal{B})$ be a separated local proximity space on (X, τ), α_β be the Alexandroff extension of β (see 0.5.1.11) and let

$$L(X, \beta, \mathcal{B}) = S_m(X, \alpha_\beta) \setminus \{\sigma_{\alpha_\beta}\}, \text{ when } \alpha_\beta \neq \beta,$$

and

$$L(X,\beta,\mathcal{B}) = S_m(X,\alpha_\beta),\ \textit{when } \alpha_\beta = \beta$$

(see 0.5.2.3 and 0.5.1.11); then

$$(\gamma_{(X,\tau)})^{-1}(X,\beta,\mathcal{B}) = [(L(X,\beta,\mathcal{B}), l_\beta)],$$

where

$$l_\beta(x) = s_{\alpha_\beta}(x),$$

for every $x \in X$ (see 0.5.2.3). Thus $(\gamma_{(X,\tau)})^{-1}(X,\beta,\mathcal{B}) \in \mathcal{C}(X,\tau)$ iff $\mathcal{B} = P(X)$.

Let $(X_i,\beta_i,\mathcal{B}_i)$, $i = 1,2$, be two separated local proximity spaces and

$$f : X_1 \longrightarrow X_2$$

be a function. Then there exists a continuous map

$$L(f) : L(X_1,\beta_1,\mathcal{B}_1) \longrightarrow L(X_2,\beta_2,\mathcal{B}_2)$$

such that

$$l_{\beta_2} \circ f = L(f) \circ l_{\beta_1}$$

iff f is a bounded p-map between $(X_1,\beta_1,\mathcal{B}_1)$ and $(X_2,\beta_2,\mathcal{B}_2)$.

Chapter 1

MVD-algebras and a new proof of Roeper's Representation Theorem

1.1 Introduction

As we have already mentioned, the *region-based theory of space* is a kind of point-free geometry and can be considered as an alternative to the well known Euclidean point-based theory of space; its main idea goes back to Whitehead [123] (see also [121, 122, 120]) and de Laguna [23], although neither Whitehead nor de Laguna presented their ideas in a detailed mathematical form. This was done by some other mathematicians and mathematically oriented philosophers who presented various versions of region-based theory of space at different levels of abstraction. Here we can mention the fundamental work of Tarski [110], who rebuilt Euclidean geometry as an extension of mereology with the primitive notion of a *ball*. Remarkable is also Grzegorczyk's paper [67]. Models of Grzegorczyk's theory are complete Boolean algebras of regular closed sets of certain topological spaces equipped with the relation of *separation* which in fact is the complement of Whitehead's contact relation. On the same line of abstraction is also the *point-free topology* [75]. Survey papers describing various aspects and historical remarks on region-based theory of space are [64, 11, 116, 94].

Let us mention that Whitehead's ideas about region-based theory of space flourished and in a sense were reinvented and applied in some areas of computer science: Qualitative Spatial Reasoning (QSR), knowledge representation, geographical information systems, formal ontologies in information systems, image processing, natural language semantics etc. The reason is that the language of region-based theory of space allows the researchers to obtain a more simple description of some qualitative spatial features and properties of space bodies. Survey papers concerning various applications are [19, 20] (see also the special issues of “Fundamenta Informaticae” [47] and “Journal of Applied Non-classical Logics” [7]). One of the most popular among the community of QSR-researchers is the system of Region Connection Calculus (RCC) introduced by Randell, Cui and Cohn [96]. RCC attracted quite intensive research in the field of region-based theory of space, both on its applied and mathematical aspects. For instance it was unknown for some time which topological models correspond adequately to RCC; this fact stimulated the investigations of a topological representation theory of RCC and RCC-like systems (see [46, 41]). Another impact of region-based theory of space is that it stimulated the appearance of a new area in logic, namely “Spatial Logics” [2], called sometimes “Logics of Space”.

The first first-order axiomatization for region-based theory of space with a detailed investigation of its connection with the point-based theory of compact Hausdorff spaces was given by de Vries in [24]. He introduced the notion of a *compingent Boolean algebra* which is a pair of a Boolean algebra B and a binary relation $\ll$ called the *compingent relation* on B. Further on, Fedorchuk [54] introduced the notion of a *Boolean δ-algebra* and showed that it is equivalent to the notion of a compingent Boolean algebra. The Fedorchuk's Boolean δ-algebras appear here under the name of *normal contact algebras*. Both authors - de Vries and Fedorchuk - stressed on the narrow connection of their theories with the theory of proximity spaces of Efremovič [50].

Another, more general than that of de Vries and Fedorchuk, first-order axiomatization for region-based theory of space was given by Roeper [99]. His theory corresponds to the point-based theory of locally compact Hausdorff spaces and his approach is in fact a successful and skilful combination of the methods of de Vries [24], Fedorchuk [54] and Leader [78], although Roeper doesn't mention this and, probably, was not aware of these results; he, however, gives prominence to the fact that the leading ideas in his paper are those from Whiteheadian region-based theory of space. Roeper [99] introduced the notion of *region-based topology* – it is a Boolean algebra with a contact relation (satisfying some additional axioms) and an additional one-place predicate of *limitedness*. His main representation theorem implies that there exists a bijective correspondence between the class of all (up to isomorphism) region-based topologies and the class of all (up to homeomorphism) locally compact Hausdorff spaces, which is a fundamental fact for our investigations in this thesis. The axioms of “region-based topology” almost coincide with the axioms of *local proximity spaces* introduced by Leader [78].

This similarity prompts us to introduce a new name for region-based topologies and to call them *local contact algebras*. The similarity between local proximity spaces and local contact algebras makes it possible to obtain (and to present in this chapter) a shorter proof of the main Representation Theorem of Roeper [99] (as well as of its special case - the de Vries Representation Theorem [24]) by application of the methods of (local) proximity spaces. The proofs of some of the steps and some of the constructions in our new exposition of Roeper's Representation Theorem can be considered as lattice-theoretic versions of certain previously known proofs and constructions in the theory of proximity spaces and local proximity spaces. The new proof of the Roeper Representation Theorem given here is similar to that from [117] but uses also some new ideas and is shorter than both previous proofs from [99] and [117]. The methods and facts established in this new proof are used many times later in the thesis. That's why the presentation of this new proof is indispensable for our exposition of the results of the thesis.

Another work devoted to region-based theory of space is Mormann's paper [85]. His system, called *enriched Boolean algebra*, is similar to normal contact algebras, but instead of the contact relation, it contains another relation between (open) regions, called *interior parthood* and denoted by " $\ll$". One of the main aims of Mormann's paper is to show that Whiteheadian theory of space can be built up on the base of the single relation of interior parthood, considered as a "purely mereological relation" ([85, p. 37]). In a discussion with Roeper (p. 52) he claims that the relations of contact and limitedness are "non-mereological". We use here without discussion Mormann's terminology, although it seems that all such relations should be considered as "*mereotopological*". Note however that Mormann's notion of interior parthood is different from the corresponding notion in local contact algebras. Mormann's definition in the intended semantics — open regions in locally compact Hausdorff spaces, is the following:

$$x \ll y \text{ iff } cl(x) \subseteq y \text{ and } cl(x) \text{ is compact, where } x, y \text{ are open regions.}$$

The difference with the corresponding definition in Roeper's paper [99] is in the requirement of compactness of $cl(x)$ which introduces some asymmetry between x and y. If we define the contact relation by the standard formula "xCy" iff "not $x \ll y^*$" then the above asymmetry implies that the contact relation is not a symmetric one as it should be. Despite this difference, the main representation theorem for Mormann's system takes the same form as Roeper's. Unfortunately, the representation theorem presented in Mormann's paper [85] is not true (see Example 1.3.2.2 here). However it becomes true if one adds an extra axiom to those which an enriched Boolean algebra has to satisfy. The obtained new notion is called an *MVD-algebra* (Mormann-Vakarelov-Dimov-algebra). We prove that MVD-algebras are equivalent to local contact algebras. Hence the representation theorem for local contact algebras is valid also for MVD-algebras. The representation result for MVD-algebras shows as well that one of the main aims of the Mormann's paper – to formalize the Whitehedian theory of space on the base of a single mereological relation, is now realized. In this way, it is demonstrated that the modified Mormann's notion of interior parthood incorporates in itself both of contact relation and boundedness, which is a quite unexpected fact.

This chapter is organized as follows. In the next Section 2 we give our new proof of the Roeper Representation Theorem. In the first subsection of Section 2, we recall the notions of *contact algebra* (introduced in [41]) and *normal contact algebra* (introduced in [24] (see also [54])), and give some basic examples. In particular, we prove here that the Boolean algebra of regular closed subsets of each Efremovič proximity space is a normal contact algebra with respect to the proximity relation. In the second subsection of Section 2, we give a new proof of de Vries' Representation Theorem for normal contact algebras. This proof is different from those given in [24, 117], uses the Stone Duality Theorem for Boolean algebras and some ideas from Leader's proof of the Smirnov Compactification Theorem (see, e.g., [87]). In the third subsection of Section 2, we introduce the notion of local contact algebra, which, though slightly different in formulation, is equivalent to Roeper's notion of "region-based topology" [99]. The changes have been made in order to fit well with Leader's definition of local proximity space [78]. It is proved here that the regular closed subsets of a local proximity space determine a local contact algebra. After that we present our new proof of Roeper's Representation Theorem for local contact algebras, which is deduced from the corresponding theorem for normal contact algebras. The main idea of the proof is a lattice-theoretic parallel with Leader's theorem for local proximity spaces [87, 78].

In Section 3 we introduce the notion of an *MVD-algebra*, which is similar to the Mormann's notion of *enriched Boolean algebra* [85]. The formal equivalence of the notions of MVD-algebras and local contact algebras is proved and a representation theorem for MVD-algebras is obtained.

The exposition of this chapter follows that of the paper [117] with the exception of some parts of the proof of Roeper's Representation Theorem.

1.2 A new proof of the Roeper Representation Theorem

1.2.1 Contact algebras

Definition 1.2.1.1. An algebraic system $(B, 0, 1, \vee, \wedge, {}^*, C)$ is called a *contact Boolean algebra* or, briefly, *contact algebra* (abbreviated as CA or C-algebra) ([41]) if the system $(B, 0, 1, \vee, \wedge, {}^*)$ is a Boolean algebra (where the operation "complement" is denoted by " $*$ ") and C is a binary relation on B, satisfying the following axioms:

(C1) If $a \neq 0$ then aCa;

(C2) If aCb then $a \neq 0$ and $b \neq 0$;

(C3) aCb implies bCa;

(C4) $aC(b \vee c)$ iff aCb or aCc.

We shall simply write (B, C) for a contact algebra. The relation C is called a *contact relation*. When B is a complete Boolean algebra, we will say that (B, C) is a *complete contact Boolean algebra* or, briefly, *complete contact algebra* (abbreviated as CCA or CC-algebra). If $D \subseteq B$ and $E \subseteq B$, we will write "DCE" for "$(\forall d \in D)(\forall e \in E)(dCe)$".

We will say that two C-algebras (B_1, C_1) and (B_2, C_2) are *CA-isomorphic* iff there exists a Boolean isomorphism $\varphi : B_1 \longrightarrow B_2$ such that, for each $a, b \in B_1$, aC_1b iff $\varphi(a)C_2\varphi(b)$. Note that in this thesis, by a "Boolean isomorphism" we understand an isomorphism in the category **BoolAlg** of Boolean algebras and Boolean homomorphisms.

A contact algebra (B, C) is called a *normal contact Boolean algebra* or, briefly, *normal contact algebra* (abbreviated as NCA or NC-algebra) ([24, 54]) if it satisfies the following axioms (we will write " $-C$" for "*not* C"):

(C5) If $a(-C)b$ then $a(-C)c$ and $b(-C)c^*$ for some $c \in B$;

(C6) If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C)a$.

A normal CA is called a *complete normal contact Boolean algebra* or, briefly, *complete normal contact algebra* (abbreviated as CNCA or CNC-algebra) if it is a CCA. The notion of normal contact algebra was introduced by Fedorchuk [54] under the name *Boolean δ-algebra* as an equivalent expression of the notion of compingent Boolean algebra of de Vries (see its definition below). We call such algebras "normal contact algebras" because they form a subclass of the class of contact algebras and naturally arise in normal Hausdorff spaces.

For any CA (B, C), we define a binary relation " $\ll_C$" on B (called *non-tangential inclusion*) by " $a \ll_C b \leftrightarrow a(-C)b^*$ ". Sometimes we will write simply " $\ll$" instead of " $\ll_C$". This relation is also known in the literature under the following names: "well-inside relation", "well below", "interior parthood", "non-tangential proper part" or "deep inclusion".

The relations C and $\ll$ are inter-definable. For example, normal contact algebras could be equivalently defined (and exactly in this way they were introduced (under the name of *compingent Boolean algebras*) by de Vries in [24]) as a pair of a Boolean algebra $B = (B, 0, 1, \vee, \wedge, ^*)$ and a binary relation $\ll$ on B subject to the following axioms:

($\ll$1) $a \ll b$ implies $a \leq b$;

($\ll$2) $0 \ll 0$;

($\ll$3) $a \leq b \ll c \leq t$ implies $a \ll t$;

($\ll$4) $a \ll c$ and $b \ll c$ implies $a \vee b \ll c$;

($\ll$5) If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$;

($\ll$6) If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$;

($\ll$7) $a \ll b$ implies $b^* \ll a^*$.

The proof of the equivalence of the two definitions of normal contact algebras is straightforward and analogous to the corresponding statement for proximity spaces (see Theorems 3.9 and 3.11 in [87]). One has just to show that xCy iff $x \not\ll y^*$.

Obviously, contact algebras could be equivalently defined as a pair of a Boolean algebra B and a binary relation $\ll$ on B subject to the axioms ($\ll$1)-($\ll$4) and ($\ll$7).

It is easy to see that axiom (C5) (resp., (C6)) can be stated equivalently in the form of ($\ll$5) (resp., ($\ll$6)).

Remark 1.2.1.2. Note that if $0 \neq 1$ then the axiom (C2) follows from the axioms (C3), (C4) and (C6). Indeed, note first that in the presence of the axiom (C4) the axiom (C2) is equivalent to the following axiom

(C2$'$) $0(-C)1$.

So, let $0 \neq 1$. We will show that $0(-C)1$. Using (C6), we get that there exists $b \neq 0$ such that $b(-C)0$. Since $b^* \neq 1$, (C6) implies that there exists $c \neq 0$ such that $c(-C)b^*$. Hence, by the axiom (C4), $0(-C)b^*$. Thus, using axioms (C3) and (C4), we get that $0(-C)(b \vee b^*)$, i.e., $0(-C)1$.

Analogously we get that if $0 \neq 1$ then the axiom ($\ll$2) follows from the axioms ($\ll$3), ($\ll$4), ($\ll$6) and ($\ll$7).

The following obvious assertion collects some easy properties of the contact relation:

Lemma 1.2.1.3. *Suppose that (B, C) is a contact algebra. Then:*
(a) If xCy, $x \leq x'$ and $y \leq y'$ then $x'Cy'$,
(b) If $x \wedge y \neq 0$ then xCy,

Fact 1.2.1.4. *Let (B, C) be a contact algebra satisfying the axiom (C6). Then the following is true:*
(C6$'$) If $x \not\leq y$ then zCx and $z(-C)y$ for some $z \in B$.

Proof. Since $x \not\leq y$, we have that $a = x \wedge y^* \neq 0$. Thus, by ($\ll$6), there exists $z \neq 0$ such that $z \ll a$. Then zCx and $z(-C)y$. □

Remark 1.2.1.5. Obviously, contact algebras satisfying the axiom ($C6'$) satisfy the axiom (C6) as well (indeed, $x \neq 1$ means that $1 \not\leq x$; hence there exists an $y \in B$ such that $yC1$ and $y(-C)x$; by (C2), $yC1$ implies $y \neq 0$). So, in the definition of a normal contact algebra, one can substitute (C6) with ($C6'$). The axiom (C6) was introduced in [24] (see also [106]).

Lemma 1.2.1.6. *Suppose that (B, C) is a contact algebra satisfying, in addition, the axiom (C6). Then $x \leq y$ iff $(\forall z)(zCx$ implies $zCy)$.*

Proof. It is obvious. □

Remark 1.2.1.7. Axiom (C6) is an extensionality axiom, since one obtains immediately from Lemma 1.2.1.6 that

$$x = y \Longleftrightarrow (\forall z \in B)[zCx \Longleftrightarrow zCy].$$

Example 1.2.1.8. Let B be a Boolean algebra. Then there exist a largest and a smallest contact relations on B; the largest one, ρ_l, is defined by

$$a\rho_l b \iff (a \neq 0 \text{ and } b \neq 0),$$

and the smallest one, ρ_s, by

$$a\rho_s b \iff a \wedge b \neq 0.$$

Note that, for $a, b \in B$,

$$a \ll_{\rho_s} b \iff a \leq b;$$

hence $a \ll_{\rho_s} a$, for any $a \in B$. Thus (B, ρ_s) is a normal contact algebra.

Example 1.2.1.9. Let (X, τ) be a topological space. We will define a contact relation $\rho_{(X,\tau)}$ on the complete Boolean algebra $RC(X)$ of all regular closed subsets of (X, τ) setting, for each $F, G \in RC(X)$,

$$F \rho_{(X,\tau)} G \text{ iff } F \cap G \neq \emptyset;$$

it is called a *standard contact relation*. So, $(RC(X, \tau), \rho_{(X,\tau)})$ is a CCA (it is called a *standard contact algebra*). We will often write simply ρ_X instead of $\rho_{(X,\tau)}$. Note that, for $F, G \in RC(X)$,

$$F \ll_{\rho_X} G \text{ iff } F \subseteq \text{int}_X(G).$$

Thus, if (X, τ) is a normal Hausdorff space then the standard contact algebra $(RC(X, \tau), \rho_{(X,\tau)})$ is a complete NCA.

Also, we define a contact relation D_X on the complete Boolean algebra $RO(X)$ of all regular open subsets of (X, τ) as follows:

$$U D_X V \text{ iff } \text{cl}(U) \cap \text{cl}(V) \neq \emptyset.$$

Then $(RO(X), D_X)$ is a CCA.

Note that $(RO(X), D_X)$ and $(RC(X), \rho_X)$ are isomorphic C-algebras. The isomorphism $\nu : (RO(X), D_X) \longrightarrow (RC(X), \rho_X)$ between them is defined by the formula

$$\nu(U) = \text{cl}(U),$$

for every $U \in RO(X)$.

We are now going to give a natural example of a normal contact algebra using proximity spaces.

We will need the following observation:

Lemma 1.2.1.10. *Let (X, δ) be an Efremovič proximity space and $A, B \subseteq X$. If $A \ll_\delta B$ then there exists a regular closed set C such that $A \ll_\delta C \ll_\delta B$.*

Proof. By (EF) (see 0.5.1.1), there exists a $D \subseteq X$ such that $A \ll D \ll B$. Applying 0.5.2.2, we obtain

$$A \ll_\delta \mathrm{int}(D) \subseteq \mathrm{cl}(\mathrm{int}(D)) \subseteq \mathrm{cl}(D) \ll_\delta B,$$

and thus, $A \ll_\delta \mathrm{cl}(\mathrm{int}(D)) \ll_\delta B$. □

Example 1.2.1.11. *Let (X, δ) be an Efremovič proximity space. Then $(RC(X), \delta)$ is a normal contact algebra.*

Proof. The verification of axioms (C1) – (C4) is straightforward; (C5) follows from Lemma 1.2.1.10. For proving ($\ll$6) (which is equivalent to (C6)), let $A \in RC(X)$ and $A \neq \emptyset$. Then there exists a point $x \in \mathrm{int}(A)$. Obviously, $x \ll_\delta \mathrm{int}_{\tau_\delta}(A)$. Applying Lemma 1.2.1.10, we obtain that there exists $B \in RC(X)$ such that $x \ll_\delta B \ll_\delta A$. So, $B \neq \emptyset$ and $B \ll_\delta A$. □

1.2.2 A new proof of de Vries' Representation Theorem for normal contact algebras

In this subsection we shall prove that each normal contact algebra can be isomorphically embedded as a dense subalgebra of a standard contact algebra of a compact Hausdorff space. Our strategy follows the proof of the Stone representation theorem for Boolean algebras. The points in a Stone space $S^a(B)$ are the maximal filters of B. In normal contact algebras, the points of the representation space will be some analogues of maximal filters, called *clusters*. We take the notion of a cluster from the theory of proximity spaces, and our definition is just the lattice-theoretic translation of the corresponding definition of a cluster (see Definition 0.5.1.6 here). Many statements about clusters in normal contact algebras have proofs which are identical (up to the aforementioned lattice-theoretical translation) to the proofs of the corresponding statements for clusters in proximity spaces. When such identical proofs exist we will refer to the corresponding statements and their proofs.

Throughout this subsection we suppose that (B, C) is a normal contact algebra.

1.2.2.1. A non-empty subset Γ of B is called a *clan in* (B, C) if the following conditions are satisfied:

(K1) If $x, y \in \Gamma$ then xCy;

(K2) If $x < y$ and $x \in \Gamma$ then $y \in \Gamma$;

(G) If $x \vee y \in \Gamma$ then $x \in \Gamma$ or $y \in \Gamma$.

A clan Γ in (B, C) is called a *cluster in* (B, C) if it satisfies the following condition:

(CLU) If xCy for every $y \in \Gamma$, then $x \in \Gamma$.

The set of all clusters in (B, C) is denoted by $\mathrm{Clust}(B, C)$ or simply by $\mathrm{Clust}(B)$.

Note that a non-empty subset Γ of B is a cluster iff it satisfies the axioms (K1), (G) and (CLU). It is not hard to see that (the proofs are similar to those given in [112]):

(a) each clan is contained in a maximal clan;

(b) each maximal clan is a cluster.

The following properties of the clusters will be helpful later:

Lemma 1.2.2.2. *Let $\Gamma \in \mathrm{Clust}(B)$, and $a, b \in B$. Then:*

(a) if aCb, then there is some $\Delta \in \mathrm{Clust}(B)$ such that $a \in \Delta$ and $b \in \Delta$;

(b) $a^* \in \Gamma$ iff $(\forall b \in B)$ ($c \in \Gamma$ and $b \vee a = 1$ imply cCb).

Proof. (a) The proof is analogous to the one of [87, Theorem 5.14].

(b) If $a^*, c \in \Gamma$ and $a \vee b = 1$, then $a^* \leq b$. It follows from (K2) that $b \in \Gamma$, and hence, cCb (by (K1)). Conversely, suppose that $(\forall b \in B)$ ($c \in \Gamma$ and $b \vee a = 1$ imply cCb). Setting $b = a^*$, we obtain that cCa^* for all $c \in \Gamma$, and thus, $a^* \in \Gamma$ (by (CLU)). □

The next theorem can be proved exactly as Theorem 5.8 of [87]:

Theorem 1.2.2.3. *A subset σ of a normal contact algebra (B, C) is a cluster iff there exists an ultrafilter u in B such that*

(1.1) $\sigma = \{a \in B : aCb \textit{ for every } b \in u\}$.

Moreover, given σ and $a_0 \in \sigma$, there exists an ultrafilter u in B satisfying (1.1) which contains a_0.

Corollary 1.2.2.4. *Let (B, C) be a normal contact algebra and u be an ultrafilter (or a basis of an ultrafilter) in B. Then there exists a unique cluster σ_u in (B, C) containing u, and*

(1.2) $\sigma_u = \{a \in B \mid aCb \textit{ for every } b \in u\}$.

Finally, the following simple result can be proved exactly as Lemma 5.6 of [87]:

Fact 1.2.2.5. *Let (B, C) be a normal contact algebra, Γ_1 and Γ_2 be two clusters in (B, C). If $\Gamma_1 \subseteq \Gamma_2$, then $\Gamma_1 = \Gamma_2$.*

Notation 1.2.2.6. Let (X, τ) be a topological space and $x \in X$. Then we set:

(1.3) $\sigma_x^X = \{F \in RC(X) \mid x \in F\}$ and $\nu_x^X = \{F \in RC(X) \mid x \in \mathrm{int}(F)\}$.

We will often write σ_x and ν_x instead of, respectively, σ_x^X and ν_x^X.

The next assertion is obvious:

Fact 1.2.2.7. *For any topological space (X, τ) and every point $x \in X$, ν_x is a filter in the Boolean algebra $RC(X)$ and σ_x is a clan in $(RC(X), \rho_X)$. If X is regular then σ_x is a cluster in the CA $(RC(X), \rho_X)$ (and it is called a* point-cluster*).*

We are now ready to present our proof of de Vries' Representation Theorem for normal contact algebras.

Theorem 1.2.2.8. (The de Vries Representation Theorem for NCAs ([24]))
(a) Each normal contact algebra (B, C) can be densely embedded (as a contact algebra) into a standard contact algebra $(RC(X), \rho_X)$, where X is a compact Hausdorff space. When B is complete this embedding becomes a CA-isomorphism;
(b) There exists a bijective correspondence between the class of all (up to isomorphism) complete normal contact algebras and the class of all (up to homeomorphism) compact Hausdorff spaces.

Proof. (a) Let (B, C) be a CNCA. Put $X = \mathrm{Clust}(B, C)$. Define a function

$$\lambda_{(B,C)} : B \longrightarrow P(X)$$

by

(1.4) $\lambda_{(B,C)}(a) = \{\sigma \in X \mid a \in \sigma\}$

in analogy to the Stone representation theorem for Boolean algebras. When there is no ambiguity, we will write simply λ_B instead of $\lambda_{(B,C)}$. Let us show that λ_B satisfies the following conditions:
(H1) $\lambda_B(0) = \emptyset$, $\lambda_B(a) = X \Longleftrightarrow a = 1$;
(H2) $\lambda_B(a \vee b) = \lambda_B(a) \cup \lambda_B(b)$.

Obviously, $\lambda_B(0) = \emptyset$ and $\lambda_B(1) = X$. Let $\lambda_B(a) = X$. Then $a \in \sigma$ for every $\sigma \in X$. Suppose that $a \neq 1$. Then $1 \not\leq a$ and hence, by $(C6')$ (see 1.2.1.4), there exists an element b of B such that $bC1$ and $b(-C)a$. By 1.2.2.2(a), there exists $\sigma \in X$ such that $b \in \sigma$. Since $a \in \sigma$, we obtain that aCb and this is a contradiction. Therefore, $a = 1$. Hence, λ_B satisfies condition (H1).

The fact that λ_B satisfies condition (H2) follows immediately from (G) and (K2) (see 1.2.2.1).

Let $\mathcal{T}$ be the topology on X having as a closed base the family

$$\{\lambda_B(a) \mid a \in B\}$$

((H1) and (H2) show that this family can be taken as a closed base of a topology). Then

(1.5) the family $\{X \setminus \lambda_B(a) \mid a \in B\}$ is an open base of $(X, \mathcal{T})$.

We put

(1.6) $\Phi^a(B, C) = (X, \mathcal{T})$.

Let $a \in B$. Then, using (H1) and (H2), we obtain that $\mathrm{cl}(X \setminus \lambda_B(a)) = \bigcap\{\lambda_B(b) \mid b \in B, X \setminus \lambda_B(a) \subseteq \lambda_B(b)\} = \bigcap\{\lambda_B(b) \mid b \in B, \lambda_B(a) \cup \lambda_B(b) = X\} = \bigcap\{\lambda_B(b) \mid b \in B, a \vee b = 1\} = \bigcap\{\lambda_B(b) \mid b \in B, a^* \leq b\} = \lambda_B(a^*)$; hence

(1.7) $\lambda_B(a^*) = \mathrm{cl}(X \setminus \lambda_B(a))$.

Then

(1.8) $X \setminus \lambda_B(a) = X \setminus \mathrm{cl}(X \setminus \lambda_B(a^*)) = \mathrm{int}(\lambda_B(a^*))$.

Therefore,

(1.9) the family $\{\mathrm{int}(\lambda_B(a)) \mid a \in B\}$ is an open base of $(X, \mathcal{T})$.

Let us show that $(X, \mathcal{T})$ is a Hausdorff space. Indeed, let $\sigma_1, \sigma_2 \in X$ and $\sigma_1 \neq \sigma_2$. Then, by 1.2.2.5, $\sigma_1 \not\subseteq \sigma_2$. Take $a_1 \in (\sigma_1 \setminus \sigma_2)$. Then there exists an $a_2 \in \sigma_2$ such that $a_1(-C)a_2$. Hence $a_1 \ll a_2^*$. There exist $b_1, b_2 \in B$ such that $a_1 \ll b_1 \ll b_2 \ll a_2^*$. Therefore, $a_1(-C)b_1^*$ and $a_2(-C)b_2$. Put $U_1 = \mathrm{int}(\lambda_B(b_1))$ and $U_2 = \mathrm{int}(\lambda_B(b_2^*)$. Then $\sigma_1 \in U_1$, $\sigma_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. So,

$$(X, \mathcal{T}) \text{ is a Hausdorff space.}$$

Further, for every $a \in B$,

(1.10) $\lambda_B(a) \in RC(X, \mathcal{T})$.

Indeed, $\lambda_B(a) = \mathrm{cl}(X \setminus \lambda_B(a^*)) = \mathrm{cl}(X \setminus \mathrm{cl}(X \setminus \lambda_B(a))) = \mathrm{cl}(\mathrm{int}(\lambda_B(a)))$. We will show that

$$\lambda_B : (B, C) \longrightarrow (RC(X), \rho_X)$$

is a dense CA-embedding. (see 1.2.1.9 for the notation ρ_X).

From (1.7) and the definition of the operation "$*$" in $RC(X)$, we obtain that $\lambda_B(a^*) = (\lambda_B(a))^*$. This, together with (H1) and (H2), shows that λ_B is a Boolean homomorphism.

To show that λ_B is injective, suppose that $a \neq b$, and let w.l.o.g. $a \not\leq b$. By $(C6')$, there is some $c \in B$ such that aCc and $b(-C)c$. Let $\sigma \in X$ be such that $a, c \in \sigma$ (see Lemma 1.2.2.2). It follows now from $b(-C)c$ and (K1) that $b \notin \sigma$. Hence $\sigma \in \lambda_B(a) \setminus \lambda_B(b)$, i.e., $\lambda_B(a) \neq \lambda_B(b)$. Therefore, λ_B is an injection.

We will now prove that $\lambda_B(B)$ is dense in $RC(X)$. So, let $F \in RC(X)$ and $F \neq \emptyset$. Since F is regular closed, we obtain that $\mathrm{int}(F) \neq \emptyset$. Let $\sigma \in \mathrm{int}(F)$. Then, by (1.9), there exists an $a \in B$ such that $\sigma \in \mathrm{int}(\lambda_B(a)) \subseteq \mathrm{int}(F)$. Hence $\lambda_B(a) \neq \emptyset$. Since $\lambda_B(a)$ is regular closed and F is closed, we obtain that $\lambda_B(a) \subseteq F$. Therefore, $\lambda_B(B)$ is dense in $RC(X)$.

Finally, Lemma 1.2.2.2 implies that for every $a, b \in B$, aCb iff $\lambda_B(a) \cap \lambda_B(b) \neq \emptyset$; hence aCb iff $\lambda_B(a)\rho_X\lambda_B(b)$. So,

(1.11) $\lambda_B : (B, C) \longrightarrow (RC(X), \rho_X)$ is a dense CA-embedding.

Using Proposition 0.3.2.3, we obtain that

(1.12) $\lambda_B : (B, C) \longrightarrow (RC(X), \rho_X)$ is a CA-isomorphism for any CNCA (B, C).

We will show that $(X, \mathfrak{T})$ is a compact space. Let $S^a(B)$ be the Stone space of the Boolean algebra B. Define a function $f : S^a(B) \longrightarrow X$ by $f(u) = \sigma_u$, for every $u \in S^a(B)$ (see 1.2.2.4 for the notation σ_u). We will show that f is a continuous surjection. This will imply that $(X, \mathfrak{T})$ is a compact space, since $S^a(B)$ is such one. So, let $u \in S^a(B)$ and $X \setminus \lambda_B(a)$ be a basic neighborhood of $f(u) = \sigma_u$. Then $a \notin \sigma_u$. Hence, by 1.2.2.4, there exists $b \in u$ such that $a(-C)b$. Let $\lambda_B^S(b) = \{v \in S^a(B) \mid b \in v\}$. Then $\lambda_B^S(b)$ is a neighborhood of u in $S^a(B)$. Let $v \in \lambda_B^S(b)$. Then $b \in v$ and since $a(-C)b$, we obtain that $a \notin f(v)$. Hence $f(v) \notin \lambda_B(a)$. Therefore, $f(\lambda_B^S(b)) \subseteq X \setminus \lambda_B(a)$. So, f is a continuous map. As it follows immediately from 1.2.2.3, f is a surjection. Hence,

(1.13) $(X, \mathfrak{T})$ is a compact Hausdorff space.

(b) For every compact Hausdorff space (Y,τ), we put

(1.14) $\Phi^t(Y,\tau) = (RC(Y,\tau), \rho_Y)$

(see 1.2.1.9 for the notation). Then $\Phi^t(Y,\tau)$ is a CNCA.

Let (X,τ) be a compact Hausdorff space and

$$t_{(X,\tau)} : (X,\tau) \longrightarrow \Phi^a(\Phi^t(X,\tau))$$

be defined by

(1.15) $t_{(X,\tau)}(x) = \{F \in RC(X,\tau) \mid x \in F\}\ (= \sigma_x), \forall x \in X.$

We will write simply t_X instead of $t_{(X,\tau)}$.

We have that

$$\Phi^a(\Phi^t(X,\tau)) = \mathrm{Clust}(RC(X,\tau), \rho_{(X,\tau)})$$

(in the sequel, we will write simply X instead of (X,τ)).

By 1.2.2.7, the map t_X is defined correctly. Let $x, y \in X$ and $x \neq y$. Then, obviously, $\sigma_x \neq \sigma_y$. Hence t_X is an injection. Let σ be a cluster in $(RC(X), \rho_X)$. Then, by 1.2.2.3, there exists an ultrafilter $\mathfrak{U}$ in $RC(X)$ such that

(1.16) $\sigma = \{F \in RC(X) \mid F \cap G \neq \emptyset, \forall G \in \mathfrak{U}\}.$

Since $F \wedge G \subseteq F \cap G$, for every $F, G \in RC(X)$, $\mathfrak{U}$ is a family of closed subsets of the compact Hausdorff space (X,τ) having the finite intersection property. Hence $P = \bigcap\{G \mid G \in \mathfrak{U}\} \neq \emptyset$. Moreover, the set P has only one point. Indeed, suppose that $x, y \in P$ and $x \neq y$. Then there exist $F, G \in RC(X)$ such that $x \in F$, $y \in G$ and $F \cap G = \emptyset$. Using (1.16), we obtain that $F, G \in \sigma$ and hence $F \rho_X G$, i.e., $F \cap G \neq \emptyset$ — a contradiction. Thus P has only one point, which will be denoted by x_σ. Now (1.16) implies that $\sigma_{x_\sigma} \subseteq \sigma$. Hence, by 1.2.2.5,

(1.17) $\sigma = \sigma_{x_\sigma}$, i.e., $\sigma = t_X(x_\sigma)$.

So, t_X is a bijection. We have also proved that every cluster in $(RC(X), \rho_X)$ is a point-cluster. Having this in mind, we obtain that, for every $F \in RC(X)$, $t_X(F) = \{\sigma_x \mid x \in F\} = \{\sigma_x \mid F \in \sigma_x\} = \{\sigma \in \mathrm{Clust}(RC(X)) \mid F \in \sigma\} = \lambda_{RC(X)}(F)$. Since $RC(X)$ and $\{\lambda_{RC(X)}(F) \mid F \in RC(X)\}$ are closed bases of X and $\Phi^a(\Phi^t(X))$ respectively, we conclude that

(1.18) t_X is a homeomorphism.

This fact, (1.12) and the propositions 1.2.2.9, 1.2.2.10 (which are proved below) imply our assertion. □

Proposition 1.2.2.9. *Let* (A, C) *and* (B, C') *be NCAs and*

$$\varphi : (A, C) \longrightarrow (B, C')$$

be an NCA-isomorphism. Then the map

$$f_\varphi : \Phi^a(A, C) \longrightarrow \Phi^a(B, C'),\ \sigma \mapsto \{\varphi(a) \mid a \in \sigma\}\ (= \varphi(\sigma)),$$

is a homeomorphism.

Proof. Obviously, the map f_φ is well-defined and is a bijection. We need only to show that it is continuous (because it is a map between compact Hausdorff spaces). Set $X = \Phi^a(A, C)$ and $Y = \Phi^a(B, C')$. Let $\sigma_0 \in X$, $\sigma_0' = f_\varphi(\sigma_0)$, $b \in B \setminus \sigma_0'$ and $V = Y \setminus \lambda_B(b)$. Then $\sigma_0' \in V$. Let $a = \varphi^{-1}(b)$ and set $U = X \setminus \lambda_A(a)$. Then $\sigma_0 \in U$. It rests to show that $f_\varphi(U) \subseteq V$. Let $\sigma \in U$. Then $a \notin \sigma$. Hence $b = \varphi(a) \notin \varphi(\sigma)$. Thus $b \notin f_\varphi(\sigma)$, i.e., $f_\varphi(\sigma) \in V$. □

Proposition 1.2.2.10. *Let* X *and* Y *be compact Hausdorff spaces and*

$$f : X \longrightarrow Y$$

be a homeomorphism. Then the map

$$\varphi_f : \Phi^t(X) \longrightarrow \Phi^t(Y),\ F \mapsto f(F),$$

is an NCA-isomorphism.

Proof. It is obvious. □

1.2.3 Local contact algebras and the proof of Roeper's Representation Theorem

The following notion is a lattice-theoretical counterpart of Leader's notion of *local proximity* ([78]):

Definition 1.2.3.1. ([99]) An algebraic system $\underline{B}_l = (B, 0, 1, \vee, \wedge, ^*, \rho, \mathbb{B})$ is called a *local contact Boolean algebra* or, briefly, *local contact algebra* (abbreviated as LCA or LC-algebra) if $(B, 0, 1, \vee, \wedge, ^*)$ is a Boolean algebra, ρ is a binary relation on B such that (B, ρ) is a CA, and $\mathbb{B}$ is an ideal (possibly non proper) of B, satisfying the following axioms:

(BC1) If $a \in \mathbb{B}$, $c \in B$ and $a \ll_\rho c$ then $a \ll_\rho b \ll_\rho c$ for some $b \in \mathbb{B}$ (see 1.2.1.1 for "$\ll_\rho$");

(BC2) If $a\rho b$ then there exists an element c of $\mathbb{B}$ such that $a\rho(c \wedge b)$;

(BC3) If $a \neq 0$ then there exists $b \in \mathbb{B} \setminus \{0\}$ such that $b \ll_\rho a$.

We shall simply write $(B, \rho, \mathbb{B})$ for a local contact algebra. We will say that the elements of $\mathbb{B}$ are *bounded* and the elements of $B \setminus \mathbb{B}$ are *unbounded*. When B is a complete Boolean algebra, the LCA $(B, \rho, \mathbb{B})$ is called a *complete local contact Boolean algebra* or, briefly, *complete local contact algebra* (abbreviated as CLCA or CLC-algebra).

We will say that two local contact algebras $(B, \rho, \mathbb{B})$ and $(B_1, \rho_1, \mathbb{B}_1)$ are *LCA-isomorphic* if there exists a Boolean isomorphism $\varphi : B \longrightarrow B_1$ such that, for $a, b \in B$, $a\rho b$ iff $\varphi(a)\rho_1\varphi(b)$, and $\varphi(a) \in \mathbb{B}_1$ iff $a \in \mathbb{B}$. A map $\varphi : (B, \rho, \mathbb{B}) \longrightarrow (B_1, \rho_1, \mathbb{B}_1)$ is called an *LCA-embedding* if $\varphi : B \longrightarrow B_1$ is an injective Boolean homomorphism (i.e., Boolean monomorphism) and, moreover, for any $a, b \in B$, $a\rho b$ iff $\varphi(a)\rho_1\varphi(b)$, and $\varphi(a) \in \mathbb{B}_1$ iff $a \in \mathbb{B}$.

Remark 1.2.3.2. Note that if $(B, \rho, \mathbb{B})$ is a local contact algebra and $1 \in \mathbb{B}$ then (B, ρ) is a normal contact algebra. Conversely, any normal contact algebra (B, C) can be regarded as a local contact algebra of the form (B, C, B).

Example 1.2.3.3. Let $(X, \beta, \mathcal{B})$ be a separated local proximity space. Then the triple $(RC(X), \beta, \mathcal{B} \cap RC(X))$ is a local contact algebra.

Proof. It is clear that only the axioms (BC1)-(BC3) need to be checked. The first one follows immediately from the axiom (LP1) (see 0.5.1.9) and the analogue of Lemma 1.2.1.10. Let us show that (BC3) is fulfilled. Take a non-empty regular closed set F. Then there exists a point $x \in \mathrm{int}(F)$. Since $\{x\} \in \mathcal{B}$ and $x \ll F$, the axiom (LP1) implies that there is a $G \in \mathcal{B}$ such that $x \ll G \ll F$. As in Lemma 1.2.1.10, we can find a $G_1 \in RC(X)$ with $x \ll G_1 \ll G$. Then $G_1 \in \mathcal{B}$, $G_1 \neq \emptyset$ and $G_1 \ll F$, as required.

It remains to be shown that the axiom (BC2) is fulfilled. Let $A, B \in RC(X)$ and $B\beta A$. By the Leader Theorem 0.5.2.5, there exists a unique locally compact extension Y of X such that $X \subseteq Y$ and $\gamma_X(Y) = (X, \beta, \mathcal{B})$. Then we have that

$\mathrm{cl}_Y(A)\cap\mathrm{cl}_Y(B)\neq\emptyset$. Let $y\in\mathrm{cl}_Y(A)\cap\mathrm{cl}_Y(B)$. Since Y is locally compact, there exists an open in Y set U with compact closure such that $y\in U$. We have that $y\in\mathrm{cl}_Y(A)=\mathrm{cl}_Y(\mathrm{int}_X(A))$ (because $A\in RC(X)$). Hence, setting $D=\mathrm{cl}_X(U\cap\mathrm{int}_X(A))$, we obtain that $D\in RC(X)$, $\mathrm{cl}_Y(D)\subseteq\mathrm{cl}_Y(U)$ and hence $\mathrm{cl}_Y(D)$ is compact in Y, i.e., $D\in\mathcal{B}$. Further, using [53, 1.3.D(a)], we have $y\in U\cap\mathrm{cl}_Y(A)=U\cap\mathrm{cl}_Y(\mathrm{int}_X(A))\subseteq\mathrm{cl}_Y(U\cap\mathrm{cl}_Y(\mathrm{int}_X(A)))=\mathrm{cl}_Y(U\cap\mathrm{int}_X(A))=\mathrm{cl}_Y(D)$, i.e., $\mathrm{cl}_Y(D)\cap\mathrm{cl}_Y(B)\neq\emptyset$. Therefore, $D\beta B$, $D\in\mathcal{B}\cap RC(X)$ and $D\subseteq A$. This completes the proof. □

The following lemmas are lattice-theoretical counterparts of some theorems from Leader's paper [78].

Definition 1.2.3.4. Let $(B,\rho,\mathbb{B})$ be a local contact algebra. Define a binary relation C_ρ on B by

(1.19) $aC_\rho b$ iff $(a\rho b$ or $a,b\in B\setminus\mathbb{B})$.

It is called the *Alexandroff extension of* ρ. Sometimes we will write $C_{(\rho,\mathbb{B})}$ or even $C_{(B,\rho,\mathbb{B})}$ instead of C_ρ.

Lemma 1.2.3.5. *Let* $(B,\rho,\mathbb{B})$ *be a local contact algebra. Then* (B,C_ρ)*, where* C_ρ *is the Alexandroff extension of* ρ*, is a normal contact algebra.*

Proof. We set, for brevity, $C=C_\rho$. The axioms $(C1)-(C3)$ (see Definition 1.2.1.1) follow directly from the properties of ρ and $\mathbb{B}$ and the definition of C. Let us check that the axiom $(C4)$ is fulfilled.

Let $xC(y\vee z)$. Suppose first that $x\rho(y\vee z)$. Then $x\rho y$ or $x\rho z$ and hence xCy or xCz. If $x\notin\mathbb{B}$ and $(y\vee z)\notin\mathbb{B}$ then $y\notin\mathbb{B}$ or $z\notin\mathbb{B}$; hence xCy or xCz. Conversely, let xCy or xCz. If $x\rho y$ or $x\rho z$ then $x\rho(y\vee z)$ and hence $xC(y\vee z)$. It remains to regard the case when $x\notin\mathbb{B}$, $y\notin\mathbb{B}$ and $z\notin\mathbb{B}$. Then $(y\vee z)\notin\mathbb{B}$ and hence $xC(y\vee z)$. So, the axiom $(C4)$ is fulfilled.

We shall now prove that the axiom $(\ll 5)$ (see 1.2.1.1) (which is equivalent to the axiom $(C5)$) is fulfilled. Recall that we write $x\ll_C y$ (resp., $x\ll_\rho y$) when $x(-C)y^*$ (resp., $x(-\rho)y^*$). Note that

$$x(-C)y\Longleftrightarrow(x(-\rho)y\text{ and }(x\in\mathbb{B}\text{ or }y\in\mathbb{B})).$$

Hence,

$$\text{if }x\in\mathbb{B}\text{ (or }y^*\in\mathbb{B})\text{ and }x\ll_\rho y,\text{ then }x\ll_C y.$$

Let $x \ll_C z$. Then $x(-C)z^*$ and hence $x(-\rho)z^*$ and ($x \in \mathbb{B}$ or $z^* \in \mathbb{B}$). Therefore $x \ll_\rho z$ and $z^* \ll_\rho x^*$. If $x \in \mathbb{B}$ then, by (BC1) (see Definition 1.2.3.1), there exists an $y \in \mathbb{B}$ such that $x \ll_\rho y \ll_\rho z$. Thus $x \ll_C y \ll_C z$. If $z^* \in \mathbb{B}$ then, by (BC1), there exists an element t of $\mathbb{B}$ such that $z^* \ll_\rho t \ll_\rho x^*$. This implies that $z^* \ll_C t \ll_C x^*$. So, $z^*(-C)t^*$ and $t(-C)x$, i.e., $x \ll_C t^* \ll_C z$. Therefore, the axiom ($C5$) is fulfilled.

Let's, finally, verify that the axiom ($C6$) is fulfilled. Let $x \neq 1$. Then $x^* \neq 0$. By (BC3), there exists an $y \in \mathbb{B} \setminus \{0\}$ such that $y \ll_\rho x^*$. This implies that $y \ll_C x^*$, i.e., $y(-C)x$ and $y \neq 0$. So, (B, C) is a normal contact algebra. □

Definition 1.2.3.6. Let $(B, \rho, \mathbb{B})$ be a local contact algebra. We will say that σ is a *cluster in* $(B, \rho, \mathbb{B})$ if σ is a cluster in the NCA (B, C_ρ) (see Definition 1.2.3.4 and Lemma 1.2.3.5). A cluster σ in $(B, \rho, \mathbb{B})$ (resp., an ultrafilter u in B) is called *bounded* if $\sigma \cap \mathbb{B} \neq \emptyset$ (resp., $u \cap \mathbb{B} \neq \emptyset$). The set of all bounded clusters in $(B, \rho, \mathbb{B})$ will be denoted by $\mathrm{BClust}(B, \rho, \mathbb{B})$.

Lemma 1.2.3.7. *Let $(B, \rho, \mathbb{B})$ be a local contact algebra and let $1 \notin \mathbb{B}$. Then*

$$\sigma_\infty^{(B,\rho,\mathbb{B})} = \{b \in B \mid b \notin \mathbb{B}\}$$

is a cluster in $(B, \rho, \mathbb{B})$. (Sometimes we will simply write σ_∞ or σ_∞^B instead of $\sigma_\infty^{(B,\rho,\mathbb{B})}$.)

Proof. By Lemma 1.2.3.5, we have that (B, C_ρ) is a normal contact algebra. We shall simply write "C" instead of "C_ρ", and "σ" instead of "σ_∞".

Let $x, y \in \sigma$. Then $x \notin \mathbb{B}$ and $y \notin \mathbb{B}$. Hence, by the definition of C (see Definition 1.2.3.4), we obtain that xCy. So the axiom (K1) (see 1.2.2.1) is fulfilled.

The axiom (G) follows directly from (I3) (see 0.3.1.5) (recall that $\mathbb{B}$ is an ideal).

For showing that the axiom (CLU) is also fulfilled, let xCy for every $y \in \sigma$. We will prove that $x \notin \mathbb{B}$, i.e., that $x \in \sigma$. So, suppose that $x \in \mathbb{B}$. Then $x \neq 1$. Hence $x \ll_\rho 1$. By (BC1), there exists an element z of $\mathbb{B}$ such that $x \ll_\rho z \ll_\rho 1$. Then $x(-\rho)z^*$. Since $1 \notin \mathbb{B}$, we obtain that $z^* \notin \mathbb{B}$. Thus $z^* \in \sigma$. This implies, by our assumption, that xCz^*. Therefore, by the definition of the relation "C" (see Definition 1.2.3.4), we conclude that $x \notin \mathbb{B}$ (because we have that $x(-\rho)z^*$), which is a contradiction.

So, σ is a cluster in (B, C). □

Proposition 1.2.3.8. ([99, 117]) *Let (X, τ) be a locally compact Hausdorff space. Then:*

(a) the triple $(RC(X,\tau), \rho_{(X,\tau)}, CR(X,\tau))$ *(see Example 1.2.1.9 for* $\rho_{(X,\tau)}$*) is a complete local contact algebra; it is called a* standard local contact algebra*;*

(b) for every $x \in X$*,* σ_x *is a bounded cluster in the standard local contact algebra* $(RC(X,\tau), \rho_{(X,\tau)}, CR(X,\tau))$*.*

Proof. (a) By Theorem 0.5.2.5, $\gamma_X((X, id_X)) = (X, \beta_{(X,id_X)}, \mathcal{B}_{(X,id_X)})$, where $\mathcal{B}_{(X,id_X)} = \{A \subseteq X \mid \mathrm{cl}_X(A) \text{ is compact}\}$ and, for $A, B \subseteq X$, $A\beta_{(X,id_X)}B$ iff $\mathrm{cl}_X(A) \cap \mathrm{cl}_X(B) \neq \emptyset$, is a separated local proximity space. We can now use Example 1.2.3.3 for finishing the proof.

(b) This is obvious. □

Proposition 1.2.3.9. *Let* L *be a locally compact Hausdorff space and* X *be a dense subspace of* L*. Then the local contact algebra* $(RC(L), \rho_L, CR(L))$ *is isomorphic to the local contact algebra* $(RC(X), \beta_{(L,i_{X,L})}, \mathcal{B}_{(L,i_{X,L})} \cap RC(X))$ *(see Theorem 0.5.2.5 for the notation). In particular, if* L *is a compact Hausdorff space then* $(RC(L), \rho_L)$ *and* $(RC(X), \beta_{(L,i_{X,L})})$ *are isomorphic normal contact algebras.*

Proof. We will write i_X instead of $i_{X,L}$.

First of all we have, by 1.2.3.8, that $(RC(L), \rho_L, CR(L))$ is a local contact algebra. Since L is a locally compact Hausdorff extension of X, the Leader Theorem 0.5.2.5 tells us that $\gamma_X((L, i_X)) = (X, \beta_{(L,i_X)}, \mathcal{B}_{(L,i_X)})$ is a separated local proximity space. We obtain now, using Example 1.2.3.3, that $(RC(X), \beta_{(L,i_X)}, \mathcal{B}_{(L,i_X)} \cap RC(X))$ is a local contact algebra. By Lemma 0.4.2.2, the function $e : RC(X) \longrightarrow RC(L)$, defined by $e(F) = \mathrm{cl}_L(F)$, is a Boolean isomorphism. Now, the definition of the family $\mathcal{B}_{(L,i_X)}$ (see Theorem 0.5.2.5) gives us that $e(F) \in CR(L)$ iff $F \in \mathcal{B}_{(L,i_X)} \cap RC(X)$. On the other hand, from the definition of $\beta_{(L,i_X)}$ (see Theorem 0.5.2.5) we obtain immediately that, for $F, G \in RC(X)$, $F\beta_{(L,i_X)}G$ iff $e(F)\rho_L f(G)$. So, e is an isomorphism between the local contact algebras $(RC(X), \beta_{(L,i_X)}, \mathcal{B}_{(L,i_X)} \cap RC(X))$ and $(RC(L), \rho_L, CR(L))$.

When L is compact, we obviously have that $CR(L) = RC(L)$ and $X \in \mathcal{B}_{(L,i_X)} \cap RC(X)$, so that the last statement of our proposition follows from its first statement and Remark 1.2.3.2. □

Theorem 1.2.3.10. (The Roeper Representation Theorem for LCAs ([99]))

(a) Each LCA $(B, \zeta, \mathbb{B})$ *can be densely embedded into a standard local contact algebra* $(RC(L), \rho_L, CR(L))$*, where* L *is a locally compact Hausdorff space. When* B *is complete this embedding becomes a complete isomorphism.*

(b) There exists a bijective correspondence between the class of all (up to isomorphism) complete local contact algebras and the class of all (up to homeomorphism) locally compact Hausdorff spaces.

Proof. (a) Let (X, τ) be a locally compact Hausdorff space. We put

(1.20) $\Psi^t(X, \tau) = (RC(X, \tau), \rho_{(X,\tau)}, CR(X, \tau))$

(see 1.2.3.8 for the notation). By 1.2.3.8, $\Psi^t(X, \tau)$ is a CLCA.

Let $(B, \rho, \mathbb{B})$ be a local contact algebra. Let $C = C_\rho$ be the Alexandroff extension of ρ (see 1.2.3.4). Then, by 1.2.3.5, (B, C) is a normal contact algebra. Put $(X, \mathcal{T}) = \Phi^a(B, C)$ (see Theorem 1.2.2.8 for Φ^a).

If $1 \in \mathbb{B}$ then $\mathbb{B} = B$, $\rho = C$ and $(B, \rho, \mathbb{B}) = (B, \rho, B) = (B, \rho)$ is a normal contact algebra (see Remark 1.2.3.2). So, we put

(1.21) $\Psi^a(B, \rho, \mathbb{B}) = (X, \mathcal{T}) = \Phi^a(B, C)$, when $1 \in \mathbb{B}$,

and our assertion follows from Theorem 1.2.2.8(a).

Let now $1 \notin \mathbb{B}$. Then, by Lemma 1.2.3.7, the set $\sigma_\infty = \{x \in B \mid x \notin \mathbb{B}\}$ is a cluster in (B, C) and, hence, $\sigma_\infty \in X$. Let

(1.22) $L = X \setminus \{\sigma_\infty\}$.

Then, using 1.2.2.5, we get that

(1.23) L is the set of all bounded clusters of $(B, \rho, \mathbb{B})$

(sometimes we will write $L_{(B,\rho,\mathbb{B})}$ or L_B instead of L). Let the topology $\tau (= \tau_B = \tau_{(B,\rho,\mathbb{B})})$ on L be the subspace topology, i.e., $\tau = \mathcal{T}|_L$. Then (L, τ) is a locally compact Hausdorff space. So, we put

(1.24) $\Psi^a(B, \rho, \mathbb{B}) = (L, \tau)$, when $1 \notin \mathbb{B}$.

Put

(1.25) $\lambda^l_{(B,\rho,\mathbb{B})}(a) = \lambda_{(B,C_\rho)}(a) \cap L$,

for each $a \in B$. We will simply write λ^l_B (or even λ^l or $\lambda_{(B,\rho,\mathbb{B})}$) instead of $\lambda^l_{(B,\rho,\mathbb{B})}$ when this does not lead to ambiguity.

We will now show that:

(I) L is a dense subset of X;

(II) λ^l_B is a dense Boolean monomorphism of the Boolean algebra B into the Boolean algebra $RC(L,\tau)$ (and, hence, when B is complete, then λ^l_B is a Boolean isomorphism onto the Boolean algebra $RC(L,\tau)$);

(III) $b \in \mathbb{B}$ iff $\lambda^l(b) \in CR(L)$;

(IV) $a\rho b$ iff $\lambda^l(a) \cap \lambda^l(b) \neq \emptyset$.

In other words, λ^l will be an LCA-embedding of the local contact algebra $(B,\rho,\mathbb{B})$ into the standard local contact algebra $(RC(L),\rho_L,CR(L))$; when B is complete, this embedding will be an isomorphism onto the standard CLCA $(RC(L),\rho_L,CR(L))$.

To prove (I), recall that $\{X \setminus \lambda_B(a) \mid a \in B\}$ is an open base of $(X,\mathcal{T})$. As it follows from the definition of σ_∞, for any $a \in B$, $\sigma_\infty \in X \setminus \lambda_B(a)$ iff $a \in \mathbb{B}$. So, let $a \in \mathbb{B}$. Since $1 \notin \mathbb{B}$, we have that $1 \not\leq a$. Then, by (C6$'$), there is an element c of B such that $cC1$ and $c(-C)a$. Thus $c \neq 0$ and, using (BC3), we can find an element b of $\mathbb{B}\setminus\{0\}$ such that $b \ll_\rho c$. Then $b(-C)a$. Therefore, $bC1$, $b \in \mathbb{B}$ and $b(-C)a$. Now, Lemma 1.2.2.2 implies that there is a cluster σ in (B,C) such that $b \in \sigma$; hence $\sigma \neq \sigma_\infty$, i.e., $\sigma \in L$. Since $b(-C)a$, we obtain that $a \notin \sigma$. Therefore, $\sigma \in L \cap (X \setminus \lambda_B(a))$. This shows that L is a dense subset of X.

Let's prove (II). We have, by (I) above and by 0.4.2.2, that the function $r : RC(X,\mathcal{T}) \longrightarrow RC(L,\tau)$, defined by $r(A) = A \cap L$ for every $A \in RC(X)$, is a Boolean isomorphism. Since $\lambda^l_B = r \circ \lambda_B$, we obtain, using Theorem 1.2.2.8, that λ^l_B is a dense Boolean monomorphism of the Boolean algebra B into the Boolean algebra $RC(L,\tau)$ (and, hence, when B is complete, then λ^l_B is a Boolean isomorphism)

To establish (III), recall that, for $a \in B$, $\sigma_\infty \in \lambda_B(a) \leftrightarrow a \in \sigma_\infty \leftrightarrow a \notin \mathbb{B}$ and hence $\lambda_B(a) \subseteq L \leftrightarrow a \in \mathbb{B}$; further, for any $a \notin \mathbb{B}$, we have that $\sigma_\infty \in \lambda_B(a) = \mathrm{cl}_X(\lambda^l(a))$ (see 0.4.2.2); hence $\lambda^l(a)$ is compact iff $\lambda_B(a) \subseteq L$ iff $a \in \mathbb{B}$.

Finally, we will show that (IV) takes place. Let $a,b \in B$ and $a\rho b$. Then, by (BC2), there exist $a_1,b_1 \in \mathbb{B}$ such that $a_1 \leq a$, $b_1 \leq b$ and $a_1\rho b_1$. Then a_1Cb_1 and hence $\lambda_B(a_1) \cap \lambda_B(b_1) \neq \emptyset$. Since, by (III) above, $\lambda_B(a_1) = \lambda^l(a_1) \subseteq \lambda_B(a) \cap L = \lambda^l(a)$ and $\lambda_B(b_1) = \lambda^l(b_1) \subseteq \lambda_B(b) \cap L = \lambda^l(b)$, we obtain that $\lambda^l(a) \cap \lambda^l(b) \neq \emptyset$.

Let's prove the implication in the converse direction. Take $a,c \in B$ for which $\lambda^l(a) \cap \lambda^l(c) \neq \emptyset$. Then there exists a cluster $\sigma \in \lambda_B(a) \cap \lambda_B(c) \cap L$. Since $\sigma \neq \sigma_\infty$, there exists an element b_0 of $\mathbb{B}$ belonging to σ. Then $b_0 \neq 1$ and hence $b_0 \ll_\rho 1$. By (BC1), there exists an element b of $\mathbb{B}$ such that $b_0 \ll_\rho b \ll_\rho 1$. Hence $b \in \sigma$ and $b_0(-\rho)b^*$. By the definition of C, we obtain that $b_0(-C)b^*$, i.e., $b^* \notin \sigma$. So, $\sigma \in \mathrm{int}_X(\lambda_B(b))$, where $b \in \mathbb{B}$. We will now show that $\sigma \in \lambda_B(a) \wedge \lambda_B(b) = \lambda_B(a \wedge b)$

and $\sigma \in \lambda_B(c) \wedge \lambda_B(b) = \lambda_B(c \wedge b)$. This will imply that $a \wedge b \in \sigma$ and $c \wedge b \in \sigma$ and hence $(a \wedge b)C(c \wedge b)$; since $a \wedge b, c \wedge b \in \mathbb{B}$, we will obtain that $(a \wedge b)\rho(c \wedge b)$ and, therefore, $a\rho c$. So, let's show that $\sigma \in \lambda_B(a) \wedge \lambda_B(b) = \mathrm{cl}_X(\mathrm{int}_X(\lambda_B(a) \cap \lambda_B(b)))$. Supposing that this is not the case, we can find an open neighborhood U of σ such that $U \cap \mathrm{int}_X(\lambda_B(a) \cap \lambda_B(b)) = \emptyset$. Put $V = U \cap \mathrm{int}_X(\lambda_B(b))$. Then V is also an open neighborhood of σ and $V \cap \mathrm{int}_X(\lambda_B(a)) = \emptyset$. This is a contradiction because $\sigma \in \lambda_B(a) = \mathrm{cl}_X(\mathrm{int}_X(\lambda_B(a)))$. So, we have proved that $\sigma \in \lambda_B(a) \wedge \lambda_B(b)$. Analogously, we obtain that $\sigma \in \lambda_B(c) \wedge \lambda_B(b)$. This finishes the proof of (IV).

Hence,

(1.26) $X = \alpha L$,

i.e., X is the Alexandroff (i.e., one-point) compactification of L and

(1.27) $\lambda_B^l : (B, \rho, \mathbb{B}) \longrightarrow (RC(L), \rho_L, CR(L))$ is a dense LCA-embedding;

thus,

(1.28) if $(B, \rho, \mathbb{B})$ is a CLCA then λ_B^l is an LCA-isomorphism.

For unifying both cases (i.e., when $1 \in \mathbb{B}$, and when $1 \notin \mathbb{B}$), we will introduce some new notation. For every LCA $(B, \rho, \mathbb{B})$ and every $a \in B$, set

(1.29) $\lambda^g_{(B,\rho,\mathbb{B})}(a) = \lambda_{(B,C_\rho)}(a) \cap \Psi^a(B, \rho, \mathbb{B})$.

We will write simply λ_B^g instead of $\lambda^g_{(B,\rho,\mathbb{B})}$ when this does not lead to ambiguity. Thus, when $1 \in \mathbb{B}$, we have that $\lambda_B^g = \lambda_B$, and if $1 \notin \mathbb{B}$ then $\lambda_B^g = \lambda_B^l$. Hence,

(1.30) $\lambda_B^g : (B, \rho, \mathbb{B}) \longrightarrow (\Psi^t \circ \Psi^a)(B, \rho, \mathbb{B})$ is a dense LCA-embedding,

and

(1.31) when $(B, \rho, \mathbb{B})$ is a CLCA then λ_B^g is an LCA-isomorphism.

(b) Let (L, τ) be a locally compact Hausdorff space and (L, τ) be non compact. Put $B = RC(L, \tau)$, $\mathbb{B} = CR(L, \tau)$ and $\rho = \rho_L$. Then $(B, \rho, \mathbb{B}) = \Psi^t(L, \tau)$ and $1 \notin \mathbb{B}$ (here $1 = L$). We will show that the map

(1.32) $t_{(L,\tau)} : (L, \tau) \longrightarrow \Psi^a(\Psi^t(L, \tau))$,

defined by $t_{(L,\tau)}(x) = \{F \in RC(L, \tau) \mid x \in F\} (= \sigma_x^B)$, for all $x \in L$, is a homeomorphism (we will often write simply t_L instead of $t_{(L,\tau)}$). Put $(X, \mathcal{T}) = \Phi^a(B, C_\rho)$. Then

$X = \mathrm{Clust}(B, C_\rho)$ and if $L' = X \setminus \{\sigma_\infty^B\}$ and $\tau' = \mathfrak{T}|_{L'}$ then $(L', \tau') = \Psi^a(B, \rho, \mathbb{B})$ (see (1.24)). It is easy to see that, for every $x \in L$, $\sigma_x^B = \{F \in RC(L) \mid x \in F\}$ is a bounded cluster in (B, C_ρ). Hence t_L is defined correctly.

Let $\alpha L = L \cup \{\infty\}$ be the one-point compactification of L. Then, by 0.4.2.2, $RC(L)$ and $RC(\alpha L)$ are isomorphic complete Boolean algebras and the isomorphism e between them is defined by the formula $e(F) = \mathrm{cl}_{\alpha L}(F)$, for every $F \in RC(L)$ (note that $e^{-1}(G) = r(G) = G \cap L$, for every $G \in RC(\alpha L)$). It is easy to see (using the notation of 0.5.2.5) that $\beta_{\alpha L, i_{L,\alpha L}} = C_{\rho_L}$ on $RC(L)$. Hence, by (the proof of) 1.2.3.9, the function e is an NCA-isomorphism between $(RC(L), C_{\rho_L})$ and $(RC(\alpha L), \rho_{\alpha L})$. Since $\Phi^a(\Phi^t(\alpha L)) = \Phi^a(RC(\alpha L), \rho_{\alpha L})$, we get, by 1.2.2.9 that the function

(1.33) $f_r : \Phi^a(\Phi^t(\alpha L)) \longrightarrow \Phi^a(RC(L), C_{\rho_L})$,

defined by $f_r(\sigma) = \{r(G) \mid G \in \sigma\} = \{G \cap L \mid G \in \sigma\}$ for every $\sigma \in \Phi^a(\Phi^t(\alpha L))$, is a homeomorphism. By (1.18), the function $t_{\alpha L} : \alpha L \longrightarrow \Phi^a(\Phi^t(\alpha L))$, defined by $t_{\alpha L}(x) = \{G \in RC(\alpha L) \mid x \in G\} (= \sigma_x)$, for all $x \in \alpha L$, is a homeomorphism. Hence $f = f_r \circ t_{\alpha L} : \alpha L \longrightarrow \Phi^a(RC(L), C_{\rho_L})$ is a homeomorphism. Recall that $\alpha L = L \cup \{\infty\}$, where $\infty \notin L$. Obviously $f(\infty)) = f_r(\{G \in RC(\alpha L) \mid \infty \in G\} = \sigma_\infty^B$. Thus the restriction f' of f to L is a homeomorphism between L and L'. From the definitions of f' and t_L we get that these maps coincide. So, we have proved that t_L is a homeomorphism.

In the case when (L, τ) is compact, we define $t_{L,\tau} : (L, \tau) \longrightarrow \Psi^a(\Psi^t(L, \tau))$ as in (1.15) (using the definitions of Ψ^a, Φ^a, Ψ^t and Φ^t) and thus, by (1.18), t_L is a homeomorphism.

This fact, (1.31) and the propositions 1.2.3.11, 1.2.3.12 (which are proved below) imply our assertion. □

Proposition 1.2.3.11. *Let $(A, \rho, \mathbb{B})$ and $(A', \rho', \mathbb{B}')$ be LCAs and*

$$\varphi : (A, \rho, \mathbb{B}) \longrightarrow (A', \rho', \mathbb{B}')$$

be an LCA-isomorphism. Then the map

$$f_\varphi : \Psi^a(A, \rho, \mathbb{B}) \longrightarrow \Psi^a(A', \rho', \mathbb{B}'),\ \sigma \mapsto \{\varphi(a) \mid a \in \sigma\},$$

is a homeomorphism.

Proof. Clearly, the map

$$\varphi_C : (A, C_\rho) \longrightarrow (A', C_{\rho'}),\ a \mapsto \varphi(a)$$

is an NCA-isomorphism. Then, by 1.2.2.9, the map

$$f_{\varphi_C} : \Phi^a(A, C_\rho) \longrightarrow \Phi^a(A', C_{\rho'})$$

is a homeomorphism. Since φ is an LCA-isomorphism, we have that $\varphi(\mathbb{B}) = \mathbb{B}'$. Thus $\varphi(A \setminus \mathbb{B}) = A' \setminus \mathbb{B}'$, i.e.,

$$f_{\varphi_C}(\sigma_\infty^A) = \sigma_\infty^{A'}.$$

Now, the definitions of f_φ and Ψ^a imply that the map f_φ is a homeomorphism. □

Proposition 1.2.3.12. *Let X and Y be locally compact Hausdorff spaces and*

$$f : X \longrightarrow Y$$

be a homeomorphism. Then the map

$$\varphi_f : \Psi^t(X) \longrightarrow \Psi^t(Y),\ F \mapsto f(F),$$

is an LCA-isomorphism.

Proof. This is obvious. □

Remarks 1.2.3.13. Let $(B, \rho, \mathbb{B})$ be an LCA, $L = L_B = \Psi^a(B, \rho, \mathbb{B})$, $X = \Phi^a(B, C_\rho)$, $\lambda_B^g : (B, \rho, \mathbb{B}) \longrightarrow \Psi^t(L)$ be defined by (1.29), and $\lambda_B : (B, C_\rho) \longrightarrow \Phi^t(X)$ be defined by (1.4). Then, using the notation of Theorems 1.2.3.10 and 1.2.2.8, we get the following:

(a) For every $b \in B$,

(1.34) $\mathrm{int}_{L_B}(\lambda_B^g(b)) = L_B \cap \mathrm{int}_X(\lambda_B(b))$.

Further, using (1.8) and (1.34), we obtain readily that for every $b \in B$,

(1.35) $L \setminus \lambda_B^g(b) = \mathrm{int}_L(\lambda_B^g(b^*))$.

(b) The next assertion specifies (1.9):

(1.36) $\{\mathrm{int}_{\Psi^a(B,\rho,\mathbb{B})}(\lambda_B^g(a)) \mid a \in \mathbb{B}\}$ is an open base of $\Psi^a(B, \rho, \mathbb{B})$.

(c) Note that (1.11) and (IV) (see the proof of Theorem 1.2.3.10) imply that

(1.37) $a\rho b$ iff there exists $\sigma \in \Psi^a(B, \rho, \mathbb{B})$ such that $a, b \in \sigma$.

(d) For every $F \in RC(L)$,

(1.38) $F = \bigvee\{\lambda_B^g(b) \mid b \in \mathbb{B}, \lambda_B^g(b) \ll_{\rho_L} F\}$

holds. Indeed, it is obvious that the right part of (1.38), which we will denote by G, is contained in the left part. Suppose that $F \neq G$. Then, since F is a regular closed set, $U = \mathrm{int}_L(F) \setminus G \neq \emptyset$. Clearly, U is open in L and thus, by (1.36), there exists $b \in \mathbb{B}$ such that $b \neq 0$ and $\lambda_B^g(b) \subseteq U$. Then $\lambda_B^g(b) \ll_{\rho_L} F$ (because of 1.2.1.9 and the fact that $U \subseteq \mathrm{int}_L(F)$), and hence $\lambda_B^g(b) \subseteq G$, a contradiction. Therefore, $F = G$.

Now, using Theorem 1.2.3.10(a), we get, in particular, that for every $a \in B$,

(1.39) $a = \bigvee\{b \in \mathbb{B} \mid b \ll_\rho a\}$.

1.3 MVD-algebras

1.3.1 Mormann's Enriched Boolean algebras

As we have already mentioned above, normal contact algebras could be equivalently defined as a pair of a Boolean algebra $B = (B, 0, 1, \vee, \wedge, ^*)$ and a binary relation $\ll$ subject to the following axioms:
($\ll$1) $x \ll y$ implies $x \leq y$;
($\ll$2) $0 \ll 0$;
($\ll$3) $x \leq y \ll z \leq t$ implies $x \ll t$;
($\ll$4) $x \ll z$ and $y \ll z$ implies $x \vee y \ll z$;
($\ll$5) If $x \ll z$ then $x \ll y \ll z$ for some $y \in B$;
($\ll$6) If $x \neq 0$ then there exists $y \neq 0$ such that $y \ll x$;
($\ll$7) $x \ll y$ implies $y^* \ll x^*$.

In [85], T. Mormann introduces the notion of an *enriched Boolean algebra* as a pair of a Boolean algebra B and a binary relation $\ll$ (called by him *interior parthood*) for which ($\ll$1)-($\ll$6) hold and the axiom
($\ll 4^*$) $x \ll y$ and $x \ll z$ imply $x \ll y \wedge z$
is fulfilled.

To be precise, he writes "$(\forall x \in B)(0 \ll x)$" instead of ($\ll$2) and replaces ($\ll$5) and ($\ll$6) with the following axiom
($\ll$5-6) $x \ll z$ and $x \neq z$ together imply $x \ll y \ll z$ for some $y \neq x$,
but, obviously, our expression of the axioms of enriched Boolean algebras is equivalent to that given by Mormann (indeed, let $x \ll z$ and $x \neq z$; then, by ($\ll$1), $z \wedge x^* \neq 0$ and,

by ($\ll$6), there exists a $t \neq 0$ such that $t \ll (z \wedge x^*)$; hence, by ($\ll$3) and ($\ll$1), $t \ll z$ and $t \leq x^*$; by ($\ll$5), there exists $u \in B$ with $x \ll u \ll z$; setting $y = u \vee t$, we obtain, using ($\ll$3) and ($\ll$4), that $x \ll y \ll z$; obviously $x \neq y$; so, ($\ll$5-6) follows from axioms ($\ll$1)-($\ll$6); conversely, the axioms ($\ll$5-6) and ($\ll$2) imply, in an obvious way, the axioms ($\ll$5) and ($\ll$6). Thus, the difference between our normal contact algebras and Mormann's enriched Boolean algebras is that our axiom ($\ll$7) is replaced by the weaker axiom ($\ll$ 4*) (because, having ($\ll$7), one can derive ($\ll$ 4*) from ($\ll$ 4)).

Note that the following extensionality axiom is fulfilled in the enriched Boolean algebras:

$$(x = y) \iff [\forall z : (z \ll x) \iff (z \ll y)].$$

Indeed, it is enough to show that $(x \leq y) \iff [\forall z : (z \ll x) \Rightarrow (z \ll y)]$. In the direction ($\Longrightarrow$), this follows from ($\ll$3). For proving the direction ($\Longleftarrow$), suppose that $x \not\leq y$. Then $z' = x \wedge y^* \neq 0$. Hence, by ($\ll$6), there exists an element $z \neq 0$ such that $z \ll z'$. Then, by ($\ll$3), $z \ll x$ and $z \ll y^*$. But $z \ll x$ implies that $z \ll y$. Thus we obtain, by ($\ll$ 4*), that $z \ll (y \wedge y^*)$, i.e., by ($\ll$1), $z = 0$, which is a contradiction.

1.3.2 MVD-algebras

In [85] Mormann affirms that for any enriched complete Boolean algebra $(B, \ll)$ there exists a locally compact Hausdorff space L such that $(B, \ll)$ is isomorphic to the enriched complete Boolean algebra $(RO(L), \ll_L)$, where $RO(L)$ is the complete Boolean algebra of regular open subsets of L and, for any $U, V \in RO(L)$, $U \ll_L V$ iff $\mathrm{cl}(U)$ is compact and $\mathrm{cl}(U) \subseteq V$; conversely, for any locally compact Hausdorff space L, the pair $(RO(L), \ll_L)$ is an enriched complete Boolean algebra. Since the map $\nu : RO(L) \longrightarrow RC(L)$, defined by $\nu(U) = \mathrm{cl}(U)$, is an isomorphism between the complete Boolean algebras $RO(L)$ and $RC(L)$, and, for $U, V \in RO(L)$, $U \ll_L V$ iff $\nu(U) \subseteq \mathrm{int}(\nu(V))$ and $\nu(U)$ is compact, we can say that $(B, \ll)$ is isomorphic to the enriched Boolean algebra $(RC(L), \ll_L)$, where, for all $F, G \in RC(L)$, $F \ll_L G$ iff F is compact and $F \subseteq \mathrm{int}(G)$ (we hope that the use of the same notation ($\ll_L$) with different meanings will cause no confusion). Trying to prove Mormann's representation theorem, we arrived to the following notion:

Definition 1.3.2.1. A pair $M = (B, \ll)$ is called an *MVD-algebra* if it is an enriched Boolean algebra and satisfies the following axiom

(MVD) If $x \ll 1$ then $y^* \ll x^*$ implies $x \ll y$.

When B is a complete Boolean algebra, we will say that M is a *complete MVD-algebra.*

It follows immediately from the corresponding definitions that normal contact algebras coincide with those MVD-algebras which satisfy the additional axiom

$(\ll 2')$ $1 \ll 1$.

We will show below that the notion of MVD-algebra is equivalent to the notion of local contact algebra and hence we will obtain a representation theorem for MVD-algebras (see Theorem 1.3.2.5 below) using Theorem 1.2.3.10. This representation theorem sounds exactly as Mormann's representation theorem [85], gives the same semantics, so that it should imply that complete MVD-algebras and enriched complete Boolean algebras are equivalent notions. This is, however, not the case, as the next simple example shows; hence, Mormann's representation theorem [85] is not true.

Example 1.3.2.2. There exists an enriched complete Boolean algebra which is not an MVD-algebra.

Proof. Let X be a non-empty and non-discrete T_1-space. We define a relation " $\ll$ " on the complete Boolean algebra $(P(X), \subseteq)$ by

$$A \ll B \iff \mathrm{cl}(A) \subseteq B.$$

Then it is easy to verify that $(P(X), \ll)$ is an enriched complete Boolean algebra. We shall show that $(P(X), \ll)$ is not an MVD-algebra, i.e., it does not satisfy the axiom (MVD) (from Definition 1.3.2.1). Indeed, let x be a non-isolated point of X. Put $B = A = X \setminus \{x\}$. Then, obviously, $A \ll X$ and $X \setminus B \ll X \setminus A$ (since $X \setminus B = \{x\}$ is a closed subset of X) but $A \not\ll B$, because A is not closed in X. □

Theorem 1.3.2.3. *The notions of local contact algebra and MVD-algebra are equivalent.*

Proof. Denote by $\mathcal{LCA}$ the class of all local contact algebras and by $\mathcal{MA}$ the class of all MVD-algebras. We shall define two functions

$$f : \mathcal{LCA} \longrightarrow \mathcal{MA} \text{ and } g : \mathcal{MA} \longrightarrow \mathcal{LCA}$$

and we will show that $f \circ g = id_{\mathcal{MA}}$ and $g \circ f = id_{\mathcal{LCA}}$.

Let $(B, \zeta, \mathbb{B})$ be a local contact algebra. We have that $x \ll_\zeta y$ iff $x(-\zeta)y^*$. Put $x \ll_M y$ iff $x \in \mathbb{B}$ and $x \ll_\zeta y$. We will prove that $(B, \ll_M)$ is an MVD-algebra and we will set $f((B, \zeta, \mathbb{B})) = (B, \ll_M)$.

For proving ($\ll$1), let $x \ll_M y$; then $x(-\zeta)y^*$ and hence $x \wedge y^* = 0$; this implies that $x \leq y$.

Since $0 \in \mathbb{B}$ and $0 \ll_\zeta 0$, we obtain that $0 \ll_M 0$, i.e., the axiom ($\ll$2) is fulfilled.

Let's verify the axiom ($\ll$3). Let $x \leq y \ll_M z \leq t$. Then $y \in \mathbb{B}$, $y(-\zeta)z^*$ and $x \vee y = y$. This implies, by (I2) and (C4) (see 0.3.1.5 and 1.2.1.1 for (I2) and (C4), and recall that, by the definition of a local contact algebra, $\mathbb{B}$ is an ideal of B and (B, ζ) is a CA), that $x \in \mathbb{B}$ and $x(-\zeta)z^*$. Since $z \leq t$, we have that $z^* \vee t^* = z^*$ and hence, by (C4), $x(-\zeta)t^*$. So, $x \in \mathbb{B}$ and $x \ll_\zeta t$. Therefore, $x \ll_M t$.

For checking ($\ll$4), let $x \ll_M z$ and $y \ll_M z$. Then $x, y \in \mathbb{B}$, $x(-\zeta)z^*$ and $y(-\zeta)z^*$. Hence, by (I3) and (C4), $x \vee y \in \mathbb{B}$ and $(x \vee y)(-\zeta)z^*$. Therefore, $x \vee y \ll_M z$.

Let's prove that ($\ll$5) is fulfilled. Let $x \ll_M z$. Then $x \in \mathbb{B}$ and $x \ll_\zeta z$. By (BC1), there exists an $y \in \mathbb{B}$ such that $x \ll_\zeta y \ll_\zeta z$. This means that $x \ll_M y \ll_M z$.

For verifying ($\ll$6), let $x \neq 0$. Then, by (BC3), there exists an $y \in \mathbb{B} \setminus \{0\}$ such that $y \ll_\zeta x$. Therefore $y \ll_M x$ and $y \neq 0$.

Let's show that ($\ll 4^*$) is fulfilled. Let $x \ll_M y$ and $x \ll_M z$. Then $x \in \mathbb{B}$, $x(-\zeta)y^*$ and $x(-\zeta)z^*$. By (C4), we obtain that $x(-\zeta)(y^* \vee z^*)$, i.e., $x(-\zeta)(y \wedge z)^*$. Hence $x \ll_M y \wedge z$.

For proving that (MVD) is fulfilled, let $x \ll_M 1$ and $y^* \ll_M x^*$. Then $x \in \mathbb{B}$ and $x(-\zeta)y^*$. Thus $x \ll_M y$.

So, $(W, \ll_M)$ is an MVD-algebra.

Let now $(B, \ll)$ be an MVD-algebra. Put

$$g((B, \ll)) = (B, \zeta_M, \mathbb{B}_M),$$

where

$$\mathbb{B}_M = \{x \in B : x \ll 1\}$$

and, for $x, y \in B$,

$$x\zeta_M y \text{ iff there exists } z \in \mathbb{B}_M \text{ such that } (z \wedge x) \not\ll (z \wedge y)^*.$$

We shall prove that $(B, \zeta_M, \mathbb{B}_M)$ is a local contact algebra, i.e., that the definition of the function g is correct. In the proof of this claim, we will put, for short, $\mathbb{B} = \mathbb{B}_M$ and $\zeta = \zeta_M$.

First of all, let's note that, for $x, y \in B$,

$$x\zeta y \text{ iff there exists } z \in \mathbb{B} \text{ such that } (z \wedge x) \not\ll (z \wedge y)^* \text{ and } (z \wedge y) \not\ll (z \wedge x)^*.$$

Indeed, if $x\zeta y$ then there exists $z \in \mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$. Since $z \wedge x \in \mathbb{B}$, (MVD) implies that $(z \wedge y) \not\ll (z \wedge x)^*$. The converse is clear.

From ($\ll$2) and ($\ll$3) we obtain that $0 \in \mathbb{B}$, i.e., the axiom (I1) is fulfilled. ($\ll$3) implies that (I2) is fulfilled and, by ($\ll$4), (I3) is also fulfilled.

For verifying (C1), let $x \neq 0$. Then ($\ll$6) implies that there exists an element z of $B \setminus \{0\}$ such that $z \ll x$. Hence, by ($\ll$3) and ($\ll$1), $z \in \mathbb{B}$, $z \wedge x = z$ and $z \not\ll z^*$. This means that $x\zeta x$.

Obviously, ($\ll$2) and ($\ll$3) imply that $0 \ll x$ for any $x \in B$. Using this fact, we will now establish the validity of (C2). Indeed, let $x\zeta y$. Then there exists an element z of $\mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$ and $(z \wedge y) \not\ll (z \wedge x)^*$. So, supposing that $x = 0$ or $y = 0$, we would obtain that $0 \not\ll (z \wedge y)^*$ or $0 \not\ll (z \wedge x)^*$, which is a contradiction. Hence, $x, y \neq 0$.

It is clear that (C3) follows directly from the observation which we have made after the definition of the relation ζ. Let's show that (C4) takes place.

Let $x\zeta(y \vee z)$. Then there exists an element t of $\mathbb{B}$ such that $(t \wedge x) \not\ll (t \wedge (y \vee z))^*$. Using ($\ll 4^*$), we obtain that

$$(t \wedge x) \not\ll (t \wedge y)^* \text{ or } (t \wedge x) \not\ll (t \wedge z)^*$$

If $(t \wedge x) \not\ll (t \wedge y)^*$ then $x\zeta y$. If $(t \wedge x) \not\ll (t \wedge z)^*$ then $x\zeta z$. So, $x\zeta(y \vee z)$ implies that $x\zeta y$ or $x\zeta z$.

Let now $x\zeta y$ or $x\zeta z$. Suppose, for example, that $x\zeta y$. Then there exists $u \in \mathbb{B}$ such that $(u \wedge x) \not\ll (u \wedge y)^*$. Using ($\ll$3), we obtain immediately that $(u \wedge x) \not\ll (u \wedge y)^* \wedge (u \wedge z)^*$, i.e., $(u \wedge x) \not\ll (u \wedge (y \vee z))^*$. Hence we obtain that $x\zeta(y \vee z)$. When $x\zeta z$, the proof is analogous. Therefore, we have shown that the axiom (C4) is fulfilled.

Let's verify that the axiom (BC1) is fulfilled. Let $x \in \mathbb{B}$, $z \in B$ and $x \ll_\zeta z$ (i.e., $x(-\zeta)z^*$). Since $x \ll 1$, there exists (by ($\ll$5)) an $u \in B$ such that $x \ll u \ll 1$. Then $u \in \mathbb{B}$. By the definition of the relation ζ, we have

$$x(-\zeta)z^* \text{ iff (for every } t \in \mathbb{B})(t \wedge x \ll (t \wedge z^*)^*).$$

Putting $t = u$, we obtain (since, by ($\ll$1), $x \leq u$) that $x \ll (u \wedge z^*)^*$, i.e., $x \ll (u^* \vee z)$. Since $x \ll u$, ($\ll 4^*$) implies that $x \ll (u \wedge z)$. Using again ($\ll$5), we find an $y \in B$ such that $x \ll y \ll u \wedge z$. Then $y \in \mathbb{B}$ and $x \ll y \ll z$. Noting that

$$(a \ll b) \rightarrow (a \ll_\zeta b)$$

(indeed, for every $t \in \mathbb{B}$, one has $t \wedge a \leq a \ll b \leq t^* \vee b = (t \wedge b^*)^*$ and hence $t \wedge a \ll (t \wedge b^*)^*$, which implies that $a(-\zeta)b^*$, i.e., $a \ll_\zeta b$), we conclude that $x \ll_\zeta y \ll_\zeta z$.

We shall now verify the axiom (BC2). Let $x\zeta y$. Then, by the definition of the relation ζ, there exists an element z of $\mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$. This implies directly that $x\zeta(z \wedge y)$ (take just the same z).

Finally, (BC3) follows immediately from ($\ll$6) and the fact, proved above, that $y \ll x$ implies $y \ll_\zeta x$. Therefore, $(B, \zeta, \mathbb{B})$ is indeed a local contact algebra.

Let us now show that $g \circ f = id_{\mathcal{LCA}}$. Take a local contact algebra $(W, \zeta, \mathbb{B})$. Then $f((W, \zeta, \mathbb{B})) = (W, \ll_M)$. Let $g((W, \ll_M)) = (W, \zeta_M, \mathbb{B}_M)$ (see the corresponding definitions above). We will show that $\mathbb{B}_M = \mathbb{B}$ and $\zeta_M = \zeta$. By our definitions, we have that $x \ll_M y$ iff $x \in \mathbb{B}$ and $x \ll_\zeta y$ (where, as usual, $x \ll_\zeta y$ means that $x(-\zeta)y^*$); further, we have $\mathbb{B}_M = \{x \in W : x \ll_M 1\}$ and $x\zeta_M y$ iff there exists an element z of $\mathbb{B}_M$ such that $(z \wedge x) \not\ll_M (z \wedge y)^*$.

If $x \in \mathbb{B}_M$ then $x \ll_M 1$ and hence, by the definition of $\ll_M$, $x \in \mathbb{B}$. Thus $\mathbb{B}_M \subseteq \mathbb{B}$. Conversely, if $x \in \mathbb{B}$ then $x \ll_\zeta 1$ (because $x(-\zeta)0$) and hence, by the definition of $\ll_M$, $x \ll_M 1$, i.e., $x \in \mathbb{B}_M$. Therefore, $\mathbb{B} \subseteq \mathbb{B}_M$. So, $\mathbb{B}_M = \mathbb{B}$.

Let $x\zeta y$. By (BC2) and (C3), there exist $u, v \in \mathbb{B}$ such that $u \leq x$, $v \leq y$ and $u\zeta v$. Then, by (I3), $z = u \vee v \in \mathbb{B}$. Since $z \wedge x \geq u$ and $z \wedge y \geq v$, we obtain, by (C4), that $(z \wedge x)\zeta(z \wedge y)$. Thus $(z \wedge x) \not\ll_\zeta (z \wedge y)^*$ and $(z \wedge y) \not\ll_\zeta (z \wedge x)^*$. Since $z \wedge x, z \wedge y \in \mathbb{B}$ (by (I2)), we obtain, by the definition of $\ll_M$, that $(z \wedge x) \not\ll_M (z \wedge y)^*$. Now, using the fact that $\mathbb{B} = \mathbb{B}_M$ (as we have proved above) and the definition of the relation ζ_M, we get that $x\zeta_M y$.

Let now $x\zeta_M y$. Then there exists an element z of $\mathbb{B}_M$ such that $(z \wedge x) \not\ll_M (z \wedge y)^*$. Since $z \wedge x, z \wedge y \in \mathbb{B}_M$ and $\mathbb{B}_M = \mathbb{B}$, we obtain, by the definition of the relation $\ll_M$, that $(z \wedge x) \not\ll_\zeta (z \wedge y)^*$, i.e., $(z \wedge x)\zeta(z \wedge y)$ and hence $x\zeta y$. We have proved that $\zeta = \zeta_M$. So, $g \circ f = id_{\mathcal{LCA}}$.

Finally, we will show that $f \circ g = id_{\mathcal{MA}}$.

Let $(W, \ll)$ be an MVD-algebra, $g((W, \ll)) = (W, \zeta, \mathbb{B})$ and $f((W, \zeta, \mathbb{B})) = (W, \ll_M)$ (see the corresponding definitions above). We have to prove that $\ll = \ll_M$. We have, by our definitions, that $\mathbb{B} = \{x \in W : x \ll 1\}$, $x\zeta y$ iff there exists an element z of $\mathbb{B}$ such that $(z \wedge x) \not\ll (z \wedge y)^*$; further, we have $x \ll_M y \Longleftrightarrow x \in \mathbb{B}$ and $x \ll_\zeta y$ (where, as usual, $x \ll_\zeta y$ means that $x(-\zeta)y^*$). Obviously, $x(-\zeta)y \Longleftrightarrow$ (for every $z \in \mathbb{B}$) $[(z \wedge x) \ll (z \wedge y)^*]$. Thus we obtain that $x \ll_\zeta y \Longleftrightarrow x(-\zeta)y^* \Longleftrightarrow$ (for every

$z \in \mathbb{B})\ [(z \wedge x) \ll (z \wedge y^*)^*] \Longleftrightarrow$ (for every $z \in \mathbb{B})\ [(z \wedge x) \ll (z^* \vee y)]$. So,

(1.40) $x \ll_\zeta y \leftrightarrow (\forall z \in \mathbb{B})[(z \wedge x) \ll (z^* \vee y)$.

Let now $x \ll y$. Then $x \ll 1$ and hence $x \in \mathbb{B}$. For every $z \in \mathbb{B}$, we have $z \wedge x \leq x \ll y \leq y \vee z^*$, so that, by ($\ll$3), $z \wedge x \ll z^* \vee y$. Hence, (1.40) implies that $x \ll_\zeta y$. Since $x \in \mathbb{B}$, we conclude that $x \ll_M y$.

Conversely, let $x \ll_M y$. Then, by the definition of the relation $\ll_M$, $x \in \mathbb{B}$ and $x \ll_\zeta y$. We have to prove that $x \ll y$.

If $x = 1$ then, obviously, $y = 1$. Since $x \in \mathbb{B}$, we obtain, by the definition of $\mathbb{B}$, that $x \ll 1$, i.e., $1 \ll 1$. Thus $x \ll y$.

Let $x \neq 1$. Since $x \ll 1$, ($\ll$5-6) (which is equivalent to ($\ll$5) and ($\ll$6), as we have proved above) implies that there exists a $z \neq x$ such that $x \ll z \ll 1$. Then $z \in \mathbb{B}$ and $x \leq z$, so that $z \wedge x = x$. Now, (1.40) implies (since $z \in \mathbb{B}$ and $x \ll_\zeta y$) that $x \ll (z^* \vee y)$. Since $x \ll z$, ($\ll 4^*$) implies that $x \ll (z^* \vee y) \wedge z$, i.e., $x \ll y \wedge z$. Applying ($\ll$3), we obtain, finally, that $x \ll y$. So, $\ll = \ll_M$. Therefore $f \circ g = id_{\mathcal{MA}}$.

We have proved that f and g are bijections. □

Proposition 1.3.2.4. *Let L be a locally compact Hausdorff space. Then*

$$(RC(L), \ll_L),$$

where, for all $F, G \in RC(L)$, $F \ll_L G$ iff F is compact and $F \subseteq \mathrm{int}(G)$, is an MVD-algebra. All such MVD-algebras will be called standard MVD-algebras.

Proof. It is straightforward to verify that the axioms of MVD-algebras hold. Axiom ($\ll$5) is the most tricky. It follows from Proposition 0.4.2.3. □

Theorem 1.3.2.5. *(a) Each MVD-algebra $(W, \ll)$ can be embedded into a standard MVD-algebra $(RC(L), \ll_L)$, where L is a locally compact Hausdorff space. When W is complete this embedding becomes a complete isomorphism.*

(b) There exists a bijective correspondence between the class of all (up to isomorphism) complete MVD-algebras and the class of all (up to homeomorphism) locally compact Hausdorff spaces.

Proof. We have, by Theorem 1.3.2.3, that the function $g : \mathcal{MA} \longrightarrow \mathcal{LCA}$, where $g((W, \ll)) = (W, \zeta, \mathbb{B})$, is a bijection. Moreover, in the proof of Theorem 1.3.2.3, we have shown that $x \ll y$ iff $x \in \mathbb{B}$ and $x \ll_\zeta y$, where $x \ll_\zeta y$ iff $x(-\zeta)y^*$. Now all follows from Theorem 1.2.3.10. □

Chapter 2

Some generalizations of de Vries' and Fedorchuk's Duality Theorems

2.1 Introduction

The structure of this chapter is the following. In the second section, we formulate and prove a theorem which is a generalization of de Vries' Duality Theorem [24]; with it we extend the de Vries Duality from the category of compact Hausdorff spaces and continuous maps to the category of locally compact Hausdorff spaces and continuous maps. In the third section, using the results obtained in the second section, we describe the products and sums (= coproducts) of complete local contact algebras and develop a completion theory for local contact algebras. In the fourth section, we extend the de Vries Duality [24] to the category of locally compact Hausdorff spaces and perfect maps. In the fifth section, we generalize the Fedorchuk Duality and Equivalence Theorems [54]. In this section many other duality theorems are obtained. Some of them are new even in the compact case, e.g., the duality theorem for the category of compact Hausdorff spaces and open maps. In the last sixth section, we characterize the embeddings, the injective and the surjective maps by means of their dual morphisms, and construct the dual objects of the open and regular closed subsets of a space X by means of the dual object of X.

Let us note that each section begins with a introduction in which a more detailed description of the content of the corresponding section is given.

The exposition of this chapter is based on the papers [27, 28, 29, 31, 34].

2.2 An extension of de Vries' Duality to the category HLC of locally compact Hausdorff spaces and continuous maps

2.2.1 Introduction

It is natural to try to extend de Vries' Duality Theorem to the category **HLC** of locally compact Hausdorff spaces and continuous maps. An important step in this direction was done by P. Roeper [99] and in Chapter 1 we made a new exposition of his results. Here, using Roeper's Theorem, we obtain a duality between the category **HLC** and the category **DHLC** of complete LC-algebras and appropriate morphisms between them; it is an extension of de Vries' duality mentioned above; the dual object of a locally compact Hausdorff space X is the LCA $(RC(X), \rho_X, CR(X))$ which will be called *the Roeper triple of the space* X. Let us note that the famous Gelfand duality [60, 61, 62, 63] also gives an algebraical description of (locally) compact Hausdorff spaces but it is not in the spirit of the ideas of Whitehead and de Laguna.

The exposition of this section is based on the paper [29].

2.2.2 The formulation of the Duality Theorem for the category HLC and some preparatory results for its proof

Definition 2.2.2.1. Let $(A, \rho, \mathbb{B})$ be an LCA. An ideal I of A is called a *δ-ideal* if $I \subseteq \mathbb{B}$ and for any $a \in I$ there exists $b \in I$ such that $a \ll_\rho b$. If I_1 and I_2 are two δ-ideals of $(A, \rho, \mathbb{B})$ then we put $I_1 \leq I_2$ iff $I_1 \subseteq I_2$. We will denote by $(I(A, \rho, \mathbb{B}), \leq)$ the poset of all δ-ideals of $(A, \rho, \mathbb{B})$.

The next assertion is obvious.

Fact 2.2.2.2. *Let $(A, \rho, \mathbb{B})$ be an LCA. Then, for every $a \in A$, the set*

$$I_a = \{b \in \mathbb{B} \mid b \ll_\rho a\}$$

is a δ-ideal. Such δ-ideals will be called principal δ-ideals*.*

Fact 2.2.2.3. *Let $(A, \rho, \mathbb{B})$ be an LCA. Then the poset $(I(A, \rho, \mathbb{B}), \leq)$ of all δ-ideals of $(A, \rho, \mathbb{B})$ (see 2.2.2.1) is a frame.*

Proof. It is well known that the set $Idl(A)$ of all ideals of a distributive lattice forms a frame under the inclusion ordering (see 0.3.2.7). It is easy to see that the join in

$(Idl(A), \subseteq)$ of a family of δ-ideals is a δ-ideal and hence it is the join of this family in $(I(A, \rho, \mathbb{B}), \leq)$. The meet in $(Idl(A), \subseteq)$ of a finite family of δ-ideals is also a δ-ideal and hence it is the meet of this family in $(I(A, \rho, \mathbb{B}), \leq)$. Therefore, $(I(A, \rho, \mathbb{B}), \leq)$ is a frame. $\square$

Theorem 2.2.2.4. *Let $(A, \rho, \mathbb{B})$ be an LCA, $X = \Psi^a(A, \rho, \mathbb{B})$ and $\mathcal{O}(X)$ be the frame of all open subsets of X. Then there exists a frame isomorphism*

$$\iota : (I(A, \rho, \mathbb{B}), \leq) \longrightarrow (\mathcal{O}(X), \subseteq),$$

where $(I(A, \rho, \mathbb{B}), \leq)$ is the frame of all δ-ideals of $(A, \rho, \mathbb{B})$. The isomorphism ι sends the set $PI(A, \rho, \mathbb{B})$ of all principal δ-ideals of $(A, \rho, \mathbb{B})$ onto the set of those regular open subsets of X whose complements are in $\lambda_A^g(A)$. In particular, if $(A, \rho, \mathbb{B})$ is a CLCA, then $\iota(PI(A, \rho, \mathbb{B})) = RO(X)$.

Proof. Let I be a δ-ideal. Put

$$\iota(I) = \bigcup\{\lambda_A^g(a) \mid a \in I\}.$$

Then $\iota(I)$ is an open subset of X. Indeed, for every $a \in I$ there exists $b \in I$ such that $a \ll b$. Then $\lambda_A^g(a) \subseteq \mathrm{int}_X(\lambda_A^g(b)) \subseteq \lambda_A^g(b) \subseteq \iota(I)$. Hence $\iota(I)$ is an open subset of X. Therefore ι is a function from $I(A, \rho, \mathbb{B})$ to $\mathcal{O}(X)$. Let $U \in \mathcal{O}(X)$. Set

$$\mathbb{B}_U = \{b \in \mathbb{B} \mid \lambda_A^g(b) \subseteq U\}.$$

Then, using (1.36), regularity of X and (III), it is easy to see that $\mathbb{B}_U$ is a δ-ideal of $(A, \rho, \mathbb{B})$ and $\iota(\mathbb{B}_U) = U$. Hence, ι is a surjection. We will show that ι is an injection as well. Indeed, let $I_1, I_2 \in I(A, \rho, \mathbb{B})$ and $\iota(I_1) = \iota(I_2)$. Set $\iota(I_1) = W$ and put $\mathbb{B}_W = \{b \in \mathbb{B} \mid \lambda_A^g(b) \subseteq W\}$. Then, obviously, $I_1 \subseteq \mathbb{B}_W$. Further, if $b \in \mathbb{B}_W$ then $\lambda_A^g(b) \subseteq W$ and $\lambda_A^g(b)$ is compact. Since I_1 is a δ-ideal, $\Omega = \{\mathrm{int}(\lambda_A^g(a)) \mid a \in I_1\}$ is an open cover of W and, hence, of $\lambda_A^g(b)$. Thus there exists a finite subfamily $\{\mathrm{int}(\lambda_A^g(a_1)), \ldots, \mathrm{int}(\lambda_A^g(a_k))\}$ of Ω such that $\lambda_A^g(b) \subseteq \bigcup\{\lambda_A^g(a_i) \mid i = 1, \ldots, k\} = \lambda_A^g(\bigvee\{a_i \mid i = 1, \ldots, k\})$. This implies that $b \leq \bigvee\{a_i \mid i = 1, \ldots, k\}$ and hence $b \in I_1$. So, we have proved that $I_1 = \mathbb{B}_W$. Analogously we can show that $I_2 = \mathbb{B}_W$. Thus $I_1 = I_2$. Therefore, ι is a bijection. It is obvious that if $I_1, I_2 \in I(A, \rho, \mathbb{B})$ and $I_1 \leq I_2$ then $\iota(I_1) \subseteq \iota(I_2)$. Conversely, if $\iota(I_1) \subseteq \iota(I_2)$ then $I_1 \leq I_2$. Indeed, if $\iota(I_i) = W_i$, $i = 1, 2$, then, as we have already seen, $I_i = \mathbb{B}_{W_i}$, $i = 1, 2$; since $W_1 \subseteq W_2$ implies that $\mathbb{B}_{W_1} \subseteq \mathbb{B}_{W_2}$, we get that $I_1 \leq I_2$. So, $\iota : (I(A, \rho, \mathbb{B}), \leq) \longrightarrow (\mathcal{O}(X), \subseteq)$ is an isomorphism of posets. This implies that ι is also a frame isomorphism.

Let U be a regular open subset of X, $F = X \setminus U$ and let $a \in A$ be such that $F = \lambda_A^g(a)$. Put $\mathbb{B}_U = \{b \in \mathbb{B} \mid \lambda_A^g(b) \subseteq U\}$. Then, as we have already seen, $\mathbb{B}_U$ is a δ-ideal and $\iota(\mathbb{B}_U) = U$. Since $F \in RC(X)$, we have that $U = X \setminus F = \mathrm{int}(F^*) = \mathrm{int}(\lambda_A^g(a^*))$. Thus $\mathbb{B}_U = \{b \in \mathbb{B} \mid b \ll_\rho a^*\}$. Hence $\mathbb{B}_U$ is a principal δ-ideal.

Conversely, if I is a principal δ-ideal then $U = \iota(I)$ is a regular open set in X such that $X \setminus U \in \lambda_A^g(A)$. Indeed, let $a \in A$, $I = \{b \in \mathbb{B} \mid b \ll_\rho a\}$ and $U = \iota(I)$. It is enough to prove that $X \setminus U = \lambda_A^g(a^*)$. If $b \in I$ then $b(-\rho)a^*$ and hence $\lambda_A^g(b) \cap \lambda_A^g(a^*) = \emptyset$. Thus $U \subseteq X \setminus \lambda_A^g(a^*)$. If $\sigma \in X \setminus \lambda_A^g(a^*)$ then, by (1.36), there exists $b \in \mathbb{B}$ such that $\sigma \in \lambda_A^g(b) \subseteq X \setminus \lambda_A^g(a^*)$. Since, by (1.8) and (1.35), $X \setminus \lambda_A^g(a^*) = \mathrm{int}_X(\lambda_A^g(a))$, we get that $b \ll_\rho a$. Therefore $b \in I$ and hence $\sigma \in U$. This means that $X \setminus \lambda_A^g(a^*) \subseteq U$. $\square$

Definition 2.2.2.5. (De Vries [24]) Let **DHC** be the category whose objects are all complete NC-algebras and whose morphisms are all functions $\varphi : (A, C) \longrightarrow (B, C')$ between the objects of **DHC** satisfying the conditions:

(DVAL1) $\varphi(0) = 0$;
(DVAL2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;
(DVAL3) If $a, b \in A$ and $a \ll_C b$, then $(\varphi(a^*))^* \ll_{C'} \varphi(b)$;
(DVAL4) $\varphi(a) = \bigvee\{\varphi(b) \mid b \ll_C a\}$, for every $a \in A$,

and let the composition "$\diamond$" of two morphisms $\varphi_1 : (A_1, C_1) \longrightarrow (A_2, C_2)$ and $\varphi_2 : (A_2, C_2) \longrightarrow (A_3, C_3)$ of **DHC** be defined by the formula

(2.1) $\varphi_2 \diamond \varphi_1 = (\varphi_2 \circ \varphi_1)\check{}$,

where, for every function $\psi : (A, C) \longrightarrow (B, C')$ between two objects of **DHC**, $\psi\check{} : (A, C) \longrightarrow (B, C')$ is defined as follows:

(2.2) $\psi\check{}(a) = \bigvee\{\psi(b) \mid b \ll_C a\}$,

for every $a \in A$.

De Vries [24] proved the following duality theorem:

Theorem 2.2.2.6. ([24]) *The categories* **HC** *and* **DHC** *are dually equivalent.*

Sketch of the proof. First we define a contravariant functor

$$\Phi^t : \mathbf{HC} \longrightarrow \mathbf{DHC}$$

by (1.14) on the objects of **HC** (i.e.,

$$\Phi^t(X,\tau) = (RC(X,\tau), \rho_X),$$

for every $X \in |\mathbf{HC}|$), and by

$$\Phi^t(f)(G) = \mathrm{cl}(f^{-1}(\mathrm{int}(G))),$$

for every $f \in \mathbf{HC}(X,Y)$ and every $G \in RC(Y)$. Further, we define another contravariant functor

$$\Phi^a : \mathbf{DHC} \longrightarrow \mathbf{HC}$$

by (1.6) on the objects of **DHC**, and

$$\Phi^a(\varphi)(\sigma') = \{a \in A \mid \text{ if } b \in A \text{ and } b \ll_C a^* \text{ then } (\varphi(b))^* \in \sigma'\},$$

for every $\varphi \in \mathbf{DHC}((A,C),(B,C'))$ and for every $\sigma' \in \mathrm{Clust}(B,C')$. Then we show that

$$\lambda : Id_{\mathbf{DHC}} \longrightarrow \Phi^t \circ \Phi^a,$$

where

$$\lambda(A,C) = \lambda_{(A,C)}$$

(see (1.4) and (1.12) for $\lambda_{(A,C)}$) for every $(A,C) \in |\mathbf{DHC}|$, is a natural isomorphism. Also,

$$t : Id_{\mathbf{HC}} \longrightarrow \Phi^a \circ \Phi^t,$$

where

$$t(X) = t_X$$

(see (1.15) for t_X) for every $X \in |\mathbf{HC}|$, is a natural isomorphism. Thus, the categories **HC** and **DHC** are dually equivalent. □

In [24], de Vries uses the regular open sets instead of regular closed sets, as we do, so that we present here the translations of his definitions for the case of regular closed sets.

We are now going to generalize de Vries' Duality Theorem.

Definition 2.2.2.7. Let **DHLC** be the category whose objects are all complete LC-algebras and whose morphisms are all functions $\varphi : (A,\rho,\mathbb{B}) \longrightarrow (B,\eta,\mathbb{B}')$ between the objects of **DHLC** satisfying the following conditions:

(DLC1) $\varphi(0) = 0$;

(DLC2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;

(DLC3) If $a \in \mathbb{B}, b \in A$ and $a \ll_\rho b$, then $(\varphi(a^*))^* \ll_\eta \varphi(b)$;

(DLC4) For every $b \in \mathbb{B}'$ there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$;

(DLC5) $\varphi(a) = \bigvee\{\varphi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$, for every $a \in A$;

let the composition "$\diamond$" of two morphisms $\varphi_1 : (A_1, \rho_1, \mathbb{B}_1) \longrightarrow (A_2, \rho_2, \mathbb{B}_2)$ and $\varphi_2 : (A_2, \rho_2, \mathbb{B}_2) \longrightarrow (A_3, \rho_3, \mathbb{B}_3)$ of **DHLC** be defined by the formula

(2.3) $\quad \varphi_2 \diamond \varphi_1 = (\varphi_2 \circ \varphi_1)\check{}$,

where, for every function $\psi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ between two objects of **DHLC**, $\psi\check{} : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ is defined as follows:

(2.4) $\quad \psi\check{}(a) = \bigvee\{\psi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$,

for every $a \in A$.

By $\mathbf{D_1HC}$ we denote the full subcategory of **DHLC** having as objects all CNC-algebras (i.e., those CLC-algebras $(A, \rho, \mathbb{B})$ for which $\mathbb{B} = A$).

(We used here the same notation as in 2.2.2.5 for the composition between the morphisms of the category **DHLC** and for the functions of the type $\psi\check{}$ because the NC-algebras can be regarded as those LC-algebras $(A, \rho, \mathbb{B})$ for which $A = \mathbb{B}$, and hence the right sides of the formulas (2.4) and (2.2) coincide in the case of NC-algebras.)

The fact that **DHLC** is indeed a category will be proved in the next subsection.

Remark 2.2.2.8. It is easy to show that condition (DLC3) in 2.2.2.7 can be replaced by the following one:

(DLC3′) If $a, b \in \mathbb{B}$ and $a \ll_\rho b$, then $(\varphi(a^*))^* \ll_\eta \varphi(b)$.

Indeed, it is clear that condition (DLC3) implies condition (DLC3′). Conversely, if $a \in \mathbb{B}$, $b \in A$ and $a \ll_\rho b$, then, by (BC1), there exists $c \in \mathbb{B}$ such that $a \ll_\rho c \ll_\rho b$. Now, using (DLC3′), we get that $(\varphi(a^*))^* \ll_\eta \varphi(c)$. Since, by condition (DLC2), φ is a monotone function, we get that $(\varphi(a^*))^* \ll_\eta \varphi(b)$. So, conditions (DLC2) and (DLC3′) imply condition (DLC3).

As we will see later, condition (DLC3) can be even replaced with the following stronger condition:

(DLC3S) If $a, b \in A$ and $a \ll_\rho b$, then $(\varphi(a^*))^* \ll_\eta \varphi(b)$.

We will now prove a simple lemma, where some immediate consequences of the axioms (DLC1)-(DLC5) are listed:

Lemma 2.2.2.9. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be CLC-algebras and $\varphi : A \longrightarrow B$ be a function between them. Then:*

(a) If φ satisfies condition (DLC2) then φ is an order preserving function;
(b) If φ satisfies conditions (DLC1) and (DLC2) then $\varphi(a^*) \leq (\varphi(a))^*$, for every $a \in A$; hence, if φ satisfies conditions (DLC1)-(DLC3), then for every $a \in \mathbb{B}$ and every $b \in A$ such that $a \ll_\rho b$, $\varphi(a) \ll_\eta \varphi(b)$;
(c) If φ satisfies conditions (DLC2) and (DLC4) (or (DLC1) and (DLC3)) then we have that $\varphi(1_A) = 1_B$;
(d) If φ satisfies condition (DLC2) then $\varphi\check{}$ satisfies conditions (DLC2) and (DLC5) (see (2.4) for $\varphi\check{}$);
(e) If φ satisfies condition (DLC5) then $\varphi = \varphi\check{}$;
(f) If φ satisfies condition (DLC2) then $(\varphi\check{})\check{} = \varphi\check{}$;
(g) If φ is a monotone function then, for every $a \in A$, $\varphi\check{}(a) \leq \varphi(a)$.

Proof. The properties (a), (b), (e) and (g) are clearly fulfilled, and (f) follows from (d) and (e).

(c) Using consecutively (a) and (DLC4), we get that $\varphi(1_A) \geq \bigvee\{\varphi(b) \mid b \in \mathbb{B}\} \geq \bigvee\{b' \mid b' \in \mathbb{B}'\} = 1_B$ (the last equality follows from (1.39)). Thus, $\varphi(1_A) = 1_B$. The assertion in brackets can be obtained easily applying (DLC3) and (DLC1) to the inequality $0_A \ll_\rho 0_A$.

(d) By (a) and (g), for every $a \in A$, $\varphi\check{}(a) \leq \varphi(a)$. Let $a \in A$. If $c \in \mathbb{B}$ and $c \ll_\rho a$ then there exists $d_c \in \mathbb{B}$ such that $c \ll_\rho d_c \ll_\rho a$; hence $\varphi(c) \leq \varphi\check{}(d_c)$. Now, $\varphi\check{}(a) = \bigvee\{\varphi(c) \mid c \in \mathbb{B}, c \ll_\rho a\} \leq \bigvee\{\varphi\check{}(d_c) \mid c \in \mathbb{B}, c \ll_\rho a\} \leq \bigvee\{\varphi\check{}(e) \mid e \in \mathbb{B}, e \ll_\rho a\} \leq \bigvee\{\varphi(e) \mid e \in \mathbb{B}, e \ll_\rho a\} = \varphi\check{}(a)$. Thus, $\varphi\check{}(a) = \bigvee\{\varphi\check{}(e) \mid e \in \mathbb{B}, e \ll_\rho a\}$. So, $\varphi\check{}$ satisfies (DLC5). Further, let $a, b \in A$. Then $\varphi\check{}(a) \wedge \varphi\check{}(b) = \bigvee\{\varphi(d) \wedge \varphi(e) \mid d, e \in \mathbb{B}, d \ll_\rho a, e \ll_\rho b\} = \bigvee\{\varphi(d \wedge e) \mid d, e \in \mathbb{B}, d \ll_\rho a, e \ll_\rho b\} = \bigvee\{\varphi(c) \mid c \in \mathbb{B}, c \ll_\rho a \wedge b\} = \varphi\check{}(a \wedge b)$. So, (DLC2) is fulfilled. □

Obviously, the assertions (a), (b) and (c) of the above lemma remain true also in the case when $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ are LC-algebras.

As we shall prove in the next subsection, condition (DLC3) in 2.2.2.7 can be also replaced by any of the following two constrains:

(LC3) If, for $i = 1, 2$, $a_i \in \mathbb{B}$, $b_i \in A$ and $a_i \ll_\rho b_i$, then $\varphi(a_1 \vee a_2) \ll_\eta \varphi(b_1) \vee \varphi(b_2)$.

(LC3S) If, for $i = 1, 2$, $a_i, b_i \in A$ and $a_i \ll_\rho b_i$, then $\varphi(a_1 \vee a_2) \ll_\eta \varphi(b_1) \vee \varphi(b_2)$.

Now, we will only show that the following proposition holds:

Proposition 2.2.2.10. *Conditions (DLC1)-(DLC3) imply condition (LC3).*

Proof. Let, for $i = 1, 2$, $a_i \in \mathbb{B}$, $b_i \in A$ and $a_i \ll_\rho b_i$. Then, by (DLC3), $(\varphi(a_i^*))^* \ll_\eta \varphi(b_i)$, where $i = 1, 2$. Hence $(\varphi(b_i))^* \ll_\eta \varphi(a_i^*)$, for $i = 1, 2$. Thus, using consecutively (DLC2) and 2.2.2.9(b), we get that $(\varphi(b_1))^* \wedge (\varphi(b_2))^* \ll_\eta \varphi(a_1^*) \wedge \varphi(a_2^*) = \varphi(a_1^* \wedge a_2^*) = \varphi((a_1 \vee a_2)^*) \leq (\varphi(a_1 \vee a_2))^*$. Therefore, $\varphi(a_1 \vee a_2) \ll_\eta \varphi(b_1) \vee \varphi(b_2)$. $\square$

Remark 2.2.2.11. Obviously, condition (LC3) implies the second assertion of Lemma 2.2.2.9(b).

Theorem 2.2.2.12. (The Duality Theorem for the category **HLC**) *The categories* **HLC** *and* **DHLC** *are dually equivalent.*

The proof of this theorem will be presented in the next subsection. Now we will only give a brief description of the main steps of the proof. We first define two contravariant functors

(2.5) $\Lambda^t : \mathbf{HLC} \longrightarrow \mathbf{DHLC}$ and $\Lambda^a : \mathbf{DHLC} \longrightarrow \mathbf{HLC}$.

Their definitions on the objects of the corresponding categories are the following:

(2.6) $\Lambda^t(X) = \Psi^t(X)$

for every $X \in |\mathbf{HLC}|$, and

(2.7) $\Lambda^a(A, \rho, \mathbb{B}) = \Psi^a(A, \rho, \mathbb{B})$,

for every $(A, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$ (see (1.20), (1.21) and (1.24) for Ψ^t and Ψ^a). Further, the definitions of the contravariant functors Λ^t and Λ^a on the morphisms are as follows:

(2.8) $\Lambda^t(f)(G) = \mathrm{cl}(f^{-1}(\mathrm{int}(G)))$,

for every $f \in \mathbf{HLC}(X, Y)$ and every $G \in RC(Y)$, and

(2.9) $\Lambda^a(\varphi)(\sigma') \cap \mathbb{B} = \{a \in \mathbb{B} \mid \text{if } b \in A \text{ and } a \ll_\rho b \text{ then } \varphi(b) \in \sigma'\}$

for every $\varphi \in \mathbf{DHLC}((A,\rho,\mathbb{B}),(B,\eta,\mathbb{B}'))$ and for every $\sigma' \in \Lambda^a(B,\eta,\mathbb{B}')$. Finally, we show that

$$\lambda^g : Id_{\mathbf{DHLC}} \longrightarrow \Lambda^t \circ \Lambda^a, \text{ where } \lambda^g(A,\rho,\mathbb{B}) = \lambda^g_A$$

for every $(A,\rho,\mathbb{B}) \in |\mathbf{DHLC}|$ (see (1.29) for λ^g_A), and

$$t^l : Id_{\mathbf{HLC}} \longrightarrow \Lambda^a \circ \Lambda^t, \text{ where } t^l(X) = t_X$$

for every $X \in |\mathbf{HLC}|$ (see (1.32) and (1.15) for t_X), are natural isomorphisms, completing in this way the proof of the theorem.

We are now going to show that our Theorem 2.2.2.12 implies the de Vries Duality Theorem. It is clear that the categories $\mathbf{D_1HC}$ and $\mathbf{DHC}$ (see 2.2.2.7 and 2.2.2.5 for the notation) are isomorphic (it can be even said that they are identical). Hence, using Theorems 1.2.3.10 and 2.2.2.12, we get that the categories $\mathbf{DHC}$ and $\mathbf{HC}$ are dually equivalent. Moreover, the definitions of the corresponding duality functors coincide. Indeed, it is obvious that the definitions of the contravariant functor Φ^t and the restriction of the contravariant functor Λ^t to the subcategory $\mathbf{HC}$ of the category $\mathbf{HLC}$ coincide. Further, we need to show that the contravariant functor Φ^a and the restriction of the contravariant functor Λ^a to the subcategory $\mathbf{D_1HC}$ of the category $\mathbf{DHLC}$ coincide. Let $\varphi \in \mathbf{D_1HC}((A,\rho,A),(B,\eta,B))(= \mathbf{DHC}((A,\rho),(B,\eta)))$ and $\sigma' \in \Psi^a(B,\eta,B)$. Then set $\sigma_\Phi = \Phi^a(\varphi)(\sigma') = \{a \in A \mid (\forall b \in A)[(b \ll a^*) \rightarrow ((\varphi(b))^* \in \sigma')]\}$. Obviously, $\sigma_\Phi = \{a \in A \mid (\forall b \in A)[(a \ll b) \rightarrow ((\varphi(b^*))^* \in \sigma')]\}$. Set $\sigma_\Lambda = \Lambda^a(\varphi)(\sigma') = \{a \in A \mid (\forall b \in A)[(a \ll b) \rightarrow (\varphi(b) \in \sigma')]\}$. Then $\sigma_\Lambda \subseteq \sigma_\Phi$. Indeed, by 2.2.2.9(b), $\varphi(b^*) \leq (\varphi(b))^*$; hence $\varphi(b) \leq (\varphi(b^*))^*$; therefore, if $a \in \sigma_\Lambda$ then $a \in \sigma_\Phi$. Now, 1.2.2.5 implies that $\sigma_\Lambda = \sigma_\Phi$. Thus, the de Vries Duality Theorem is a corollary of Theorem 2.2.2.12.

Finally, we will need the following definitions and assertions:

Definition 2.2.2.13. In analogy to the corresponding definitions in the theory of proximity spaces (see, e.g., [87]), we say that:

(a) a subset ξ of an NCA (B,C) is called an *end* if the following conditions are satisfied:

(E1) for any $b,c \in \xi$ there exists $a \in \xi$ such that $a \neq 0$, $a \ll b$ and $a \ll c$;

(E2) if $a,b \in B$ and $a \ll b$ then either $a^* \in \xi$ or $b \in \xi$;

(b) a subset v of an NCA (B,C) is called a *round filter* if it is a filter and for every $b \in v$ there exists $a \in v$ such that $a \ll b$.

The next two theorems (and their proofs) are analogous to the Theorems 6.7 and 6.11 in [87] (and their proofs), respectively:

Theorem 2.2.2.14. *Let (B, C) be a normal contact algebra and ξ be an end in (B, C). Then ξ is a maximal round filter in (B, C).*

Theorem 2.2.2.15. *Let (B, C) be a normal contact algebra and $\sigma \subseteq B$. Then $\sigma \in \mathrm{Clust}(B, C)$ iff $d(\sigma) = \{b \in B \mid b^* \notin \sigma\}$ is an end in (B, C).*

Corollary 2.2.2.16. *Let (B, C) be a normal contact algebra, $\sigma \in \mathrm{Clust}(B, C)$, $a \in B$ and $a \notin \sigma$. Then there exists $b \in B$ such that $b \notin \sigma$ and $a \ll b$.*

Proof. Put $\xi = d(\sigma)(= \{c \in B \mid c^* \notin \sigma\})$. Then, by 2.2.2.15 and 2.2.2.14, ξ is a round filter in (B, C). Since $a \notin \sigma$, we obtain that $a^* \in \xi$. Hence, there exists $b^* \in \xi$ such that $b^* \ll a^*$. Then $b \notin \sigma$ and $a \ll b$. □

2.2.3 The proof of the Duality Theorem for the category HLC

The proof of our Theorem 2.2.2.12 will be divided in several lemmas, propositions, facts and remarks. The plan is as follows: we begin with some preparatory assertions; after that we show that **DHLC** is indeed a category; the crucial step in the proof of this fact is to show that any function between CLC-algebras, which satisfies conditions (DLC1)-(DLC5), satisfies condition (DLC3S) as well; this statement is obtained as a corollary of some other assertions which are also used later on in the last portion of the proof where the construction of the desired duality between the categories **DHLC** and **HLC** is presented.

We start with some simple, but important, assertions about (bounded) clusters in LC-algebras.

Proposition 2.2.3.1. *Let $(B, \rho, \mathbb{B})$ be an LCA and σ be a cluster in it (i.e., in (B, C_ρ) (see 1.2.3.6)). Then the following holds: if $\sigma \cap \mathbb{B} \neq \emptyset$ then there exists $b \in \mathbb{B}$ such that $b^* \notin \sigma$.*

Proof. Let $b_0 \in \sigma \cap \mathbb{B}$. Since $b_0 \ll_\rho 1$, (BC1) implies that there exists $b \in \mathbb{B}$ such that $b_0 \ll_\rho b$. Then $b_0(-\rho)b^*$ and since $b_0 \in \mathbb{B}$, we obtain that $b_0(-C_\rho)b^*$. Thus $b^* \notin \sigma$. □

Proposition 2.2.3.2. *Let $(A, \rho, \mathbb{B})$ be an LCA. If u is an ultrafilter in A and $\sigma_u \cap \mathbb{B} \neq \emptyset$, then $u \cap \mathbb{B} \neq \emptyset$.*

Proof. By 2.2.3.1, there exists $a \in \mathbb{B}$ such that $a^* \notin \sigma_u$. Then $a \in u \cap \mathbb{B}$. □

Proposition 2.2.3.3. *Let $(A, \rho, \mathbb{B})$ be an LCA and σ be a bounded cluster in it. Then:*

(a) If $a \in \sigma$ then there exists $c \in \mathbb{B} \cap \sigma$ such that $c \leq a$;

(b) $(\forall a \in A)[(a \notin \sigma) \leftrightarrow ((\exists b \in \mathbb{B} \cap \sigma)(b \ll_\rho a^*))]$;

(c) $\sigma = \{a \in A \mid a\rho(\sigma \cap \mathbb{B})\}$.

Proof. (a) Let $a \in \sigma$. By 1.2.2.3, there exists an ultrafilter u in A such that $a \in u$ and $\sigma = \sigma_u$. Then $u \subseteq \sigma$. Since $\sigma \cap \mathbb{B} \neq \emptyset$, 2.2.3.2 implies that there exists $a_1 \in u \cap \mathbb{B}$. Set $c = a \wedge a_1$. Then $c \in u \cap \mathbb{B} \subseteq \sigma \cap \mathbb{B}$ and $c \leq a$.

(b) Let $a \notin \sigma$. Then, by 2.2.2.16, there exists $c \in A$ such that $c \notin \sigma$ and $a \ll_{C_\rho} c$. Then $c^* \ll_\rho a^*$ and $c^* \in \sigma$. Hence, by (a), there exists $b \in \mathbb{B} \cap \sigma$ such that $b \ll_\rho a^*$. Conversely, if there exists $b \in \mathbb{B} \cap \sigma$ such that $b \ll_\rho a^*$, then $b \ll_{C_\rho} a^*$. Therefore, $a \notin \sigma$.

(c) This is just another form of (b). □

Corollary 2.2.3.4. *Let $(A, \rho, \mathbb{B})$ be an LCA and σ_1, σ_2 be two clusters in $(A, \rho, \mathbb{B})$ such that $\mathbb{B} \cap \sigma_1 = \mathbb{B} \cap \sigma_2$. Then $\sigma_1 = \sigma_2$.*

Proof. By 1.2.3.7, $\sigma_\infty = A \setminus \mathbb{B}$ is a cluster in (A, C_ρ). Hence, if $\mathbb{B} \cap \sigma_1 = \mathbb{B} \cap \sigma_2 = \emptyset$, then $\sigma_i \subseteq \sigma_\infty$, for $i = 1, 2$. Now, 1.2.2.5 implies that $\sigma_1 = \sigma_\infty = \sigma_2$.

Let $\mathbb{B} \cap \sigma_1 \neq \emptyset$. Then our assertion follows from 2.2.3.3(c). □

Recall that if A is a lattice then an element $p \in A \setminus \{1\}$ is called a *prime element of A* if for each $a, b \in A$, $a \wedge b \leq p$ implies that $a \leq p$ or $b \leq p$. We will now show that if $(A, \rho, \mathbb{B})$ is a CLCA then the prime elements of $I(A, \rho, \mathbb{B})$ are in a bijective correspondence with the bounded clusters in $(A, \rho, \mathbb{B})$. The existence of such a bijection follows immediately from Roeper's Theorem 1.2.3.10, our Theorem 2.2.2.4 and localic duality (see, e.g., [75]). We will present here an explicit formula for this bijection which will be very useful later on.

Proposition 2.2.3.5. *Let σ be a bounded cluster in an LCA $(A, \rho, \mathbb{B})$. Then $I = \mathbb{B} \setminus \sigma$ is a prime element of the frame $I(A, \rho, \mathbb{B})$ (see 2.2.2.1 for the notation).*

Proof. We have that $I \neq \mathbb{B}$ because $\sigma \cap \mathbb{B} \neq \emptyset$. Since σ is an upper set, we get that I is a lower set. Let $a, b \in I$. Suppose that $a \vee b \in \sigma$. Then $a \in \sigma$ or $b \in \sigma$, a contradiction. Hence, $a \vee b \in I$. So, I is an ideal. Let $a \in I$. Then $a \notin \sigma$. By 2.2.2.16, there exists $c \in A$ such that $c \notin \sigma$ and $a \ll_{C_\rho} c$. Thus $a \ll_\rho c$. By (BC1), there exists $b \in \mathbb{B}$ such that $a \ll_\rho b \ll_\rho c$. Then $b \notin \sigma$, i.e., $b \in I$ and $a \ll_\rho b$. So, I is a δ-ideal. Let

$J_1, J_2 \in I(A, \rho, \mathbb{B})$ and $J_1 \cap J_2 \subseteq I$. Suppose that, for $i = 1, 2$, there exists $a_i \in J_i \setminus I$. Since, for $i = 1, 2$, J_i is a δ-ideal, there exists $b_i \in J_i$ such that $a_i \ll_\rho b_i$, and hence $a_i \ll_{C_\rho} b_i$; thus $b_i^* \notin \sigma$. Then $b_1^* \vee b_2^* \notin \sigma$. Therefore, $b_1 \wedge b_2 \in \sigma \cap I$. Since $\sigma \cap I = \emptyset$, we get a contradiction. All this shows that I is a prime element of the frame $I(A, \rho, \mathbb{B})$. □

Proposition 2.2.3.6. *Let $(A, \rho, \mathbb{B})$ be an LCA and I be a prime element of the frame $I(A, \rho, \mathbb{B})$. Then the set $V = \{a \in \mathbb{B} \mid (\exists b \in \mathbb{B} \setminus I)(b \ll_\rho a)\}$ is a filter in $\mathbb{B}$.*

Proof. Set $S = \mathbb{B} \setminus I$. Then S is a non-void upper set in $\mathbb{B}$. Thus, $V \subseteq S$. If $a \in S$ then, by (BC1), there exists $b \in \mathbb{B}$ such that $a \ll_\rho b$. Then $b \in V$, i.e., $V \neq \emptyset$. Obviously, $0 \notin V$ and V is an upper set in $\mathbb{B}$. Let $a, b \in V$ and suppose that $a \wedge b \notin V$. Then, for every $c \in S$, $c \not\ll_\rho a \wedge b$. Hence $I_a \cap I_b \subseteq I$ (see 2.2.2.2 for the notation). Thus, $I_a \subseteq I$ or $I_b \subseteq I$. Let, e.g., $I_a \subseteq I$. Since $a \in V$, there exists $c \in S$ such that $c \ll_\rho a$. Then $c \in I_a \cap S \subseteq I \cap S = \emptyset$, a contradiction. Therefore, $a \wedge b \in V$. So, V is a filter in $\mathbb{B}$. □

Proposition 2.2.3.7. *Let $(A, \rho, \mathbb{B})$ be an LCA and I be a prime element of the frame $I(A, \rho, \mathbb{B})$. Then there exists a unique cluster σ in $(A, \rho, \mathbb{B})$ such that $\sigma \cap \mathbb{B} = \mathbb{B} \setminus I$; moreover, $\sigma = \{a \in A \mid a\rho(\mathbb{B} \setminus I)\}$. (In this case we will say that σ is* generated by *I.)*

Proof. By 2.2.3.6, the set V defined there is a filter in $\mathbb{B}$. Hence, $V \neq \emptyset$ and V is a filter-base in A. Let F be the filter in A generated by the filter-base V. Then $F \cap \mathbb{B} = V$ and hence $F \cap I = \emptyset$. Now, the famous Stone Separation Theorem (see, e.g., [75]) implies that there exists an ultrafilter u in A such that $F \subseteq u$ and $u \cap I = \emptyset$. Set $\sigma = \sigma_u$ (see 1.2.2.4 for σ_u). Then σ is a cluster in $(A, \rho, \mathbb{B})$ (i.e., σ is a cluster in the NCA (A, C_ρ)) and $\sigma = \{a \in A \mid aC_\rho b$ for every $b \in u\}$. Since $F \subseteq u \subseteq \sigma$, we have that $V \subseteq \sigma \cap \mathbb{B}$. Let us show that $\sigma \cap I = \emptyset$. Indeed, suppose that $a \in \sigma \cap I$. Since I is a δ-ideal, there exists $b \in I$ such that $a \ll_\rho b$. Then, obviously, $b \in \sigma \cap I$. We have that $a(-\rho)b^*$ and $a \in \mathbb{B}$. Hence $a(-C_\rho)b^*$. Since $u \subseteq \sigma$ and $a \in \sigma$, we get that $b^* \notin u$. Thus $b \in u$, i.e., $b \in u \cap I$, a contradiction. So, $\sigma \cap I = \emptyset$, i.e., $\sigma \cap \mathbb{B} \subseteq \mathbb{B} \setminus I$. Set $S = \mathbb{B} \setminus I$. We will now prove that $\sigma \cap \mathbb{B} = S$. Indeed, suppose that there exists $a \in S \setminus \sigma$. Then there exists $b \in u$ such that $a(-\rho)b$. Hence $a \ll_\rho b^*$. Now, (BC1) implies that there exists $c \in \mathbb{B}$ such that $a \ll_\rho c \ll_\rho b^*$. Then $c \in V$ and $c \leq b^*$. Thus $c \wedge b = 0$, which means that $c \notin u$. Since $V \subseteq u$ and $c \in V$, we get a contradiction. So, $\sigma \cap \mathbb{B} = S$. Finally, 2.2.3.4 implies the uniqueness of σ and the formula $\sigma = \{a \in A \mid a\rho(\mathbb{B} \setminus I)\}$ follows from 2.2.3.3(c). □

Corollary 2.2.3.8. *Let $(A, \rho, \mathbb{B})$ be an LCA. Then there exists a bijective correspondence between the bounded clusters in $(A, \rho, \mathbb{B})$ and the prime elements of the frame $I(A, \rho, \mathbb{B})$ (see 2.2.2.1 for the notation).*

Proof. It follows from 2.2.3.5 and 2.2.3.7. □

Remark 2.2.3.9. If $(A, \rho, \mathbb{B})$ is an LCA, then it is easy to see that every prime ideal J of $\mathbb{B}$ (i.e., J is an ideal, $J \neq \mathbb{B}$ and $(\forall a, b \in \mathbb{B})[(a \wedge b \in J) \to (a \in J \text{ or } b \in J)]$) which is a δ-ideal (shortly, prime δ-ideal) is a prime element of the frame $I(A, \rho, \mathbb{B})$. However, in contrast to the case of ideals of a lattice (where the prime elements of the frame of all ideals of this lattice are precisely the prime ideals of the lattice), the prime elements of the frame $I(A, \rho, \mathbb{B})$ need not be prime δ-ideals of $\mathbb{B}$. Indeed, let I be a prime element of $I(A, \rho, \mathbb{B})$ and a prime ideal of $\mathbb{B}$; then $\mathbb{B} \setminus I$ is a filter in $\mathbb{B}$; thus the cluster σ generated by I (see 2.2.3.7) has the property that $\sigma \cap \mathbb{B}$ is a filter. Let $X = \Psi^a(A, \rho, \mathbb{B})$ and suppose that A is complete. Then $\lambda_A^g(\mathbb{B}) = CR(X)$ and $\sigma \in X$. Let $F, G \in CR(X)$ and $\sigma \in F \cap G$. There exist $a, b \in \mathbb{B}$ such that $F = \lambda_A^g(a)$ and $G = \lambda_A^g(b)$. Thus $a, b \in \sigma \cap \mathbb{B}$. Therefore $a \wedge b \in \sigma$. Then $\sigma \in \lambda_A^g(a \wedge b) = \lambda_A^g(a) \wedge \lambda_A^g(b) = F \wedge G$. Hence, $\operatorname{int}(F \cap G) \neq \emptyset$. So, if $F, G \in CR(X)$ and $\sigma \in F \cap G$ then $\operatorname{int}(F \cap G) \neq \emptyset$. The points of the real line $\mathbb{R}$ with its natural topology have not this property. (Indeed, if $x \in \mathbb{R}$ and $F = [x, x+1]$, $G = [x-1, x]$, then $F, G \in CR(\mathbb{R})$, $x \in F \cap G$ but $\operatorname{int}_{\mathbb{R}}(F \cap G) = \emptyset$.) Thus the CLCA $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$ is such that no one prime element of the frame $I(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$ is a prime ideal of $CR(\mathbb{R})$.

Notation 2.2.3.10. Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras, $\varphi : A \longrightarrow B$ be a function and σ' be a cluster in $(B, \eta, \mathbb{B}')$. Then we set:

- $S_{\sigma'} = \{a \in \mathbb{B} \mid (\forall b \in A)[(a \ll_\rho b) \to (\varphi(b) \in \sigma')]\}$;
- $V_{\sigma'} = \{a \in \mathbb{B} \mid (\exists b \in S_{\sigma'})(b \ll_\rho a)\}$;
- $J_{\sigma'} = \mathbb{B} \setminus S_{\sigma'}$.

Fact 2.2.3.11. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras and $\varphi : A \longrightarrow B$ be a function satisfying conditions (DLC1)-(DLC3). Then, for every cluster σ' in $(B, \eta, \mathbb{B}')$, $S_{\sigma'} = \{a \in \mathbb{B} \mid (\forall b \in A)[(a \ll_\rho b) \to ((\varphi(b^*))^* \in \sigma')]\}$.*

Proof. Let $a \in S_{\sigma'}$, $b \in A$ and $a \ll_\rho b$. Then $\varphi(b) \in \sigma'$. Since, by 2.2.2.9(b), $\varphi(b) \leq (\varphi(b^*))^*$, we get that $(\varphi(b^*))^* \in \sigma'$. Thus $a \in R = \{a \in \mathbb{B} \mid (\forall b \in A)[(a \ll_\rho$

$b) \to ((\varphi(b^*))^* \in \sigma')]\}$. Conversely, let $a \in R$, $b \in A$ and $a \ll_\rho b$. Then, by (BC1), there exists $c \in \mathbb{B}$ such that $a \ll_\rho c \ll_\rho b$. Since $a \in R$ and $a \ll_\rho c$, we get that $(\varphi(c^*))^* \in \sigma'$. Further, by (DLC3), $(\varphi(c^*))^* \ll_\eta \varphi(b)$. Hence, $\varphi(b) \in \sigma'$. Therefore, $a \in S_{\sigma'}$. So, $S_{\sigma'} = R$. □

Lemma 2.2.3.12. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras, $\varphi : A \longrightarrow B$ be a function satisfying conditions (DLC1)-(DLC3) (or conditions (DLC1), (DLC2), (LC3)), and σ' be a cluster in $(B, \eta, \mathbb{B}')$. Then $J_{\sigma'}$ is a δ-ideal of $(A, \rho, \mathbb{B})$. If σ' is a bounded cluster in $(B, \eta, \mathbb{B}')$ and φ satisfies, in addition, condition (DLC4), then $J_{\sigma'}$ is a prime element of the frame $I(A, \rho, \mathbb{B})$ (see 2.2.2.1 for the notation).*

Proof. Obviously, $J_{\sigma'} = \{a \in \mathbb{B} \mid (\exists b \in A)[(a \ll_\rho b) \wedge (\varphi(b) \notin \sigma')]\}$. Since $0 \ll_\rho 0$, (DLC1) implies that $0 \in J_{\sigma'}$. It is clear that $J_{\sigma'}$ is a lower set. Let $a_1, a_2 \in J_{\sigma'}$. Then, for $i = 1, 2$, there exists $b_i \in A$ such that $a_i \ll_\rho b_i$ and $\varphi(b_i) \notin \sigma'$. Since, for $i = 1, 2$, $a_i \in \mathbb{B}$, there exists $c_i \in \mathbb{B}$ such that $a_i \ll_\rho c_i \ll_\rho b_i$ (by condition (BC1) in 1.2.3.1); then $\varphi(c_i) \notin \sigma'$. Set $c = c_1 \vee c_2$. Now, by 2.2.2.10 (resp., by (LC3)), $\varphi(c) = \varphi(c_1 \vee c_2) \ll_\eta \varphi(b_1) \vee \varphi(b_2)$. Since $\varphi(b_1) \vee \varphi(b_2) \notin \sigma'$, we get that $\varphi(c) \notin \sigma'$. Therefore, $a_1 \vee a_2 \in J_{\sigma'}$. All this shows that $J_{\sigma'}$ is an ideal of A. Let $a \in J_{\sigma'}$. Then there exists $b \in A$ such that $a \ll_\rho b$ and $\varphi(b) \notin \sigma'$. Using again condition (BC1), we get that there exists $c \in \mathbb{B}$ such that $a \ll_\rho c \ll_\rho b$. Then $c \in J_{\sigma'}$ and $a \ll_\rho c$. So, $J_{\sigma'}$ is a δ-ideal of $(A, \rho, \mathbb{B})$.

Let now $\sigma' \cap \mathbb{B}' \neq \emptyset$ and φ satisfies, in addition, condition (DLC4). Then $J_{\sigma'} \neq \mathbb{B}$. Indeed, there exists $b \in \sigma' \cap \mathbb{B}'$. Then, by (DLC4), there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$; hence $\varphi(a) \in \sigma'$. This implies that $a \in S_{\sigma'}$. Thus $J_{\sigma'} \neq \mathbb{B}$. Let $J_1 \cap J_2 \subseteq J_{\sigma'}$. Suppose that there exists $a_i \in J_i \setminus J_{\sigma'}$, $i = 1, 2$. Since J_1, J_2 are δ-ideals, there exists $b_i \in J_i$ such that $a_i \ll_\rho b_i$, $i = 1, 2$. There exists $c_i \in \mathbb{B}$ such that $a_i \ll_\rho c_i \ll_\rho b_i$, $i = 1, 2$. Since $a_i \in S_{\sigma'}$, we have that $\varphi(c_i) \in \sigma'$, $i = 1, 2$. By 2.2.2.9(b) (respectively, 2.2.2.11), we get that $\varphi(c_i) \ll_\eta \varphi(b_i)$, $i = 1, 2$. Now, 2.2.3.3(a) implies that there exists $d_i \in \mathbb{B}' \cap \sigma'$ such that $d_i \ll_\eta \varphi(b_i)$, $i = 1, 2$. Then $d_i \ll_{C_\eta} \varphi(b_i)$, $i = 1, 2$. Thus $(\varphi(b_i))^* \notin \sigma'$, $i = 1, 2$. This implies that $\varphi(b_1) \wedge \varphi(b_2) \in \sigma'$, i.e., by (DLC2), $\varphi(b_1 \wedge b_2) \in \sigma'$. We have, however, that $b_1 \wedge b_2 \in J_{\sigma'}$. Hence, there exists $d \in A$ such that $b_1 \wedge b_2 \ll_\rho d$ and $\varphi(d) \notin \sigma'$. Since $\varphi(b_1 \wedge b_2) \leq \varphi(d)$ and $\varphi(b_1 \wedge b_2) \in \sigma'$, we get that $\varphi(d) \in \sigma'$, a contradiction. Therefore, $J_1 \subseteq J_{\sigma'}$ or $J_2 \subseteq J_{\sigma'}$. All this shows that $J_{\sigma'}$ is a prime element of the frame $I(A, \rho, \mathbb{B})$. □

Lemma 2.2.3.13. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras, $\varphi : A \longrightarrow B$ be a function satisfying conditions (DLC1)-(DLC4) (or conditions (DLC1), (DLC2), (LC3), (DLC4)), and σ' be a bounded cluster in $(B, \eta, \mathbb{B}')$. Then $V_{\sigma'}$ is a filter in $(\mathbb{B}, \leq)$.*

Proof. It follows immediately from 2.2.3.12 and 2.2.3.6. □

Lemma 2.2.3.14. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras, $\varphi : A \longrightarrow B$ be a function satisfying conditions (DLC1)-(DLC4) (or conditions (DLC1), (DLC2), (LC3), (DLC4)), and σ' be a bounded cluster in $(B, \eta, \mathbb{B}')$. Then there exists a unique cluster σ in $(A, \rho, \mathbb{B})$ such that $\sigma \cap \mathbb{B} = S_{\sigma'}$; moreover, $\sigma = \{a \in A \mid a\rho S_{\sigma'}\}$.*

Proof. It follows from 2.2.3.12 and 2.2.3.7. □

Notation 2.2.3.15. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras and $\varphi : A \longrightarrow B$ be a function. We set, for every $a \in A$,*

$$D_\varphi(a) = \bigcup \{I_{\varphi(b)} \mid b \in \mathbb{B}, b \ll_\rho a\}$$

(see 2.2.2.2 for I_c).

Proposition 2.2.3.16. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras and $\varphi : A \longrightarrow B$ be a monotone function. Then, for every $a \in A$, $D_\varphi(a)$ is a δ-ideal of $(B, \eta, \mathbb{B}')$.*

Proof. Let $a \in A$. We will prove that $D_\varphi(a) = \bigvee \{I_{\varphi(b)} \mid b \in \mathbb{B}, b \ll_\rho a\}$, where the join is taken in the frame $I(B, \eta, \mathbb{B}')$ (see 2.2.2.3). Then, by 2.2.2.3, $D_\varphi(a)$ will be a δ-ideal.

Set $I = \bigvee \{I_{\varphi(b)} \mid b \in \mathbb{B}, b \ll_\rho a\}$. The ideal I is generated by $D_\varphi(a)$. Hence, $D_\varphi(a) \subseteq I$. Conversely, let $c \in I$. Then there exists $n \in \mathbb{N}^+$ and, for each $i = 1, \ldots, n$, there exist $b_i \in \mathbb{B}$ and $c_i \in \mathbb{B}'$ such that $b_i \ll_\rho a$, $c_i \ll_\eta \varphi(b_i)$ and $c = \bigvee \{c_i \mid i = 1, \ldots, n\}$. Set $b = \bigvee \{b_i \mid i = 1, \ldots, n\}$. Then $b \in \mathbb{B}$, $b \ll_\rho a$ and $c \ll_\eta \bigvee \{\varphi(b_i) \mid i = 1, \ldots, n\} \leq \varphi(b)$. Hence $c \ll_\eta \varphi(b)$, and since $c \in \mathbb{B}'$, we get that $c \in I_{\varphi(b)}$, where $b \ll_\rho a$. Thus $c \in D_\varphi(a)$. □

Lemma 2.2.3.17. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be LC-algebras, $\varphi : A \longrightarrow B$ be a function satisfying conditions (DLC1)-(DLC4) (or conditions (DLC1), (DLC2), (LC3), (DLC4)), $X = \Psi^a(A, \rho, \mathbb{B})$ and $Y = \Psi^a(B, \eta, \mathbb{B}')$ (see Theorem 1.2.3.10 for Ψ^a). For every $\sigma' \in Y$, set $f_\varphi(\sigma') = \sigma$, where σ is the unique bounded cluster in $(A, \rho, \mathbb{B})$ such that $\sigma \cap \mathbb{B} = S_{\sigma'}$ (see 2.2.3.14 for σ). Then $f_\varphi : Y \longrightarrow X$ is a continuous function and*

(2.10) $\forall a \in \mathbb{B},\ f_\varphi^{-1}(\mathrm{int}(\lambda_A^g(a)) = \iota_B(D_\varphi(a))$

(see 2.2.2.4 for ι).

Proof. We will first show that the formula (2.10) holds. So, let $a \in \mathbb{B}$. Since $D_\varphi(a) = \bigcup\{I_{\varphi(b)} \mid b \in \mathbb{B}, b \ll_\rho a\}$, we get that

$$\iota_B(D_\varphi(a)) = \bigcup\{\lambda^g_B(c) \mid (c \in \mathbb{B}') \wedge (\exists b \in \mathbb{B})[(b \ll_\rho a) \wedge (c \ll_\eta \varphi(b))]\}.$$

Let $\sigma' \in f^{-1}_\varphi(\mathrm{int}(\lambda^g_A(a))$. Then $f_\varphi(\sigma') = \sigma \in \mathrm{int}(\lambda^g_A(a))$. Hence $a^* \notin \sigma$. Now, by 2.2.2.16, there exists $a_1 \in A$ such that $a^* \ll_{C_\rho} a_1^*$ and $a_1^* \notin \sigma$. We get that $a_1 \ll_{C_\rho} a$ and $a_1 \in \sigma \cap \mathbb{B} = S_{\sigma'}$. Since $a_1 \ll_\rho a$, there exist $a_2, b \in \mathbb{B}$ such that $a_1 \ll_\rho a_2 \ll_\rho b \ll_\rho a$. Then, by the definition of the set $S_{\sigma'}$, $\varphi(a_2) \in \sigma'$. By 2.2.2.9(b) (resp., by 2.2.2.11), $\varphi(a_2) \ll_\eta \varphi(b)$. Now, 2.2.3.3(a) implies that there exists $c \in \mathbb{B}' \cap \sigma'$ such that $c \ll_\eta \varphi(b)$. Thus $\sigma' \in \lambda^g_B(c)$, where $c \in \mathbb{B}'$, $c \ll_\eta \varphi(b)$ and $b \ll_\rho a$. This means that $\sigma' \in \iota_B(D_\varphi(a))$. Hence, $f^{-1}_\varphi(\mathrm{int}(\lambda^g_A(a)) \subseteq \iota_B(D_\varphi(a))$.

Conversely, let $\sigma' \in \iota_B(D_\varphi(a))$ and $\sigma = f_\varphi(\sigma')$. Then there exist $b \in \mathbb{B}$ and $c \in \mathbb{B}'$ such that $b \ll_\rho a$, $c \ll_\eta \varphi(b)$ and $\sigma' \in \lambda^g_B(c)$. Thus $c \in \sigma'$ and hence $\varphi(b) \in \sigma'$. This implies that $b \in S_{\sigma'} = \mathbb{B} \cap \sigma$. Since $b \in \mathbb{B}$ and $b \ll_\rho a$, we get that $b \ll_{C_\rho} a$, i.e., $b(-C_\rho)a^*$. Thus $a^* \notin \sigma$. This means that $f_\varphi(\sigma') = \sigma \in \mathrm{int}(\lambda^g_A(a))$. Therefore, $\sigma' \in f^{-1}_\varphi(\mathrm{int}(\lambda^g_A(a))$. We have proved that $f^{-1}_\varphi(\mathrm{int}(\lambda^g_A(a)) \supseteq \iota_B(D_\varphi(a))$.

So, the formula (2.10) is established. Now, by (1.36), $\{\mathrm{int}\lambda^g_A(a) \mid a \in \mathbb{B}\}$ is a base of X, and, for every $a \in A$, $D_\varphi(a)$ is a δ-ideal (see 2.2.3.16). Hence, 2.2.2.4 implies that, for every $a \in A$, $\iota_B(D_\varphi(a))$ is an open subset of Y. Thus, by formula (2.10), f_φ is a continuous function. $\square$

Lemma 2.2.3.18. *Let $f \in \mathbf{HLC}(X, Y)$. Define a function $\varphi_f : \Psi^t(Y) \longrightarrow \Psi^t(X)$ by the formula:*

(2.11) $\forall\, G \in RC(Y),\ \varphi_f(G) = cl_X(f^{-1}(\mathrm{int}_Y(G)))$

(see Theorem 1.2.3.10 for Ψ^t). Then the function φ_f satisfies conditions (DLC1)-(DLC5) from 2.2.2.7 and, moreover, it satisfies conditions (DLC3S) and (LC3S).

Proof. Obviously, condition (DLC1) is fulfilled. For proving condition (DLC2), recall that (see [24]) if U and V are two open subsets of a topological space Z then

(2.12) $\mathrm{int}(\mathrm{cl}(U \cap V)) = \mathrm{int}(\mathrm{cl}(U) \cap \mathrm{cl}(V))$.

Let $F, G \in RC(Y)$. Using the fact that $\mathrm{int}(F \cap G)$ is a regular open set, we get that $\mathrm{int}(F \cap G) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(F \cap G)))$. Thus

$$\varphi_f(F \wedge G) = \mathrm{cl}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(F \cap G))))) = \mathrm{cl}(f^{-1}(\mathrm{int}(F \cap G))).$$

Now, setting $U = f^{-1}(\mathrm{int}(F))$ and $V = f^{-1}(\mathrm{int}(G))$, we obtain, using (2.12), that

$$\varphi_f(F) \wedge \varphi_f(G) = \mathrm{cl}(U) \wedge \mathrm{cl}(V) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U) \cap \mathrm{cl}(V))) =$$

$$= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U \cap V))) = \mathrm{cl}(U \cap V) = \mathrm{cl}(f^{-1}(\mathrm{int}(F \cap G))).$$

Therefore, $\varphi_f(F \wedge G) = \varphi_f(F) \wedge \varphi_f(G)$. So, (DLC2) is fulfilled.

We will now show that not only condition (DLC3) is true, but even condition (DLC3S) holds. Indeed, let $F, G \in RC(Y)$ and $F \ll_{\rho_Y} G$. Then $F \subseteq \mathrm{int}(G)$ and $(\varphi_f(F^*))^* = (\mathrm{cl}(f^{-1}(\mathrm{int}(F^*))))^* = (\mathrm{cl}(f^{-1}(Y \setminus F)))^* = (\mathrm{cl}(X \setminus f^{-1}(F)))^* = \mathrm{cl}(\mathrm{int}(f^{-1}(F))) \subseteq f^{-1}(F) \subseteq f^{-1}(\mathrm{int}(G)) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(\mathrm{int}(G)))) = \mathrm{int}(\varphi_f(G))$. Hence $(\varphi_f(F^*))^* \ll_{\rho_X} \varphi_f(G)$, i.e., condition (DLC3S) is fulfilled.

For verifying (DLC4), let $H \in CR(X)$. Then $f(H)$ is compact. Since Y is locally compact, there exists $F \in CR(Y)$ such that $f(H) \subseteq \mathrm{int}(F)$. Now we obtain that $H \subseteq f^{-1}(\mathrm{int}(F)) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(\mathrm{int}(F)))) = \mathrm{int}(\varphi_f(F))$, i.e., $H \ll_{\rho_X} \varphi_f(F)$. Hence, condition (DLC4) holds.

Let $F \in RC(Y)$. For establishing condition (DLC5), we have to show that $\varphi_f(F) = \bigvee\{\varphi_f(G) \mid G \in CR(Y), G \subseteq \mathrm{int}(F)\}$. Since Y is locally compact and regular, we have that $\mathrm{int}(F) = \bigcup\{\mathrm{int}(G) \mid G \in CR(Y), G \subseteq \mathrm{int}(F)\}$. Recall that $\varphi_f(F) = \mathrm{cl}(f^{-1}(\mathrm{int}(F)))$. Further, it is obvious that if $G \subseteq \mathrm{int}(F)$ then $\mathrm{cl}(f^{-1}(\mathrm{int}(G))) \subseteq f^{-1}(G) \subseteq f^{-1}(\mathrm{int}(F))$. Now, it is easy to see that the desired equality is fulfilled. So, condition (DLC5) is verified.

Finally, we will show that condition (LC3S) is fulfilled as well. Let, for $i = 1, 2$, $F_i, G_i \in RC(Y)$ and $F_i \subseteq \mathrm{int}(G_i)$. We have to show that $\varphi_f(F_1 \cup F_2) \subseteq \mathrm{int}(\varphi_f(G_1) \cup \varphi_f(G_2))$. Indeed, $\varphi_f(F_1 \cup F_2) = \mathrm{cl}(f^{-1}(\mathrm{int}(F_1 \cup F_2))) \subseteq f^{-1}(F_1 \cup F_2) \subseteq f^{-1}(\mathrm{int}(G_1) \cup \mathrm{int}(G_2)) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(\mathrm{int}(G_1) \cup \mathrm{int}(G_2)))) = \mathrm{int}(\varphi_f(G_1) \cup \varphi_f(G_2))$. So, condition (LC3S) is verified. □

Lemma 2.2.3.19. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be CLC-algebras, $\varphi : A \longrightarrow B$ be a function satisfying conditions (DLC1)-(DLC5) (or conditions (DLC1), (DLC2), (LC3), (DLC4), (DLC5)), $X = \Psi^a(A, \rho, \mathbb{B})$ and $Y = \Psi^a(B, \eta, \mathbb{B}')$ (see Theorem 1.2.3.10 for Ψ^a). Let $f = f_\varphi$ (see 2.2.3.17 for f_φ) and $\varphi' = \varphi_f$ (see 2.2.3.18 for φ_f). Then $\lambda^g_B \circ \varphi = \varphi' \circ \lambda^g_A$ (see Theorem 1.2.3.10 for λ^g_A and λ^g_B).*

Proof. Note that, by 2.2.3.17, $f : Y \longrightarrow X$ is a continuous function. Hence, 2.2.3.18 implies that the function φ' satisfies conditions (DLC1)-(DLC5).

Let us now consider the case when $a \in \mathbb{B}$. We have to show that $\lambda^g_B(\varphi(a)) = \varphi'(\lambda^g_A(a))$. By the definitions of φ' and f, and the formula (2.10), we obtain that $\varphi'(\lambda^g_A(a)) = \mathrm{cl}(f_\varphi^{-1}(\mathrm{int}(\lambda^g_A(a)))) = \mathrm{cl}(\iota_B(D_\varphi(a))) = \mathrm{cl}(\bigcup\{\lambda^g_B(b) \mid b \in D_\varphi(a)\} = \bigvee\{\lambda^g_B(b) \mid b \in D_\varphi(a)\} = \lambda^g_B(\bigvee\{b \mid b \in D_\varphi(a)\})$ (since, by Theorem 1.2.3.10, λ^g_B is an LCA-isomorphism). Hence, we have to prove that $\varphi(a) = \bigvee\{b \mid b \in D_\varphi(a)\}) = \bigvee D_\varphi(a)$. By condition (DLC5), we have that $\varphi(a) = \bigvee\{\varphi(c) \mid c \in \mathbb{B}, c \ll_\rho a\}$. Since, for every $c \in A$, $\varphi(c) = \bigvee\{b \in \mathbb{B}' \mid b \ll_\eta \varphi(c)\}$, we get that $\varphi(a) = \bigvee\{b \in \mathbb{B}' \mid (\exists c \in \mathbb{B})[(c \ll_\rho a) \wedge (b \ll_\eta \varphi(c))]\}$. By definition, $D_\varphi(a) = \bigcup\{I_{\varphi(c)} \mid c \in \mathbb{B}, c \ll_\rho a\}$. Thus $(b \in D_\varphi(a)) \leftrightarrow [(b \in \mathbb{B}') \wedge ((\exists c \in \mathbb{B})((c \ll_\rho a) \wedge (b \ll_\eta \varphi(c))))]$. This shows that $\varphi(a) = \bigvee D_\varphi(a)$. So, we have proved that $\lambda^g_B(\varphi(a)) = \varphi'(\lambda^g_A(a))$ for every $a \in \mathbb{B}$.

Let now $a \in A$. Then, by condition (DLC5), $\varphi(a) = \bigvee\{\varphi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$. Hence, using the fact that λ^g_B is an LCA-isomorphism and the formula proved in the preceding paragraph, we get that $\lambda^g_B(\varphi(a)) = \bigvee\{\lambda^g_B(\varphi(b)) \mid b \in \mathbb{B}, b \ll_\rho a\} = \bigvee\{\varphi'(\lambda^g_A(b)) \mid b \in \mathbb{B}, b \ll_\rho a\}$. Further, since the function φ' satisfies condition (DLC5), we have that for every $G \in RC(X)$, $\varphi'(G) = \bigvee\{\varphi'(F) \mid F \in CR(X), F \ll_{\rho_X} G\}$. Now using the fact that λ^g_A is an LCA-isomorphism between local contact algebras $(A, \rho, \mathbb{B})$ and $(RC(X), \rho_X, CR(X))$, we get that $\varphi'(\lambda^g_A(a)) = \bigvee\{\varphi'(\lambda^g_A(b)) \mid b \in \mathbb{B}, b \ll_\rho a\} = \lambda^g_B(\varphi(a))$. So, the desired equality is established. □

Lemma 2.2.3.20. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be CLC-algebras and $\varphi : A \longrightarrow B$ be a function satisfying conditions (DLC1)-(DLC5) (or conditions (DLC1), (DLC2), (LC3), (DLC4), (DLC5)). Then φ satisfies conditions (DLC3S) and (LC3S) as well.*

Proof. Let $f = f_\varphi$ (see 2.2.3.17 for f_φ) and $\varphi' = \varphi_f$ (see 2.2.3.18 for φ_f). Then, by 2.2.3.19, $\lambda^g_B \circ \varphi = \varphi' \circ \lambda^g_A$. Since the function φ' satisfies conditions (DLC3S) and (LC3S) (by 2.2.3.18) and the functions λ^g_A and λ^g_B are LCA-isomorphisms, we get that the function φ satisfies conditions (DLC3S) and (LC3S) as well. □

The above lemma implies the following fact mentioned in the previous subsection:

Corollary 2.2.3.21. *Condition (DLC3) in 2.2.2.7 can be replaced by any of the conditions (DLC3S), (LC3) and (LC3S) (i.e., we obtain equivalent systems of axioms by these replacements).*

Lemma 2.2.3.22. *Let $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$ be CLC-algebras and $\psi : A \longrightarrow B$ satisfy conditions (DLC2), (DLC4). Then, for every $a \in A$, $\psi\check{}(a) = \bigvee\{\psi(b) \mid b \in A, b \ll_{C_\rho} a\}$. (See (2.4) for $\psi\check{}$.)*

Proof. Set, for every $a \in A$, $\psi\tilde{}(a) = \bigvee\{\psi(b) \mid b \in A, b \ll_{C_\rho} a\}$. Obviously, if $b \in \mathbb{B}$ and $b \ll_\rho a$ then $b \ll_{C_\rho} a$. Thus, $\psi\check{}(a) \leq \psi\tilde{}(a)$.

Let now $b \in A$ and $b \ll_{C_\rho} a$. Then $b \ll_\rho a$. We have, by (1.39), that $\psi(b) = \bigvee\{c' \in \mathbb{B}' \mid c' \ll_\eta \psi(b)\}$. Let $c' \in \mathbb{B}'$ and $c' \ll_\eta \psi(b)$. Then, by (DLC4), there exists $c \in \mathbb{B}$ such that $c' \leq \psi(c)$. Now, (DLC2) implies that $c' \leq \psi(b \wedge c)$. Set $d = b \wedge c$. Then $d \in \mathbb{B}$, $d \leq b \ll_\rho a$ (and, hence, $d \ll_\rho a$), $c' \leq \psi(d) \leq \psi\check{}(a)$. Thus, $\psi(b) \leq \psi\check{}(a)$. We conclude that $\psi\tilde{}(a) \leq \psi\check{}(a)$. So, $\psi\check{}(a) = \psi\tilde{}(a)$. □

Lemma 2.2.3.23. *Let $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ be a function between CLC-algebras and let φ satisfy conditions (DLC1)-(DLC4). Then the function $\varphi\check{}$ (see (2.4)) satisfies conditions (DLC1)-(DLC5).*

Proof. Obviously, for every $a \in A$, $\varphi\check{}(a) \leq \varphi(a)$. Hence, $\varphi\check{}(0) = 0$, i.e., (DLC1) is fulfilled. For (DLC2) and (DLC5) see 2.2.2.9(d).

Let $a \in \mathbb{B}, b \in A$ and $a \ll_\rho b$. Then, by (BC1), there exist $c, d \in \mathbb{B}$ such that $a \ll_\rho c \ll_\rho d \ll_\rho b$. Thus $a \ll_{C_\rho} c$ and hence $c^* \ll_{C_\rho} a^*$. Now, using 2.2.3.22, we obtain that $\varphi(c^*) \leq \varphi\check{}(a^*)$. Since $\varphi(d) \leq \varphi\check{}(b)$, we get that $(\varphi\check{}(a^*))^* \leq (\varphi(c^*))^* \ll_\eta \varphi(d) \leq \varphi\check{}(b)$. Therefore, $(\varphi\check{}(a^*))^* \ll_\eta \varphi\check{}(b)$. So, (DLC3) is fulfilled.

For verifying (DLC4), let $b \in \mathbb{B}$. Then there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$. By (BC1), there exists $a_1 \in \mathbb{B}$ with $a \ll_\rho a_1$. Then $b \leq \varphi(a) \leq \varphi\check{}(a_1)$. Thus, $\varphi\check{}$ satisfies condition (DLC4). □

Lemma 2.2.3.24. *Let $\varphi_i : (A_i, \rho_i, \mathbb{B}_i) \longrightarrow (A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1})$, where $i = 1, 2$, be two functions between CLC-algebras. Then:*

(a) $(\varphi_2\check{} \circ \varphi_1)\check{} = (\varphi_2 \circ \varphi_1)\check{}$;

(b) If φ_1 and φ_2 are monotone functions, then $(\varphi_2 \circ \varphi_1\check{})\check{} = (\varphi_2 \circ \varphi_1)\check{}$;

(c) If φ_1 and φ_2 satisfy conditions (DLC1)-(DLC5) then the function $\varphi_2 \circ \varphi_1$ satisfies conditions (DLC1)-(DLC4) and even condition (DLC3S).

Proof. We will write, for $i = 1, 2$, "$\ll_i$" instead of "$\ll_{\rho_i}$". We also set $\varphi = \varphi_2 \circ \varphi_1$.

(a) Let $a \in A_1$. Then $(\varphi_2\check{} \circ \varphi_1)\check{}(a) = \bigvee\{\varphi_2\check{}(\varphi_1(b)) \mid b \in \mathbb{B}_1, b \ll_1 a\} = \bigvee\{\bigvee\{\varphi(c) \mid c \in \mathbb{B}_1, c \ll_1 b\} \mid b \in \mathbb{B}_1, b \ll_1 a\} = \bigvee\{\varphi(c) \mid c \in \mathbb{B}_1, c \ll_1 a\} = \varphi\check{}(a)$.

(b) Let $a \in A_1$. Then $L = (\varphi_2 \circ \varphi_1\check{})\check{}(a) = \bigvee\{\varphi_2(\varphi_1\check{}(b)) \mid b \in \mathbb{B}_1, b \ll_1 a\} = \bigvee\{\varphi_2(\bigvee\{\varphi_1(c) \mid c \in \mathbb{B}_1, c \ll_1 b\}) \mid b \in \mathbb{B}_1, b \ll_1 a\}$ and $R = \varphi\check{}(a) = \bigvee\{\varphi(c) \mid c \in \mathbb{B}_1, c \ll_1 a\}$. Let $c \in \mathbb{B}_1$ and $c \ll_1 a$. Then, by (BC1), there exists $b \in \mathbb{B}_1$ such that $c \ll_1 b \ll_1 a$. This shows that $R \leq L$. Conversely, let $b \in \mathbb{B}_1$ and $b \ll_1 a$. Then, for every $c \in \mathbb{B}_1$ such that $c \ll_1 b$, we have that $\varphi_1(c) \leq \varphi_1(b)$. Hence $\bigvee\{\varphi_1(c) \mid c \in \mathbb{B}_1, c \ll_1 b\} \leq \varphi_1(b)$. Then $\varphi_2(\bigvee\{\varphi_1(c) \mid c \in \mathbb{B}_1, c \ll_1 b\}) \leq \varphi(b)$ and $b \ll_1 a$. This implies that $L \leq R$. So, $L = R$. Hence, $(\varphi_2 \circ \varphi_1\check{})\check{} = (\varphi_2 \circ \varphi_1)\check{}$.

(c) Obviously, the function φ satisfies conditions (DLC1), (DLC2) and (DLC4). For proving that φ satisfies condition (DLC3S), let $a, b \in A_1$ and $a \ll_1 b$. Since the functions φ_1 and φ_2 satisfy condition (DLC3S) (by 2.2.3.20), we obtain that $(\varphi_1(a^*))^* \ll_2 \varphi_1(b)$ and $(\varphi_2(\varphi_1(a^*)))^* \ll_3 \varphi_2(\varphi_1(b))$, i.e., $(\varphi(a^*))^* \ll_3 \varphi(b)$. Hence, the function φ satisfies condition (DLC3S). □

Proposition 2.2.3.25. **DHLC** *is a category.*

Proof. It is clear that for every CLCA $(A, \rho, \mathbb{B})$, the usual identity function $id_A : A \longrightarrow A$ satisfies conditions (DLC1)-(DLC5); moreover, using 2.2.2.9(e), we get that if $(B, \eta, \mathbb{B}')$ and $(B_1, \eta_1, \mathbb{B}'_1)$ are CLC-algebras, and $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ and $\psi : (B_1, \eta_1, \mathbb{B}'_1) \longrightarrow (A, \rho, \mathbb{B})$ are functions satisfying condition (DLC5), then $id_A \diamond \psi = \psi$ and $\varphi \diamond id_A = \varphi$. So, id_A is the **DHLC**-identity on $(A, \rho, \mathbb{B})$.

Let $\varphi_i : (A_i, \rho_i, \mathbb{B}_i) \longrightarrow (A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1})$, where $i = 1, 2$, be two functions between CLC-algebras, and let φ_1 and φ_2 satisfy conditions (DLC1)-(DLC5). We will show that the function $\varphi_2 \diamond \varphi_1$ satisfies conditions (DLC1)-(DLC5).

Set $\varphi = \varphi_2 \circ \varphi_1$. Then, by 2.2.3.24(c), the function φ satisfies conditions (DLC1)-(DLC4). Now, 2.2.3.23 implies that the function $\varphi\check{}$ satisfies conditions (DLC1)-(DLC5). Since $\varphi_2 \diamond \varphi_1 = \varphi\check{}$, we get that the function $\varphi_2 \diamond \varphi_1$ satisfies conditions (DLC1)-(DLC5).

Finally, we will show that the composition in **DHLC** is associative. Let, for $i = 1, 2, 3$, $\varphi_i : (A_i, \rho_i, \mathbb{B}_i) \longrightarrow (A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1})$ be a function between CLC-algebras satisfying conditions (DLC1)-(DLC5). We will show that $(\varphi_3 \diamond \varphi_2) \diamond \varphi_1 = \varphi_3 \diamond (\varphi_2 \diamond \varphi_1)$. Using 2.2.3.24, we get that $(\varphi_3 \diamond \varphi_2) \diamond \varphi_1 = ((\varphi_3 \circ \varphi_2)\check{} \circ \varphi_1)\check{} = ((\varphi_3 \circ \varphi_2) \circ \varphi_1)\check{}$ and $\varphi_3 \diamond (\varphi_2 \diamond \varphi_1) = (\varphi_3 \circ (\varphi_2 \circ \varphi_1)\check{})\check{} = (\varphi_3 \circ (\varphi_2 \circ \varphi_1))\check{}$. Thus, the associativity of the composition in **DHLC** is proved.

All this shows that **DHLC** is a category. □

Lemma 2.2.3.26. *Let $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ be a function between CLC-algebras satisfying conditions (DLC1)-(DLC4). Then $f_\varphi = f_{\varphi\check{}}$ (see 2.2.3.17 for f_φ and (2.4) for $\varphi\check{}$).*

Proof. Let $X = \Psi^a(A, \rho, \mathbb{B})$ and $Y = \Psi^a(B, \eta, \mathbb{B}')$. By 2.2.3.23, the function $\varphi\check{}$ satisfies conditions (DLC1)-(DLC4). Hence, we can apply 2.2.3.17 in order to construct two (continuous) functions $f_\varphi, f_{\varphi\check{}} : Y \longrightarrow X$. Let $\sigma' \in Y$. Set $\sigma = f_\varphi(\sigma')$ and $\sigma_1 = f_{\varphi\check{}}(\sigma')$. By 2.2.3.4, for proving that $\sigma = \sigma_1$, it is enough to show that $\sigma \cap \mathbb{B} = \sigma_1 \cap \mathbb{B}$, where $\sigma \cap \mathbb{B} = \{a \in \mathbb{B} \mid (\forall b \in A)[(a \ll_\rho b) \to (\varphi(b) \in \sigma')]\}$ and $\sigma_1 \cap \mathbb{B} = \{a \in \mathbb{B} \mid (\forall b \in A)[(a \ll_\rho b) \to (\varphi\check{}(b) \in \sigma')]\}$ (see 2.2.3.17).

Let $a \in \sigma_1 \cap \mathbb{B}$, $b \in A$ and $a \ll_\rho b$. Then $\varphi\check{}(b) \in \sigma'$. Since $\varphi\check{}(b) \leq \varphi(b)$ (by 2.2.2.9(g)), we get that $\varphi(b) \in \sigma'$. So, $\sigma \cap \mathbb{B} \supseteq \sigma_1 \cap \mathbb{B}$. Conversely, let $a \in \sigma \cap \mathbb{B}$, $b \in A$ and $a \ll_\rho b$. By (BC1), there exists $c \in \mathbb{B}$ such that $a \ll_\rho c \ll_\rho b$. Then $\varphi(c) \in \sigma'$ and $\varphi(c) \leq \varphi\check{}(b)$. Hence, $\varphi\check{}(b) \in \sigma'$. So, $\sigma \cap \mathbb{B} \subseteq \sigma_1 \cap \mathbb{B}$. Therefore, $\sigma = \sigma_1$. This shows that $f_\varphi = f_{\varphi\check{}}$. □

Proposition 2.2.3.27. *For every $(A, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$, set*

$$\Lambda^a(A, \rho, \mathbb{B}) = \Psi^a(A, \rho, \mathbb{B})$$

(see Theorem 1.2.3.10 for Ψ^a), and for every $\varphi \in \mathbf{DHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$, define

$$\Lambda^a(\varphi) : \Lambda^a(B, \eta, \mathbb{B}') \longrightarrow \Lambda^a(A, \rho, \mathbb{B})$$

by the formula $\Lambda^a(\varphi) = f_\varphi$, where f_φ is the function defined in 2.2.3.17. Then Λ^a : $\mathbf{DHLC} \longrightarrow \mathbf{HLC}$ is a contravariant functor.

Proof. By Theorem 1.2.3.10, if $(A, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$ then $\Lambda^a(A, \rho, \mathbb{B}) \in |\mathbf{HLC}|$, and, by 2.2.3.17, if φ is a $\mathbf{DHLC}$-morphism between $(A, \rho, \mathbb{B})$ and $(B, \eta, \mathbb{B}')$, then $\Lambda^a(\varphi)$ is a $\mathbf{HLC}$-morphism between $\Lambda^a(B, \eta, \mathbb{B}')$ and $\Lambda^a(A, \rho, \mathbb{B})$. Further, let $(A, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$ and set $X = \Lambda^a(A, \rho, \mathbb{B})$, $f = \Lambda^a(id_A)$. We have to show that $f = id_X$. Indeed, let $\sigma' \in X$. Set $\sigma = f(\sigma')$. We will prove that $\sigma' \cap \mathbb{B} = \sigma \cap \mathbb{B}$; then 2.2.3.4 will imply that $\sigma = \sigma'$. We have, by the definition of σ, that $\sigma \cap \mathbb{B} = \{a \in \mathbb{B} \mid (\forall b \in A)[(a \ll_\rho b) \to (b \in \sigma')]\}$. Obviously, $\sigma' \cap \mathbb{B} \subseteq \sigma \cap \mathbb{B}$. Conversely, let $a \in \mathbb{B} \cap \sigma$. Suppose that $a \notin \sigma'$. Then, by 2.2.2.16, there exists $b \in A$ such that $a \ll_{C_\rho} b$ and $b \notin \sigma'$. Since $a \ll_\rho b$, we have that $b \in \sigma'$, a contradiction. Hence, $\sigma' \cap \mathbb{B} \supseteq \sigma \cap \mathbb{B}$. So, $f = id_X$.

Let $\varphi_i \in \mathbf{DHLC}((A_i, \rho_i, \mathbb{B}_i), (A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1}))$, where $i = 1, 2$. Set, for $i = 1, 2, 3$, $X_i = \Lambda^a(A_i, \rho_i, \mathbb{B}_i)$, and, for $i = 1, 2$, $f_i = \Lambda^a(\varphi_i)$. We will write, for $i = 1, 2, 3$, "$\ll_i$" instead of "$\ll_{\rho_i}$". Let $\varphi = \varphi_2 \circ \varphi_1$ and $f = f_1 \circ f_2$. We have to show that $\Lambda^a(\varphi_2 \diamond \varphi_1) = f$. By 2.2.3.24(c), the function φ satisfies conditions (DLC1)-(DLC4). Thus, by 2.2.3.17, the function $g = f_\varphi : X_3 \longrightarrow X_1$ is well-defined. We will show that $g = f$. Let $\sigma_3 \in X_3$ and $\sigma = g(\sigma_3)$. Then $\sigma \cap \mathbb{B}_1 = \{a \in \mathbb{B}_1 \mid (\forall b \in A_1)[(a \ll_1 b) \to (\varphi(b) \in \sigma_3)]\}$. Let $\sigma_2 = f_2(\sigma_3)$ and $\sigma_1 = f_1(\sigma_2)$. For proving that $\sigma = \sigma_1$, it is enough to show (by

2.2.3.4) that $\sigma\cap\mathbb{B}_1=\sigma_1\cap\mathbb{B}_1$. We have that $\sigma_2\cap\mathbb{B}_2=\{a\in\mathbb{B}_2\mid(\forall b\in A_2)[(a\ll_2 b)\to(\varphi_2(b)\in\sigma_3)]\}=\{a\in\mathbb{B}_2\mid(\forall b\in A_2)[(a\ll_2 b)\to((\varphi_2(b^*))^*\in\sigma_3)\}$ and $\sigma_1\cap\mathbb{B}_1=\{a\in\mathbb{B}_1\mid(\forall b\in A_1)[(a\ll_1 b)\to(\varphi_1(b)\in\sigma_2)]\}=\{a\in\mathbb{B}_1\mid(\forall b\in A_1)[(a\ll_1 b)\to((\varphi_1(b^*))^*\in\sigma_2)]\}$ (see 2.2.3.17 and 2.2.3.11). Let $a\in\sigma_1\cap\mathbb{B}_1$, $b\in A_1$ and $a\ll_1 b$. Then, by (BC1), there exists $c\in\mathbb{B}_1$ such that $a\ll_1 c\ll_1 b$. Then $\varphi_1(c)\in\sigma_2$ and, by 2.2.2.9(b), $\varphi_1(c)\ll_2\varphi_1(b)$. Hence, by 2.2.3.3(a), there exists $d_2\in\mathbb{B}_2\cap\sigma_2$ such that $d_2\ll_2\varphi_1(b)$. Then $\varphi(b)\in\sigma_3$. So, $a\in\sigma\cap\mathbb{B}_1$. Thus, $\sigma_1\cap\mathbb{B}_1\subseteq\sigma\cap\mathbb{B}_1$. Conversely, let $a\in\sigma\cap\mathbb{B}_1$. Suppose that $a\notin\sigma_1\cap\mathbb{B}_1$. Then there exists $b\in A_1$ such that $a\ll_1 b$ and $(\varphi_1(b^*))^*\notin\sigma_2$. There exists $c\in\mathbb{B}_1$ such that $a\ll_1 c\ll_1 b$. Since $\varphi_1(b^*)\in\sigma_2$ and, by (DLC3S), $\varphi_1(b^*)\ll_2\varphi_1(c^*)$, 2.2.3.3(a) implies that there exists $d_1\in\sigma_2\cap\mathbb{B}_2$ such that $d_1\ll_2\varphi_1(c^*)$. Thus $\varphi(c^*)\in\sigma_3$. Further, there exists $d\in\mathbb{B}_1$ such that $a\ll_1 d\ll_1 c$. Then $\varphi(d)\in\sigma_3$ and, by (DLC3S) (see 2.2.3.24(c)), $\varphi(d)\ll_3\varphi(c)$. Using again 2.2.3.3(a), we get that there exists $e\in\sigma_3\cap\mathbb{B}_3$ such that $e\ll_3\varphi(c)$. Thus $e\ll_{C_{\rho_3}}\varphi(c)$, i.e., $e(-C_{\rho_3})(\varphi(c))^*$. Then, by 2.2.2.9(b), $e(-C_{\rho_3})\varphi(c^*)$. Therefore $\varphi(c^*)\notin\sigma_3$, a contradiction. It shows that $a\in\sigma_1\cap\mathbb{B}_1$. So, $\sigma_1\cap\mathbb{B}_1\supseteq\sigma\cap\mathbb{B}_1$. We have proved that $\sigma=\sigma_1$. So, $g=f$. Since $\varphi_2\diamond\varphi_1=\check{\varphi}$, 2.2.3.26 implies that $\Lambda^a(\varphi_2\diamond\varphi_1)=g$. Therefore, $\Lambda^a(\varphi_2\diamond\varphi_1)=f$. □

Proposition 2.2.3.28. *For every $X\in|\mathbf{HLC}|$, set $\Lambda^t(X)=\Psi^t(X)$ (see Theorem 1.2.3.10 for Ψ^t), and for every $f\in\mathbf{HLC}(X,Y)$, define $\Lambda^t(f):\Lambda^t(Y)\longrightarrow\Lambda^t(X)$ by the formula $\Lambda^t(f)=\varphi_f$, where φ_f is the function defined in 2.2.3.18. Then Λ^t : $\mathbf{HLC}\longrightarrow\mathbf{DHLC}$ is a contravariant functor.*

Proof. By Theorem 1.2.3.10, if $X\in|\mathbf{HLC}|$ then $\Lambda^t(X)\in|\mathbf{DHLC}|$, and, by 2.2.3.18, if $f\in\mathbf{HLC}(X,Y)$ then $\Lambda^t(f)\in\mathbf{DHLC}(\Lambda^t(Y),\Lambda^t(X))$. Further, it is obvious that Λ^t preserves identity morphisms.

Let $f\in\mathbf{HLC}(X,Y)$ and $g\in\mathbf{HLC}(Y,Z)$. We will prove that $\Lambda^t(g\circ f)=\Lambda^t(f)\diamond\Lambda^t(g)$. Set $h=g\circ f$. We have that $\varphi_h=\Lambda^t(h)$, $\varphi_f=\Lambda^t(f)$ and $\varphi_g=\Lambda^t(g)$ (see 2.2.3.18 for φ_f etc.). Let $F\in RC(Z)$. Then $\varphi_h(F)=\mathrm{cl}(h^{-1}(\mathrm{int}(F)))=\mathrm{cl}(f^{-1}(g^{-1}(\mathrm{int}(F))))$ and $(\varphi_f\circ\varphi_g)\check{}(F)=\bigvee\{\varphi_f(\varphi_g(G))\mid G\in CR(Z),\ G\ll_{\rho_Z}F\}$. If $G\in CR(Z)$ (or even $G\in RC(Z)$) and $G\subseteq\mathrm{int}(F)$, then

$$\varphi_f(\varphi_g(G))\subseteq f^{-1}(\mathrm{cl}(g^{-1}(\mathrm{int}(G))))\subseteq f^{-1}(g^{-1}(G))\subseteq f^{-1}(g^{-1}(\mathrm{int}(F))).$$

Thus $(\varphi_f\circ\varphi_g)\check{}(F)\subseteq\varphi_h(F)$. Further, since $\mathrm{int}(F)=\bigcup\{\mathrm{int}(G)\mid G\in CR(Z),G\subseteq\mathrm{int}(F)\}$, we get that $g^{-1}(\mathrm{int}(F))\subseteq\bigcup\{\mathrm{int}(\varphi_g(G))\mid G\in CR(Z),G\subseteq\mathrm{int}(F)\}$. Hence

$$\varphi_h(F)\subseteq\mathrm{cl}(\bigcup\{f^{-1}(\mathrm{int}(\varphi_g(G)))\mid G\in CR(Z),G\subseteq\mathrm{int}(F)\})\subseteq$$

$$\subseteq \mathrm{cl}(\bigcup\{\varphi_f(\varphi_g(G)) \mid G \in CR(Z), G \subseteq \mathrm{int}(F)\}) = (\varphi_f \circ \varphi_g)\check{}(F).$$

Therefore, $\varphi_h = \varphi_f \diamond \varphi_g$. So, $\Lambda^t : \mathbf{HLC} \longrightarrow \mathbf{DHLC}$ is a contravariant functor. $\square$

Proposition 2.2.3.29. *The identity functor $Id_{\mathbf{DHLC}}$ and the functor $\Lambda^t \circ \Lambda^a$ are naturally isomorphic.*

Proof. Recall that for every $(A, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$, the function

$$\lambda_A^g : (A, \rho, \mathbb{B}) \longrightarrow (\Lambda^t \circ \Lambda^a)(A, \rho, \mathbb{B})$$

is an LCA-isomorphism (see (1.31)). We will show that $\lambda^g : Id_{\mathbf{DHLC}} \longrightarrow \Lambda^t \circ \Lambda^a$, where for every $(A, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$, $\lambda^g(A, \rho, \mathbb{B}) = \lambda_A^g$, is a natural isomorphism. (Note that, clearly, every LCA-isomorphism is a **DHLC**-isomorphism.)

Let $\varphi \in \mathbf{DHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$. We have to show that $\lambda_B^g \diamond \varphi = (\Lambda^t \circ \Lambda^a)(\varphi) \diamond \lambda_A^g$, i.e., that $(\lambda_B^g \circ \varphi)\check{} = ((\Lambda^t \circ \Lambda^a)(\varphi) \circ \lambda_A^g)\check{}$. Since, by 2.2.3.19 and the definitions of the contravariant functors Λ^t and Λ^a, we have that

(2.13) $\lambda_B^g \circ \varphi = (\Lambda^t \circ \Lambda^a)(\varphi) \circ \lambda_A^g$,

our assertion follows immediately. So, λ^g is a natural isomorphism. $\square$

Proposition 2.2.3.30. *The identity functor $Id_{\mathbf{HLC}}$ and the functor $\Lambda^a \circ \Lambda^t$ are naturally isomorphic.*

Proof. Recall that, for every $X \in |\mathbf{HLC}|$, the map $t_X : X \longrightarrow (\Lambda^a \circ \Lambda^t)(X)$, where $t_X(x) = \sigma_x$ for every $x \in X$, is a homeomorphism (see (1.32) and (1.15) for t_X, and (1.3) for σ_x). We will show that $t^l : Id_{\mathbf{HLC}} \longrightarrow \Lambda^a \circ \Lambda^t$, where for every $X \in |\mathbf{HLC}|$, $t^l(X) = t_X$, is a natural isomorphism.

Let $f \in \mathbf{HLC}(X, Y)$ and $f' = (\Lambda^a \circ \Lambda^t)(f)$. We have to prove that $t_Y \circ f = f' \circ t_X$, i.e., that for every $x \in X$, $\sigma_{f(x)} = f'(\sigma_x)$. By 2.2.3.4, it is enough to show that $\sigma_{f(x)} \cap CR(Y) = f'(\sigma_x) \cap CR(Y)$.

We have, by the definition of Λ^t, that $\Lambda^t(f) = \varphi_f$, where, for every $G \in RC(Y)$, $\varphi_f(G) = \mathrm{cl}(f^{-1}(\mathrm{int}(G)))$. Hence $f'(\sigma_x) \cap CR(Y) = \{F \in CR(Y) \mid (\forall G \in RC(Y))[(F \subseteq \mathrm{int}(G)) \rightarrow (\varphi_f(G) \in \sigma_x)]\} = \{F \in CR(Y) \mid (\forall G \in RC(Y))[(F \subseteq \mathrm{int}(G)) \rightarrow (x \in \mathrm{cl}(f^{-1}(\mathrm{int}(G))))]\}$. Let $F \in \sigma_{f(x)} \cap CR(Y)$. Then $f(x) \in F$. If $G \in RC(Y)$ and $F \subseteq \mathrm{int}(G)$, then $f(x) \in \mathrm{int}(G)$. Thus $x \in f^{-1}(\mathrm{int}(G)) \subseteq \mathrm{cl}(f^{-1}(\mathrm{int}(G)))$. Therefore $F \in f'(\sigma_x) \cap CR(Y)$. So, $\sigma_{f(x)} \cap CR(Y) \subseteq f'(\sigma_x) \cap CR(Y)$. Conversely, let $F \in f'(\sigma_x) \cap CR(Y)$. Suppose that $f(x) \notin F$. Then, by 0.4.2.3, there exists $G \in CR(Y)$ such that

$F \subseteq \mathrm{int}(G) \subseteq G \subseteq Y \setminus \{f(x)\}$. Thus $x \notin f^{-1}(G)$. Since $\mathrm{cl}(f^{-1}(\mathrm{int}(G))) \subseteq f^{-1}(G)$, we get that $x \notin \mathrm{cl}(f^{-1}(\mathrm{int}(G)))$, a contradiction. Hence $f(x) \in F$, i.e., $F \in \sigma_{f(x)}$. Thus $\sigma_{f(x)} \cap CR(Y) \supseteq f'(\sigma_x) \cap CR(Y)$. Therefore, for every $x \in X$, $\sigma_{f(x)} = f'(\sigma_x)$. This means that $t_Y \circ f = f' \circ t_X$. So, t^l is a natural isomorphism. □

It is clear now that Theorem 2.2.2.12 follows from 2.2.3.25, 2.2.3.27, 2.2.3.28, 2.2.3.29 and 2.2.3.30. So, the proof of Theorem 2.2.2.12 is complete.

2.2.4 A Duality Theorem for the category of connected locally compact Hausdorff spaces and continuous maps

We will now derive from our Theorem 2.2.2.12 a corollary concerning the subcategory of the category **HLC** whose objects are connected spaces.

Definition 2.2.4.1. A CA (B, C) is said to be *connected* if it satisfies the following axiom:

(CON) If $a \neq 0, 1$ then aCa^*.

An LCA $(B, \rho, \mathbb{B})$ is called *connected* if the CA (B, ρ) is connected.

Remark 2.2.4.2. The axiom (CON) is equivalent to the following one:

(CON') If $a, b \neq 0$ and $a \vee b = 1$ then aCb.

The following obvious fact was noted in [14].

Fact 2.2.4.3. ([14]) *Let (X, τ) be a topological space. Then the standard contact algebra $(RC(X, \tau), \rho_{(X,\tau)})$ is connected iff the space (X, τ) is connected.*

Notation 2.2.4.4. If **K** is a category whose objects form a subclass of the class of all topological spaces (resp., contact algebras) then we will denote by **KCon** the full subcategory of **K** whose objects are all "connected" **K**-objects, where "connected" is understood in the usual sense when the objects of **K** are topological spaces and in the sense of 2.2.4.1 when the objects of **K** are contact algebras.

So, we denote by:

- **HLCCon** the full subcategory of the category **HLC** whose objects are all connected locally compact Hausdorff spaces,
- **DHLCCon** the full subcategory of the category **DHLC** whose objects are all connected CLC-algebras.

Theorem 2.2.4.5. *The categories* **HLCCon** *and* **DHLCCon** *are dually equivalent.*

Proof. It follows immediately from Theorem 2.2.2.12 and Fact 2.2.4.3. □

2.3 Products, sums and completions of LCAs

2.3.1 Introduction

In this section we will apply our Theorem 2.2.2.12 for obtaining an explicit description of the products and sums of complete local contact algebras. Also, using again Theorem 2.2.2.12, we will develop a completion theory for local contact algebras.

The structure of the section is the following. In the second subsection, an explicit description of the products of arbitrary families of complete local contact algebras (= CLC-algebras) in the category **DHLC** dual to the category **HLC** is given. Note that the Duality Theorem 2.2.2.12 implies that the category **DHLC** has products because its dual category **HLC** has sums (i.e., coproducts). However, one needs to do some extra work in order to obtain a direct description of these products.

In the third subsection, we obtain some direct descriptions of the **DHLC**-sums of finite families of complete LC-algebras and the **DHC**-sums of arbitrary families of complete NC-algebras (where **DHC** is the de Vries category dual to the category **HC**) using the de Vries Duality Theorem [24] (see Theorem 2.2.2.6 here) and our Theorem 2.2.2.12.

In the fourth subsection, a completion theory for LC-algebras is developed. We give a definition of a completion of an LCA, and we show that any LCA has a unique completion. The fact that any LCA can be embedded into a complete LCA was established in [41] but we worked there with another definition of a completion and there were no uniqueness of that completion.

The exposition of this section is based on the papers [31] and [34].

2.3.2 A description of DHLC-products of complete local contact algebras

In this subsection we will describe the **DHLC**-products of arbitrary families of CLC-algebras. Note that the products in the category **DHLC** surely exist because its dual category **HLC** of all locally compact Hausdorff spaces and continuous maps has sums.

Definition 2.3.2.1. Let Γ be a set and $\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$ be a family of LC-algebras. Let

$$A = \prod\{A_\gamma \mid \gamma \in \Gamma\}$$

be the product of the Boolean algebras $\{A_\gamma \mid \gamma \in \Gamma\}$ in the category **BoolAlg** of Boolean algebras and Boolean homomorphisms (i.e., A is the Cartesian product of the family $\{A_\gamma \mid \gamma \in \Gamma\}$, construed as a Boolean algebra with respect to the coordinate-wise operations). Let

$$\mathbb{B} = \{(b_\gamma)_{\gamma\in\Gamma} \in \prod\{\mathbb{B}_\gamma \mid \gamma \in \Gamma\} \mid |\{\gamma \in \Gamma \mid b_\gamma \neq 0\}| < \aleph_0\},$$

where $\prod\{\mathbb{B}_\gamma \mid \gamma \in \Gamma\}$ is the Cartesian product of the family $\{\mathbb{B}_\gamma \mid \gamma \in \Gamma\}$ (in other words, $\mathbb{B}$ is the σ-product of the family $\{\mathbb{B}_\gamma \mid \gamma \in \Gamma\}$ with base point $0 = (0_\gamma)_{\gamma\in\Gamma}$). For any two points $a = (a_\gamma)_{\gamma\in\Gamma} \in A$ and $b = (b_\gamma)_{\gamma\in\Gamma} \in A$, set

$$a\rho b \text{ iff there exists } \gamma \in \Gamma \text{ such that } a_\gamma \rho_\gamma b_\gamma.$$

Then the triple $(A, \rho, \mathbb{B})$ is called a *product of the family* $\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$ *of LC-algebras*. We will write

$$(A, \rho, \mathbb{B}) = \prod\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}.$$

Fact 2.3.2.2. *The product* $(A, \rho, \mathbb{B})$ *of a family* $\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$ *of LC-algebras (resp., of CLCAs) is an LCA (resp., a CLCA).*

Proof. The proof is straightforward. □

Recall that (see, e.g., [1]) if **C** is a category, a source

$$\mathcal{P} = \{p_\gamma : P \longrightarrow A_\gamma \mid \gamma \in \Gamma\}$$

of **C**-morphisms (i.e., a **C**-*source*) is called a *product* provided that for every **C**-source $\mathcal{S} = \{f_\gamma : A \longrightarrow A_\gamma \mid \gamma \in \Gamma\}$ with the same codomain as $\mathcal{P}$ there exists a unique **C**-morphism $f : A \longrightarrow P$ with

$$f_\gamma = p_\gamma \circ f, \ \forall \gamma \in \Gamma$$

(i.e., briefly, $\mathcal{S} = \mathcal{P} \circ f$); a product with codomain $\{A_\gamma \mid \gamma \in \Gamma\}$ is called a *product (or,* **C***-product) of the family* $\{A_\gamma \mid \gamma \in \Gamma\}$ of **C**-objects. For any family $\{A_\gamma \mid \gamma \in \Gamma\}$ of **C**-objects, products of $\{A_\gamma \mid \gamma \in \Gamma\}$ are essentially unique; i.e., if $\mathcal{P} = \{p_\gamma : P \longrightarrow A_\gamma \mid \gamma \in \Gamma\}$ is a product of $\{A_\gamma \mid \gamma \in \Gamma\}$, then the following hold:

(1) for each product $\mathcal{Q} = \{q_\gamma : Q \longrightarrow A_\gamma \mid \gamma \in \Gamma\}$ there exists an isomorphism $h : Q \longrightarrow P$ in $\mathbf{C}$ with $\mathcal{Q} = \mathcal{P} \circ h$,
(2) for each $\mathbf{C}$-isomorphism $g : A \longrightarrow P$ the source $\mathcal{P} \circ g$ is a product of $\{A_\gamma \mid \gamma \in \Gamma\}$.

The notion of a *coproduct (or* $\mathbf{C}$*-coproduct, or, even,* $\mathbf{C}$*-sum or sum) of a family* $\{A_\gamma \mid \gamma \in \Gamma\}$ of $\mathbf{C}$-objects is defined analogously.

Proposition 2.3.2.3. *Let Γ be a set and $\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$ be a family of CLC-algebras. Then the source $\{\pi_\gamma : (A, \rho, \mathbb{B}) \longrightarrow (A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$, where*

$$(A, \rho, \mathbb{B}) = \prod\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$$

(see Definition 2.3.2.1) and, for every $a = (a_\gamma)_{\gamma\in\Gamma} \in A$ and every $\gamma \in \Gamma$,

$$\pi_\gamma(a) = a_\gamma,$$

is a product of the family $\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$ in the category $\mathbf{DHLC}$.

Proof. By Fact 2.3.2.2, $(A, \rho, \mathbb{B})$ is a CLCA. It is easy to see that, for every $\gamma \in \Gamma$, π_γ is a $\mathbf{DHLC}$-morphism.

Let $X_\gamma = \Lambda^a(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma)$ for every $\gamma \in \Gamma$, and let

$$X = \bigoplus\{X_\gamma \mid \gamma \in \Gamma\}$$

be the topological sum of the family $\{X_\gamma \mid \gamma \in \Gamma\}$. Then the sink of inclusions

$$\{i_\gamma : X_\gamma \longrightarrow X \mid \gamma \in \Gamma\}$$

is a coproduct in the category $\mathbf{HLC}$ (briefly, $\mathbf{HLC}$-coproduct) of the family $\{X_\gamma \mid \gamma \in \Gamma\}$. Since Λ^t is a duality (by Theorem 2.2.2.12), the source

$$\mathcal{P} = \{\Lambda^t(i_\gamma) : \Lambda^t(X) \longrightarrow \Lambda^t(X_\gamma) \mid \gamma \in \Gamma\}$$

is a $\mathbf{DHLC}$-product of the family $\{\Lambda^t(X_\gamma) \mid \gamma \in \Gamma\}$. Then, clearly, the source

$$\mathcal{Q} = \{(\lambda^g_{A_\gamma})^{-1} \diamond \Lambda^t(i_\gamma) : \Lambda^t(X) \longrightarrow (A_\gamma, \rho_g, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$$

is a $\mathbf{DHLC}$-product of the family $\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$. Set, for each $\gamma \in \Gamma$,

$$\alpha_\gamma = (\lambda^g_{A_\gamma})^{-1} \diamond \Lambda^t(i_\gamma).$$

We will show that there exists a $\mathbf{DHLC}$-isomorphism $\alpha : \Lambda^t(X) \longrightarrow (A, \rho, \mathbb{B})$ such that, for any $\gamma \in \Gamma$, $\pi_\gamma \diamond \alpha = \alpha_\gamma$. Obviously, this will imply that the source $\{\pi_\gamma : (A, \rho, \mathbb{B}) \longrightarrow$

$(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$ is a **DHLC**-product of the family $\{(A_\gamma, \rho_\gamma, \mathbb{B}_\gamma) \mid \gamma \in \Gamma\}$. Set, for every $F \in RC(X)$ and any $\gamma \in \Gamma$, $F_\gamma = F \cap X_\gamma$. Then $F_\gamma \in RC(X_\gamma)$ for every $\gamma \in \Gamma$. Define the map

$$\alpha : RC(X) \longrightarrow A$$

by $\alpha(F) = ((\lambda^g_{A_\gamma})^{-1}(F_\gamma))_{\gamma \in \Gamma}$, for every $F \in RC(X)$. Since $\Lambda^t(X) = RC(X)$ and $\Lambda^t(X_\gamma) = RC(X_\gamma)$, it is easy to see that the map α is a **DHLC**-isomorphism between $\Lambda^t(X)$ and $(A, \rho, \mathbb{B})$. Further, for any $\gamma \in \Gamma$ and any $F \in RC(X)$, $\Lambda^t(i_\gamma)(F) = \mathrm{cl}_{X_\gamma}(i_\gamma^{-1}(\mathrm{int}_X(F)))$ (see Theorem 2.2.2.12). We get that $\Lambda^t(i_\gamma)(F) = F_\gamma$ which implies easily that $\pi_\gamma \circ \alpha = \alpha_\gamma$, for every $\gamma \in \Gamma$. Thus, by (DLC5), $\pi_\gamma \diamond \alpha = \alpha_\gamma$, for every $\gamma \in \Gamma$. $\square$

2.3.3 A description of DHLC-sums of CLCAs and DHC-sums of CNCAs

In this subsection we will describe the **DHLC**-sums of finite families of complete local contact algebras and the **DHC**-sums of arbitrarily many complete normal contact algebras using the notion of a *sum of a family of Boolean algebras* (see [65]) which is known also as a *free product* (see [77]). (We will denote the sum of a family $\{A_\gamma \mid \gamma \in \Gamma\}$ of Boolean algebras by

$$\bigoplus_{\gamma \in \Gamma} A_\gamma$$

(as in [77]).) Note that the sums (resp., finite sums) in the category **DHC** (resp., **DHLC**) surely exist because the dual category **HC** (resp., **HLC**) of all compact (resp., locally compact) Hausdorff spaces and continuous maps has products (resp., finite products).

Let us recall the definition of the notion of a *sum of a family* $(A_i)_{i \in I}$ *of Boolean algebras* (see, e.g. [77]): a pair

$$(A, (e_i)_{i \in I})$$

is a *sum of* $(A_i)_{i \in I}$ if A is a Boolean algebra, each e_i is a homomorphism from A_i into A and, for every family $(f_i)_{i \in I}$ of homomorphisms from A_i into any Boolean algebra B, there is a unique homomorphism $f : A \longrightarrow B$ such that $f \circ e_i = f_i$ for $i \in I$. (Hence, the sum of a family of Boolean algebras is, in fact, the **BoolAlg**-coproduct of this family.) It is well known that every family of Boolean algebras has, up to isomorphism, a unique sum. Recall, as well, that a family $(B_i)_{i \in I}$ of subalgebras of a Boolean algebra A is *independent* if, for arbitrary $n \in \mathbb{N}^+$, pairwise distinct $i(1), \ldots, i(n) \in I$ and non-zero elements $b_{i(k)}$ of $B_{i(k)}$, for $k = 1, \ldots, n$,

$$b_{i(1)} \wedge \ldots \wedge b_{i(n)} > 0$$

holds in A. The following characterization of the sums is well-known (see, e.g., [77, Proposition 11.4]):

Proposition 2.3.3.1. *Let A be a Boolean algebra and, for $i \in I$,*

$$e_i : A_i \longrightarrow A$$

a homomorphism; assume that no A_i is trivial. The pair

$$(A, (e_i)_{i \in I})$$

is a sum of $(A_i)_{i \in I}$ iff each of (a) through (c) holds:

(a) each $e_i : A_i \longrightarrow A$ is an injection,

(b) $(e_i(A_i))_{i \in I}$ is an independent family of subalgebras of A,

(c) A is generated by $\bigcup_{i \in I} e_i(A_i)$.

Moreover, if $(A, (e_i)_{i \in I})$ is a sum of $(A_i)_{i \in I}$ then

(d) $e_i(A_i) \cap e_j(A_j) = \{0, 1\}$, for $i \neq j$.

We start with a proposition which should be known, although I was not able to find it in the literature. Recall that a topological space X is called *semiregular* if $RO(X)$ is a base of X. By a *completion* of a Boolean algebra A, we will understand the *MacNeille completion* of A (i.e., the *minimal completion* of A) (recall that a pair (φ, A') is a minimal completion of A if A' is a complete Boolean algebra, $\varphi : A \longrightarrow A'$ is a monomorphism and $\varphi(A)$ is a dense subalgebra of A').

Proposition 2.3.3.2. *Let $\{X_\gamma \mid \gamma \in \Gamma\}$ be a family of semiregular topological spaces and $X = \prod\{X_\gamma \mid \gamma \in \Gamma\}$. Then the Boolean algebra $RC(X)$ is isomorphic to the completion of $\bigoplus_{\gamma \in \Gamma} RC(X_\gamma)$.*

Proof. Let, for every $\gamma \in \Gamma$,

$$\pi_\gamma : X \longrightarrow X_\gamma$$

be the projection. Using the fact that π_γ is an open map (and, thus, the formulae $\mathrm{cl}(\pi_\gamma^{-1}(M)) = \pi_\gamma^{-1}(\mathrm{cl}(M))$ and $\mathrm{int}(\pi_\gamma^{-1}(M)) = \pi_\gamma^{-1}(\mathrm{int}(M))$ hold for every $M \subseteq X_\gamma$) (see, e.g., [53]), it is easy to show, that the map

$$\varphi_\gamma : RC(X_\gamma) \longrightarrow RC(X),\ F \mapsto \pi_\gamma^{-1}(F),$$

is a complete monomorphism for every $\gamma \in \Gamma$. Set

$$A_\gamma = \varphi_\gamma(RC(X_\gamma)),$$

for every $\gamma \in \Gamma$, and let A be the subalgebra of $RC(X)$ generated by $\bigcup\{A_\gamma \mid \gamma \in \Gamma\}$. It is easy to check that, for every finite non-empty subset Γ_0 of Γ, we have that if $a_\gamma \in A_\gamma \setminus \{0\}$ for every $\gamma \in \Gamma_0$, then $\bigwedge\{a_\gamma \mid \gamma \in \Gamma_0\} \neq 0$ (i.e., the family $\{A_\gamma \mid \gamma \in \Gamma\}$ is an independent family). Thus, by 2.3.3.1, we get that

$$A = \bigoplus_{\gamma \in \Gamma} RC(X_\gamma).$$

Since $RO(X_\gamma)$ is a base of X_γ, for every $\gamma \in \Gamma$, we obtain that A is a dense subalgebra of $RC(X)$. Thus, $RC(X)$ is the completion of A. □

The proof of this proposition shows that the following is even true:

Corollary 2.3.3.3. *Let $\{X_\gamma \mid \gamma \in \Gamma\}$ be a family of semiregular topological spaces and $X = \prod\{X_\gamma \mid \gamma \in \Gamma\}$. Let, for every $\gamma \in \Gamma$, B_γ be a subalgebra of $RC(X_\gamma)$ such that $\{\mathrm{int}(F) \mid F \in B_\gamma\}$ is a base of X_γ. Then the Boolean algebra $RC(X)$ is isomorphic to the completion of $\bigoplus_{\gamma \in \Gamma} B_\gamma$.*

Definition 2.3.3.4. Let $n \in \mathbb{N}^+$ and let, for every $i = 1, \ldots, n$, $(A_i, \rho_i, \mathbb{B}_i)$ be a CLCA. Let

$$(A, (\varphi_i)_{i=1}^n) = \bigoplus_{i=1}^n A_i,$$

where, for every $i \in \{1, \ldots, n\}$,

$$\varphi_i : A_i \longrightarrow A$$

is the canonical complete monomorphism, and let $\tilde{A}$ be the completion of A. We can suppose, without loss of generality, that $A \subseteq \tilde{A}$. Set

$$E = \{\bigwedge_{i=1}^n \varphi_i(a_i) \mid a_i \in \mathbb{B}_i\}$$

and let $\widetilde{\mathbb{B}}$ be the ideal of $\tilde{A}$ generated by E (thus,

$$\widetilde{\mathbb{B}} = \{x \in \tilde{A} \mid x \leq e_1 \vee \ldots \vee e_n \text{ for some } n \in \mathbb{N}^+ \text{ and some } e_1, \ldots, e_n \in E\}).$$

For every two elements $a = \bigwedge_{i=1}^n \varphi_i(a_i)$ and $b = \bigwedge_{i=1}^n \varphi_i(b_i)$ of E, set

$$a\tilde{\rho}b \Leftrightarrow (a_i \rho_i b_i, \forall i \in \{1, \ldots, n\}).$$

Further, for every two elements c and d of $\widetilde{\mathbb{B}}$, set

$$c(-\tilde{\rho})d \Leftrightarrow (\exists k, l \in \mathbb{N}^+ \text{ and } \exists c_1, \dots, c_k, d_1, \dots, d_l \in E \text{ such that}$$

$$c \leq \bigvee_{i=1}^{k} c_i,\ d \leq \bigvee_{j=1}^{l} d_j \text{ and } c_i(-\tilde{\rho})d_j,\ \forall i = 1, \dots, k \text{ and } \forall j = 1, \dots, l).$$

Finally, for every two elements a and b of $\tilde{A}$, set

$$a\tilde{\rho}b \Leftrightarrow (\exists c, d \in \widetilde{\mathbb{B}} \text{ such that } c \leq a,\ d \leq b \text{ and } c\tilde{\rho}d).$$

Then the triple $(\tilde{A}, \tilde{\rho}, \widetilde{\mathbb{B}})$ will be denoted by $\bigoplus_{i=1}^{n}(A_i, \rho_i, \mathbb{B}_i)$.

Theorem 2.3.3.5. *Let $n \in \mathbb{N}^+$ and $\{(A_i, \rho_i, \mathbb{B}_i) \mid i = 1, \dots, n\}$ be a family of CLCAs. Then $\bigoplus_{i=1}^{n}(A_i, \rho_i, \mathbb{B}_i)$ is a* **DHLC***-sum of the family $\{(A_i, \rho_i, \mathbb{B}_i) \mid i = 1, \dots, n\}$.*

Proof. As the Duality Theorem 2.2.2.12 shows, for every $i \in \{1, \dots, n\}$ there exists a $X_i \in |\mathbf{HLC}|$ such that the CLCAs $(RC(X_i), \rho_{X_i}, CR(X_i))$ and $(A_i, \rho_i, \mathbb{B}_i)$ are LCA-isomorphic. Let

$$X = \prod_{i=1}^{n} X_i.$$

Then we have, in the notation of Definition 2.3.3.4, that the Boolean algebras $RC(X)$ and $\tilde{A}$ are isomorphic (see Proposition 2.3.3.2). Also, again in the notation of Definition 2.3.3.4, $(A, (\varphi_i)_{i=1}^{n})$ is isomorphic to $(\bigoplus_{i=1}^{n} RC(X_i), (\psi_i)_{i=1}^{n})$, where

$$\psi_i : RC(X_i) \longrightarrow RC(X),\ F \mapsto \pi_i^{-1}(F),$$

and

$$\pi_i : X \longrightarrow X_i$$

is the projection, for every $i \in \{1, \dots, n\}$ (this follows from Proposition 2.3.3.1). Thus, the set E from Definition 2.3.3.4 corresponds to the following set:

$$E' = \{\bigwedge_{i=1}^{n} \psi_i(F_i) \mid F_i \in CR(X_i)\}.$$

Let $F \in E'$. Then there exist $F_i \in CR(X_i)$, for $i = 1, \dots, n$, such that $F = \bigwedge_{i=1}^{n} \psi_i(F_i)$. Set

$$U_i = \operatorname{int}_{X_i}(F_i),\ \text{for } i = 1, \dots, n.$$

Then

$F = \bigwedge_{i=1}^{n} \pi_i^{-1}(F_i) = \mathrm{cl}_X(\bigcap_{i=1}^{n} \operatorname{int}_X(\pi_i^{-1}(F_i))) = \mathrm{cl}_X(\bigcap_{i=1}^{n} \pi_i^{-1}(U_i)) = \mathrm{cl}(\prod_{i=1}^{n} U_i) =$

$\prod_{i=1}^n F_i$ (note that we used [53, 1.4.C,2.3.3] here). Hence, for every $F, G \in E'$, where $F = \prod_{i=1}^n F_i$ and $G = \prod_{i=1}^n G_i$, we have that

$$F\rho_X G \Leftrightarrow F \cap G \neq \emptyset \Leftrightarrow (F_i \cap G_i \neq \emptyset, \forall i = 1, \ldots, n) \Leftrightarrow (F_i \rho_{X_i} G_i, \forall i = 1, \ldots, n).$$

Further, since $\{\prod_{i=1}^n U_i \mid U_i \in RO(X_i), \forall i = 1, \ldots, n\}$ is a base of X and X is regular, we obtain that $CR(X)$ coincides with the ideal of $RC(X)$ generated by E'. The fact that every two disjoint compact subsets of X can be separated by open sets implies that if $F, G \in CR(X)$ then $F(-\rho_X)G$ (i.e., $F \cap G = \emptyset$) iff there exists finitely many elements $F_1, \ldots, F_k, G_1, \ldots, G_l \in E'$ such that $F \subseteq \bigcup_{i=1}^k F_i$, $G \subseteq \bigcup_{i=1}^l G_i$ and $F_i \cap G_j = \emptyset$ (i.e., $F_i(-\rho_X)G_j$) for all $i = 1, \ldots, k$ and all $j = 1, \ldots, l$. Finally, since $(RC(X), \rho_X, CR(X))$ is an LCA (see 1.2.3.8), we have (by (BC2)) that for any $F', G' \in RC(X)$, $F'\rho_X G' \Leftrightarrow \exists F, G \in CR(X)$ such that $F \subseteq F'$, $G \subseteq G'$ and $F\rho_X G$. All this shows that the triple $(\tilde{A}, \tilde{\rho}, \widetilde{\mathbb{B}})$ from 2.3.3.4 is an LCA which is LCA-isomorphic to $(RC(X), \rho_X, CR(X))$. Now, using Theorem 2.2.2.12 and the facts that $\Psi^t(X) = (RC(X), \rho_X, CR(X))$, $\Psi^t(X_i) = (RC(X_i), \rho_{X_i}, CR(X_i))$ for all $i = 1, \ldots, n$, and X is a **HLC**-product of the family $\{X_i \mid i = 1, \ldots, n\}$, we get that $(RC(X), \rho_X, CR(X))$ is a **DHLC**-sum of the family $\{(RC(X_i), \rho_{X_i}, CR(X_i)) \mid i = 1, \ldots, n\}$. Thus $(\tilde{A}, \tilde{\rho}, \widetilde{\mathbb{B}})$ is a **DHLC**-sum of the family $\{(A_i, \rho_i, \mathbb{B}_i) \mid i = 1, \ldots, n\}$. $\square$

Definition 2.3.3.6. Let J be a set and let, for every $j \in J$, (A_j, ρ_j) be a CNCA. Let

$$(A, (\varphi_j)_{j\in J}) = \bigoplus_{j\in J} A_j,$$

where, for every $j \in J$,

$$\varphi_j : A_j \longrightarrow A$$

is the canonical complete monomorphism, and let $\tilde{A}$ be the completion of A. We can suppose, without loss of generality, that $A \subseteq \tilde{A}$. Set

$$E = \{\bigwedge_{i\in I} \varphi_i(a_i) \mid I \subseteq J, |I| < \aleph_0, a_i \in A_i, \forall i \in I\}.$$

For every two elements $a = \bigwedge_{i\in I_1} \varphi_i(a_i)$ and $b = \bigwedge_{i\in I_2} \varphi_i(b_i)$ of E, set

$$a\tilde{\rho}b \Leftrightarrow (a_i\rho_i b_i, \forall i \in I_1 \cap I_2).$$

Further, for every two elements c and d of $\tilde{A}$, set

$$c(-\tilde{\rho})d \Leftrightarrow (\exists k, l \in \mathbb{N}^+ \text{ and } \exists c_1, \ldots, c_k, d_1, \ldots, d_l \in E \text{ such that}$$

$$c \leq \bigvee_{i=1}^{k} c_i,\ d \leq \bigvee_{j=1}^{l} d_j \text{ and } c_i(-\tilde{\rho})d_j,\ \forall i = 1,\dots,k \text{ and } \forall j = 1,\dots,l).$$

Then the pair $(\tilde{A}, \tilde{\rho})$ will be denoted by $\bigoplus_{j \in J}(A_j, \rho_j)$.

Theorem 2.3.3.7. *Let $\{(A_j, \rho_j) \mid j \in J\}$ be a family of CNCAs. Then $\bigoplus_{j \in J}(A_j, \rho_j)$ is a* **DHC**-*sum of the family $\{(A_j, \rho_j) \mid j \in J\}$.*

Proof. The proof is similar to that one of Theorem 2.3.3.5. In it de Vries' Duality Theorem 2.2.2.6 instead of Theorem 2.2.2.12 has to be used. $\square$

2.3.4 A completion theory for LC-algebras

In this section we will use the technique developed in the first section of this chapter for the proof of Theorem 2.2.2.12 in order to obtain a completion theory for LC-algebras, where both the existence and the uniqueness of the LCA-completion are proved.

Definition 2.3.4.1. Let $(A, \rho, \mathbb{B})$ be an LCA and $\mathcal{B}$ be a subset of $\mathbb{B}$. Then $\mathcal{B}$ is called a *dV-dense subset* of $(A, \rho, \mathbb{B})$ if for each $a, c \in \mathbb{B}$ such that $a \ll_\rho c$ there exists $b \in \mathcal{B}$ with $a \leq b \leq c$.

Fact 2.3.4.2. *If $(A, \rho, \mathbb{B})$ is an LCA and $\mathcal{B}$ is a subset of $\mathbb{B}$ then $\mathcal{B}$ is a dV-dense subset of $(A, \rho, \mathbb{B})$ iff for each $a, c \in \mathbb{B}$ such that $a \ll_\rho c$ there exists $b \in \mathcal{B}$ with $a \ll_\rho b \ll_\rho c$.*

Proof. ($\Rightarrow$) Let $a, c \in \mathbb{B}$ and $a \ll_\rho c$. Then, by (BC1), there exists $d, e \in \mathbb{B}$ with $a \ll_\rho d \ll_\rho e \ll_\rho c$. Now, there exists $b \in \mathcal{B}$ such that $d \leq b \leq e$. Therefore $a \ll_\rho b \ll_\rho c$.
($\Leftarrow$) This is clear. $\square$

Definition 2.3.4.3. Let $(A, \rho, \mathbb{B})$ be an LCA. A pair $(\varphi, (A', \rho', \mathbb{B}'))$ is called an *LCA-completion* of the LCA $(A, \rho, \mathbb{B})$ if $(A', \rho', \mathbb{B}')$ is a CLCA, φ is an LCA-embedding of $(A, \rho, \mathbb{B})$ into $(A', \rho', \mathbb{B}')$, and $\varphi(\mathbb{B})$ is a dV-dense subset of $(A', \rho', \mathbb{B}')$.

Two LCA-completions $(\varphi, (A', \rho', \mathbb{B}'))$ and $(\psi, (A'', \rho'', \mathbb{B}''))$ of a local contact algebra $(A, \rho, \mathbb{B})$ are said to be *equivalent* if there exists an LCA-isomorphism

$$\eta : (A', \rho', \mathbb{B}') \longrightarrow (A'', \rho'', \mathbb{B}'')$$

such that $\psi = \eta \circ \varphi$.

Note that condition (BC3) implies that every dV-dense subset of an LCA $(A, \rho, \mathbb{B})$ is a dense subset of A. Hence, if $(\varphi, (A', \rho', \mathbb{B}'))$ is an LCA-completion of the LCA $(A, \rho, \mathbb{B})$ then (φ, A') is a minimal completion of the Boolean algebra A.

Let us start with a simple lemma.

Lemma 2.3.4.4. *Let $(\varphi, (B, \eta, \mathbb{B}'))$ be an LCA-completion of an LCA $(A, \rho, \mathbb{B})$ and let us suppose, for simplicity, that $A \subseteq B$ and $\varphi(a) = a$ for every $a \in A$. Then:*

(a) $\mathbb{B}' = \downarrow_B (\mathbb{B})$ and $\mathbb{B}' \cap A = \mathbb{B}$;

(b) If J is a δ-ideal of $(B, \eta, \mathbb{B}')$ then $J \cap A$ is a δ-ideal of $(A, \rho, \mathbb{B})$ and $\downarrow_B (J \cap A) = J$;

(c) If J is a δ-ideal of $(A, \rho, \mathbb{B})$ then $\downarrow_B (J)$ is a δ-ideal of $(B, \eta, \mathbb{B}')$ and also $A \cap \downarrow_B (J) = J$;

(d) If J is a prime element of $I(B, \eta, \mathbb{B}')$ then $J \cap A$ is a prime element of the frame $I(A, \rho, \mathbb{B})$;

(e) If J is a prime element of $I(A, \rho, \mathbb{B})$ then $\downarrow_B (J)$ is a prime element of $(B, \eta, \mathbb{B}')$.

Proof. (a) Let $b \in \mathbb{B}'$. Then, by condition (BC1), there exists $c \in \mathbb{B}'$ such that $b \ll_\eta c$ (because $b \ll_\eta 1$). Since $\mathbb{B}$ is a dV-dense subset of $(B, \eta, \mathbb{B}')$, there exists $a \in \mathbb{B}$ such that $b \leq a \leq c$. Hence $\mathbb{B}' \subseteq \downarrow_B (\mathbb{B})$. Since $\mathbb{B} \subseteq \mathbb{B}'$ and $\mathbb{B}'$ is an ideal of B, we get that $\downarrow_B (\mathbb{B}) \subseteq \mathbb{B}'$. Hence, $\mathbb{B}' = \downarrow_B (\mathbb{B})$.

Obviously, $\mathbb{B} \subseteq \mathbb{B}' \cap A$. If $a \in \mathbb{B}' \cap A$ then, as above, there exists $b \in \mathbb{B}$ such that $a \leq b$. Thus $a \in \mathbb{B}$. Hence $\mathbb{B}' \cap A = \mathbb{B}$.

(b) We have that $J \cap A \subseteq \mathbb{B}' \cap A = \mathbb{B}$. Let $a \in J \cap A$. Then there exists $b \in J$ such that $a \ll_\eta b$. Since $\mathbb{B}$ is a dV-dense subset of $(B, \eta, \mathbb{B}')$, we get that there exists $c \in \mathbb{B}$ such that $a \ll_\eta c \ll_\eta b$ (see Fact 2.3.4.2). Then $c \in J \cap A$ and $a \ll_\rho c$. So, $J \cap A$ is a δ-ideal of $(A, \rho, \mathbb{B})$. The last argument shows as well that $J \subseteq \downarrow_B (J \cap A)$. Since, clearly, $\downarrow_B (J \cap A) \subseteq J$, we get that $\downarrow_B (J \cap A) = J$.

(c) Let J be a δ-ideal of $(A, \rho, \mathbb{B})$. Set $J' = \downarrow_B (J)$. Clearly, J' is an ideal of B. Let $a \in J'$. Then there exists $b, c \in J$ such that $a \leq b \ll_\rho c$. Thus $a \ll_\eta c$ and $c \in J'$. Hence J' is a δ-ideal of $(B, \eta, \mathbb{B}')$. Obviously, $J \subseteq A \cap \downarrow_B (J)$. Conversely, let $a \in A \cap \downarrow_B (J)$. Then there exists $b \in J$ such that $a \leq b$. Thus $a \in J$. So, $A \cap \downarrow_B (J) = J$.

(d) Let J be a prime element of $I(B, \eta, \mathbb{B}')$. Then, by (b), $J \cap A \in I(A, \rho, \mathbb{B})$. Let $J_1, J_2 \in I(A, \rho, \mathbb{B})$ and $J_1 \cap J_2 \subseteq J \cap A$. Then $\downarrow_B (J_1) \cap \downarrow_B (J_2) = \downarrow_B (J_1 \cap J_2) \subseteq \downarrow_B (J \cap A)$. Since, by (c), $\downarrow_B (J_i) \in I(B, \eta, \mathbb{B}')$, for $i = 1, 2$, and, by (b), $\downarrow_B (J \cap A) = J$, we get that $\downarrow_B (J_1) \subseteq J$ or $\downarrow_B (J_2) \subseteq J$. Then $A \cap \downarrow_B (J_1) \subseteq A \cap J$ or $A \cap \downarrow_B (J_2) \subseteq A \cap J$. Thus, by (c), $J_1 \subseteq J \cap A$ or $J_2 \subseteq J \cap A$. Hence, $J \cap A$ is a prime element of $I(A, \rho, \mathbb{B})$.

(e) Let J be a prime element of $I(A,\rho,\mathbb{B})$. Let $J_1, J_2 \in I(B,\eta,\mathbb{B}')$ and $J_1 \cap J_2 \subseteq \downarrow_B (J)$. Then, by (c), $A \cap J_1 \cap J_2 \subseteq A \cap \downarrow_B (J) = J$. Hence, by (b), $A \cap J_1 \subseteq J$ or $A \cap J_2 \subseteq J$. Thus, by (b), $J_1 \subseteq \downarrow_B (J)$ or $J_2 \subseteq \downarrow_B (J)$. Therefore, $\downarrow_B (J)$ is a prime element of $I(B,\eta,\mathbb{B}')$. □

Theorem 2.3.4.5. *Every local contact algebra $(A,\rho,\mathbb{B})$ has a unique (up to equivalence) LCA-completion.*

Proof. Let $(A,\rho,\mathbb{B})$ be an LCA. Then, by Roeper's Theorem 1.2.3.10, there exists a locally compact Hausdorff space X and a dense LCA-embedding

$$\lambda_A^g : (A,\rho,\mathbb{B}) \longrightarrow (RC(X), \rho_X, CR(X))$$

such that $\{\mathrm{int}(\lambda_A^g(a)) \mid a \in \mathbb{B}\}$ is a base of X. Since $\mathbb{B}$ is closed under finite joins, we get easily (using the compactness of the elements of $CR(X)$) that $\lambda_A^g(\mathbb{B})$ is a dV-dense subset of the CLCA $(RC(X), \rho_X, CR(X))$. Hence the pair

$$(\lambda_A^g, (RC(X), \rho_X, CR(X)))$$

is an LCA-completion of the LCA $(A,\rho,\mathbb{B})$.

We will now prove the uniqueness (up to equivalence) of the LCA-completion. Let $(\varphi, (B,\eta,\mathbb{B}'))$ be an LCA-completion of the LCA $(A,\rho,\mathbb{B})$. Then, as we have already mentioned, (φ, B) is a minimal completion of A, i.e., the Boolean algebra B is determined uniquely (up to isomorphism) by the Boolean algebra A. We can suppose without loss of generality that $A \subseteq B$ and $\varphi(a) = a$, for every $a \in A$. Thus A is a Boolean subalgebra of B.

As we have already shown (see Lemma 2.3.4.4(a)), $\mathbb{B}' = \downarrow_B (\mathbb{B})$, i.e., the set $\mathbb{B}'$ is uniquely determined by the set $\mathbb{B}$.

We have that $\eta_{|A} = \rho$. We will show that the relation η on B is uniquely determined by the relation ρ on A. There are two cases.

Case 1. Let $a_1 \in \mathbb{B}'$ and $b_1 \in B$. We will prove that $a_1 \ll_\eta b_1$ iff there exist $a, b \in \mathbb{B}$ such that $a_1 \leq a \ll_\rho b \leq b_1$. By (BC1), it is enough to prove this for $b_1 \in \mathbb{B}'$.

So, let $a_1, b_1 \in \mathbb{B}'$ and $a_1 \ll_\eta b_1$. Then, using dV-density of $\mathbb{B}$ in $(B,\eta,\mathbb{B}')$ and Fact 2.3.4.2, we get that there exist $a, b \in \mathbb{B}$ such that $a_1 \leq a \ll_\eta b \leq b_1$. Then $a \ll_\rho b$.

The converse assertion is clear because, for every $a, b \in A$, $a \ll_\rho b$ iff $a \ll_\eta b$.

Case 2. Let $a_1 \in B \setminus \mathbb{B}'$ and $b_1 \in B$. We will prove that $a_1 \ll_\eta b_1$ iff (for every prime element J of $I(A,\rho,\mathbb{B})$) [(there exists $a \in \downarrow_B (\mathbb{B}) \setminus \downarrow_B (J)$ such that $a \ll_\eta a_1^*$) or (there exists $b \in\downarrow_B (\mathbb{B})\setminus \downarrow_B (J)$ such that $b \ll_\eta b_1)]$. Note that the inequalities $a \ll_\eta a_1^*$ and $b \ll_\eta b_1$ from the above formula are already expressed in Case 1 in a form which depends only of $(A, \rho, \mathbb{B})$ (because $a, b \in \mathbb{B}'$). Hence, Case 1 and Case 2 will imply that the relation η on B is uniquely determined by the relation ρ on A.

So, let $a_1 \in B \setminus \mathbb{B}'$ and $b_1 \in B$. Then using Proposition 2.2.3.5, Proposition 2.2.3.7 and Lemma 2.3.4.4, we get that $a_1 \ll_\eta b_1$ iff $a_1(-\eta)b_1^*$ iff [(for every $\sigma \in \Lambda^a(B, \eta, \mathbb{B}'))(\{a_1, b_1^*\} \not\subseteq \sigma)$] iff (for every prime element J' of $I(B, \eta, \mathbb{B}'))$[(there exists $a \in \mathbb{B}' \setminus J'$ such that $a(-\eta)a_1)$ or (there exists $b \in \mathbb{B}' \setminus J'$ such that $b(-\eta)b_1^*)$] iff (for every prime element J of $I(A, \rho, \mathbb{B})$) [(there exists $a \in\downarrow_B (\mathbb{B})\setminus \downarrow_B (J)$ such that $a \ll_\eta a_1^*$) or (there exists $b \in \downarrow_B (\mathbb{B})\setminus \downarrow_B (J)$ such that $b \ll_\eta b_1)$].

Let now $(\varphi_1, (A_1, \rho_1, \mathbb{B}_1))$ and $(\varphi_2, (A_2, \rho_2, \mathbb{B}_2))$ be two LCA-completions of an LCA $(A, \rho, \mathbb{B})$. Then, since (φ_i, A_i), for $i = 1, 2$, are minimal completions of A, there exists a Boolean isomorphism $\varphi : A_1 \longrightarrow A_2$ such that $\varphi \circ \varphi_1 = \varphi_2$. The preceding considerations imply that $\mathbb{B}_i =\downarrow_{A_i} (\varphi_i(\mathbb{B}))$, for $i = 1, 2$. From this we easily get that $\varphi(\mathbb{B}_1) = \mathbb{B}_2$. Further, for $a_i \in \mathbb{B}_i, b_i \in A_i$, $i = 1, 2$, we have that $a_i \ll_{\rho_i} b_i$ iff there exists $a_i', b_i' \in \mathbb{B}$ such that $a_i' \ll_\rho b_i'$, $a_i \leq \varphi_i(a_i')$ and $\varphi_i(b_i') \leq b_i$, for $i = 1, 2$. Finally, for $a_i \in A_i \setminus \mathbb{B}_i, b_i \in A_i$, $i = 1, 2$, we have that $a_i \ll_{\rho_i} b_i$ iff (for every prime element J of $I(A, \rho, \mathbb{B})$) [(there exists $a_i' \in \mathbb{B}_i\setminus \downarrow_{A_i} (\varphi_i(J))$ such that $a_i' \ll_{\rho_i} a_i^*$) or (there exists $b_i' \in \mathbb{B}_i\setminus \downarrow_{A_i} (\varphi_i(J))$ such that $b_i' \ll_{\rho_i} b_i)$]. Having in mind these formulas, it is easy to conclude that φ is an LCA-isomorphism. Hence the LCA-completions $(\varphi_1, (A_1, \rho_1, \mathbb{B}_1))$ and $(\varphi_2, (A_2, \rho_2, \mathbb{B}_2))$ of $(A, \rho, \mathbb{B})$ are equivalent. □

Corollary 2.3.4.6. *Let $(A, \rho, \mathbb{B})$ be a local contact algebra and $(B, \eta, \mathbb{B}')$ be a CLCA. Then $\Lambda^a(A, \rho, \mathbb{B})$ is homeomorphic to $\Lambda^a(B, \eta, \mathbb{B}')$ if and only if there exists an LCA-embedding $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ such that $\varphi(\mathbb{B})$ is a dV-dense subset of $(B, \eta, \mathbb{B}')$.*

Proof. ($\Rightarrow$) In the proof of Theorem 2.3.4.5, we have seen that the set $\lambda_A^g(\mathbb{B})$ is dV-dense in $\Lambda^t(\Lambda^a(A, \rho, \mathbb{B}))$. Since $\Lambda^t(\Lambda^a(A, \rho, \mathbb{B}))$ is LCA-isomorphic to $\Lambda^t(\Lambda^a(B, \eta, \mathbb{B}'))$ and $(B, \eta, \mathbb{B}') \cong \Lambda^t(\Lambda^a(B, \eta, \mathbb{B}'))$, we get that there exists an LCA-embedding $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ such that $\varphi(\mathbb{B})$ is a dV-dense subset of $(B, \eta, \mathbb{B}')$.

($\Leftarrow$) By the proof of Theorem 2.3.4.5, $(\lambda_A^g, \Lambda^t(\Lambda^a(A, \rho, \mathbb{B})))$ is an LCA-completion of $(A, \rho, \mathbb{B})$. Since the hypothesis of our assertion imply that the pair $(\varphi, (B, \eta, \mathbb{B}'))$ is also an LCA-completion of $(A, \rho, \mathbb{B})$, we get, by Theorem 2.3.4.5, that the CLC-algebras $\Lambda^t(\Lambda^a(A, \rho, \mathbb{B}))$ and $(B, \eta, \mathbb{B}')$ are LCA-isomorphic. Then

$$\Lambda^a(B, \eta, \mathbb{B}') \cong \Lambda^a(\Lambda^t(\Lambda^a(A, \rho, \mathbb{B}))) \cong \Lambda^a(A, \rho, \mathbb{B}).$$

□

Corollary 2.3.4.7. *Let* $(A, \rho, \mathbb{B})$ *and* $(A', \rho', \mathbb{B}')$ *be local contact algebras. Then the space* $\Lambda^a(A, \rho, \mathbb{B})$ *is homeomorphic to the space* $\Lambda^a(A', \rho', \mathbb{B}')$ *iff there exists a CLCA* $(B, \eta, \mathbb{B}'')$ *and LCA-embeddings* $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}'')$ *and* $\varphi' : (A', \rho', \mathbb{B}') \longrightarrow (B, \eta, \mathbb{B}'')$ *such that the sets* $\varphi(\mathbb{B})$ *and* $\varphi'(\mathbb{B}')$ *are dV-dense in* $(B, \eta, \mathbb{B}'')$*.*

Proof. ($\Rightarrow$) Set $(B, \eta, \mathbb{B}'') = \Lambda^t(\Lambda^a(A, \rho, \mathbb{B}))$. Then, by the hypothesis of our assertion, there exists an LCA-isomorphism $\psi : \Lambda^t(\Lambda^a(A', \rho', \mathbb{B}')) \longrightarrow (B, \eta, \mathbb{B}'')$. Now, it is clear that the maps λ^g_A and $\psi \circ \lambda^g_{A'}$ are the required LCA-embeddings.
($\Leftarrow$) By Corollary 2.3.4.6, we have that

$$\Lambda^a(A, \rho, \mathbb{B}) \cong \Lambda^a(B, \eta, \mathbb{B}'') \cong \Lambda^a(A', \rho', \mathbb{B}').$$

□

2.4 An extension of de Vries' Duality to the category of locally compact Hausdorff spaces and perfect maps

2.4.1 Introduction

In this section, a category **DPHLC** dually equivalent to the category **PHLC** of all locally compact Hausdorff spaces and all perfect maps between them will be defined. In this way, we will obtain one more generalization of the Duality Theorem of H. de Vries [24].

The structure of the section is the following. In the second subsection, we prove our Duality Theorem for the category **PHLC** using Theorem 2.2.2.12. In the third subsection, we present a proof of this duality theorem based on the de Vries Duality Theorem only (i.e., in this new proof we will not use Theorem 2.2.2.12). This new proof is used further for obtaining some new assertions. In the last forth subsection, we derive some corollaries from the Duality Theorem proved in the previous two subsections and from some theorems of de Vries' connected with his Duality Theorem; we discuss the axioms of the category **DPHLC** (and of the category **DHLC**) and its categorical properties as well.

The results of this section are based on the papers [27] and [31].

2.4.2 The Duality Theorem for the category PHLC as a corollary of Theorem 2.2.2.12

Definition 2.4.2.1. Let $\mathbf{D_1PHLC}$ be the cofull subcategory of the category **DHLC** whose morphisms are all **DHLC**-morphisms $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ satisfying the following condition:

(PAL5) If $a \in \mathbb{B}$ then $\varphi(a) \in \mathbb{B}'$.

It is obvious that the identity morphisms in the category **DHLC** satisfy condition (PAL5) and the composition of two **DHLC**-morphisms satisfying condition (PAL5) satisfies condition (PAL5) as well (use Lemma 2.2.2.9(g)). So, $\mathbf{D_1PHLC}$ is indeed a subcategory of the category **DHLC**.

We will now show, using our Theorem 2.2.2.12, that the categories $\mathbf{D_1PHLC}$ and **PHLC** are dually equivalent.

Theorem 2.4.2.2. *The categories* **PHLC** *and* $\mathbf{D_1PHLC}$ *are dually equivalent.*

Proof. We will show that the restriction Λ^a_p of the duality functor $\Lambda^a : \mathbf{DHLC} \longrightarrow \mathbf{HLC}$, constructed in Theorem 2.2.2.12, to the subcategory $\mathbf{D_1PHLC}$ of the category **DHLC** is the desired duality functor between the categories $\mathbf{D_1PHLC}$ and **PHLC**.

Let $f \in \mathbf{PHLC}(X, Y)$. Then, by Theorem 2.2.2.12, $\varphi_f = \Lambda^t(f)$ is a **DHLC**-morphism between the LCAs $(RC(Y), \rho_Y, CR(Y))$ and $(RC(X), \rho_X, CR(X))$. We will show that φ_f is a $\mathbf{D_1PHLC}$-morphism, i.e., that φ_f satisfies, in addition, condition (PAL5). So, let $G \in CR(Y)$. Then $\varphi_f(G) = \mathrm{cl}_X(f^{-1}(\mathrm{int}(G)))$ (see Theorem 2.2.2.12), and hence $\varphi_f(G) \subseteq f^{-1}(G)$. Since f is a perfect map, $f^{-1}(G)$ is a compact subset of X. Thus $\varphi_f(G) \in CR(X)$. Therefore, condition (PAL5) is fulfilled.

Let now $\varphi \in \mathbf{D_1PHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$ and $f_\varphi = \Lambda^a(\varphi)$ (see Theorem 2.2.2.12). Let $X = \Lambda^a(A, \rho, \mathbb{B})$ and $Y = \Lambda^a(B, \eta, \mathbb{B}')$. Then, by Theorem 2.2.2.12, $f_\varphi : Y \longrightarrow X$ is a continuous map. We will show that f_φ is a perfect map. Let us first prove that for every $a \in \mathbb{B}$, $f_\varphi^{-1}(\lambda^g_A(a))$ is a compact subset of Y. Indeed, let $a \in \mathbb{B}$. Then, by condition (BC1), there exists $b \in \mathbb{B}$ such that $a \ll_\rho b$. We will show that $f_\varphi^{-1}(\lambda^g_A(a)) \subseteq \lambda^g_B(\varphi(b))$. So, let $\sigma' \in f_\varphi^{-1}(\lambda^g_A(a))$. Then $f_\varphi(\sigma') \in \lambda^g_A(a)$. Hence $a \in \mathbb{B} \cap f_\varphi(\sigma')$. This implies, by Theorem 2.2.2.12, that $\varphi(b) \in \sigma'$. Thus $\sigma' \in \lambda^g_B(\varphi(b))$. Since, by (PAL5), $\varphi(b) \in \mathbb{B}'$, we have that $\lambda^g_B(\varphi(b))$ is compact. Hence $f_\varphi^{-1}(\lambda^g_A(a))$ is compact. So, for every $a \in \mathbb{B}$, $f_\varphi^{-1}(\lambda^g_A(a))$ is a compact subset of Y. Now, using the fact that the family $\{\mathrm{int}(\lambda^g_A(a)) \mid a \in \mathbb{B}\}$ is an open base of X, we conclude that $f_\varphi^{-1}(K)$ is compact for every compact subset K of X. Then the local compactness of X

implies that f_φ is a perfect map (see [53, Theorem 3.7.18]). Therefore, we have proved that $f_\varphi \in \mathbf{PHLC}(Y,X)$.

The rest follows from Theorem 2.2.2.12. $\square$

Definition 2.4.2.3. Let **DPHLC** be the category whose objects are all complete LC-algebras and whose morphisms are all functions $\varphi : (A,\rho,\mathbb{B}) \longrightarrow (B,\eta,\mathbb{B}')$ between the objects of **DPHLC** satisfying the axioms (DLC1)-(DLC4), (PAL5) and the following condition:

(PAL6) $\varphi(a) = \bigvee\{\varphi(b) \mid b \ll_{C_\rho} a\}$, for every $a \in A$;

let the composition "$*$" of two morphisms $\varphi_1 : (A_1,\rho_1,\mathbb{B}_1) \longrightarrow (A_2,\rho_2,\mathbb{B}_2)$ and $\varphi_2 : (A_2,\rho_2,\mathbb{B}_2) \longrightarrow (A_3,\rho_3,\mathbb{B}_3)$ of **DPHLC** be defined by the formula

(2.14) $\varphi_2 * \varphi_1 = (\varphi_2 \circ \varphi_1)\tilde{}$,

where, for every function $\psi : (A,\rho,\mathbb{B}) \longrightarrow (B,\eta,\mathbb{B}')$ between two objects of **DPHLC**,

$$\psi\tilde{} : (A,\rho,\mathbb{B}) \longrightarrow (B,\eta,\mathbb{B}')$$

is defined as follows:

(2.15) $\psi\tilde{}(a) = \bigvee\{\psi(b) \mid b \ll_{C_\rho} a\}$,

for every $a \in A$.

By **DVHC** we denote the full subcategory of **DPHLC** having as objects all CNC-algebras (i.e., those complete LC-algebras $(A,\rho,\mathbb{B})$ for which $\mathbb{B} = A$).

Remark 2.4.2.4. Note that the categories **DHC** and **DVHC** are isomorphic (it can be even said that they are identical) because the axiom (PAL5) is trivially fulfilled in the category **DHC** (indeed, all elements of its objects are bounded), the axiom (DLC4) follows immediately from the obvious fact that $\varphi(1) = 1$ for every **DHC**-morphism φ (see Lemma 2.2.2.9(c)), and the compositions are the same.

The fact that **DPHLC** is indeed a category will be proved now. We will show that **DPHLC** is, in fact, the category $\mathbf{D_1PHLC}$ and, thus, **DPHLC** is a category.

Using Lemma 2.2.3.22, we obtain immediately the following assertion:

Fact 2.4.2.5. *The system of axioms (DLC2), (DLC4), (PAL6) is equivalent to the system of axioms (DLC2), (DLC4), (DLC5).*

Also, the following fact holds:

Fact 2.4.2.6. *The compositions in* **DPHLC** *and* $\mathbf{D_1PHLC}$ *coincide.*

Proof. Let φ_1, φ_2 be **DPHLC**-morphisms. We have to show that $\varphi_2 * \varphi_1 = \varphi_2 \diamond \varphi_1$. Set $\psi = \varphi_2 \circ \varphi_1$. Then one obtains immediately that ψ satisfies conditions (DLC2) and (DLC4). Now we can apply Lemma 2.2.3.22. $\square$

All this proves the following assertion:

Proposition 2.4.2.7. *The categories* $\mathbf{D_1PHLC}$ *and* **DPHLC** *coincide.*

Thus we get the following result:

Theorem 2.4.2.8. *The categories* **PHLC** *and* **DPHLC** *are dually equivalent.*

Proof. It follows from Theorem 2.4.2.2 and Proposition 2.4.2.7. $\square$

So, we obtained Theorem 2.4.2.8 as a corollary of Theorem 2.2.2.12.

Theorem 2.4.2.8 implies that the category **HC** of all compact Hausdorff spaces and continuous maps and the category **DVHC** are dually equivalent (just take the restriction of the duality functor Λ^t to the subcategory **HC**). Now, using Remark 2.4.2.4, we get that the categories **HC** and **DHC** are dually equivalent. Obviously, de Vries' duality functor Φ^t coincides with the restriction of the duality functor Λ^t to the subcategory **HC**. Therefore, Theorem 2.4.2.8 is a generalization of de Vries' Duality Theorem.

2.4.3 A direct proof of the Duality Theorem for the category PHLC

In this subsection, we will present a new proof of Theorem 2.4.2.8 based on the de Vries Duality Theorem only. We will later use this new proof for obtaining another results.

Proposition 2.4.3.1. *Let X be a locally compact Hausdorff space. Then the NCAs $(RC(X), C_{\rho_X})$ and $(RC(\alpha X), \rho_{\alpha X})$ are CA-isomorphic (see 1.2.3.4 and 1.2.3.8 for the notation) and the maps $e_{X,\alpha X}$, $r_{X,\alpha X}$ are CA-isomorphisms between them (see 0.4.2.2 for the notation).*

Proof. Obviously, we can suppose that $X \subseteq \alpha X$. By 0.4.2.2, we have only to show that $AC_{\rho_X}B$ iff $\mathrm{cl}_{\alpha X}(A)\rho_{\alpha X}\mathrm{cl}_{\alpha X}(B)$, for every $A, B \in RC(X)$. This follows easily from the respective definitions. Hence, the map $e_{X,\alpha X} : (RC(X), C_{\rho_X}) \longrightarrow (RC(\alpha X, \rho_{\alpha X})$ is a CA-isomorphism. Thus the map $r_{X,\alpha X}$ is also a CA-isomorphism. $\square$

Lemma 2.4.3.2. *Let $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ be a function between two CLCAs. Let φ satisfy conditions (DLC3) and (PAL5). Then, for every $a, b \in A$, $a \ll_{C_\rho} b$ implies that $(\varphi(a^*))^* \ll_{C_\eta} \varphi(b)$. Hence, if, in addition, φ satisfies conditions (DLC1) and (DLC2), then $\varphi(a) \ll_{C_\eta} \varphi(b)$.*

Proof. Let $a, b \in A$ and $a \ll_{C_\rho} b$. Then $a \ll_\rho b$ and at least one of the elements a and b^* is bounded.

Let $a \in \mathbb{B}$. Then (DLC3) implies that $(\varphi(a^*))^* \ll_\eta \varphi(b)$. By (BC1), there exists $c \in \mathbb{B}$ such that $a \ll_\rho c$. Hence, using again (DLC3), we get that $(\varphi(a^*))^* \ll_\eta \varphi(c)$. Since $\varphi(c) \in \mathbb{B}'$ (according to (PAL5)), we obtain that $(\varphi(a^*))^* \in \mathbb{B}'$. Therefore, $(\varphi(a^*))^* \ll_{C_\eta} \varphi(b)$.

Let now $b^* \in \mathbb{B}$. Since $b^* \ll_{C_\rho} a^*$, we get, by the previous case, that $(\varphi(b))^* \ll_{C_\eta} \varphi(a^*)$. Thus $(\varphi(a^*))^* \ll_{C_\eta} \varphi(b)$.

If, in addition, φ satisfies conditions (DLC1) and (DLC2), then Lemma 2.2.2.9(b) and the just obtained result imply the last assertion of our lemma. □

2.4.3.3 (*A new proof of Theorem 2.4.2.8.*)**.** In the proof of Roeper's Theorem 1.2.3.10, two correspondences $\Psi^t : |\mathbf{PHLC}| \longrightarrow |\mathbf{DPHLC}|$ and $\Psi^a : |\mathbf{DPHLC}| \longrightarrow |\mathbf{PHLC}|$ between the objects of the categories **PHLC** and **DPHLC** were defined (see (1.20) for Ψ^t and (1.21), (1.24) for Ψ^a). We will extend them to the morphisms of these categories, constructing in this way two contravariant functors (having, for simplicity, the same names)

$$\Psi^a : \mathbf{DPHLC} \longrightarrow \mathbf{PHLC} \text{ and } \Psi^t : \mathbf{PHLC} \longrightarrow \mathbf{DPHLC}.$$

I. The definition of Ψ^t.

Let $f : (X, \tau) \longrightarrow (Y, \tau') \in \mathbf{PHLC}(X, Y)$. We set

(2.16) $\Psi^t(f) : \Psi^t(Y, \tau') \longrightarrow \Psi^t(X, \tau)$, $\;\Psi^t(f)(F) = cl_X(f^{-1}(\mathrm{int}_Y(F)))$.

Put, for the sake of brevity, $\varphi_f = \Psi^t(f)$. We have to show that φ_f is a **DPHLC**-morphism. Obviously, (DLC1) is fulfilled. Exactly as in the proof of Lemma 2.2.3.18, we can verify that φ_f satisfies conditions (DLC2)-(DLC4). The fact that φ_f satisfies condition (PAL5) can be established as in the proof of Theorem 2.4.2.2.

By 0.4.2.4, f has a continuous extension $\alpha(f) : \alpha X \longrightarrow \alpha Y$. Set $\varphi_{\alpha f} = \Phi^t(\alpha(f))$ (see Theorem 2.2.2.6 for Φ^t). Then, by Theorem 2.2.2.6, $\varphi_{\alpha f}$ is a **DHC**-morphism. We will prove that

(2.17) $r_{X,\alpha X} \circ \varphi_{\alpha f} = \varphi_f \circ r_{Y,\alpha Y}$

(see 0.4.2.2 and 2.4.3.1 for the notation), i.e., that, for every $G \in RC(Y)$, the following equality holds:

(2.18) $X \cap \varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(G)) = \varphi_f(G)$,

or, in other words, that

$$X \cap \mathrm{cl}_{\alpha X}((\alpha(f))^{-1}(\mathrm{int}_{\alpha Y}(\mathrm{cl}_{\alpha Y}(G)))) = \mathrm{cl}_X(f^{-1}(\mathrm{int}_Y(G))).$$

Since the last equality follows easily from the obvious inclusions $\mathrm{int}_Y(G) \cup \{\infty_Y\} \supseteq \mathrm{int}_{\alpha Y}(\mathrm{cl}_{\alpha Y}(G)) \supseteq \mathrm{int}_Y(G)$, (2.17) is proved. Therefore,

$$\varphi_f = r_{X,\alpha X} \circ \varphi_{\alpha f} \circ e_{Y,\alpha Y}.$$

Now, we will verify (PAL6). Let $F \in RC(Y)$; then $\mathrm{cl}_{\alpha Y}(F) \in RC(\alpha Y)$ and hence, by (DVAL4),

$$\varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(F)) = \bigvee \{\varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(G)) \mid G \in RC(Y), \mathrm{cl}_{\alpha Y}(G) \ll_{\rho_{\alpha Y}} \mathrm{cl}_{\alpha Y}(F)\}.$$

Since $r_{X,\alpha X}$ is an isomorphism, we obtain that

$$r_{X,\alpha X}(\varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(F))) =$$

$$= \bigvee \{r_{X,\alpha X}(\varphi_{\alpha f}(\mathrm{cl}_{\alpha Y}(G))) \mid G \in RC(Y), \mathrm{cl}_{\alpha Y}(G) \ll_{\rho_{\alpha Y}} \mathrm{cl}_{\alpha Y}(F)\}.$$

Thus, (2.17) and 2.4.3.1 imply that

$$\varphi_f(F) = \bigvee \{\varphi_f(G) \mid G \in RC(Y), G \ll_{C_{\rho_Y}} F\}.$$

So, (PAL6) is fulfilled.

Therefore, φ_f is a **DPHLC**-morphism.

Let $f \in \mathbf{PHLC}(X, Y)$ and $g \in \mathbf{PHLC}(Y, Z)$. We will prove that

$$\Psi^t(g \circ f) = \Psi^t(f) * \Psi^t(g).$$

Put $h = g \circ f$, $\varphi_h = \Psi^t(h)$, $\varphi_f = \Psi^t(f)$ and $\varphi_g = \Psi^t(g)$. Set also $e_X = e_{X,\alpha X}$ and $r_Z = r_{Z,\alpha Z}$. Let $\alpha(f) : \alpha X \longrightarrow \alpha Y$, $\alpha(g) : \alpha Y \longrightarrow \alpha Z$ and $\alpha(h) : \alpha X \longrightarrow \alpha Z$ be the continuous extensions of f, g and h, respectively (see 0.4.2.4). Then, obviously, $\alpha(h) = \alpha(g) \circ \alpha(f)$. Set $\varphi_{\alpha f} = \Phi^t(\alpha(f))$, $\varphi_{\alpha g} = \Phi^t(\alpha(g))$ and $\varphi_{\alpha h} = \Phi^t(\alpha(h))$. Then, by Theorem 2.2.2.6,

$$\varphi_{\alpha h} = (\varphi_{\alpha f} \circ \varphi_{\alpha g})\check{}.$$

Now, using (2.17) and 0.4.2.2, we get that

$$e_X \circ \varphi_h \circ r_Z = \varphi_{\alpha h} = (e_X \circ \varphi_f \circ \varphi_g \circ r_Z)\check{}.$$

Thus, for every $F \in RC(\alpha Z)$, we have that

$$\varphi_h(r_Z(F)) = \bigvee\{(\varphi_f \circ \varphi_g)(r_Z(G)) \mid G \ll_{\rho_{\alpha Z}} F\}.$$

Now, 0.4.2.2 and 2.4.3.1 imply that $\varphi_h = (\varphi_f \circ \varphi_g)\check{}$, i.e., $\varphi_h = \varphi_f * \varphi_g$.

So, $\Psi^t : \mathbf{PHLC} \longrightarrow \mathbf{DPHLC}$ is a contravariant functor.

II. The definition of Ψ^a.

Let $\varphi \in \mathbf{DPHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$. We define the map

$$\Psi^a(\varphi) : \Psi^a(B, \eta, \mathbb{B}') \longrightarrow \Psi^a(A, \rho, \mathbb{B})$$

by the formula

(2.19) $\Psi^a(\varphi)(\sigma') = \{a \in A \mid \text{ if } b \ll_{C_\rho} a^* \text{ then } (\varphi(b))^* \in \sigma'\}$,

for every bounded cluster σ' in (B, C_η). Set, for the sake of brevity,

$$\Psi^a(\varphi) = f_\varphi, \ X = \Psi^a(A, \rho, \mathbb{B}) \text{ and } Y = \Psi^a(B, \eta, \mathbb{B}').$$

We will show that

$$f_\varphi : Y \longrightarrow X$$

is well-defined and is a perfect map.

Let $\varphi_C : (A, C_\rho) \longrightarrow (B, C_\eta)$ be defined by $\varphi_C(a) = \varphi(a)$, for every $a \in A$. Then φ_C is a **DHC**-morphism. Indeed, (DVAL3) follows from Lemma 2.4.3.2 and the fact that φ satisfies the axiom (PAL5); the other three axioms are obviously fulfilled. Set $f_\alpha = \Phi^a(\varphi_C)$. Then $f_\alpha : \alpha Y \longrightarrow \alpha X$ (see Theorem 2.2.2.6 and the proof of Theorem 1.2.3.10). The definitions of f_φ and f_α coincide on the bounded clusters of (B, C_η) (see (2.19) and Theorem 2.2.2.6); hence, the right side of the formula (2.19) defines a cluster in (A, C_ρ) and f_α is an extension of f_φ. Thus, if we show that $f_\alpha^{-1}(\infty_X) = \{\infty_Y\}$, the map f_φ will be well-defined and will be a perfect map. Let us prove that $f_\alpha(Y) \subseteq X$, i.e., that if σ' is a bounded cluster in (B, C_η) then $\sigma = f_\alpha(\sigma') = f_\varphi(\sigma')$ is a bounded cluster in (A, C_ρ). So, let σ' be a bounded cluster in (B, C_η) and $\sigma = f_\alpha(\sigma')$. Then 2.2.3.1 implies that there exists $b \in \mathbb{B}'$ such that $b^* \notin \sigma'$. By (DLC4), there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$. Thus $(\varphi(a))^* \leq b^*$ and hence $(\varphi(a))^* \notin \sigma'$. By (BC1), there exists $a_1 \in \mathbb{B}$ such that $a \ll_\rho a_1$. Then $a \ll_{C_\rho} a_1$ and, by the definition of σ, $a_1^* \notin \sigma$. Therefore $a_1 \in \mathbb{B} \cap \sigma$, i.e., σ is a bounded cluster in (A, C_ρ). Hence $f_\varphi(Y) = f_\alpha(Y) \subseteq X$. Further, we have (by 1.2.3.7) that $\infty_X = A \setminus \mathbb{B}$ and $\infty_Y = B \setminus \mathbb{B}'$. Let us show that $f_\alpha(\infty_Y) = \infty_X$. Set $\sigma' = \infty_Y$ and $\sigma = f_\alpha(\sigma')$. Let $a \in \sigma$. Suppose that $a \in \mathbb{B}$. Then, by (BC1), there exist $a_1, a_2 \in \mathbb{B}$ such that $a \ll_\rho a_1 \ll_\rho a_2$. Thus $a \ll_{C_\rho} a_1 \ll_{C_\rho} a_2$. Hence $a_1^* \ll_{C_\rho} a^*$. Since $a \in \sigma$, the definition of σ implies that $(\varphi(a_1^*))^* \in \sigma'$. Further, we have that $a_1 \ll_\rho a_2$; thus, by (DLC3), we obtain that $(\varphi(a_1^*))^* \leq \varphi(a_2)$. Therefore, $\varphi(a_2) \in \sigma'$. Since $\varphi(a_2) \in \mathbb{B}'$ (by (PAL5)), we obtain a contradiction. Thus $\sigma \subseteq A \setminus \mathbb{B}$. Now, 1.2.3.7 and 1.2.2.5 imply that $\sigma = A \setminus \mathbb{B}$, i.e., $f_\alpha(\infty_Y) = \infty_X$. Hence $f_\alpha^{-1}(X) = Y$. This shows that f_φ is a perfect map (because f_α is such) (see [53, Proposition 3.7.4]). So, we have proved that $f_\varphi \in \mathbf{PHLC}(Y, X)$.

Let $\varphi_i \in \mathbf{DPHLC}((A_i, \rho_i, \mathbb{B}_i), (A_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1}))$ and $f_i = \Psi^a(\varphi_i)$ for $i = 1, 2$, $\varphi = \varphi_2 * \varphi_1$, $f_\varphi = \Psi^a(\varphi)$ and $X_i = \Psi^a(A_i, \rho_i, \mathbb{B}_i)$ for $i = 1, 2, 3$. We will prove that $f_\varphi = f_1 \circ f_2$. Let $\varphi_{iC} : (A_i, C_{\rho_i}) \longrightarrow (A_{i+1}, C_{\rho_{i+1}})$ be defined by $\varphi_{iC}(a) = \varphi_i(a)$ for every $a \in A_i$, where $i = 1, 2$. Then, as we know, φ_{iC} is a **DHC**-morphism, for $i = 1, 2$. Set $f_{i\alpha} = \Phi^a(\varphi_{iC})$ for $i = 1, 2$, $\psi = \varphi_{2C} \diamond \varphi_{1C}$, $f_\psi = \Phi^a(\psi)$. Let $\varphi_C : (A_1, C_{\rho_1}) \longrightarrow (A_3, C_{\rho_3})$ be defined by $\varphi_C(a) = \varphi(a)$ for every $a \in A_1$. From the respective definitions we obtain that, for every $a \in A_1$, $\psi(a) = (\varphi_{2C} \circ \varphi_{1C})\check{}(a) = (\varphi_2 \circ \varphi_1)\check{}(a) = \varphi(a)$. Thus, $\psi = \varphi_C$. Hence $f_\psi = \Phi^a(\varphi_C)$. We know that $\Phi^a(A_i, C_{\rho_i}) = \alpha X_i$, for $i = 1, 2, 3$, and $f_{i\alpha}$ is a continuous extension of f_i, for $i = 1, 2$. The equality "$\psi = \varphi_C$" implies that f_ψ is a continuous extension of f_φ. From Theorem 2.2.2.6 we get that $f_\psi = f_{1\alpha} \circ f_{2\alpha}$. Since $f_{1\alpha}^{-1}(X_1) = X_2$ and $f_{2\alpha}^{-1}(X_2) = X_3$, we conclude that $f_\varphi = f_1 \circ f_2$.

We have proved that $\Psi^a : \mathbf{DPHLC} \longrightarrow \mathbf{PHLC}$ is a contravariant functor.

III. $\Psi^a \circ \Psi^t$ is naturally isomorphic to the identity functor $Id_{\mathbf{PHLC}}$.

Recall that, for every $X \in |\mathbf{PHLC}|$, the map $t_X : X \longrightarrow (\Psi^a \circ \Psi^t)(X)$, where $t_X(x) = \sigma_x$ for every $x \in X$, is a homeomorphism (see (1.32) and (1.18)). We will show that $t^{lp} : Id_{\mathbf{PHLC}} \longrightarrow \Psi^a \circ \Psi^t$, where for every $X \in |\mathbf{PHLC}|$, $t^{lp}(X) = t_X$, is a natural isomorphism.

Let $f \in \mathbf{PHLC}(X, Y)$ and $f' = (\Psi^a \circ \Psi^t)(f)$, $X' = (\Psi^a \circ \Psi^t)(X)$, $Y' = (\Psi^a \circ \Psi^t)(Y)$. We have to prove that $t_Y \circ f = f' \circ t_X$. Let $\alpha(f) : \alpha X \longrightarrow \alpha Y$ and $\alpha(f') : \alpha X' \longrightarrow \alpha Y'$ be the continuous extensions of f and f', respectively (see 0.4.2.4). Then, by Theorem 2.2.2.6, we have that $t_{\alpha Y} \circ \alpha(f) = \alpha(f') \circ t_{\alpha X}$. Obviously, $t_{\alpha X}(\infty_X) = \{\mathrm{cl}_{\alpha X}(F) \mid F \in RC(X), \infty_X \in \mathrm{cl}_{\alpha X}(F)\} = \{\mathrm{cl}_{\alpha X}(F) \mid F \in RC(X) \setminus CR(X)\} = \sigma_\infty^{(RC(\alpha X), \rho_{\alpha X})} = \infty_{X'}$, and, analogously, $t_{\alpha Y}(\infty_Y) = \infty_{Y'}$. Using 0.4.2.2 and taking the restrictions on X, we obtain that $t_Y \circ f = f' \circ t_X$, i.e., $Id_{\mathbf{PHLC}} \cong \Psi^a \circ \Psi^t$.

IV. $\Psi^t \circ \Psi^a$ *is naturally isomorphic to the identity functor* $Id_{\mathbf{DPHLC}}$.

Recall that for every $(A, \rho, \mathbb{B}) \in |\mathbf{DPHLC}|$, the function

$$\lambda_A^g : (A, \rho, \mathbb{B}) \longrightarrow (\Psi^t \circ \Psi^a)(A, \rho, \mathbb{B})$$

is an LCA-isomorphism (see (1.31)). We will show that $\lambda^{gp} : Id_{\mathbf{DPHLC}} \longrightarrow \Psi^t \circ \Psi^a$, where for every $(A, \rho, \mathbb{B}) \in |\mathbf{DPHLC}|$, $\lambda^{gp}(A, \rho, \mathbb{B}) = \lambda_A^g$, is a natural isomorphism.

Let $\varphi \in \mathbf{DPHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$ and $\varphi' = (\Psi^t \circ \Psi^a)(\varphi)$, $X = \Psi^a(A, \rho, \mathbb{B})$, $Y = \Psi^a(B, \eta, \mathbb{B}')$. We have to prove that $\lambda_B^g * \varphi = \varphi' * \lambda_A^g$. According to (2.14) and (2.15), it is enough to show that $\lambda_B^g \circ \varphi = \varphi' \circ \lambda_A^g$. Clearly, it is enough to establish this equality on the **Set**-level, i.e., for the corresponding underlying maps. For simplicity, the morphisms and their underlying maps will have, in this proof only, one and the same names; for example, λ_A^g will stay for the **DPHLC**-morphism between $(A, \rho, \mathbb{B})$ and $(\Psi^t \circ \Psi^a)(A, \rho, \mathbb{B})$, and, also, for the underlying **Set**-map from A to $(\Psi^t \circ \Psi^a)(A)$ induced by it. Set $f = \Psi^a(\varphi)$. Hence $\varphi' = \Psi^t(f)$. Let $\varphi_C : (A, C_\rho) \longrightarrow (B, C_\eta)$ be defined by $\varphi_C(a) = \varphi(a)$ for every $a \in A$, and let $(\varphi')_C$ be defined analogously. Then φ_C and $(\varphi')_C$ are **DHC**-morphisms. Set $f_\alpha = \Phi^a(\varphi_C)$ and $(\varphi_C)' = \Phi^t(f_\alpha)$. We know that $f_\alpha : \alpha Y \longrightarrow \alpha X$ is a continuous extension of f. By the proof of Theorem 2.2.2.6, we have that

$$\lambda_B \circ \varphi_C = (\varphi_C)' \circ \lambda_A$$

(see (1.4) for λ_A and λ_B). Note that $\lambda_A : (A, C_\rho) \longrightarrow (RC(\alpha X), \rho_{\alpha X})$ and $\lambda_B : (B, C_\eta) \longrightarrow (RC(\alpha Y), \rho_{\alpha Y})$. We have that $(\varphi')_C : (RC(X), C_{\rho_X}) \longrightarrow (RC(Y), C_{\rho_Y})$ is defined by $(\varphi')_C(F) = \varphi'(F)$, for every $F \in RC(X)$. Then, by (2.17),

$$(\varphi')_C \circ r_{X,\alpha X} = r_{Y,\alpha Y} \circ (\varphi_C)'.$$

By (1.29), we have, on the **Set**-level, that

$$r_{X,\alpha X} \circ \lambda_A = \lambda_A^g \text{ and } r_{Y,\alpha Y} \circ \lambda_B = \lambda_B^g.$$

The last four equalities imply that, on the **Set**-level, $\lambda_B^g \circ \varphi = r_{Y,\alpha Y} \circ \lambda_B \circ \varphi_C = r_{Y,\alpha Y} \circ (\varphi_C)' \circ \lambda_A = (\varphi')_C \circ r_{X,\alpha X} \circ \lambda_A = \varphi' \circ \lambda_A^g$. Therefore $\lambda_B^g \circ \varphi = \varphi' \circ \lambda_A^g$. Thus $Id_{\mathbf{DPHLC}} \cong \Psi^t \circ \Psi^a$. $\square$

2.4.4 Some corollaries of the Duality Theorem for the category PHLC

Theorem 2.4.4.1. *Let φ be a* **DPHLC***-morphism. Then φ is an injection iff $\Psi^a(\varphi)$ is a surjection.*

Proof. Let $\varphi \in \mathbf{DPHLC}((A,\rho,\mathbb{B}),(B,\eta,\mathbb{B}))$ and let $\varphi_C : (A, C_\rho) \longrightarrow (B, C_\eta)$ be defined by the formula $\varphi_C(a) = \varphi(a)$, for every $a \in A$. Then, as we have seen in 2.4.3.3, φ_C is a **DHC**-morphism. Setting $f = \Psi^a(\varphi)$, we obtain that $\alpha(f) = \Phi^a(\varphi_C)$ (see again 2.4.3.3). Obviously, $\alpha(f)$ is a surjection iff f is a surjection. By a theorem of de Vries ([24, Theorem 1.7.1]), $\Phi^a(\varphi_C)$ is a surjection iff φ_C is an injection. Hence, f is a surjection iff φ is an injection. □

It is clear that if we want to build **DPHLC** as a category dually equivalent to the category **PHLC** then the axiom (PAL5) is indispensable for describing the morphisms of the category **DPHLC**. With the next simple example we show that the axiom (DLC4) cannot be dropped in Definitions 2.4.2.3 and 2.2.2.7.

Example 2.4.4.2. Let $(A,\rho,\mathbb{B})$ be a CLCA and $\mathbb{B} \neq A$. Then (A,ρ_s,A) is also a CLCA (by 1.2.1.8). Obviously, the map $i : (A,\rho,\mathbb{B}) \longrightarrow (A,\rho_s,A)$, where $i(a) = a$, for every $a \in A$, satisfies the axioms (DLC1)-(DLC3), (DLC5), (PAL5), (PAL6) but it does not satisfy the axiom (DLC4). If we suppose that our duality theorems 2.4.2.8 and 2.2.2.12 are true without the presence of the axiom (DLC4) in the Definitions 2.4.2.3 and 2.2.2.7, then we will obtain, by Theorem 2.4.4.1, that there exists a continuous map from a compact Hausdorff space onto a locally compact non-compact Hausdorff space, a contradiction.

Fact 2.4.4.3. *For every LCA $(A,\rho,\mathbb{B})$, the triple $(A,\rho_s,\mathbb{B})$ is also an LCA (see 1.2.1.8 for ρ_s); if $(A,\rho,\mathbb{B})$ is a CLCA then the map $i : (A,\rho,\mathbb{B}) \longrightarrow (A,\rho_s,\mathbb{B})$, where $i(a) = a$, for every $a \in A$, is a* **DPHLC***-morphism.*

Proof. We will first check that $(A,\rho_s,\mathbb{B})$ is an LCA. Since $a \ll_{\rho_s} a$, for every $a \in A$, the axiom (BC1) of 1.2.3.1 is clearly fulfilled. Obviously, for every $a,b \in A$, $a \ll_\rho b$ implies $a \ll_{\rho_s} b$. This implies that the axiom (BC3) is also satisfied. For checking (BC2), let $a,b \in A$ and $a\rho_s b$. Then $a \wedge b \neq 0$. Since $b = \bigvee\{c \mid c \in \mathbb{B}, c \ll_\rho b\}$, we have that $b = \bigvee\{c \mid c \in \mathbb{B}, c \wedge b^* = 0\}$. Hence $a \wedge b = \bigvee\{a \wedge c \mid c \in \mathbb{B}, c \wedge b^* = 0\}$. Thus, there exists $c \in \mathbb{B}$ such that $c \wedge b^* = 0$ and $a \wedge c \neq 0$. Therefore, there exists $c \in \mathbb{B}$ such that $a\rho_s(c \wedge b)$. So, $(A,\rho_s,\mathbb{B})$ is an LCA. The rest is clear. □

Recall that a topological space X is said to be *extremally disconnected* if for every open set $U \subseteq X$, the closure $\mathrm{cl}_X(U)$ is open in X. Clearly, a topological space X is extremally disconnected iff $RC(X)$ consists only of clopen sets.

Proposition 2.4.4.4. *Let $(A, \rho, \mathbb{B})$ be a CLCA and $X = \Psi^a(A, \rho, \mathbb{B})$. Then X is an extremally disconnected locally compact Hausdorff space iff $\rho = \rho_s$ (see 1.2.1.8 for ρ_s).*

Proof. Recall that, by (1.31), $\lambda_A^g : (A, \rho, \mathbb{B}) \longrightarrow (RC(X), \rho_X, CR(X))$ is an LCA-isomorphism.

Let X be extremally disconnected. Then, by (1.31), for every $a, b \in A$, $\lambda_A^g(a \wedge b) = \lambda_A^g(a) \wedge \lambda_A^g(b) = \mathrm{cl}(\mathrm{int}(\lambda_A^g(a) \cap \lambda_A^g(b))) = \lambda_A^g(a) \cap \lambda_A^g(b)$. Hence, using once more (1.31), we get that $a \wedge b \neq 0$ iff $a\rho b$. Thus $\rho = \rho_s$ (see 1.2.1.8).

Conversely, let $\rho = \rho_s$. Then, for every $a \in A$, $a \ll_\rho a$. Since for every $a, b \in A$, $a \ll_\rho b$ iff $\lambda_A^g(a) \subseteq \mathrm{int}_X(\lambda_A^g(b))$, we get that for every $a \in A$, $\lambda_A^g(a) \subseteq \mathrm{int}_X(\lambda_A^g(a))$, i.e., $\lambda_A^g(a)$ is a clopen set. Therefore, X is extremally disconnected. □

Note that from 2.4.4.4, 2.4.4.3 and Theorem 2.4.4.1, we obtain immediately an easy proof of the following well-known fact: every locally compact Hausdorff space X is a perfect image of an extremally disconnected locally compact Hausdorff space Y.

Theorem 2.4.4.5. *Let X and Y be two locally compact Hausdorff spaces, $\Psi^t(X) = (A, \rho, \mathbb{B})$ and $\Psi^t(Y) = (B, \eta, \mathbb{B}')$. Then a map $f : X \longrightarrow Y$ is a closed embedding iff the map $\varphi = \Psi^t(f)$ satisfies the following two conditions:*
(1) $\forall a, b \in A$ with $a \ll_{C_\rho} b$ there exists $c \in B$ such that $a \ll_{C_\rho} \varphi(c) \ll_{C_\rho} b$;
(2) $\forall a, b \in B$, $\varphi(a) \ll_{C_\rho} \varphi(b)$ iff there exist $a_1, b_1 \in B$ such that $a_1 \ll_{C_\eta} b_1$ and $\varphi(a_1) = \varphi(a)$, $\varphi(b_1) = \varphi(b)$.

Proof. Obviously, $f : X \longrightarrow Y$ is a closed embedding iff the map $\alpha(f) : \alpha X \longrightarrow \alpha Y$ is an embedding (note that every closed embedding is a perfect map and see 0.4.2.4 for $\alpha(f)$). De Vries proved (see [24, Theorem 1.7.3]) that $\alpha(f)$ is an embedding iff the following two conditions are satisfied: (a) for every $F, G \in RC(\alpha X)$ with $F \ll_{\rho_{\alpha X}} G$, there exists $H \in \Phi^t(\alpha(f))(RC(\alpha Y))$ such that $F \ll_{\rho_{\alpha X}} H \ll_{\rho_{\alpha X}} G$, and (b) for every $F, G \in RC(\alpha Y)$, $\Phi^t(\alpha(f))(F) \ll_{\rho_{\alpha X}} \Phi^t(\alpha(f))(G)$ iff there exist $F_1, G_1 \in RC(\alpha Y)$ such that $F_1 \ll_{\rho_{\alpha Y}} G_1$ and $\Phi^t(\alpha(f))(F_1) = \Phi^t(\alpha(f))(F)$, $\Phi^t(\alpha(f))(G_1) = \Phi^t(\alpha(f))(G)$. Now, using 2.4.3.1 and (2.17), it is easy to obtain that f is a closed embedding iff φ satisfies conditions (1) and (2). □

Notation 2.4.4.6. Let us denote by **PHLCCon** the full subcategory of the category **PHLC** whose objects are all connected locally compact Hausdorff spaces.

Let **DPHLCCon** be the full subcategory of the category **DPHLC** whose objects are all connected CLCAs.

Theorem 2.4.4.7. *The categories* **PHLCCon** *and* **DPHLCCon** *are dually equivalent.*

Proof. It follows immediately from Theorem 2.4.2.8 and Fact 2.2.4.3.

Remark 2.4.4.8. Let's note that the category **PHLC** has no coseparators (= cogenerators) (see 0.2.1.11 for this notion). Indeed, suppose that Z is a coseparator in **PHLC**. Let $D(\tau)$ be an infinite discrete space of cardinality τ and $f, g : D(\tau) \longrightarrow D(\tau)$ be two distinct functions with finite preimages. Then $f, g \in \mathbf{PHLC}(D(\tau), D(\tau))$ and thus there exists an $h \in \mathbf{PHLC}(D(\tau), Z)$ such that $h \circ f \neq h \circ g$. Since h is a perfect map, the preimages of all points of Z are finite. Therefore, $|Z| \geq |h(D(\tau))| = \tau$. Since all discrete spaces are objects of the category **PHLC**, we obtain that $|Z| \geq \tau$ for any cardinal τ, a contadiction. So, the category **PHLC** has no coseparators. Thus, if $V : \mathbf{DPHLC} \longrightarrow \mathbf{Set}$ is a faithful functor then the contravariant functor $V \circ \Psi^t$ is not representable (see 0.2.1.12 for this notion). (Indeed, suppose that the contravariant functor $V \circ \Psi^t$ is representable; then $V \circ \Psi^t$ will be naturally isomorphic to a contravariant hom-functor $\mathbf{PHLC}(-, X)$ for some **PHLC**-object X and thus, by 0.2.2.2, X will be a coseparator of the category **PHLC** (because $V \circ \Psi^t$ is a faithful contravariant functor), a contradiction.) Note that this implies that the (covariant) functor V is not representable as well. (Indeed, suppose that V is representable by an **DPHLC**-object A; then, using the fact that the pair (Ψ^a, Ψ^t) is a duality, we will obtain that the contravariant functor $V \circ \Psi^t$ is representable by the **PHLC**-object $\Psi^a(A)$, a contradiction.) Therefore, the pair (Ψ^a, Ψ^t) is not a natural duality (in the sense of 0.2.1.13).

Let us also mention that if $U : \mathbf{PHLC} \longrightarrow \mathbf{Set}$ is the obvious underlying functor, then the functor U is representable by the one-point space P and hence the contravariant functor $U \circ \Psi^a$ is representable by the NCA $\Psi^t(P) = (\mathbf{2}, \rho_s, \mathbf{2})$ ($= (\mathbf{2}, \rho_s)$), where $\mathbf{2}$ is the Boolean algebra $\{0, 1\}$ with $0 \neq 1$. Analogously, if $U' : \mathbf{HLC} \longrightarrow \mathbf{Set}$ is the obvious underlying functor, then the contravariant functor $U' \circ \Lambda^a$ is representable by the NCA $\Lambda^t(P) = \Psi^t(P) = (\mathbf{2}, \rho_s)$. Hence, for any CLCA $(A, \rho, \mathbb{B})$, the set $U(\Psi^a(A, \rho, \mathbb{B}))$ ($= U'(\Lambda^a(A, \rho, \mathbb{B}))$) is **Set**-isomorphic to the set $\mathbf{DPHLC}((A, \rho, \mathbb{B}), (\mathbf{2}, \rho_s, \mathbf{2}))$ ($= \mathbf{DHLC}((A, \rho, \mathbb{B}), (\mathbf{2}, \rho_s, \mathbf{2}))$) (see Lemma 2.2.3.22

and note that the elements of $\mathbf{DHLC}((A,\rho,\mathbb{B}),(\mathbf{2},\rho_s,\mathbf{2}))$ trivially satisfy the axiom (PAL5))). Another proof of this fact can be obtained with the help of Proposition 2.4.4.9 below.

Proposition 2.4.4.9. *The bounded clusters of a CLCA $(A,\rho,\mathbb{B})$ are precisely those subsets of A which are of the form*

$$\sigma^\varphi = \{a \in A \mid \varphi(a^*) = 0\},$$

where $\varphi \in \mathbf{DPHLC}((A,\rho,\mathbb{B}),(\mathbf{2},\rho_s,\mathbf{2}))$ $(= \mathbf{DHLC}((A,\rho,\mathbb{B}),(\mathbf{2},\rho_s,\mathbf{2})))$.

Proof. In this proof, we will write "$\ll$" instead of "$\ll_{\rho_s}$". We will show that the map

$$\upsilon : \mathbf{DHLC}((A,\rho,\mathbb{B}),(\mathbf{2},\rho_s,\mathbf{2})) \longrightarrow \mathrm{BClust}(A,\rho,\mathbb{B}), \quad \varphi \mapsto \sigma^\varphi$$

is a bijection. First of all, we will prove that the map υ is well defined.

Let $\varphi \in \mathbf{DHLC}((A,\rho,\mathbb{B}),(\mathbf{2},\rho_s,\mathbf{2}))$ $(= \mathbf{DPHLC}((A,\rho,\mathbb{B}),(\mathbf{2},\rho_s,\mathbf{2})))$. We will show that σ^φ is a bounded cluster in the CLCA $(A,\rho,\mathbb{B})$. Clearly, $\sigma^\varphi \neq \emptyset$ because, by (DLC1), $\varphi(0) = 0$ and thus $1 \in \sigma^\varphi$. We have to prove that $\sigma^\varphi \cap \mathbb{B} \neq \emptyset$ and the axioms (K1), (K2), (G), (CLU) are fulfilled.

(K2): Let $a < b$ and $a \in \sigma^\varphi$. Then $b^* < a^*$ and, by (DLC2), $\varphi(b^*) \leq \varphi(a^*) = 0$. Hence, $\varphi(b^*) = 0$, i.e., $b \in \sigma^\varphi$.

(K1): Let $a, b \in \sigma^\varphi$. Suppose that $a(-C_\rho)b$. Then $a(-\rho)b$ (i.e., $a \ll_\rho b^*$ and $b \ll_\rho a^*$) and $\{a,b\} \cap \mathbb{B} \neq \emptyset$. Let $a \in \mathbb{B}$. Then, using (DLC3), we get that $(\varphi(a^*))^* \ll \varphi(b^*)$, i.e., $1 \ll 0$, a contradiction. If $b \in \mathbb{B}$, we get a contradiction arguing analogously. Hence, $aC_\rho b$.

(G): Let $a \vee b \in \sigma^\varphi$. Then, using (DLC2), we get that $0 = \varphi((a \vee b)^*) = \varphi(a^* \wedge b^*) = \varphi(a^*) \wedge \varphi(b^*)$. Hence, $\varphi(a^*) = 0$ or $\varphi(b^*) = 0$. Thus, $a \in \sigma^\varphi$ or $b \in \sigma^\varphi$.

(Boundedness): By (DLC4), there exists $a \in \mathbb{B}$ such that $\varphi(a) = 1$. Then we get that $0 = \varphi(a \wedge a^*) = \varphi(a) \wedge \varphi(a^*) = \varphi(a^*)$. Therefore, $a \in \sigma^\varphi \cap \mathbb{B}$.

(CLU): Let $aC_\rho b$, for every $b \in \sigma^\varphi$. Suppose that $a \notin \sigma^\varphi$. Then $\varphi(a^*) = 1$. Now, using (DLC5), we get that there exists $b \in \mathbb{B}$ such that $b \ll_\rho a^*$ and $\varphi(b) = 1$. Then $b \ll_{C_\rho} a^*$, i.e., $a(-C_\rho)b$. Hence $b \notin \sigma^\varphi$. Thus $\varphi(b^*) = 1$. Then we get, as above, that $\varphi(b) = 0$, a contradiction. Therefore, $a \in \sigma^\varphi$.

So, $\sigma^\varphi \in \mathrm{BClust}(A,\rho,\mathbb{B})$ and thus, the map υ is well defined. Setting $B^* = \{b^* \mid b \in B\}$, for every subset B of A, we can rewrite the definition of $\upsilon(\varphi)$ $(= \sigma^\varphi)$ as follows: $\upsilon(\varphi) = (\varphi^{-1}(0))^*$. This shows that υ is an injection. We are now going to prove that υ is a surjection.

Let $\sigma \in \mathrm{BClust}(A, \rho, \mathbb{B})$. Let $\varphi : A \longrightarrow \mathbf{2}$ be defined by

$$\varphi(a) = 0 \iff a^* \in \sigma.$$

Then, clearly, $\sigma = \{a \in A \mid \varphi(a^*) = 0\}$, i.e., $\sigma = \sigma^\varphi$ ($= \upsilon(\varphi)$). We will show that $\varphi \in \mathbf{DHLC}((A, \rho, \mathbb{B}), (\mathbf{2}, \rho_s, \mathbf{2}))$, i.e., we will prove that φ satisfies the axioms (DLC1)-(DLC5).

(DLC1): Since $0^* = 1 \in \sigma$, we get that $\varphi(0) = 0$.

(DLC2): Let $\varphi(a \wedge b) = 0$. Then $(a \wedge b)^* \in \sigma$, i.e., $a^* \vee b^* \in \sigma$. Hence, by (G), $a^* \in \sigma$ or $b^* \in \sigma$. Therefore, $\varphi(a) = 0$ or $\varphi(b) = 0$. Thus $\varphi(a) \wedge \varphi(b) = 0 = \varphi(a \wedge b)$.

Let $\varphi(a \wedge b) = 1$. Then $(a \wedge b)^* \notin \sigma$, i.e., $a^* \vee b^* \notin \sigma$. Hence, by (K2), $a^* \notin \sigma$ and $b^* \notin \sigma$. Therefore, $\varphi(a) = 1 = \varphi(b)$. Thus $\varphi(a) \wedge \varphi(b) = 1 = \varphi(a \wedge b)$.

(DLC3): Let $a \in \mathbb{B}$, $b \in A$ and $a \ll_\rho b$. Let $\varphi(a^*) = 0$. Then $a \in \sigma$. Since $a(-C_\rho)b^*$, we get that $b^* \notin \sigma$. Hence $\varphi(b) = 1$. Thus $(\varphi(a^*))^* \ll \varphi(b)$. If $\varphi(a^*) = 1$ then, clearly, $(\varphi(a^*))^* \ll \varphi(b)$. Therefore, φ satisfies the axiom (DLC3).

(DLC4): We have to show that for every $b \in \mathbf{2}$ there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$. If $b = 0$ then we set $a = 0$ (since, as we have already seen, $\varphi(0) = 0$). Let $b = 1$. By Proposition 2.2.3.1, there exists $a \in \mathbb{B}$ such that $a^* \notin \sigma$. Then $\varphi(a) = 1$. Therefore, φ satisfies the axiom (DLC4).

(DLC5): Let $a \in A$. If $\varphi(a) = 0$ then, using the monotony of φ and the facts that $0 \in \mathbb{B}$, $0 \ll_\rho a$ and $\varphi(0) = 0$, we get that $\varphi(a) = \bigvee\{\varphi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$. If $\varphi(a) = 1$ then $a^* \notin \sigma$. Thus, by (CLU), there exists $c \in \sigma$ such that $a^*(-C_\rho)c$. Hence $c \ll_{C_\rho} a$. Then there exists $b \in A$ such that $c \ll_{C_\rho} b \ll_{C_\rho} a$. Since $c(-C_\rho)b^*$, we get that $b^* \notin \sigma$. Therefore $\varphi(b) = 1$. This implies that $\varphi(a) = \bigvee\{\varphi(b) \mid b \ll_{C_\rho} a\}$. Now, Lemma 2.2.3.22 shows that $\varphi(a) = \bigvee\{\varphi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$. Hence, φ satisfies the axiom (DLC4).

So, $\varphi \in \mathbf{DHLC}((A, \rho, \mathbb{B}), (\mathbf{2}, \rho_s, \mathbf{2}))$ ($= \mathbf{DPHLC}((A, \rho, \mathbb{B}), (\mathbf{2}, \rho_s, \mathbf{2}))$) and $\sigma = \upsilon(\varphi)$. All this shows that υ is a bijection. $\square$

2.5 Some generalizations of the Fedorchuk Duality and Equivalence Theorems

2.5.1 Introduction

As we have already seen, the composition of the morphisms of the category **DHC** differs from their set-theoretic composition. In 1973, Fedorchuk [54] noted that the complete

DHC-morphisms (= **DHC**-morphisms which are complete Boolean homomorphisms) have a very simple description and, moreover, the **DHC**-composition of two such morphisms coincides with their set-theoretic composition. He considered the category **DQHC** of complete compingent Boolean algebras and complete **DHC**-morphisms. It is a subcategory of the category **DHC**, and he proved that the restriction of de Vries' duality functor to it produces a duality between the category **DQHC** and the category **QHC** of compact Hausdorff spaces and quasi-open maps (a class of maps introduced by Mardešic and Papic in [82]). Moreover, he defined a category **EQHC** (whose objects are again the complete compingent Boolean algebras but the morphisms are completely different from the **DQHC**-morphisms) and proved that it is equivalent to the category **QHC**. In Chapter 1, we have mentioned that Fedorchuk [54] introduced the notion of a *Boolean δ-algebra* (which is again a Boolean algebra with an additional relation) as an equivalent expression of the notion of compingent Boolean algebra of de Vries. This new notion reflects even better the ideas of de Laguna and Whitehead because the additional relation considered by Fedorchuk corresponds exactly to their concept of "connection" (= "contact" in our terminology). The axioms defining this relation are very similar to the axioms of Efremovič proximities [50], while the axioms defining de Vries' relation are very similar to the axioms of Efremovič relation "deep inclusion" ([50]). As we have already mentioned, Fedorchuk's Boolean δ-algebras will be called here (as in [41]) "*normal contact algebras*" (briefly, NCAs). So, the regions used by Fedorchuk are again the regular open sets, the chosen algebraic structure is the same as that of de Vries but the morphisms between them differ from those used by de Vries.

In Chapter 1, we have mentioned that in 1997, Roeper [99] defined the notion of *region-based topology* and proved the following theorem (expressed in our terms): there is a bijective correspondence between all (up to homeomorphism) locally compact Hausdorff spaces and all (up to isomorphism) complete LCAs. In the same paper he introduced some morphisms between region-based topologies; this morphisms are similar to those of Fedorchuk's category **EQHC**. In this way, a category **ESHLC** was defined by Roeper in [99]; he defined also a category **SHLC** and a covariant functor F from **ESHLC** to **SHLC**, and showed that F is a full and isomorphism-dense functor, i.e., only the proof that F is faithful is missing for obtaining that F is an equivalence (see Remark 2.5.7.7 below for more details). (Note that in [99], Roeper didn't mention the term "category", and also the notation **ESHLC** and the notation **SHLC** were not used in [99] – they are introduced here.)

Our aims in the present section are the following:

(a) to extend the Fedorchuk Duality Theorem and his Equivalence Theorem to the category **SHLC** whose objects are all locally compact Hausdorff spaces, and to show that the morphisms of the category **SHLC** defined by Roeper are in fact the skeletal (in the sense of Mioduszewski and Rudolf [83]) continuous maps;

(b) to describe the images, under the obtained in (a) duality (resp., equivalence) functor, of some cofull subcategories of the category **SHLC** and to get in this way duality and equivalence theorems for these subcategories; this is done for the cofull subcategories defined by the following classes of maps: open maps, open perfect maps, quasi-open perfect maps;

(c) to find the "connected versions" of the obtained duality and equivalence theorems (i.e., their variants concerning only the connected spaces).

As far as we know, the duality and equivalence theorems for the category of all locally compact Hausdorff spaces and all open maps between them which will be obtained here (i.e., the cofull subcategory of the category **SHLC** defined by the open maps (see (b) above)) are new even in the case of compact Hausdorff spaces. Note, as well, that the Fedorchuk Duality and Equivalence Theorems can be also derived from the obtained here respective theorems for the cofull subcategory of the category **SHLC** defined by the quasi-open perfect maps (see again (b)).

The structure of this section is the following. In the second subsection, we present some preliminary results about skeletal and quasi-open maps and Boolean homomorphisms; a part of these results is well-known but we include them for completeness of our exposition. In the third subsection, we prove the main results of this section with which we extend the Fedorchuk Duality Theorem [54] to the categories of locally compact Hausdorff spaces and skeletal maps, respectively, quasi-open perfect maps. We obtain also a duality theorem for the category of locally compact Hausdorff spaces and open maps. This theorem is new even in the compact case, i.e., we obtain a de Vries' type duality theorem for the category of compact Hausdorff spaces and open maps. A duality theorem for the category of locally compact Hausdorff spaces and open perfect maps is proved as well. The fourth subsection contains the connected versions of the duality theorems obtained in the third subsection, i.e., we find the dual categories of the full subcategories of the categories of locally compact Hausdorff spaces regarded in the previous subsection determined by the requirement that the objects are connected spaces. In the last (fifth) subsection of this section, we extend the Fedorchuk Equivalence Theorem [54] to the categories of locally compact Hausdorff spaces and skeletal maps, respectively, quasi-open perfect maps. We prove as well some equivalence theorems for the categories of locally compact Hausdorff spaces and open maps, respectively, open perfect maps. Finally, we compare one of our equivalence theorems with a result obtained by Roeper in [99].

The results of this section are based on the paper [28].

2.5.2 Some preliminary results

Fact 2.5.2.1. *If A and B are Boolean algebras, $\varphi : A \longrightarrow B$ is a Boolean homomorphism, A has all meets and φ preserves them, then:*
(a) $\forall a \in A$ and $\forall b \in B$, $\varphi(a) \wedge b = 0$ iff $a \wedge \varphi_\Lambda(b) = 0$ (see 0.3.2.4 for the notation);
(b) $\forall a \in A$ and $\forall b \in B$, $\varphi_\Lambda(\varphi(a) \wedge b) = a \wedge \varphi_\Lambda(b)$.

Proof. (a) Let $a \in A$, $b \in B$ and $\varphi(a) \wedge b = 0$. Put $c = a \wedge \varphi_\Lambda(b)$. Since, by ($\Lambda$1) (see 0.3.2.4), $\varphi(c) \wedge b = \varphi(a) \wedge \varphi(\varphi_\Lambda(b)) \wedge b = \varphi(a) \wedge b = 0$, we get that $b \leq \varphi(c^*)$ and hence $\varphi_\Lambda(b) \leq c^*$ (see (Λ) in 0.3.2.4); therefore $c \leq c^*$, i.e., $c = 0$. Therefore, $a \wedge \varphi_\Lambda(b) = 0$.

Conversely, let $a \wedge \varphi_\Lambda(b) = 0$. Then $\varphi(a) \wedge \varphi(\varphi_\Lambda(b)) = 0$ and thus, by (Λ1), $\varphi(a) \wedge b = 0$.

(b) Obviously, $\varphi_\Lambda(\varphi(a) \wedge b) \leq \varphi_\Lambda(\varphi(a)) \wedge \varphi_\Lambda(b) \leq a \wedge \varphi_\Lambda(b)$ (by (Λ2) and the fact that φ_Λ is a monotone map (see 0.3.2.4)). Hence, we need only to show that $\varphi_\Lambda(\varphi(a) \wedge b) \geq a \wedge \varphi_\Lambda(b)$. By (4) (see 0.3.2.4), we have to prove that $a \wedge \varphi_\Lambda(b) \leq \bigwedge\{c \in B \mid \varphi(c) \geq \varphi(a) \wedge b\}$. Let $c \in B$ and $\varphi(c) \geq \varphi(a) \wedge b$. We will show that $a \wedge \varphi_\Lambda(b) \leq c$. Using (a) and ($\Lambda$1) (see 0.3.2.4), we obtain that: $a \wedge \varphi_\Lambda(b) \leq c \leftrightarrow c^* \wedge a \wedge \varphi_\Lambda(b) = 0 \leftrightarrow \varphi(c^* \wedge a) \wedge b = 0 \leftrightarrow (\varphi(c))^* \wedge \varphi(a) \wedge b = 0 \leftrightarrow \varphi(a) \wedge b \leq \varphi(c)$. Thus $a \wedge \varphi_\Lambda(b) \leq c$. Hence (b) is proved. $\square$

Remark 2.5.2.2. Note that every closed irreducible map $f : X \longrightarrow Y$ is quasi-open (because, for every non-empty open subset U of X, $f^\#(U)$ is a non-empty open subset of Y ([91])) (see 0.1.2.2 and 0.4.1.3).

Recall that a continuous map $f : X \longrightarrow Y$ is skeletal (see 0.4.1.3) iff $f^{-1}(\mathrm{Fr}(V))$ is nowhere dense in X, for every open subset V of Y (see [83]).

It is easy to see that a function $f : X \longrightarrow Y$ is skeletal iff $\mathrm{int}(f^{-1}(\mathrm{Fr}(V))) = \emptyset$, for every open subset V of Y.

The next assertion is known but, for completeness, we will present here its proof:

Lemma 2.5.2.3. *A function $f : X \longrightarrow Y$ is skeletal iff* $\mathrm{int}(\mathrm{cl}(f(U))) \neq \emptyset$*, for every non-empty open subset U of X.*

Proof. ($\Rightarrow$) Let U be a non-empty open subset of X. Suppose that $\mathrm{int}(\mathrm{cl}(f(U))) = \emptyset$. Set $V = Y \setminus \mathrm{cl}(f(U))$. Then $Y = \mathrm{cl}(V)$ and $\mathrm{Fr}(V) = Y \setminus V = \mathrm{cl}(f(U))$. Hence $U \subseteq f^{-1}(\mathrm{Fr}(V))$ and thus $\mathrm{int}(f^{-1}(\mathrm{Fr}(V))) \neq \emptyset$, a contradiction. Therefore, $\mathrm{int}(\mathrm{cl}(f(U))) \neq \emptyset$.

($\Leftarrow$) Let V be an open subset of Y. Suppose that $U = \mathrm{int}(f^{-1}(\mathrm{Fr}(V)))$ is a non-empty set. Then $\emptyset \neq \mathrm{int}(\mathrm{cl}(f(U))) \subseteq \mathrm{Fr}(V) = \mathrm{cl}(V) \setminus V$, which is impossible. Hence $\mathrm{int}(f^{-1}(\mathrm{Fr}(V))) = \emptyset$. So, f is a skeletal map. □

Definition 2.5.2.4. A topological space (X, τ) is said to be *π-regular* if for each non-empty $U \in \tau$ there exists a non-empty $V \in \tau$ such that $\mathrm{cl}(V) \subseteq U$.

Note that the semiregular π-regular spaces are exactly the *weakly regular* spaces of Düntsch and Winter ([46]).

Corollary 2.5.2.5. *(a) Every quasi-open map is skeletal.*
(b) Let X be a π-regular space and $f : X \longrightarrow Y$ be a closed map. Then f is quasi-open iff f is skeletal.

Proof. (a) It follows from 2.5.2.3.
(b) Let f be skeletal and closed. Take an open non-empty subset U of X. Then there exists an open non-empty subset V of X such that $\mathrm{cl}(V) \subseteq U$. Using 2.5.2.3, we obtain that $\mathrm{int}(f(U)) \supseteq \mathrm{int}(f(\mathrm{cl}(V))) = \mathrm{int}(\mathrm{cl}(f(V))) \neq \emptyset$. Therefore, f is a quasi-open map. □

For completeness, we will supply with a proof the next assertion:

Lemma 2.5.2.6. ([73, 72, 16]) *Let $f : X \longrightarrow Y$ be a continuous map. Then the following conditions are equivalent:*
(a) f is a skeletal map;
(b) For every $F \in RC(X)$, $\mathrm{cl}(f(F)) \in RC(Y)$.

Proof. (a)$\Rightarrow$(b) Let f be a skeletal map, $F \in RC(X)$ and $F \neq \emptyset$. Set $U = \mathrm{int}(F)$. Then $U \neq \emptyset$. Hence, by 2.5.2.3, $V = \mathrm{int}(\mathrm{cl}(f(U))) \neq \emptyset$. We will show that

(2.20) $\mathrm{cl}(f(F)) = \mathrm{cl}(V)$.

Note that, by the continuity of f, $\mathrm{cl}(f(F)) = \mathrm{cl}(f(U))$. Now suppose that $f(U) \not\subseteq \mathrm{cl}(V)$. Then there exists $y \in f(U) \setminus \mathrm{cl}(V)$. Hence there exists an open neighborhood O_1 of y in Y such that $O_1 \cap V = \emptyset$. Thus $\mathrm{cl}(O_1) \cap V = \emptyset$. There exists $x \in U$ such that $y = f(x)$. Since f is continuous, there exists an open neighborhood O of x in X such that $x \in O \subseteq U$ and $f(O) \subseteq O_1$. Then $\mathrm{cl}(f(O)) \subseteq \mathrm{cl}(O_1)$ and thus $\mathrm{cl}(f(O)) \cap V = \emptyset$. Since, by 2.5.2.3, $\emptyset \neq \mathrm{int}(\mathrm{cl}(f(O))) \subseteq \mathrm{cl}(f(O)) \cap \mathrm{int}(\mathrm{cl}(f(U))) = \mathrm{cl}(f(O)) \cap V = \emptyset$, we obtain a contradiction. Therefore $f(U) \subseteq \mathrm{cl}(V)$ and hence $\mathrm{cl}(f(U)) \subseteq \mathrm{cl}(V)$. Since the converse inclusion is obvious, (2.20) is established. Thus, $\mathrm{cl}(f(F)) \in RC(Y)$.

(b)$\Rightarrow$(a) Let U be a non-empty open subset of X. Then $F = \mathrm{cl}(U) \in RC(X)$. Hence $\mathrm{cl}(f(F)) \in RC(Y)$. Since $F \neq \emptyset$, we obtain that $\mathrm{int}(\mathrm{cl}(f(F))) \neq \emptyset$. Now, using the continuity of f, we get that $\mathrm{int}(\mathrm{cl}(f(U))) \neq \emptyset$. Therefore, by 2.5.2.3, f is a skeletal map. $\square$

The next lemma generalizes the well-known result of Ponomarev [91] that the regular closed sets are preserved by the closed irreducible maps.

Lemma 2.5.2.7. *Let $f : X \longrightarrow Y$ be a closed map and X be a π-regular space. Then the following conditions are equivalent:*
(a) f is a quasi-open map;
(b) For every $F \in RC(X)$, $f(F) \in RC(Y)$.

Proof. (a)$\Rightarrow$(b) It follows from 2.5.2.5(a) and 2.5.2.6.

(b)$\Rightarrow$(a) It follows from 2.5.2.5(b) and 2.5.2.6. Note that the π-regularity of X is used only in the proof of this implication. $\square$

Corollary 2.5.2.8. *If $f : X \longrightarrow Y$ is a quasi-open closed map then $f(X) \in RC(Y)$.*

Remarks 2.5.2.9. In [69], Henriksen and Jerison considered functions $f : X \longrightarrow Y$ between topological spaces for which

(2.21) $\mathrm{cl}(\mathrm{int}(f^{-1}(F))) = \mathrm{cl}(f^{-1}(\mathrm{int}(F)))$ for every $F \in RC(Y)$.

Clearly, every continuous skeletal map $f : X \longrightarrow Y$ satisfies (2.21) ([83]). Hence, by 2.5.2.5(a), every quasi-open map $f : X \longrightarrow Y$ satisfies (2.21) ([92]).

Recall that a function $f : X \longrightarrow Y$ (not necessarily continuous) which satisfies condition (5) (see 0.4.1.3) for every $V \in RO(X)$ is called a *HJ-map* in [83]. Obviously, every continuous HJ-map $f : X \longrightarrow Y$ satisfies (2.21). As it is noted in [83], the composition of two continuous HJ-maps needs not be an HJ-map, while the composition of two continuous skeletal maps is a skeletal map. It is clear that the composition of two quasi-open maps is a quasi-open map.

We will now formulate the Fedorchuk Duality Theorem [54].

2.5.2.10. Let **DQHC** be the category whose objects are all complete normal contact algebras (i.e., CNC-algebras) and whose morphisms $\varphi : (A, C) \longrightarrow (B, C')$ are all complete Boolean homomorphisms $\varphi : A \longrightarrow B$ satisfying the following condition:

(F1) For all $a, b \in A$, $\varphi(a)C'\varphi(b)$ implies aCb.

Theorem 2.5.2.11. (Fedorchuk [54]) *The categories* **QHC** *and* **DQHC** *are dually equivalent.*

2.5.3 Two generalizations of the Fedorchuk Duality Theorem

Definition 2.5.3.1. Let **DSHLC** be the category whose objects are all complete local contact algebras and whose morphisms $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ are all complete Boolean homomorphisms $\varphi : A \longrightarrow B$ satisfying the following conditions:

(L1) $\forall a, b \in A$, $\varphi(a)\eta\varphi(b)$ implies $a\rho b$;

(L2) $b \in \mathbb{B}'$ implies $\varphi_\Lambda(b) \in \mathbb{B}$ (see 0.3.2.4 for φ_Λ).

It is easy to see that in this way we have indeed defined a category.

Let us note that (L1) is equivalent to the following condition:

(EL1) $\forall a, b \in B$, $a\eta b$ implies $\varphi_\Lambda(a)\rho\varphi_\Lambda(b)$.

Proposition 2.5.3.2. *The condition (L2) in the definition of the category* **DSHLC** *can be replaced by the axiom (DLC4).*

Proof. Let $b \in \mathbb{B}'$. Then, by (L2), $\varphi_\Lambda(b) \in \mathbb{B}$. Set $a = \varphi_\Lambda(b)$. Since φ_Λ is a left adjoint to φ, we get that $b \leq \varphi(a)$. So, condition (DLC4) is checked.

Conversely, if $b \in \mathbb{B}'$ then, by (DLC4), there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$; thus $\varphi_\Lambda(b) \leq a$; therefore $\varphi_\Lambda(b) \in \mathbb{B}$, i.e., condition (L2) is fulfilled. $\square$

We are now going to prove the following theorem.

Theorem 2.5.3.3. *The categories* **SHLC** *and* **DSHLC** *are dually equivalent.*

Since, by 2.5.2.5(b), a closed map between two regular spaces is skeletal iff it is quasi-open, we get that Theorem 2.5.3.3 is a generalization of the Fedorchuk Duality Theorem.

The proof of Theorem 2.5.3.3 will be given below in the form of some steps and propositions.

Step 1. We first define two contravariant functors

$$\Psi_1^t : \mathbf{SHLC} \longrightarrow \mathbf{DSHLC} \text{ and } \Psi_1^a : \mathbf{DSHLC} \longrightarrow \mathbf{SHLC}.$$

Their definitions on the objects of the corresponding categories are the following:

$$\Psi_1^t(X) = \Psi^t(X)$$

for every $X \in |\mathbf{SHLC}|$, and

$$\Psi_1^a(A, \rho, \mathbb{B}) = \Psi^a(A, \rho, \mathbb{B}),$$

for every $(A, \rho, \mathbb{B}) \in |\mathbf{DSHLC}|$ (see (1.20) for Ψ^t and (1.21), (1.24) for Ψ^a). The definitions of the contravariant functors Ψ_1^t and Ψ_1^a on the corresponding morphisms are as follows:

$$\Psi_1^t(f)(G) = \mathrm{cl}(f^{-1}(\mathrm{int}(G)))$$

for every $f \in \mathbf{SHLC}(X, Y)$ and every $G \in RC(Y)$, and, further, for every $\varphi \in \mathbf{DSHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$ and for every bounded ultrafilter u in B (i.e., $u \cap \mathbb{B}' \neq \emptyset$), we set

(2.22) $\Psi_1^a(\varphi)(\sigma_u) = \sigma_{\varphi^{-1}(u)}$

where $\sigma_{\varphi^{-1}(u)}$ is a cluster in (A, C_ρ) (see 1.2.2.4 for the notation of the type σ_v, and note that by Theorem 1.2.2.3, any bounded cluster σ in $(B, \eta, \mathbb{B}')$ can be written in the form σ_u for some bounded ultrafilter u in B).

We are going to show that Ψ_1^t and Ψ_1^a are indeed contravariant functors between the corresponding categories. In this Step 1 we will only prove two preparatory propositions. We start with the following one.

Proposition 2.5.3.4. *The categories* **SHLC** *and* **DSHLC** *are cofull subcategories of, respectively,* **HLC** *and* **DHLC**. *The restriction of the contravariant functor* Λ^a *(respectively,* Λ^t*) to the subcategory* **DSHLC** *(resp.,* **SHLC***) coincides with* Ψ_1^a *(resp.,* Ψ_1^t*).*

Proof. Obviously, the category **SHLC** is a cofull subcategory of the category **HLC** and the restriction of the contravariant functor Λ^t to the subcategory **SHLC** coincides with Ψ_1^t.

Let $\varphi \in \mathbf{DSHLC}((A,\rho,\mathbb{B}),(B,\eta,\mathbb{B}'))$. Then it is clear that φ satisfies conditions (DLC1) and (DLC2). Let $a,b \in A$ and $a \ll_\rho b$. Then $a(-\rho)b^*$. Hence, by (L1), $\varphi(a)(-\eta)\varphi(b^*)$. Since φ is a Boolean homomorphism, we have that $\varphi(b^*) = (\varphi(b))^*$ and $(\varphi(a^*))^* = \varphi(a)$. Thus, $(\varphi(a^*))^* \ll_\eta \varphi(b)$. Therefore, condition (DLC3) is satisfied. Further, by Proposition 2.5.3.2, condition (DLC4) is fulfilled. Finally, let $a \in A$. Then $a = \bigvee\{b \in \mathbb{B} \mid b \ll_\rho a\}$. Since φ is a complete Boolean homomorphism, we conclude that $\varphi(a) = \bigvee\{\varphi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$. Thus, condition (DLC5) is satisfied. So, every **DSHLC**-morphism is a **DHLC**-morphism. Since the composition $\varphi_2 \circ \varphi_1$ of two complete Boolean homomorphisms is a complete Boolean homomorphism, Lemma 2.2.2.9(e) implies that $(\varphi_2 \circ \varphi_1)\check{} = \varphi_2 \circ \varphi_1$. Hence, $\varphi_2 \diamond \varphi_1 = \varphi_2 \circ \varphi_1$. Therefore, the category **DSHLC** is a cofull subcategory of the category **DHLC**.

Let $\varphi \in \mathbf{DSHLC}((A,\rho,\mathbb{B}),(B,\eta,\mathbb{B}'))$ and u be a bounded ultrafilter in B. We have to show that $\Psi_1^a(\varphi)(\sigma_u)$ is a bounded cluster (see (2.22)). Set

$$f = \Psi_1^a(\varphi), \quad X = \Psi_1^a(A,\rho,\mathbb{B}) \text{ and } Y = \Psi_1^a(B,\eta,\mathbb{B}').$$

Then X is the set of all bounded clusters of $(A,\rho,\mathbb{B})$ and Y is the set of all bounded clusters of $(B,\eta,\mathbb{B}')$ (see 1.2.3.6, (1.21) and (1.23)). We set $C = C_\rho$ and $C' = C_\eta$.

Let us start with the following observation:

(2.23) if $u \in \mathrm{Ult}(B)$ then $\varphi^{-1}(u) \in \mathrm{Ult}(A)$ and $\varphi_\Lambda(u)$ is a filter-base of $\varphi^{-1}(u)$.

So, let $u \in \mathrm{Ult}(B)$. Then, obviously, $\varphi^{-1}(u) \in \mathrm{Ult}(A)$. Let us show that $\varphi_\Lambda(u) \subseteq \varphi^{-1}(u)$. Let $b \in u$. Then, by (Λ1) (see 0.3.2.4), $\varphi(\varphi_\Lambda(b)) \geq b$. Hence $\varphi(\varphi_\Lambda(b) \in u$, i.e., $\varphi_\Lambda(b) \in \varphi^{-1}(u)$. Therefore, $\varphi_\Lambda(u) \subseteq \varphi^{-1}(u)$. Further, suppose that there exists $a \in \varphi^{-1}(u)$ such that $\varphi_\Lambda(b) \not\leq a$ for all $b \in u$. Then $\varphi_\Lambda(b) \wedge a^* \neq 0$ for every $b \in u$. Hence, by 2.5.2.1(a), $b \wedge \varphi(a^*) \neq 0$ for every $b \in u$. Since $u \in \mathrm{Ult}(B)$, we obtain that $\varphi(a^*) \in u$. Thus both $\varphi(a)$ and $(\varphi(a))^*$ are elements of u, a contradiction. Therefore, $\varphi_\Lambda(u)$ is a basis of the ultrafilter $\varphi^{-1}(u)$.

Obviously, (2.23) implies that

(2.24) $\forall u \in \mathrm{Ult}(B), \sigma_{\varphi^{-1}(u)} = \sigma_{\varphi_\Lambda(u)}$,

where $\sigma_{\varphi^{-1}(u)}$ and $\sigma_{\varphi_\Lambda(u)}$ are clusters in (A,C) (see 1.2.2.4 for the notation).

Let $u,v \in \mathrm{Ult}(B)$, $\sigma_u = \sigma_v$ and $\sigma = \sigma_u (= \sigma_v)$ be bounded. We will prove that $\sigma_{\varphi^{-1}(u)} = \sigma_{\varphi^{-1}(v)}$. Indeed, by 2.2.3.2, there exists $c \in u \cap \mathbb{B}'$. Let $a \in u$ and $b \in v$. Then $a \wedge c \in u \cap \mathbb{B}'$ and $(a \wedge c)C'b$. Thus $(a \wedge c)\eta b$. Hence, by (EL1), $\varphi_\Lambda(a \wedge c)\rho\varphi_\Lambda(b)$. Therefore, $\varphi_\Lambda(a \wedge c)C\varphi_\Lambda(b)$. Thus $\varphi_\Lambda(a)C\varphi_\Lambda(b)$. Since this is true for every $a \in u$ and every $b \in v$, we obtain, using (2.23) and (1.2), that $\varphi_\Lambda(u) \subseteq \sigma_{\varphi_\Lambda(v)}$. Then, by 1.2.2.4 and (2.23), $\sigma_{\varphi_\Lambda(u)} = \sigma_{\varphi_\Lambda(v)}$. Using (2.24), we get that $\sigma_{\varphi^{-1}(u)} = \sigma_{\varphi^{-1}(v)}$. Therefore, $\Psi_1^a(\varphi)$ is well defined, i.e., it doesn't depend on the choice of the generating ultrafilter (see the formula (2.22)).

Now, using (2.24), we obtain that

(2.25) if $\sigma \in Y$ and $b \in \sigma$ then $\varphi_\Lambda(b) \in f(\sigma)$.

Indeed, by 1.2.2.3, there exists $u \in \mathrm{Ult}(B)$ such that $b \in u$ and $\sigma = \sigma_u$. Thus, by (2.24), $f(\sigma) = \sigma_{\varphi_\Lambda(u)}$. Therefore $\varphi_\Lambda(b) \in f(\sigma)$. So, (2.25) is proved.

Let us show that for every $\sigma \in \mathrm{Clust}(B, C')$,

(2.26) $\sigma \cap \mathbb{B}' \neq \emptyset$ implies that $f(\sigma) \cap \mathbb{B} \neq \emptyset$.

Indeed, let $\sigma \in \mathrm{Clust}(B, C')$ and $b \in \sigma \cap \mathbb{B}'$. Then, by (2.25), $\varphi_\Lambda(b) \in f(\sigma)$. Since, by (L2), $\varphi_\Lambda(b) \in \mathbb{B}$, we obtain that $f(\sigma) \cap \mathbb{B} \neq \emptyset$.

So, if $\sigma \in Y$ then $\Psi_1^a(\varphi)(\sigma)$ is a bounded cluster, i.e., $f(Y) \subseteq X$.

We have that $\Psi_1^a(\varphi)(\sigma_u) = \sigma_{\varphi^{-1}(u)}$ and $\Lambda^a(\varphi)(\sigma_u) \cap \mathbb{B} = \{a \in \mathbb{B} \mid \text{if } b \in A \text{ and } a \ll_\rho b \text{ then } \varphi(b) \in \sigma_u\}$. According to Corollary 2.2.3.4, $\Psi_1^a(\varphi)(\sigma_u) = \Lambda^a(\varphi)(\sigma_u)$ iff $\mathbb{B} \cap \Psi_1^a(\varphi)(\sigma_u) = \mathbb{B} \cap \Lambda^a(\varphi)(\sigma_u)$. Thus, we have to show that

(2.27) $\mathbb{B} \cap \sigma_{\varphi^{-1}(u)} = \{a \in \mathbb{B} \mid \text{ if } b \in A \text{ and } a \ll_\rho b \text{ then } \varphi(b) \in \sigma_u\}$.

So, let $a \in \mathbb{B} \cap \sigma_{\varphi^{-1}(u)}$. Suppose that there exists $b \in A$ such that $a \ll_\rho b$ and $\varphi(b) \notin \sigma_u$. Then $\varphi(b) \notin u$. Hence $(\varphi(b))^* \in u$, i.e., $\varphi(b^*) \in u$. Thus $b^* \in \varphi^{-1}(u)$. Since $a(-\rho)b^*$, we get a contradiction.

Conversely, let $a \in \mathbb{B}$ and for all $b \in A$ such that $a \ll_\rho b$, we have that $\varphi(b) \in \sigma_u$. We have to prove that $a \in \sigma_{\varphi^{-1}(u)}$, i.e., that $a\rho b$ for all $b \in \varphi^{-1}(u)$. Suppose that there exists $b_0 \in \varphi^{-1}(u)$ such that $a(-\rho)b_0$. Then $\varphi(b_0) \in u$ and $a \ll_\rho b_0^*$. By (BC1), there exists $a_1 \in \mathbb{B}$ such that $a \ll_\rho a_1 \ll_\rho b_0^*$. Hence $a_1(-\rho)b_0$ and $b_0 \leq a_1^*$. Then $\varphi(b_0) \leq \varphi(a_1^*)$ and thus $\varphi(a_1^*) \in u$. Since $\varphi(a_1^*) = (\varphi(a_1))^*$, we get that $\varphi(a_1) \notin u$. We have that $\varphi(a_1) \in \sigma_u$ (because $a \ll_\rho a_1$). Let $c_0 \in u \cap \mathbb{B}'$. Then $c_0 \wedge \varphi(b_0) \in u \cap \mathbb{B}'$. Thus $\varphi(a_1)\eta(c_0 \wedge \varphi(b_0))$. Therefore $\varphi(a_1)\eta\varphi(b_0)$. Then, by (L1), we obtain that $a_1\rho b_0$, a contradiction. So, the equality (2.27) is established. This completes the proof. □

Proposition 2.5.3.5. *The cofull subcategory* $\mathbf{D_1SHLC}$ *of the category* **DHLC** *determined by the* **DHLC***-morphisms which are complete Boolean homomorphisms coincides with the category* **DSHLC**.

Proof. It is easy to see that each $\mathbf{D_1SHLC}$-morphism φ satisfies condition (L1). Indeed, let $a, b \in A$ and $\varphi(a)\eta\varphi(b)$. Suppose that $a(-\rho)b$. Then $a \ll_\rho b^*$. Thus, by (DLC3S), we get that $(\varphi(a^*))^* \ll_\eta \varphi(b^*)$, i.e., $\varphi(a)(-\eta)\varphi(b)$, a contradiction. Therefore, $a\rho b$. So, φ satisfies condition (L1). Further, by Proposition 2.5.3.2, φ satisfies condition (L2) as well. The rest follows from Proposition 2.5.3.4. $\square$

Step 2. We will show that Ψ_1^t is a contravariant functor between the categories **SHLC** and **DSHLC**. Let $f \in \mathbf{SHLC}(X, Y)$. Put

$$\varphi = \Psi_1^t(f).$$

Then, by Theorem 2.2.2.12, φ is a **DHLC**-morphism. We will first show that φ is a complete Boolean homomorphism. Let Γ be a set and $\{F_\gamma \mid \gamma \in \Gamma\} \subseteq RC(Y)$. Put $F = \mathrm{cl}(\bigcup\{F_\gamma \mid \gamma \in \Gamma\})$. (Note that $F = \mathrm{cl}(\bigcup\{\mathrm{int}(F_\gamma) \mid \gamma \in \Gamma\})$.) Then $F \in RC(Y)$ and $\bigvee\{F_\gamma \mid \gamma \in \Gamma\} = F$. Since φ is an order-preserving map, we get that $\varphi(F) \geq \bigvee\{\varphi(F_\gamma) \mid \gamma \in \Gamma\}$. We will now prove the converse inequality. We have that $\varphi(F) = \mathrm{cl}(f^{-1}(\mathrm{int}(F)))$. Let $x \in f^{-1}(\mathrm{int}(F))$. Then $f(x) \in \mathrm{int}(F)$. Hence, there exist open neighborhoods O and O' of $f(x)$ in Y such that $\mathrm{cl}(O') \subseteq O \subseteq F$. Since f is continuous, there exists an open neighborhood U of x in X such that $f(U) \subseteq O'$. Suppose that there exists an open neighborhood V of x in X such that, for every $\gamma \in \Gamma$, $V \cap \mathrm{cl}(\mathrm{int}(f^{-1}(F_\gamma))) = \emptyset$. Obviously, we can suppose that $V \subseteq U$. Since f is continuous and skeletal, we get, using 2.5.2.9 and (2.21), that $V \cap f^{-1}(\mathrm{int}(F_\gamma)) = \emptyset$, for every $\gamma \in \Gamma$. Thus, $f(V) \cap \bigcup\{\mathrm{int}(F_\gamma) \mid \gamma \in \Gamma\} = \emptyset$. Put $W = \bigcup\{\mathrm{int}(F_\gamma) \mid \gamma \in \Gamma\}$. Then $\mathrm{cl}(f(V)) \cap W = \emptyset$ and $\mathrm{cl}(f(V)) \subseteq \mathrm{cl}(f(U) \subseteq \mathrm{cl}(O') \subseteq O \subseteq F = \mathrm{cl}(W)$. Thus $\mathrm{cl}(f(V)) \subseteq \mathrm{cl}(W) \backslash W = \mathrm{Fr}(W)$. Since f is skeletal, 2.5.2.3 implies that $\mathrm{int}(\mathrm{cl}(f(V))) \neq \emptyset$ and this leads to a contradiction. Therefore, $x \in \mathrm{cl}(\bigcup\{\mathrm{cl}(\mathrm{int}(f^{-1}(F_\gamma))) \mid \gamma \in \Gamma\})$. We have proved that $\varphi(F) \leq \bigvee\{\varphi(F_\gamma) \mid \gamma \in \Gamma\}$. So, $\varphi(\bigvee\{F_\gamma \mid \gamma \in \Gamma\}) = \bigvee\{\varphi(F_\gamma) \mid \gamma \in \Gamma\}$.

Let $F \in RC(Y)$. Then $(\varphi(F))^* = (\mathrm{cl}(f^{-1}(\mathrm{int}(F))))^* = (\mathrm{cl}(f^{-1}(Y \setminus F^*)))^* = (\mathrm{cl}(X \setminus f^{-1}(F^*)))^* = \mathrm{cl}(X \setminus \mathrm{cl}(X \setminus f^{-1}(F^*))) = \mathrm{cl}(\mathrm{int}(f^{-1}(F^*)))$. So, using again 2.5.2.9 and (2.21), we get that $\varphi(F^*) = (\varphi(F))^*$. Since, obviously, φ preserves the zero and the unit elements, φ is a complete Boolean homomorphism. Now, Proposition 2.5.3.5 implies that $\Psi_1^t(f)$ is a **DSHLC**-morphism.

Hence we get, using Proposition 2.5.3.4, that

$$\Psi_1^t : \mathbf{SHLC} \longrightarrow \mathbf{DSHLC}$$

is a contravariant functor.

Step 3. We will show that Ψ_1^a is a contravariant functor between the categories **DSHLC** and **SHLC**. Let $\varphi \in \mathbf{DSHLC}((A,\rho,\mathbb{B}),(B,\eta,\mathbb{B}'))$, $f = \Psi_1^a(\varphi)$, $X = \Psi_1^a(A,\rho,\mathbb{B})$ and $Y = \Psi_1^a(B,\eta,\mathbb{B}')$. Then, using Proposition 2.5.3.4, we get that $f = \Lambda^a(\varphi)$. Thus, by Theorem 2.2.2.12,

(2.28) $f : Y \longrightarrow X$ is a continuous function.

We will now show that f is a skeletal map, i.e., that $\mathrm{int}_X(f(\mathrm{cl}(U))) \neq \emptyset$ for every non-empty open subset U of Y (see 2.5.2.3). Note that, by (1.36) and (1.31), it is enough to prove that $\mathrm{int}_X(f(\lambda_B^g(b)) \neq \emptyset$, for every $b \in \mathbb{B}' \setminus \{0\}$.

We will first show that for every $b \in \mathbb{B}'$,

(2.29) $f(\lambda_B^g(b)) = \lambda_A^g(\varphi_\Lambda(b))$

(note that $b \in \mathbb{B}'$ implies that $\lambda_B(b) \subseteq Y$ and $\varphi_\Lambda(b) \in \mathbb{B}$ (by (L2)); thus we have also that $\lambda_A(\varphi_\Lambda(b)) \subseteq X$; hence (2.29) can be written as $f(\lambda_B(b)) = \lambda_A(\varphi_\Lambda(b)))$. Since $\varphi(0) = 0$, we have, by 0.3.2.4, that $\varphi_\Lambda(0) = 0$ and $\varphi_\Lambda(b) \neq 0$ for any $b \neq 0$. Hence, (2.29) is true for $b = 0$.

Let $b \in \mathbb{B}' \setminus \{0\}$ and $\sigma \in f(\lambda_B(b))$. Then there exists $\sigma' \in \lambda_B(b)$ such that $f(\sigma') = \sigma$. Hence $b \in \sigma'$ and thus, by (2.25), $\varphi_\Lambda(b) \in f(\sigma') = \sigma$. Therefore we get that $\sigma \in \lambda_A(\varphi_\Lambda(b))$. So, $f(\lambda_B(b)) \subseteq \lambda_A(\varphi_\Lambda(b))$. Conversely, let $b \in \mathbb{B}' \setminus \{0\}$ and $\sigma \in \lambda_A(\varphi_\Lambda(b))$, i.e., $\varphi_\Lambda(b) \in \sigma$. Then, by 1.2.2.3, there exists $u \in \mathrm{Ult}(A)$ such that $\varphi_\Lambda(b) \in u \subseteq \sigma$, and hence, by 1.2.2.4, $\sigma = \sigma_u$. Let us show that $\varphi(u) \cup \{b\}$ has the finite intersection property. Since $\varphi(u)$ is closed under finite meets, it is enough to prove that $b \wedge \varphi(a) \neq 0, \forall a \in u$. Indeed, suppose that there exists $a_0 \in u$ such that $b \wedge \varphi(a_0) = 0$. Then, by 2.5.2.1(a), we will have that $\varphi_\Lambda(b) \wedge a_0 = 0$. This is, however, impossible, since $\varphi_\Lambda(b) \in u$. So, there exists an ultrafilter v in B such that $v \supseteq \varphi(u) \cup \{b\}$. Set $\sigma' = \sigma_v$. Then σ' is a cluster in (B, C') (see 1.2.2.3) and since $v \subseteq \sigma'$, we have that $b \in \sigma'$. Hence $\sigma' \in \lambda_B(b)$. Further, $f(\sigma') = \sigma$. Indeed, since $\varphi(u) \subseteq v$, we have that $u \subseteq \varphi^{-1}(v)$; thus $u = \varphi^{-1}(v)$ and hence $\sigma = \sigma_u = \sigma_{\varphi^{-1}(v)} = f(\sigma_v) = f(\sigma')$. Therefore $\sigma = f(\sigma') \in f(\lambda_B(b))$. So, (2.29) is proved.

Now, suppose that there exists $b \in \mathbb{B}' \setminus \{0\}$ such that $\mathrm{int}_X(f(\lambda_B^g(b))) = \emptyset$. Then $X \setminus f(\lambda_B^g(b))$ is dense in X. Using (2.29), we obtain that $X \setminus \lambda_A^g(\varphi_\Lambda(b))$ is dense in X. Thus, by (1.35), $\mathrm{int}(\lambda_A^g((\varphi_\Lambda(b))^*))$ is dense in X. Hence $\lambda_A^g((\varphi_\Lambda(b))^*) = \mathrm{cl}(\mathrm{int}(\lambda_A^g((\varphi_\Lambda(b))^*))) = X$. Therefore, by (1.31), $(\varphi_\Lambda(b))^* = 1$. Then $\varphi_\Lambda(b) = 0$ and hence $b = 0$ (by 0.3.2.4), a contradiction. Hence,

(2.30) $f : Y \longrightarrow X$ is a skeletal map.

So, we have proved that $\Psi_1^a(\varphi) \in \mathbf{SHLC}(\Psi_1^a(B,\eta,\mathbb{B}'), \Psi_1^a(A,\rho,\mathbb{B}))$.

Thus, using Proposition 2.5.3.4, we get that

$$\Psi_1^a : \mathbf{DSHLC} \longrightarrow \mathbf{SHLC}$$

is a contravariant functor.

Now, Theorem 2.5.3.3 follows from Proposition 2.5.3.4 and Theorem 2.2.2.12. □

We will now obtain one more generalization of the Fedorchuk Duality Theorem 2.5.2.11.

Recall that we have denoted by **PSHLC** the category of all locally compact Hausdorff spaces and all skeletal perfect maps between them (see 0.1.2.6). Note that, by 2.5.2.5(b), the morphisms of the category **PSHLC** are precisely the quasi-open perfect maps (because the perfect maps are closed maps and the regular spaces are π-regular).

Definition 2.5.3.6. Let **DPSHLC** be the category whose objects are all complete local contact algebras (see 1.2.3.1) and whose morphisms are all **DSHLC**-morphisms $\varphi : (A,\rho,\mathbb{B}) \longrightarrow (B,\eta,\mathbb{B}')$ satisfying the following condition:

(L3) $a \in \mathbb{B}$ implies $\varphi(a) \in \mathbb{B}'$.

It is easy to see that in this way we have indeed defined a category. Obviously, **PSHLC** (resp., **DPSHLC**) is a subcategory of the category **SHLC** (resp., **DSHLC**).

Theorem 2.5.3.7. *The categories* **PSHLC** *and* **DPSHLC** *are dually equivalent.*

Proof. We will show that the restrictions

$$\Psi_p^a : \mathbf{DPSHLC} \longrightarrow \mathbf{PSHLC} \ \text{ and } \Psi_p^t : \mathbf{PSHLC} \longrightarrow \mathbf{DPSHLC}$$

of the contravariant functors Ψ_1^a and Ψ_1^t defined in the proof of Theorem 2.5.3.3 are the desired duality functors.

Let $f \in \mathbf{PSHLC}((X,\tau),(Y,\tau'))$. Since f is a perfect map, we obtain that $\varphi = \Psi_p^t(f)$ satisfies condition (L3) (using [53, Theorem 3.7.2]). Hence, φ is well defined. Therefore the contravariant functor $\Psi_p^t : \mathbf{PSHLC} \longrightarrow \mathbf{DPSHLC}$ is well defined.

Let $\varphi \in \mathbf{DPSHLC}((A,\rho,\mathbb{B}),(B,\eta,\mathbb{B}'))$ and set $f = \Psi_p^a(\varphi)$, i.e.,

$$f : \Psi_p^a(B,\eta,\mathbb{B}') \longrightarrow \Psi_p^a(A,\rho,\mathbb{B}).$$

Then, by Theorem 2.5.3.3, f is a continuous skeletal map. We have to show that f is a perfect map. Put $C = C_\rho$ and $C' = C_\eta$ (see 1.2.3.4 for the notation). Then, by 1.2.3.4,

(A, C) and (B, C') are CNCA's. Denote by φ_c the map φ considered as a function from (A, C) into (B, C'). We will show that φ_c satisfies condition (F1) (see 2.5.2.10).

For verifying condition (F1), let $a, b \in A$ and let $\varphi_c(a)C'\varphi_c(b)$. Then either $\varphi_c(a)\eta\varphi_c(b)$ or $\varphi_c(a), \varphi_c(b) \notin \mathbb{B}'$. If $\varphi_c(a)\eta\varphi_c(b)$ then, by (L1), $a\rho b$; hence aCb. If $\varphi_c(a), \varphi_c(b) \notin \mathbb{B}'$ then, by (L3), $a, b \notin \mathbb{B}$. Hence aCb. So, (F1) is verified. Therefore,

(2.31) $\varphi_c : (A, C_\rho) \longrightarrow (B, C_\eta)$ satisfies condition (F1).

Set $X = \Psi^a(A, C, A)$ and $Y = \Psi^a(B, C', B)$ (see (1.21)). Then X and Y are compact Hausdorff spaces. Let $f_c = \Psi_1^a(\varphi_c)$, i.e.,

(2.32) $f_c : Y \longrightarrow X$ is defined by $f_c(\sigma_u) = \sigma_{\varphi_c^{-1}(u)}$, for every $u \in \mathrm{Ult}(B)$.

We will consider three cases now.

(a) Let $1_A \notin \mathbb{B}$ and $1_B \notin \mathbb{B}'$. Then $\Psi_p^a(B, \eta, \mathbb{B}') = L_B = Y \setminus \{\sigma_\infty^B\}$ and $\Psi_p^a(A, \rho, \mathbb{B})) = L_A = X \setminus \{\sigma_\infty^A\}$ (see 1.2.3.7 and (1.23)).

We will show that $f_c^{-1}(\sigma_\infty^A) = \{\sigma_\infty^B\}$ (see 1.2.3.7 for the notation). We first prove that $f_c(\sigma_\infty^B) = \sigma_\infty^A$. Let $u \in \mathrm{Ult}(B)$ be such that $u \subset \sigma_\infty^B$ and $\sigma_\infty^B = \sigma_u$ (see 1.2.2.3). Then $f_c(\sigma_\infty^B) = \sigma_{\varphi_c^{-1}(u)}$. We will show that $\varphi_c^{-1}(u) \subset \sigma_\infty^A$. Indeed, let $a \in \varphi_c^{-1}(u)$. Then $\varphi_c(a) \in u \subset B \setminus \mathbb{B}'$. Hence $\varphi_c(a) \notin \mathbb{B}'$. Thus, by (L3), $a \notin \mathbb{B}$. So, $\varphi_c^{-1}(u) \subset A \setminus \mathbb{B} = \sigma_\infty^A$ (see 1.2.3.7). Then, by 1.2.3.7 and 1.2.2.4, $\sigma_\infty^A = \sigma_{\varphi_c^{-1}(u)}$. Therefore, $f_c(\sigma_\infty^B) = \sigma_\infty^A$. Since L_A and L_B consist of bounded clusters (see (1.23)), (2.26) implies that $f_c(L_B) \subseteq L_A$. Therefore,

(2.33) $f_c^{-1}(\sigma_\infty^A) = \{\sigma_\infty^B\}$.

This shows that $f_c^{-1}(L_A) = L_B$. Since f_c is a perfect map, we obtain (by [53, Proposition 3.7.4]) that

(2.34) $(f_c)_{L_A} : L_B \longrightarrow L_A$ is a perfect map.

Obviously, f is the restriction of f_c to L_B. Hence $f = (f_c)_{L_A}$, i.e., f is a perfect map.

(b) Let $1_A \notin \mathbb{B}$ and $1_B \in \mathbb{B}'$. Then $C' = \eta$, $\Psi_p^a(A, \rho, \mathbb{B}) = X \setminus \{\sigma_\infty^A\} = L_A$ and $\Psi_p^a(B, \eta, \mathbb{B}') = Y$. Thus (2.26) implies that $f_c(Y) \subset L_A$. Therefore, the restriction $f : Y \longrightarrow L_A$ of f_c is a perfect map.

(c) Let $1_A \in \mathbb{B}$. Then, by (L3), $1_B \in \mathbb{B}'$. Hence $C = \rho$, $C' = \eta$, $\Psi_p^a(B, \eta, \mathbb{B}') = Y$, $\Psi_p^a(A, \rho, \mathbb{B}) = X$. Thus $f = f_c$. Hence, $f : Y \longrightarrow X$ is a skeletal perfect map.

We have considered all possible cases. Therefore, Ψ_p^a is well defined on the objects and morphisms of the category **DPSHLC**.

Note that, using (1.31), we obtain that λ_B^g is a **DPSHLC**-isomorphism. The rest follows from Theorem 2.5.3.3. □

2.5.4 Duality Theorems for the categories OHLC, POHLC and OHC

Definition 2.5.4.1. Let **DOHLC** be the category whose objects are all complete local contact algebras and whose morphisms are all **DSHLC**-morphisms $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ satisfying the following condition:

(LO) $\forall a \in A$ and $\forall b \in \mathbb{B}'$, $\varphi_\Lambda(b)\rho a$ implies $b\eta\varphi(a)$.

It is easy to see that in this way we have indeed defined a category. Obviously, **DOHLC** (resp., **OHLC**) is a (non-full) subcategory of the category **DSHLC** (resp., **SHLC**).

Theorem 2.5.4.2. *The categories* **OHLC** *and* **DOHLC** *are dually equivalent.*

Proof. We will show that the restrictions

$$\Psi_o^a : \mathbf{DOHLC} \longrightarrow \mathbf{OHLC} \ \text{ and } \Psi_o^t : \mathbf{OHLC} \longrightarrow \mathbf{DOHLC}$$

of the contravariant functors Ψ_1^a and Ψ_1^t defined in the proof of Theorem 2.5.3.3 are the desired duality functors.

Let $f \in \mathbf{OHLC}((X, \tau), (Y, \tau'))$. Set

$$\varphi = \Psi_o^t(f).$$

Then, since f is an open map, [53, 1.4.C] implies that for every $F \in RC(Y)$,

$$f^{-1}(F) = f^{-1}(\mathrm{cl}(\mathrm{int}(F))) = \mathrm{cl}(f^{-1}(\mathrm{int}(F))) = \varphi(F).$$

Hence,

(2.35) $\Psi_o^t(f) : \Psi_o^t(Y, \tau') \longrightarrow \Psi_o^t(X, \tau)$ is defined by $\Psi_o^t(f)(F) = f^{-1}(F)$,

for all $F \in \Psi_o^t(Y, \tau')$. Further, by the proof of Theorem 2.5.3.3, φ is an **DSHLC**-morphism. We will show that φ satisfies condition (LO). For doing this, we will first show that $\varphi_\Lambda : RC(X) \longrightarrow RC(Y)$ is defined by the formula $\varphi_\Lambda(F) = \mathrm{cl}(f(F))$, for every $F \in RC(X)$. Indeed, using 2.5.2.6, we can define a map

(2.36) $\psi : \Psi^t(X,\tau) \longrightarrow \Psi^t(Y,\tau')$ by $\psi(G) = \mathrm{cl}(f(G))$, for every $G \in \Psi^t(X,\tau)$.

Obviously, ψ is an order-preserving map. Since f is a continuous map, we have that for every $F \in RC(Y)$, $\psi(\varphi(F)) = \mathrm{cl}(f(\mathrm{cl}(f^{-1}(\mathrm{int}(F))))) = \mathrm{cl}(f(f^{-1}(\mathrm{int}(F)))) \subseteq \mathrm{cl}(\mathrm{int}(F)) = F$, and, similarly, for every $G \in RC(X)$, $\varphi(\psi(G)) = \varphi(\mathrm{cl}(f(G))) = \mathrm{cl}(\mathrm{int}(f^{-1}(\mathrm{cl}(f(G))))) \supseteq \mathrm{cl}(\mathrm{int}(G)) = G$. Hence ψ is a left adjoint to φ (see 0.3.2.4), i.e.,

(2.37) $\psi = \varphi_\Lambda$.

Let now $F \in RC(Y)$, $G \in CR(X)$ and $F\rho_Y\varphi_\Lambda(G)$; then $F \cap f(G) \neq \emptyset$ and hence $f^{-1}(F) \cap G \neq \emptyset$; therefore, $\varphi(F)\rho_X G$. So, the axiom (LO) is fulfilled. Hence, $\Psi^t_o(f)$ is an **DOHLC**-morphism. Therefore, the contravariant functor Ψ^t_o is well defined.

Let $\varphi \in \mathbf{DOHLC}((A,\rho,\mathbb{B}),(B,\eta,\mathbb{B}'))$. Put $C = C_\rho$ and $C' = C_\eta$ (see 1.2.3.4 for the notation). Then, by 1.2.3.4, (A,C) and (B,C') are CNCA's.

Set $X = \Psi^a_o(A,\rho,\mathbb{B})$, $Y = \Psi^a_o(B,\eta,\mathbb{B}')$ and $f = \Psi^a_o(\varphi)$. Then, by the proof of Theorem 2.5.3.3, $f : Y \longrightarrow X$ is a continuous skeletal map. We are now going to show that f is an open map. By (1.36), it is enough to prove that, for every $b \in \mathbb{B}'$, $f(\mathrm{int}_Y(\lambda_B(b)))$ is an open subset of X (note that $\lambda_B(b) = \lambda^g_B(b)$ because $b \in \mathbb{B}'$).

Let us first note that if σ is a cluster in (B,C') then

(2.38) $b_1^*, b_2^* \notin \sigma$ implies that $b_1 \wedge b_2 \in \sigma$ and $(b_1 \wedge b_2)^* \notin \sigma$.

Indeed, if $b_1^*, b_2^* \notin \sigma$ then, by (G), $b_1^* \vee b_2^* \notin \sigma$, i.e., $(b_1 \wedge b_2)^* \notin \sigma$; hence $b_1 \wedge b_2 \in \sigma$.

Note also that, using (1.31) and 1.2.1.9, one can easily show that for all $a, b \in A$,

(2.39) $a \ll_\rho b$ implies that $\lambda^g_A(a) \subseteq \mathrm{int}_X(\lambda^g_A(b))$.

Let now $b \in \mathbb{B}'$. Let $\sigma \in f(\mathrm{int}_Y(\lambda_B(b)))$. Then there exists $\sigma' \in \mathrm{int}_Y(\lambda_B(b))$ such that $\sigma = f(\sigma')$. By (1.35), $b^* \notin \sigma'$. Then 2.2.2.16 implies that there exists $c_1 \in B$ such that $b^* \ll_{C'} c_1^*$ and $c_1^* \notin \sigma'$. Since σ' is a bounded cluster in (B,C'), (2.2.3.1) implies that there exists $c_2 \in \mathbb{B}'$ such that $c_2^* \notin \sigma'$. Put $b_1 = c_1 \wedge c_2$. Then $b_1 \in \mathbb{B}' \cap \sigma'$ (by (2.38)), $b_1^* \notin \sigma'$ (by (2.38)) and $b^* \ll_{C'} b_1^*$ (by ($\ll$3) (see 1.2.1.1)). Thus $b_1 \ll_{C'} b$. Therefore, by (1.35) and (2.39), $\sigma' \in \mathrm{int}_Y(\lambda_B(b_1)) \subseteq \lambda_B(b_1) \subseteq \mathrm{int}_Y(\lambda_B(b))$. By 1.2.2.3, there exists $u \in \mathrm{Ult}(B)$ such that $b_1 \in u \subseteq \sigma'$ and $\sigma' = \sigma_u$. Put $a = \varphi_\Lambda(b_1)$. Then, by (2.25), $a \in f(\sigma') = \sigma$. Suppose that $a^* \in \sigma$. We will show that this implies that $\varphi(a^*) \in \sigma'$.

Indeed, suppose that $\varphi(a^*) \notin \sigma'$. Then there exists $c_3 \in u$ such that $\varphi(a^*)(-C')c_3$. Set $b_2 = c_2 \wedge c_3$. Then $b_2 \in u \cap \mathbb{B}'$ and $\varphi(a^*)(-C')b_2$. Since $C' = C_\eta$, we obtain, by 1.2.3.4, that $\varphi(a^*)(-\eta)b_2$. Using condition (LO), we get that $a^*(-\rho)\varphi_\Lambda(b_2)$. Since $\varphi_\Lambda(b_2) \in \mathbb{B}$ (by (L2)), we obtain that $a^*(-C)\varphi_\Lambda(b_2)$ (see again 1.2.3.4). By (Λ1), $\varphi(\varphi_\Lambda(b_2)) \geq b_2$; thus $\varphi(\varphi_\Lambda(b_2)) \in u$. Hence $\varphi_\Lambda(b_2) \in \varphi^{-1}(u)$. Since $\sigma = f(\sigma') = \sigma_{\varphi^{-1}(u)}$ and $a^* \in \sigma$, we have that a^*Cc, for every $c \in \varphi^{-1}(u)$. Therefore $a^*C\varphi_\Lambda(b_2)$, a contradiction. Hence, $\varphi(a^*) \in \sigma'$, i.e., $(\varphi(\varphi_\Lambda(b_1)))^* \in \sigma'$. Since, by ($\Lambda$1), $b_1^* \geq (\varphi(\varphi_\Lambda(b_1)))^*$, we obtain that $b_1^* \in \sigma'$, a contradiction. Thus, $a^* \notin \sigma$. Then, using (1.35), (2.39) and (2.29), we obtain that $\sigma \in \mathrm{int}_X(\lambda_A(a)) \subseteq \lambda_A(a) = \lambda_A(\varphi_\Lambda(b_1)) = f(\lambda_B(b_1)) \subseteq f(\mathrm{int}_Y(\lambda_B(b)))$. Therefore, $f(\mathrm{int}_Y(\lambda_B(b)))$ is an open set in X. Thus, f is an open map. Hence Ψ_o^a is well defined.

Further, it is easy to see that λ_B^g is an **DOHLC**-isomorphism (use (1.31)). The rest follows from Theorem 2.5.3.3. □

Definition 2.5.4.3. Let **DOHC** be the category whose objects are all complete normal contact algebras and whose morphisms are all **DQHC**-morphisms $\varphi : (A, C) \longrightarrow (B, C')$ satisfying the following condition:

(CO) For all $a \in A$ and all $b \in B$, $aC\varphi_\Lambda(b)$ implies $\varphi(a)C'b$ (see 0.3.2.4 for φ_Λ).

It is easy to see that in this way we have indeed defined a category. The category **DOHC** (resp., **OHC**) is a (non-full) subcategory of the category **DQHC** (resp., **QHC**).

Theorem 2.5.4.4. *The categories* **OHC** *and* **DOHC** *are dually equivalent.*

Proof. By 1.2.3.2, the class of normal contact algebras coincides with the class of local contact algebras of the form (B, ρ, B) (i.e., those for which $\mathbb{B} = B$). Hence, for normal contact algebras, conditions (LO) and (CO) are identical. Now, using 1.2.2.8(b) (see also its proof), we get that the restriction of the contravariant functor Ψ_o^a, defined in the proof of Theorem 2.5.4.2, to the subcategory **DOHC** of the category **DOHLC** is the desired duality functor. □

Definition 2.5.4.5. Let **DPOHLC** be the category whose objects are all complete local contact algebras (see 1.2.3.1) and whose morphisms are all **DPSHLC**-morphisms satisfying condition (LO).

It is easy to see that in this way we have indeed defined a category. Clearly, **DPOHLC** (respectively, **POHLC**) is a subcategory of the category **DPSHLC** (respectively, **PSHLC**).

Theorem 2.5.4.6. *The categories* **POHLC** *and* **DPOHLC** *are dually equivalent.*

Proof. It follows from Theorems 2.5.3.7 and 2.5.4.2. □

Note that since the morphisms of the category **POHLC** are closed maps, the proof of Theorem 2.5.4.2 (see there the text immediately after (2.35)) shows that in the definition of the category **DPOHLC** (see 2.5.4.5) we can replace condition (LO) with the following one:

(LO') $\forall a \in A$ and $\forall b \in B$, $a\rho\varphi_\Lambda(b)$ implies $\varphi(a)\eta b$.

2.5.5 Duality Theorems for categories of connected spaces

Notation 2.5.5.1. Following the notation rules given in 2.2.4.4, we denote by:

• **PSHLCCon** the full subcategory of the category **PSHLC** having as objects all connected locally compact Hausdorff spaces;

• **DPSHLCCon** the full subcategory of the category **DPSHLC** having as objects all connected CLCA's.

Analogously, we introduce the notation **POHLCCon**, **DOHCCon**, **OHCCon**, **DPOHLCCon**, **SHCCon** and **DSHCCon** for the "connected versions" of the corresponding categories.

Theorem 2.5.5.2. *The categories* **PSHLCCon** *and* **DPSHLCCon** *are dually equivalent; in particular, the categories* **SHCCon** *and* **DSHCCon** *are dually equivalent.*

Proof. It follows immediately from 2.2.4.3, Theorem 2.5.3.7 and Theorem 2.5.2.11. □

Theorem 2.5.5.3. *The categories* **POHLCCon** *and* **DPOHLCCon** *are dually equivalent; in particular, the categories* **OHCCon** *and* **DOHCCon** *are dually equivalent.*

Proof. It follows immediately from 2.2.4.3, Theorem 2.5.4.4 and Theorem 2.5.4.6. □

Analogously we can formulate and prove the connected versions of the theorems 2.5.3.3 and 2.5.4.2.

2.5.6 Two generalizations of the Fedorchuk Equivalence Theorem

Definition 2.5.6.1. ([54]) Let **EQHC** be the category whose objects are all complete normal contact algebras and whose morphisms $\psi : (A, C) \longrightarrow (B, C')$ are all functions $\psi : A \longrightarrow B$ satisfying the following conditions:

(EF1) for every $a \in A$, $\psi(a) = 0$ iff $a = 0$;
(EF2) ψ preserves all joins;
(EF3) if $a \in A$, $b \in B$ and $b \leq \psi(a)$ then there exists $c \in A$ such that $c \leq a$ and $\psi(c) = b$;
(EF4) for every $a, b \in A$, aCb implies that $\psi(a)C'\psi(b)$.

In [54], V. V. Fedorchuk proved the following theorem:

Theorem 2.5.6.2. ([54]) *The categories* **QHC** *and* **EQHC** *are equivalent.*

We will now present a generalization of this theorem.

Definition 2.5.6.3. Let **ESHLC** be the category whose objects are all complete local contact algebras and whose morphisms $\psi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ are all functions $\psi : A \longrightarrow B$ satisfying conditions (EF1)-(EF3) (see Definition 2.5.6.1) and the following two constraints:

(EL4) for every $a, b \in A$, $a\rho b$ implies that $\psi(a)\eta\psi(b)$;
(EL5) if $a \in \mathbb{B}$ then $\psi(a) \in \mathbb{B}'$.

It is easy to see that in this way we have indeed defined a category.

The proof of the following theorem is similar to that of Theorem 2.5.6.2.

Theorem 2.5.6.4. *The categories* **SHLC** *and* **ESHLC** *are equivalent.*

Proof. Since the categories **SHLC** and **DSHLC** are dually equivalent (by Theorem 2.5.3.3), it is enough to show that the categories **ESHLC** and **DSHLC** are dually equivalent.

Let us define a contravariant functor

$$D_p : \mathbf{ESHLC} \longrightarrow \mathbf{DSHLC}.$$

Let D_p be the identity on the objects of the category **ESHLC** and let, for every $\psi \in \mathbf{ESHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$,

$$D_p(\psi) = \psi_P,$$

where ψ_P is the right adjoint of ψ (see 0.3.2.4 and (EF2)). Setting $\varphi = \psi_P$, we have to show that

$$\varphi \in \mathbf{DSHLC}((B, \eta, \mathbb{B}'), (A, \rho, \mathbb{B})).$$

As it is proved in [54], φ is a complete Boolean homomorphism. For completeness of our exposition, we will present here the Fedorchuk's proof. Note first that $\psi = \varphi_\Lambda$. By 0.3.2.4, φ preserves all meets in B. Since, by (EF1), $\psi(0) = 0$, we have that $\varphi(0) = \varphi(\psi(0))$; if $\varphi(0) > 0$ then, by (EF1) and 0.3.2.4, $0 = \psi(0) = \psi(\varphi(\psi(0))) > 0$, a contradiction. Hence $\varphi(0) = 0$. Further, since $\psi(1) \leq 1 \iff 1 \leq \varphi(1)$, we get that $\varphi(1) = 1$. Finally, $\varphi(b^*) = (\varphi(b))^*$, for every $b \in B$. Indeed, let $b \in B$. Set $a = \varphi(b) \wedge \varphi(b^*)$. Then, by 0.3.2.4, $\psi(a) \leq \psi(\varphi(b)) \wedge \psi(\varphi(b^*)) \leq b \wedge b^* = 0$. Hence $\psi(a) = 0$. Therefore, by (EF1), $a = 0$, i.e., $\varphi(b) \wedge \varphi(b^*) = 0$. Set now $c = \varphi(b) \vee \varphi(b^*)$ and suppose that $c < 1$. Then $c^* \neq 0$. Since $0 = c^* \wedge c = (c^* \wedge \varphi(b)) \vee (c^* \wedge \varphi(b^*))$, we have that $c^* \wedge \varphi(b) = 0 = c^* \wedge \varphi(b^*)$. By (EF1), $\psi(c^*) \neq 0$. Obviously, $\psi(c^*) = (\psi(c^*) \wedge b) \vee (\psi(c^*) \wedge b^*)$. Therefore, at least one of the elements $\psi(c^*) \wedge b$ and $\psi(c^*) \wedge b^*$ is different from 0. Let $\psi(c^*) \wedge b \neq 0$. By (EF3), the inequality $\psi(c^*) \wedge b \leq \psi(c^*)$ implies that there exists $d \in A$ such that $d \leq c^*$ and $\psi(d) = \psi(c^*) \wedge b$. Since $\psi(d) \neq 0$, we get, by (EF1), that $d \neq 0$. Further, $\psi(d) \leq b$ implies that $d \leq \varphi(b)$. Then $d \leq c^* \wedge \varphi(b) = 0$, i.e., $d = 0$, a contradiction. Analogously, we obtain a contradiction if $\psi(c^*) \wedge b^* \neq 0$. So, $c = 1$, i.e., $\varphi(b) \vee \varphi(b^*) = 1$. Hence, we have proved that $\varphi(b^*) = (\varphi(b))^*$. All this shows that φ is a complete Boolean homomorphism.

Since conditions (L1) and (EL1) (see 2.5.3.1) are equivalent and $\psi = \varphi_\Lambda$, (EL4) implies that φ satisfies condition (L1). Obviously, (EL5) implies that φ satisfies condition (L2) (see 2.5.3.1). So, φ is a **DSHLC**-morphism. Now, from $D_p(id) = id$ and the formula $(\psi_2 \circ \psi_1)_P = (\psi_1)_P \circ (\psi_2)_P$, we obtain that D_p is a contravariant functor.

Let us define a contravariant functor

$$D_l : \mathbf{DSHLC} \longrightarrow \mathbf{ESHLC}.$$

Let D_l be the identity on the objects of the category **DSHLC** and let, for every $\varphi \in \mathbf{DSHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$,

$$D_l(\varphi) = \varphi_\Lambda,$$

where φ_Λ is the left adjoint of φ (see 0.3.2.4). Setting $\psi = \varphi_\Lambda$, we have to show that

$$\psi \in \mathbf{ESHLC}((B, \eta, \mathbb{B}'), (A, \rho, \mathbb{B})).$$

Since $0 \leq \varphi(0)$ implies that $\psi(0) \leq 0$, we get that $\psi(0) = 0$. If $\psi(b) = 0$ then $\psi(b) \leq 0$ and hence $b \leq \varphi(0) = 0$, i.e., $b = 0$. Therefore, ψ satisfies condition (EF1). Further, conditions (EF2), (EL4) and (EL5) are clearly satisfied by ψ. Finally, let

$a \leq \psi(b)$. Set $c = b \wedge \varphi(a)$. Then $c \leq b$ and, by 2.5.2.1(b), $\psi(c) = a \wedge \psi(b) = a$. Therefore, ψ satisfies condition (EF3). So, ψ is an **ESHLC**-morphism. Now, it is clear that D_l is a contravariant functor. Since the compositions of D_p and D_l are the identity functors, we get that D_p is a duality. Put now

$$E_F^a = \Psi^a \circ D_p \text{ and } E_F^t = D_l \circ \Psi^t.$$

Then

$$E_F^a : \mathbf{ESHLC} \longrightarrow \mathbf{SHLC} \text{ and } E_F^t : \mathbf{SHLC} \longrightarrow \mathbf{ESHLC}$$

are the required equivalences. □

We are now going to obtain one more generalization of the Fedorchuk Equivalence Theorem 2.5.6.2.

Definition 2.5.6.5. Let **EPSHLC** be the category whose objects are all complete local contact algebras (see 1.2.3.1) and whose morphisms are all **ESHLC**-morphisms $\psi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ satisfying the following condition:

(EL6) if $b \in \mathbb{B}'$ then $\psi_P(b) \in \mathbb{B}$ (where ψ_P is the right adjoint of ψ (see 0.3.2.4)).

It is easy to see that in this way we have indeed defined a category.

Theorem 2.5.6.6. *The categories* **PSHLC** *and* **EPSHLC** *are equivalent.*

Proof. Using Theorem 2.5.3.7, it is enough to show that the categories **DPSHLC** and **EPSHLC** are dually equivalent. We will show that the restriction of the contravariant functor D_p (defined in the proof of Theorem 2.5.6.4) to the category **EPSHLC** is the required duality functor.

Let $\psi \in \mathbf{EPSHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$. Then, by (EL6), ψ_P satisfies condition (L3). Hence, by the proof of Theorem 2.5.6.4, $D_p(\psi)$ is a **DPSHLC**-morphism. Further, let us consider the restriction of the contravariant functor D_l (defined in the proof of Theorem 2.5.6.4) to the category **DPSHLC**. If φ is a **DPSHLC**-morphism then, by (L3), φ_Λ satisfies condition (EL6). Hence $D_l(\varphi)$ is an **EPSHLC**-morphism. Therefore, D_p is a duality. □

2.5.7 Equivalence Theorems for the categories OHLC, OHC, POHLC

Definition 2.5.7.1. Let **EOHLC** be the category whose objects are all complete local contact algebras and whose morphisms are all **ESHLC**-morphisms $\psi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ satisfying the following condition:

(EL7) if $b \in B$, $a \in \mathbb{B}$ and $\psi(a)\eta b$ then $a\rho\psi_P(b)$ (where ψ_P is the right adjoint of ψ (see 0.3.2.4)).

It is easy to see that in this way we have indeed defined a category.

Theorem 2.5.7.2. *The categories* **OHLC** *and* **EOHLC** *are equivalent.*

Proof. It is clear that if ψ satisfies condition (EL7) then ψ_P satisfies condition (LO) (see 2.5.4.1) and if φ satisfies condition (LO) then φ_Λ satisfies (EL7). Now, using Theorem 2.5.4.2, we argue as in the proof of Theorem 2.5.6.6. □

Definition 2.5.7.3. Let **EOC** be the category whose objects are all complete normal contact algebras and whose morphisms are all **EQHC**-morphisms $\psi : (A, C) \longrightarrow (B, C')$ satisfying the following condition:

(EC7) if $a \in A$, $b \in B$ and $\psi(a)\, C'\, b$ then $a\, C\, \psi_P(b)$ (where ψ_P is the right adjoint of ψ (see 0.3.2.4)).

It is easy to see that in this way we have indeed defined a category.

Theorem 2.5.7.4. *The categories* **OHC** *and* **EOC** *are equivalent.*

Proof. It follows directly from Theorem 2.5.7.2. □

Definition 2.5.7.5. Let **EPOHLC** be the category whose objects are all complete local contact algebras and whose morphisms are all **EPSHLC**-morphisms satisfying condition (EL7).

It is easy to see that in this way we have indeed defined a category.

Theorem 2.5.7.6. *The categories* **POHLC** *and* **EPOHLC** *are equivalent.*

Proof. It follows from the proofs of 2.5.6.6 and 2.5.7.2. □

Remark 2.5.7.7. A great part of our Theorem 2.5.6.4 is formulated (in another form) and proved in Roeper's paper [99]. Let us state precisely what is done there (using our notation). Roeper defines the notion of *mereological mapping*: such is any function $\psi : B \longrightarrow A$, where A and B are complete Boolean algebras, which satisfies the following conditions: (i) $\psi(b) = 0$ iff $b = 0$; (ii) $a \leq b$ implies $\psi(a) \leq \psi(b)$; (iii) if $0 \neq a \leq \psi(b)$, where $b \in B$ and $a \in A$, then there exists $b' \in B$ such that $0 \neq b' \leq b$ and $\psi(b') \leq a$. It is shown that any mereological mapping preserves all joins in B. Further, a mapping ψ of a CLCA $(B, \eta, \mathbb{B}')$ to another CLCA $(A, \rho, \mathbb{B})$ is called: (a) *continuous* if $a\eta b$ implies $\psi(a)\rho\psi(b)$, and (b) *bounded* if $\psi(b) \in \mathbb{B}$ when $b \in \mathbb{B}'$. It is shown that every continuous and bounded mereological mapping $\psi : (B, \eta, \mathbb{B}') \longrightarrow (A, \rho, \mathbb{B})$ generates a function $f_\psi : \Psi^a(B, \eta, \mathbb{B}') \longrightarrow \Psi^a(A, \rho, \mathbb{B})$, defined by the formula $f_\psi(\sigma_u) = \sigma_{\psi(u)}$, for every $u \in \mathrm{Ult}(B)$; the function f_ψ is continuous (in topological sense) and is such that $\mathrm{cl}(f_\psi(F))$ is regular closed when F is regular closed. It is proved that if $f : \Psi^a(B, \eta, \mathbb{B}') \longrightarrow \Psi^a(A, \rho, \mathbb{B})$ is a continuous function such that $\mathrm{cl}(f(F))$ is regular closed when F is regular closed then there exists a continuous and bounded mereological function $\psi : (B, \eta, \mathbb{B}') \longrightarrow (A, \rho, \mathbb{B})$ such that $f = f_\psi$. Finally, a mereological function $\psi : (B, \eta, \mathbb{B}') \longrightarrow (A, \rho, \mathbb{B})$ is called *topological* if $\psi(1_B) = 1_A$, $\psi(a)\rho\psi(b)$ iff $a\eta b$, and $\psi(b) \in \mathbb{B}$ iff $b \in \mathbb{B}'$; it is shown that if ψ is topological then f_ψ is a homeomorphism and if $f : \Psi^a(B, \eta, \mathbb{B}') \longrightarrow \Psi^a(A, \rho, \mathbb{B})$ is a homeomorphism then there exists a topological function $\psi : (B, \eta, \mathbb{B}') \longrightarrow (A, \rho, \mathbb{B})$ such that $f = f_\psi$.

It is easy to see that a function $\psi : B \longrightarrow A$ is mereological iff it satisfies conditions (EF1)-(EF3) (see Definition 2.5.6.1); ψ is continuous (respectively, bounded) iff it satisfies condition (EL4) (respectively, (EL5)). Further, Lemma 2.5.2.6 shows that a continuous map $f : X \longrightarrow Y$ satisfies Roeper's condition "$\mathrm{cl}(f(F)) \in RC(Y)$ when $F \in RC(X)$" iff f is a skeletal map. Therefore, our covariant functor E^a_F : **ESHLC** $\longrightarrow$ **SHLC** (see the proof of Theorem 2.5.6.4) was defined in [99] in another but equivalent form and it was shown there that E^a_F is full and isomorphism-dense; however, in [99] it was not shown that E^a_F is faithful.

2.6 The dual objects and the dual morphisms of some special subspaces and some special maps

2.6.1 Introduction

As it was shown by M. Stone [108], the dual objects of the closed subsets of a zero-dimensional compact Hausdorff space (briefly, *Stone space*) X are the quotients of the dual object of X. Also, M. Stone [108] proved that if $f : X \longrightarrow Y$ is a continuous map between two Stone spaces, then f is an injection (resp., surjection) iff its dual morphism is a surjection (resp., injection). Some similar results were obtained by de Vries [24] as well. Our investigations here are in this direction. In the second subsection of this section, we characterize the injective and surjective morphisms of the categories **HLC**, **PHLC**, **SHLC** and some of their subcategories discussed in the previous two sections of this chapter by means of some corresponding properties of their dual morphisms determined by the duality described in Theorem 2.2.2.12. In this way we generalize the corresponding results of de Vries [24] about the category **HC**. We characterize as well the homeomorphic embeddings, dense embeddings, LCA-embeddings, and closed embeddings by means of the corresponding properties of their dual maps. Our Theorem 2.6.2.12, in which we characterize LCA-embeddings, generalizes a theorem of Fedorchuk [54, Theorem 6].

In the third subsection, the dual object $\Lambda^t(M)$ of an open (respectively, regular open, clopen, compact open, regular closed, etc.) subset M of a locally compact Hausdorff space X is directly described by means of the dual object $\Lambda^t(X)$ of X. Some of these results (e.g., for regular closed sets) seem to be new even in the compact case.

The results of this section are based on the paper [31].

2.6.2 Characterizations of the embeddings, surjective and injective maps by means of their dual maps

In this subsection we will characterize the injective and surjective morphisms of the category **HLC** and its subcategories **PHLC**, **SHLC** and **OHLC**, by means of corresponding properties of their dual morphisms determined by the duality described in Theorem 2.2.2.12. In this way we will generalize some results of de Vries [24]. We will characterize as well the homeomorphic embeddings, dense embeddings, LCA-embeddings, etc. by means of their dual morphisms. Our result about LCA-embeddings (see Theorem 2.6.2.12) generalizes a theorem of Fedorchuk [54, Theorem 6].

Recall that if $X \in |\mathbf{HLC}|$ then $\Lambda^t(X) = (RC(X), \rho_X, CR(X))$.

We start with a simple observation.

Proposition 2.6.2.1. *Let $f \in \mathbf{HLC}(X, Y)$, $(A, \rho, \mathbb{B}) = \Lambda^t(X)$, $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then the following conditions are equivalent:*

(a) φ is an injection,

(b) $\varphi_{|\mathbb{B}'}$ is an injection,

(c) $\mathrm{cl}_Y(f(X)) = Y$.

Proof. We have that $\varphi : RC(Y) \longrightarrow RC(X)$ and $\varphi(G) = \mathrm{cl}(f^{-1}(\mathrm{int}(G)))$, for every $G \in RC(Y)$ (see Theorem 2.2.2.12).

Obviously, if φ is an injection then $\varphi_{|\mathbb{B}'}$ is an injection.

Let $\varphi_{|\mathbb{B}'}$ be an injection, $G \in CR(Y)$ and $G \neq \emptyset$. Then $\varphi(G) \neq \emptyset$, i.e., we obtain that $f^{-1}(\mathrm{int}(G)) \neq \emptyset$. This means that $f(X) \cap \mathrm{int}(G) \neq \emptyset$. Thus $\mathrm{cl}(f(X)) = Y$.

Finally, let $\mathrm{cl}(f(X)) = Y$, $G, H \in RC(Y)$, $G \neq H$ and $\varphi(G) = \varphi(H)$. Then, by the continuity of f,

$$\mathrm{cl}_Y(f(\mathrm{cl}_X(f^{-1}(\mathrm{int}_Y(G))))) = \mathrm{cl}_Y(f((f^{-1}(\mathrm{int}_Y(G)))) = \mathrm{cl}_Y(f(X) \cap \mathrm{int}_Y(G)) = G$$

and, analogously, $\mathrm{cl}_Y(f(\mathrm{cl}_X(f^{-1}(\mathrm{int}_Y(H))))) = H$. Hence $G = H$, a contradiction. So, φ is an injection. □

Theorem 2.6.2.2. *Let $f \in \mathbf{HLC}(X, Y)$, $(A, \rho, \mathbb{B}) = \Lambda^t(X)$, $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is an injection iff $\varphi : (B, \eta, \mathbb{B}') \longrightarrow (A, \rho, \mathbb{B})$ satisfies the following condition:*

(InHLC) *For any $a, b \in \mathbb{B}$, $a(-\rho)b$ implies that there exist $c, d \in \mathbb{B}'$ such that $c \ll_\eta d$, $a \leq \varphi(c)$ and $\varphi(d)(-\rho)b$.*

Proof. Let f be an injection. We will show that φ satisfies condition (InHLC). Let $F, G \in CR(X)$ and $F \cap G = \emptyset$. Since f is an injection, we get that $f(F) \cap f(G) = \emptyset$. Using the fact that $f(F)$ and $f(G)$ are compact sets, we get that there exist $F', G' \in CR(Y)$ such that $f(F) \subseteq \mathrm{int}(F') \subseteq F' \subseteq \mathrm{int}(G') \subseteq G' \subseteq Y \setminus f(G)$. Then, clearly, $F \subseteq f^{-1}(\mathrm{int}(F')) \subseteq \varphi(F')$ and $G \cap \varphi(G') = \emptyset$ (because $\varphi(G') \subseteq f^{-1}(G')$ and $f^{-1}(G') \cap G = \emptyset$). Therefore, φ satisfies condition (InHLC).

Let now φ satisfies condition (InHLC). We will prove that f is an injection. Let $x, y \in X$ and $x \neq y$. Then there exist disjoint $F_x, F_y \in CR(X)$ such that $x \in F_x$ and $y \in F_y$. Now, by condition (InHLC), there exist $G_x, G_y \in CR(Y)$ such that $G_x \subseteq \mathrm{int}(G_y)$, $F_x \subseteq \mathrm{cl}(f^{-1}(\mathrm{int}(G_x)))$ and $\mathrm{cl}(f^{-1}(\mathrm{int}(G_y))) \cap F_y = \emptyset$. Since $F_x \subseteq f^{-1}(G_x)$, we get that $f(x) \in G_x$. Further, we have that $F_y \cap f^{-1}(\mathrm{int}(G_y)) = \emptyset$. Thus $f(F_y) \cap \mathrm{int}(G_y) = \emptyset$. Then $f(F_y) \cap G_x = \emptyset$, and therefore, $f(x) \neq f(y)$. Hence, f is an injection. □

Theorem 2.6.2.3. *Let $f \in \mathbf{HLC}(X, Y)$, $(A, \rho, \mathbb{B}) = \Lambda^t(X)$, $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is a surjection iff $\varphi : (B, \eta, \mathbb{B}') \longrightarrow (A, \rho, \mathbb{B})$ satisfies the following condition:*

(SuHLC) *For any bounded ultrafilter v in $(B, \eta, \mathbb{B}')$ there exists a bounded ultrafilter u in $(A, \rho, \mathbb{B})$ such that $\forall b \in \mathbb{B}'$, $(b\eta v) \leftrightarrow ((\forall b' \in \mathbb{B}')[(b \ll_\eta b') \rightarrow (\varphi(b')\rho u)])$.*

Proof. Let f be a surjection and v be a bounded ultrafilter in $(B, \eta, \mathbb{B}')$. Recall that that $(B, \eta, \mathbb{B}') = (RC(Y), \rho_Y, CR(Y))$. Obviously, $\bigcap v$ is an one-point set. Let

$\{y\} = \bigcap v$. Since f is a surjection, there exists $x \in X$ such that $f(x) = y$. Recall our notation (see 1.2.2.6)

$$\sigma_x = \{F \in RC(X) \mid x \in F\} \text{ and } \nu_x = \{F \in RC(X) \mid x \in \mathrm{int}(F)\}.$$

There exists an ultrafilter u in $(A, \rho, \mathbb{B})$ such that $u \supseteq \nu_x$. Then it is easy to see that $u \subseteq \sigma_x$. Hence $\bigcap u = \{x\}$. Let now $G, H \in CR(Y)$. Clearly, if $y \in G$ then $G\rho_Y v$ (i.e., $G\eta v$). Conversely, let $G\rho_Y v$. If $y \notin G$ then, using [53, Corollary 3.1.5], we get that there exists $G' \in v$ such that $G \cap G' = \emptyset$, a contradiction. Hence, $G\rho_Y v$ iff $y \in G$. In an analogous way we obtain that $\mathrm{cl}(f^{-1}(\mathrm{int}(H)))\rho_X u$ iff $x \in \mathrm{cl}(f^{-1}(\mathrm{int}(H)))$. So, we have to show that $y \in G$ iff for all $H \in CR(Y)$, $(G \subseteq \mathrm{int}(H)) \to (x \in \mathrm{cl}(f^{-1}(\mathrm{int}(H))))$. Let $y \in G$ and $G \subseteq \mathrm{int}(H)$ for some $H \in CR(Y)$. Then $\mathrm{cl}(f^{-1}(\mathrm{int}(H))) \supseteq f^{-1}(G) \supseteq f^{-1}(y)$ and thus $x \in \mathrm{cl}(f^{-1}(\mathrm{int}(H)))$. Conversely, let $x \in \mathrm{cl}(f^{-1}(\mathrm{int}(H)))$ for every $H \in CR(Y)$ such that $G \subseteq \mathrm{int}(H)$. Suppose that $y \notin G$. Then there exists $H \in CR(Y)$ such that $G \subseteq \mathrm{int}(H) \subseteq H \subseteq Y \setminus \{y\}$. Then $\mathrm{cl}(f^{-1}(\mathrm{int}(H))) \subseteq f^{-1}(H) \subseteq X \setminus f^{-1}(y)$. Thus $x \notin \mathrm{cl}(f^{-1}(\mathrm{int}(H)))$, a contradiction. So, φ satisfies condition (SuHLC).

Let now φ satisfies condition (SuHLC). We will show that $f' = \Lambda^a(\varphi)$ is a surjection. This will imply that f is a surjection. Let $\sigma' \in Y' = \Lambda^a(B, \eta, \mathbb{B}')$. Then there exists a bounded ultrafilter v in $(B, \eta, \mathbb{B}')$ such that $\sigma' = \sigma_v$ (see Theorem 1.2.2.3). By condition (SuHLC), there exists a bounded ultrafilter u in $(A, \rho, \mathbb{B})$ such that $\forall b \in \mathbb{B}'$, $(b\eta v) \leftrightarrow ((\forall b' \in \mathbb{B}')[(b \ll_\eta b') \to (\varphi(b')\rho u)])$. Then it is easy to see that $f'(\sigma_u) = \sigma_v$ (see Theorem 2.2.2.12 for the definition of the map f'). Hence, f' is a surjection. □

The next theorem coincides, in fact, with our Theorem 2.4.4.1 which was proved with the help of a theorem of de Vries [24]. We will now give a direct proof of it using only our Proposition 2.6.2.1.

Theorem 2.6.2.4. *Let $f \in$ **PHLC**(X, Y) and $\varphi = \Lambda^t(f)$. Then f is a surjection iff φ is an injection.*

Proof. If f is a surjection then Proposition 2.6.2.1 implies that φ is an injection. Let now φ be an injection. Then, by Proposition 2.6.2.1, $\mathrm{cl}(f(X)) = Y$. Since f is a closed map, we get that f is a surjection. □

Obviously, Theorem 2.6.2.2 implies the following result:

Theorem 2.6.2.5. *Let $f \in$ **PHLC**(X, Y), $(A, \rho, \mathbb{B}) = \Lambda^t(X)$, $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is an injection iff φ satisfies condition (InHLC) (see Theorem 2.6.2.2).*

Note that a characterization of the injective **PHLC**-morphisms was given in our Theorem 2.4.4.5 which was derived from a theorem of de Vries [24]. It seems that the new condition looks better. Note also that Theorem 2.6.2.5 is new even in the compact case.

Theorem 2.6.2.6. *Let $f \in \mathbf{SHLC}(X,Y)$, $(A,\rho,\mathbb{B}) = \Lambda^t(X)$, $(B,\eta,\mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is a surjection if and only if $\varphi : (B,\eta,\mathbb{B}') \longrightarrow (A,\rho,\mathbb{B})$ satisfies the following condition:*

(SuSkeLC) *For every bounded ultrafilter v in $(B,\eta,\mathbb{B}')$ there exists a bounded ultrafilter u in $(A,\rho,\mathbb{B})$ such that $\varphi^{-1}(u)\eta v$.*

Proof. Let $f : X \longrightarrow Y$ be a surjective continuous skeletal map between two locally compact Hausdorff spaces and $\varphi = \Psi_1^t(f)$ (note that, by Proposition 2.5.3.4, $\Psi_1^t(f) = \Lambda^t(f)$). Then $\varphi : RC(Y) \longrightarrow RC(X)$ and $\varphi_\Lambda(F) = \mathrm{cl}(f(F))$, for every $F \in RC(X)$ (see (2.36) and (2.37)). Let v be a bounded ultrafilter in $RC(Y)$. Then there exists $G_0 \in CR(Y) \cap v$. Hence there exists $y \in \bigcap\{G \mid G \in v\}$. Since f is a surjection, there exists $x \in X$ such that $f(x) = y$. Let u be an ultrafilter in $RC(X)$ which contains ν_x (see 1.2.2.6 for ν_x). Then, obviously, u is a bounded ultrafilter in $(RC(X), \rho_X, CR(X))$. It is easy to see that $u \subseteq \sigma_x$ (see 1.2.2.6 for σ_x). Hence $y \in \varphi_\Lambda(F)$, for every $F \in u$. This means that $\varphi_\Lambda(u)\rho_Y v$. Since $\varphi_\Lambda(u)$ is a filter-base of $\varphi^{-1}(u)$ (see (2.23)), we get that $\varphi^{-1}(u)\rho_Y v$. Therefore, φ satisfies condition (SuSkeLC).

Let φ satisfies condition (SuSkeLC). Set $f' = \Psi_1^a(\varphi)$. Let $X' = \Lambda^a(A,\rho,\mathbb{B})$, $Y' = \Lambda^a(B,\eta,\mathbb{B}')$ and $\sigma \in Y'$. Then σ is a bounded cluster in $(B,\eta,\mathbb{B}')$. Hence there exists a bounded ultrafilter v in $(B,\eta,\mathbb{B}')$ such that $\sigma = \sigma_v$. By (SuSkeLC), there exists a bounded ultrafilter u in $(A,\rho,\mathbb{B})$ such that $\varphi^{-1}(u)\eta v$. Thus $\varphi^{-1}(u)C_\eta v$. Therefore, by (2.22), $f'(\sigma_u) = \sigma_{\varphi^{-1}(u)} = \sigma_v = \sigma$. So, f' is a surjection. Then, by Theorem 2.5.3.3, f is also a surjection. □

Obviously, in condition (SuSkeLC), "$\varphi^{-1}(u)\eta v$" can be replaced by "$\varphi_\Lambda(u)\eta v$".

Note that it is easy to see that condition (SuSkeLC) implies condition (SuHLC) when φ is a **DSHLC**-morphism. This provides with a new proof the sufficiency part of Theorem 2.6.2.6.

Theorem 2.6.2.7. *Let $f \in \mathbf{SHLC}(X,Y)$, $(A,\rho,\mathbb{B}) = \Lambda^t(X)$, $(B,\eta,\mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is an injection if and only if $\varphi : (B,\eta,\mathbb{B}') \longrightarrow (A,\rho,\mathbb{B})$ satisfies the following condition:*

(InSkeLC) $\forall a,b \in \mathbb{B}$, *$\varphi_\Lambda(a)\eta\varphi_\Lambda(b)$ implies $a\rho b$ (here φ_Λ is the left adjoint of φ).*

Proof. Let $f : X \longrightarrow Y$ be an injective continuous skeletal map. The function $\varphi_\Lambda : RC(X) \longrightarrow RC(Y)$ is defined by $\varphi_\Lambda(F) = \mathrm{cl}(f(F))$, for every $F \in RC(X)$ (see (2.36) and (2.37)). Hence, for $F \in CR(X)$, $\varphi_\Lambda(F) = f(F)$. Since f is an injection, it becomes obvious that φ satisfies condition (InSkeLC).

Let φ satisfies condition (InSkeLC). We will show that f is an injection. Let $x, y \in X$ and $x \neq y$. Then there exist disjoint $F_x, F_y \in CR(X)$ such that $x \in F_x$ and $y \in F_y$. If $f(x) = f(y)$ then $f(F_x) \cap f(F_y) \neq \emptyset$, i.e., $\varphi_\Lambda(F_x)\eta\varphi_\Lambda(F_y)$, and, hence, by (InSkeLC), $F_x \cap F_y \neq \emptyset$, a contradiction. Thus, $f(x) \neq f(y)$. □

Again, it is easy to see that condition (InSkeLC) implies condition (InHLC) when φ is a **DSHLC**-morphism.

Theorem 2.6.2.8. *Let $f \in \mathbf{OHLC}(X, Y)$, $(A, \rho, \mathbb{B}) = \Lambda^t(X)$, $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is an injection if and only if φ is a surjection.*

Proof. Let φ be a surjection. Let $a, b \in \mathbb{B}$ and $\varphi_\Lambda(a)\eta\varphi_\Lambda(b)$. Then, by condition (LO) (see Definition 2.5.4.1), $\varphi(\varphi_\Lambda(a))\rho b$. Since surjectivity of φ implies that $\varphi(\varphi_\Lambda(a)) = a$, we get that $a\rho b$. Therefore, φ satisfies condition (InSkeLC). Hence, by Theorem 2.6.2.7, f is an injection.

Let $f : X \longrightarrow Y$ be an injective open map. Then $\varphi(G) = f^{-1}(G)$, for every $G \in RC(Y)$ (see (2.35)). For every $F \in RC(X)$ we have, by Corollary 2.5.2.5(a) and Lemma 2.5.2.6, that $\mathrm{cl}(f(F)) \in RC(Y)$. Set $G = \mathrm{cl}(f(F))$. Then, by [53, 1.4.C], $f^{-1}(G) = \mathrm{cl}(f^{-1}(f(F)))$ (because f is an open map), and the injectivity of f implies that $f^{-1}(G) = F$. Hence $\varphi(G) = F$. Therefore, φ is a surjection. □

If $f \in \mathbf{OHLC}(X, Y)$ then, obviously, $f \in \mathbf{SHLC}(X, Y)$ and thus for determining when f is a surjection, we can use Theorem 2.6.2.6; therefore, f is a surjection iff it satisfies condition (SuSkeLC).

Now we will be occupied with the homeomorphic embeddings. We will call them *embeddings* for short.

Theorem 2.6.2.9. *Let $f \in \mathbf{HLC}(X, Y)$, $(A, \rho, \mathbb{B}) = \Lambda^t(X)$, $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is a dense embedding iff φ is a Boolean isomorphism satisfying the following condition:*

(LO″) $\forall b \in B$ and $\forall a \in \mathbb{B}$, $\varphi^{-1}(a)\eta b$ implies $a\rho\varphi(b)$.

Proof. Let f be a dense embedding of X in Y. Then $f(X)$ is a locally compact dense subspace of Y and hence it is open in Y. Thus f is an open injection. Therefore, by Theorem 2.5.4.2 and Proposition 2.5.3.4, φ is a complete homomorphism satisfying condition (LO) (see Definition 2.5.4.1). Put $Z = f(X)$ and let $i : Z \longrightarrow Y$ be the embedding of Z in Y. Then $\psi = \Lambda^t(i) : RC(Y) \longrightarrow RC(Z)$ is defined by the formula $\psi(F) = cl_Z(Z \cap \mathrm{int}_Y(F)) = F \cap Z$, for every $F \in RC(Y)$. Hence, by Lemma 0.4.2.2, ψ is a Boolean isomorphism. Since $f = i \circ f_{\restriction X}$, we obtain that φ is a Boolean isomorphism as well. Then $\varphi_\Lambda = \varphi^{-1}$ and thus condition (LO) coincides with condition (LO$''$) (because the only one difference between these two conditions is that φ^{-1} in (LO$''$) is replaced with φ_Λ in (LO)). So, φ is a Boolean isomorphism satisfying condition (LO$''$).

Conversely, let φ be a Boolean isomorphism satisfying condition (LO$''$). Then φ is a complete homomorphism satisfying condition (LO). Obviously, condition (DLC3S) implies condition (L1). By Proposition 2.5.3.2, condition (DLC4) implies condition (L2). Hence, φ is a **DOHLC**-morphism (see Definition 2.5.4.1). Thus, by Theorem 2.5.4.2, f is an open map. Since, by Theorem 2.6.2.8, f is an injection, we get that f is an embedding. Finally, by Proposition 2.6.2.1, $f(X)$ is dense in Y. □

Remark 2.6.2.10. Note that, in the notation of Theorem 2.6.2.9, f is a closed embedding iff φ satisfies conditions (PAL5) (see Definition 2.4.2.1) and (InHLC) (see Theorem 2.6.2.2); this follows from Theorems 2.4.2.2 and 2.6.2.2.

Proposition 2.6.2.11. *Let $f \in$ **HLC**(X, Y), $(A, \rho, \mathbb{B}) = \Lambda^t(X)$, $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$ and $\varphi = \Lambda^t(f)$. Then f is an embedding iff there exists a complete LCA $(A_1, \rho_1, \mathbb{B}_1)$ and* **DHLC**-*morphisms $\varphi_1 : (A_1, \rho_1, \mathbb{B}_1) \longrightarrow (A, \rho, \mathbb{B})$ and $\varphi_2 : (B, \eta, \mathbb{B}') \longrightarrow (A_1, \rho_1, \mathbb{B}_1)$ such that $\varphi = \varphi_1 \circ \varphi_2$, φ_1 is a Boolean isomorphism satisfying condition (LO$''$) and φ_2 satisfies conditions (PAL5) and (InHLC).*

Proof. Obviously, f is an embedding iff $f = i \circ f_1$ where f_1 is a dense embedding and i is a closed embedding. (Indeed, when f is an embedding then let $f_1 : X \longrightarrow \mathrm{cl}_Y(f(X))$ be the restriction of f and $i : \mathrm{cl}_Y(f(X)) \longrightarrow Y$ be the inclusion map; the converse is also clear.) Setting $\varphi_1 = \Lambda^t(f_1)$ and $\varphi_2 = \Lambda^t(i)$, we get, by Theorem 2.2.2.12, that $\varphi = \varphi_1 \circ \varphi_2$. Now our assertion follows from Theorem 2.6.2.9 and Remark 2.6.2.10. □

Now we are going to characterize those **DSHLC**-morphisms which are LCA-embeddings. This will imply a generalization of a theorem of Fedorchuk [54, Theorem 6].

Recall that a continuous mapping $f : X \longrightarrow Y$ is said to be *semi-open* ([119]) if for every point $y \in f(X)$ there exists a point $x \in f^{-1}(y)$ such that, for every $M \subseteq X$, $x \in \mathrm{int}_X(M)$ implies that $y \in \mathrm{int}_{f(X)}(f(M))$.

Theorem 2.6.2.12. *Let $f \in \mathbf{SHLC}(Y, X)$. Then $\varphi = \Lambda^t(f)$ is an LCA-embedding iff f is a semi-open perfect surjection.*

Proof. Note that, by Theorem 2.5.3.3 and Proposition 2.5.3.4, φ is a complete Boolean homomorphism. Recall also that when the map f is closed then it is quasi-open (see Corollary 2.5.2.5(b)).

Let $(A, \rho, \mathbb{B}) = \Lambda^t(X)$ and $(B, \eta, \mathbb{B}') = \Lambda^t(Y)$. Then $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$. Set $C = C_\rho$ and $C' = C_\eta$ (see Definition 1.2.3.4 for the notation). Then, by Lemma 1.2.3.5, (A, C) and (B, C') are CNCA's. By (1.26) and (1.22), $\Lambda^a(A, C) = \alpha X = X \cup \{\sigma_\infty^A\}$ and $\Lambda^a(B, C') = \alpha Y = Y \cup \{\sigma_\infty^B\}$.

Let φ be an LCA-embedding, i.e., $\varphi : A \longrightarrow B$ is a Boolean embedding such that, for any $a, b \in A$, $a\rho b$ iff $\varphi(a)\eta\varphi(b)$, and $a \in \mathbb{B}$ iff $\varphi(a) \in \mathbb{B}'$; hence φ satisfies condition (PAL5) (see Definition 2.4.2.1). Then, by Theorems 2.6.2.4 and 2.4.2.2, f is a perfect surjection. It remains to show that f is semi-open. Denote by φ_c the map φ regarded as a function from (A, C) to (B, C'). Obviously, φ_c satisfies condition (F1) (see 2.5.2.10). We will show that φ_c is an NCA-embedding. Indeed, for any $a, b \in A$, we have that aCb iff $a\rho b$ or $a, b \notin \mathbb{B}$; since φ is an LCA-embedding, we obtain that aCb iff $\varphi_c(a)C'\varphi_c(b)$. So, φ_c is an NCA-embedding and a **DQHC**-morphism (see Theorem 2.5.2.11). Then, by Theorem 6 of Fedorchuk's paper [54], $f_c = \Lambda^a(\varphi_c) : \alpha Y \longrightarrow \alpha X$ is a semi-open map. If $1_A \notin \mathbb{B}$ and $1_B \notin \mathbb{B}'$ then $f_c^{-1}(\sigma_\infty^A) = \{\sigma_\infty^B\}$ (see (2.33)) and since $f = (f_c)_{|Y}$, we obtain that f is semi-open. Further, if $1_A \in \mathbb{B}$ and $1_B \in \mathbb{B}'$ then the fact that f is semi-open is obvious. Since only these two cases are possible in the given situation, we have proved that f is a perfect quasi-open semi-open surjection.

Conversely, let f be a perfect semi-open surjection. Then, by Theorem 2.6.2.4, φ is an injection. Hence $\varphi_\Lambda \circ \varphi = id_A$. Thus, if $\varphi(a) \in \mathbb{B}'$ then, by (L2) (see Theorem 2.5.3.3 and Definition 2.5.3.1), $a = \varphi_\Lambda(\varphi(a)) \in \mathbb{B}$. Since f is perfect, Theorem 2.4.2.2 implies that φ satisfies condition (PAL5). Using it, we obtain that $a \in \mathbb{B}$ iff $\varphi(a) \in \mathbb{B}'$. Since (L1) takes place (see Theorem 2.5.3.3 and Definition 2.5.3.1), it remains only to prove that $a\rho b$ implies $\varphi(a)\eta\varphi(b)$, for all $a, b \in A$. Let $F, G \in RC(X)$, $F \cap G \neq \emptyset$ and $x \in F \cap G$. Set $U = \mathrm{int}(F)$ and $V = \mathrm{int}(G)$. Then $x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$. Since f is a semi-open surjection, there exists $y \in f^{-1}(x)$ such that, for every $M \subseteq Y$, $y \in \mathrm{int}_Y(M)$ implies that $x \in \mathrm{int}_X(f(M))$. We will show that $y \in \mathrm{cl}(f^{-1}(U)) \cap \mathrm{cl}(f^{-1}(V))$. Indeed, suppose that $y \notin \mathrm{cl}(f^{-1}(U))$. Then there exists an open neighborhood Oy of y such that $Oy \cap f^{-1}(U) = \emptyset$. Thus $f(Oy) \cap U = \emptyset$. Since $x \in \mathrm{cl}(U)$ and $x \in \mathrm{int}(f(Oy))$, we obtain a contradiction. Hence $y \in \mathrm{cl}(f^{-1}(U))$. Analogously we show that $y \in \mathrm{cl}(f^{-1}(V))$.

Therefore, $y \in \mathrm{cl}(f^{-1}(U)) \cap \mathrm{cl}(f^{-1}(V)) = \varphi(F) \cap \varphi(G)$. So, we get that $a\rho b$ implies $\varphi(a)\eta\varphi(b)$. Therefore, φ is an LCA-embedding. $\square$

2.6.3 The construction of the dual objects of the open and regular closed sets

In Theorem 2.2.2.4, we proved that the frame of all open subsets of a locally compact Hausdorff space X is isomorphic to the frame of δ-ideals of $\Lambda^t(X)$. Here we determine the types of δ-ideals which correspond to clopen, regular open and compact open subsets of $X \in |\mathbf{HLC}|$. This is by analogy with the results of M. Stone from [108]. Moreover, we describe explicitly the dual objects of the open subsets of a locally compact Hausdorff space X using only the dual object of X and the corresponding δ-ideal. We also show how the dual object $\Lambda^t(F)$ of a regular closed subset F of a locally compact Hausdorff space X can be constructed by means of the dual object $\Lambda^t(X)$ of X.

By analogy with the Stone's terminology from [107, 108], a δ-ideal J of an LCA $(A, \rho, \mathbb{B})$ will be called a *simple δ-ideal* if it has a complement in the frame $I(A, \rho, \mathbb{B})$, i.e., if $J \vee \neg J = \mathbb{B}$ (here $\neg J$ is the pseudocomplement of J in the frame $I(A, \rho, \mathbb{B})$) (see Definition 2.2.2.1 for the notation); further, the regular elements of the frame $I(A, \rho, \mathbb{B})$ (i.e., those $J \in I(A, \rho, \mathbb{B})$ for which $\neg\neg J = J$) will be called *normal δ-ideals*.

Corollary 2.6.3.1. *Let $(A, \rho, \mathbb{B})$ be a CLCA, $(X, \mathcal{O}) = \Lambda^a(A, \rho, \mathbb{B})$, J be a δ-ideal of $(A, \rho, \mathbb{B})$ and $U = \iota(J)$ (see Theorem 2.2.2.4 for ι). Then:*

(a) U is a clopen set iff J is a simple δ-ideal of $(A, \rho, \mathbb{B})$;

(b) U is a regular open set $\iff$ J is a normal δ-ideal of $(A, \rho, \mathbb{B})$ $\iff$ J is a principal δ-ideal of $(A, \rho, \mathbb{B})$;

(c) U is a compact open set iff J is a principal ideal of $\mathbb{B}$.

Proof. Since the map ι is a frame isomorphism (see Theorem 2.2.2.4), it preserves and reflects the regular elements and the elements which have a complement. Note also that the pseudocomplement $\neg U$ of U in the frame $(\mathcal{O}, \subseteq)$ is the set $\mathrm{int}(X \setminus U)$.

(a) Clearly, U is a clopen set iff it has a complement in the frame $(\mathcal{O}, \subseteq)$ iff J is a simple δ-ideal.

(b) Obviously, U is a regular open set iff it is a regular element of the frame $(\mathcal{O}, \subseteq)$. Thus our assertion follows from the second statement in Theorem 2.2.2.4.

(c) We have that U is a compact open set $\iff U = \lambda^g_A(a)$ for some $a \in \mathbb{B}$ such that $a \ll_\rho a \iff U = \lambda^g_A(a)$ for some $a \in \mathbb{B}$ such that the set $\{b \in \mathbb{B} \mid b \leq a\}$ is a δ-ideal of $(A, \rho, \mathbb{B}) \iff J = \{b \in \mathbb{B} \mid b \leq a\}$ for some $a \in \mathbb{B}$. □

We have seen that the open sets correspond to the δ-ideals. Now we are going to describe explicitly the dual objects of the open subsets of a locally compact Hausdorff space X using only the dual object of X and the corresponding δ-ideal.

Recall that if A is a Boolean algebra and $a \in A$ then the set $\downarrow(a)$ endowed with the same meets and joins as in A and with complements b' defined by the formula

$$b' = b^* \wedge a,$$

for every $b \leq a$, is a Boolean algebra; it is denoted by

$$A|a.$$

If $J = \downarrow(a^*)$ then $A|a$ is isomorphic to the factor algebra A/J; the isomorphism $h : A|a \longrightarrow A/J$ is the following: $h(b) = [b]$, for every $b \leq a$ (see, e.g., [102]).

Theorem 2.6.3.2. *Let X be a locally compact Hausdorff space and U be an open subset of X. Let $a_U = \mathrm{cl}_X(U)$, $I = \{F \in CR(X) \mid F \subseteq U\}$ and $B = RC(X)|a_U$. For every $a, b \in B$, set $a\eta b$ iff there exist $c, d \in I$ such that $c\rho_X d$ (i.e., $c \cap d \neq \emptyset$), $c \leq a$ and $d \leq b$. Then (B, η, I) is LCA-isomorphic to $\Lambda^t(U)$.*

Proof. Set, for short, $(A, \rho, \mathbb{B}) = \Lambda^t(X)$ (i.e., $(A, \rho, \mathbb{B}) = (RC(X), \rho_X, CR(X))$) and let the map $\varphi : A \longrightarrow B$ be defined by the formula $\varphi(a) = a \wedge a_U$, for every $a \in A$. Then, obviously, B is a complete Boolean algebra and φ is a surjective complete Boolean homomorphism.

It is easy to see that I is a δ-ideal of $(A, \rho, \mathbb{B})$ and $a_U = \bigvee_A I$.

If $I = \{0\}$ then $U = \emptyset$, $a_U = 0$, $B = \{0\}$; hence, in this case the assertion of the theorem is true. Thus, let us assume that $I \neq \{0\}$.

We will first check that (B, η, I) is a CLCA, i.e., that conditions (C1)-(C4) and (BC1)-(BC3) (see Definitions 1.2.1.1 and 1.2.3.1) are fulfilled. Note that, for every $a, b \in B$, $a\eta b$ implies that $a\rho b$; thus, if $a \ll_\rho b$ then $a(-\rho)b^*$ and hence $a(-\rho)(b^* \wedge a_U)$, which implies that $a \ll_\eta b$.

Let $b \in B \setminus \{0\}$. Then $b = \bigvee\{c \wedge b \mid c \in I\}$. Thus there exists $c \in I$ such that $c \wedge b \neq 0$. We get that $d = c \wedge b \in I$, $d \leq b$ and $d\rho d$. Therefore $b\eta b$. So, the axiom (C1) is fulfilled. Using the same notation, we get that there exists $a \in \mathbb{B} \setminus \{0\}$ such that $a \ll_\rho d$. Then $a \in I \setminus \{0\}$ and $a \ll_\eta b$. Therefore, the axiom (BC3) is checked as well. Clearly, the axioms (C2), (C3) and (BC2) are satisfied. Let $a, b_1, b_2 \in B$ and $a\eta(b_1 \vee b_2)$. Then there exist $c, d \in I$ such that $c \leq a$, $d \leq b_1 \vee b_2$ and $c\rho d$. Since $d = (d \wedge b_1) \vee (d \wedge b_2)$, we get that either $c\rho(d \wedge b_1)$ or $c\rho(d \wedge b_2)$. Clearly, this implies that either $a\eta b_1$ or $a\eta b_2$. The converse implication is obvious. So, we obtain that the axiom (C4) is also fulfilled.

Let $a \in I$, $b \in B$ and $a \ll_\eta b$. Then $a(-\eta)(b^* \wedge a_U)$. Thus, for every $c \in I$ such that $c \leq b^*$, we have that $a(-\rho)c$. Since $a \in I$ and I is a δ-ideal of $(A, \rho, \mathbb{B})$, we get that there exists $c \in I$ such that $a \ll_\rho c \ll_\rho a_U$. Then $c \wedge b^* \leq b^*$ and $c \wedge b^* \in I$. Thus $a(-\rho)(c \wedge b^*)$, i.e $a \ll_\rho (c^* \vee b)$. Combining this fact with the inequality $a \ll_\rho c$, we get that $a \ll_\rho (c \wedge (c^* \vee b))$, i.e., $a \ll_\rho (b \wedge c)$. Then there exists $d \in \mathbb{B}$ such that $a \ll_\rho d \ll_\rho (c \wedge b)$. Since $c \wedge b \in I$, we get that $d \in I$. Therefore, $a \ll_\rho d \ll_\rho b$. This implies that $a \ll_\eta d \ll_\eta b$. Thus, the axiom (BC1) is checked.

So, we have proved that $(B, \eta, \mathbb{B}')$ is a CLCA.

We will show that φ is a **DOHLC**-morphism, i.e., that φ satisfies axioms (L1), (L2) and (LO) (see Theorems 2.5.3.3 and 2.5.4.2). Note first that, for every $a \in B$,

(2.40) $\varphi_\Lambda(a) = a$.

This observation shows that φ satisfies conditions (L2) and (EL1) (note that condition (EL1) is equivalent to the condition (L1)). Let us prove that the axiom (LO) is fulfilled as well. Let $a \in A$, $b \in I$ and $\varphi_\Lambda(b)\rho a$. Then $a\rho b$. We have to show that $b\eta\varphi(a)$, i.e., that $b\eta(a \wedge a_U)$. Suppose that $b(-\eta)(a \wedge a_U)$. Then, for every $c \in I$ such that $c \leq a$, we have that $b(-\rho)c$. Since I is a δ-ideal of $(A, \rho, \mathbb{B})$, there exists $d \in I$ such that $b \ll_\rho d \ll_\rho a_U$. Then $d \wedge a \in I$ and $d \leq a$. Hence $b(-\rho)(d \wedge a)$, i.e., $b \ll_\rho (d^* \vee a^*)$. Since $b \ll_\rho d$, we get that $b \ll_\rho (d \wedge (d^* \vee a^*))$. Thus $b \ll_\rho a^*$, i.e., $b(-\rho)a$, a contradiction. Therefore, condition (LO) is checked. So, φ is a **DOHLC**-morphism. Thus, if we set $f = \Lambda^a(\varphi)$, then Theorem 2.5.4.2 and Proposition 2.5.3.4 imply that f is an open mapping.

Set $X' = \Lambda^a(A, \rho, \mathbb{B})$, $U' = \Lambda^a(B, \eta, I)$ and $V = \iota(I)$ (i.e., by Theorem 2.2.2.4, $V = \bigcup\{\lambda_A^g(b) \mid b \in I\}$). Then V is open in X' (see Theorem 2.2.2.4) and $\mathrm{cl}(V) = \lambda_A^g(a_U)$. The fact that I is a δ-ideal of $(A, \rho, \mathbb{B})$ implies that $\{\mathrm{int}(\lambda_A^g(b)) \mid b \in I\}$ is an open cover of V. Since φ is a surjection, we obtain, by Theorem 2.6.2.8, that $f : U' \longrightarrow X'$ is an open injection and hence f is a homeomorphism between U' and $f(U')$. Let us show that $f(U') = V$. Recall that the function f is defined by the formula $f(\sigma_u) = \sigma_{\varphi^{-1}(u)} (= \sigma_{\varphi_\Lambda(u)})$, where u is a bounded ultrafilter in (B, η, I) (see

Theorem 2.5.3.3 and Proposition 2.5.3.4). Now, if $\sigma_u \in U'$ then there exists $a \in I \cap u$. Since $\varphi(a) = a \wedge a_U = a$, we get that $a \in \varphi^{-1}(u)$. Thus $f(\sigma_u) \in \lambda_A^g(a) \subseteq V$. Hence, $f(U') \subseteq V$. Conversely, if $\sigma' \in V$ then there exists $a \in \sigma' \cap I$. Thus, there exists an ultrafilter v in A such that $a \in v$ and $\sigma' = \sigma_v$. Obviously, $v \cap I$ is a filter-base of v (because $a \in v \cap I$ and I is an ideal). It is clear that $u = v \cap B$ is a bounded filter in (B, η, I). Moreover, u is an ultrafilter in B. Indeed, let $c \in B = \downarrow_A (a_U)$. If $c \in v$ then $c \in u$. If $c^* \in v$ then $a \wedge c^* \in v \cap B$ and thus $c' = c^* \wedge a_U \in u$. Hence, u is a bounded ultrafilter in (B, η, I). Since $\varphi_\Lambda(u) = u$ and u is a filter-base of v, we get that $f(\sigma_u) = \sigma_{\varphi_\Lambda(u)} = \sigma_v = \sigma'$. Therefore, $f(U') = V$. Hence, U' is homeomorphic to V. We will now show that U is homeomorphic to V. Since the map $t_X : X \longrightarrow X'$, $x \mapsto \sigma_x$, is a homeomorphism (see the proof of Theorem 1.2.3.10), it is enough to show that $t_X(U) = V$. Recall that $V = \bigcup\{\lambda_A^g(a) \mid a \in I\} = \{\sigma \in X' \mid \sigma \cap I \neq \emptyset\}$. Let now $x \in U$. Then there exists $F \in I$ such that $x \in F$. Then $F \in \sigma_x$ and thus $\sigma_x \in V$, i.e., $t_X(x) \in V$. Hence $t_X(U) \subseteq V$. Conversely, let $\sigma \in V$. Then there exists $F \in \sigma \cap I$. Since $V \subseteq X'$ and t_X is a surjection, there exists $x \in X$ such that $\sigma = t_X(x) = \sigma_x$. This implies that $x \in F$ and thus $x \in U$. So, $V \subseteq t_X(U)$. Hence, U is homeomorphic to V. Therefore, U is homeomorphic to U'. Now, by Theorem 2.2.2.12, $\Lambda^t(U)$ is LCA-isomorphic to (B, η, I). □

We will now show how one can construct the dual object $\Lambda^t(F)$ of a regular closed subset F of a locally compact Hausdorff space X using only F and the dual object $\Lambda^t(X)$ of X.

Theorem 2.6.3.3. *Let X be a locally compact Hausdorff space and $F \in RC(X)$. Set $B = RC(X)|F$, $\mathbb{B}' = \{G \wedge F \mid G \in CR(X)\}$ and let, for every $a, b \in B$, $a\eta b$ iff $a\rho_X b$ (i.e., $a \cap b \neq \emptyset$). Then $(B, \eta, \mathbb{B}')$ is LCA-isomorphic to $\Lambda^t(F)$.*

Proof. Set, for short, $(A, \rho, \mathbb{B}) = \Lambda^t(X)$ (i.e., $(A, \rho, \mathbb{B}) = (RC(X), \rho_X, CR(X))$) and let $\varphi : A \longrightarrow B$ be defined by the formula $\varphi(G) = G \wedge F$, for every $G \in A$. Then B is a complete Boolean algebra, φ is a complete Boolean homomorphism and $\varphi_\Lambda(a) = a$, for every $a \in B$. We will show that $\psi = (\varphi_\Lambda)_{\restriction B} : B \longrightarrow \varphi_\Lambda(B)$ is a Boolean isomorphism between B and $RC(F)$. Since $F \in RC(X)$, we have, as it is well known, that $RC(F) \subseteq RC(X)$ and $RC(F) = \{G \wedge F \mid G \in RC(X)\}$; moreover, $RC(F) = RC(X)|F$. Hence $\psi : B \longrightarrow RC(F)$ is a Boolean isomorphism. Further, for any $a, b \in B$, we have that $a\eta b \iff \varphi_\Lambda(a)\rho\varphi_\Lambda(b) \iff \psi(a)\rho_F\psi(b)$. Finally, for any $a \in B$, we have that $a \in \mathbb{B}' \iff \varphi_\Lambda(a) \in \mathbb{B} \iff \varphi_\Lambda(a)$ is compact $\iff \psi(a) \in CR(F)$. Therefore, $(B, \eta, \mathbb{B}')$ is a CLCA because $(RC(F), \rho_F, CR(F))$ is such, and they are isomorphic.

Let $X' = \Lambda^a(A, \rho, \mathbb{B})$, $G = \Lambda^a(B, \eta, \mathbb{B}')$, $f = \Lambda^a(\varphi)$ and $F' = \lambda_A^g(F)$. We will show that $f : G \longrightarrow X'$ is a homeomorphic embedding and $f(G) = F'$. Note that φ satisfies conditions (L1), (L2), (PAL5) and condition (InSkeLC), and hence, by Theorems 2.5.3.3, 2.6.2.7 and 2.4.2.8, f is a quasi-open perfect injection, i.e., f is a homeomorphic embedding. From (2.29) we get that, for every $b \in \mathbb{B}'$, $f(\lambda_B^g(b)) = \lambda_A^g(\varphi_\Lambda(b)) = \lambda_A^g(b) \subseteq F'$. Since $G = \bigcup\{\lambda_B^g(b) \mid b \in \mathbb{B}'\}$, we obtain that $f(G) \subseteq F'$. Let $\sigma \in \mathrm{int}_{X'}(F')$. Then there exists $b \in \mathbb{B}$ such that $\sigma \in \mathrm{int}_{X'}(\lambda_A^g(b)) \subseteq \lambda_A^g(b) \subseteq \mathrm{int}_{X'}(F')$. Hence $b \in \mathbb{B}'$. Using again (2.29), we get that $\sigma \in f(\lambda_B^g(b))$, i.e., $\sigma \in f(G)$. Thus $\mathrm{int}_{X'}(F') \subseteq f(G)$. Since $f(G)$ is closed in X', we conclude that $f(G) \supseteq F'$. Therefore, $f(G) = F'$. So, G is homeomorphic to F'. We will show that F and F' are homeomorphic. Since the map $t_X : X \longrightarrow X'$, $x \mapsto \sigma_x$, is a homeomorphism (see the proof of Theorem 1.2.3.10), it is enough to show that $t_X(F) = F'$, i.e., that $t_X(F) = \lambda_A^g(F)$. Let $x \in F$. Then $F \in \sigma_x$ and thus $\sigma_x \in \lambda_A^g(F)$. So, $t_X(F) \subseteq \lambda_A^g(F)$. Conversely, let $\sigma \in \lambda_A^g(F)$. Then there exists $x \in X$ such that $\sigma = \sigma_x$. Hence $F \in \sigma_x$ and thus $x \in F$. Therefore, $\lambda_A^g(F) \subseteq t_X(F)$. So, F and F' are homeomorphic. This implies that F and G are homeomorphic. Then, by Theorem 2.2.2.12, $\Lambda^t(F)$ is LCA-isomorphic to $(B, \eta, \mathbb{B}')$. □

We will finish with mentioning some assertions about isolated points and a characterization of extremally disconnected locally compact Hausdorff spaces. All these statements have easy proofs which will be omitted.

Proposition 2.6.3.4. *Let $(A, \rho, \mathbb{B})$ be an LCA, $X = \Lambda^a(A, \rho, \mathbb{B})$ and $a \in A$. Then a is an atom of A iff $\lambda_A^g(a)$ is an isolated point of the space X. Also, for every isolated point x of X there exists an $a \in \mathbb{B}$ such that a is an atom of $\mathbb{B}$ (equivalently, of A) and $\{x\} = \lambda_A^g(a)$.*

Proposition 2.6.3.5. *Let $(A, \rho, \mathbb{B})$ be an LCA and $X = \Lambda^a(A, \rho, \mathbb{B})$. Then X is a discrete space iff $\mathbb{B}$ coincides with the set of all finite sums of the atoms of A.*

Proposition 2.6.3.6. *Let $(A, \rho, \mathbb{B})$ be an LCA and $X = \Lambda^a(A, \rho, \mathbb{B})$. Then the set of all isolated points of X is dense in X iff A is an atomic Boolean algebra iff $\mathbb{B}$ is an atomic 0-pseudolattice.*

Proposition 2.6.3.7. *Let $(A, \rho, \mathbb{B})$ be a CLCA and $X = \Lambda^a(A, \rho, \mathbb{B})$. Then X is an extremally disconnected space iff $a \ll_\rho a$, for every $a \in A$.*

Chapter 3

Some generalizations of the Stone Duality Theorem

3.1 Introduction

In this chapter we develop further the ideas from the previous chapters (see Remark 3.2.1.2 below) and obtain some extensions of the famous Stone Duality Theorem [108]. Recall that in 1937, M. Stone [108] proved that there exists a bijective correspondence S_l between the class of all (up to homeomorphism) zero-dimensional locally compact Hausdorff spaces (briefly, *Boolean spaces*) and the class of all (up to isomorphism) generalized Boolean algebras (briefly, GBAs) (or, equivalently, Boolean rings with or without unit). In the class of compact Boolean spaces (briefly, *Stone spaces*) this bijection can be extended to a duality S^t : **Stone** $\longrightarrow$ **BoolAlg** between the category **Stone** of Stone spaces and continuous maps and the category **BoolAlg** of Boolean algebras and Boolean homomorphisms; this is the classical Stone Duality Theorem. In 1964, H. P. Doctor [45] showed that the Stone bijection S_l can be even extended to a duality between the category **PBoolSp** of all Boolean spaces and all perfect maps between them and the category **GenBoolAlg** of all GBAs and suitable morphisms between them. It is natural to ask whether there exists such an extension over the category **BoolSp** of all Boolean spaces and all continuous functions between them. Let us mention that it is even not easy to obtain a duality for the category **PBoolSp**. Indeed, to every Boolean space X, M. Stone juxtaposed the generalized Boolean algebra $KO(X)$ of all compact open subsets of X and reconstructed from it the space X (up to homeomorphism). If $f : X \longrightarrow Y$ is a continuous map between two Stone spaces then its dual map $\varphi = S^t(f) : CO(Y) \longrightarrow CO(X)$ is defined by the formula $\varphi(G) = f^{-1}(G)$, for every $G \in CO(Y)$. If, however, $f : X \longrightarrow Y$ is a continuous map between two Boolean spaces and at least the space X is not compact then the preimages $f^{-1}(G)$ of the elements G of $KO(Y)$ are not obliged to be elements of the set $KO(X)$. These preimages will belong to $KO(X)$ iff the map f is perfect; then it is natural to expect that the category of GBAs and pseudolattice homomorphisms preserving zero elements (or, equivalently, the category **BoolRng** of Boolean rings and ring homomorphisms) will be the dual category of the category **PBoolSp** of Boolean spaces and perfect maps. However it is not the case. For example, if X and Y are two non-empty non-compact Boolean spaces and the 0-pseudolattice homomorphism $\varphi_0 : KO(Y) \longrightarrow KO(X)$ is defined by $\varphi_0(G) = 0(= \emptyset)$ for every $G \in KO(Y)$, then there is no function $f : X \longrightarrow Y$ such that $\varphi_0(G) = f^{-1}(G)$, for every $G \in KO(Y)$. Hence, even in the case of perfect maps, the mentioned homomorphisms are too much. In fact, as it is proved by D. Hofmann [70], the category **BoolRng** is dually equivalent to the category **pStone** of pointed Stone spaces and continuous maps preserving the fixed points. Thus, if one looks for a dual category to the category **PBoolSp**, having GBAs as objects, then this category has to have as morphisms some subclass of the class of pseudolattice homomorphisms preserving zero elements. Such a category was described by H. P. Doctor [45] and here it is named **GenBoolAlg** (see Theorem 3.2.2.17 below where two duality functors $\Theta_g^t : \mathbf{PBoolSp} \longrightarrow \mathbf{GenBoolAlg}$ and $\Theta_g^a : \mathbf{GenBoolAlg} \longrightarrow \mathbf{PBoolSp}$ are defined). If we want to find a dual category to the category **BoolSp** then it is clear that in this case the preimages of the compact open sets are clopen sets but they are not obliged to be compact sets. In [108], M. Stone proved that clopen subsets of a Boolean space X correspond to simple ideals of the GBA $KO(X)$ (i.e., those ideals of $KO(X)$ which have a complement in the frame $Idl(KO(X))$ of all ideals of $KO(X)$). Therefore one has to use the simple ideals of GBAs. As it is proved by M. Stone, the set of all simple ideals of a GBA forms a Boolean algebra. Here we describe the objects of the desired dual category to the category **BoolSp** as pairs (B, I), where B is a Boolean algebra and I is a dense (proper or non proper) ideal of it, satisfying a condition of completeness type; this condition is the following: for every simple ideal J of I, the join $\bigvee_B J$ exists; it is fulfilled for every pair (B, B), where B is a Boolean algebra because, as it is shown by M. Stone, an ideal of a Boolean algebra is simple iff it is principal. In this way we build a category named **ZLBA** and we prove that it is dually equivalent to the category **BoolSp** (see Theorem 3.2.2.7 where two duality functors $\Theta_d^t : \mathbf{BoolSp} \longrightarrow \mathbf{ZLBA}$ and $\Theta_d^a : \mathbf{ZLBA} \longrightarrow \mathbf{BoolSp}$ are defined). The idea of the creation of the category

ZLBA comes from the ideas and results presented here in Chapter 2 (see Remark 3.2.1.2 below). However, the proof that the categories **BoolSp** and **ZLBA** are dually equivalent will be carried out independently from the results presented in Chapter 2 because this is the more economical way (see [25] for a proof based on the results of Chapter 2). In fact, we first construct a category **LBA** containing as a subcategory the category **ZLBA** and find a contravariant adjunction between the categories **LBA** and **BoolSp**; it leads to the mentioned above duality between the categories **BoolSp** and **ZLBA**. The restriction of this duality on the category **PBoolSp** leads us to the definition of two categories **PZLBA** and **PLBA** dual to the category **PBoolSp**. It is then easy to see that the category **PLBA** is equivalent to the Doctor's category **GenBoolAlg**. In this way we obtain a new proof of Doctor's Duality Theorem [45]. Finally, we define two subcategories **DZHLC** and **DPZHLC** of the category **DHLC**, which was constructed here in Chapter 2 as a dual category to the category **HLC** of locally compact Hausdorff spaces and continuous maps; these subcategories are dual, respectively, to the categories **BoolSp** and **PBoolSp**. We obtain also many other results. The main of them are listed below, where we describe the structure of the chapter.

In the second section we present, after some preliminary observations, the results which we discussed above.

In the third section, we prove some Stone-type duality theorems for some subcategories of the category **BoolSp**. These theorems are new even in the compact case (see Theorems 3.3.1.2, 3.3.1.4(b),(c), 3.3.1.6, 3.3.2.1(b), 3.3.2.3(b), 3.3.2.6). They concern the cofull subcategories **SBoolSp**, **QPBoolSp**, **OBoolSp** and **POBoolSp** of the category **BoolSp** determined, respectively, by the skeletal maps, by the quasi-open perfect maps, by the open maps, and by the open perfect maps. Since the categories **QPBoolSp** and **POBoolSp** are cofull subcategories simultaneously of the categories **BoolSp** and **PBoolSp**, we find their images by the both functors Θ^t_d and Θ^t_g (see Corollary 3.3.1.4(b), Theorem 3.3.1.6 and Corollary 3.3.2.6). For the compact case, these theorems give the following results: (a) The category **QStone** of compact zero-dimensional Hausdorff spaces and quasi-open maps is dually equivalent to the category **CBool** of Boolean algebras and complete Boolean homomorphisms (see Corollary 3.3.1.4(c)), and (b) The category **OStone** of compact zero-dimensional Hausdorff spaces and open maps is dually equivalent to the category **OBool** of Boolean algebras and Boolean homomorphisms φ having lower adjoint ψ (i.e., the pair (ψ, φ)

forms a Galois connection) (see Corollary 3.3.2.4(b)). Let us notice also the following result (see Theorem 3.3.1.6): the category **QPBoolSp** is dually equivalent to the cofull subcategory **QGBA** of the category **GenBoolAlg** whose morphisms, in addition, preserve all meets that happen to exist. Note also that Theorem 3.3.1.2 and Corollary 3.3.1.4(b),(c) are zero-dimensional analogues of the Fedorchuk Duality Theorem [54] and its generalization presented in Chapter 2. From the mentioned above Corollary 3.3.1.4(c) and Fedorchuk's Duality Theorem [54], we obtain, as an immediate application, the following assertion which is a special case of a much more general theorem of Monk [84]: a Boolean homomorphism can be extended to a complete homomorphism between the corresponding minimal completions iff it is a complete homomorphism.

In the fourth section we characterize the dual maps of the injective and surjective morphisms of the category **BoolSp** and its subcategories **PBoolSp**, **OBoolSp**. Such investigations were done by M. Stone in [108] for surjective continuous maps and for closed embeddings. Analogous results are obtained here for the homeomorphic embeddings and dense embeddings.

In the last fifth section, the connections between the dual object of a space $X \in |\mathbf{BoolSp}|$ and the dual objects of the closed, regular closed and open subsets of X are found. It seems that the obtained result for regular closed subsets is new even in the compact case.

3.2 An extension of the Stone Duality to the category BoolSp of Boolean spaces and continuous maps

In this section we obtain some generalizations of the Stone Duality Theorem [108]. In it we introduce the notions of *local Boolean algebra* and *prime local Boolean algebra*. Using them, a category **LBA** is constructed and a contravariant adjunction between it and the category **BoolSp** of *Boolean spaces* (= zero-dimensional locally compact Hausdorff spaces) and continuous maps is obtained. The fixed objects of this adjunction give us a duality between the category **BoolSp** and the subcategory **ZLBA** of the category **LBA**. As it was already mentioned, H. P. Doctor [45] introduced a category **GenBoolAlg** and proved that it is dual to the category **PBoolSp** of Boolean spaces and perfect maps. Here two new categories **PZLBA** and **PLBA** dual to the category **PBoolSp** are described and a new proof of the Doctor Duality Theorem is given. The restrictions of the obtained duality functors to the category **Stone** coincide with the Stone duality functor $S^t : \mathbf{Stone} \longrightarrow \mathbf{BoolAlg}$. We describe as well two subcategories **DZHLC** and **DPZHLC** of the category **DHLC**, constructed here in Chapter 2, which are dual, respectively, to the categories **BoolSp** and **PBoolSp**.

3.2.1 Local Boolean algebras and the category LBA

Definition 3.2.1.1. A pair (A, I), where A is a Boolean algebra and I is an ideal of A (possibly non proper) which is dense in A (shortly, dense ideal), is called a *local Boolean algebra* (abbreviated as LBA). An LBA (A, I) is called a *prime local Boolean algebra* (abbreviated as PLBA) if $I = A$ or I is a prime ideal of A. Two LBAs (A, I) and (B, J) are said to be *LBA-isomorphic* (or, simply, *isomorphic*) if there exists a Boolean isomorphism $\varphi : A \longrightarrow B$ such that $\varphi(I) = J$.

Let **LBA** be the category whose objects are all LBAs and whose morphisms are all functions $\varphi : (A, I) \longrightarrow (B, J)$ between the objects of **LBA** such that $\varphi : A \longrightarrow B$ is a Boolean homomorphism satisfying the following condition:

(LBA) For every $b \in J$ there exists $a \in I$ such that $b \leq \varphi(a)$;

let the composition between the morphisms of **LBA** be the usual composition between functions, and the **LBA**-identities be the identity functions.

Remark 3.2.1.2. Note that a pair (B, I) is an LBA iff (B, ρ_s, I) is an LCA (see Example 1.2.1.8 for the notation ρ_s). Indeed, since $a \ll_{\rho_s} b$ iff $a \leq b$, for every $a, b \in B$ (see 1.2.1.8), we obtain immediately that:

1) if (B, I) is an LBA, then (B, ρ_s, I) satisfies condition (BC1) automatically, and conditions (BC2) and (BC3) follow directly from the fact that I is dense in B (see 1.2.3.1 for (BC1)-(BC3)), and

2) if (B, ρ_s, I) is an LCA then, by (BC3), I is dense in B.

Remark 3.2.1.3. Note that by Stone's result about the existence of a bijective correspondence between the ideals and open sets, any LBA (A, I) determines a pair (X, L) (we will write $(X, L) = p(A, I)$), where $X = S^a(A)$ (and hence X is a Stone space) and $L = \bigcup\{\lambda_A^S(a) \mid a \in I\}$ (and thus L is an open subset of X). Moreover, since I is dense in A, it is easy to see that L is dense in X (see, e.g., Lemma 3.2.1.9 below). Therefore, X is a 0-dimensional compactification of L. Clearly, by the results of M. Stone, X is the one-point compactification of L iff I is a prime ideal iff (A, I) is a PLBA and $I \neq A$.

Remark 3.2.1.4. Note that two LBAs (A, I) and (B, J) are **LBA**-isomorphic iff they are LBA-isomorphic. Indeed, let $\varphi : (A, I) \longrightarrow (B, J)$ be an **LBA**-isomorphism. Then, obviously, $\varphi : A \longrightarrow B$ is a Boolean isomorphism. We have to show that $\varphi(I) = J$. Let $\psi \in \mathbf{LBA}((B, J), (A, I))$ be such that $\varphi \circ \psi = id_B$ and $\psi \circ \varphi = id_A$. Let $a \in I$. Then, by condition (LBA), there exists $b \in J$ such that $a \leq \psi(b)$. Thus $\varphi(a) \leq b$; this implies that $\varphi(a) \in J$. So, $\varphi(I) \subseteq J$. Analogously, we get that $\psi(J) \subseteq I$. Let $b \in J$. Then $a = \psi(b) \in I$ and $\varphi(a) = b$. Hence, $\varphi(I) = J$. Therefore, (A, I) and (B, J) are LBA-isomorphic. The converse implication is obvious.

Remark 3.2.1.5. Note that a prime (= maximal) ideal I of a Boolean algebra A is a dense subset of A iff I is a non-principal ideal of A. For proving this, observe first that if I is a prime ideal, $a \in A \setminus \{1\}$ and $I \leq a$ then $a \in I$. (Indeed, if $a \notin I$ then $a^* \in I$ and hence $a^* \leq a$, i.e., $a = 1$.) Let now I be dense in A. Suppose that $I =\downarrow (a)$ for some $a \in A \setminus \{1\}$. Then $a^* \neq 0$. There exists $b \in I \setminus \{0\}$ such that $b \leq a^*$. Since $b \leq a$, we get that $b = 0$, a contradiction. Hence, I is a non-principal ideal. Conversely, let I be a non-principal ideal and $b \in A \setminus \{0\}$. Suppose that $b \wedge a = 0$, for every $a \in I$. Then $I \leq b^*$. Hence $I =\downarrow (b^*)$, a contradiction. Thus, there exists $a \in I$ such that $a \wedge b \neq 0$. Then $a \wedge b \in I \setminus \{0\}$ and $a \wedge b \leq b$. Therefore, I is a dense subset of A.

Recall that a distributive 0-pseudolattice A is called a *generalized Boolean algebra* (briefly, *GBA*) if it satisfies the following condition:

(GBA) for every $a \in A$ and every $b, c \in A$ such that $b \leq a \leq c$ there exists $x \in A$ with $a \wedge x = b$ and $a \vee x = c$ (i.e., x is the *relative complement of a in the interval* $[b, c]$).

Fact 3.2.1.6. *(a) A distributive 0-pseudolattice A is a generalized Boolean algebra iff every principal ideal of A is simple.*

(b) If A is a generalized Boolean algebra then the correspondence

$$e_A : A \longrightarrow Si(A),\ a \mapsto \downarrow (a),$$

is a dense 0-pseudolattice embedding of A in the Boolean algebra $Si(A)$ and the pair $(Si(A), e_A(A))$ is an LBA.

(c)(M. Stone [107]) *An ideal of a Boolean algebra is simple iff it is principal.*

Proof. (a) ($\Rightarrow$) Let A be a generalized Boolean algebra and $a \in A$. We have to prove that $\downarrow (a) \vee \neg(\downarrow (a)) = A$. Let $b \in A$. Then $c = a \wedge b \in [0, b]$. Hence there exists $d \in A$ such that $d \wedge c = 0$ and $d \vee c = b$. Thus $d \leq b$, i.e., $d \wedge b = d$. Therefore,

$d \wedge a = d \wedge b \wedge a = d \wedge c = 0$. We obtain that $d \in \neg(\downarrow(a))$, $c \in \downarrow(a)$ and $c \vee d = b$. So, $\downarrow(a) \vee \neg(\downarrow(a)) = A$.

($\Leftarrow$) Let $a, b, c \in A$ and $a \in [b, c]$. Since $\downarrow(a) \vee \neg(\downarrow(a)) = A$, we get that there exists $y \in \neg(\downarrow(a))$ such that $c = a \vee y$. Set $x = y \vee b$. Then $x \wedge a = (y \vee b) \wedge a = b \wedge a = b$ and $x \vee a = y \vee b \vee a = y \vee a = c$. So, A is a generalized Boolean algebra.

(b) By (a), for every $a \in A$, $\downarrow(a) \in Si(A)$. Further, it is easy to see that e_A is a 0-pseudolattice embedding and $I = e_A(A)$ is dense in $Si(A)$. Let us show that I is an ideal of $Si(A)$. Since I is closed under finite joins, it is enough to prove that I is a lower set. Let $J \in Si(A)$, $a \in A$ and $J \subseteq \downarrow(a)$. We need to show that J is a principal ideal of A. Since $J \in Si(A)$, there exist $b \in J$ and $c \in \neg J$ such that $a = b \vee c$. We will prove that $J = \downarrow(b)$. Note first that if $b' \in J$ and $a = b' \vee c$ then $b = b'$. Indeed, we have that $b' = a \wedge b' = (b \vee c) \wedge b' = b \wedge b'$ and $b = a \wedge b = (b' \vee c) \wedge b = b \wedge b'$; thus $b = b'$. Let now $d \in J$. Then $d \leq a$ and hence $a = a \vee d = (b \vee d) \vee c$. Since $b \vee d \in J$, we get that $b \vee d = b$, i.e., $d \leq b$. So, $J = \downarrow(b)$, and hence $J \in I$. Thus $(Si(A), e_A(A))$ is an LBA.

(c) Let B be a Boolean algebra and $J \in Si(B)$. Then there exist $a \in J$ and $b \in \neg J$ such that $1 = a \vee b$. Now we obtain, as in the proof of (b), that $J = \downarrow(a)$. So, every simple ideal of B is principal. Thus, using (a), we complete the proof. $\square$

Notation 3.2.1.7. Let I be a proper ideal of a Boolean algebra A. We set

$$B_A(I) = I \cup \{a^* \mid a \in I\}.$$

When there is no ambiguity, we will often write $B(I)$ instead of $B_A(I)$.

It is clear that $B_A(I)$ is a Boolean subalgebra of A and I is a prime ideal of $B_A(I)$ (see, e.g., [48]).

Fact 3.2.1.8. *Let (A, I) be an LBA. Then:*

(a) I is a generalized Boolean algebra;

(b) If (B, J) is a PLBA and there exists a poset-isomorphism $\psi : J \longrightarrow I$ then ψ can be uniquely extended to a Boolean embedding $\varphi : B \longrightarrow A$ (and $\varphi(B) = B_A(I)$); in particular, if (A, I) is also a PLBA then φ is a Boolean isomorphism and an isomorphism between LBAs (A, I) and (B, J);

(c) There exists a bijective correspondence between the class of all (up to isomorphism) generalized Boolean algebras and the class of all (up to isomorphism) PLBAs.

Proof. (a) Obviously, for every $a \in I$, $\neg_I(\downarrow(a)) = I \cap \downarrow_A (a^*)$; then, clearly,

$$\downarrow(a) \vee \neg_I(\downarrow(a)) = I.$$

Now apply 3.2.1.6(a).

(b) By [102, Theorem 12.5], ψ can be uniquely extended to a Boolean isomorphism $\psi' : B \longrightarrow B_A(I)$. Now, define $\varphi : B \longrightarrow A$ by $\varphi(b) = \psi'(b)$, for every $b \in B$.

(c) For every PLBA (A, I), set $f(A, I) = I$. Then, by (a), I is a generalized Boolean algebra. Conversely, if I is a generalized Boolean algebra then there exists a dense embedding $e : I \longrightarrow Si(I)$ (see Fact 3.2.1.6(b)). Thus, setting $g(I) = (B_{Si(I)}(e(I)), e(I))$, we get that $g(I)$ is a PLBA. Now, using (b), we obtain that for every PLBA (A, I), $g(f(A, I))$ is isomorphic to (A, I). Finally, it is clear that for every generalized Boolean algebra I, $f(g(I))$ is isomorphic to I. □

We will need a simple lemma.

Lemma 3.2.1.9. *Let A be a Boolean algebra, $M \subseteq A$, $X = S^a(A)$ and*

$$L_M = \{u \in X \mid u \cap M \neq \emptyset\}$$

(sometimes we will write L_M^A instead of L_M). Then:

(a) $L_M = \bigcup\{\lambda_A^S(a) \mid a \in M\}$;

(b) L_M is an open subset of X and hence the subspace L_M of X is a zero-dimensional locally compact Hausdorff space; $L_M \neq \emptyset$ iff $M \not\subseteq \{0\}$;

(c) $\lambda_A^S(M) \subseteq KO(L_M)$ (where $\lambda_A^S(M) = \{\lambda_A^S(a) \mid a \in M\}$);

(d) If M is dense in A then L_M is dense in X;

(e) If M is a lower set and L_M is dense in X then M is dense in A;

(f) If L_M is dense in X then the map

$$(3.1) \quad \lambda_{(A,M)} : A \longrightarrow CO(L_M),\ a \mapsto L_M \cap \lambda_A^S(a),$$

is a Boolean monomorphism;

(g) If M is an ideal of A then $\lambda_A^S(M) = KO(L_M)$ and hence $\lambda_A^S(M)$ is a base of L_M;

(h) If (A, M) is an LBA then $\lambda_{(A,M)} : A \longrightarrow CO(L_M)$ is a dense Boolean embedding;

(i) If $M_1, M_2 \subseteq A$ then $L_{M_1} = L_{M_2}$ iff the ideals of A generated by M_1 and M_2 coincide.

Proof. Assertions (a)-(c) and (i) are obvious, and (h) follows from (b), (d), (f), (g).

(d) It is enough to prove that $\lambda_A^S(a) \cap L_M \neq \emptyset$, for every $a \in A \setminus \{0\}$. So, let $a \in A \setminus \{0\}$. Then there exists $b \in M \setminus \{0\}$ such that $b \leq a$. There exists $u \in X$ such that $b \in u$. Then $a \in u$. Thus $u \in \lambda_A^S(a) \cap L_M$.

(e) Let M be a lower set and L_M be dense in X. Let $a \in A \setminus \{0\}$. Then, by (b), $\lambda_A^S(a) \cap L_M$ is an open non-empty subset of X. Hence, there exists $b \in A \setminus \{0\}$ such that $\lambda_A^S(b) \subseteq L_M \cap \lambda_A^S(a)$. Let $u \in \lambda_A^S(b)$. Then there exists $c \in u \cap M$. Since M is a lower set, we get that $b \wedge c \in u \cap M$. Thus $b \wedge c \neq 0$, $b \wedge c \in M$ and $b \wedge c \leq a$ (because, by the Stone Duality Theorem, $b \leq a$). Therefore, M is dense in A.

(f) Since, by the Stone Duality Theorem, the map

$$\lambda_A^S : A \longrightarrow CO(X), \quad a \mapsto \lambda_A^S(a),$$

is a Boolean isomorphism, it is clear that the map $\lambda_{(A,M)}$ is a Boolean homomorphism. Further, since L_M is dense in X, we have that if $a \in A \setminus \{0\}$ then $\lambda_{(A,M)}(a) \neq \emptyset$. Therefore, $\lambda_{(A,M)}$ is a Boolean monomorphism.

(g) Let M be an ideal of A and $U \in KO(L_M)$. For every $u \in U$ there exists $b_u \in M \cap u$, and hence $u \in \lambda_A^S(b_u) \subseteq L_M$. Thus $U \subseteq \bigcup\{\lambda_A^S(b_u) \mid u \in U\}$. Since U is compact, there exist $\{u_i \in U \mid i = 1, \ldots, n\}$, where n is some natural number, such that $U \subseteq \bigcup\{\lambda_A^S(b_{u_i}) \mid i = 1, \ldots, n\}$. Let $b_0 = \bigvee\{b_{u_i} \mid i = 1, \ldots, n\}$. Then $b_0 \in M$ and $\lambda_A^S(b_0) = \bigcup\{\lambda_A^S(b_{u_i}) \mid i = 1, \ldots, n\} \supseteq U$. Now, for every $u \in U$ there exists $a_u \in A$ such that $u \in \lambda_A^S(a_u) \subseteq U$ and thus $\lambda_A^S(a_u) \subseteq \lambda_A^S(b_0)$. Therefore, for every $u \in U$, $a_u \leq b_0$, and hence, $a_u \in M$. Using again the compactness of U, we get that there exists $a_0 \in M$ such that $U = \lambda_A^S(a_0)$. So, $\lambda_A^S(M) \supseteq KO(L_M)$. This fact together with (c) imply that $\lambda_A^S(M) = KO(L_M)$. □

Notation 3.2.1.10. Let X be a topological space. For every $x \in X$, we set

$$u_x^{CO(X)} = \{F \in CO(X) \mid x \in F\}.$$

When there is no ambiguity, we will write "u_x^C" instead of "$u_x^{CO(X)}$".

Theorem 3.2.1.11. *There exists a contravariant adjunction (see 0.2.1.8 for this notion)*

$$(\Theta^a, \Theta^t, \lambda^C, t^C) : \mathbf{LBA} \longrightarrow \mathbf{BoolSp},$$

where **BoolSp** *is the category of locally compact zero-dimensional Hausdorff spaces and continuous maps.*

Proof. We will first define two contravariant functors

$$\Theta^a : \mathbf{LBA} \longrightarrow \mathbf{BoolSp} \text{ and } \Theta^t : \mathbf{BoolSp} \longrightarrow \mathbf{LBA}.$$

Let $X \in |\mathbf{BoolSp}|$. Define

(3.2) $\Theta^t(X) = (CO(X), KO(X))$.

Obviously, $\Theta^t(X)$ is an LBA.

Let $f \in \mathbf{BoolSp}(X, Y)$. Define $\Theta^t(f) : \Theta^t(Y) \longrightarrow \Theta^t(X)$ by the formula

(3.3) $\Theta^t(f)(G) = f^{-1}(G), \ \forall G \in CO(Y)$.

Set

$$\varphi_f = \Theta^t(f).$$

Clearly, φ_f is a Boolean homomorphism between $CO(Y)$ and $CO(X)$. If $F \in KO(X)$ then $f(F)$ is a compact subset of Y. Since $KO(Y)$ is an open base of the space Y and $KO(Y)$ is closed under finite unions, we get that there exists $G \in KO(Y)$ such that $f(F) \subseteq G$. Then $F \subseteq f^{-1}(G) = \varphi_f(G)$. So, φ_f satisfies condition (LBA). Therefore φ_f is an **LBA**-morphism, i.e., $\Theta^t(f)$ is well-defined.

Now we get easily that Θ^t is a contravariant functor.

For every LBA (A, I), set

(3.4) $\Theta^a(A, I) = L_I^A$

(see Lemma 3.2.1.9 for the notation L_I^A). Then Lemma 3.2.1.9 implies that $L = \Theta^a(A, I)$ is a zero-dimensional locally compact Hausdorff space and $\lambda_{(A,I)}(I)$ is an open base of L (see (3.1) for the notation $\lambda_{(A,I)}$). So, $\Theta^a(A, I) \in |\mathbf{BoolSp}|$.

Let $\varphi \in \mathbf{LBA}((A, I), (B, J))$. We define the map

$$\Theta^a(\varphi) : \Theta^a(B, J) \longrightarrow \Theta^a(A, I)$$

by the formula

(3.5) $\Theta^a(\varphi)(u') = \varphi^{-1}(u'), \ \forall u' \in \Theta^a(B, J)$.

Set

$$f_\varphi = \Theta^a(\varphi), \quad L = \Theta^a(A, I) \text{ and } M = \Theta^a(B, J).$$

By Lemma 3.2.1.9, if (A', I') is a LBA then the set $\Theta^a(A', I')$ consists of all *bounded ultrafilters in* (A', I') (i.e., those ultrafilters u in A' for which $u \cap I' \neq \emptyset$). Since any **LBA**-morphism is a Boolean homomorphism, we get that the inverse image of an ultrafilter is an ultrafilter.

So, let $u' \in M$. Then u' is a bounded ultrafilter in (B, J). Set $u = f_\varphi(u')$. Then, as we have seen, u is an ultrafilter in A. We have to show that u is bounded in (A, I). Indeed, since u' is bounded, there exists $b \in u' \cap J$. By condition (LBA), there exists $a \in I$ such that $\varphi(a) \geq b$. Then $\varphi(a) \in u'$, and hence, $a \in u$. Thus $a \in u \cap I$. Therefore, $f_\varphi : M \longrightarrow L$.

We will show that f_φ is a continuous function. Let $u' \in M$ and $u = f_\varphi(u')$. Let $a \in A$ and $u \in \lambda_{(A,I)}(a)(= \mathrm{int}(\lambda_{(A,I)}(a)))$. Then $a \in u$. Hence $\varphi(a) \in u'$, i.e., $u' \in \lambda_{B,J}(\varphi(a))$. We will prove that

(3.6) $\quad f_\varphi(\lambda_{(B,J)}(\varphi(a))) \subseteq \lambda_{(A,I)}(a).$

Indeed, let $v' \in \lambda_{(B,J)}(\varphi(a))$. Then $\varphi(a) \in v'$. Thus $a \in f_\varphi(v')$, i.e., $f_\varphi(v') \in \lambda_{(B,J)}(a)$. So, (3.6) is proved. Since $\{\lambda_{(A,I)}(a) \mid a \in A\}$ is an open base of L, we get that f_φ is a continuous function. So,

$$\Theta^a(\varphi) \in \mathbf{BoolSp}(\Theta^a(B, J), \Theta^a(A, I)).$$

Now it becomes obvious that Θ^a is a contravariant functor.

Let $X \in |\mathbf{BoolSp}|$. Then it is easy to see that for every $x \in X$, u_x^C (see 3.2.1.10 for this notation) is an ultrafilter in $CO(X)$ and hence, using the fact that u_x^C contains always elements of $KO(X)$, we get that $u_x^C \in \Theta^a(CO(X), KO(X))$. We will show that the map

(3.7) $\quad t_X^C : X \longrightarrow \Theta^a(\Theta^t(X)),\ x \mapsto u_x^C,$

is a homeomorphism. Set $L = \Theta^a(\Theta^t(X))$ and $A = CO(X)$, $I = KO(X)$. We will prove that t_X^C is a continuous map. Let $x \in X$, $F \in I$ and $u_x^C \in \lambda_{(A,I)}(F)$. Then $F \in u_x^C$ and hence, $x \in F$. It is enough to show that $t_X^C(F) \subseteq \lambda_{(A,I)}(F)$. Let $y \in F$. Then $F \in u_y^C = t_X^C(y)$. Hence $t_X^C(y) \in \lambda_{(A,I)}(F)$. So, $t_X^C(F) \subseteq \lambda_{(A,I)}(F)$. Since $\lambda_{(A,I)}(I)$ is an open base of L, we get that t_X^C is a continuous map. Let us show that t_X^C is a bijection. Let $u \in L$. Then u is a bounded ultrafilter in (A, I). Hence, there exists $F \in u \cap I$. Since F is compact, we get that $\bigcap u \neq \emptyset$. Suppose that $x, y \in \bigcap u$ and $x \neq y$. Then there exist $F_x, F_y \in I$ such that $x \in F_x$, $y \in F_y$ and $F_x \cap F_y = \emptyset$. Since, clearly, $F_x, F_y \in u$, we get a contradiction. So, $\bigcap u = \{x\}$ for some $x \in X$. It is clear now that $u = u_x^C$, i.e., $u = t_X^C(x)$ and $u \neq t_X^C(y)$, for $y \in X \setminus \{x\}$. So, t_X^C is a bijection. For showing that $(t_X^C)^{-1}$ is a continuous function, let $u_x^C \in L$. Then

$(t_X^C)^{-1}(u_x^C) = x$. Let $F \in I$ and $x \in F$. Then $F \in u_x^C$ and thus $u_x^C \in \lambda_{(A,I)}(F)$. We will prove that $(t_X^C)^{-1}(\lambda_{(A,I)}(F)) \subseteq F$. Since I is a base of X, this will imply that $(t_X^C)^{-1}$ is a continuous function. So, let $y \in (t_X^C)^{-1}(\lambda_{(A,I)}(F))$. Then $t_X^C(y) \in \lambda_{(A,I)}(F)$, i.e., $F \in u_y^C$. Then $y \in F$. Therefore, t_X^C is a homeomorphism.

We will show that

$$t^C : Id_{\mathbf{BoolSp}} \longrightarrow \Theta^a \circ \Theta^t,$$

defined by $t^C(X) = t_X^C, \ \forall X \in |\mathbf{BoolSp}|$, is a natural isomorphism.

Let $f \in \mathbf{BoolSp}(X,Y)$ and $\hat{f} = \Theta^a(\Theta^t(f))$. We have to show that $\hat{f} \circ t_X^C = t_Y^C \circ f$. Let $x \in X$. Then $\hat{f}(t_X^C(x)) = \hat{f}(u_x^{CO(X)})$ and $(t_Y^C \circ f)(x) = u_{f(x)}^{CO(Y)}$. Set $y = f(x)$, $u_x = u_x^{CO(X)}$ and $u_y = u_{f(x)}^{CO(Y)}$. We will prove that

$$\hat{f}(u_x) = u_y.$$

Let $\varphi = \Theta^t(f)$. Then $\hat{f} = \Theta^a(\varphi)(= f_\varphi)$. Hence,

$\hat{f}(u_x) = \varphi^{-1}(u_x) = \{G \in CO(Y) \mid \varphi(G) \in u_x\} = \{G \in CO(Y) \mid x \in \varphi(G)\} = \{G \in CO(Y) \mid x \in f^{-1}(G)\} = \{G \in CO(Y) \mid f(x) \in G\} = u_y$.

So, t^C is a natural isomorphism.

Let (A,I) be an LBA and $L = \Theta^a(A,I)$. Then, by Lemma 3.2.1.9(h),

$$\lambda_{(A,I)} : A \longrightarrow CO(L)$$

is a dense Boolean embedding. Also, by Lemma 3.2.1.9(g), $\lambda_{(A,I)}(I) = KO(L)$. We denote by $\lambda_{(A,I)}^C$ the map

$$\lambda_{(A,I)}^C : (A,I) \longrightarrow (CO(L), KO(L)),$$

where $\lambda_{(A,I)}^C(a) = \lambda_{(A,I)}(a)$, for every $a \in A$; we will write sometimes "λ_A^C" instead of "$\lambda_{(A,I)}^C$". Note that

$$\lambda_{(A,I)}^C : (A,I) \longrightarrow \Theta^t(\Theta^a(A,I)).$$

We will prove that

$$\lambda^C : Id_{\mathbf{LBA}} \longrightarrow \Theta^t \circ \Theta^a, \text{ where } \lambda^C(A,I) = \lambda_{(A,I)}^C, \ \forall (A,I) \in |\mathbf{LBA}|,$$

is a natural transformation.

Let $\varphi \in \mathbf{LBA}((A,I),(B,J))$ and $\hat{\varphi} = \Theta^t(\Theta^a(\varphi))$. We have to prove that $\lambda_B^C \circ \varphi = \hat{\varphi} \circ \lambda_A^C$. Set $f = \Theta^a(\varphi)$ and $M = \Theta^a(B,J)$. Then $\hat{\varphi} = \Theta^t(f)(= \varphi_f)$. Let $a \in A$. Then $\hat{\varphi}(\lambda_A^C(a)) = f^{-1}(\lambda_A^C(a)) = \{u \in M \mid f(u) \in \lambda_A^C(a)\} = \{u \in M \mid a \in f(u)\} =$

$\{u \in M \mid a \in \varphi^{-1}(u)\} = \{u \in M \mid \varphi(a) \in u\} = \lambda^C_B(\varphi(a))$. So, λ^C is a natural transformation.

Let us show that

$$\Theta^t(t^C_X) \circ \lambda^C_{\Theta^t(X)} = id_{\Theta^t(X)},$$

for every $X \in |\mathbf{BoolSp}|$. Indeed, let $X \in |\mathbf{BoolSp}|$ and $Y = \Theta^a(\Theta^t(X))$. Then $\Theta^t(t^C_X) : \Theta^t(Y) \longrightarrow \Theta^t(X)$, $G \mapsto (t^C_X)^{-1}(G)$, for every $G \in \Theta^t(Y) = (CO(Y), KO(Y))$. Let $F \in CO(X)$. Then $(\Theta^t(t^C_X) \circ \lambda^C_{\Theta^t(X)})(F) = (t^C_X)^{-1}(\lambda^C_{\Theta^t(X)}(F)) = H$. We have to show that $F = H$. Since $t^C_X(H) = \lambda^C_{\Theta^t(X)}(F)$, we get that $\{u^C_x \mid x \in H\} = \{u \in Y \mid F \in u\}$. Thus $x \in H \iff F \in u^C_x \iff x \in F$. Therefore, $F = H$.

Finally, we will prove that

$$\Theta^a(\lambda^C_{(A,I)}) \circ t^C_{\Theta^a(A,I)} = id_{\Theta^a(A,I)}$$

for every $(A, I) \in |\mathbf{LBA}|$. So, let $(A, I) \in |\mathbf{LBA}|$ and $X = \Theta^a(A, I)$. We have that

$$f = \Theta^a(\lambda^C_{(A,I)}) : \Theta^a(CO(X), KO(X)) \longrightarrow X$$

is defined by $u \mapsto (\lambda^C_{(A,I)})^{-1}(u)$, for every bounded ultrafilter u in $(CO(X), KO(X))$. Let $x \in X$. Then $f(t^C_X(x)) = f(u^C_x) = (\lambda^C_{(A,I)})^{-1}(u^C_x) = y$. We have to show that $x = y$. Indeed, for every $a \in A$, we get that $a \in y \iff a \in (\lambda^C_{(A,I)})^{-1}(u^C_x) \iff \lambda^C_{(A,I)}(a) \in u^C_x \iff x \in \lambda^C_{(A,I)}(a) \iff a \in x$. Therefore, $x = y$.

We have proved that $(\Theta^a, \Theta^t, \lambda^C, t^C)$ is a contravariant adjunction between the categories **LBA** and **BoolSp**. Moreover, we have even shown that t^C is a natural isomorphism. Hence Θ^t is a full and faithful contravariant functor and, thus, it reflects isomorphisms. □

3.2.2 The generalizations of the Stone Duality Theorem

Definition 3.2.2.1. An LBA (A, I) is called a *ZLB-algebra* (briefly, *ZLBA*) if, for every $J \in Si(I)$, the join $\bigvee_A J$ $(= \bigvee_A\{a \mid a \in J\})$ exists.

Let **ZLBA** be the full subcategory of the category **LBA** having as objects all ZLBAs.

Example 3.2.2.2. Let B be a Boolean algebra. Then the pair (B, B) is a ZLBA. This follows from Fact 3.2.1.6(c).

Remark 3.2.2.3. Note that if A and B are Boolean algebras then any Boolean homomorphism $\varphi : A \longrightarrow B$ is a **ZLBA**-morphism between the ZLBAs (A, A) and (B, B).

Hence, the full subcategory $\mathbf{B}$ of the category $\mathbf{ZLBA}$ whose objects are all ZLBAs of the form (A, A) is isomorphic (it can be even said that it coincides) with the category $\mathbf{BoolAlg}$ of Boolean algebras and Boolean homomorphisms.

We will need the following result of M. Stone [108]:

Proposition 3.2.2.4. (M. Stone [108, Theorem 5(3)]) *Let $X \in |\mathbf{BoolSp}|$. Then the map*

$$\Sigma : Si(KO(X)) \longrightarrow CO(X), \quad J \mapsto \bigvee_{RC(X)} J,$$

is a Boolean isomorphism.

Proof. For completeness of our exposition, we will verify this fact. Let $J \in Si(KO(X))$. Set $U = \bigcup\{F \mid F \in J\}$ and $V = \bigcup\{G \mid G \in \neg J\}$. Obviously, U and V are disjoint open subsets of X. We will show that $U \cup V = X$. Indeed, let $x \in X$. Then there exists $H \in KO(X)$ such that $x \in H$. Since $J \vee \neg J = KO(X)$, we get that there exist $F \in J$ and $G \in \neg J$ such that $H = F \cup G$. Thus $x \in F$ or $x \in G$, and hence, $x \in U$ or $x \in V$. So, U is a clopen subset of X. Thus $U \in CO(X)$ and $U = \bigvee_{RC(X)} J = \bigvee_{CO(X)} J$. Conversely, it is easy to see that if $U \in CO(X)$ then $J = \{F \in KO(X) \mid F \subseteq U\} \in Si(KO(X))$. This implies easily that Σ is a Boolean isomorphism. □

Proposition 3.2.2.5. *Let (A, I) be an LBA and $L = \Theta^a(A, I)$. Then (A, I) is a ZLBA iff $\lambda_{(A,I)}(A) = CO(L)$ (see (3.1) for the notation $\lambda_{(A,I)}$).*

Proof. Let (A, I) be a ZLBA. We will prove that $\lambda_{(A,I)}(A) = CO(L)$. Let $U \in CO(L)$ and $J' = \{F \in KO(L) \mid F \subseteq U\}$. Then J' is a simple ideal of $KO(L)$ and $\bigvee_{RC(L)} J' = U$. Since the restriction $\varphi : I \longrightarrow KO(L)$ of $\lambda_{(A,I)}$ is a 0-pseudolattice isomorphism, we get that $J = \varphi^{-1}(J')$ is a simple ideal of I. Set $b_J = \bigvee_A J$ and $C = \lambda_{(A,I)}(A)$ (note that the join $\bigvee_A J$ exists because (A, I) is a ZLBA). Now, the restriction $\psi : A \longrightarrow C$ of $\lambda_{(A,I)}$ is a Boolean isomorphism, and hence $\lambda_{(A,I)}(b_J) = \psi(b_J) = \psi(\bigvee_A J) = \bigvee_C \psi(J) = \bigvee_C J'$. The fact that C is a dense Boolean subalgebra of the Boolean algebra $CO(L)$, and hence of $RC(L)$, implies that C is a regular subalgebra of $RC(L)$. Thus $\bigvee_C J' = \bigvee_{RC(L)} J' = U$. Therefore, $\lambda_{(A,I)}(b_J) = U$. So, we have proved that $\lambda_{(A,I)}(A) = CO(L)$.

Let now (A, I) be an LBA and $\lambda_{(A,I)}(A) = CO(L)$. Set, for short, $\psi = \lambda_{(A,I)}$. Then the map $\psi : A \longrightarrow CO(L)$ is a Boolean isomorphism. Let $J \in Si(I)$. Since the restriction of ψ to I is a 0-pseudolattice isomorphism between I and $KO(L)$, we get that $\psi(J) \in Si(KO(L))$. Then, by the proof of 3.2.2.4, $U = \bigcup\{F \mid F \in \psi(J)\}(= \bigcup\{\psi(a) \mid a \in J\})$ is a clopen subset of L. Therefore, the join $\bigvee_{CO(L)}\{\psi(a) \mid a \in J\}$ exists. Since $\psi^{-1} : CO(L) \longrightarrow A$ is a Boolean isomorphism, we obtain that $\psi^{-1}(U) = \psi^{-1}(\bigvee_{CO(L)}\{\psi(a) \mid a \in J\}) = \bigvee_A\{\psi^{-1}(\psi(a)) \mid a \in J\} = \bigvee_A\{a \mid a \in J\}$. Hence, the join $\bigvee_A J$ exists. Thus, (A, I) is a ZLBA. $\square$

Remark 3.2.2.6. Note that, by Lemma 3.2.1.9(h), if (A, I) is an LBA then $\lambda_{(A,I)}(A)$ is isomorphic to A and is a Boolean subalgebra of $CO(L)$, where $L = \Theta^a(A, I)$ (see (3.4) for the notation Θ^a). If (A, I) is an LBA and $(X, L) = p(A, I)$ (see Remark 3.2.1.3 for this notation) then, by Lemma 3.2.1.9(a), $L = \Theta^a(A, I))$; thus, by Proposition 3.2.2.5, (A, I) is a ZLBA iff A is mapped isomorphically by $\lambda_{(A,I)}$ to $CO(L)$; since the Banaschewski compactification $\beta_0 L$ of L (see [8] and [48, Theorem 13.1]) is constructed as $S^a(CO(L))$ (i.e., it is *the Stonification of* L), we get that (A, I) is a ZLBA iff $p(A, I) = (\beta_0 L, L)$, where L is defined as in Remark 3.2.1.3 (i.e., $L = \bigcup\{\lambda_A^S(a) \mid a \in I\}$).

Theorem 3.2.2.7. *The categories* **BoolSp** *and* **ZLBA** *are dually equivalent.*

Proof. In Theorem 3.2.1.11, we constructed a contravariant adjunction

$$(\Theta^a, \Theta^t, \lambda^C, t^C)$$

between the categories **LBA** and **BoolSp**, where t^C was even a natural isomorphism. Let us check that the functor Θ^t is in fact a functor from the category **BoolSp** to the category **ZLBA**. Indeed, let $X \in |\mathbf{BoolSp}|$. Then $\Theta^t(X) = (CO(X), KO(X))$. As it follows from 3.2.2.4, for every $J \in Si(KO(X))$, $\bigvee_{CO(X)} J$ exists. Hence, $\Theta^t(X) \in |\mathbf{ZLBA}|$. So, the restriction

(3.8) $\quad \Theta_d^t : \mathbf{BoolSp} \longrightarrow \mathbf{ZLBA}$

of the contravariant functor $\Theta^t : \mathbf{BoolSp} \longrightarrow \mathbf{LBA}$ is well-defined. Further, by Proposition 3.2.2.5, the natural transformation λ^C becomes a natural isomorphism exactly on the subcategory **ZLBA** of the category **LBA**. We will denote by

(3.9) $\quad \Theta_d^a : \mathbf{ZLBA} \longrightarrow \mathbf{BoolSp}$

the restriction of the contravariant functor Θ^a to the category **ZLBA**. All this shows that there is a duality between the categories **BoolSp** and **ZLBA**. $\square$

Corollary 3.2.2.8. (The Stone Duality Theorem [108]) *The categories* **BoolAlg** *and* **Stone** *are dually equivalent.*

Proof. Obviously, the restriction of the contravariant functor Θ^a_d to the subcategory **B** of the category **ZLBA** (see 3.2.2.3 for the notation **B**) produces a duality between the categories **B** and **Stone**. $\square$

Corollary 3.2.2.9. *For every ZLBA (A, I), the map*

$$\Sigma_{(A,I)} : Si(I) \longrightarrow A, \quad J \mapsto \bigvee_A \{a \mid a \in J\}$$

is a Boolean isomorphism.

Proof. Let $L = \Theta^a_d(A, I)$ (see (3.9) for the notation Θ^a_d). Then, as it was shown in the proof of Theorem 3.2.2.7, the map

$$\lambda^C_A : (A, I) \longrightarrow (CO(L), KO(L)),$$

where $\lambda^C_A(a) = \lambda_{(A,I)}(a)$ for every $a \in A$, is a **ZLBA**-isomorphism. By 3.2.2.4, the map

$$\Sigma = \Sigma_{(CO(L),KO(L))} : Si(KO(L)) \longrightarrow CO(L), \quad J \mapsto \bigvee_{CO(L)} J,$$

is a Boolean isomorphism. Define a map $\lambda'_A : Si(I) \longrightarrow Si(KO(L))$ by the formula $\lambda'_A(J) = \lambda^C_A(J)$, for every $J \in Si(I)$. Then, obviously, λ'_A is a Boolean isomorphism and $\Sigma_{(A,I)} = (\lambda^C_A)^{-1} \circ \Sigma \circ \lambda'_A$. Thus $\Sigma_{(A,I)}$ is a Boolean isomorphism. $\square$

Definition 3.2.2.10. Let **PZLBA** be the cofull subcategory of the category **ZLBA** whose morphisms $\varphi : (A, I) \longrightarrow (B, J)$ satisfy the following additional condition:

(PLBA) $\varphi(I) \subseteq J$.

Theorem 3.2.2.11. *The category* **PBoolSp** *of all locally compact Hausdorff zero-dimensional spaces and all perfect maps between them is dually equivalent to the category* **PZLBA**.

Proof. Let $f \in$ **PBoolSp**(X, Y). Then, as we have seen in the proof of Theorem 3.2.2.7, $\Theta^t_d(f) : \Theta^t_d(Y) \longrightarrow \Theta^t_d(X)$ is defined by the formula $\Theta^t_d(f)(G) = f^{-1}(G), \forall G \in CO(Y)$. Set $\varphi_f = \Theta^t_d(f)$. Since f is a perfect map, we have that for any $K \in KO(Y)$, $\varphi_f(K) = f^{-1}(K) \in KO(X)$. Hence, φ_f satisfies condition (PLBA). Thus, φ_f is a

PZLBA-morphism. So, the restriction Θ^t_p of the duality functor Θ^t_d to the subcategory **PBoolSp** of the category **BoolSp** is a contravariant functor from **PBoolSp** to **PZLBA**, i.e.,

(3.10) $\Theta^t_p : \mathbf{PBoolSp} \longrightarrow \mathbf{PZLBA}$.

Let $\varphi \in \mathbf{PZLBA}((A, I), (B, J))$. The map $\Theta^a_d(\varphi) : \Theta^a_d(B, J) \longrightarrow \Theta^a_d(A, I)$ was defined in Theorem 3.2.2.7 by the formula $\Theta^a_d(\varphi)(u') = \varphi^{-1}(u')$, $\forall u' \in \Theta^a_d(B, J)$. Set $f_\varphi = \Theta^a_d(\varphi)$, $L = \Theta^a_d(A, I)$ and $M = \Theta^a_d(B, J)$.

Let $a \in I$. We will show that $f^{-1}_\varphi(\lambda_{(A,I)}(a))$ is compact. We have, by (PLBA), that $\varphi(a) \in J$. Let us prove that

(3.11) $\lambda_{(B,J)}(\varphi(a)) = f^{-1}_\varphi(\lambda_{(A,I)}(a))$.

Let $u' \in f^{-1}_\varphi(\lambda_{(A,I)}(a))$. Then $u = f_\varphi(u') \in \lambda_{(A,I)}(a)$, i.e., $a \in u$. Thus $\varphi(a) \in u'$, and hence $u' \in \lambda_{(B,J)}(\varphi(a))$. Therefore, $\lambda_{(B,J)}(\varphi(a)) \supseteq f^{-1}_\varphi(\lambda_{(A,I)}(a))$. Now, (3.6) implies that $\lambda_{(B,J)}(\varphi(a)) = f^{-1}_\varphi(\lambda_{(A,I)}(a))$. Since $\lambda_{(B,J)}(\varphi(a))$ is compact, we get that $f^{-1}_\varphi(\lambda_{(A,I)}(a))$ is compact. Let now K be a compact subset of L. Since $\lambda_{(A,I)}(I)$ is an open base of L and $\lambda_{(A,I)}(I)$ is closed under finite unions, we get that there exists $a \in I$ such that $K \subseteq \lambda_{(A,I)}(a)$. Then $f^{-1}_\varphi(K) \subseteq f^{-1}_\varphi(\lambda_{(A,I)}(a))$, and hence, as a closed subset of a compact set, $f^{-1}_\varphi(K)$ is compact. This implies that f_φ is a perfect map (see, e.g.,[53]). Therefore, the restriction Θ^a_p of the duality functor Θ^a_d to the subcategory **PZLBA** of the category **ZLBA** is a contravariant functor from **PZLBA** to **PBoolSp**, i.e.,

(3.12) $\Theta^a_p : \mathbf{PZLBA} \longrightarrow \mathbf{PBoolSp}$.

The rest follows from Theorem 3.2.2.7. □

The above theorem can be stated in a better form. We will do this now.

Definition 3.2.2.12. Let **PLBA** be the subcategory of the category **LBA** whose objects are all PLBAs and whose morphisms are all **LBA**-morphisms $\varphi : (A, I) \longrightarrow (B, J)$ between the objects of **PLBA** satisfying condition (PLBA).

Remark 3.2.2.13. It is obvious that **PLBA** is indeed a category. Note also that any Boolean homomorphism $\varphi : A \longrightarrow B$ is a **PLBA**-morphism between the PLBAs (A, A) and (B, B). Hence, the full subcategory **B** of the category **PLBA** whose objects are all PLBAs of the form (A, A) is isomorphic (it can be even said that it coincides) with the category **BoolAlg** of Boolean algebras and Boolean homomorphisms. (Obviously, the category **B** introduced here coincides also with the category **B** from Remark 3.2.2.3, and that's why we don't introduce a new notation for it.)

Theorem 3.2.2.14. *The categories* **PBoolSp** *and* **PLBA** *are dually equivalent.*

Proof. In virtue of Theorem 3.2.2.11, it is enough to show that the categories **PLBA** and **PZLBA** are equivalent.

Let (B, I) be a ZLBA. Set $A = B_B(I)$ (see 3.2.1.7 for the notation). Then, obviously, (A, I) is a PLBA. Set

$$E^z(B, I) = (A, I).$$

If $\varphi \in \mathbf{PZLBA}((B_1, I_1), (B_2, I_2))$ then let $E^z(\varphi)$ be the restriction of φ to $E^z(B_1, I_1)$. Then, clearly, $E^z(\varphi) \in \mathbf{PLBA}(E^z(B_1, I_1), E^z(B_2, I_2))$. It is evident that E^z is a (covariant) functor from **PZLBA** to **PLBA**, i.e.,

$$E^z : \mathbf{PZLBA} \longrightarrow \mathbf{PLBA}.$$

Let (A, I) be a PLBA. Then, by 3.2.1.8(a), I is a generalized Boolean algebra. Hence, according to 3.2.1.6(b), the map $e_I : I \longrightarrow Si(I)$, where $e_I(a) = \downarrow(a)$, is a dense embedding of I in the Boolean algebra $Si(I)$ and the pair $(Si(I), e_I(I))$ is an LBA. Set $I' = e_I(I)$ and

$$E^p(A, I) = (Si(I), I').$$

Then, for every $J \in Si(I)$, $\bigvee_{Si(I)} e_I(J) = \bigvee_{Si(I)} \{\downarrow(a) \mid a \in J\} = J$. This implies that $(Si(I), I') \in |\mathbf{PZLBA}|$.

Let $\varphi \in \mathbf{PLBA}((A_1, I_1), (A_2, I_2))$. Let the map

$$\varphi' = E^p(\varphi)$$

be defined by the formula

$$\varphi'(J_1) = \bigcup \{\downarrow(\varphi(a)) \mid a \in J_1\},$$

for every $J_1 \in Si(I_1)$. We will show that φ' is a **PZLBA**-morphism between $E^p(A_1, I_1)$ and $E^p(A_2, I_2)$. Obviously, $\varphi'(\{0\}) = \{0\}$ and, thanks to conditions (LBA) and (PLBA), $\varphi'(I_1) = I_2$. Let $J_1 \in Si(I_1)$. Set $J_2 = \varphi'(J_1)$. Then condition (PLBA) and the fact that φ is a homomorphism imply that J_2 is an ideal of I_2. Let us show that $J_2 \vee \neg J_2 = I_2$. Indeed, let $a_2 \in I_2$. Then condition (LBA) implies that there exists

$a_1 \in I_1$ such that $a_2 \leq \varphi(a_1)$. Since $J_1 \vee \neg J_1 = I_1$, there exist $a_1' \in J_1$ and $a_1'' \in \neg J_1$ such that $a_1 = a_1' \vee a_1''$. Then $a_2 = (\varphi(a_1') \wedge a_2) \vee (\varphi(a_1'') \wedge a_2)$. Obviously, $(\varphi(a_1') \wedge a_2) \in J_2$. We will prove that $(\varphi(a_1'') \wedge a_2) \in \neg J_2$. It is enough to show that $\varphi(a_1'') \in \neg J_2$. Let $b_2 \in J_2$. Then, by the definition of J_2, there exists $b_1 \in J_1$ such that $b_2 \leq \varphi(b_1)$. Since $b_1 \wedge a_1'' = 0$, we get that $\varphi(b_1) \wedge \varphi(a_1'') = 0$. Thus $\varphi(a_1'') \wedge b_2 = 0$. Therefore, $\varphi(a_1'') \in \neg J_2$. So, $J_2 \in Si(I_2)$. Note that this implies that $\varphi'(J_1) = \bigvee_{Si(I_2)}\{\downarrow (\varphi(a)) \mid a \in J_1\}$. The above arguments show also that $\varphi'(\neg J_1) \subseteq \neg\varphi'(J_1)$, for every $J_1 \in Si(I_1)$. In fact, there is an equality here, i.e., $\varphi'(\neg J_1) = \neg\varphi'(J_1)$. Indeed, let $b_2 \in \neg\varphi'(J_1)$. Then $b_2 \wedge a_2 = 0$, for every $a_2 \in \varphi'(J_1)$. By condition (LBA), there exists $a_1 \in I_1$ such that $b_2 \leq \varphi(a_1)$. We have again that there exist $a_1' \in J_1$ and $a_1'' \in \neg J_1$ such that $a_1 = a_1' \vee a_1''$. Then $b_2 = (\varphi(a_1') \wedge b_2) \vee (\varphi(a_1'') \wedge b_2) = \varphi(a_1'') \wedge b_2$. Thus, $b_2 \leq \varphi(a_1'')$. This shows that $b_2 \in \varphi'(\neg J_1)$. Further, if $J, J' \in Si(I_1)$ then $\varphi'(J) \wedge \varphi'(J') = \varphi'(J) \cap \varphi'(J') = \bigcup\{\downarrow (a) \wedge \downarrow (b) \mid a \in J, b \in J'\} = \bigcup\{\downarrow (a) \mid a \in J \cap J'\} = \varphi'(J \cap J') = \varphi'(J \wedge J')$. Therefore, $\varphi' : Si(I_1) \longrightarrow Si(I_2)$ is a Boolean homomorphism. Since, for every $a \in I_1$, $\varphi'(\downarrow (a)) = \downarrow (\varphi(a))$, we have that $e_{I_2} \circ \varphi_{|I_1} = \varphi' \circ e_{I_1}$. This shows that $\varphi' \in \mathbf{PZLBA}(E^p(A_1, I_1), E^p(A_2, I_2))$. Now one can easily see that E^p is a (covariant) functor between the categories **PLBA** and **PZLBA**, i.e.,

$$E^p : \mathbf{PLBA} \longrightarrow \mathbf{PZLBA}.$$

Finally, we have to verify that the compositions $E^p \circ E^z$ and $E^z \circ E^p$ are naturally isomorphic to the corresponding identity functors.

Let us start with the composition $E^z \circ E^p$.

Let (A, I) be a PLBA. Then, as we have seen above, the map $e_I : I \longrightarrow Si(I)$, where $e_I(a) = \downarrow (a)$, is a dense embedding of I in the Boolean algebra $Si(I)$ and the pair $(Si(I), e_I(I))$ is an LBA. Now 3.2.1.8(b) implies that the map $(e_I)_{\restriction I} : I \longrightarrow e_I(I)$ can be extended to a Boolean isomorphism $e_{(A,I)} : A \longrightarrow B_{Si(I)}(e_I(I))$. (Note that $A = I \cup I^*$ and $B_{Si(I)}(e_I(I)) = e_I(I) \cup (e_I(I))^*$, so that the map $e_{(A,I)}$ is defined by the following formula: for every $a \in I$, $e_{(A,I)}(a^*) = (e_I(a))^*$.) Set $I' = e_I(I)$ and $A' = e_{(A,I)}(A)$. Then the map $e_{(A,I)} : (A, I) \longrightarrow (A', I')$ is a **PLBA**-isomorphism. Note that $(A', I') = (E^z \circ E^p)(A, I)$. Hence, $e_{(A,I)} : (A, I) \longrightarrow (E^z \circ E^p)(A, I)$ is a **PLBA**-isomorphism. We will show that $e : Id_{\mathbf{PLBA}} \longrightarrow E^z \circ E^p$, defined by $e(A, I) = e_{(A,I)}$ for every $(A, I) \in |\mathbf{PLBA}|$, is the required natural isomorphism. Indeed, if $\varphi \in \mathbf{PLBA}((A, I), (B, J))$ and $\varphi' = (E^z \circ E^p)(\varphi)$ then we have to prove that $e_{(B,J)} \circ \varphi = \varphi' \circ e_{(A,I)}$. Clearly, for doing this it is enough to show that $e_J \circ (\varphi_{|I}) = (\varphi')_{|e_I(I)} \circ e_I$. Since this is obvious, we obtain that the functors $Id_{\mathbf{PLBA}}$ and $E^z \circ E^p$ are naturally isomorphic.

Let us proceed with the composition $E^p \circ E^z$. Let (B, I) be a ZLBA. Then, by Corollary 3.2.2.9, the map $\Sigma_{(B,I)} : Si(I) \longrightarrow B$, where $\Sigma_{(B,I)}(J) = \bigvee_B\{a \mid a \in J\}$ for every $J \in Si(I)$, is a Boolean isomorphism. We will show that $s : Id_{\mathbf{PZLBA}} \longrightarrow E^p \circ E^z$, defined by $s(B, I) = (\Sigma_{(B,I)})^{-1}$ for every $(B, I) \in |\mathbf{PZLBA}|$, is the required natural isomorphism. Indeed, if $\varphi \in \mathbf{PZLBA}((A, I), (B, J))$ and $\varphi' = (E^p \circ E^z)(\varphi)$ then we have to prove that $\Sigma_{(B,J)} \circ \varphi' = \varphi \circ \Sigma_{(A,I)}$. Let $I_1 \in Si(I)$. Then $(\varphi \circ \Sigma_{(A,I)})(I_1) = \varphi(\bigvee_A I_1)$ and $(\Sigma_{(B,J)} \circ \varphi')(I_1) = \Sigma_{(B,J)}(\varphi'(I_1)) = \Sigma_{(B,J)}(\bigvee_{Si(J)}\{\downarrow (\varphi(a)) \mid a \in I_1\}) = \bigvee_B\{\Sigma_{(B,J)}(\downarrow (\varphi(a))) \mid a \in I_1\} = \bigvee_B \varphi(I_1)$. So, we have to prove that $\varphi(\bigvee_A I_1) = \bigvee_B \varphi(I_1)$. Set $b = \varphi(\bigvee_A I_1)$ and $c = \bigvee_B \varphi(I_1)$. Since $a \leq \bigvee_A I_1$, for every $a \in I_1$, we have that $\varphi(a) \leq b$ for every $a \in I_1$. Hence $c \leq b$. We will now prove that $b \leq c$. Since J is dense in B, we get that $b = \bigvee_B\{d \in J \mid d \leq b\}$. By condition (LBA), for every $d \in J$ there exists $e_d \in I$ such that $d \leq \varphi(e_d)$. So, let $d \in J$ and $d \leq b$. Since $I_1 \vee \neg I_1 = I$, there exist $e_d^1 \in I_1$ and $e_d^2 \in \neg I_1$ such that $e_d = e_d^1 \vee e_d^2$. Now we obtain that $d \leq \varphi(e_d) \wedge b = \varphi(e_d \wedge \bigvee_A I_1) = \varphi(\bigvee_A\{e_d \wedge a \mid a \in I_1\}) = \varphi(\bigvee_A\{e_d^1 \wedge a \mid a \in I_1\}) = \varphi(e_d^1 \wedge \bigvee_A I_1) \leq \varphi(e_d^1) \leq c$. Thus $b = \bigvee_B\{d \in J \mid d \leq b\} \leq c$. So, the functors $Id_{\mathbf{PZLBA}}$ and $E^p \circ E^z$ are naturally isomorphic. □

Corollary 3.2.2.15. *There exists a bijective correspondence between the classes of all (up to* **PLBA***-isomorphism) PLBAs, all (up to* **ZLBA***-isomorphism) ZLBAs and all (up to homeomorphism) locally compact zero-dimensional Hausdorff spaces.*

We can even express Theorem 3.2.2.14 in a more simple form; in this way we will obtain a new proof of the Doctor Duality Theorem [45].

Definition 3.2.2.16. ([45]) Let **GenBoolAlg** be the category whose objects are all generalized Boolean algebras and whose morphisms are all 0-pseudolattice homomorphisms $\varphi : I \longrightarrow J$ between its objects satisfying condition (LBA) (i.e., $\forall b \in J\ \exists a \in I$ such that $b \leq \varphi(a)$).

Theorem 3.2.2.17. ([45]) *The categories* **PBoolSp** *and* **GenBoolAlg** *are dually equivalent.*

Proof. By virtue of Theorem 3.2.2.14, it is enough to show that the categories **PLBA** and **GenBoolAlg** are equivalent.

Define a functor

$$E^l : \mathbf{PLBA} \longrightarrow \mathbf{GenBoolAlg}$$

by setting

$$E^l(A, I) = I,$$

for every $(A, I) \in |\mathbf{PLBA}|$, and for every $\varphi \in \mathbf{PLBA}((A, I), (B, J))$, put

$$E^l(\varphi) = \varphi_{|I} : I \longrightarrow J.$$

Using Fact 3.2.1.8(a) and condition (PLBA), we get that E^l is a well-defined functor.

Define a functor

$$E^g : \mathbf{GenBoolAlg} \longrightarrow \mathbf{PLBA}$$

by setting

$$E^g(I) = (B_{Si(I)}(e_I(I)), e_I(I))$$

for every $I \in |\mathbf{GenBoolAlg}|$ (see 3.2.1.6(b) and 3.2.1.7 for the notation), and for every $\varphi \in \mathbf{GenBoolAlg}(I, J)$ define

$$E^g(\varphi) : B_{Si(I)}(e_I(I)) \longrightarrow B_{Si(J)}(e_J(J))$$

to be the obvious extension of the map $\varphi_e : e_I(I) \longrightarrow e_J(J)$ defined by

$$\varphi_e(\downarrow(a)) = \downarrow(\varphi(a)).$$

Then, using Facts 3.2.1.6(a) and 3.2.1.8(b), it is easy to see that E^g is a well-defined functor.

Finally, it is almost obvious that the compositions $E^g \circ E^l$ and $E^l \circ E^g$ are naturally isomorphic to the corresponding identity functors. So, the functors

$$\Theta_g^t = E^l \circ E^z \circ \Theta_p^t : \mathbf{PBoolSp} \longrightarrow \mathbf{GenBoolAlg}$$

and

$$\Theta_g^a = \Theta_p^a \circ E^p \circ E^g : \mathbf{GenBoolAlg} \longrightarrow \mathbf{PBoolSp}$$

(see Theorems 3.2.2.14 and 3.2.2.11 for these notation) are the desired duality functors. Note that

$$\Theta_g^t(X) = KO(X),$$

for every $X \in |\mathbf{PBoolSp}|$, and if $f \in \mathbf{PBoolSp}(X, Y)$ then

$$\varphi = \Theta_g^t(f) : KO(Y) \longrightarrow KO(X)$$

is defined by the formula

$$\varphi(G) = f^{-1}(G),$$

for every $G \in KO(Y)$. $\square$

The definition of the functor Θ_g^t is very simple but that of Θ_g^a is more complicated. We will recall the definition of the contravariant functor

$$\Theta_s^a : \mathbf{GenBoolAlg} \longrightarrow \mathbf{PBoolSp}$$

of H. P. Doctor [45] where the original Stone construction (see [108]) of the dual space of a GBA is used. Then the pair (Θ_g^t, Θ_s^a) will be a duality between the categories **GenBoolAlg** and **PBoolSp**. This will imply that Θ_g^a and Θ_s^a are naturally isomorphic; hence, we will obtain that the spaces $\Theta_g^a(I)$ and $\Theta_s^a(I)$ are homeomorphic for any GBA I (the last assertion can be proved directly as well).

Let I be a GBA. Set $\Theta_s^a(I)$ to be the set X of all prime ideals of I endowed with a topology $\mathcal{O}$ having as an open base the set $\{\gamma_I(b) \mid b \in I\}$ where, for every $b \in I$, $\gamma_I(b) = \{i \in X \mid b \notin i\}$ (see M. Stone [108]). Then, as it is proved in [108], $(X, \mathcal{O})$ is a Boolean space and

$$\gamma_I : I \longrightarrow KO(X, \mathcal{O}), \quad b \mapsto \gamma_I(b),$$

is a 0-pseudolattice isomorphism and hence, a **GenBoolAlg**-isomorphism. If $\varphi \in$ **GenBoolAlg**(I, J) then set $X = \Theta_s^a(I)$, $Y = \Theta_s^a(J)$ and define a map $f = f_\varphi : Y \longrightarrow X$ by the formula $f(j) = \varphi^{-1}(j)$, for every $j \in Y$. Since φ is a **GenBoolAlg**-morphism, we get that this definition is correct and, for every $b \in I$,

(3.13) $f_\varphi^{-1}(\gamma_I(b)) = \gamma_J(\varphi(b))$.

This implies easily that f is a perfect map. It becomes now clear that Θ_s^a is a contravariant functor, and also, it is not difficult to show that the pair (Θ_g^t, Θ_s^a) is a duality between the categories **GenBoolAlg** and **PBoolSp** (see [45]).

Corollary 3.2.2.18. (M. Stone [108]) *There exists a bijective correspondence between the class of all (up to 0-pseudolattice isomorphism) generalized Boolean algebras and all (up to homeomorphism) locally compact zero-dimensional Hausdorff spaces.*

Note that in [107], M. Stone proved that there exists a bijective correspondence between generalized Boolean algebras and Boolean rings (with or without unit).

Obviously, the categories **BoolSp** and **PBoolSp** are subcategories of the category **HLC**. In the next theorem we will find the subcategories of the category **DHLC** which are dual to the categories **BoolSp** and **PBoolSp**.

Definition 3.2.2.19. Let **DZHLC** (resp., **DPZHLC**) be the full subcategory of the category **DHLC** (resp., $\mathbf{D_1PHLC}$) (see Definition 2.4.2.1) having as objects all CLCAs $(A, \rho, \mathbb{B})$ such that if $a, b \in \mathbb{B}$ and $a \ll_\rho b$ then there exists $c \in \mathbb{B}$ with $c \ll_\rho c$ and $a \leq c \leq b$.

Theorem 3.2.2.20. *The following categories are dually equivalent:*
(a) **BoolSp** *and* **DZHLC***;*
(b) **PBoolSp** *and* **DPZHLC**.

Proof. We will show that the contravariant functors

$$\Lambda_z^t = (\Lambda^t)_{|\mathbf{BoolSp}} \ \text{ and } \ \Lambda_z^a = (\Lambda^a)_{|\mathbf{DZHLC}}$$

are the required duality functors (see (2.5) for Λ^t and Λ^a) for the first pair of categories. Indeed, if $X \in |\mathbf{BoolSp}|$ then

$$\Lambda^t(X) = (RC(X), \rho_X, CR(X))$$

and, obviously, $(RC(X), \rho_X, CR(X)) \in |\mathbf{DZHLC}|$.

Conversely, if $(A, \rho, \mathbb{B}) \in |\mathbf{DZHLC}|$ then $X = \Lambda^a(A, \rho, \mathbb{B})$ is a locally compact Hausdorff space. For proving that X is a zero-dimensional space, let $x \in X$ and U be an open neighborhood of x. Then there exist open sets V, W in X such that $x \in V \subseteq \mathrm{cl}(V) \subseteq W \subseteq \mathrm{cl}(W) \subseteq U$ and $\mathrm{cl}(V)$, $\mathrm{cl}(W)$ are compacts. Then there exist $a, b \in \mathbb{B}$ such that $\lambda_A^g(a) = \mathrm{cl}(V)$ and $\lambda_A^g(b) = \mathrm{cl}(W)$ (see (1.29) for the notation λ_A^g) . Obviously, $a \ll_\rho b$. Thus, there exists $c \in \mathbb{B}$ such that $c \ll_\rho c$ and $a \leq c \leq b$. Then $F = \lambda_A^g(c)$ is a clopen subset of X and $x \in F \subseteq U$. So, X is zero-dimensional. Now, all follows from Theorem 2.2.2.12.

The restrictions of the obtained above duality functors to the categories of the second pair give, according to Theorem 2.4.2.2, the desired second duality. □

3.3 Duality Theorems for some cofull subcategories of the category BoolSp

3.3.1 A Duality Theorem for the category of Boolean spaces and skeletal maps

Recall that a homomorphism φ between two Boolean algebras is called *complete* if it preserves all joins (and, consequently, all meets) that happen to exist; this means that if $\{a_i\}$ is a family of elements in the domain of φ with join a, then the family $\{\varphi(a_i)\}$ has a join and that join is equal to $\varphi(a)$.

Recall that we denoted by **SBoolSp** the category of zero-dimensional locally compact Hausdorff spaces and continuous skeletal maps (see 0.4.1.3 for the definition of a skeletal map).

Definition 3.3.1.1. Let **SZLBA** be the cofull subcategory of the category **ZLBA** whose morphisms are, in addition, complete homomorphisms.

Theorem 3.3.1.2. *The categories* **SBoolSp** *and* **SZLBA** *are dually equivalent.*

Proof. Having in mind Theorem 3.2.2.7, it is enough to prove that if f is a morphism of the category **SBoolSp** then $\Theta^t_d(f)$ is complete, and if φ is a **SZLBA**-morphism then $\Theta^a_d(\varphi)$ is a skeletal map.

So, let $f \in \mathbf{SBoolSp}(X, Y)$ and $\varphi = \Theta^t_d(f)$. Then

$$\varphi : (CO(Y), KO(Y)) \longrightarrow (CO(X), KO(X))$$

and $\varphi(G) = f^{-1}(G)$, for all $G \in CO(Y)$. Let $\{G_\gamma \mid \gamma \in \Gamma\} \subseteq CO(Y)$ and let this family have a join G in $CO(Y)$. Set $W = \bigcup\{G_\gamma \mid \gamma \in \Gamma\}$. Since Y is zero-dimensional, we get easily that $G = \mathrm{cl}(W)$. Thus $\varphi(G) \supseteq \mathrm{cl}(\bigcup\{\varphi(G_\gamma) \mid \gamma \in \Gamma\}) = F$. Let $x \in f^{-1}(G)(= \varphi(G))$. Then $f(x) \in G$ and there exists a neighborhood U of x such that $f(U) \subseteq G$. Suppose that $x \notin F$. Then there exists a neighborhood V of x such that $V \subseteq U$ and $V \cap f^{-1}(G_\gamma) = \emptyset$ for all $\gamma \in \Gamma$. Thus $f(V) \cap W = \emptyset$. Then $\mathrm{cl}(f(V)) \cap W = \emptyset$. Since $\mathrm{cl}(f(V)) \subseteq \mathrm{cl}(f(U)) \subseteq G = \mathrm{cl}(W)$, we get that $\mathrm{cl}(f(V)) \subseteq \mathrm{cl}(W) \setminus W (= \mathrm{Fr}(W))$. This leads to a contradiction because f is skeletal and thus $\mathrm{int}(\mathrm{cl}(f(V))) \neq \emptyset$ (see 2.5.2.3). So, $\varphi(G) = f^{-1}(G) = F$. Since $\varphi(G)$ is clopen, we get that $\varphi(G)$ is the join of the family $\{\varphi(G_\gamma) \mid \gamma \in \Gamma\}$ in $CO(X)$. Therefore, φ is complete.

Let now $\varphi \in \mathbf{SZLBA}((A, I), (B, J))$ and $f = \Theta^a_d(\varphi)$. Set $X = \Theta^a_d(A, I)$ and $Y = \Theta^a_d(B, J)$. Then $f : Y \longrightarrow X$. Since $KO(Y)$ is an open base of Y, for proving that f is skeletal it is enough to show that for every $G \in KO(Y) \setminus \{\emptyset\}$, $\mathrm{int}(f(G)) \neq \emptyset$. So, let $G \in KO(Y) \setminus \{\emptyset\}$. Then there exists $b \in J \setminus \{0\}$ such that $G = \lambda_{(B,J)}(b)$. Suppose that $\bigwedge\{c \in A \mid b \leq \varphi(c)\} = 0$. Then, using the completeness of φ, we get that $0 = \varphi(0) = \bigwedge\{\varphi(c) \mid c \in A, b \leq \varphi(c)\} \geq b$. Since $b \neq 0$, we get a contradiction. Hence there exists $a \in A \setminus \{0\}$ such that $a \leq c$ for all $c \in A$ for which $b \leq \varphi(c)$. We will prove that $\lambda_{(A,I)}(a) \subseteq f(\lambda_{(B,J)}(b))(= f(G))$. This will imply that $\mathrm{int}(f(G)) \neq \emptyset$. Let $u \in \lambda_{(A,I)}(a)$. Then $a \in u$. Suppose that there exists $c \in u$ such that $b \wedge \varphi(c) = 0$.

Then $b \leq \varphi(c^*)$. Thus $a \leq c^*$, i.e., $a \wedge c = 0$. Since $a, c \in u$, we get a contradiction. Therefore, the set $\{b\} \cup \varphi(u)$ is a filter-base. Hence there exists an ultrafilter v in B such that $\{b\} \cup \varphi(u) \subseteq v$. Then $b \in v$ and $u \subseteq \varphi^{-1}(v)$. Thus $u = \varphi^{-1}(v)$, i.e., $f(v) = u$. So, $u \in f(\lambda_{(B,J)}(b))$. □

Remarks 3.3.1.3. Note that in the definition of the category **SZLBA** the requirement that the morphisms $\varphi : (A, I) \longrightarrow (B, J)$ are complete can be replaced by the following condition:

(SkeZLBA) For every $b \in J \setminus \{0\}$ there exists $a \in I \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Indeed, the proof of the above theorem shows the sufficiency of this condition and its necessity can be established as follows. Let $f \in \mathbf{SBoolSp}(X, Y)$ and $\varphi = \Theta^t(f)$. Then $\varphi : (CO(Y), KO(Y)) \longrightarrow (CO(X), KO(X))$ and $\varphi(G) = f^{-1}(G)$, for all $G \in CO(Y)$. Let $F \in KO(X) \setminus \{\emptyset\}$. Then $\mathrm{int}(f(F)) \neq \emptyset$. Hence there exists $G \in KO(Y) \setminus \{\emptyset\}$ such that $G \subseteq \mathrm{int}(f(F))$. Let $H \in CO(Y)$ and $F \subseteq f^{-1}(H)$. Then $G \subseteq \mathrm{int}(f(F)) \subseteq f(F) \subseteq H$. So, condition (SkeZLBA) is satisfied.

Moreover, condition (SkeZLBA) can be replaced by the following one:

(CEP) For every $b \in B \setminus \{0\}$ there exists $a \in A \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Indeed, if $b \in B \setminus \{0\}$ then, by the density of J in B, there exists $b_1 \in I \setminus \{0\}$ such that $b_1 \leq b$. Now, applying (SkeZLBA) for b_1, we get an $a \in I \setminus \{0\}$ which satisfies also the requirements of (CEP) about b. Conversely, if $b \in J \setminus \{0\}$ then, by (CEP), there exists $a \in A \setminus \{0\}$ such that $(\forall c \in A)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$; but, by condition (LBA) (see 3.2.1.1), there exists $a_1 \in I$ such that $b \leq \varphi(a_1)$; thus $a \leq a_1$; since I is an ideal, we get that $a \in I$; so, condition (SkeZLBA) is satisfied.

The assertion (c) of the next corollary is a zero-dimensional analogue of the Fedorchuk Duality Theorem [54] (see Theorem 2.5.2.11 here).

Corollary 3.3.1.4. *(a) Let f be a* **PBoolSp**-*morphism. Then f is a quasi-open map iff $\Theta^t(f)$ is complete. In particular, if f is a* **Stone**-*morphism then f is a quasi-open map iff $S^t(f)$ is complete.*

(b) The cofull subcategory **QPBoolSp** *of the category* **PBoolSp** *whose morphisms are, in addition, quasi-open maps, is dually equivalent to the cofull subcategory* **QPZLBA** *of the category* **PZLBA** *whose morphisms are, in addition, complete homomorphisms;*

(c) The category **QStone** *of compact zero-dimensional Hausdorff spaces and quasi-open maps is dually equivalent to the category* **CBool** *of Boolean algebras and complete Boolean homomorphisms.*

Proof. The assertion (a) follows from the proof of Theorem 3.3.1.2 and Corollary 2.5.2.5. The assertions (b) and (c) follow from (a) and Theorem 3.3.1.2. □

The last corollary together with Fedorchuk's Duality Theorem [54] imply the following assertion in which the equivalence $(a) \iff (b)$ is a special case of a much more general theorem due to Monk [84].

Corollary 3.3.1.5. *Let $\varphi \in$* **BoolAlg**(A, B) *and A', B' be minimal completions of A and B respectively. We can suppose that $A \subseteq A'$ and $B \subseteq B'$. Then the following conditions are equivalent:*

(a) φ can be extended to a complete homomorphism $\psi : A' \longrightarrow B'$;

(b) φ is a complete homomorphism;

(c) φ satisfies condition (CEP) (see 3.3.1.3 above).

Proof. (a)⇒(b) This is obvious.
(b)⇒(c) This was already established in the proof of Theorem 3.3.1.2 (see also 3.3.1.3).
(c)⇒(a) Obviously, $\varphi \in \mathbf{ZLBA}((A, A), (B, B))$. Then, by 3.3.1.3 and Theorem 3.3.1.2, condition (CEP) implies that $f = \Theta^a_d(\varphi)(= S^a(\varphi))$ is a skeletal map. Since f is closed, we get that f is a quasi-open map between $Y = \Theta^a_d(B, B)(= S^a(B))$ and $X = \Theta^a_d(A, A)(= S^a(A))$. Now, by Fedorchuk's Duality Theorem [54], the map $\psi : RC(X) \longrightarrow RC(Y)$, $F \mapsto \mathrm{cl}(f^{-1}(\mathrm{int}(F)))$, is a complete homomorphism. Obviously, for every $F \in CO(X)$, $\psi(F) = f^{-1}(F) = \varphi'(F)$ (here $\varphi' = \Theta^t_d(\Theta^a_d(\varphi))$). Then the existence of a natural isomorphism between the composition $\Theta^t_d \circ \Theta^a_d$ and the identity functor (see Theorem 3.2.2.7), and the fact that $RC(X)$ and $RC(Y)$ are minimal completions of, respectively, A and B, imply our assertion. □

Now, using Theorem 3.2.2.17, we will present in a simpler form the result established in Corollary 3.3.1.4(b).

Theorem 3.3.1.6. *The category* **QPBoolSp** *is dually equivalent to the cofull subcategory* **QGBA** *of the category* **GenBoolAlg** *whose morphisms, in addition, preserve all meets that happen to exist.*

Proof. Having in mind Theorem 3.2.2.17 and Corollary 3.3.1.4(b), it is enough to show that the functor $E^l \circ E^z$ (see 3.2.2.17 and 3.2.2.14) maps **QPZLBA** to **QGBA** and the functor $E^p \circ E^g$ (see again 3.2.2.17 and 3.2.2.14) maps **QGBA** to **QPZLBA** because with this we will obtain that the categories **QPZLBA** and **QGBA** are equivalent. Obviously, if $\varphi' : (A, I) \longrightarrow (B, J)$ is a **QPZLBA**-morphism then $\varphi = (E^l \circ E^z)(\varphi') = (\varphi')_{|I} : I \longrightarrow J$ preserves all meets in I that happen to exist (indeed, since I is an ideal of A, every meet in I of elements of I is also a meet in A). Conversely, let $\varphi : I \longrightarrow J$ be a **QGBA**-morphism. We will show that φ satisfies the following condition:

(QGBPL) For every $b \in J \setminus \{0\}$ there exists $a \in I \setminus \{0\}$ such that $(\forall c \in I)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$.

Indeed, let $b \in J \setminus \{0\}$. Suppose that $\bigwedge_I \{c \in I \mid b \leq \varphi(c)\} = 0$. Then, using the completeness of φ, we get that $0 = \varphi(0) = \bigwedge\{\varphi(c) \mid c \in I, b \leq \varphi(c)\} \geq b$. Since $b \neq 0$, we get a contradiction. Hence there exists $a \in I \setminus \{0\}$ such that $a \leq c$ for all $c \in I$ for which $b \leq \varphi(c)$.

Let $\varphi' = (E^p \circ E^g)(\varphi)$. We will show that the map φ' satisfies condition (SkeZLBA). We have that $\varphi' : (SI(I), e_I(I)) \longrightarrow (Si(J), e_J(J))$. Let $J_1 \in e_J(J) \setminus \{0\}$. Then there exists $b \in J \setminus \{0\}$ such that $J_1 = \downarrow(b)$. By (QGBPL), there exists $a \in I \setminus \{0\}$ such that $(\forall c \in I)[(b \leq \varphi(c)) \rightarrow (a \leq c)]$. Let $I_1 \in Si(I)$ and $J_1 \subseteq \varphi'(I_1)$. Then, by the definition of the map φ' (see Theorem 3.2.2.17), we have that $\downarrow(b) \subseteq \bigcup\{\downarrow(\varphi(c)) \mid c \in I_1\}$. Thus there exists $c \in I_1$ such that $b \leq \varphi(c)$. Since $c \in I$, we get that $a \leq c$. Therefore, $\downarrow(a) \subseteq I_1$. So, the map φ' satisfies condition (SkeZLBA). Now 3.3.1.3 implies that φ' is a complete homomorphism. Thus φ' is a **QPZLBA**-morphism. □

Remark 3.3.1.7. The proof of Theorem 3.3.1.6 shows that in the definition of the category **QGBA** the requirement that its morphisms $\varphi : I \longrightarrow J$ preserve all meets that happen to exist can be replaced by the condition (QGBPL) introduced above.

3.3.2 A Duality Theorem for the category of Boolean spaces and open maps

Theorem 3.3.2.1. *(a) Let $f \in$ **BoolSp**(X, Y), $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $\Theta^t(Y) = (B, J)$. Then the map f is open iff there exists a map $\psi : I \longrightarrow J$ which satisfies the following conditions:*

(OZL1) *For every $b \in J$ and every $a \in I$, $(a \wedge \varphi(b) = 0) \rightarrow (\psi(a) \wedge b = 0)$, and*

(OZL2) *For every $a \in I$, $\varphi(\psi(a)) \geq a$*

(such a map ψ will be called a lower pre-adjoint *of φ).*

(b) The cofull subcategory **OBoolSp** *of the category* **BoolSp** *whose morphisms are the open maps is dually equivalent to the cofull subcategory* **OZLBA** *of the category* **ZLBA** *whose morphisms have, in addition, lower pre-adjoints.*

Proof. (a) Let $f \in \mathbf{BoolSp}(X, Y)$ be an open map. For every $F \in KO(X)(= I)$ set $\psi(F) = f(F)$. Then, clearly, $\psi(F) \in KO(Y)(= J)$ and $\varphi(\psi(F)) = f^{-1}(f(F)) \supseteq F$. Hence, condition (OZL2) is satisfied. Let $F \in KO(X)$, $G \in KO(Y)$ and $F \wedge \varphi(G) = 0$. Then $F \cap f^{-1}(G) = \emptyset$. Thus $f(F) \cap G = \emptyset$, i.e., $\psi(F) \wedge G = 0$. Therefore, condition (OZL1) is satisfied as well.

Let now φ has a lower pre-adjoint. We will show that $f' = \Theta^a(\varphi)$ is an open map. This will imply that f is open. Let $X' = \Theta^a(\Theta^t(X))$ and $Y' = \Theta^a(\Theta^t(Y))$. Since $\lambda_{(A,I)}(I)$ is an open base of X', it is enough to show that $f'(\lambda_{(A,I)}(a))$ is an open set, for every $a \in I$. So, let $a \in I$. We will prove that $f'(\lambda_{(A,I)}(a)) = \lambda_{(B,J)}(\psi(a))$. Let $u \in \lambda_{(A,I)}(a)$. Then $a \in u$. Let $v = f'(u)$, i.e., $v = \varphi^{-1}(u)$. By (OZL2), $\varphi(\psi(a)) \geq a$ and hence $\varphi(\psi(a)) \in u$. Thus $\psi(a) \in \varphi^{-1}(u) = v$, i.e., $f'(u) \in \lambda_{(B,J)}(\psi(a))$. Therefore $f'(\lambda_{(A,I)}(a)) \subseteq \lambda_{(B,J)}(\psi(a))$. Conversely, let $v \in \lambda_{(B,J)}(\psi(a))$. Then $\psi(a) \in v$. Suppose that there exists $b \in v$ such that $a \wedge \varphi(b) = 0$. Since v is a bounded ultrafilter, there exists $b_0 \in v \cap J$. Then $b_1 = b \wedge b_0 \in J \cap v$ and $a \wedge \varphi(b_1) = 0$. Now, condition (OZL1) implies that $\psi(a) \wedge b_1 = 0$, which is a contradiction. Hence, the set $\{a\} \cup \varphi(v)$ is a filter-base. Thus there exists an ultrafilter $u \supseteq \{a\} \cup \varphi(v)$. Then $a \in u \cap I$ and $v \subseteq \varphi^{-1}(u)$. Therefore, $v = \varphi^{-1}(u) = f(u)$. This shows that $f'(\lambda_{(A,I)}(a)) \supseteq \lambda_{(B,J)}(\psi(a))$. Hence, f' is an open map.

(b) It follows from (a) and Theorem 3.2.2.7. □

Remarks 3.3.2.2. Note that condition (OZL2) implies condition (LBA). Indeed, in the notation of Theorem 3.3.2.1, if $a \in I$ then $b = \psi(a) \in J$ and $\varphi(b) \geq a$. Further, condition (OZL1) implies that (again in the notation of Theorem 3.3.2.1) $\psi(0) = 0$. Indeed, $0 \wedge \varphi(\psi(0)) = 0$ implies that $\psi(0) \wedge \psi(0) = 0$, i.e., that $\psi(0) = 0$.

Theorem 3.3.2.3. *(a) Let* $f \in \mathbf{PBoolSp}(X, Y)$, $(A, I) = \Theta^t(X)$, $(B, J) = \Theta^t(Y)$ *and* $\varphi = \Theta^t(f)$. *Then the map* f *is open iff* $\varphi : B \longrightarrow A$ *has a lower adjoint* $\psi : A \longrightarrow B$.
(b) The cofull subcategory **POBoolSp** *of the category* **PBoolSp** *whose morphisms are open perfect maps is dually equivalent to the cofull subcategory* **POZLBA** *of the category* **PZLBA** *whose morphisms have, in addition, lower adjoints.*

Proof. (a) Let $f \in \mathbf{PBoolSp}(X, Y)$ and f is open. Then set $\psi(F) = f(F)$, for every $F \in CO(X)(= A)$. Then, since f is open and closed map, $\psi : A \longrightarrow B(=$

$CO(Y))$. Obviously, $\varphi(\psi(F)) = f^{-1}(f(F)) \supseteq F$ for every $F \in CO(X)$ and $\psi(\varphi(G)) = f(f^{-1}(G)) \subseteq G$ for every $G \in CO(Y)$. Hence ψ is a lower adjoint of φ. Conversely, let φ has a lower adjoint ψ. Then $\psi(I) \subseteq J$. Indeed, let $a \in I$. Then, by condition (LBA), there exists $b \in J$ such that $a \leq \varphi(b)$. Then $\psi(a) \leq \psi(\varphi(b)) \leq b \in J$. Thus, $\psi(a) \in J$. Further, condition (OZL2) is clearly fulfilled as well as condition (OZL1) (see Fact 2.5.2.1). So, $(\psi)_{|I}$ is a lower pre-adjoint of φ. Then, by Theorem 3.3.2.1(a), $f : X \longrightarrow Y$ is an open map.

(b) It follows from (a) and Theorem 3.2.2.11. □

Corollary 3.3.2.4. *(a) Let $f \in \mathbf{Stone}(X, Y)$, $\varphi = S^t(f)$, $A = S^t(X)$ and $B = S^t(Y)$. Then the map f is open iff $\varphi : B \longrightarrow A$ has a lower adjoint $\psi : A \longrightarrow B$.*

(b) The category **OStone** *of compact zero-dimensional Hausdorff spaces and open maps is dually equivalent to the category* **OBool** *of Boolean algebras and Boolean homomorphisms having lower adjoints.*

Proof. It follows immediately from Theorem 3.3.2.3. □

Definition 3.3.2.5. Let $\varphi \in \mathbf{GenBoolAlg}(J, I)$. If $\psi : I \longrightarrow J$ is a map which satisfies conditions (OZL1) and (OZL2) (see 3.3.2.1) then ψ is called a *lower preadjoint* of φ.

Let **OGBA** be the cofull subcategory of the category **GenBoolAlg** whose morphisms have, in addition, lower preadjoints.

(Note that we distinguish between “preadjoint” and “pre-adjoint” (see Theorem 3.3.2.1).

Corollary 3.3.2.6. *The categories* **POBoolSp** *and* **OGBA** *are dually equivalent.*

Proof. It follows from Theorems 3.2.2.17, 3.3.2.1 and 3.3.2.3. Indeed, it is enough to show that the categories **OGBA** and **POZLBA** are equivalent. From the proof of Theorem 3.3.2.3, it follows that if φ' is an **POZLBA**-morphism then $\varphi = E^l(E^z(\varphi'))$ has a lower preadjoint. Conversely, if φ is an **OGBA**-morphism then $\varphi' = E^p(E^g(\varphi))$ can be regarded as an extension of φ. This implies immediately that φ' has a lower pre-adjoint. Now, Theorem 3.3.2.1 implies that $f = \Theta^a(\varphi')$ is an open map. Thus, by Theorem 3.3.2.3, φ' has a lower adjoint. □

3.4 Characterizations of the dual morphisms of the embeddings, surjections and injections

In this section we will investigate the following problem: characterize the dual morphisms of the injective and surjective morphisms of the category **BoolSp** and its subcategories **PBoolSp**, **OBoolSp**. Such a problem was regarded by M. Stone in [108] for surjective continuous maps and for closed embeddings (i.e., for injective morphisms of the category **PBoolSp**). An analogous problem will be investigated for the homeomorphic embeddings and dense embeddings.

3.4.1 Characterizations of the dual morphisms of the injective maps

We start with a simple observation.

Proposition 3.4.1.1. *Let* $f \in \mathbf{BoolSp}(X,Y)$, $(A,I) = \Theta^t(X)$, $(B,J) = \Theta^t(Y)$ *and* $\varphi = \Theta^t(f)$. *Then* φ *is an injection* $\iff$ $\varphi_{|J}$ *is an injection* $\iff$ $\mathrm{cl}_Y(f(X)) = Y$.

Proof. We have that $(A,I) = (CO(X), KO(X))$, $(B,J) = (CO(Y), KO(Y))$ and $\varphi : CO(Y) \longrightarrow CO(X)$.

Obviously, if φ is an injection then $\varphi_{|J}$ is an injection.

Let $\varphi_{|J}$ be an injection, $G \in KO(Y)$ and $G \neq \emptyset$. Then $\varphi(G) \neq \emptyset$, i.e., $f^{-1}(G) \neq \emptyset$. This means that $f(X) \cap G \neq \emptyset$. Thus $\mathrm{cl}(f(X)) = Y$.

Finally, let $\mathrm{cl}(f(X)) = Y$, $G \in CO(Y)$ and $G \neq \emptyset$. Then $G \cap f(X) \neq \emptyset$ and thus $\varphi(G) = f^{-1}(G) \neq \emptyset$. So, φ is an injection. □

Proposition 3.4.1.2. *Let* $f \in \mathbf{BoolSp}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$, $(B,J) = \Theta^t(Y)$ *and* $\varphi(B) \supseteq I$ *(or* $\varphi(J) \supseteq I$*). Then* f *is an injection.*

Proof. Suppose that there exist $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$. Then there exists $U \in KO(X)$ such that $x \in U \subseteq X \setminus \{y\}$. There exists $V \in CO(Y)$ (or, respectively, $V \in KO(Y)$) with $\varphi(V) = U$, i.e., $f^{-1}(V) = U$. Then $f(U) = f(X) \cap V$ and $f^{-1}(f(U)) = f^{-1}(V) = U$. Since $f(y) = f(x) \in f(U)$, we get that $y \in U$, a contradiction. Thus, f is an injection. □

Theorem 3.4.1.3. *Let* $f \in \mathbf{BoolSp}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ *and* $(B,J) = \Theta^t(Y)$. *Then* f *is an injection iff* $\varphi : (B,J) \longrightarrow (A,I)$ *satisfies the following condition:*

(InZLC) *For any* $a, b \in I$ *such that* $a \wedge b = 0$ *there exists* $a', b' \in J$ *with* $a' \wedge b' = 0$, $\varphi(a') \geq a$ *and* $\varphi(b') \geq b$.

Proof. Let $f : X \longrightarrow Y$ be an injection. We have that $(A, I) = (CO(X), KO(X))$, $(B, J) = (CO(Y), KO(Y))$ and $\varphi : CO(Y) \longrightarrow CO(X)$, $G \mapsto f^{-1}(G)$. Let $F_1, F_2 \in KO(X)$ and $F_1 \cap F_2 = \emptyset$. Since f is a continuous injection, we get that $f(F_1)$ and $f(F_2)$ are disjoint compact subsets of Y. Using the fact that $KO(Y)$ is a base of Y, we get that there exist disjoint $G_1, G_2 \in KO(Y)$ such that $f(F_i) \subseteq G_i$, $i = 1, 2$. Then $F_i \subseteq f^{-1}(G_i)$, i.e., $F_i \subseteq \varphi(G_i)$, $i = 1, 2$. Hence, φ satisfies condition (InZLC).

Let now φ satisfies condition (InZLC). We will prove that f is an injection. Let $x, y \in X$ and $x \neq y$. Then there exist disjoint $F_x, F_y \in KO(X)$ such that $x \in F_x$ and $y \in F_y$. Now, by condition (InZLC), there exist $G_x, G_y \in KO(Y)$ such that $G_x \cap G_y = \emptyset$, $f^{-1}(G_x) \supseteq F_x$ and $f^{-1}(G_y) \supseteq F_y$. Then $f(x) \in G_x$ and $f(y) \in G_y$. Thus $f(x) \neq f(y)$. $\square$

Corollary 3.4.1.4. *The cofull subcategory* **InjBoolSp** *of the category* **BoolSp** *whose morphisms are, in addition, injective maps, is dually equivalent to the cofull subcategory* **DInjBoolSp** *of the category* **ZLBA** *whose morphism satisfy, in addition, condition (InZLC).*

Proof. It follows from Theorems 3.4.1.3 and 3.2.2.7. $\square$

In the sequel, we will not formulate corollaries like that because they follow directly from the respective characterization of injectivity or surjectivity and the corresponding duality theorems.

Remark 3.4.1.5. Let us show how Theorem 3.4.1.3 implies Proposition 3.4.1.2. Let $\varphi(B) \supseteq I$. Then $\varphi(J) \supseteq I$. Indeed, let $a \in I$; then, by condition (LBA), there exists $b_1 \in J$ such that $\varphi(b_1) \geq a$; since there exists $b_2 \in B$ with $\varphi(b_2) = a$, we get that $\varphi(b_1 \wedge b_2) = a$ and $b_1 \wedge b_2 \in J$. Hence, $\varphi(J) \supseteq I$. Let now $a, b \in I$ and $a \wedge b = 0$. There exist $a_1, b_1 \in J$ such that $\varphi(a_1) = a$ and $\varphi(b_1) = b$. Then $\varphi(a_1 \wedge b_1^*) = a \wedge b^* = a$, $a_1 \wedge b_1^* \in J$ and $(a_1 \wedge b_1^*) \wedge b_1 = 0$. Therefore, φ satisfies condition (InZLC).

3.4.2 Characterizations of the dual morphisms of the surjective maps

In the next theorem we will assume that the ideals and prime ideals could be non-proper.

Theorem 3.4.2.1. *Let $f \in \mathbf{BoolSp}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ and $(B,J) = \Theta^t(Y)$. Then the following conditions are equivalent:*

(a) f is a surjection;

(b) $\varphi : B \longrightarrow A$ is an injection and for every bounded ultrafilter v in (B,J) there exists $a \in I$ such that $a \wedge \varphi(v) \neq 0$ (i.e., $a \wedge \varphi(b) \neq 0$ for any $b \in v$);

(c) $\varphi : B \longrightarrow A$ is an injection and for every prime ideal J_1 of J, we have that $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$ implies $J_1 = J$ (where $I_{\varphi(b)} = \{a \in I \mid a \leq \varphi(b)\}$);

(d) $\varphi : B \longrightarrow A$ is an injection and for every ideal J_1 of J, $[(\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I) \rightarrow (J_1 = J)]$.

Proof. (a)$\Rightarrow$(b) Let $f(X) = Y$. Then, by Proposition 3.4.1.1, φ is an injection. Further, by (3.7), the bounded ultrafilters in $(B,J) = (CO(Y), KO(Y))$ are of the form u_y^C (see 3.2.1.10 for this notation) and analogously for (A,I). So, let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. This implies that $\varphi(u_y^C) \subseteq u_x^C$. There exists $F \in KO(X) \cap u_x^C$. Then $F \cap f^{-1}(G) \neq \emptyset$, for every $G \in u_y^C$, i.e., $F \wedge \varphi(u_y^C) \neq 0$.

(b)$\Rightarrow$(c) Let J_1 be a prime ideal of J. Let $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$. Suppose that $J_1 \neq J$. Then $v_1 = \{b \in B \mid b \wedge (J \setminus J_1) \neq 0\}$ is a bounded ultrafilter in (B,J) and $v_1 \cap J = J \setminus J_1$. This follows from the more general Proposition 2.2.3.7 but we will supply it with a new direct proof. So, it is clear that $J \setminus J_1$ is a filter in J, and hence $J \setminus J_1 \subseteq v_1$; also, $J \setminus J_1 \neq \emptyset$ and v_1 is an upper set. We will show that $v_1 \cap J = J \setminus J_1$. Since $J \setminus J_1 \subseteq v_1$, it is enough to prove that $v_1 \cap J_1 = \emptyset$. Let $d \in J_1$. There exists $e \in J \setminus J_1$. If $d^* \wedge e \in J_1$ then $e = (e \wedge d) \vee (e \wedge d^*) \in J_1$, a contradiction. Hence $c = d^* \wedge e \in J \setminus J_1$ and $d \wedge c = 0$. Therefore, $d \notin v_1$. So, $v_1 \cap J_1 = \emptyset$ and thus $v_1 \cap J = J \setminus J_1$. Further, if $b_1 \in v_1$ and $b_2 \in J \setminus J_1$ then $b_1 \wedge b_2 \in J \setminus J_1$. Indeed, if $b = b_1 \wedge b_2 \in J_1$ then $b \notin v_1$ and hence there exists $c \in J \setminus J_1$ such that $b \wedge c = 0$, i.e., $b_1 \wedge (b_2 \wedge c) = 0$; since $c \wedge b_2 \in J \setminus J_1$, we get a contradiction. Let now $b_1, b_2 \in v_1$. We will show that $b_1 \wedge b_2 \in v_1$ and this will imply that v_1 is a filter in B. Let $c \in J \setminus J_1$. Then $b_1 \wedge c, b_2 \wedge c \in J \setminus J_1$ and thus $(b_1 \wedge c) \wedge (b_2 \wedge c) \in J \setminus J_1$; hence $(b_1 \wedge b_2) \wedge c \neq 0$. Therefore, $b_1 \wedge b_2 \in v_1$. Finally, for showing that the filter v_1 is an ultrafilter, suppose that there exists $b \in B$ such that $b \notin v_1$ and $b^* \notin v_1$. Then there exist $c, d \in J \setminus J_1$ such that $b \wedge c = 0$ and $b^* \wedge d = 0$. Since $c \wedge d \in J \setminus J_1$, we have that $c \wedge d \neq 0$. On the other hand, $d \leq b$ and hence $c \wedge d \leq c \wedge b = 0$, i.e., $c \wedge d = 0$, a contradiction. Therefore, v_1 is a bounded ultrafilter in (B,J) and $v_1 \cap J = J \setminus J_1$. By (b), there exists $a \in I$ such that $a \wedge \varphi(v_1) \neq 0$. Since $a \in I$ and $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$, there exist $b_1, \ldots, b_k \in J_1$

and $a_1, \ldots, a_k \in I$ (where $k \in \mathbb{N}^+$) such that $a = \bigvee\{a_i \mid i = 1, \ldots, k\}$ and $a_i \leq \varphi(b_i)$, $i = 1, \ldots, k$. Set $b = \bigvee\{b_i \mid i = 1, \ldots, k\}$. Then $a \leq \varphi(b)$ and $b \in J_1$. Since φ is an injection, we have that $\varphi(v_1 \cap J) = \varphi(J \setminus J_1) = \varphi(J) \setminus \varphi(J_1)$. Thus $\varphi(b) \notin \varphi(v_1 \cap J)$ (because $b \in J_1$). Since $a \leq \varphi(b)$, we get that $\varphi(b) \wedge \varphi(v_1) \neq 0$. The injectivity of φ implies that $b \wedge v_1 \neq 0$. Thus $b \in v_1 \cap J_1$, a contradiction. Hence, $J_1 = J$.

(c)$\Rightarrow$(a) Suppose that $f(X) \neq Y$. Then there exists $y \in Y \setminus f(X)$. Set $U = Y \setminus \{y\}$. Thus $f(X) \subseteq U$. Set $J_1 = \{G \in KO(Y) \mid G \subseteq U\}$. Then J_1 is a prime ideal of $J(= KO(Y))$. (Indeed, if $G_1, G_2 \in KO(Y)$ and $y \notin G_1 \cap G_2$ then either $y \notin G_1$ or $y \notin G_2$; hence, $G_1 \in J_1$ or $G_2 \in J_1$.) Obviously, $J_1 \neq J$. We will prove that $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$, which, by (c), will lead to a contradiction. So, let $F \in KO(X)$. Then $f(F) \subseteq U$. Since $f(F)$ is compact, there exists $G \in KO(Y)$ such that $f(F) \subseteq G \subseteq U$. Then $G \in J_1$ and $F \subseteq f^{-1}(G) = \varphi(G)$. Thus $F \in I_{\varphi(G)}$. Therefore, $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$. So, $f(X) = Y$.

(a)$\Rightarrow$(d) Let $f(X) = Y$. Then, by Proposition 3.4.1.1, φ is an injection. Let J_1 be an ideal of J such that $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I$. Suppose that $J_1 \neq J$. Set $U = \bigcup\{G \mid G \in J_1\}$. Then $U \neq Y$. (Indeed, if $U = Y$ then every $H \in KO(Y)(= J)$ will be covered by a finite number of elements of J_1; since J_1 is an ideal, we will get that $H \in J_1$.) Since f is a surjection, we get that $V = f^{-1}(U) \neq X$. Set $I_V = \{F \in I \mid F \subseteq V\}$. Then, obviously, I_V is a proper ideal of I. Let $G \in J_1$ and $F \in I_{\varphi(G)}$. Then $F \subseteq \varphi(G) = f^{-1}(G) \subseteq f^{-1}(U) = V$. Thus $\bigvee\{I_{\varphi(b)} \mid b \in J_1\} \subseteq I_V$. Since $I_V \neq I$, we get a contradiction. Therefore, $J_1 = J$.

(d)$\Rightarrow$(c) It is obvious. $\square$

Remark 3.4.2.2. In [108, Theorem 7] M. Stone proved a result which is equivalent to our assertion that (a)$\Leftrightarrow$(d) in the previous theorem. More precisely, M. Stone proved the following (in our notation): the map f is a surjection iff the map $\psi = \varphi_{|J} : J \longrightarrow A$ is a 0-pseudolattice monomorphism and for every ideal J_1 of J, $[(\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I) \leftrightarrow (J_1 = J)]$. The Stone's condition "$(J_1 = J) \rightarrow (\bigvee\{I_{\varphi(b)} \mid b \in J_1\} = I)$", i.e., "$\bigvee\{I_{\varphi(b)} \mid b \in J\} = I$", is equivalent (as it is easy to see) to our condition (LBA) (see 3.2.1.1) which is automatically satisfied by the morphisms of the category **ZLBA** and thus it appears in our Theorem 3.4.2.1 in another form. Further, when φ is an injection then, obviously, $\psi = \varphi_{|J}$ is an injection; in the converse direction we have the following: the map ψ can be extended to a homomorphism $\varphi : B \longrightarrow A$ (by the result proved below) and then φ is obliged to be an injection (indeed, if $b \in B \setminus \{0\}$ and $\varphi(b) = 0$ then the density of J in B implies that there exists $c \in J \setminus \{0\}$ such that

$c \leq b$; then $\psi(c) = \varphi(c) = 0$, a contradiction). So, our condition (d) is equivalent to the cited above Stone condition from [108, Theorem 7].

Proposition 3.4.2.3. *Let (A, I) be a ZLBA, (B, J) be an LBA and $\psi : J \longrightarrow A$ be a 0-pseudolattice homomorphism satisfying condition (LBA) (i.e., $\forall a \in I\ \exists b \in J$ such that $a \leq \psi(b)$). Then ψ can be extended to a homomorphic map $\varphi : B \longrightarrow A$.*

Proof. For every $a \in A$ and every $b \in B$, set $I_a = \{c \in I \mid c \leq a\}$ and $J_b = \{c \in J \mid c \leq b\}$. It is easy to see that I_a and J_b are simple ideals of I and J respectively. Note also that $\neg I_a = I_{a^*}$ and analogously for J_b.

Let $b \in B$. Since J is dense in B, we have that $b = \bigvee J_b$. We will show that $I(b) = \bigvee\{I_{\psi(c)} \mid c \in J_b\}$ is a simple ideal of I. It is easy to see that $I(b) = \bigcup\{I_{\psi(c)} \mid c \in J_b\}$. Let now $a \in I$. Then, by condition (LBA), there exists $c \in J$ such that $a \leq \psi(c)$. We have that $c = (c \wedge b) \vee (c \wedge b^*)$, $c_1 = c \wedge b \in J_b$, $c_2 = c \wedge b^* \in \neg J_b$ and $c = c_1 \vee c_2$. Thus $a \leq \psi(c) = \psi(c_1) \vee \psi(c_2)$. We obtain that $a = a_1 \vee a_2$, where $a_1 = a \wedge \psi(c_1)$ and $a_2 = a \wedge \psi(c_2)$. Obviously, $a_1 \in I(b)$. We will show that $a_2 \in \neg I(b)$. Indeed, let $a' \in I(b)$; then there exists $d \in J_b$ such that $a' \leq \psi(d)$. Since $c_2 \in \neg J_b$, we get that $d \wedge c_2 = 0$. Thus $\psi(d) \wedge \psi(c_2) = 0$. Hence $a' \wedge a_2 \leq \psi(d) \wedge a \wedge \psi(c_2) = 0$. Therefore, for every $a' \in I(b)$ we have that $a_2 \wedge a' = 0$. This means that $a_2 \in \neg I(b)$. Therefore, $I(b) \vee \neg I(b) = I$, i.e., $I(b)$ is a simple ideal. Since (A, I) is a ZLBA, we get that $\bigvee I(b)$ exists in A. We set now $\varphi(b) = \bigvee I(b)$. Obviously, $\varphi(0) = 0$. Further, $\varphi(1) = \bigvee I(1)$. We have that $I(1) = \bigcup\{I_{\psi(c)} \mid c \in J\}$. Applying condition (LBA), we get that $I(1) = I$. Now, using the density of I in A, we obtain that $\varphi(1) = 1$. Finally, the fact that φ preserves finite meets and finite joins can be easily proved. Hence $\varphi : B \longrightarrow A$ is a Boolean homomorphism and the definition of φ together with the density of I in A imply that φ extends ψ. $\square$.

Remark 3.4.2.4. Note that 3.4.2.2 and 3.4.2.3 imply that in Theorem 3.4.2.1 we can obtain new conditions equivalent to the condition (a) by replacing in (b), (c) and (d) the phrase "φ is an injection" by the phrase "$\varphi_{|J}$ is an injection".

3.4.3 Characterizations of the dual morphisms of some special maps

Theorem 3.4.3.1. *Let $f \in$ **OBoolSp**(X, Y), $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$, $(B, J) = \Theta^t(Y)$. Then f is an injection $\iff \varphi(J) \supseteq I \iff \varphi(B) \supseteq I$.*

Proof. Note that, by Remark 3.4.1.5, conditions "$\varphi(J) \supseteq I$" and "$\varphi(B) \supseteq I$" are equivalent.

Let f be an injection and $F \in KO(X)$. Then $f(F) \in KO(Y)$ and $f^{-1}(f(F)) = F$. Hence, $\varphi(J) \supseteq I$. Conversely, let $\varphi(J) \supseteq I$. Then, by 3.4.1.2, we get that f is an injection. $\square$

Theorem 3.4.3.2. *Let $f \in \mathbf{PBoolSp}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ and $(B,J) = \Theta^t(Y)$. Then f is a surjection $\iff$ φ is an injection $\iff$ $\varphi_{|J}$ is an injection.*

Proof. By Proposition 3.4.1.1, if f is a surjection then φ is an injection. Hence $\varphi_{|J}$ is an injection.

Let now $\varphi_{|J}$ be an injection. Then, by Proposition 3.4.1.1, $\mathrm{cl}(f(X)) = Y$. Since f is a closed map, we get that f is a surjection. $\square$

Theorem 3.4.3.3. *Let $f \in \mathbf{PBoolSp}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ and $(B,J) = \Theta^t(Y)$. Then f is an injection iff $\varphi(J) = I$.*

Proof. Let f be an injection. Then $f_{\restriction X} : X \longrightarrow f(X)$ is a homeomorphism. Let $F' \in KO(X)$. Then $F = f(F')$ is compact. Since F is open in $f(X)$, there exists an open set U in Y such that $U \cap f(X) = F$. Then there exists $G \in KO(Y)$ such that $F \subseteq G \subseteq U$. Then, clearly, $f^{-1}(G) = f^{-1}(F) = F'$. Hence $\varphi(G) = F'$. Therefore, $\varphi(J) \supseteq I$. Since f is perfect, we have that $\varphi(J) \subseteq I$. Thus $\varphi(J) = I$. Conversely, let $\varphi(J) = I$. Then Proposition 3.4.1.2 implies that f is an injection. $\square$

Obviously, the last two theorems imply the well-known Stone's results that a **Stone**-morphism f is an injection (resp., a surjection) iff $\varphi = S^t(f)$ is a surjection (resp., an injection).

Now we will be occupied with the homeomorphic embeddings. We will call them shortly *embeddings*.

Theorem 3.4.3.4. *Let $f \in \mathbf{BoolSp}(X,Y)$, $\varphi = \Theta^t(f)$, $(A,I) = \Theta^t(X)$ and $(B,J) = \Theta^t(Y)$. Then f is a dense embedding iff φ is an injection and $\varphi(J) \supseteq I$.*

Proof. Let f be a dense embedding. Then $f(X)$ is open in Y and thus f is an open injection. Now, Theorem 3.4.3.1 implies that $\varphi(J) \supseteq I$. Since $\mathrm{cl}(f(X)) = Y$, we get, by 3.4.1.1, that φ is an injection.

Conversely, let φ be an injection and $\varphi(J) \supseteq I$. Then, by 3.4.1.1, $\mathrm{cl}(f(X)) = Y$. We will show that φ has a lower pre-adjoint. Indeed, for every $a \in I$ there exists a unique $b_a \in J$ such that $\varphi(b_a) = a$. Let $\psi : I \longrightarrow J$ be defined by $\psi(a) = b_a$ for every $a \in I$. Then, obviously, $\varphi(\psi(a)) = a$, for every $a \in I$. Thus condition (OZL2) (see 3.3.2.1) is satisfied. Further, let $a \in I$, $b \in J$ and $a \wedge \varphi(b) = 0$. Since $a = \varphi(\psi(a))$, we get that $\varphi(\psi(a) \wedge b) = 0$. This implies, by the injectivity of φ, that $\psi(a) \wedge b = 0$. So, condition (OZL1) (see 3.3.2.1) is also satisfied. Therefore, ψ is a lower pre-adjoint of φ. Hence, by Theorem 3.3.2.1, f is an open map. Now, using the condition $\varphi(J) \supseteq I$, we get, by Theorem 3.4.3.1, that f is an injection. Hence, f is a dense embedding. □

Theorem 3.4.3.5. (M. Stone [108]) *Let $f \in \mathbf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $(B, J) = \Theta^t(Y)$. Then f is a closed embedding iff $\varphi(J) = I$.*

Proof. Let f be a closed embedding. Then f is a perfect injection. Hence, by Theorem 3.4.3.3, $\varphi(J) = I$.

Conversely, let $\varphi(J) = I$. Then, by Theorem 3.2.2.11, f is a perfect map. Using Proposition 3.4.1.2, we get that f is an injection. Hence, f is a closed embedding. □

Proposition 3.4.3.6. *Let $f \in \mathbf{BoolSp}(X, Y)$, $\varphi = \Theta^t(f)$, $(A, I) = \Theta^t(X)$ and $\Theta^t(Y)$ $= (B, J)$. Then f is an embedding iff there exists a ZLBA (A_1, I_1) and two* **ZLBA**-*morphisms $\varphi_1 : (A_1, I_1) \longrightarrow (A, I)$ and $\varphi_2 : (B, J) \longrightarrow (A_1, I_1)$ such that $\varphi = \varphi_1 \circ \varphi_2$, φ_1 is an injection, $\varphi_1(I_1) \supseteq I$ and $\varphi_2(J) = I_1$.*

Proof. Obviously, f is an embedding iff $f = i \circ f_1$ where f_1 is a dense embedding and i is a closed embedding. (Indeed, when f is an embedding then let $f_1 : X \longrightarrow \mathrm{cl}_Y(f(X))$ be the restriction of f and $i : \mathrm{cl}_Y(f(X)) \longrightarrow Y$ be the inclusion map; the converse is also clear.) Setting $\varphi_1 = \Theta^t_d(f_1)$ and $\varphi_2 = \Theta^t_d(i)$, we get, by Theorem 3.2.2.7, that $\varphi = \varphi_1 \circ \varphi_2$. Now our assertion follows from Theorems 3.4.3.4 and 3.4.3.5. □

3.5 The construction of the dual objects of the closed, regular closed and open subsets

3.5.1 The dual objects of the open subsets

The next theorem is the well-known result of M. Stone [108] (written in our terms and notation) that the open sets correspond to the ideals.

Theorem 3.5.1.1. (Stone [108]) *Let I be a GBA and $(X, \mathcal{O}) = \Theta^a_s(I)$. Then there exists a frame isomorphism*

$$\iota_s : (Idl(I), \leq) \longrightarrow (\mathcal{O}, \subseteq),\ J \mapsto \bigcup\{\gamma_I(a) \mid a \in J\}.$$

If $U \in \mathcal{O}$ then

$$J = \iota_s^{-1}(U)) = \{b \in I \mid \gamma_I(b) \subseteq U\},$$

J is isomorphic to the ideal

$$J_U = \{F \in KO(X) \mid F \subseteq U\}$$

of $KO(X)$ $(= \Theta_g^t(X))$ and $J_U = KO(U)$, i.e., $J_U = \Theta_g^t(U)$.

Corollary 3.5.1.2. *Let (A, I) be a ZLBA and $(X, \mathcal{O}) = \Theta^a(A, I) (= \Theta_g^a(I))$. Then there exists a frame isomorphism*

$$\iota : (Idl(I), \leq) \longrightarrow (\mathcal{O}, \subseteq), \ J \mapsto \bigcup\{\lambda_{(A,I)}(a) \mid a \in J\}.$$

If $U \in \mathcal{O}$ then

$$J = \iota^{-1}(U) = \{b \in I \mid \lambda_{(A,I)}(b) \subseteq U\},$$

J is isomorphic to the ideal

$$J_U = \{F \in KO(X) \mid F \subseteq U\}$$

of $KO(X)$ $(= \Theta_g^t(X))$ and $J_U = KO(U)$, i.e., $J_U = \Theta_g^t(U)$.

Corollary 3.5.1.3. (M. Stone [108, Theorem 5]) *Let I be a GBA, $(X, \mathcal{O}) = \Theta_s^a(I)$, J be an ideal of I and $U = \iota_s(J)$. Then:*

(a) U is a clopen set $\iff$ J is a simple ideal of I;

(b) U is a regular open set iff J is a normal ideal of I;

(c) U is a compact open set iff J is a principal ideal of I.

If (A, I) is an LBA and $a \in A$ then the ideal

$$I_a = \{b \in I \mid b \leq a\}$$

of I will be called an *A-principal ideal of I*.

Corollary 3.5.1.4. *Let (A, I) be a ZLBA, $(X, \mathcal{O}) = \Theta^a(A, I)$ $(= \Theta_g^a(I))$, J be an ideal of I and $U = \iota(J)$. Then:*

(a) U is a clopen set $\iff$ J is a simple ideal of I $\iff$ J is an A-principal ideal;

(b) U is a regular open set iff J is a normal ideal of I;

(c) U is a compact open set iff J is a principal ideal of I.

Proof. We need only to prove the second assertion in (a). By Proposition 3.2.2.5, we have that $\lambda_{(A,I)}(A) = CO(X)$. Let U be a clopen set. There exists $a \in A$ such that $U = \lambda_{(A,I)}(a)$. Then $J = \iota^{-1}(U) = \{b \in I \mid \lambda_{(A,I)}(b) \subseteq U\} = \{b \in I \mid \lambda_{(A,I)}(b) \subseteq \lambda_{(A,I)}(a)\} = \{b \in I \mid b \leq a\}$, i.e., J is an A-principal ideal. Conversely, let J be an A-principal ideal. Then there exists $a \in A$ such that $J = \{b \in I \mid b \leq a\}$. Since I is dense in A, we get that $a = \bigvee J$. Using again Proposition 3.2.2.5, we get that $\lambda_{(A,I)}(a) = \bigvee_{CO(X)}\{\lambda_{(A,I)}(b) \mid b \in J\} = \bigvee_{RC(X)}\{\lambda_{(A,I)}(b) \mid b \in J\} = \mathrm{cl}_X(\bigcup\{\lambda_{(A,I)}(b) \mid b \in J\}) = \mathrm{cl}_X(U)$. If there exists $x \in \lambda_{(A,I)}(a) \setminus U$ then there exists $b \in I$ such that $x \in \lambda_{(A,I)}(b) \subseteq \lambda_{(A,I)}(a)$ (since $\lambda_{(A,I)}(a)$ is open). Thus $b \leq a$, i.e., $b \in J$, a contradiction. Therefore, $U = \lambda_{(A,I)}(a)$, i.e., U is a clopen set. □

The above results show that if $X \in |\mathbf{BoolSp}|$ and U is an open subset of X then $\iota^{-1}(U)$ (or, equivalently, $\iota_s^{-1}(U)$) is **GenBoolAlg**-isomorphic to $\Theta_g^t(U)$. Then the dual object $\Theta_d^t(U)$ of U can be obtained with the help of the following fact which was proved in Subsection 3.2.2: if $Y \in |\mathbf{BoolSp}|$ and $I = \Theta_g^t(Y)$ then $\Theta_d^t(Y) = (Si(I), e_I(I))$.

3.5.2 The dual objects of the closed and regular closed subsets

Now, for every $X \in |\mathbf{BoolSp}|$, we will find the connections between the dual objects $\Theta_g^t(F)$ of the closed or regular closed subsets F of X and the dual object $\Theta_g^t(X)$ of X. The obtained result for regular closed subsets of X seems to be new even in the compact case.

Theorem 3.5.2.1. *Let $I, J \in |\mathbf{GenBoolAlg}|$, $X = \Theta_g^a(I)$ and $F = \Theta_g^a(J)$. Then:*

(a)(M. Stone [108, Theorem 4(4)]) *F is homeomorphic to a closed subset of X iff there exists a 0-pseudolattice epimorphism $\varphi : I \longrightarrow J$ (i.e., iff J is a quotient of I);*

(b) F is homeomorphic to a regular closed subset of X if and only if there exists a 0-pseudolattice epimorphism $\varphi : I \longrightarrow J$ which preserves all meets that happen to exist in I.

Proof. (a) Let F be homeomorphic to a closed subset of X, i.e there exists a closed embedding $f : F \longrightarrow X$. Then, by Theorem 3.4.3.5, $\varphi' = \Theta_g^t(f) : \Theta_g^t(X) \longrightarrow \Theta_g^t(F)$ is a surjective 0-pseudolattice homomorphism. Thus, by the duality, there exists a surjective 0-pseudolattice homomorphism $\varphi : I \longrightarrow J$.

Conversely, if $\varphi : I \longrightarrow J$ is a surjective 0-pseudolattice homomorphism then, by Theorem 3.4.3.5, F is homeomorphic to a closed subset of X.

(b) Having in mind the assertion (a) above and Theorem 3.3.1.6, it is enough to show that if $f : F \longrightarrow X$ is a closed injection then $f(F) \in RC(X)$ iff f is a quasi-open map. This can be easily shown (using Corollary 2.5.2.8), so that the proof of assertion (b) is complete. $\square$

We will finish with mentioning some assertions about isolated points. All these statements have easy proofs which will be omitted.

Proposition 3.5.2.2. *Let (A, I) be a ZLBA, $X = \Theta^a(A, I)$ and $a \in A$. Then a is an atom of A iff $\lambda_{(A,I)}(a)$ is an isolated point of the space X. Also, for every isolated point x of X there exists an $a \in I$ such that a is an atom of I (equivalently, of A) and $\{x\} = \lambda_{(A,I)}(a)$.*

Proposition 3.5.2.3. *Let (A, I) be a ZLBA and $X = \Theta^a(A, I)(= \Theta^a_g(I))$. Then X is a discrete space $\Longleftrightarrow$ the elements of I are either atoms of I or finite sums of atoms of I.*

Proposition 3.5.2.4. (M. Stone [108]) *Let (A, I) be a ZLBA and $X = \Theta^a(A, I)(= \Theta^a_g(I))$. Then X is an extremally disconnected space iff A is a complete Boolean algebra.*

Proposition 3.5.2.5. *Let (A, I) be a ZLBA and $X = \Theta^a(A, I)(= \Theta^a_g(I))$. Then the set of all isolated points of X is dense in X iff A is an atomic Boolean algebra iff I is an atomic 0-pseudolattice.*

Chapter 4

Some applications in General Topology

4.1 Construction of all locally compact Hausdorff extensions of completely regular T_2-spaces by means of non-symmetric proximities

4.1.1 Introduction

In this section all extensions are assumed to be Hausdorff topological spaces.

In 1952, Ju. M. Smirnov [103] showed with his celebrated Compactification Theorem that the ordered set of all (up to equivalence) compact Hausdorff extensions of a Tychonoff space (X,τ) is isomorphic to the ordered set of all EF-proximities on (X,τ). In 1967, S. Leader [78] described the ordered set of all (up to equivalence) locally compact Hausdorff extensions of a Tychonoff space by means of *separated local proximities* in which both the *boundedness* and the *basic proximity* are primitive terms. In this way he generalized Smirnov Compactification Theorem on the base of a notion which is, so to say, two-sorted. Further on, the locally compact Hausdorff extensions were described also by V. Zaharov (see [124] and [125]) (through some special vector lattices of functions) and by G. Dimov and D. Doitchinov [35] (by using the notion of *supertopological space*). The natural question *whether these extensions have a purely proximity-type description* was posed in [26], where also an affirmative answer was obtained on the base of the notion of *LC-proximity.* However, this answer is not completely satisfactory, because the LC-proximity is *a pair* $\alpha = (\delta, \Sigma)$ of an R-proximity δ and a family Σ of δ-round filters satisfying some conditions. Hence, although everything is expressed only in proximity-type terms, two components are involved in the definition of α. In the present section, we give a completely satisfactory (as we hope) answer to the above question. This is done by introducing the notion of *lc-proximity*, which is some kind of non-symmetric proximity similar to the Pervin proximity. The symmetric lc-proximities coincide with EF-proximities and correspond to the compact extensions. In this way, the Smirnov Compactification Theorem obtains a purely proximity-type generalization. The idea for defining lc-proximities comes from our definition of an *MVD-algebra* (see Definition 1.3.2.1) which, in turn, is based on some ideas from the Mormann's paper [85]. The notions of *lc-map* and *perfect lc-map* are introduced as well. The first is equivalent to the Leader notion of *equicontinuous map* (see 0.5.1.9) and, through the second, a characterization of the surjective maps between Tychonoff spaces having a perfect lifting over arbitrary, but fixed, Hausdorff local compactifications of their domain and range spaces is obtained.

The exposition of this section is based on the paper [40].

4.1.2 lc-proximities and lc-maps

When $(X, \beta, \mathcal{B})$ is a separated local proximity space, Theorem 0.5.2.5 shows that the family $\mathcal{B}$ consists of those subsets of the topological space (X, τ_β) whose closures in the corresponding locally compact Hausdorff extension $Y = L(X, \beta, \mathcal{B})$ are compact; hence the elements of the family $\mathcal{B}$ can be described as the traces on X of those subsets of Y which are far from the point at infinity of Y (in the Alexandroff one-point compactification of Y). In the next definition, which is the main one in the present section, we formalize this observation using the empty set as the point at infinity[2] — the formula (4.3) in the proof of our Theorem 4.1.2.8 will make this clear.

Notation 4.1.2.1. The category of separated local proximity spaces and bounded p-maps (see 0.5.1.9 for the corresponding definitions) will be denoted by $\mathbf{LP}$.

Definition 4.1.2.2. Let X be a set. A binary relation δ on $P(X)$ is called an *lc-proximity* on X if it satisfies the following conditions:
(LCP1) $\emptyset(-\delta)X$ ("$-\delta$" means "not δ");
(LCP2) $\{x\}(-\delta)\emptyset$, for every $x \in X$;
(LCP3) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;
(LCP4) $(A \cup B)\delta C$ iff $A\delta C$ or $B\delta C$;
(LCP5) If $A(-\delta)B$, then there exists a $C \subseteq X$ such that $A(-\delta)C$ and $(X \setminus C)(-\delta)B$;

[2] A similar idea was used in the well-known paper [55] of J. Fell in the construction of a new hyper-topology, now known as *Fell topology*.

(LCP6) If $A(-\delta)\emptyset$ and $A\delta B$ then $B\delta A$;
(LCP7) $\{x\}\delta\{y\}$ iff $x = y$.

The pair (X, δ), where X is a set and δ is an lc-proximity on X, is called an *lc-proximity space.*

Proposition 4.1.2.3. *Let X be a set and δ be a binary relation on $P(X)$. Let us write $A \ll_\delta B$ (or simply $A \ll B$) when $A(-\delta)(X \setminus B)$. Then δ is an lc-proximity on X if and only if it satisfies the following conditions:*
(LCO1) $F \ll G$ implies $F \subseteq G$;
(LCO2) $\emptyset \ll \emptyset$;
(LCO3) $F_1 \subseteq F \ll G \subseteq G_1$ implies $F_1 \ll G_1$;
(LCO4) $F \ll H$ and $G \ll H$ implies $(F \cup G) \ll H$;
(LCO5) $F \ll G$ and $F \ll H$ imply $F \ll (G \cap H)$;
(LCO6) If $F \ll H$ then $F \ll G \ll H$ for some $G \subseteq X$;
(LCO7) $\{x\} \ll X$, for every $x \in X$;
(LCO8) If $x \neq y$ then $\{x\} \ll (X \setminus \{y\})$;
(LCO9) If $F \ll X$ and $(X \setminus G) \ll (X \setminus F)$ then $F \ll G$.

Proof. The proof is straightforward. □

The following fact follows immediately from (LCP3) and (LCP1).

Fact 4.1.2.4. *Let δ be an lc-proximity on the set X. Then $\emptyset(-\delta)A$, for every $A \subseteq X$.*

It is easy to prove the following assertion:

Proposition 4.1.2.5. *Let X be a set and δ be an lc-proximity on X. Then the following conditions are equivalent:*
(a) δ is an EF-proximity;
(b) δ is symmetric (i.e., for any $A, B \subseteq X$, $A\delta B$ iff $B\delta A$);
(c) $A(-\delta)\emptyset$, for every $A \subseteq X$;
(d) $X(-\delta)\emptyset$.

Fact 4.1.2.6. *Every lc-proximity δ on a set X generates a topology τ_δ on X whose closure operator coincides with cl_δ (where cl_δ is defined as in 0.5.1.2).*

Notation 4.1.2.7. *For every Tychonoff space (X, τ), we set*

$$\mathcal{LCP}(X, \tau) = \{(X, \delta) : (X, \delta) \text{ is an lc-proximity space and } \tau_\delta = \tau\}.$$

Theorem 4.1.2.8. *Let (X, τ) be a Tychonoff space. Then there exists a bijection between the sets $\mathcal{LP}(X, \tau)$ (see 0.5.1.10 for this notation) and $\mathcal{LCP}(X, \tau)$.*

Proof. Let

$$\Phi : \mathcal{LP}(X, \tau) \longrightarrow \mathcal{LCP}(X, \tau)$$

be defined by $\Phi(X, \beta, \mathcal{B}) = (X, \delta)$, for every $(X, \beta, \mathcal{B}) \in \mathcal{LP}(X, \tau)$, where, for $A, B \subseteq X$,

(4.1) $A\delta B$ iff $A\beta B$ or $A \notin \mathcal{B}$.

Hence,

(4.2) $A(-\delta)B$ iff $A(-\beta)B$ and $A \in \mathcal{B}$.

It is easy to show that Φ is well defined.

Define now

$$\Psi : \mathcal{LCP}(X, \tau) \longrightarrow \mathcal{LP}(X, \tau)$$

by $\Psi(X, \delta) = (X, \beta, \mathcal{B})$, for every $(X, \delta) \in \mathcal{LCP}(X, \tau)$, where

(4.3) $\mathcal{B} = \{B \subseteq X : B(-\delta)\emptyset\}$

and, for $A, B \subseteq X$,

(4.4) $A\beta B$ iff there exists a $C \in \mathcal{B}$ such that $(C \cap A)\delta(C \cap B)$.

Hence,

(4.5) $A(-\beta)B$ iff for every $C \in \mathcal{B}$ we have that $(C \cap A)(-\delta)(C \cap B)$.

Observe that

(4.6) $A(-\delta)B$ implies that $A(-\beta)B$.

Let's show that Ψ is is well defined.

We first prove that $\tau_\beta = \tau_\delta$ (and hence $\tau_\beta = \tau$). Let $A \subseteq X$. We have to show that $\mathrm{cl}_\beta(A) = \mathrm{cl}_\delta(A)$. Let $x\beta A$. Then (4.6) implies that $x\delta A$. Hence, $\mathrm{cl}_\beta(A) \subseteq \mathrm{cl}_\delta(A)$. Conversely, let $x\delta A$. Suppose that $x(-\beta)A$. Then, for every $C \in \mathcal{B}$ such that $x \in C$, we will have that $x(-\delta)(C \cap A)$. By (LCP2), $x(-\delta)\emptyset$. Now, (LCP5) implies that there exists an $H \subseteq X$ such that $x(-\delta)H$ and $(X \setminus H)(-\delta)\emptyset$. Put $B = X \setminus H$. Then $B \in \mathcal{B}$, $x(-\delta)(X \setminus B)$ and hence $x \in B$ (by (LCP7) and (LCP3)). Therefore, by (4.5), $x(-\delta)(A \cap B)$ (since $x(-\beta)A$ and $x \in B \in \mathcal{B}$). Further, from $x(-\delta)(X \setminus B)$, we obtain, using (LCP3), that $x(-\delta)(A\setminus B)$. Applying (LCP3) once more, we get $x(-\delta)A$ (because $x(-\delta)(A\cap B)$), which is a contradiction. Hence $x\beta A$. So, $\mathrm{cl}_\beta(A)\supseteq \mathrm{cl}_\delta(A)$. Therefore, $\mathrm{cl}_\beta(A)=\mathrm{cl}_\delta(A)$.

Further on, it is easy to prove that $\mathcal{B}$ is a boundedness in X and β is a separated basic proximity on X. Let's now show that the axiom (LP1) from 0.5.1.9 is fulfilled.

Let $A\in\mathcal{B}$, $B\subseteq X$ and $A\ll_\beta B$ (i.e., $A(-\beta)(X\setminus B)$). We have that $A(-\delta)\emptyset$. Hence, by (LCP5), there exists an $A'\subseteq X$ such that $A(-\delta)A'$ and $(X\setminus A')(-\delta)\emptyset$. Put $B'=X\setminus A'$. Then $B'\in\mathcal{B}$ and $A(-\delta)(X\setminus B')$. From the last expression and (LCP7), (LCP3), (LCP4), we get that $A\subseteq B'$, i.e., $A\cap B'=A$. Since $A(-\beta)(X\setminus B)$ and $B'\in\mathcal{B}$, we obtain, by (4.5), that $(A\cap B')(-\delta)(B'\cap(X\setminus B))$. Hence $A(-\delta)(B'\setminus B)$. Then from $A(-\delta)(X\setminus B')$ and (LCP3), we get that $A(-\delta)((B'\setminus B))\cup(X\setminus B'))$, i.e., $A(-\delta)(X\setminus(B\cap B'))$. Therefore, applying (LCP5), we can find an $H\subseteq X$ such that $A(-\delta)H$ and $(X\setminus H)(-\delta)(X\setminus(B\cap B'))$. Hence $X\setminus H\in\mathcal{B}$ and $(X\setminus H)(-\delta)((X\setminus B)\cup(X\setminus B'))$. Using (LCP3), we get that $(X\setminus H)(-\delta)(X\setminus B)$. Then, by (4.6), $(X\setminus H)(-\beta)(X\setminus B)$ and $A(-\beta)H$. Put $C=X\setminus H$. Then $C\in\mathcal{B}$, $A(-\beta)(X\setminus C)$ and $C(-\beta)(X\setminus B)$, i.e., $A\ll_\beta C\ll_\beta B$ and $C\in\mathcal{B}$. So, the axiom (LP1) from 0.5.1.9 is fulfilled.

It is easy to see that the axiom (LP2) from 0.5.1.9 is also fulfilled. Hence,

$$(X,\beta,\mathcal{B})\in\mathcal{LP}(X,\tau).$$

Therefore, Ψ is is well defined.

We will show now that

(4.7) $\quad \Psi\circ\Phi=id_{\mathcal{LP}(X,\tau)}$.

Let $(X,\beta,\mathcal{B})\in\mathcal{LP}(X,\tau)$. Then $\Phi(X,\beta,\mathcal{B})=(X,\delta)$, where δ is defined by the formula (4.1). Further, $\Psi(X,\delta)=(X,\beta',\mathcal{B}')$, where $\mathcal{B}'$ and β' are defined as in (4.3) and (4.4) (adding only primes to $\mathcal{B}$ and β in the formulas). It is easy to see that $\mathcal{B}=\mathcal{B}'$. So, it rests to prove that $\beta'=\beta$. Let $A,B\subseteq X$ and $A\beta B$. Then, by (LP2) from 0.5.1.9, there exist $A_1,B_1\in\mathcal{B}$ such that $A_1\subseteq A$, $B_1\subseteq B$ and $A_1\beta B_1$. Put $C=A_1\cup B_1$. Then $C\in\mathcal{B}$ and $(A\cap C)\beta(B\cap C)$. Hence, by (4.1), $(A\cap C)\delta(B\cap C)$. Since $\mathcal{B}=\mathcal{B}'$, we obtain that $C\in\mathcal{B}'$. Therefore, by (4.4), $A\beta' B$. Conversely, let $A\beta' B$. Then there exists a $C\in\mathcal{B}'$ such that $(A\cap C)\delta(B\cap C)$. Since $\mathcal{B}'=\mathcal{B}$, we obtain that $A\cap C\in\mathcal{B}$. Hence, by (4.1), $A\beta B$. Therefore, $\beta=\beta'$. So, we have proved that $\Psi\circ\Phi=id_{\mathcal{LP}(X,\tau)}$.

Finally, we will show that

(4.8) $\Phi \circ \Psi = id_{\mathcal{LCP}(X,\tau)}$.

Let $(X,\delta) \in \mathcal{LCP}(X,\tau)$. Then $\Psi(X,\delta) = (X,\beta,\mathcal{B})$, where $\mathcal{B}$ and β are defined by the formulas (4.3) and (4.4). Further, $\Phi(X,\beta,\mathcal{B}) = (X,\delta')$, where δ' is defined as in (4.1) (adding only primes to δ in the formula). We have to prove that $\delta' = \delta$. Let $A, B \subseteq X$ and $A\delta B$. Suppose that $A(-\delta')B$. Then, by (4.2), $A(-\beta)B$ and $A \in \mathcal{B}$. Hence, by (4.3), $A(-\delta)\emptyset$. Then (LCP5) implies that there exists an $H \subseteq X$ such that $A(-\delta)H$ and $(X \setminus H)(-\delta)\emptyset$. Put $C = X \setminus H$. Then $C(-\delta)\emptyset$ and $A(-\delta)(X \setminus C)$. Hence $A \subseteq C$ and $C \in \mathcal{B}$. Since $A(-\beta)B$, we obtain, by the formula (4.5), that $A(-\delta)(B \cap C)$. Together with the fact that $A(-\delta)(X \setminus C)$, we get, using (LCP3), that $A(-\delta)B$, which is a contradiction. Therefore, $A\delta B$ implies that $A\delta' B$. Conversely, let $A\delta' B$. Then, by (4.1), $A\beta B$ or $A \notin \mathcal{B}$ (i.e., by (4.3), $A\delta\emptyset$). If $A\delta\emptyset$ then (LCP3) implies that $A\delta B$, what was our aim. If $A\beta B$ then, by (4.4), there exists a $C \in \mathcal{B}$ such that $(A \cap C)\delta(B \cap C)$. Hense, using (LCP3) and (LCP4), we get that $A\delta B$. So, $A\delta' B$ implies $A\delta B$. We have proved that $\delta = \delta'$. So, $\Phi \circ \Psi = id_{\mathcal{LCP}(X,\tau)}$. □

Definition 4.1.2.9. Let (X,δ) and (Y,δ') be two lc-proximity spaces. A function $f : X \longrightarrow Y$ will be called an *lc-map* if the following two conditions are fulfilled:

(lcm1) if $A \subseteq X$ and $A(-\delta)\emptyset$ then $f(A)(-\delta')\emptyset$;

(lcm2) for every $A, B \subseteq X$, if $A(-\delta)\emptyset$ and $A\delta B$ then $f(A)\delta' f(B)$.

4.1.2.10. It is easy to see that the identity map on an lc-proximity space is always an lc-map and the composition of two lc-maps is an lc-map. Hence a category of all lc-proximity spaces and all lc-maps can be defined. It will be denoted by **LCP**.

The following assertion is obvious:

Fact 4.1.2.11. *If f is an lc-map between lc-proximity spaces (X,δ) and (Y,δ') then f is a continuous map between (X,τ_δ) and $(Y,\tau_{\delta'})$.*

Theorem 4.1.2.12. *The categories* **LP** *and* **LCP** *are isomorphic (see 4.1.2.1 and 4.1.2.10 for the notation).*

Proof. Let

$$F : \mathbf{LP} \longrightarrow \mathbf{LCP}$$

be defined as follows: on the objects of the category **LP** it coincides with the map Φ, introduced in the proof of Theorem 4.1.2.8, and

$$F(f) = f : F(X, \beta, \mathcal{B}) \longrightarrow F(Y, \beta', \mathcal{B}')$$

on the morphisms $f \in \mathbf{LP}((X, \beta, \mathcal{B}), (Y, \beta', \mathcal{B}'))$ (i.e., f and $F(f)$ coincide as maps between the sets X and Y). It is easy to show that F is is well defined and that F is a functor.

Let

$$G : \mathbf{LCP} \longrightarrow \mathbf{LP}$$

be defined as follows: on the objects of the category **LCP** it coincides with the map Ψ, introduced in the proof of Theorem 4.1.2.8, and on the morphisms $f \in \mathbf{LCP}((X, \delta), (Y, \delta'))$,

$$G(f) = f : G(X, \delta) \longrightarrow G(Y, \delta')$$

(i.e., f and $G(f)$ coincide as maps between the sets X and Y). It is easy to show that G is is well defined and that G is a functor.

Finally, from the definitions of F and G, we obtain, using (4.7) and (4.8), that $F \circ G = Id_{\mathbf{LCP}}$ and $G \circ F = Id_{\mathbf{LP}}$. Therefore, F is an isomorphism between the categoies **LP** and **LCP**. □

Definition 4.1.2.13. Let δ and δ_1 be two lc-proximities on the set X. We will write $\delta \leq \delta_1$ iff the identity map $id : (X, \delta_1) \longrightarrow (X, \delta)$ is an lc-map.
It is easy to see that this relation is an order on the set of all lc-proximities on X.

Combining Theorem 4.1.2.12 with Leader's Theorem 0.5.2.5, we obtain our main result:

Theorem 4.1.2.14. *Let (X, τ) be a Tychonoff space. Then there exists an isomorphism $\Gamma_{(X,\tau)}$ between the ordered sets $(\mathcal{LC}(X, \tau), \leq)$ and $(\mathcal{LCP}(X, \tau), \leq)$ (see 0.4.1.2 and 4.1.2.7 for the notation), i.e.,*

$$\Gamma_{(X,\tau)} : (\mathcal{LC}(X, \tau), \leq) \longrightarrow (\mathcal{LCP}(X, \tau), \leq).$$

Namely, if (Y, π) is a locally compact Hausdorff extension of X then

$$\Gamma_{(X,\tau)}([(Y, \pi)]) = (X, \delta_\pi)$$

where, for $A, B \subseteq X$,

$$A\delta_\pi B \iff [\mathrm{cl}_Y(\pi(A)) \cap \mathrm{cl}_Y(\pi(B)) \neq \emptyset \text{ or } \mathrm{cl}_Y(\pi(A)) \text{ is not a compact subset of } Y].$$

Put

$$\Lambda_{(X,\tau)} = \Gamma^{-1}_{(X,\tau)}.$$

Then, for any lc-proximity δ on (X, τ), $\Lambda_{(X,\tau)}(\delta)$ is an equivalence class of Hausdorff local compactifications of X; denoting by $(\lambda(X, \delta), \pi_\delta)$ an element of this class, we get that

$$\Lambda_{(X,\tau)}(\delta) = [(\lambda(X, \delta), \pi_\delta)].$$

If (X, δ) and (X', δ') are two lc-proximity spaces and $f : X \longrightarrow X'$ is a function, then we have: $f : (X, \delta) \longrightarrow (X', \delta')$ is an lc-map iff there exists a (unique) continuous map

$$\Lambda(f) : \lambda(X, \delta) \longrightarrow \lambda(X', \delta')$$

such that $\pi_{\delta'} \circ f = \Lambda(f) \circ \pi_\delta$.

Proof. Put $\Gamma_{(X,\tau)} = \Phi \circ \gamma_{(X,\tau)}$ (see Theorem 0.5.2.5 and the proof of Theorem 4.1.2.8 for the notation). Now all follows from Theorem 0.5.2.5 and Theorem 4.1.2.12.

Note that we can set $\lambda(X, \delta) = L(\Psi(X, \delta))$ and $\pi_\delta = l_\beta$, where $\Psi(X, \delta) = (X, \beta, \mathcal{B})$ (see Theorem 0.5.2.5 and the proof of Theorem 4.1.2.8 for the notation). □

4.1.3 Perfect lc-maps

In Theorem 4.1.2.12 we have seen that lc-maps correspond precisely to bounded p-maps. However, the condition (lcm2) in their definition (see 4.1.2.9) looks somehow strange. We will investigate now those maps whose definition is "more natural".

Definition 4.1.3.1. Let (X, δ) and (Y, δ') be two lc-proximity spaces. A function $f : X \longrightarrow Y$ will be called a *perfect lc-map* if the following two conditions are fulfilled:

(lcm1) if $A \subseteq X$ and $A(-\delta)\emptyset$ then $f(A)(-\delta')\emptyset$;

(plcm2) for every $A, B \subseteq X$, $A\delta B$ implies $f(A)\delta' f(B)$.

Proposition 4.1.3.2. *Let (X, δ) and (Y, δ') be two lc-proximity spaces and $f : X \longrightarrow Y$ be a function between the sets X and Y. Then the following conditions are equivalent:*

(1) f is a perfect lc-map;

(2) f is an lc-map such that if $C \subseteq Y$ and $C(-\delta')\emptyset$ then $f^{-1}(C)(-\delta)\emptyset$;

(3) f satisfies the condition (lcm1) from Definition 4.1.3.1 and the following one:

(plcm2') *for every $C, D \subseteq Y$, $C(-\delta')D$ implies $f^{-1}(C)(-\delta)f^{-1}(D)$.*

Proof. (1) $\Rightarrow$ (2). Let $f : (X, \delta) \longrightarrow (Y, \delta')$ be a perfect lc-map. Then, obviously, it is an lc-map. Let $C \subseteq Y$ and $C(-\delta')\emptyset$. We will prove that $f^{-1}(C)(-\delta)\emptyset$. Indeed, suppose that $f^{-1}(C)\delta\emptyset$. Then, by (plcm2), $f(f^{-1}(C))\delta'\emptyset$. Since $C \supseteq f(f^{-1}(C))$, we obtain that $C\delta'\emptyset$, which is a contradiction. Hence, $f^{-1}(C)(-\delta)\emptyset$.

(2) $\Rightarrow$ (3). Let $f : (X, \delta) \longrightarrow (Y, \delta')$ be an lc-map such that if $C \subseteq Y$ and $C(-\delta')\emptyset$ then $f^{-1}(C)(-\delta)\emptyset$. Let $C, D \subseteq Y$ and $C(-\delta')D$. We shall prove that $f^{-1}(C)(-\delta)f^{-1}(D)$. Indeed, suppose that $f^{-1}(C)\delta f^{-1}(D)$. By (LCP3), $C(-\delta')D$ implies that $C(-\delta')\emptyset$. Hence $f^{-1}(C)(-\delta)\emptyset$. From this we get, according to our assumption, that $f(f^{-1}(C))\delta' f(f^{-1}(D))$ (because f is an lc-map). Now, from the obvious inclusions $f(f^{-1}(C)) \subseteq C$ and $f(f^{-1}(D)) \subseteq D$, we get that $C\delta' D$, which is a contradiction. Hence $f^{-1}(C)(-\delta)f^{-1}(D)$.

(3) $\Rightarrow$ (1). Let $f : (X, \delta) \longrightarrow (Y, \delta')$ satisfies the conditions (lcm1) and (plcm2'). We shall show that f satisfies also the condition (plcm2). Indeed, let $A, B \subseteq X$ and $A\delta B$. Suppose that $f(A)(-\delta')f(B)$. Then, by (plcm2'), we obtain that

$$f^{-1}(f(A))(-\delta)f^{-1}(f(B)).$$

Since $A \subseteq f^{-1}(f(A))$ and $B \subseteq f^{-1}(f(B))$, we get that $A(-\delta)B$, which is a contradiction. Hence $f(A)\delta' f(B)$. Therefore, f is a perfect lc-map. □

Lemma 4.1.3.3. *Let (X, δ) and (Y, δ') be two lc-proximity spaces, $f : (X, \delta) \longrightarrow (Y, \delta')$ be an lc-map and $\Lambda(f) : \lambda(X, \delta) \longrightarrow \lambda(Y, \delta')$ be the unique continuous map such that $\pi_{\delta'} \circ f = \Lambda(f) \circ \pi_\delta$ (see Theorem 4.1.2.14 for the notation and the existence of $\Lambda(f)$). If $\Lambda(f)$ is a perfect map then f is a perfect lc-map.*

Proof. Since f is an lc-map, we need only to show (by Proposition 4.1.3.2(2)) that if $C \subseteq Y$ and $C(-\delta')\emptyset$ then $f^{-1}(C)(-\delta)\emptyset$. So, let $C \subseteq Y$ and $C(-\delta')\emptyset$. Then, by Theorem 4.1.2.14, $\mathrm{cl}_{\lambda(Y,\delta')}(\pi_{\delta'}(C))$ is a compact subset of $\lambda(Y, \delta')$. Since $\Lambda(f)$ is a perfect map, we get that $(\Lambda(f))^{-1}(\mathrm{cl}_{\lambda(Y,\delta')}(\pi_{\delta'}(C)))$ is a compact subset of $\lambda(X, \delta)$. From the continuity of $\Lambda(f)$ and the equality $\pi_{\delta'} \circ f = \Lambda(f) \circ \pi_\delta$, we obtain that

$$(\Lambda(f))^{-1}(\mathrm{cl}_{\lambda(Y,\delta)}(\pi_{\delta'}(C))) \supseteq \mathrm{cl}_{\lambda(X,\delta)}((\Lambda(f))^{-1}(\pi_{\delta'}(C))) \supseteq \mathrm{cl}_{\lambda(X,\delta)}(\pi_\delta(f^{-1}(C))).$$

Hence, $\mathrm{cl}_{\lambda(X,\delta)}(\pi_\delta(f^{-1}(C)))$ is a compact subset of $\lambda(X, \delta)$. This means, by Theorem 4.1.2.14, that $f^{-1}(C)(-\delta)\emptyset$. Therefore, f is a perfect lc-map. □

Theorem 4.1.3.4. *Let (X,δ) and (Y,δ') be two lc-proximity spaces and $f:(X,\delta)\longrightarrow(Y,\delta')$ be a surjective lc-map. Then the map $\Lambda(f):\lambda(X,\delta)\longrightarrow\lambda(Y,\delta')$ (see Theorem 4.1.2.14 for the notation and for the existence of $\Lambda(f)$) is a perfect map if and only if f is a perfect lc-map.*

Proof. ($\Rightarrow$) If $\Lambda(f)$ is a perfect map then, by Lemma 4.1.3.3, f is a perfect lc-map.

($\Leftarrow$) Let f be a perfect lc-map. We shall prove that $\Lambda(f):\lambda(X,\delta)\longrightarrow\lambda(Y,\delta')$ is a perfect map.

If $X(-\delta)\emptyset$ then $Y(-\delta')\emptyset$ (since $f(X)=Y$ and f is an lc-map) and both δ and δ' are EF-proximities (see 4.1.2.5). Then $\Lambda(f)$ is a continuous map between compact Hausdorff spaces and hence it is a perfect map.

Let now $X\delta\emptyset$. Then $Y\delta'\emptyset$, because f is a perfect lc-map. Let $\Psi(X,\delta)=(X,\beta,\mathcal{B})$ and $\Psi(Y,\delta')=(Y,\beta',\mathcal{B}')$ (see the proof of Theorem 4.1.2.8 for the notation). Then $(X,\beta,\mathcal{B})$ and $(Y,\beta',\mathcal{B}')$ are separated local proximity spaces on (X,τ_δ) and $(Y,\tau_{\delta'})$, respectively. Let α and α' be the Alexandroff extensions of β and β', respectively (see 0.5.1.11). We will show that $f:(X,\alpha)\longrightarrow(Y,\alpha')$ is a proximity mapping. Let $A,B\subseteq X$ and $A\alpha B$. Then $A\beta B$ or $A,B\notin\mathcal{B}$. Suppose that $A\beta B$. Since f is an lc-map, we get from (the proof of) Theorem 4.1.2.12, that f is a bounded p-map and hence we obtain that $f(A)\beta' f(B)$. This means that $f(A)\alpha' f(B)$. So, let us regard the case when $A,B\notin\mathcal{B}$. Then, by (4.3) (see the proof of Theorem 4.1.2.8), $A\delta\emptyset$ and $B\delta\emptyset$. Since f is a perfect lc-map, we get (from (plcm2) of 4.1.3.1) that $f(A)\delta'\emptyset$ and $f(B)\delta'\emptyset$. Hence, by (4.3), $f(A),f(B)\notin\mathcal{B}'$. This implies that $f(A)\alpha' f(B)$. Therefore f is a surjective proximity mapping between the Efremovič proximity spaces (X,α) and (Y,α'). Then, by 0.5.2.4, there exists a continuous surjective mapping $S_mf:S_m(X,\alpha)\longrightarrow S_m(Y,\alpha')$ such that $S_mf\circ s_\alpha=s_{\alpha'}\circ f$. It is defined by the formula $S_mf(\sigma)=\{C\subseteq Y : C\alpha' f(A)$ for every $A\in\sigma\}$, for every $\sigma\in S_m(X,\alpha)$ (i.e., for every cluster σ in (X,α)). We will show that $(S_mf)^{-1}(\sigma_{\alpha'})=\{\sigma_\alpha\}$ (see 0.5.1.11 for the notation). Indeed, let $A\in\sigma_\alpha$. Then $A\notin\mathcal{B}$, i.e., $A\delta\emptyset$. Hence $f(A)\delta'\emptyset$. This means that $f(A)\notin\mathcal{B}'$. Let now $C\in\sigma_{\alpha'}$, i.e., $C\notin\mathcal{B}'$. Then $C\alpha' f(A)$, for every $A\in\sigma_\alpha$. Therefore, $\sigma_{\alpha'}\subseteq S_mf(\sigma_\alpha)$. Since $\sigma_{\alpha'}$ and $S_mf(\sigma_\alpha)$ are both clusters in (Y,α'), we obtain that $\sigma_{\alpha'}=S_mf(\sigma_\alpha)$. Further, we will show that σ_α is the unique cluster in (X,α) whose image by S_mf is $\sigma_{\alpha'}$. Indeed, let $\sigma\in S_m(X,\alpha)\setminus\{\sigma_\alpha\}$. Then σ is a bounded cluster, i.e., there exists an $A_0\in\mathcal{B}$ such that $A_0\in\sigma$. We have to prove that $S_mf(\sigma)$ is a bounded cluster in (Y,α'). For doing this it is enough to show that $f(A_0)\in S_mf(\sigma)$. Let $A\in\sigma$. Then $A_0\in\mathcal{B}$ and $A_0\alpha A$. Hence $A_0\beta A$. Since f is a bounded p-map, we obtain that $f(A_0)\beta' f(A)$.

Therefore, $f(A_0)\alpha' f(A)$ for every $A \in \sigma$. This means that $f(A_0) \in S_m f(\sigma)$. Hence, $S_m f(\sigma) \neq \sigma_{\alpha'}$. So, we have proved that $(S_m f)^{-1}(\sigma_{\alpha'}) = \{\sigma_\alpha\}$. This implies that $(S_m f)^{-1}(S_m(Y, \alpha') \setminus \{\sigma_{\alpha'}\}) = S_m(X, \alpha) \setminus \{\sigma_\alpha\}$. Now, according to Theorem 0.5.2.5, we obtain that $(S_m f)^{-1}(L(Y, \beta', \mathcal{B}')) = L(X, \beta, \mathcal{B})$. Since $S_m f$ is a perfect map, we get that its restriction to $L(X, \beta, \mathcal{B})$ is also a perfect map. This restriction, however, coincides with $\Lambda(f)$ (note that $L(X, \beta, \mathcal{B}) = \lambda(X, \delta)$ and $\pi_\delta = l_\beta$ (see the proof of Theorem 4.1.2.14)). Hence, we have proved that $\Lambda(f)$ is a perfect map. □

4.2 Open and other kinds of map extensions over local compactifications

4.2.1 Introduction

In 1959, V. I. Ponomarev [90] proved that if $f : X \longrightarrow Y$ is a perfect open surjection between two normal Hausdorff spaces X and Y then its extension $\beta f : \beta X \longrightarrow \beta Y$ over Stone-Čech compactifications of these spaces is an open map; also, he obtained a more general variant of this theorem, which concerns multi-valued mappings. V. I. Ponomarev posed the following problem: characterize those continuous maps $f : X \longrightarrow Y$ between two Tychonoff spaces for which the map βf is open. In 1960, A. D. Taĭmanov [6] improved Ponomarev's theorem cited above by replacing "perfect" with "closed" (and A. V. Arhangel'skiĭ [6] generalized Taĭmanov's result for multi-valued mappings). Later on, V. Z. Poljakov [89] described the maps between two Tychonoff spaces X and Y which have an *open* extension over arbitrary, but fixed, Hausdorff compactifications (cX, c_X) and (cY, c_Y) of X and Y respectively. His work is based on the famous Smirnov Compactification Theorem (see 0.5.2.3 here); with the help of this theorem, Ju. M. Smirnov [103] described the maps between two Tychonoff spaces which can be extended *continuously* over arbitrary, but fixed, compactifications of these spaces (see Theorem 0.5.2.4 here). Let us also recall that S. Leader [78] generalized Smirnov Compactification Theorem and characterized the maps having a continuous lifting over arbitrary, but fixed, Hausdorff local compactifications (= locally compact extensions) of their domain and range spaces (see Theorem 0.5.2.5 here).

In this section we generalize Poljakov's and Leader's theorems and obtain some other results of this type. We regard the following kinds of map extensions over Hausdorff local compactifications: open, quasi-open, perfect, skeletal, injective, surjective. We characterize the functions between Tychonoff spaces which have extensions of the kinds listed above over arbitrary, but fixed, local compactifications (see Theorem 4.2.3.7). The characterizations of all these maps are obtained here with the help of a strengthening of the Leader Local Compactification Theorem 0.5.2.5 (see Theorems 4.2.2.2 and 4.2.3.1 below). We give a de Vries-type formulation of the Leader Theorem (i.e., we describe axiomatically the restrictions of Leader's *local proximities* on the Boolean algebra $RC(X)$ of all regular closed subsets of a Tychonoff space X) and prove this new assertion independently of the Leader Theorem using only our generalization (see Theorem 2.2.2.12) of de Vries Duality Theorem [24]. This permits us to use our general results obtained in Chapter 2. Finally, based on our variant of Leader's Theorem, we characterize in the language of *local contact algebras* only (i.e., without mentioning the points of the space) the poset $(\mathcal{LC}(X), \leq)$ of all, up to equivalence, Hausdorff local compactifications of X, where X is a locally compact Hausdorff space (see Theorem 4.2.2.12); the algebras which correspond to the Alexandroff (one-point) compactification and to the Stone-Čech compactification of a locally compact Hausdorff space are described explicitly (see Theorem 4.2.2.13). Let us also mention that in the previous section we characterized, using the language of non-symmetric proximities, the surjective continuous maps which have a perfect extension over arbitrary, but fixed, Hausdorff local compactifications.

The exposition of this section is based on the paper [32].

4.2.2 A de Vries-type revision of the Leader Local Compactification Theorem

In this subsection we will obtain a strengthening of the Leader Local Compactification Theorem 0.5.2.5; it is similar to de Vries' ([24]) strengthening of the Smirnov Compactification Theorem 0.5.2.3.

Definition 4.2.2.1. Let (X, τ) be a Tychonoff space. An LCA $(RC(X, \tau), \rho, \mathbb{B})$ is said to be *admissible for* (X, τ) if it satisfies the following conditions:

(A1) if $F, G \in RC(X)$ and $F \cap G \neq \emptyset$ then $F \rho G$;

(A2) if $F \in RC(X)$ and $x \in \mathrm{int}_X(F)$ then there exists $G \in \mathbb{B}$ such that $x \in \mathrm{int}_X(G)$ and $G \ll_\rho F$.

The set of all LCAs $(RC(X, \tau), \rho, \mathbb{B})$ which are admissible for (X, τ) will be denoted by

$$\mathcal{L}_{ad}(X, \tau)$$

(or simply by $\mathcal{L}_{ad}(X)$). If $(RC(X), \rho_i, \mathbb{B}_i) \in \mathcal{L}_{ad}(X)$, where $i = 1, 2$, then we set

$$(RC(X), \rho_1, \mathbb{B}_1) \preceq_{ad} (RC(X), \rho_2, \mathbb{B}_2) \iff \rho_2 \subseteq \rho_1 \text{ and } \mathbb{B}_2 \subseteq \mathbb{B}_1.$$

Obviously, $(\mathcal{L}_{ad}(X, \tau), \preceq_{ad})$ is a poset.

Theorem 4.2.2.2. *Let (X, τ) be a Tychonoff space. Then the posets $(\mathcal{LC}(X, \tau), \leq)$ and $(\mathcal{L}_{ad}(X, \tau), \preceq_{ad})$ are isomorphic.*

Proof. Let (Y, f) be a locally compact Hausdorff extensions of X. Set

(4.9) $\mathbb{B}_{(Y,f)} = f^{-1}(CR(Y))$ and let $F\eta_{(Y,f)}G \iff \mathrm{cl}_Y(f(F)) \cap \mathrm{cl}_Y(f(G)) \neq \emptyset$,

for every $F, G \in RC(X)$. Note that, by 0.4.2.2,

$$\mathbb{B}_{(Y,f)} = \{F \in RC(X) \mid \mathrm{cl}_Y(f(F)) \text{ is compact}\}.$$

Hence $\mathbb{B}_{(Y,f)} \subseteq RC(X)$. We will show that $(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}) \in \mathcal{L}_{ad}(X)$. We have, by 0.4.2.2, that the map

(4.10) $r_{(Y,f)} : (RC(Y), \rho_Y, CR(Y)) \longrightarrow (RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}),\ G \mapsto f^{-1}(G)$,

is a Boolean isomorphism and, for every $F, G \in RC(Y)$, the following is fulfilled:

$$F\rho_Y G \iff r_{(Y,f)}(F)\eta_{(Y,f)}r_{(Y,f)}(G),$$

and

$$F \in CR(Y) \iff r_{(Y,f)}(F) \in \mathbb{B}_{(Y,f)}.$$

Hence $(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)})$ is an LCA and $r_{(Y,f)}$ is an LCA-isomorphism. Clearly, condition (A1) is fulfilled. Let now $F \in RC(X)$. Set $U = \mathrm{int}_X(F)$ and let $x \in U$. There exists an open subset V of Y such that $V \cap f(X) = f(U)$. Since Y is a locally compact Hausdorff space, there exists an $H \in CR(Y)$ with $f(x) \in \mathrm{int}_Y(H) \subseteq H \subseteq V$. Let $G = f^{-1}(H)$. Then $G \in \mathbb{B}_{(Y,f)}$ and, obviously, $x \in \mathrm{int}_X(G)$ and $G \ll_{\eta_{(Y,f)}} F$. So, condition (A2) is also checked. Hence $(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}) \in \mathcal{L}_{ad}(X)$. It is clear that if (Y_1, f_1) is a locally compact Hausdorff extensions of X equivalent to the extension (Y, f), then $(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}) = (RC(X), \eta_{(Y_1,f_1)}, \mathbb{B}_{(Y_1,f_1)})$. Therefore, a map

(4.11) $\alpha_X : \mathcal{LC}(X) \longrightarrow \mathcal{L}_{ad}(X),\ [(Y, f)] \mapsto (RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)})$,

is well-defined (see 0.4.1.1 and 0.4.1.2 for the notation).

Set, for short, $A = RC(X)$. Let $(A, \rho, \mathbb{B}) \in \mathcal{L}_{ad}(X)$ and $Y = \Lambda^a(A, \rho, \mathbb{B})$. Then, by Roeper's Theorem 1.2.3.10, Y is a locally compact Hausdorff space. Let us show that for every $x \in X$, we have that $\sigma_x \in Y$ (where $\sigma_x = \{F \in A \mid x \in F\}$). By Fact 1.2.2.7, ν_x is a filter in the Boolean algebra A. Hence there exists an ultrafilter u in A such that $\nu_x \subseteq u$. It is easy to see that $u \subseteq \sigma_x$. Let $\sigma = \{F \in A \mid FC_\rho u\}$ (i.e., $\sigma = \sigma_u$). Since, by (A2), $\nu_x \cap \mathbb{B} \neq \emptyset$, we get that $\sigma \in Y$. We will show that $\sigma_x = \sigma$. Indeed, let $F \in \sigma_x$ and $G \in u$. Then $x \in F \cap G$. Thus, by (A1), $F\rho G$. This implies that $FC_\rho u$, i.e., that $F \in \sigma$. So, $\sigma_x \subseteq \sigma$. Now, suppose that there exists $F \in \sigma$ such that $x \notin F$. Then $x \in X \setminus F = \mathrm{int}_X(F^*)$. Thus, by (A2), there exists $G \in \mathbb{B}$ such that $x \in \mathrm{int}_X(G)$ and $G \ll_\rho F^*$. Therefore $G \in \nu_x$ and $G(-\rho)F$. Since $G \in \mathbb{B}$, we get that $F(-C_\rho)G$, a contradiction. So, we have proved that $\sigma_x = \sigma$ and, thus, $\sigma_x \in Y$ for every $x \in X$. Define

(4.12) $f_{(\rho,\mathbb{B})} : X \longrightarrow Y,\ x \mapsto \sigma_x.$

Set, for short, $f = f_{(\rho,\mathbb{B})}$. Then $\mathrm{cl}_Y(f(X)) = Y$. Indeed, for every $F \in \mathbb{B} \setminus \{\emptyset\}$ and for every $x \in \mathrm{int}_X(F)$, we have that $\sigma_x \in f(X) \cap \mathrm{int}_Y(\lambda_A^g(F))$. Hence $\mathrm{cl}_Y(f(X)) = Y$. We will now show that f is a homeomorphic embedding. It is clear that f is an injection. Further, let $x \in X$, $F \in \mathbb{B}$ and $\sigma_x \in \mathrm{int}_Y(\lambda_A^g(F))$. Since $\mathrm{int}_Y(\lambda_A^g(F)) = Y \setminus \lambda_A^g(F^*)$, we get that $\sigma_x \notin \lambda_A^g(F^*)$. Thus $F^* \notin \sigma_x$. This implies that $x \notin F^*$, i.e., $x \in X \setminus F^* = \mathrm{int}_X(F)$. Moreover, $f(\mathrm{int}_X(F)) \subseteq \mathrm{int}_Y(\lambda_A^g(F))$. Indeed, if $y \in \mathrm{int}_X(F)$ then $y \notin F^*$; thus $F^* \notin \sigma_y$, i.e., $\sigma_y \notin \lambda_A^g(F^*)$; this implies that $\sigma_y \in \mathrm{int}_Y(\lambda_A^g(F))$. All this shows that f is a continuous function. Set $g = (f_{\restriction X})^{-1}$, where $f_{\restriction X} : X \longrightarrow f(X)$ is the restriction of f. We will prove that g is a continuous function. Let $x \in X$, $F \in A$ and $x \in \mathrm{int}_X(F)$. We have that $x = g(\sigma_x)$ and $\sigma_x \in f(X) \cap \mathrm{int}_Y(\lambda_A^g(F))$. Let $\sigma_y \in \mathrm{int}_Y(\lambda_A^g(F))$. Then $\sigma_y \in Y \setminus \lambda_A^g(F^*)$, i.e., $y \notin F^*$; thus $y \in X \setminus F^* = \mathrm{int}_X(F)$. Therefore, $g(f(X) \cap \mathrm{int}_Y(\lambda_A^g(F))) \subseteq \mathrm{int}_X(F)$. So, g is a continuous function. All this shows that (Y, f) is a locally compact Hausdorff extension of X. We now set:

(4.13) $\beta_X : \mathcal{L}_{ad}(X) \longrightarrow \mathcal{LC}(X),\ (RC(X), \rho, \mathbb{B}) \mapsto [(\Lambda^a(RC(X), \rho, \mathbb{B}), f_{(\rho,\mathbb{B})})].$

We will show that

(4.14) $\alpha_X \circ \beta_X = id_{\mathcal{L}_{ad}(X)}$ and $\beta_X \circ \alpha_X = id_{\mathcal{LC}(X)}$.

Let $[(Y, f)] \in \mathcal{LC}(X)$. Then $\beta_X(\alpha_X([(Y, f)])) = \beta_X(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}) = [(\Lambda^a(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}), f_{(\eta_{(Y,f)},\mathbb{B}_{(Y,f)})})]$. Set, for short, $\eta = \eta_{(Y,f)}$, $\mathbb{B} = \mathbb{B}_{(Y,f)}$, $g = f_{(\eta_{(Y,f)},\mathbb{B}_{(Y,f)})}$, $Z = \Lambda^a(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)})$ and $r_{(Y,f)} = r_f$. We have to show that

$[(Y,f)] = [(Z,g)]$. Since r_f is an LCA-isomorphism, we get that $h = \Lambda^a(r_f) : Z \longrightarrow \Lambda^a(\Lambda^t(Y))$ is a homeomorphism. Set $Y' = \Lambda^a(\Lambda^t(Y))$. By Roeper's Theorem 1.2.3.10, the map $t_Y : Y \longrightarrow Y'$, $y \mapsto \sigma_y$ is a homeomorphism. Let $h' = (t_Y)^{-1} \circ h$. Then $h' : Z \longrightarrow Y$ is a homeomorphism. We will prove that $h' \circ g = f$ and this will imply that $[(Y,f)] = [(Z,g)]$. Let $x \in X$ and u be an ultrafilter containing the filter ν_x. Then, as we have shown above, $\sigma_x = \sigma_u$. Hence, by Theorem 2.2.2.12, $h(\sigma_x) = h(\sigma_u) = \sigma_{e_f(u)}$, where $e_f = (r_f)^{-1}$. Thus $h'(g(x)) = h'(\sigma_x) = (t_Y)^{-1}(h(\sigma_x)) = (t_Y)^{-1}(\sigma_{e_f(u)})$. Note that, by 0.4.2.2, $e_f(F) = \mathrm{cl}_Y(f(F))$, for every $F \in RC(X)$. Since $e_f : RC(X) \longrightarrow RC(Y)$ is a Boolean isomorphism, we get that $e_f(u)$ is an ultrafilter in $RC(Y)$ containing $\nu^Y_{f(x)}$. Thus $\sigma_{e_f(u)} = \sigma^Y_{f(x)}$. Hence $(t_Y)^{-1}(\sigma_{e_f(u)}) = f(x)$. So, $h' \circ g = f$. Therefore, $\beta_X \circ \alpha_X = id_{\mathcal{LC}(X)}$.

Let $(RC(X), \rho, \mathbb{B}) \in \mathcal{L}_{ad}(X)$ and $Y = \Lambda^a(RC(X), \rho, \mathbb{B})$. Recall that we have set $A = RC(X)$. We have that $\beta_X(A, \rho, \mathbb{B}) = [(Y, f_{(\rho,\mathbb{B})})]$. Set $f = f_{(\rho,\mathbb{B})}$. Then $\alpha_X(\beta_X(A,\rho,\mathbb{B})) = (A, \eta_{(Y,f)}, \mathbb{B}_{(Y,f)})$. By Roeper's Theorem 1.2.3.10, we have that $\lambda^g_A : (A,\rho,\mathbb{B}) \longrightarrow (RC(Y), \rho_Y, CR(Y))$ is an LCA-isomorphism. We will show that $f^{-1}(\lambda^g_A(F)) = F$, for every $F \in RC(X)$. Indeed, if $x \in F$ then $F \in \sigma_x$, and thus $\sigma_x \in \lambda^g_A(F)$; hence $f(F) \subseteq \lambda^g_A(F)$, i.e., $F \subseteq f^{-1}(\lambda^g_A(F))$. If $x \in f^{-1}(\lambda^g_A(F))$ then $f(x) \in \lambda^g_A(F)$, i.e., $\sigma_x \in \lambda^g_A(F)$; therefore $F \in \sigma_x$, which means that $x \in F$. So, $f^{-1}(\lambda^g_A(F)) = F$, for every $F \in RC(X)$. Since $CR(Y) = \{\lambda^g_A(F) \mid F \in \mathbb{B}\}$, we get that $f^{-1}(CR(Y)) = \mathbb{B}$. Thus $\mathbb{B}_{(Y,f)} = \mathbb{B}$. Further, by 0.4.2.2, $\mathrm{cl}_Y(f(F)) = \lambda^g_A(F)$, for every $F \in RC(X)$. Since, for every $F, G \in RC(X)$, $F\rho G \iff \lambda^g_A(F) \cap \lambda^g_A(G) \neq \emptyset$, we get that $\rho = \eta_{(Y,f)}$. Therefore, $\alpha_X \circ \beta_X = id_{\mathcal{L}_{ad}(X)}$.

We will now prove that α_X and β_X are monotone functions.

Let $[(Y_i, f_i)] \in \mathcal{LC}(X)$, where $1 = 1, 2$, and $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. Then there exists a continuous map $g : Y_2 \longrightarrow Y_1$ such that $g \circ f_2 = f_1$. Let $\alpha_X([(Y_i, f_i)]) = (RC(X), \eta_{(Y_i,f_i)}, \mathbb{B}_{(Y_i,f_i)})$, where $i = 1, 2$. Set $\eta_i = \eta_{(Y_i,f_i)}$ and $\mathbb{B}_i = \mathbb{B}_{(Y_i,f_i)}$, $i = 1, 2$. We have to show that $\eta_2 \subseteq \eta_1$ and $\mathbb{B}_2 \subseteq \mathbb{B}_1$. Let $F \in \mathbb{B}_2$. Then $\mathrm{cl}_{Y_2}(f_2(F))$ is compact. Hence $g(\mathrm{cl}_{Y_2}(f_2(F)))$ is compact. We have that $f_1(F) = g(f_2(F)) \subseteq g(\mathrm{cl}_{Y_2}(f_2(F))) \subseteq \mathrm{cl}_{Y_1}(g(f_2(F))) = \mathrm{cl}_{Y_1}(f_1(F))$. Thus $\mathrm{cl}_{Y_1}(f_1(F)) = g(\mathrm{cl}_{Y_2}(f_2(F)))$, i.e., $\mathrm{cl}_{Y_1}(f_1(F))$ is compact. Therefore $F \in \mathbb{B}_1$. So, we have proved that $\mathbb{B}_2 \subseteq \mathbb{B}_1$. Let $F, G \in RC(X)$ and $F\eta_2 G$. Then there exists $y \in \mathrm{cl}_{Y_2}(f_2(F)) \cap \mathrm{cl}_{Y_2}(f_2(G))$. Since $g(\mathrm{cl}_{Y_2}(f_2(F))) \subseteq \mathrm{cl}_{Y_1}(f_1(F))$ and, analogously, $g(\mathrm{cl}_{Y_2}(f_2(G))) \subseteq \mathrm{cl}_{Y_1}(f_1(G))$, we get that $g(y) \in \mathrm{cl}_{Y_1}(f_1(F)) \cap \mathrm{cl}_{Y_1}(f_1(G))$. Thus $F\eta_1 G$. Therefore, $\eta_2 \subseteq \eta_1$. All this shows that $\alpha_X([(Y_1, f_1)]) \preceq_{ad} \alpha_X([(Y_2, f_2)])$. Hence, α_X is a monotone function.

Let now $(RC(X), \rho_i, \mathbb{B}_i) \in \mathcal{L}_{ad}(X)$, where $i = 1, 2$, and $(RC(X), \rho_1, \mathbb{B}_1) \preceq_{ad} (RC(X), \rho_2, \mathbb{B}_2)$. Set, for short, $Y_i = \Lambda^a(RC(X), \rho_i, \mathbb{B}_i)$ and $f_i = f_{(\rho_i, \mathbb{B}_i)}$, $i = 1, 2$. Then $\beta_X(RC(X), \rho_i, \mathbb{B}_i) = [(Y_i, f_i)]$, $i = 1, 2$. We will show that $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. We have that $f_i : X \longrightarrow Y_i$ is defined by $f_i(x) = \sigma_x$, for every $x \in X$ and $i = 1, 2$. We also have that $\mathbb{B}_2 \subseteq \mathbb{B}_1$ and $\rho_2 \subseteq \rho_1$. Let us regard the following function $\varphi : (RC(X), \rho_1, \mathbb{B}_1) \longrightarrow (RC(X), \rho_2, \mathbb{B}_2)$, $F \mapsto F$. We will prove that φ is a **DHLC**-morphism. Clearly, φ satisfies conditions (DLC1) and (DLC2). The fact that $\rho_2 \subseteq \rho_1$ implies immediately that φ satisfies also condition (DLC3). Further, condition (DLC4) follows from the inclusion $\mathbb{B}_2 \subseteq \mathbb{B}_1$. Let $F \in RC(X)$. Then $F = \bigvee\{G \in \mathbb{B}_1 \mid G \ll_{\rho_1} F\}$ and thus $\varphi(F) = \bigvee\{\varphi(G) \mid G \in \mathbb{B}_1, G \ll_{\rho_1} F\}$. This shows that φ satisfies condition (DLC5). So, φ is a **DHLC**-morphism. Then, by Theorem 2.2.2.12, $g = \Lambda^a(\varphi) : Y_2 \longrightarrow Y_1$ is a continuous map. We will prove that $g \circ f_2 = f_1$, i.e., that for every $x \in X$, $g(\sigma_x) = \sigma_x$. So, let $x \in X$. We have, by (2.9), that $g(\sigma_x) \cap \mathbb{B}_1 = \{F \in \mathbb{B}_1 \mid (\forall G \in RC(X))[(F \ll_{\rho_1} G) \rightarrow (x \in G)]\}$. We will show that $g(\sigma_x) \cap \mathbb{B}_1 = \sigma_x \cap \mathbb{B}_1$. This will imply, by 2.2.3.4, that $g(\sigma_x) = \sigma_x$. Let $F \in \sigma_x \cap \mathbb{B}_1$. Then $x \in F$ and thus $F \in g(\sigma_x) \cap \mathbb{B}_1$. Conversely, suppose that there exists $H \in g(\sigma_x) \cap \mathbb{B}_1$ such that $x \notin H$. Then $x \in X \setminus H = \mathrm{int}_X(H^*)$. By (A2), there exists $G \in \mathbb{B}_1$ with $x \in \mathrm{int}_X(G)$ and $G \ll_{\rho_1} H^*$. We get that $H \ll_{\rho_1} G^*$ and $x \notin G^*$, a contradiction. Therefore, $g(\sigma_x) = \sigma_x$. Thus $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. So, β_X is also a monotone function. Since $\beta_X = (\alpha_X)^{-1}$, we get that α_X is an isomorphism. □

Definition 4.2.2.3. Let (X, τ) be a Tychonoff space. An NCA $(RC(X, \tau), C)$ is said to be *admissible for* (X, τ) if the LCA $(RC(X, \tau), C, RC(X, \tau)) \in \mathcal{L}_{ad}(X, \tau)$. The set of all NCAs which are admissible for (X, τ) will be denoted by $\mathcal{K}_{ad}(X, \tau)$ (or simply by $\mathcal{K}_{ad}(X)$). Note that $\mathcal{K}_{ad}(X)$ is, in fact, a subset of $\mathcal{L}_{ad}(X)$. The restriction on $\mathcal{K}_{ad}(X)$ of the order $\preceq_{ad}$, defined on $\mathcal{L}_{ad}(X)$, will be denoted again by $\preceq_{ad}$.

Corollary 4.2.2.4. (de Vries [24]) *For every Tychonoff space X, there exists an isomorphism between the posets $(\mathcal{C}(X), \leq)$ and $(\mathcal{K}_{ad}(X), \preceq_{ad})$ (see 0.4.1.2 for the notation).*

Proof. It follows immediately from Theorem 4.2.2.2. □

The first part of the Leader Local Compactification Theorem 0.5.2.5 follows from our Theorem 4.2.2.2 and the next three lemmas.

Lemma 4.2.2.5. *Let $(X, \beta_i, \mathcal{B}_i)$, $i = 1, 2$, be two separated local proximity spaces on a Tychonoff space (X, τ), $\mathcal{B}_1 \cap RC(X) = \mathcal{B}_2 \cap RC(X)$ and $(\beta_1)_{|RC(X)} = (\beta_2)_{|RC(X)}$ (i.e., for every $F, G \in RC(X)$, $F\beta_1 G \iff F\beta_2 G$). Then $\beta_1 = \beta_2$ and $\mathcal{B}_1 = \mathcal{B}_2$.*

Proof. Set $\mathbb{B} = \mathcal{B}_1 \cap RC(X)$ and $\mathcal{B} = \{M \subseteq X \mid \exists F \in \mathbb{B}$ such that $M \subseteq F\}$. Then $\mathcal{B} \subseteq \mathcal{B}_i$, for $i = 1, 2$. Let $M \in \mathcal{B}_1$. Then there exist $N, K \in \mathcal{B}_1$ such that $M \ll_{\beta_1} N \ll_{\beta_1} K$; hence $M \subseteq \mathrm{int}(N) \subseteq \mathrm{cl}(N) \subseteq K$. Thus, $M \subseteq F = \mathrm{cl}(\mathrm{int}(N))$ and $F \in \mathbb{B}$. Therefore, $M \in \mathcal{B}$. Hence $\mathcal{B}_1 = \mathcal{B}$. Analogously, $\mathcal{B}_2 = \mathcal{B}$. Thus $\mathcal{B}_1 = \mathcal{B}_2$.

Set $\rho = (\beta_1)_{|RC(X)}$ and let, $\forall M, N \subseteq X$, $M(-\beta)N \iff \forall B \in \mathcal{B}\ \exists F, G \in RC(X)$ such that $M \cap B \subseteq \mathrm{int}_X(F)$, $N \cap B \subseteq \mathrm{int}_X(G)$ and $F(-\rho)G$. We will show that $\beta_i = \beta$, for $i = 1, 2$.

Let $M, N \subseteq X$, $M(-\beta_1)N$ and $B \in \mathcal{B}$. Set $M' = M \cap B$ and $N' = N \cap B$. Then $M'(-\beta_1)N'$ and $M' \in \mathcal{B}_1$. Thus, there exist $F, F_1 \in \mathbb{B}$ such that $M' \ll_{\beta_1} F \ll_{\beta_1} F_1 \ll_{\beta_1} X \setminus N'$. Put $G = F_1^*$ (i.e., $G = \mathrm{cl}(X \setminus F_1)$). Then $M' \subseteq \mathrm{int}(F)$, $N' \subseteq \mathrm{int}(G)$ and $F(-\rho)G$. Hence, $M(-\beta)N$. So, we get that $\beta \subseteq \beta_1$. Conversely, let $M(-\beta)N$. Suppose that $M\beta_1 N$. Then there exists $B_1 \in \mathcal{B}$ such that $(M \cap B_1)\beta_1 N$; also, there exists $B_2 \in \mathcal{B}$ with $(M \cap B_1)\beta_1(N \cap B_2)$. Setting $B = B_1 \cup B_2$, we get that $B \in \mathcal{B}$ and $(M \cap B)\beta_1(N \cap B)$. Thus $M\beta N$, a contradiction. Therefore, $M(-\beta_1)N$. So, $\beta_1 = \beta$. Analogously, we get that $\beta_2 = \beta$. Hence $\beta_1 = \beta_2$. □

Lemma 4.2.2.6. *Let $(X, \beta, \mathcal{B})$ be a separated local proximity space. Set $\tau = \tau_{(X,\beta,\mathcal{B})}$. Let $\rho = \beta_{|RC(X,\tau)}$ and $\mathbb{B} = \mathcal{B} \cap RC(X, \tau)$. Then $(RC(X, \tau), \rho, \mathbb{B}) \in \mathcal{L}_{ad}(X, \tau)$.*

Proof. The fact that $(RC(X, \tau), \rho, \mathbb{B})$ is an LCA was proved in Example 1.2.3.3. The rest can be easily checked. □

Lemma 4.2.2.7. *Let (X, τ) be a Tychonoff space and $(RC(X), \rho, \mathbb{B}) \in \mathcal{L}_{ad}(X)$. Set*

$$\mathcal{B} = \{M \subseteq X \mid \exists B \in \mathbb{B} \text{ such that } M \subseteq B\},$$

and, for every $M, N \subseteq X$, put

$$M(-\beta)N \iff$$

$\forall B \in \mathcal{B}\ \exists F, G \in RC(X)$ such that $M \cap B \subseteq \mathrm{int}_X(F)$, $N \cap B \subseteq \mathrm{int}_X(G)$ and $F(-\rho)G$.

Let $Y = \beta_X(RC(X), \rho, \mathbb{B})$ (see (4.13) for β_X) and $f = f_{(\rho,\mathbb{B})}$ (see (4.12) for $f_{(\rho,\mathbb{B})}$). Then $\mathcal{B} = \{M \subseteq X \mid \mathrm{cl}_Y(f(M))$ is compact$\}$ and $\forall M, N \subseteq X$, $M(-\beta)N \iff \mathrm{cl}_Y(f(M)) \cap \mathrm{cl}_Y(f(N)) = \emptyset$.

The triple $(X, \beta, \mathcal{B})$ *is a separated local proximity space on* (X, τ), $\beta_{|RC(X)} = \rho$ *and* $\mathbb{B} = RC(X) \cap \mathcal{B}$*; moreover,* $(X, \beta, \mathcal{B})$ *is the unique separated local proximity space on* (X, τ) *having these properties.*

Proof. Since Y is locally compact and $\mathbb{B} = \{F \in RC(X) \mid \mathrm{cl}_Y(f(F)) \text{ is compact}\}$ (see the proof of Theorem 4.2.2.2), we get easily that $\mathcal{B} = \{M \subseteq X \mid \mathrm{cl}_Y(f(M))$ is compact$\}$. Using this equality and the fact that $\forall F, G \subseteq RC(X)$, $F(-\rho)G \iff \mathrm{cl}_Y(f(F)) \cap \mathrm{cl}_Y(f(G)) = \emptyset$ (see again the proof of Theorem 4.2.2.2), it is not difficult to show that $\forall M, N \subseteq X$, $M(-\beta)N \iff \mathrm{cl}_Y(f(M)) \cap \mathrm{cl}_Y(f(N)) = \emptyset$. Now it is easy to check that $(X, \beta, \mathcal{B})$ is a separated local proximity space on (X, τ), $\mathbb{B} = RC(X) \cap \mathcal{B}$ and $\beta_{|RC(X)} = \rho$. The uniqueness of $(X, \beta, \mathcal{B})$ follows from Lemma 4.2.2.5. □

Lemma 4.2.2.7 shows that the separated local proximity space $(X, \beta, \mathcal{B})$ constructed in it coincides with $\gamma_X([(Y, f)])$ (see Theorem 0.5.2.5 for γ_X).

Definition 4.2.2.8. Let X be a locally compact Hausdorff space. We will denote by

$$\mathcal{L}_a(X)$$

the set of all LCAs of the form $(RC(X), \rho, \mathbb{B})$ which satisfy the following conditions:

(LA1) $\rho_X \subseteq \rho$;

(LA2) $CR(X) \subseteq \mathbb{B}$;

(LA3) for every $F \in RC(X)$ and every $G \in CR(X)$, $F\rho G$ implies $F \cap G \neq \emptyset$.

If $(A, \rho_i, \mathbb{B}_i) \in \mathcal{L}_a(X)$, where $i = 1, 2$, we set

$$(A, \rho_1, \mathbb{B}_1) \preceq_l (A, \rho_2, \mathbb{B}_2) \iff (\rho_2 \subseteq \rho_1 \text{ and } \mathbb{B}_2 \subseteq \mathbb{B}_1).$$

Theorem 4.2.2.9. *Let* (X, τ) *be a locally compact Hausdorff space. Then there exists an isomorphism*

$$\mu : (\mathcal{LC}(X), \leq) \longrightarrow (\mathcal{L}_a(X), \preceq_l)$$

between the posets $(\mathcal{LC}(X), \leq)$ *and* $(\mathcal{L}_a(X), \preceq_l)$.

Proof. Obviously, if we prove that an LCA $(RC(X), \rho, \mathbb{B})$ belongs to $\mathcal{L}_a(X)$ iff it is admissible for X, then our theorem will follow from Theorem 4.2.2.2.

Let $(RC(X), \rho, \mathbb{B})$ be admissible for X. Let $H \in CR(X)$. Then, by (A2), for every $x \in H$ there exists $G_x \in \mathbb{B}$ such that $x \in \mathrm{int}_X(G_x)$ (indeed, set $F = X$ in (A2)). Since H is compact, we get that H is a subset of a union of finitely many elements of $\mathbb{B}$. Thus $H \in \mathbb{B}$. So, condition (LA2) is fulfilled. Let now $F \in RC(X)$, $G \in CR(X)$

and $F\rho G$. Suppose that $F \cap G = \emptyset$. Then $G \subseteq X \setminus F = \mathrm{int}_X(F^*)$. Thus, by (A2), for every $x \in G$ there exists $G_x \in \mathbb{B}$ such that $x \in \mathrm{int}_X(G_x)$ and $G_x \ll_\rho F^*$. Then the compactness of G implies that there exist $n \in \mathbb{N}^+$ and $x_1, \ldots, x_n \in G$ such that $G \subseteq G_1 = \bigcup_{i=1}^n G_{x_i}$. Clearly, by ($\ll$4) (see 1.2.1.1), $G_1 \ll_\rho F^*$. Therefore, $G \ll_\rho F^*$, i.e., $G(-\rho)F$, a contradiction. Hence $F \cap G \neq \emptyset$. So, condition (LA3) is checked. Since conditions (A1) and (LA1) coincide, we get that $(RC(X), \rho, \mathbb{B}) \in \mathcal{L}_a(X)$.

Conversely, let $(RC(X), \rho, \mathbb{B}) \in \mathcal{L}_a(X)$. Let $F \in RC(X)$ and $x \in \mathrm{int}_X(F)$. Since X is locally compact, there exists $G \in CR(X)$ such that $x \in \mathrm{int}_X(G) \subseteq G \subseteq \mathrm{int}_X(F)$. Then $G \cap F^* = \emptyset$. Thus, by (LA3), $G(-\rho)F^*$. Hence $G \ll_\rho F$. Clearly, by (LA2), we have that $G \in \mathbb{B}$. Therefore, condition (A2) is verified. This shows that $(RC(X), \rho, \mathbb{B})$ is admissible for X. $\square$

Remark 4.2.2.10. The proof of Theorem 4.2.2.9 shows that if X is a Tychonoff space, then any admissible for X LCA $(RC(X), \rho, \mathbb{B})$ satisfies conditions (LA1)-(LA3).

Notation 4.2.2.11. If $(A, \rho, \mathbb{B})$ is a CLCA then we will write $\rho \subseteq_{\mathbb{B}} C$ provided that C is a normal contact relation on A satisfying the following conditions:
(RC1) $\rho \subseteq C$, and
(RC2) for every $a \in A$ and every $b \in \mathbb{B}$, aCb implies $a\rho b$.

If $\rho \subseteq_{\mathbb{B}} C_1$ and $\rho \subseteq_{\mathbb{B}} C_2$ then we will write

$$C_1 \preceq_c C_2 \iff C_2 \subseteq C_1.$$

Let (X, τ) be a locally compact Hausdorff space. We will denote by

$$\mathcal{K}_a(X)$$

the set of all normal contact relations C on $RC(X)$ such that $\rho_X \subseteq_{CR(X)} C$ (i.e., C satisfies conditions (LA1) and (LA3) with ρ replaced by C).

Corollary 4.2.2.12. *Let (X, τ) be a locally compact Hausdorff space. Then there exists an isomorphism*

$$\mu_c : (\mathcal{C}(X), \leq) \longrightarrow (\mathcal{K}_a(X), \preceq_c).$$

Proof. Let $C \in \mathcal{K}_a(X)$. Then $(RC(X), C, RC(X)) \in \mathcal{L}_a(X)$ because condition (LA2) is obviously fulfilled. Thus, all follows from Theorem 4.2.2.9. $\square$

Proposition 4.2.2.13. *Let (X,τ) be a locally compact non-compact Hausdorff space. Then $C_{(RC(X),\rho_X,CR(X))}$ (briefly, C_{ρ_X}) (see 1.2.3.4 for this notation) is the smallest element of the poset $(\mathcal{K}_a(X),\preceq_c)$; hence, if $(\alpha X,\alpha)$ is the Alexandroff (one-point) compactification of X then $\mu_c([(\alpha X,\alpha)])=C_{\rho_X}$ (see Corollary 4.2.2.12 for μ_c). Further, the poset $(\mathcal{K}_a(X),\preceq_c)$ has a greatest element $C_{\beta\rho_X}$; it is defined as follows: for every $F,G\in RC(X)$, $F(-C_{\beta\rho_X})G$ iff there exists a set $\{H_d\in RC(X)\mid d\in\mathbb{D}\}$ such that:*
(1) $F\ll_{\rho_X}H_d\ll_{\rho_X}G^*$ (i.e., $F\subseteq \mathrm{int}_X(H_d)\subseteq H_d\subseteq X\setminus G$), for all $d\in\mathbb{D}$, and
(2) for any two elements d_1,d_2 of $\mathbb{D}$, $d_1<d_2$ implies that $H_{d_1}\ll_{\rho_X}H_{d_2}$ (i.e., $H_{d_1}\subseteq \mathrm{int}_X(H_{d_2})$).
Hence, if $(\beta X,\beta)$ is the Stone-Čech compactification of X then $\mu_c([(\beta X,\beta)])=C_{\beta\rho_X}$.

Proof. It is straightforward. □

Remark 4.2.2.14. The definition of the relation $C_{\beta\rho_X}$ in Proposition 4.2.2.13 is given in the language of contact relations. It is clear that if we use the fact that all happens in a topological space X then we can define the relation $C_{\beta\rho_X}$ by setting for every $F,G\in RC(X)$, $F(-C_{\beta\rho_X})G$ iff F and G are completely separated.

Proposition 4.2.2.15. *Let X be a locally compact non-compact Hausdorff space. Let $\{C_m\mid m\in M\}$ be a subset of $\mathcal{K}_a(X)$ (see 4.2.2.12 for $\mathcal{K}_a(X)$). For every $F,G\in RC(X)$, put $F(-C)G$ iff there exists a set $\{H_d\in RC(X)\mid d\in\mathbb{D}\}$ such that:*
(1) $F\ll_{C_m}H_d\ll_{C_m}G^*$, for all $d\in\mathbb{D}$ and for each $m\in M$, and
(2) for any two elements d_1,d_2 of $\mathbb{D}$, $d_1<d_2$ implies that $H_{d_1}\ll_{C_m}H_{d_2}$, for every $m\in M$.
Then C is the supremum in $(\mathcal{K}_a(X),\preceq_c)$ of the set $\{C_m\mid m\in M\}$.

Proof. The proof is straightforward. □

4.2.3 Map extensions over local compactifications

Theorem 4.2.3.1. *Let, for $i=1,2$, (X_i,τ_i) be a Tychonoff space, $(RC(X_i),\eta_i,\mathbb{B}_i)\in\mathcal{L}_{ad}(X_i,\tau_i)$,*
$Y_i=\Lambda^a(RC(X_i),\eta_i,\mathbb{B}_i)$ and $f_i=f_{(\eta_i,\mathbb{B}_i)}$ (see (4.12)). Let $f:X_1\longrightarrow X_2$ be a continuous function. Then there exists a continuous function $g:Y_1\longrightarrow Y_2$ such that $g\circ f_1=f_2\circ f$ iff f satisfies the following conditions:
(REQ1) *For every $F,G\in RC(X_2)$, $\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F)))\eta_1\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(G)))$ implies that $F\eta_2G$;*

(REQ2) *For every $F \in \mathbb{B}_1$ there exists $G \in \mathbb{B}_2$ such that $f(F) \subseteq G$.*

First Proof. Note that, by (4.13) and (4.14), we have that, for $i = 1, 2$, (Y_i, f_i) is a Hausdorff local compactification of X_i, $[(Y_i, f_i)] = \beta_{X_i}(RC(X_i), \eta_i, \mathbb{B}_i)$ and $\alpha_{X_i}([(Y_i, f_i)]) = (RC(X_i), \eta_i, \mathbb{B}_i)$. Set, for $i = 1, 2$,

$$\mathcal{B}_i = \{M \subseteq X_i \mid \exists B \in \mathbb{B}_i \text{ such that } M \subseteq B\}.$$

For every $M, N \subseteq X_i$ and $i = 1, 2$, set

$$M(-\eta'_i)N \iff$$

$[\forall B \in \mathcal{B}_i\ \exists F, G \in RC(X_i)$ such that $M \cap B \subseteq \mathrm{int}_{X_i}(F),\ N \cap B \subseteq \mathrm{int}_{X_i}(G),\ F(-\eta_i)G]$.

Then, by 4.2.2.7, for $i = 1, 2$, $(X_i, \eta'_i, \mathcal{B}_i)$ is the unique separated local proximity space such that $\mathcal{B}_i \cap RC(X_i) = \mathbb{B}_i$ and $(\eta'_i)_{|RC(X_i)} = \eta_i$. Thus, by the proofs of Theorems 4.2.2.2 and 0.5.2.5, $[(Y_i, f_i)] = \gamma^{-1}_{(X_i,\tau_i)}(X_i, \eta'_i, \mathcal{B}_i)$, where $i = 1, 2$. So, if we show that $f : (X_1, \eta'_1, \mathcal{B}_1) \longrightarrow (X_2, \eta'_2, \mathcal{B}_2)$ is equicontinuous iff it satisfies conditions (REQ1) and (REQ2), our assertion will follow from Leader's Theorem 0.5.2.5.

It is easy to see that f satisfies condition (EQ2) iff it satisfies condition (REQ2). Let f be an equicontinuous function, $F_1, F_2 \in RC(X_2)$ and

$$\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_1)))\eta_1\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_2))).$$

Then $\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_1)))\eta'_1\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_2)))$ and thus

$$f(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_1))))\eta'_2 f(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_2)))).$$

Since, for $i = 1, 2$, $f(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_i))) \subseteq \mathrm{cl}_{X_2} f((\mathrm{int}_{X_1}(f^{-1}(F_i)))) \subseteq F_i$, we get that $F_1\eta'_2F_2$ and, therefore, $F_1\eta_2F_2$. Hence, f satisfies condition (REQ1). So, every equicontinuous function satisfies conditions (REQ1) and (REQ2). Conversely, let f satisfies conditions (REQ1) and (REQ2), $M, N \subseteq X_1$ and $M\eta'_1N$. Then there exists $B \in \mathcal{B}_1$ such that for every $H_1, H_2 \in RC(X_1)$ with $M \cap B \subseteq \mathrm{int}_{X_1}(H_1)$ and $N \cap B \subseteq \mathrm{int}_{X_1}(H_2)$, $H_1\eta_1H_2$ holds. Suppose that $f(M)(-\eta'_2)f(N)$. Then, for every $C \in \mathcal{B}_2$ there exist $F, G \in RC(X_2)$ such that $f(M) \cap C \subseteq \mathrm{int}_{X_2}(F)$, $f(N) \cap C \subseteq \mathrm{int}_{X_2}(G)$ and $F(-\eta_2)G$. Since condition (REQ2) implies condition (EQ2), we have that $f(B) \in \mathcal{B}_2$. Thus there exist $F, G \in RC(X_2)$ such that $f(M) \cap f(B) \subseteq \mathrm{int}_{X_2}(F)$, $f(N) \cap f(B) \subseteq \mathrm{int}_{X_2}(G)$ and $F(-\eta_2)G$. Then $M \cap B \subseteq \mathrm{int}_{X_1}(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F))))$ and $N \cap B \subseteq \mathrm{int}_{X_1}(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(G))))$. Hence, by (REQ1), $F\eta_2G$ holds, a contradiction. Therefore, $f(M)\eta'_2f(N)$. Thus, f is an equicontinuous function.

Second Proof. In the first proof we used the Leader Local Compactification Theorem 0.5.2.5. We will now give another proof which does not use Leader's theorem. Hence, by the First Proof, it will imply the second part of the Leader Theorem 0.5.2.5. The more important thing is that the method of this new proof will be used later on for the proof of our Theorem 4.2.3.7 (which is the main result of this section).
($\Rightarrow$) Let there exists a continuous function $g : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$. Then, using the notation of (4.10), we have, by the proof of Theorem 4.2.2.2, that the maps $r_i = r_{(Y_i,f_i)}$ are LCA-isomorphisms, $i = 1, 2$. Set, for $i = 1, 2$, $e_i = (r_i)^{-1}$ and $\rho_i = \rho_{Y_i}$. Then, by 0.4.2.2, for every $F \in RC(X_i)$ and $i = 1, 2$, $e_i(F) = \mathrm{cl}_{Y_i}(f_i(F))$. Let $\varphi_g = \Lambda^t(g)$ (see Theorem 2.2.2.12), i.e.,

(4.15) $\varphi_g : (RC(Y_2), \rho_2, CR(Y_2)) \longrightarrow (RC(Y_1), \rho_1, CR(Y_1)),\ G \mapsto \mathrm{cl}_{Y_1}(g^{-1}(\mathrm{int}_{Y_2}(G)))$.

Set also

(4.16) $\varphi_f = r_1 \circ \varphi_g \circ e_2 : (RC(X_2), \eta_2, \mathbb{B}_2) \longrightarrow (RC(X_1), \eta_1, \mathbb{B}_1)$.

We will prove that

(4.17) $\varphi_f(G) = \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G))),$ for every $G \in RC(X_2)$.

Indeed, let $G \in RC(X_2)$. Then

$$\varphi_f(G) = (f_1)^{-1}(\mathrm{cl}_{Y_1}(g^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))))) = \mathrm{cl}_{X_1}((f_1)^{-1}(g^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))))).$$

It is easy to see that

$$(f_2)^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))) = \mathrm{int}_{X_2}(G).$$

Thus $(f_1)^{-1}(g^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G))))) = \{x \in X_1 \mid (g \circ f_1)(x) \in \mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))\} = \{x \in X_1 \mid f_2(f(x)) \in \mathrm{int}_{Y_2}(\mathrm{cl}(f_2(G)))\} = \{x \in X_1 \mid f(x) \in (f_2)^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}(f_2(G))))\} = \{x \in X_1 \mid f(x) \in \mathrm{int}_{X_2}(G)\} = f^{-1}(\mathrm{int}_{X_2}(G))$. Now it becomes clear that (4.17) holds.

Since, by Theorem 2.2.2.12, φ_g is a **DHLC**-morphism, we get that φ_f is a **DHLC**-morphism. Let $F \in \mathbb{B}_1$. Then, by (DLC4), there exists $G \in \mathbb{B}_2$ such that $F \subseteq \varphi_f(G)$; thus $f(F) \subseteq f(\mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G))) \subseteq \mathrm{cl}_{X_2}(f(f^{-1}(\mathrm{int}_{X_2}(G)))) \subseteq G$. Hence, condition (REQ2) is checked. Further, let $F, G \in RC(X_2)$ and $F \ll_{\eta_2} G$. Then, by condition (DLC3S), $(\varphi_f(F^*))^* \ll_{\eta_1} \varphi_f(G)$. Now, using the fact that $(\varphi_f(H^*))^* = \mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(H)))$, for every $H \in RC(X_2)$, it is easy to see that condition (REQ1) is also fulfilled.

($\Leftarrow$) Let f be a continuous function satisfying conditions (REQ1) and (REQ2). Set φ_f : $(RC(X_2), \eta_2, \mathbb{B}_2) \longrightarrow (RC(X_1), \eta_1, \mathbb{B}_1)$, $G \mapsto \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G)))$. Using conditions (A1) and (A2) and the given data for f, it is easy to check that φ_f is a **DHLC**-morphism. Putting $g = \Lambda^a(\varphi_f)$, we get, by Theorem 2.2.2.12, that g is a continuous function and $g : Y_1 \longrightarrow Y_2$. We will show that $g \circ f_1 = f_2 \circ f$. Let $x \in X_1$. Then $g(f_1(x)) = g(\sigma_x)$ and $f_2(f(x)) = \sigma_{f(x)}$. By Theorem 2.2.2.12, we have that $g(\sigma_x) \cap \mathbb{B}_2 = \{G \in \mathbb{B}_2 \mid \forall F \in RC(X_2), (G \ll_{\eta_2} F) \to (x \in \varphi_f(F))\}$. We will prove that $\{G \in \mathbb{B}_2 \mid \forall F \in RC(X_2), (G \ll_{\eta_2} F) \to (x \in \varphi_f(F))\} = \{G \in \mathbb{B}_2 \mid f(x) \in G\}$. This will imply, by 2.2.3.4, the desired equality. So, let $G \in \mathbb{B}_2$ and $f(x) \in G$. Let $F \in RC(X_2)$ and $G \ll_{\eta_2} F$. Since the LCA $(RC(X_2), \eta_2, \mathbb{B}_2)$ is admissible for X_2, it satisfies conditions (A1) and (A2) of Definition 4.2.2.1. Now, using condition (A1), we get that $G \ll_{\rho_{X_2}} F$, i.e., that $G \subseteq \mathrm{int}_{X_2}(F)$. Thus we obtain that $x \in f^{-1}(G) \subseteq f^{-1}(\mathrm{int}_{X_2}(F) \subseteq \varphi_f(F)$. Conversely, let $G \in \mathbb{B}_2 \cap g(\sigma_x)$. Suppose that $f(x) \notin G$. Then $f(x) \in X_2 \setminus G = \mathrm{int}_{X_2}(G^*)$. By condition (A2), there exists $F \in \mathbb{B}_2$ such that $f(x) \in \mathrm{int}_{X_2}(F)$ and $F \ll_{\eta_2} G^*$. Then $G \ll_{\eta_2} F^*$. Hence $x \in \varphi_f(F^*)$. Since $f(x) \in \mathrm{int}_{X_2}(F) = X_2 \setminus F^*$, we get a contradiction. Therefore, $f(x) \in G$. Thus $g \circ f_1 = f_2 \circ f$. □

Theorem 4.2.3.1 implies immediately the following corollary:

Corollary 4.2.3.2. *Let, for $i = 1, 2$, X_i be a Tychonoff space, Y_i be a Hausdorff local compactification of X_i and let's assume, for simplicity of notation, that $X_i \subseteq Y_i$. Let $f : X_1 \longrightarrow X_2$ be a continuous function. Then f has a continuous extension $g : Y_1 \longrightarrow Y_2$ iff f satisfies the following conditions:*

(REQ1') *For every $F, G \in RC(X_2)$, $\mathrm{cl}_{Y_1}(\mathrm{int}_{X_1}(f^{-1}(F))) \cap \mathrm{cl}_{Y_1}(\mathrm{int}_{X_1}(f^{-1}(G))) \neq \emptyset$ implies that $\mathrm{cl}_{Y_2}(F) \cap \mathrm{cl}_{Y_2}(G) \neq \emptyset$;*

(REQ2') *For every $F \in RC(X_1)$ such that $\mathrm{cl}_{Y_1}(F)$ is compact, we have that $\mathrm{cl}_{Y_2}(f(F))$ is compact.*

We will need a result of A. Blaszczyk [15] (for completeness, we will present here a proof of it). Let us start with a lemma.

Lemma 4.2.3.3. *A continuous map $f : X \longrightarrow Y$, where X and Y are topological spaces, is skeletal iff for every open subset V of Y such that $\mathrm{cl}_Y(V)$ is open, $\mathrm{cl}_X(f^{-1}(V)) = f^{-1}(\mathrm{cl}_Y(V))$ holds.*

Proof. ($\Rightarrow$) Let f be a skeletal continuous map and V be an open subset of Y such that $\mathrm{cl}_Y(V)$ is open. Let $x \in f^{-1}(\mathrm{cl}_Y(V))$. Then $f(x) \in \mathrm{cl}_Y(V)$. Since f is continuous, there exists an open neighborhood U of x in X such that $f(U) \subseteq \mathrm{cl}_Y(V)$. Suppose that $x \notin \mathrm{cl}_X(f^{-1}(V))$. Then there exists an open neighborhood W of x in X such that $W \subseteq U$ and $W \cap f^{-1}(V) = \emptyset$. We obtain that $\mathrm{cl}_Y(f(W)) \cap V = \emptyset$ and $\mathrm{cl}_Y(f(W)) \subseteq \mathrm{cl}_Y(f(U)) \subseteq \mathrm{cl}_Y(V)$. Since, by Lemma 2.5.2.3, $\mathrm{int}_Y(\mathrm{cl}_Y(f(W))) \neq \emptyset$, we get a contradiction. Thus $f^{-1}(\mathrm{cl}_Y(V)) \subseteq \mathrm{cl}_X(f^{-1}(V))$. The converse inclusion follows from the continuity of f. Hence $f^{-1}(\mathrm{cl}_Y(V)) = \mathrm{cl}_X(f^{-1}(V))$.

($\Leftarrow$) Suppose that there exists an open subset U of X such that $\mathrm{int}_Y(\mathrm{cl}_Y(f(U))) = \emptyset$ and $U \neq \emptyset$. Then, clearly, $V = Y\ \mathrm{cl}_Y(f(U))$ is an open dense subset of Y. Hence $\mathrm{cl}_Y(V)$ is open in Y. Thus $\mathrm{cl}_X(f^{-1}(V)) = f^{-1}(\mathrm{cl}_Y(V)) = f^{-1}(Y) = X$ holds. Therefore $X = \mathrm{cl}_X(f^{-1}(V)) = \mathrm{cl}_X(f^{-1}(Y\ \mathrm{cl}_Y(f(U)))) = \mathrm{cl}_X(X\ f^{-1}(\mathrm{cl}_Y(f(U))))$. Since $U \subseteq f^{-1}(\mathrm{cl}_Y(f(U)))$, we get that $X\ U \supseteq \mathrm{cl}_X(X\ f^{-1}(\mathrm{cl}_Y(f(U)))) = X$, a contradiction. Hence, f is a skeletal map. $\square$

Clearly, the proof of Lemma 4.2.3.3 shows that the following assertion is also true:

Lemma 4.2.3.4. ([15]) *A continuous map $f : X \longrightarrow Y$, where X and Y are topological spaces, is skeletal iff for every open dense subset V of Y, $\mathrm{cl}_X(f^{-1}(V)) = X$ holds.*

Lemma 4.2.3.5. *Let, for $i = 1, 2$, (X_i, τ_i) be a topological space, (Y_i, f_i) be some extension of (X_i, τ_i), and $f : X_1 \longrightarrow X_2$, $g : Y_1 \longrightarrow Y_2$ be two continuous functions such that $g \circ f_1 = f_2 \circ f$. Then g is skeletal iff f is skeletal.*

Proof. ($\Rightarrow$) Let g be skeletal and V be an open dense subset of X_2. Set $U = Ex_{Y_2}(V)$, i.e., $U = Y_2 \setminus \mathrm{cl}_{Y_2}(f_2(X_2 \setminus V))$. Then U is an open dense subset of Y_2 and $f_2^{-1}(U) = V$. Hence, by Lemma 4.2.3.4, $g^{-1}(U)$ is a dense open subset of Y_1. We will prove that $f_1^{-1}(g^{-1}(U)) \subseteq f^{-1}(V)$. Indeed, let $x \in f_1^{-1}(g^{-1}(U))$. Then $g(f_1(x)) \in U$, i.e., $f_2(f(x)) \in U$. Thus $f(x) \in f_2^{-1}(U) = V$. So, $f_1^{-1}(g^{-1}(U)) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is dense in X_1. Therefore, by Lemma 4.2.3.4, f is a skeletal map.

($\Leftarrow$) Let f be a skeletal map and U be a dense open subset of Y_2. Set $V = f_2^{-1}(U)$. Then V is an open dense subset of X_2. Thus, by Lemma 4.2.3.4, $f^{-1}(V)$ is a dense subset of X_1. We will prove that $f^{-1}(V) \subseteq f_1^{-1}(g^{-1}(U))$. Indeed, let $x \in f^{-1}(V)$. Then $f(x) \in V = f_2^{-1}(U)$. Thus $f_2(f(x)) \in U$, i.e., $g(f_1(x)) \in U$. So, $f^{-1}(V) \subseteq f_1^{-1}(g^{-1}(U))$. This implies that $g^{-1}(U)$ is dense in Y_1. Now, Lemma 4.2.3.4 shows that g is a skeletal map. $\square$

4.2.3.6. It is natural to write

$$\text{“}f:(X_1,RC(X_1),\eta_1,\mathbb{B}_1)\longrightarrow(X_2,RC(X_2),\eta_2,\mathbb{B}_2)\text{”}$$

when X_1 and X_2 are Tychonoff spaces, $(RC(X_i),\eta_i,\mathbb{B}_i)\in\mathcal{L}_{ad}(X_i)$, for $i=1,2$, and $f:X_1\longrightarrow X_2$ is a continuous function. Then, in an analogy with Leader's equicontinuous functions (see the Leader Theorem 0.5.2.5), the functions

$$f:(X_1,RC(X_1),\eta_1,\mathbb{B}_1)\longrightarrow(X_2,RC(X_2),\eta_2,\mathbb{B}_2)$$

which satisfy conditions (REQ1) and (REQ2) will be called *R-equicontinuous functions.*

We are now ready to prove the main result of this section:

Theorem 4.2.3.7. *Let $f:(X_1,RC(X_1),\eta_1,\mathbb{B}_1)\longrightarrow(X_2,RC(X_2),\eta_2,\mathbb{B}_2)$ be an R-equicontinuous function, $Y_i=\Lambda^a(RC(X_i),\eta_i,\mathbb{B}_i)$ and $f_i=f_{(\eta_i,\mathbb{B}_i)}$ (see (4.12)) for $i=1,2$, and $g:Y_1\longrightarrow Y_2$ be a continuous function such that $g\circ f_1=f_2\circ f$ (its existence and uniqueness are guaranteed by Theorem 4.2.3.1). Then:*

(a) g is skeletal iff f is skeletal;

(b) g is an open map iff f is a skeletal map and satisfies the following condition:

(O) $\forall F\in\mathbb{B}_1$ *and* $\forall G\in RC(X_1),(F\ll_{\eta_1}G)\rightarrow(\mathrm{cl}_{X_2}(f(F))\ll_{\eta_2}\mathrm{cl}_{X_2}(f(G)))$*;*

(b′) g is an open map iff f satisfies the following condition:

(O1) $\forall F\in\mathbb{B}_1$ *and* $\forall G\in RC(X_1),(F\ll_{\eta_1}G)\rightarrow(f(F)\ll_{\eta_2'}\mathrm{cl}_{X_2}(f(G)))$*, where η_2' is the local proximity on (X_2,τ_2) generated by (Y_2,f_2) (see Theorem 0.5.2.5);*

(b″) g is an open map iff f satisfies the following condition:

(O2) $\forall A\subseteq X_1$ *such that there exists $F\in\mathbb{B}_1$ with $A\subseteq F$, and $\forall B\subseteq X_1$, $(A\ll_{\eta_1'}B)\rightarrow(f(A)\ll_{\eta_2'}\mathrm{cl}_{X_2}(f(B)))$, where, for $i=1,2$, η_i' is the local proximity on (X_i,τ_i) generated by (Y_i,f_i) (see Theorem 0.5.2.5);*

(c) g is a perfect map iff f satisfies the following condition:

(P) *For every $G\in\mathbb{B}_2$, $\mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G)))\in\mathbb{B}_1$ holds;*

(d) $\mathrm{cl}_{Y_2}(g(Y_1))=Y_2$ iff $\mathrm{cl}_{X_2}(f(X_1))=X_2$;

(e) g is an injection iff f satisfies the following condition:

(I) *For every $F_1,F_2\in\mathbb{B}_1$ such that $F_1(-\eta_1)F_2$ there exist $G_1,G_2\in\mathbb{B}_2$ with $G_1\ll_{\eta_2}G_2$, $F_1\subseteq\mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G_1)))$ and $\mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G_2)))(-\eta_1)F_2$;*

(f) g is an open injection iff f satisfies condition (O1) *(or, equivalently, f is skeletal and satisfies condition (O)) and the following one:*

(OI) $\forall F \in RC(X_1)\ \exists G \in RC(X_2)$ *such that* $F = \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G)))$*;*

(g) g *is a perfect surjection iff* f *satisfies condition* (P) *and* $\mathrm{cl}_{X_2}(f(X_1)) = X_2$*.*

Proof. Note that, by (4.13) and (4.14), we have that, for $i = 1, 2$, (Y_i, f_i) is a Hausdorff local compactification of X_i, $[(Y_i, f_i)] = \beta_{X_i}(RC(X_i), \eta_i, \mathbb{B}_i)$ and $\alpha_{X_i}([(Y_i, f_i)]) = (RC(X_i), \eta_i, \mathbb{B}_i)$.

Set $\varphi_g = \Lambda^t(g)$ (see Theorem 2.2.2.12). Then

$$\varphi_g : RC(Y_2) \longrightarrow RC(Y_1),\quad G \mapsto \mathrm{cl}_{Y_1}(g^{-1}(\mathrm{int}_{Y_2}(G))).$$

Set also

$$\varphi_f : RC(X_2) \longrightarrow RC(X_1),\quad F \mapsto \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(F))).$$

Then, (4.15), (4.16) and (4.17) imply that

(4.18) $\varphi_f = r_1 \circ \varphi_g \circ e_2$,

where, for $i = 1, 2$,

$$r_i : (RC(Y_i), \rho_{Y_i}, CR(Y_i)) \longrightarrow (RC(X_i), \eta_i, \mathbb{B}_i),\ G \mapsto f_i^{-1}(G),$$

and

$$e_i = r_i^{-1}.$$

Note that, according to the proof of Theorem 4.2.2.2, the maps r_i, e_i, where $i = 1, 2$, are LCA-isomorphisms.

(a) It follows from Lemma 4.2.3.5.

(b) Since every open map is skeletal, we get, using (a), that if g is an open map then f is skeletal. So, we can suppose that f is skeletal. Then, as it follows from the proof of Theorem 2.5.3.3, φ_f is a complete Boolean homomorphism. Thus, by (2.37), the map φ_f has a left adjoint

$$\varphi^f : RC(X_1) \longrightarrow RC(X_2),\ F \mapsto \mathrm{cl}_{X_2}(f(F)).$$

Hence, (4.16) and 2.5.4.2 imply that g is an open map iff the map f (is skeletal and) satisfies the following condition:

(O$'$) $\forall F \in \mathbb{B}_1$ and $\forall G \in RC(X_2), (\varphi^f(F)\eta_2 G) \to (F\eta_1\varphi_f(G))$.

It is easy to see that condition (O$'$) is equivalent to the following one:

(O$''$) $\forall F \in \mathbb{B}_1$ and $\forall G \in RC(X_2), (F \ll_{\eta_1} \varphi_f(G)) \to (\varphi^f(F) \ll_{\eta_2} G)$.

We will prove that condition (O$''$) is equivalent to condition (O). Indeed, let condition (O$''$) is satisfied, $F \in \mathbb{B}_1$, $G \in RC(X_1)$ and $F \ll_{\eta_1} G$. Set $H = \varphi^f(G)$. Then $H \in RC(X_2)$ and $\varphi_f(H) = \varphi_f(\varphi^f(G)) \supseteq G$. Therefore $F \ll_{\eta_1} \varphi_f(H)$. Now, by (O$''$), $\varphi^f(F) \ll_{\eta_2} H$ holds, i.e., $\varphi^f(F) \ll_{\eta_2} \varphi^f(G)$. So, condition (O) is satisfied. Conversely, let f satisfies condition (O), $F \in \mathbb{B}_1$, $G \in RC(X_2)$ and $F \ll_{\eta_1} \varphi_f(G)$. Then, by (O), $\varphi^f(F) \ll_{\eta_2} \varphi^f(\varphi_f(G))$. Since $\varphi^f(\varphi_f(G)) \subseteq G$, we get that $\varphi^f(F) \ll_{\eta_2} G$. Thus, condition (O$''$) is fulfilled. This completes the proof of (b).

(b$'$) Having in mind Lemma 2.5.2.6 and the fact that η_2' is a Lodato proximity, we need only to show that if f satisfies condition (O1) then f is a skeletal map. So, let f satisfies condition (O1), V be an open dense subset of X_2 and $G = \mathrm{cl}_{X_1}(f^{-1}(V))$. Then $G \in RC(X_1)$. Suppose that $G \neq X_1$. Then there exists $x \in X_1 \setminus G$. Clearly, $f_1(x) \notin \mathrm{cl}_{Y_1}(f_1(G))$. Since Y_1 is locally compact and Hausdorff, we get that there exists $F \in \mathbb{B}_1$ such that $x \in F$ and $F(-\eta_1)G$. Thus $F \ll_{\eta_1} G^*$. Therefore, by (O1), $f(F) \ll_{\eta_2'} \mathrm{cl}_{X_2}(f(G^*))$. Set $U = X_1 \setminus G$. Since f is continuous, we have that $H = \mathrm{cl}_{X_2}(f(G^*)) = \mathrm{cl}_{X_2}(f(U)) \subseteq \mathrm{cl}_{X_2}(f(X_1 \setminus f^{-1}(V))) \subseteq \mathrm{cl}_{X_2}(f(X_1) \setminus V) \subseteq X_2 \setminus V$. Thus $H^* = \mathrm{cl}_{X_2}(X_2 \setminus H) \supseteq \mathrm{cl}_{X_2}(V) = X_2$. Since η_2' is a Lodato proximity, we get that $f(F)(-\eta_2')X_2$, a contradiction. Therefore, $f^{-1}(V)$ is dense in X_1. Then Lemma 4.2.3.4 implies that f is a skeletal map.

(b$''$) Obviously, condition (O2) implies condition (O1). We will show that condition (O1) implies condition (O2). Set

$$\mathbb{B}_1' = \{A \subseteq X_1 \mid \exists F \in \mathbb{B}_1 \text{ such that } A \subseteq F\}.$$

Let f satisfies condition (O1), $A \in \mathbb{B}_1', B \subseteq X_1$ and $A \ll_{\eta_1'} B$. Then $A(-\eta_1')(X_1 \setminus B)$. Thus $\mathrm{cl}_{Y_1}(f_1(A)) \cap \mathrm{cl}_{Y_1}(f_1(X_1 \setminus B)) = \emptyset$. Since $A \in \mathbb{B}_1'$, we have that $\mathrm{cl}_{Y_1}(f_1(A))$ is a compact subset of Y_1. Using the fact that Y_1 is a locally compact Hausdorff space, we get that there exist $F \in \mathbb{B}_1$ and $U \in RO(X_1)$ such that $A \subseteq F$, $X_1 \setminus B \subseteq U$ and $F(-\eta_1')U$. Set $G = X_1 \setminus U$. Then $G \in RC(X_1)$ and $F \ll_{\eta_1} G$. Thus, by (O1), $f(F) \ll_{\eta_2'} \mathrm{cl}_{X_2}(f(G))$. Since $G \subseteq B$, we get that $f(A) \ll_{\eta_2'} \mathrm{cl}_{X_2}(f(B))$. So, f satisfies condition (O2).

(c) By [53, Theorem 3.7.18], g is a perfect map iff φ_g satisfies the following condition: for every $G \in CR(Y_2)$, $\varphi_g(G) \in CR(Y_1)$ holds. Having in mind the proof of Theorem 4.2.2.2 and (4.18), we get that g is a perfect map iff f satisfies condition (P).

(d) This is obvious.

(e) Using again (4.18), our assertion follows from Theorem 2.6.2.2.

(f) It follows from (b), (4.18), and Theorem 2.6.2.8.

(g) It follows from (c) and (d). □

Now, Corollary 2.5.2.5(b) and Theorem 4.2.3.7 imply the following two corollaries:

Corollary 4.2.3.8. *Let (X_1, δ_1) and (X_2, δ_2) be two Efremovič proximity spaces. Let $(cX_i, c_i) = S_m(\delta_i)$ (see 0.5.2.3 for this notation) be the Hausdorff compactification of (X_i, τ_{δ_i}) corresponding, by the Smirnov Compactification Theorem 0.5.2.3, to the Efremovič proximity δ_i, where $i = 1, 2$. Further, let $f : (X_1, \delta_1) \longrightarrow (X_2, \delta_2)$ be a proximally continuous function, and $g = S_m f : cX_1 \longrightarrow cX_2$ be the continuous function such that $g \circ c_1 = c_2 \circ f$ (see 0.5.2.4 for its existence). Then:*

(a) g is quasi-open iff f is skeletal;

(b)(V. Z. Poljakov [89]) *g is an open map iff f satisfies the following condition:*

(OC) *For every $A, B \subseteq X_1$ such that $A \ll_{\delta_1} B$, $f(A) \ll_{\delta_2} \mathrm{cl}_{X_2}(f(B))$ holds.*

Proof. (a) It follows from Corollary 2.5.2.5(b) and Theorem 4.2.3.7(a).

(b) Obviously, in our hypothesis, condition (OC) implies condition (O2) and conversely. Thus our assertion follows from Theorem 4.2.3.7(b$''$). □

Corollary 4.2.3.9. *Let X_1, X_2 be two Tychonoff spaces, $f : X_1 \longrightarrow X_2$ be a continuous function and $\beta f : \beta X_1 \longrightarrow \beta X_2$ be the extension of f to the Stone-Čech compactifications of X_1 and X_2. Then:*

(a) βf is quasi-open iff f is skeletal;

(b) βf is an open map iff f satisfies the following condition:

(OB) *For every $A, B \subseteq X_1$ which are completely separated in X_1, $f(A)$ and $X_2 \setminus \mathrm{cl}_{X_2}(f(X_1 \setminus B))$ are completely separated in X_2;*

(c)(V. Z. Poljakov [89]) *If X_1 and X_2 are normal spaces then βf is open iff for every $A, B \subseteq X_1$ such that $\mathrm{cl}_{X_1}(A) \subseteq \mathrm{int}_{X_1}(B)$, $\mathrm{cl}_{X_2}(f(A)) \subseteq \mathrm{int}_{X_2}(\mathrm{cl}_{X_2}(f(B)))$ holds;*

(d)(A. D. Taĭmanov [6]) *If X_1 and X_2 are normal spaces and f is an open and closed map then βf is open.*

Proof. (a) It follows from Corollary 4.2.3.8(a).

(b) Let δ_β be the proximity on a Tychonoff space X that corresponds, by Theorem 0.5.2.3, to βX. Then, as it is well known, for every $A, B \subseteq X$, $A(-\delta_\beta)B$ iff A and B are completely separated. Having this in mind, we can easily see that our assertion follows from Corollary 4.2.3.8(b).

(c) As it is well known, the famous Urysohn Lemma implies that two subsets of a normal space X are completely separated iff their closures are disjoint. Thus, using the notation from the proof of (b), for every $A, B \subseteq X$, $A \ll_{\delta_\beta} B$ iff $\mathrm{cl}_X(A) \cap \mathrm{cl}_X(X \setminus B) = \emptyset$ iff $\mathrm{cl}_X(A) \subseteq \mathrm{int}_X(B)$. Now it becomes clear that our assertion follows from Corollary 4.2.3.8(b).

(d) Let $A, B \subseteq X_1$ and $\mathrm{cl}_{X_1}(A) \subseteq \mathrm{int}_{X_1}(B)$. Then, using [53, 1.4.C], we get that $\mathrm{cl}_{X_2}(f(A)) = f(\mathrm{cl}_{X_1}(A)) \subseteq f(\mathrm{int}_{X_1}(B)) \subseteq \mathrm{int}_{X_2}(f(B)) \subseteq \mathrm{int}_{X_2}(\mathrm{cl}_{X_2}(f(B)))$. Thus, our assertion follows from (c). $\square$

Remark 4.2.3.10. In [89], after establishing the general result 4.2.3.8(b), V. Z. Poljakov writes (in the notations of Corollary 4.2.3.9) that βf is open iff for every two completely separated subsets A and B of X_1, the sets $f(A)$ and $\{y \in X_2 \mid f^{-1}(y) \subseteq B\}$ are completely separated in X_2. Since $\{y \in X_2 \mid f^{-1}(y) \subseteq B\} = f^{\#}(B) = X_2 \setminus f(X_1 \setminus B)$, we get that Poljakov's condition implies condition (OB) and thus it is sufficient for the openness of βf. It is, however, not necessary. Indeed, let $f : \mathbb{Q} \longrightarrow \beta\mathbb{Q}$ be the inclusion map (supposing, for simplicity, that $\mathbb{Q} \subseteq \beta\mathbb{Q}$). Then $\beta f : \beta\mathbb{Q} \longrightarrow \beta\mathbb{Q}$ is the identity map and hence it is an open map. Let $A, B \subseteq \mathbb{Q}$ and A, B be completely separated in $\mathbb{Q}$. Then, by Poljakov's condition, the sets $f(A)$ and $f^{\#}(B)$ are completely separated in $\beta\mathbb{Q}$, i.e.,

$$\mathrm{cl}_{\beta\mathbb{Q}}(f(A)) \cap \mathrm{cl}_{\beta\mathbb{Q}}(f^{\#}(B)) = \emptyset.$$

Since $f^{\#}(B) = f(B) \cup (\beta\mathbb{Q} \setminus \mathbb{Q})$, we get that $\mathrm{cl}_{\beta\mathbb{Q}}(f^{\#}(B)) = \beta\mathbb{Q}$. Thus

$$\mathrm{cl}_{\beta\mathbb{Q}}(f(A)) \cap \mathrm{cl}_{\beta\mathbb{Q}}(f^{\#}(B)) \neq \emptyset,$$

a contradiction. Hence, the map f does not satisfy Poljakov's condition.

Finally, it is easy to see that Theorem 4.2.3.7 implies the following result:

Corollary 4.2.3.11. *Let, for $i = 1, 2$, X_i be a Tychonoff space, Y_i be a Hausdorff local compactification of X_i and let's assume, for simplicity of notation, that $X_i \subseteq Y_i$. Let $f : X_1 \longrightarrow X_2$ be a continuous function having a continuous extension $g : Y_1 \longrightarrow Y_2$. Then:*

(a) g is skeletal iff f is skeletal;

(b) g is an open map iff f satisfies the following condition:

(O′) $\forall F, G \in RC(X_1)$ *such that* $\mathrm{cl}_{Y_1}(F)$ *is compact, we have that*

$$(\mathrm{cl}_{Y_1}(F) \subseteq Ex_{Y_1}(G)) \rightarrow (\mathrm{cl}_{Y_2}(f(F)) \subseteq Ex_{Y_2}(\mathrm{cl}_{X_2}(f(G))));$$

(c) g is a perfect map iff f satisfies the following condition:

(P′) *If $G \in RC(X_2)$ and $\mathrm{cl}_{Y_2}(G)$ is compact then $\mathrm{cl}_{Y_1}(f^{-1}(\mathrm{int}_{X_2}(G)))$ is compact;*

(d) $\mathrm{cl}_{Y_2}(g(Y_1)) = Y_2$ iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(e) g is an injection iff f satisfies the following condition:

(I′) *For every $F_1, F_2 \in RC(X_1)$ such that $\mathrm{cl}_{Y_1}(F_1)$ and $\mathrm{cl}_{Y_1}(F_2)$ are disjoint compact subsets of Y_1, there exist $G_1, G_2 \in RC(X_2)$ such that $\mathrm{cl}_{Y_2}(G_1)$ and $\mathrm{cl}_{Y_2}(G_2)$ are compact subsets of Y_2, $\mathrm{cl}_{Y_2}(G_1) \subseteq Ex_{Y_2}(G_2)$, $F_1 \subseteq \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G_1)))$ and $\mathrm{cl}_{Y_1}(f^{-1}(\mathrm{int}_{X_2}(G_2))) \cap \mathrm{cl}_{Y_1}(F_2) = \emptyset$;*

(f) g is an open injection iff f satisfies condition (O′) *and the following one:*

(OI) *$\forall F \in RC(X_1)\ \exists G \in RC(X_2)$ such that $F = \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G)))$;*

(g) g is a perfect surjection iff f satisfies condition (P′) *and $\mathrm{cl}_{X_2}(f(X_1)) = X_2$.*

4.3 Open and other kinds of map extensions over zero-dimensional local compactifications

4.3.1 Introduction

In [8], B. Banaschewski proved that every zero-dimensional Hausdorff space X has a zero-dimensional Hausdorff compactification $\beta_0 X$ with the following remarkable property: every continuous map $f : X \longrightarrow Y$, where Y is a zero-dimensional Hausdorff compact space, can be extended to a continuous map $\beta_0 f : \beta_0 X \longrightarrow Y$; in particular, $\beta_0 X$ is the maximal zero-dimensional Hausdorff compactification of X. As far as I know, there are no descriptions of the maps f for which the extension $\beta_0 f$ is open or quasi-open. In this section we solve the following more general problem: let $f : X \longrightarrow Y$ be a map between two zero-dimensional Hausdorff spaces and (lX, l_X), (lY, l_Y) be Hausdorff zero-dimensional locally compact extensions of X and Y, respectively; find the necessary and sufficient conditions which has to satisfy the map f in order to have an "extension" $g : lX \longrightarrow lY$ (i.e., $g \circ l_X = l_Y \circ f$) which is a map with some special properties (we consider the following properties: continuous, open, perfect, quasi-open, skeletal, injective, surjective, dense embedding). In [78], S. Leader solved such a problem for continuous extensions over Hausdorff local compactifications using the language of local proximities. Hence, if one can describe the local proximities which correspond to zero-dimensional Hausdorff local compactifications then the above problem will be solved for continuous extensions. Recently, G. Bezhanishvili

[13], solving an old problem of L. Esakia, described the Efremovič proximities which correspond (in the sense of the famous Smirnov Compactification Theorem 0.5.2.3) to the zero-dimensional Hausdorff compactifications (and called them *zero-dimensional Efremovič proximities*). We extend here his result to the Leader local proximities, i.e., we describe the local proximities which correspond to the Hausdorff zero-dimensional local compactifications and call them *zero-dimensional local proximities* (see Theorem 4.3.3.2). We do not use, however, these zero-dimensional local proximities for solving our problem. We introduce a simpler notion (namely, the *admissibe ZLB-algebra*) for doing this. Ph. Dwinger [48] proved, using Stone's Duality Theorem [108], that the ordered set of all, up to equivalence, zero-dimensional Hausdorff compactifications of a zero-dimensional Hausdorff space is isomorphic to the ordered by inclusion set of all *Boolean bases* of X (i.e., of those bases of X which are Boolean subalgebras of the Boolean algebra $CO(X)$ of all clopen (= closed and open) subsets of X). This description is much simpler than that by the Efremovič proximities. It was rediscovered by K. D. Magill Jr. and J. A. Glasenapp [81] and applied very successfully to the study of the poset of all, up to equivalence, zero-dimensional Hausdorff compactifications of a zero-dimensional Hausdorff space. We extend the cited above Dwinger Theorem [48] to the zero-dimensional Hausdorff *local compactifications* (see Theorem 4.3.2.4 below) with the help of our generalization of the Stone Duality Theorem proved in Chapter 3 (see Theorem 3.2.2.7) and the notion of an "admissible ZLB-algebra" introduced here. We obtain our solution of the problem formulated above in the language of the admissible ZLB-algebras (see Theorem 4.3.4.7). As a corollary, we characterize the maps $f : X \longrightarrow Y$ between two Hausdorff zero-dimensional spaces X and Y for which the extension $\beta_0 f : \beta_o X \longrightarrow \beta_0 Y$ is open or quasi-open (see Corollary 4.3.4.8). Of course, one can pass from admissible ZLB-algebras to zero-dimensional local proximities and conversely (see Theorem 4.3.3.4 below; it generalizes an analogous result about the connection between Boolean bases and zero-dimensional Efremovič proximities obtained in [13]).

For the notions and notation not defined here see [48, 53, 75, 87].

The exposition of this section is based on the paper [30].

4.3.2 A generalization of Dwinger's Theorem

In the next assertion we recall (for a convenience of the reader) some results from Chapter 3; they will be used in this section.

Proposition 4.3.2.1. *Let (A, I) be a ZLBA. Set $X = \{u \in Ult(A) \mid u \cap I \neq \emptyset\}$. Set, for every $a \in A$, $\lambda_A^C(a) = \{u \in X \mid a \in u\}$. Let τ be the topology on X having as an open base the family $\{\lambda_A^C(a) \mid a \in I\}$. Then (X, τ) is a zero-dimensional locally compact Hausdorff space, $\lambda_A^C(A) = CO(X)$, $\lambda_A^C(I) = KO(X)$ and $\lambda_A^C : A \longrightarrow CO(X)$ is a Boolean isomorphism; hence, $\lambda_A^C : (A, I) \longrightarrow (CO(X), KO(X))$ is a* **ZLBA**-*isomorphism. We set $\Theta^a(A, I) = (X, \tau)$. The space $\Theta^a(A, I)$ is compact iff $A = I$.*

Definition 4.3.2.2. Let X be a zero-dimensional Hausdorff space. Then:

(a) A ZLBA (A, I) is called *admissible for* X if A is a Boolean subalgebra of the Boolean algebra $CO(X)$ and I is an open base of X.

(b) The set of all admissible for X ZLB-algebras will be denoted by

$$\mathcal{ZA}(X).$$

(c) If $(A_1, I_1), (A_2, I_2) \in \mathcal{ZA}(X)$ then we set

$$(A_1, I_1) \preceq_0 (A_2, I_2)$$

if A_1 is a Boolean subalgebra of A_2 and for every $V \in I_2$ there exists $U \in I_1$ such that $V \subseteq U$.

Notation 4.3.2.3. The set of all (up to equivalence) zero-dimensional locally compact Hausdorff extensions of a zero-dimensional Hausdorff space X will be denoted by

$$\mathcal{L}_0(X).$$

The order on $\mathcal{L}_0(X)$ induced by the order " $\leq$" on $\mathcal{LC}(X)$ will be denoted again by " $\leq$".

Theorem 4.3.2.4. *Let X be a zero-dimensional Hausdorff space. Then the ordered sets $(\mathcal{L}_0(X), \leq)$ and $(\mathcal{ZA}(X), \preceq_0)$ are isomorphic; at that, the zero-dimensional compact Hausdorff extensions of X correspond to the admissible for X (Z)LB-algebras of the form (A, A).*

Proof. Let (Y, f) be a locally compact Hausdorff zero-dimensional extension of X. Set

(4.19) $A_{(Y,f)} = f^{-1}(CO(Y))$ and $I_{(Y,f)} = f^{-1}(KO(Y))$.

Note that

$$A_{(Y,f)} = \{F \in CO(X) \mid \mathrm{cl}_Y(f(F)) \text{ is open in } Y\}$$

and

$$I_{(Y,f)} = \{F \in A_{(Y,f)} \mid \mathrm{cl}_Y(f(F)) \text{ is compact}\}.$$

We will show that $(A_{(Y,f)}, I_{(Y,f)}) \in \mathcal{ZA}(X)$. Obviously, the map

(4.20) $r^0_{(Y,f)} : (CO(Y), KO(Y)) \longrightarrow (A_{(Y,f)}, I_{(Y,f)}),\ G \mapsto f^{-1}(G),$

is a Boolean isomorphism such that $r^0_{(Y,f)}(KO(Y)) = I_{(Y,f)}$. Hence $(A_{(Y,f)}, I_{(Y,f)})$ is a ZLBA and $r^0_{(Y,f)}$ is an LBA-isomorphism. It is easy to see that $I_{(Y,f)}$ is a base of X (because Y is locally compact). Hence $(A_{(Y,f)}, I_{(Y,f)}) \in \mathcal{ZA}(X)$. It is clear that if (Y_1, f_1) is a locally compact Hausdorff zero-dimensional extension of X equivalent to the extension (Y, f), then $(A_{(Y,f)}, I_{(Y,f)}) = (A_{(Y_1,f_1)}, I_{(Y_1,f_1)})$. Therefore, the map

(4.21) $\alpha^0_X : \mathcal{L}_0(X) \longrightarrow \mathcal{ZA}(X),\ [(Y,f)] \mapsto (A_{(Y,f)}, I_{(Y,f)}),$

is well-defined. Note that, by (4.19), $A_{(Y,f)} = I_{(Y,f)}$ iff (Y, f) is a compact Hausdorff zero-dimensional extension of X.

Let $(A, I) \in \mathcal{ZA}(X)$ and $Y = \Theta^a(A, I)$. Then Y is a locally compact Hausdorff zero-dimensional space. For every $x \in X$, set

(4.22) $u_{x,A} = \{F \in A \mid x \in F\}.$

Since I is a base of X, we get that $u_{x,A}$ is an ultrafilter in A and $u_{x,A} \cap I \neq \emptyset$, i.e., $u_{x,A} \in Y$. Define

(4.23) $f_{(A,I)} : X \longrightarrow Y,\ x \mapsto u_{x,A}.$

Set, for short, $f = f_{(A,I)}$. Obviously, $\mathrm{cl}_Y(f(X)) = Y$. It is easy to see that f is a homeomorphic embedding. Hence (Y, f) is a locally compact Hausdorff zero-dimensional extension of X. We now set:

(4.24) $\beta^0_X : \mathcal{ZA}(X) \longrightarrow \mathcal{L}_0(X),\ (A, I) \mapsto [(\Theta^a(A, I), f_{(A,I)})].$

Note that, by Proposition 4.3.2.1, $\Theta^a(A, I)$ is a compact Hausdorff zero-dimensional space iff $A = I$.

We will show that

(4.25) $\alpha^0_X \circ \beta^0_X = id_{\mathcal{ZA}(X)}$ and $\beta^0_X \circ \alpha^0_X = id_{\mathcal{L}_0(X)}$.

Let $[(Y, f)] \in \mathcal{L}_0(X)$. Set, for short, $A = A_{(Y,f)}$, $I = I_{(Y,f)}$, $g = f_{(A,I)}$, $Z = \Theta^a(A, I)$ and $\varphi = r^0_{(Y,f)}$. Then $\beta^0_X(\alpha^0_X([(Y, f)])) = \beta^0_X(A, I) = [(Z, g)]$. We have to show that $[(Y, f)] = [(Z, g)]$. Since φ is an LBA-isomorphism, we get that

$$h = \Theta^a(\varphi) : Z \longrightarrow \Theta^a(\Theta^t(Y))$$

is a homeomorphism. Set $Y' = \Theta^a(\Theta^t(Y))$. By the proof of Theorem 3.2.1.11, the map

$$t_Y^C : Y \longrightarrow Y',\ y \mapsto u_y^{CO(Y)},$$

is a homeomorphism. Let

$$h' = (t_Y^C)^{-1} \circ h.$$

Then $h' : Z \longrightarrow Y$ is a homeomorphism. We will prove that $h' \circ g = f$ and this will imply that $[(Y, f)] = [(Z, g)]$. Let $x \in X$. Then $h'(g(x)) = h'(u_{x,A}) = (t_Y^C)^{-1}(h(u_{x,A})) = (t_Y^C)^{-1}(\varphi^{-1}(u_{x,A}))$. We have that $u_{x,A} = \{f^{-1}(F) \mid F \in CO(Y), x \in f^{-1}(F)\} = \{\varphi(F) \mid F \in CO(Y), f(x) \in F\}$. Thus $\varphi^{-1}(u_{x,A}) = \{F \in CO(Y) \mid f(x) \in F\} = u_{f(x)}^{CO(Y)}$. Hence $(t_Y^C)^{-1}(\varphi^{-1}(u_{x,A})) = f(x)$. So, $h' \circ g = f$. Therefore, $\beta_X^0 \circ \alpha_X^0 = id_{\mathcal{L}_0(X)}$.

Let $(A, I) \in \mathcal{ZA}(X)$ and $Y = \Theta^a(A, I)$. Set $f = f_{(A,I)}$, $B = A_{(Y,f)}$ and $J = I_{(Y,f)}$. Then $\alpha_X^0(\beta_X^0(A, I)) = (B, J)$. By Proposition 4.3.2.1, we have that $\lambda_A^C : (A, I) \longrightarrow (CO(Y), KO(Y))$ is an LBA-isomorphism. Hence $\lambda_A^C(A) = CO(Y)$ and $\lambda_A^C(I) = KO(Y)$. We will show that $f^{-1}(\lambda_A^C(F)) = F$, for every $F \in A$. Recall that $\lambda_A^C(F) = \{u \in Y \mid F \in u\}$. Now we have that if $F \in A$ then $f^{-1}(\lambda_A^C(F)) = \{x \in X \mid f(x) \in \lambda_A^C(F)\} = \{x \in X \mid u_{x,A} \in \lambda_A^C(F)\} = \{x \in X \mid F \in u_{x,A}\} = \{x \in X \mid x \in F\} = F$. Thus

(4.26) $B = f^{-1}(CO(Y)) = A$ and $J = f^{-1}(KO(Y)) = I$.

Therefore, $\alpha_X^0 \circ \beta_X^0 = id_{\mathcal{ZA}(X)}$.

We will now prove that α_X^0 and β_X^0 are monotone maps.

Let $[(Y_i, f_i)] \in \mathcal{L}_0(X)$, where $i = 1, 2$, and $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. Then there exists a continuous map $g : Y_2 \longrightarrow Y_1$ such that $g \circ f_2 = f_1$. Set $A_i = A_{(Y_i,f_i)}$ and $I_i = I_{(Y_i,f_i)}$, $i = 1, 2$. Then $\alpha_X^0([(Y_i, f_i)]) = (A_i, I_i)$, where $i = 1, 2$. We have to show that $A_1 \subseteq A_2$ and for every $V \in I_2$ there exists $U \in I_1$ such that $V \subseteq U$. Let $F \in A_1$. Then $F' = \mathrm{cl}_{Y_1}(f_1(F)) \in CO(Y_1)$ and, hence, $G' = g^{-1}(F') \in CO(Y_2)$. Thus $(f_2)^{-1}(G') \in A_2$. Since $(f_2)^{-1}(G') = (f_2)^{-1}(g^{-1}(F')) = (f_2)^{-1}(g^{-1}(\mathrm{cl}_{Y_1}(f_1(F)))) = (f_1)^{-1}(\mathrm{cl}_{Y_1}(f_1(F))) = F$, we get that $F \in A_2$. Therefore, $A_1 \subseteq A_2$. Further, let $V \in I_2$. Then $V' = \mathrm{cl}_{Y_2}(f_2(V)) \in KO(Y_2)$. Thus $g(V')$ is a compact subset of Y_1. Hence there exists $U \in I_1$ such that $g(V') \subseteq \mathrm{cl}_{Y_1}(f_1(U))$. Then $V \subseteq (f_2)^{-1}(g^{-1}(g(\mathrm{cl}_{Y_2}(f_2(V))))) = (f_1)^{-1}(g(V')) \subseteq (f_1)^{-1}(\mathrm{cl}_{Y_1}(f_1(U))) = U$. So, $\alpha_X^0([(Y_1, f_1)]) \preceq_0 \alpha_X^0([(Y_2, f_2)])$. Hence, α_X^0 is a monotone function.

Let now $(A_i, I_i) \in \mathcal{ZA}(X)$, where $i = 1, 2$, and $(A_1, I_1) \preceq_0 (A_2, I_2)$. Set, for short, $Y_i = \Theta^a(A_i, I_i)$ and $f_i = f_{(A_i,I_i)}$, $i = 1, 2$. Then $\beta_X^0(A_i, I_i) = [(Y_i, f_i)]$, $i = 1, 2$.

We will show that $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. We have that, for $i = 1, 2$, $f_i : X \longrightarrow Y_i$ is defined by $f_i(x) = u_{x,A_i}$, for every $x \in X$. We also have that $A_1 \subseteq A_2$ and for every $V \in I_2$ there exists $U \in I_1$ such that $V \subseteq U$. Let us consider the function $\varphi : (A_1, I_1) \longrightarrow (A_2, I_2)$, $F \mapsto F$. Obviously, φ is a **ZLBA**-morphism. Then

$$g = \Theta^a(\varphi) : Y_2 \longrightarrow Y_1$$

is a continuous map. We will prove that $g \circ f_2 = f_1$, i.e., that for every $x \in X$, $g(u_{x,A_2}) = u_{x,A_1}$. So, let $x \in X$. We have that $u_{x,A_2} = \{F \in A_2 \mid x \in F\}$ and $g(u_{x,A_2}) = \varphi^{-1}(u_{x,A_2})$. Clearly, $\varphi^{-1}(u_{x,A_2}) = \{F \in A_1 \cap A_2 \mid x \in F\}$. Since $A_1 \subseteq A_2$, we get that $\varphi^{-1}(u_{x,A_2}) = \{F \in A_1 \mid x \in F\} = u_{x,A_1}$. So, $g \circ f_2 = f_1$. Thus $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. Therefore, β_X^0 is also a monotone function. Since $\beta_X^0 = (\alpha_X^0)^{-1}$, we get that α_X^0 (as well as β_X^0) is an isomorphism. $\square$

Definition 4.3.2.5. Let (X, τ) be a zero-dimensional Hausdorff space. A Boolean algebra A is called *admissible for* (X, τ) (or, a *Boolean base of* (X, τ)) if A is a Boolean subalgebra of the Boolean algebra $CO(X)$ and A is an open base of (X, τ). The set of all admissible Boolean algebras for (X, τ) will be denoted by

$$\mathcal{BA}(X, \tau).$$

Notation 4.3.2.6. The set of all (up to equivalence) zero-dimensional compact Hausdorff extensions of a zero-dimensional Hausdorff space (X, τ) will be denoted by

$$\mathcal{K}_0(X, \tau).$$

The order on $\mathcal{K}_0(X, \tau)$ induced by the order " $\leq$ " on the set $\mathcal{L}_0(X, \tau)$ (defined above) will be denoted again by " $\leq$ ".

Corollary 4.3.2.7. (Ph. Dwinger [48]) *Let (X, τ) be a zero-dimensional Hausdorff space. Then the ordered sets $(\mathcal{K}_0(X, \tau), \leq)$ and $(\mathcal{BA}(X, \tau), \subseteq)$ are isomorphic.*

Proof. Clearly, a Boolean algebra A is admissible for X iff the ZLBA (A, A) is admissible for X. Also, if A_1, A_2 are two admissible for X Boolean algebras then $A_1 \subseteq A_2$ iff $(A_1, A_1) \preceq_0 (A_2, A_2)$. Since the admissible (Z)LB-algebras of the form (A, A) and only they correspond to the zero-dimensional compact Hausdorff extensions of X, it becomes obvious that our assertion follows from Theorem 4.3.2.4. $\square$

4.3.3 Zero-dimensional local proximities

With the next definition we generalize the notion of a *zero-dimensional proximity* introduced in [13].

Definition 4.3.3.1. A local proximity space $(X, \delta, \mathcal{B})$ is called *zero-dimensional* if for every $A, B \in \mathcal{B}$ with $A \ll B$ there exists $C \subseteq X$ such that $A \subseteq C \subseteq B$ and $C \ll C$.

The set of all separated zero-dimensional local proximity spaces on a Tychonoff space (X, τ) (i.e., those which are compatible with the topology τ on X (see 0.5.1.9)) will be denoted by

$$\mathcal{LP}_0(X, \tau).$$

The restriction of the order relation $\preceq$ on $\mathcal{LP}(X, \tau)$ (see 0.5.1.10) to the set $\mathcal{LP}_0(X, \tau)$ will be denoted again by $\preceq$.

Theorem 4.3.3.2. *Let (X, τ) be a zero-dimensional Hausdorff space. Then the ordered sets $(\mathcal{LP}_0(X, \tau), \preceq)$ and $(\mathcal{L}_0(X), \leq)$ are isomorphic (see 4.3.3.1 and 4.3.2.3 for the notation).*

Proof. Having in mind Leader's Theorem 0.5.2.5, we need only to show that if $[(Y, f)] \in \mathcal{LC}(X)$ and $\gamma_X([(Y, f)]) = (X, \delta, \mathcal{B})$ then Y is a zero-dimensional space iff $(X, \delta, \mathcal{B}) \in \mathcal{LP}_0(X)$.

So, let Y be a zero-dimensional space. Then, by Theorem 0.5.2.5, $\mathcal{B} = \{B \subseteq X \mid \mathrm{cl}_Y(f(B)) \text{ is compact}\}$, and for every $A, B \subseteq X$, $A\delta B$ iff $\mathrm{cl}_Y(f(A)) \cap \mathrm{cl}_Y(f(B)) \neq \emptyset$. Let $A, B \in \mathcal{B}$ and $A \ll B$. Then $\mathrm{cl}_Y(f(A)) \cap \mathrm{cl}_Y(f(X \setminus B)) = \emptyset$. Since $\mathrm{cl}_Y(f(A))$ is compact and Y is zero-dimensional, there exists $U \in CO(Y)$ such that $\mathrm{cl}_Y(f(A)) \subseteq U \subseteq Y \setminus \mathrm{cl}_Y(f(X \setminus B))$. Set $V = f^{-1}(U)$. Then $A \subseteq V \subseteq \mathrm{int}_X(B)$, $\mathrm{cl}_Y(f(V)) = U$ and $\mathrm{cl}_Y(f(X \setminus V)) = Y \setminus U$. Thus $V \ll V$ and $A \subseteq V \subseteq B$. Therefore, $(X, \delta, \mathcal{B}) \in \mathcal{LP}_0(X)$.

Conversely, let $(X, \delta, \mathcal{B}) \in \mathcal{LP}_0(X)$. We will prove that Y is a zero-dimensional space. We have, by Theorem 0.5.2.5, that the formulas written in the preceding paragraph for $\mathcal{B}$ and δ hold. Let $y \in Y$ and U be an open neighborhood of y. Since Y is locally compact and Hausdorff, there exist $F_1, F_2 \in CR(Y)$ such that $y \in F_1 \subseteq \mathrm{int}_Y(F_2) \subseteq F_2 \subseteq U$. Let $A_i = f^{-1}(F_i)$, $i = 1, 2$. Then $\mathrm{cl}_Y(f(A_i)) = F_i$, and hence $A_i \in \mathcal{B}$, for $i = 1, 2$. Also, $A_1 \ll A_2$. Thus there exists $C \in \mathcal{B}$ such that $A_1 \subseteq C \subseteq A_2$ and $C \ll C$. It is easy to see that $F_1 \subseteq \mathrm{cl}_Y(f(C)) \subseteq F_2$ and that $\mathrm{cl}_Y(f(C)) \in CO(Y)$. Therefore, Y is a zero-dimensional space. □

By Theorem 0.5.2.5, for every Tychonoff space (X,τ), the local proximity spaces of the form $(X,\delta,P(X))$ on (X,τ) and only they correspond to the Hausdorff compactifications of (X,τ). Obviously, a pair (X,δ), for which the triple $(X,\delta,P(X))$ is a local proximity space, is an Efremovič proximity space. An Efremovič proximity which is a zero-dimensional local proximity is called a *zero-dimensional proximity.* This notion was recently introduced by G. Bezhanishvili [13]. Let us denote by

$$\mathcal{P}_0(X,\tau)$$

the set of all zero-dimensional proximities on a zero-dimensional Hausdorff space (X,τ). Then our Theorem 4.3.3.2 implies immediately the following theorem of G. Bezhanishvili [13]:

Corollary 4.3.3.3. (G. Bezhanishvili [13]) *Let (X,τ) be a zero-dimensional Hausdorff space. Then there exists an isomorphism between the ordered sets $(\mathcal{K}_0(X,\tau),\leq)$ and $(\mathcal{P}_0(X,\tau),\preceq)$ (see Notation 4.3.2.6 for $\mathcal{K}_0(X,\tau)$).*

The connection between the zero-dimensional local proximities on a zero-dimensional Hausdorff space X and the admissible for X ZLB-algebras is clarified in the next result:

Theorem 4.3.3.4. *Let (X,τ) be a zero-dimensional Hausdorff space. Then:*

(a) Let $(A,I)\in\mathcal{ZA}(X,\tau)$. Set $\mathcal{B}=\{M\subseteq X\mid \exists B\in I$ such that $M\subseteq B\}$, and for every $M,N\in\mathcal{B}$, let $M\delta N \iff (\forall F\in I)[(M\subseteq F)\to(F\cap N\neq\emptyset)]$; further, for every $K,L\subseteq X$, let $K\delta L \iff [\exists M,N\in\mathcal{B}$ such that $M\subseteq K, N\subseteq L$ and $M\delta N]$. Then $(X,\delta,\mathcal{B})\in\mathcal{LP}_0(X,\tau)$. Set

$$L_X(A,I)=(X,\delta,\mathcal{B}).$$

(b) Let $(X,\delta,\mathcal{B})\in\mathcal{LP}_0(X,\tau)$. Set $A=\{F\subseteq X\mid F\ll F\}$ and $I=A\cap\mathcal{B}$. Then $(A,I)\in\mathcal{ZA}(X,\tau)$. Set $l_X(X,\delta,\mathcal{B})=(A,I)$.

(c) $\beta_X^0=(\gamma_X)^{-1}\circ L_X$ and, for every $(X,\delta,\mathcal{B})\in\mathcal{LP}_0(X,\tau)$, $(\beta_X^0\circ l_X)(X,\delta,\mathcal{B})=(\gamma_X)^{-1}(X,\delta,\mathcal{B})$ (see Theorem 0.5.2.5, (4.24), as well as (a) and (b) here for the notation);

(d) The correspondence $L_X:(\mathcal{ZA}(X,\tau),\preceq_0)\longrightarrow(\mathcal{LP}_0(X,\tau),\preceq)$ is an isomorphism (between posets) and $L_X^{-1}=l_X$.

Proof. It follows from Theorems 4.3.2.4, 4.3.3.2 and 0.5.2.5. □

The above assertion is a generalization of the analogous result of G. Bezhanishvili [13] concerning the connection between the zero-dimensional proximities on a zero-dimensional Hausdorff space X and the Boolean bases of X.

4.3.4 Map extensions over zero-dimensional local compactifications

Theorem 4.3.4.1. *Let, for $i = 1, 2$, (X_i, τ_i) be a zero-dimensional Hausdorff space, $(A_i, I_i) \in \mathcal{ZA}(X_i)$, $Y_i = \Theta^a(A_i, I_i)$, $f_i = f_{(A_i,I_i)}$ (see (4.23) for $f_{(A_i,I_i)}$) and $f : X_1 \longrightarrow X_2$ be a function. Then there exists a continuous function $g : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$ iff f satisfies the following conditions:*

(ZEQ1) *For every $G \in A_2$, $f^{-1}(G) \in A_1$ holds;*

(ZEQ2) *For every $F \in I_1$ there exists $G \in I_2$ such that $f(F) \subseteq G$.*

Proof. Note first that, according to the proof of Theorem 4.3.2.4, for $i = 1, 2$, (Y_i, f_i) is a zero-dimensional Hausdorff local compactification of X_i, $\beta^0_{X_i}(A_i, I_i) = [(Y_i, f_i)]$ and $\alpha^0_{X_i}([(Y_i, f_i)]) = (A_i, I_i)$.

($\Rightarrow$) Let $g : Y_1 \longrightarrow Y_2$ be a continuous function such that $g \circ f_1 = f_2 \circ f$. By (4.26) and Lemma 0.4.2.2, we have that the maps

(4.27) $r^c_i : CO(Y_i) \longrightarrow A_i,\ G \mapsto (f_i)^{-1}(G), \quad e^c_i : A_i \longrightarrow CO(Y_i),\ F \mapsto \mathrm{cl}_{Y_i}(f_i(F)),$

where $i = 1, 2$, are Boolean isomorphisms; moreover, since $r^c_i(KO(Y_i)) = I_i$ and $e^c_i(I_i) = KO(Y_i)$, we get that

(4.28) $r^c_i : (CO(Y_i), KO(Y_i)) \longrightarrow (A_i, I_i)$ and $e^c_i : (A_i, I_i) \longrightarrow (CO(Y_i), KO(Y_i))$,

where $i = 1, 2$, are LBA-isomorphisms. Set

(4.29) $\psi_g : CO(Y_2) \longrightarrow CO(Y_1),\ G \mapsto g^{-1}(G),$ and $\psi_f = r^c_1 \circ \psi_g \circ e^c_2$.

Then $\psi_f : A_2 \longrightarrow A_1$. We will prove that

(4.30) $\psi_f(G) = f^{-1}(G),$ for every $G \in A_2$.

Indeed, let $G \in A_2$. Then $\psi_f(G) = (r^c_1 \circ \psi_g \circ e^c_2)(G) = (f_1)^{-1}(g^{-1}(\mathrm{cl}_{Y_2}(f_2(G)))) = \{x \in X_1 \mid (g \circ f_1)(x) \in \mathrm{cl}_{Y_2}(f_2(G))\} = \{x \in X_1 \mid f_2(f(x)) \in \mathrm{cl}_{Y_2}(f_2(G))\} = \{x \in X_1 \mid f(x) \in (f_2)^{-1}(\mathrm{cl}_{Y_2}(f_2(G)))\} = \{x \in X_1 \mid f(x) \in G\} = f^{-1}(G)$. This shows that condition

(ZEQ1) is fulfilled. Since, by (3.3), $\psi_g = \Theta^t(g)$, we get that ψ_g is a **ZLBA**-morphism. Thus ψ_f is a **ZLBA**-morphism. Therefore, for every $F \in I_1$ there exists $G \in I_2$ such that $f^{-1}(G) \supseteq F$. Hence, condition (ZEQ2) is also satisfied.

($\Leftarrow$) Let f satisfy conditions (ZEQ1) and (ZEQ2). Set

$$\psi_f : A_2 \longrightarrow A_1, \ \ G \mapsto f^{-1}(G).$$

Then $\psi_f : (A_2, I_2) \longrightarrow (A_1, I_1)$ is a **ZLBA**-morphism. Put $g = \Theta^a(\psi_f)$. Then $g : Y_1 \longrightarrow Y_2$ and g is a continuous function (see Theorem 3.2.1.11). We will show that $g \circ f_1 = f_2 \circ f$. Let $x \in X_1$. Then, by (4.23) and Theorem 3.2.1.11, $g(f_1(x)) = g(u_{x,A_1}) = (\psi_f)^{-1}(u_{x,A_1}) = \{G \in A_2 \mid \psi_f(G) \in u_{x,A_1}\} = \{G \in A_2 \mid x \in f^{-1}(G)\} = \{G \in A_2 \mid f(x) \in G\} = u_{f(x),A_2} = f_2(f(x))$. Thus, $g \circ f_1 = f_2 \circ f$. □

Corollary 4.3.4.2. *Let, for $i = 1, 2$, X_i be a zero-dimensional Hausdorff space, (Y_i, f_i) be a zero-dimensional Hausdorff local compactification of X_i, and $f : X_1 \longrightarrow X_2$ be a function. Then there exists a continuous function $g : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$ iff f satisfies the following conditions:*

(ZEQ1') *For every $G \in f_2^{-1}(CO(Y_2))$, $f^{-1}(G) \in f_1^{-1}(CO(Y_1))$ holds;*

(ZEQ2') *For every $F \in f_1^{-1}(KO(Y_1))$ there exists $G \in f_2^{-1}(KO(Y_2))$ such that $f(F) \subseteq G$.*

Corollary 4.3.4.3. *Let (X_i, τ_i), $i = 1, 2$, be two zero-dimensional Hausdorff spaces, $A_i \in \mathcal{BA}(X_i)$, $(Y_i, f_i) = \beta^0_{X_i}(A_i, A_i)$ (see (4.24) for $\beta^0_{X_i}$), where $i = 1, 2$, and $f : X_1 \longrightarrow X_2$ be a function. Then there exists a continuous function $g : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$ iff f satisfies condition* (ZEQ1).

Proof. It follows from Theorem 4.3.4.1 because for ZLB-algebras of the form (A_i, A_i), where $i = 1, 2$, condition (ZEQ2) is always fulfilled. □

4.3.4.4. Clearly, Theorem 4.3.2.7 implies (see [48]) that every zero-dimensional Hausdorff space X has a greatest zero-dimensional Hausdorff compactification which corresponds to the admissible for X Boolean algebra $CO(X)$. This compactification was discovered by B. Banaschewski [8]; it is denoted by

$$(\beta_0 X, \beta_0^X) \ \ \text{(or, simply, by } (\beta_0 X, \beta_0))$$

and it is called the *Banaschewski compactification* of X.

Using our Corollary 4.3.4.3, one obtains immediately the main property of the Banaschewski compactification:

Corollary 4.3.4.5. (B. Banaschewski [8]) *Let (X_i, τ_i), $i = 1, 2$, be zero-dimensional Hausdorff spaces and (cX_2, c) be a zero-dimensional Hausdorff compactification of X_2. Then for every continuous function $f : X_1 \longrightarrow X_2$ there exists a continuous function $g : \beta_0 X_1 \longrightarrow cX_2$ such that $g \circ \beta_0 = c \circ f$.*

Proof. Since $\beta_0 X_1$ corresponds to the admissible for X_1 Boolean algebra $CO(X_1)$, condition (ZEQ1) is clearly fulfilled when f is a continuous function. Now apply Corollary 4.3.4.3. □

If, in the above Corollary 4.3.4.5, we have that $cX_2 = \beta_0 X_2$, then the map g will be denoted by

$$\beta_0 f.$$

4.3.4.6. It is natural to write

$$\text{“}f : (X_1, A_1, I_1) \longrightarrow (X_2, A_2, I_2)\text{”}$$

when, for $i = 1, 2$, X_i is a zero-dimensional Hausdorff space, $(A_i, I_i) \in \mathcal{ZA}(X_i)$ and $f : X_1 \longrightarrow X_2$ is a function. Then, by analogy with Leader's equicontinuous functions (see the Leader Theorem 0.5.2.5), the functions

$$f : (X_1, A_1, I_1) \longrightarrow (X_2, A_2, I_2)$$

which satisfy conditions (ZEQ1) and (ZEQ2) will be called *0-equicontinuous functions*.

Since I_2 is a base of X_2, we obtain that every 0-equcontinuous function is a continuous function.

Theorem 4.3.4.7. *Let $f : (X_1, A_1, I_1) \longrightarrow (X_2, A_2, I_2)$ be a 0-equicontinuous function, $Y_i = \Theta^a(A_i, I_i)$, $f_i = f_{(A_i,I_i)}$ (see (4.23) for $f_{(A_i,I_i)}$) and $g : Y_1 \longrightarrow Y_2$ be a continuous function such that $g \circ f_1 = f_2 \circ f$ (its existence and uniqueness are guaranteed by Theorem 4.3.4.1). Then:*

(a) g is skeletal iff f is skeletal;

(b) g is an open map iff f satisfies the following condition:

(ZO) *For every $F \in I_1$, $\mathrm{cl}_{X_2}(f(F)) \in I_2$ holds;*

(c) g is a perfect map iff f satisfies the following condition:

(ZP) *For every* $G \in I_2$, $f^{-1}(G) \in I_1$ *holds (i.e., briefly,* $f^{-1}(I_2) \subseteq I_1$*);*

(d) $\mathrm{cl}_{Y_2}(g(Y_1)) = Y_2$ *iff* $\mathrm{cl}_{X_2}(f(X_1)) = X_2$*;*

(e) g *is an injection iff* f *satisfies the following condition:*

(ZI) *For every* $F_1, F_2 \in I_1$ *such that* $F_1 \cap F_2 = \emptyset$*, there exist* $G_1, G_2 \in I_2$ *with* $G_1 \cap G_2 = \emptyset$ *and* $f(F_i) \subseteq G_i$*,* $i = 1, 2$*;*

(f) g *is an open injection iff* f *satisfies condition* (ZO) *and* $I_1 \subseteq f^{-1}(I_2)$ *(i.e., for every* $F \in I_1$ *there exists* $G \in I_2$ *such that* $F = f^{-1}(G)$*);*

(g) g *is a closed injection iff* $f^{-1}(I_2) = I_1$*;*

(h) g *is a perfect surjection iff* f *satisfies condition* (ZP) *and* $\mathrm{cl}_{X_2}(f(X_1)) = X_2$*;*

(i) g *is a dense embedding iff* $\mathrm{cl}_{X_2}(f(X_1)) = X_2$ *and* $I_1 \subseteq f^{-1}(I_2)$*.*

Proof. Note first that, according to the proof of Theorem 4.3.2.4, for $i = 1, 2$, (Y_i, f_i) is a zero-dimensional Hausdorff local compactification of X_i, $\beta^0_{X_i}(A_i, I_i) = [(Y_i, f_i)]$ and $\alpha^0_{X_i}([(Y_i, f_i)]) = (A_i, I_i)$.

Set $\psi_g = \Theta^t(g)$ (see (3.3)). Then

$$\psi_g : CO(Y_2) \longrightarrow CO(Y_1), \quad G \mapsto g^{-1}(G).$$

Set also

$$\psi_f : A_2 \longrightarrow A_1, \quad G \mapsto f^{-1}(G).$$

Let r_i^c and e_i^c, $i = 1, 2$, be defined by (4.27). Then, (4.29) and (4.30) imply that

(4.31) $\psi_f = r_1^c \circ \psi_g \circ e_2^c$ and, hence, $\psi_g = e_1^c \circ \psi_f \circ r_2^c$.

(a) It follows from Lemma 4.2.3.5.

(b) We have, by (4.19), that $I_i = (f_i)^{-1}(KO(Y_i))$, for $i = 1, 2$. Thus, for every $F \in I_i$, where $i \in \{1, 2\}$, $\mathrm{cl}_{Y_i}(f_i(F)) \in KO(Y_i)$ holds.

Let g be an open map and $F \in I_1$. Then $G = \mathrm{cl}_{Y_1}(f_1(F)) \in KO(Y_1)$. Thus $g(G) \in KO(Y_2)$. Since G is compact, we have that $g(G) = \mathrm{cl}_{Y_2}(g(f_1(F))) = \mathrm{cl}_{Y_2}(f_2(f(F))) = \mathrm{cl}_{Y_2}(f_2(\mathrm{cl}_{X_2}(f(F))))$. Therefore, $\mathrm{cl}_{X_2}(f(F)) = (f_2)^{-1}(g(G))$, i.e., $\mathrm{cl}_{X_2}(f(F)) \in I_2$. So, condition (ZO) is fulfilled.

Conversely, let f satisfies condition (ZO). Since $KO(Y_1)$ is an open base of Y_1, for showing that g is an open map, it is enough to prove that for every $G \in KO(Y_1)$, $g(G) = \mathrm{cl}_{Y_2}(f_2(\mathrm{cl}_{X_2}(f(F))))$ holds, where $F = (f_1)^{-1}(G)$ and thus $F \in I_1$. Obviously, $G = \mathrm{cl}_{Y_1}(f_1(F))$. Using again the fact that G is compact, we get that

$g(G) = g(\mathrm{cl}_{Y_1}(f_1(F))) = \mathrm{cl}_{Y_2}(g(f_1(F))) = \mathrm{cl}_{Y_2}(f_2(f(F))) = \mathrm{cl}_{Y_2}(f_2(\mathrm{cl}_{X_2}(f(F))))$. So, g is an open map.

(c) Since Y_2 is a locally compact Hausdorff space and $KO(Y_2)$ is a base of Y_2, we get, using [53, Theorem 3.7.18], that g is a perfect map iff $g^{-1}(G) \in KO(Y_1)$ for every $G \in KO(Y_2)$. Now, using (4.31), we get that: (g is a perfect map) $\iff$ (for every $G \in I_2$, $f^{-1}(G) \in I_1$ holds) $\iff$ (f satisfies condition (ZP)).

(d) This is obvious.

(e) Having in mind (4.28), (4.29) and (4.30), our assertion follows from Theorem 3.4.1.3. A *direct proof* follows. Let g be an injection and $F_1, F_2 \in I_1$, $F_1 \cap F_2 = \emptyset$ and $F_i' = \mathrm{cl}_{Y_1}(f_1(F_i))$, where $i = 1, 2$. Then, by (4.19) and (4.28), F_1' and F_2' are disjoint compact open subsets of Y_1. Hence $g(F_1')$ and $g(F_2')$ are disjoint compact subsets of Y_2. Since $KO(Y_2)$ is a base of Y_2, there exist $G_1', G_2' \in KO(Y_2)$ which are disjoint and $g(F_i') \subseteq G_i'$ for $i = 1, 2$. Setting $G_i = f_2^{-1}(G_i')$ for $i = 1, 2$, we get easily (using (4.19), (4.29) and (4.30)) that $G_1, G_2 \in I_2$, $G_1 \cap G_2 = \emptyset$ and $f(F_i) \subseteq G_i$, $i = 1, 2$. So, condition (ZI) is fulfilled. Conversely, let condition (ZI) be satisfied and $y_1, y_2 \in Y_1$, $y_1 \neq y_2$. Then there exist $F_1', F_2' \in KO(Y_1)$ which are disjoint and $y_i \in F_i'$ for $i = 1, 2$. Setting $F_i = f_1^{-1}(F_i')$ for $i = 1, 2$, we get that $F_1, F_2 \in I_1$ and $F_1 \cap F_2 = \emptyset$. Thus, by (ZI), there exist $G_1, G_2 \in I_2$ with $G_1 \cap G_2 = \emptyset$ and $f(F_i) \subseteq G_i$, $i = 1, 2$. Set $G_i' = \mathrm{cl}_{Y_2}(f_2(G_i))$ for $i = 1, 2$. Then, using continuity of g, we get that $g(y_i) \in G_i'$ for $i = 1, 2$. Since, by (4.28), $G_1' \cap G_2' = \emptyset$, we obtain that $g(y_1) \neq g(y_2)$. Therefore, g is an injection.

(f) It follows from (b), (4.28), (4.29), (4.30), and Theorem 3.4.3.1. We will give a *direct proof* as well. Let g be an open injection. Then, by (b), f satisfies condition (ZO). Let $F \in I_1$ and $F' = \mathrm{cl}_{Y_1}(f_1(F))$. By (4.19), F' is a compact open subset of Y_1. Since g is an open map, we get that $G' = g(F') \in KO(Y_2)$. Further, the injectivity of g implies that $F' = g^{-1}(G')$. Setting $G = f_2^{-1}(G')$, we get that $G \in I_2$ (by (4.19)). Now, using (4.28), (4.29) and (4.30), we obtain that $F = f^{-1}(G)$. So, $f^{-1}(I_2) \supseteq I_1$.

Conversely, let $f^{-1}(I_2) \supseteq I_1$ and f satisfies condition (ZO). Then, by (b), g is an open map. Suppose that there exist $y_1, y_2 \in Y_1$ such that $y_1 \neq y_2$ and $g(y_1) = g(y_2)$. Then there exists $F' \in KO(Y_1)$ such that $y_1 \in F' \subseteq Y_1 \setminus \{y_2\}$. Setting $F = f_1^{-1}(F')$, we get that $F \in I_1$ (see (4.19)). Thus there exists $G \in I_2$ such that $F = f^{-1}(G)$. Let $G' = \mathrm{cl}_{Y_2}(f_2(G))$. Then $G' \in KO(Y_2)$. Note that (4.28), (4.29) and (4.30) imply that $e_1^c \circ \psi_f = \psi_g \circ e_2^c$. Therefore $g^{-1}(G') = F'$. We get that $y_2 \in F'$, a contradiction. Hence, g is an injection.

(g) It follows from (c), (4.28), (4.29), (4.30), and Theorem 3.4.3.3. A *direct proof* will be given now. Let $f^{-1}(I_2) = I_1$. Then, by (c), g is a closed map. Further, the last paragraph of the proof of (f) shows that g is an injection. So, g is a closed injection. Conversely, let g be a closed injection. Then g is a perfect map and (c) implies that $f^{-1}(I_2) \subseteq I_1$. Hence, we need only to show that $f^{-1}(I_2) \supseteq I_1$. Let $F \in I_1$ and $F' = \mathrm{cl}_{Y_1}(f_1(F))$. By (4.19), F' is a compact open subset of Y_1. Since $g_{\upharpoonright Y_1} : Y_1 \longrightarrow g(Y_1)$ is a homeomorphism, we get that $H = g(F')$ is an open subset of $g(Y_1)$. Hence there exists an open subset U of Y_2 such that $U \cap g(Y_1) = H$. Now, using the compactness of H, we obtain that there exists a $G' \in KO(Y_2)$ such that $H \subseteq G' \subseteq U$. Then, obviously, $g^{-1}(G') = F'$. Setting $G = f_2^{-1}(G')$, we get that $G \in I_2$ and $F = f^{-1}(G)$. So, $f^{-1}(I_2) \supseteq I_1$.

(h) It follows from (c) and (d).

(i) It follows from (d), Theorem 3.4.3.4 and Proposition 3.4.1.1. We will also give a *direct proof* of this fact. Obviously, if g is a dense embedding then $g(Y_1)$ is an open subset of Y_2 (because Y_1 is locally compact); thus g is an open mapping and we can apply (f) and (d). Conversely, if $\mathrm{cl}_{X_2}(f(X_1)) = X_2$ and $I_1 \subseteq f^{-1}(I_2)$, then, by (d), $g(Y_1)$ is a dense subset of Y_2. We will show that f satisfies condition (ZO). Let $F_1 \in I_1$. Then there exists $F_2 \in I_2$ such that $F_1 = f^{-1}(F_2)$. Thus $\mathrm{cl}_{X_2}(f(F_1)) \subseteq F_2$. Suppose that $G_2 = F_2 \setminus \mathrm{cl}_{X_2}(f(F_1)) \neq \emptyset$. Since G_2 is open, there exists $x_2 \in G_2 \cap f(X_1)$. Then there exists $x_1 \in X_1$ such that $f(x_1) = x_2 \in F_2$. Thus $x_1 \in F_1$ and hence $x_2 \notin G_2$, a contradiction. Therefore, $\mathrm{cl}_{X_2}(f(F_1)) = F_2$. Thus, $\mathrm{cl}_{X_2}(f(F_1)) \in I_2$. So, condition (ZO) is fulfilled. Now, using (f), we get that g is an open injection. All this shows that g is a dense embedding. □

Corollary 4.3.4.8. *Let X_1, X_2 be two zero-dimensional Hausdorff spaces and $f : X_1 \longrightarrow X_2$ be a continuous function. Then:*

(a) $\beta_0 f$ is quasi-open iff f is skeletal;

(b) $\beta_0 f$ is an open map iff f satisfies the following condition:

(ZOB) *For every $F \in CO(X_1)$, $\mathrm{cl}_{X_2}(f(F)) \in CO(X_2)$ holds;*

(c) $\beta_0 f$ is a surjection iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(d) $\beta_0 f$ is an injection iff for every $F \in CO(X_1)$ there exists $G \in CO(X_2)$ such that $F = f^{-1}(G)$.

Proof. (a) It follows from Theorem 4.3.4.7(a) and Corollary 2.5.2.5(b).

(b) Since $(\beta_0 X_i, \beta_0^{X_i}) = \beta^0_{X_i}(CO(X_i), CO(X_i))$ (see 4.3.4.4 and (4.24)), condition (ZO) in Theorem 4.3.4.7 transforms in condition (ZOB). Now all follows from Theorem 4.3.4.7(b).

(c) It follows from Theorem 4.3.4.7(d).

(d) Obviously, $f^{-1}(CO(X_2)) \subseteq CO(X_1)$. Thus, using the fact that $\beta_0 f$ is a closed map, we get, by Theorem 4.3.4.7(g), that $\beta_0 f$ is an injection iff $f^{-1}(CO(X_2)) \supseteq CO(X_1)$. □

Corollary 4.3.4.9. *Let X_1, X_2 be two zero-dimensional Hausdorff spaces, $f : X_1 \longrightarrow X_2$ be a continuous function, $\mathcal{B}$ be a Boolean base of X_2, (cX_2, c) be the Hausdorff zero-dimensional compactification of X_2 corresponding to $\mathcal{B}$ (see Corollary 4.3.2.7) and $g : \beta_0 X_1 \longrightarrow cX_2$ be a continuous function such that $g \circ \beta_0 = c \circ f$ (its existence and uniqueness are guaranteed by Theorem 4.3.4.5). Then:*

(a) g is quasi-open iff f is skeletal;

(b) g is an open map iff f satisfies the following condition:

(ZOC) *For every $F \in CO(X_1)$, $\mathrm{cl}_{X_2}(f(F)) \in \mathcal{B}$ holds;*

(c) g is a surjection iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(d) g is an injection iff for every $F \in CO(X_1)$ there exists $G \in \mathcal{B}$ such that $F = f^{-1}(G)$.

Proof. It is analogous to the proof of Corollary 4.3.4.8. □

Corollary 4.3.4.10. *Let, for $i = 1, 2$, X_i be a zero-dimensional Hausdorff space, (Y_i, f_i) be a zero-dimensional Hausdorff local compactification of X_i, $f : X_1 \longrightarrow X_2$ be a continuous function for which there exists a continuous function $g : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$. Then:*

(a) g is skeletal iff f is skeletal;

(b) g is an open map iff f satisfies the following condition:

(ZO') *If $F \in f_1^{-1}(KO(Y_1))$, then $\mathrm{cl}_{Y_2}(f_2(f(F))) \in KO(Y_2)$;*

(c) g is a perfect map iff f satisfies the following condition:

(ZP') *If $G \in f_2^{-1}(KO(Y_2))$, then $f^{-1}(G) \in f_1^{-1}(KO(Y_1))$;*

(d) $\mathrm{cl}_{Y_2}(g(Y_1)) = Y_2$ iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(e) g is an injection iff f satisfies the following condition:

(ZI') *For every $F_1, F_2 \in f_1^{-1}(KO(Y_1))$ such that $F_1 \cap F_2 = \emptyset$, there exist $G_1, G_2 \in f_2^{-1}(KO(Y_2))$ with $G_1 \cap G_2 = \emptyset$ and $f(F_i) \subseteq G_i$, $i = 1, 2$;*

(f) g is an open injection iff f satisfies condition (ZO') *and for every $F \in f_1^{-1}(KO(Y_1))$ there exists $G \in f_2^{-1}(KO(Y_2))$ such that $F = f^{-1}(G)$;*

(g) g is a closed injection iff $(F \in f_1^{-1}(KO(Y_1))) \iff (\exists G \in f_2^{-1}(KO(Y_2))$ such that $F = f^{-1}(G))$;

(h) g is a perfect surjection iff f satisfies condition (ZP') *and $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;*

(i) g is a dense embedding iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$ and for every $F \in f_1^{-1}(KO(Y_1))$ there exists $G \in f_2^{-1}(KO(Y_2))$ such that $F = f^{-1}(G)$.

4.4 A Whiteheadian-type description of Euclidean spaces, spheres, tori and Tychonoff cubes

4.4.1 Introduction

A description of the dual object of the real line under the localic duality (i.e., a description of the frame (or locale) determined by the topology of the real line) without the help of the real line was given by Fourman and Hyland [56] (see, also, Grayson [66] and Johnstone [75, IV.1.1-IV.1.3]), assuming the set of rationals as given.

In this section we construct directly the dual objects of Euclidean spaces, spheres, tori and Tychonoff cubes under the de Vries duality [24] and the duality described in our Theorem 2.2.2.12, i.e., we construct the complete LC-algebras isomorphic to the Roeper triples of these spaces without the help of the corresponding spaces, assuming only the set of natural numbers as given. Let us note explicitly that, as it follows from the results of de Vries [24], Roeper [99] and our Theorem 2.2.2.12, the Euclidean spaces, spheres, tori and Tychonoff cubes can be completely reconstructed as topological spaces from the algebraical objects which we will describe in this section. Therefore, our results can be regarded as a mathematical realization of the original philosophical ideas of Whitehead [121, 123] and de Laguna [23] about Euclidean spaces; this realization is in accordance with the Grzegorczyk's [67] and Roeper's [99] mathematical interpretations of these ideas.

The exposition of this section is based on the paper [34].

4.4.2 A Whiteheadian-type description of Euclidean spaces

Notation 4.4.2.1. We will denote by $\mathbb{Z}$ the set of all integers with the natural order, by $\mathbb{I}'$ – the open interval $(0,1)$ with its natural topology, by $\mathbb{N}$ the set of natural numbers, and by $\mathbb{J}$ the subspace of the real line consisting of all irrational numbers. We set

$$\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\},\ \mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{N} \text{ and } \ \mathbb{J}_2 = \mathbb{I}' \setminus \mathbb{D}.$$

If $(X, <)$ is a linearly ordered set and $x \in X$, then we set

$$succ(x) = \{y \in X \mid x < y\},\ pred(x) = \{y \in X \mid y < x\};$$

also, we denote by x^+ the *successor of* x (when it exists) and by x^- – the *predecessor of* x (when it exists). If M is a set, then we will denote by $P(M)$ the power set Boolean algebra of M.

Now we will construct a CLCA $(\tilde{A}, \tilde{\sigma}, \widetilde{\mathbb{B}})$ and we will show that it is LCA-isomorphic to $\Psi^t(\mathbb{R})$ (see (1.20)).

4.4.2.2. The construction of $(\tilde{A}, \tilde{\sigma}, \widetilde{\mathbb{B}})$**.** Let

$$A_i = P(\mathbb{Z}_0),$$

for every $i \in \mathbb{N}^+$. Thus, if $i \in \mathbb{N}^+$ and $a_i \in A_i$, t hen a_i is a subset of $\mathbb{Z}_0$ and its cardinality will be denoted by $|a_i|$. Let

$$(A, (\varphi_i)_{i\in\mathbb{N}^+})$$

be the sum of Boolean algebras $\{A_i \mid i \in \mathbb{N}^+\}$; then, by Proposition 2.3.3.1, for every $i \in \mathbb{N}^+$,

$$\varphi_i : A_i \longrightarrow A$$

is a monomorphism, the family $\{\varphi_i(A_i) \mid i \in \mathbb{N}^+\}$ is an independent family and the set $\bigcup_{i\in\mathbb{N}^+} \varphi_i(A_i)$ generates A. Let

$$\tilde{A} \text{ be the completion of } A.$$

We can suppose, without loss of generality, that

$$A \subseteq \tilde{A}.$$

The following subset of A will be important for us:

(4.32) $B_0 = \{\varphi_1(a_1) \wedge \ldots \wedge \varphi_k(a_k) \mid k \in \mathbb{N}^+, (\forall i = 1, \ldots, k)(a_i \in A_i \text{ and } |a_i| = 1)\}$.

If $b \in B_0$ and $b = \varphi_1(a_1) \wedge \ldots \wedge \varphi_k(a_k)$, where $a_k = \{p\}$, then we set

(4.33) $b_- = \varphi_1(a_1) \wedge \varphi_2(a_2) \wedge \ldots \wedge \varphi_{k-1}(a_{k-1}) \wedge \varphi_k(\{p^-\})$.

For every $b \in B_0$, where $b = \varphi_1(a_1) \wedge \ldots \wedge \varphi_k(a_k)$, and every $n \in \mathbb{N}^+$, we set

(4.34) $q_{bn} = (b_- \wedge \varphi_{k+1}(succ(n))) \vee (b \wedge \varphi_{k+1}(pred(-n)))$.

Now we set

(4.35) $B_1 = \{q_{bn} \mid b \in B_0, n \in \mathbb{N}^+\}$.

Let

$$\widetilde{\mathbb{B}}$$

be the ideal of $\tilde{A}$ generated by the set $B_0 \cup B_1$. Now, we will define a relation $\tilde{\sigma}$ on $\tilde{A}$. It will be, by definition, a symmetric relation.

Let $r, r' \in \mathbb{N}^+$, $b, b' \in B_0$, $b = \varphi_1(a_1) \wedge \ldots \wedge \varphi_k(a_k)$, $b' = \varphi_1(a'_1) \wedge \ldots \wedge \varphi_l(a'_l)$ and $a_k = \{n\}$, $a'_k = \{m\}$. We can suppose, without loss of generality, that $k \leq l$. If $k < l$, then let $a'_{k+1} = \{p\}$. Now we set

(4.36) $b\tilde{\sigma}b' \Leftrightarrow [(a_i = a'_i,\ \forall i \in \{1, \ldots, k-1\})$

$$\text{and } (\begin{cases} m \in \{n^-, n, n^+\}, & \text{if } k = l \\ m = n, & \text{if } k < l \end{cases})],$$

and

(4.37) $q_{br}\tilde{\sigma}q_{b'r'} \Leftrightarrow [(a_i = a'_i,\ \forall i \in \{1, \ldots, k-1\})$ and

$$(\begin{cases} m = n, & \text{if } l = k \\ (m = n \text{ and } p \leq -r) \text{ or } (m = n^- \text{ and } p > r), & \text{if } l = k+1 \\ (m = n \text{ and } p < -r) \text{ or } (m = n^- \text{ and } p > r), & \text{if } l > k+1 \end{cases})].$$

Let $r \in \mathbb{N}^+$, $b, b' \in B_0$, $b = \varphi_1(a_1) \wedge \ldots \wedge \varphi_k(a_k)$, $b' = \varphi_1(a'_1) \wedge \ldots \wedge \varphi_l(a'_l)$ and $a_k = \{n\}$, $a'_k = \{m\}$. If $k < l$, then let $a'_{k+1} = \{p\}$. Now, if $k > l$, we set

(4.38) $q_{br}\tilde{\sigma}b' \Leftrightarrow (a_i = a'_i,\ \forall i \in \{1, \ldots, l\})$;

if $k \leq l$, we set

(4.39) $q_{br}\tilde{\sigma}b' \Leftrightarrow [(a_i = a'_i,\ \forall i \in \{1, \ldots, k-1\})$ and

$$(\begin{cases} m \in \{n^-, n\}, & \text{if } l = k \\ (p \geq r \text{ and } m = n^-) \text{ or } (p \leq -r \text{ and } m = n), & \text{if } l = k+1 \\ (p > r \text{ and } m = n^-) \text{ or } (p < -r \text{ and } m = n), & \text{if } l > k+1 \end{cases})].$$

Further, for every two elements c and d of $\widetilde{\mathbb{B}}$, set

(4.40) $c(-\tilde{\sigma})d \Leftrightarrow (\exists k, l \in \mathbb{N}^+$ and $\exists c_1, \ldots, c_k, d_1, \ldots, d_l \in B_0 \cup B_1$ such that

$$c \leq \bigvee_{i=1}^{k} c_i, \ d \leq \bigvee_{j=1}^{l} d_j \text{ and } c_i(-\tilde{\sigma})d_j, \ \forall i = 1, \ldots, k \text{ and } \forall j = 1, \ldots, l).$$

Finally, for every two elements a and b of $\tilde{A}$, set

(4.41) $a\tilde{\sigma}b \Leftrightarrow (\exists c, d \in \widetilde{\mathbb{B}}$ such that $c \leq a, \ d \leq b$ and $c\tilde{\sigma}d)$.

Theorem 4.4.2.3. *The triple $(\tilde{A}, \tilde{\sigma}, \widetilde{\mathbb{B}})$, constructed in 4.4.2.2, is a CLCA; it is LCA-isomorphic to the CLCA $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$. Thus, the triple $(\tilde{A}, \tilde{\sigma}, \widetilde{\mathbb{B}})$ completely determines the real line $\mathbb{R}$ with its natural topology.*

Proof. In this proof, we will use the notation introduced in 4.4.2.2.

Let $\mathbb{Z}_0$ be endowed with the discrete topology. Then $RC(\mathbb{Z}_0) = P(\mathbb{Z}_0)$ and Proposition 2.3.3.2 shows that the algebra $\tilde{A}$, constructed in 4.4.2.2, is isomorphic to $RC(\mathbb{Z}_0^{\mathbb{N}^+})$. Since the space $\mathbb{Z}_0^{\mathbb{N}^+}$ is homeomorphic to $\mathbb{J}$ (see, e.g., [53]), we get, by Lemma 0.4.2.2, that $\tilde{A}$ is isomorphic to $RC(\mathbb{R})$. Clearly, $RC(\mathbb{J})$ can be endowed with an LCA-structure LCA-isomorphic to the LCA $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$. Then, using the homeomorphism between $\mathbb{J}$ and $\mathbb{Z}_0^{\mathbb{N}^+}$, we can transfer this structure to $RC(\mathbb{Z}_0^{\mathbb{N}^+})$ and, hence, to $\tilde{A}$. For technical reasons, this plan will be slightly modified. We will use the homeomorphism between $\mathbb{Z}_0^{\mathbb{N}^+}$ and $\mathbb{J}_2$ described in [4]. Since $\mathbb{J}_2$ is dense in the open interval $\mathbb{I}'$, and $\mathbb{I}'$ is homeomorphic to $\mathbb{R}$, we can use $\mathbb{J}_2$ instead of $\mathbb{J}$ for realizing the desired transfer. So, we start with the description (given by P. S. Alexandroff [4]) of the homeomorphism

$$f : \mathbb{Z}_0^{\mathbb{N}^+} \longrightarrow \mathbb{J}_2.$$

Let, for every $j \in \mathbb{N}^+$, $\Delta_j = [1 - \frac{1}{2^j}, 1 - \frac{1}{2^{j+1}}]$ and let, for every $j \in \mathbb{Z}^-$, $\Delta_j = [2^{j-1}, 2^j]$. Set $\delta_1 = \{\Delta_j \mid j \in \mathbb{Z}_0\}$. Further, for every $\Delta_j \in \delta_1$, where $\Delta_j = [a_j, b_j]$, set $d_j = b_j - a_j$ and $\Delta_{jk} = [b_j - \frac{d_j}{2^k}, b_j - \frac{d_j}{2^{k+1}}]$ when $k \in \mathbb{N}^+$, $\Delta_{jk} = [a_j + d_j.2^{k-1}, a_j + d_j.2^k]$ when $k \in \mathbb{Z}^-$. Let $\delta_2 = \{\Delta_{jk} \mid j, k \in \mathbb{Z}_0\}$. In the next step we construct analogously the family δ_3, and so on. Set

$$\delta = \bigcup\{\delta_i \mid i \in \mathbb{N}^+\}.$$

It is easy to see that the set of all end-points of the elements of the family δ coincides with the set $\mathbb{D}$. Now we define the function $f : \mathbb{Z}_0^{\mathbb{N}^+} \longrightarrow \mathbb{J}_2$ by the formula

$$f(n_1, n_2, \ldots, n_k, \ldots) = \Delta_{n_1} \cap \Delta_{n_1 n_2} \cap \ldots \cap \Delta_{n_1 n_2 \ldots n_k} \cap \ldots.$$

One can prove that the definition of f is correct and that f is a homeomorphism. Set $X_i = \mathbb{Z}_0$, for every $i \in \mathbb{N}^+$. Let $X = \prod\{X_i \mid i \in \mathbb{N}^+\}$ and let

$$\pi_i : X \longrightarrow X_i,$$

where $i \in \mathbb{N}^+$, be the projection. Then, for every $k \in \mathbb{N}^+$ and every $n_i \in X_i$, where $i = 1, \ldots, k$, we have that (writing, for short, "$\pi_i^{-1}(n_i)$" instead of "$\pi_i^{-1}(\{n_i\})$")

(4.42) $f(\bigcap_{i=1}^{k} \pi_i^{-1}(n_i)) = \Delta_{n_1 n_2 \ldots n_k} \cap \mathbb{J}_2.$

Let

$$\psi_i : RC(X_i) \longrightarrow RC(X), \quad F \mapsto \pi_i^{-1}(F),$$

where $i \in \mathbb{N}^+$; then, as we have seen in the proof of Proposition 2.3.3.2, ψ_i is a complete monomorphism. Set

$$A'_i = \psi_i(RC(X_i)).$$

Since X_i is a discrete space, we have that $A_i = RC(X_i)$ and $A'_i \subseteq CO(X)$, for all $i \in \mathbb{N}^+$. Thus, for the elements of the subset $\bigcup_{i \in \mathbb{N}^+} A'_i$ of $RC(X)$, the Boolean operation "meet in $RC(X)$" coincides with the set-theoretic operation "intersection" between the subsets of X, and the same for the Boolean complement in $RC(X)$ and the set-theoretic complement in X. We also have that the Boolean algebras A_i and A'_i are isomorphic. Let

$$A'$$

be the subalgebra of $P(X)$ generated by $\bigcup_{i \in \mathbb{N}^+} A'_i$. Then A' is isomorphic to A. Note that A' is a subalgebra of $CO(X)$. Also, A' is a dense subalgebra of $RC(X)$; therefore, $RC(X)$ is the completion of A'. Thus, $\tilde{A}$ is isomorphic to $RC(X)$. So, without loss of generality, we can think that $\tilde{A}$ is $RC(X)$, A is A', $\varphi_i = \psi_i$ and hence $\varphi_i(A_i)$ is A'_i, for $i \in \mathbb{N}^+$. We will now construct an LCA $(RC(X), \sigma, \mathbb{B})$ LCA-isomorphic to $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$. Then, identifying $RC(X)$ with $\tilde{A}$, we will show that $\sigma = \tilde{\sigma}$ and $\mathbb{B} = \widetilde{\mathbb{B}}$.

Let

$$\mathbb{B}_2 = \{M \in RC(\mathbb{J}_2) \mid \mathrm{cl}_{\mathbb{I}'}(M) \text{ is compact}\}.$$

For every two elements M and N of $RC(\mathbb{J}_2)$, set

$$M \rho_2 N \Leftrightarrow \mathrm{cl}_{\mathbb{I}'}(M) \cap \mathrm{cl}_{\mathbb{I}'}(N) \neq \emptyset.$$

Then, using Lemma 0.4.2.2, we get that the triple $(RC(\mathbb{J}_2), \rho_2, \mathbb{B}_2)$ is LCA-isomorphic to the LCA $(RC(\mathbb{I}'), \rho_{\mathbb{I}'}, CR(\mathbb{I}'))$ (which, in turn, is LCA-isomorphic to the LCA $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$). Now, for every two elements $F, G \in RC(X)$, we set

(4.43) $F\sigma G \Leftrightarrow f(F)\rho_2 f(G)$.

Also, we put

(4.44) $\mathbb{B} = \{f^{-1}(M) \mid M \in \mathbb{B}_2\}$.

Obviously, $(RC(X), \sigma, \mathbb{B})$ is LCA-isomorphic to $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$. In the rest of this proof, we will show that the definitions of $\mathbb{B}$ and σ given above agree with the corresponding definitions of $\widetilde{\mathbb{B}}$ and $\tilde{\sigma}$ given in 4.4.2.2.

Note first that the subset B'_0 of A', which corresponds to the subset B_0 of A described in 4.4.2.2, is the following:

(4.45) $B'_0 = \{\bigcap_{i=1}^{k} \pi_i^{-1}(n_i) \mid k \in \mathbb{N}^+, (\forall i = 1, \dots, k)(n_i \in X_i)\}$.

Let $F, G \in B'_0$ and $F = \bigcap_{i=1}^{k} \pi_i^{-1}(n_i)$, $G = \bigcap_{i=1}^{l} \pi_i^{-1}(m_i)$. We can suppose, without loss of generality, that $k \leq l$. Then, by (4.42) and Lemma 0.4.2.2, $\mathrm{cl}_{\mathbb{I}'}(f(F)) = \Delta_{n_1 n_2 \dots n_k}$ and $\mathrm{cl}_{\mathbb{I}'}(f(G)) = \Delta_{m_1 m_2 \dots m_l}$. If $k = l$, then, clearly, $\Delta_{n_1 n_2 \dots n_k} \cap \Delta_{m_1 m_2 \dots m_k} \neq \emptyset$ iff ($n_i = m_i$, for all $i = 1, \dots, k-1$, and $m_k \in \{n_k^-, n_k, n_k^+\}$). If $k < l$, then, obviously, $\Delta_{n_1 n_2 \dots n_k} \cap \Delta_{m_1 m_2 \dots m_l} \neq \emptyset$ iff ($n_i = m_i$, for all $i = 1, \dots, k$). Then, using (4.43) and the formula (4.36), we get that σ and $\tilde{\sigma}$ agree on B'_0 (or, equivalently, on B_0).

Let $F \in B'_0$, $F = \bigcap_{i=1}^{k} \pi_i^{-1}(n_i)$ and $n \in \mathbb{N}^+$. Then the element Q_{Fn} of A' corresponding to the element q_{bn} of A, where $b \in B_0$ corresponds to F, is the following:

$$Q_{Fn} = [(\bigcap_{i=1}^{k-1} \pi_i^{-1}(n_i)) \cap \pi_k^{-1}(n_k^-) \cap \pi_{k+1}^{-1}(succ(n))] \cup [F \cap \pi_{k+1}^{-1}(pred(-n))].$$

Clearly,

(4.46) $$Q_{Fn} = [\bigcup_{s \in succ(n)} (\bigcap_{i=1}^{k-1} \pi_i^{-1}(n_i) \cap \pi_k^{-1}(n_k^-) \cap \pi_{k+1}^{-1}(s))] \cup$$

$$[\bigcup_{s \in pred(-n)} (\bigcap_{i=1}^{k} \pi_i^{-1}(n_i) \cap \pi_{k+1}^{-1}(s))].$$

(It is easy to see, as well, that in the formula (4.46) the sign of the union can be replaced everywhere with the sign of the join in $RC(X)$.) Thus,

(4.47) $f(Q_{Fn}) = [(\bigcup_{s\in succ(n)} \Delta_{n_1n_2\dots n_{k-1}n_k^- s}) \cup (\bigcup_{s\in pred(-n)} \Delta_{n_1n_2\dots n_k s})] \cap \mathbb{J}_2.$

Let d be the left end-point of the closed interval $\Delta_{n_1n_2\dots n_k}$. Then it is easy to see that

(4.48) $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fn})) = [d-\varepsilon_n, d+\varepsilon'_n]$,

where ε_n and ε'_n depend from n and also from $n_1,\dots,n_k$ (for simplicity, we don't reflect this dependence on the notation), but for fixed $n_1,\dots,n_k$, we have that $\varepsilon_n > \varepsilon_{n+1} > 0$, $\varepsilon'_n > \varepsilon'_{n+1} > 0$, for all $n \in \mathbb{N}^+$, and $\lim_{n\to\infty} \varepsilon_n = 0$, $\lim_{n\to\infty} \varepsilon'_n = 0$; also, the closed interval $[d-\varepsilon_n, d+\varepsilon'_n]$ lies in the open interval having as end-points the middles of the closed intervals $\Delta_{n_1n_2\dots n_{k-1}n_k^-}$ and $\Delta_{n_1n_2\dots n_k}$. Since the family $\{D\cap\mathbb{J}_2 \mid D\in\delta\}$ is a base of $\mathbb{J}_2$ and every element of $\mathbb{D}$ appears as a left end-point of some element of the family δ, we get that the family

$$\mathcal{B} = \{\mathrm{int}_{\mathbb{I}'}(\mathrm{cl}_{\mathbb{I}'}((f(F))), \mathrm{int}_{\mathbb{I}'}(\mathrm{cl}_{\mathbb{I}'}((f(Q_{Fn}))) \mid n\in\mathbb{N}^+, F\in B'_0\}$$

is a base of $\mathbb{I}'$. Also, if

$$\mathbf{B} = \{\mathrm{cl}_{\mathbb{I}'}((f(F)), \mathrm{cl}_{\mathbb{I}'}((f(Q_{Fn})) \mid n\in\mathbb{N}^+, F\in B'_0\},$$

then $\mathbf{B} = \{\mathrm{cl}_{\mathbb{I}'}(U) \mid U\in\mathcal{B}\}$ and $\mathbf{B}\subseteq CR(\mathbb{I}')$. Hence, $\mathbf{B}$ generates the ideal $CR(\mathbb{I}')$ of $RC(\mathbb{I}')$. Clearly, the family

(4.49) $B'_1 = \{Q_{Fn} \mid F\in B'_0, n\in\mathbb{N}^+\}$

corresponds to the subset B_1 of A constructed in 4.4.2.2. Since $\mathbf{B} = \{\mathrm{cl}_{\mathbb{I}'}(G) \mid G\in f(B'_0\cup B'_1)\}$, we get that the subset $f(B'_0\cup B'_1)$ of $RC(\mathbb{J}_2)$ generates the ideal $\mathbb{B}_2$ of $RC(\mathbb{J}_2)$. Thus, the subset $B'_0\cup B'_1$ of $RC(X)$ generates the ideal $\mathbb{B}$ of $RC(X)$. Therefore, $\mathbb{B}$ corresponds to $\widetilde{\mathbb{B}}$; we can even write that $\mathbb{B} = \widetilde{\mathbb{B}}$.

Let now $r, r'\in\mathbb{N}^+$, $F, F'\in B'_0$, $F = \pi_1^{-1}(n_1)\cap\ldots\cap\pi_k^{-1}(n_k)$ and $F' = \pi_1^{-1}(n'_1)\cap\ldots\cap\pi_l^{-1}(n'_l)$. We can suppose, without loss of generality, that $k\le l$. Let d and d' be the left end-points of the closed intervals $\Delta_{n_1n_2\dots n_k}$ and $\Delta_{n'_1n'_2\dots n'_l}$, respectively. Then, using (4.48), we get that $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr})) = [d-\varepsilon_r, d+\varepsilon'_r]$ and $\mathrm{cl}_{\mathbb{I}'}(f(Q_{F'r'})) = [d'-\varepsilon_{r'}, d'+\varepsilon'_{r'}]$. If $k = l$, then it is easy to see that $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr}))\cap\mathrm{cl}_{\mathbb{I}'}(f(Q_{F'r'}))\neq\emptyset$ iff ($n_i = n'_i$, for all $i = 1,\dots,k$). If $l = k+1$, then one readily checks that $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr}))\cap\mathrm{cl}_{\mathbb{I}'}(f(Q_{F'r'}))\neq\emptyset$ iff [($n_i = n'_i$, for all $i = 1,\dots,k-1$) and (($n_k = n'_k$ and $n'_{k+1}\le -r$) or ($n'_k = (n_k)^-$ and

$n'_{k+1} > r))]$. Finally, if $l > k+1$, then $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr})) \cap \mathrm{cl}_{\mathbb{I}'}(f(Q_{F'r'})) \neq \emptyset$ iff $[(n_i = n'_i$, for all $i = 1, \ldots, k-1)$ and $((n_k = n'_k$ and $n'_{k+1} < -r)$ or $(n'_k = (n_k)^-$ and $n'_{k+1} > r))]$. All this shows that the relations σ and $\tilde{\sigma}$ agree on B'_1 (or, equivalently, on B_1).

Let $r \in \mathbb{N}^+$, $F, F' \in B'_0$, $F = \pi_1^{-1}(n_1) \cap \ldots \cap \pi_k^{-1}(n_k)$ and $F' = \pi_1^{-1}(n'_1) \cap \ldots \cap \pi_l^{-1}(n'_l)$. If $l < k$, then we get that $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr})) \cap \mathrm{cl}_{\mathbb{I}'}(f(F')) \neq \emptyset$ iff $(n_i = n'_i$, for all $i = 1, \ldots, l)$. If $l = k$, then $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr})) \cap \mathrm{cl}_{\mathbb{I}'}(f(F')) \neq \emptyset$ iff $(n_i = n'_i$, for all $i = 1, \ldots, k-1$, and $n'_k \in \{n_k^-, n_k\})$. If $l = k+1$, then $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr})) \cap \mathrm{cl}_{\mathbb{I}'}(f(F')) \neq \emptyset$ iff $[(n_i = n'_i$, for all $i = 1, \ldots, k-1)$, and $((n'_k = n_k^-$ and $n'_{k+1} \geq r)$ or $(n'_k = n_k$ and $n'_{k+1} \leq -r))]$. Finally, if $l > k+1$, then $\mathrm{cl}_{\mathbb{I}'}(f(Q_{Fr})) \cap \mathrm{cl}_{\mathbb{I}'}(f(F')) \neq \emptyset$ iff $[(n_i = n'_i$, for all $i = 1, \ldots, k-1)$, and $((n'_k = n_k^-$ and $n'_{k+1} > r)$ or $(n'_k = n_k$ and $n'_{k+1} < -r))]$. We get that the relations σ and $\tilde{\sigma}$ agree on $B'_0 \cup B'_1$ (or, equivalently, on $B_0 \cup B_1$).

Now, using the facts that $\mathcal{B}$ is a base of $\mathbb{I}'$, $\mathbb{I}'$ is a regular space, and $\mathrm{cl}_{\mathbb{I}'}(f(F))$ is a compact set for all $F \in \mathbb{B}$, we get that for all $F, G \in \mathbb{B}$, $\mathrm{cl}_{\mathbb{I}'}(f(F)) \cap \mathrm{cl}_{\mathbb{I}'}(f(G)) = \emptyset$ iff (there exist $F_1, \ldots, F_k, G_1, \ldots, G_l \in B'_0 \cup B'_1$ such that $F \subseteq \bigcup_{i=1}^k F_i$, $G \subseteq \bigcup_{j=1}^l G_j$ and $\mathrm{cl}_{\mathbb{I}'}(f(F_i)) \cap \mathrm{cl}_{\mathbb{I}'}(f(G_j)) = \emptyset$ for all $i = 1, \ldots, k$ and all $j = 1, \ldots, l$). This shows that the relations σ and $\tilde{\sigma}$ agree on $\mathbb{B}$ (or, equivalently, on $\widetilde{\mathbb{B}}$).

Finally, as in every LCA, for every $F, G \in RC(X)$, we have that $F\sigma G$ iff (there exist $F', G' \in \mathbb{B}$ such that $F' \subseteq F$, $G' \subseteq G$ and $F'\sigma G'$). Therefore, the relations σ and $\tilde{\sigma}$ agree on $RC(X)$ (or, equivalently, on $\tilde{A}$). $\square$

Theorem 4.4.2.4. *For every $n \in \mathbb{N}^+$, the CLCA $(RC(\mathbb{R}^n), \rho_{\mathbb{R}^n}, CR(\mathbb{R}^n))$ $(= \Psi^t(\mathbb{R}^n))$ is LCA-isomorphic to the* **DHLC***-sum $(\tilde{A}_n, \tilde{\sigma}_n, \widetilde{\mathbb{B}}_n)$ of n copies of the CLCA $(\tilde{A}, \tilde{\sigma}, \widetilde{\mathbb{B}})$, constructed in 4.4.2.2; thus, the CLCA $(\tilde{A}_n, \tilde{\sigma}_n, \widetilde{\mathbb{B}}_n)$ completely determines the Euclidean space $\mathbb{R}^n$ with its natural topology. For every $n \in \mathbb{N}^+$, the Boolean algebras $\tilde{A}_n$ and $\tilde{A}$ are isomorphic.*

Proof. Since $\mathbb{J}^n$ is homeomorphic to $\mathbb{J}$ and is dense in $\mathbb{R}^n$, we get that $RC(\mathbb{R}^n)$ is isomorphic to $RC(\mathbb{J})$, and thus, to $\tilde{A}$ (see 4.4.2.2 and the proof of Theorem 4.4.2.3). Now all follows from Theorems 4.4.2.3 and 2.3.3.5. $\square$

We will now present the description of the CLCA $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$ in two new forms; the notation used in them permits to obtain a more compact form of the definitions of the corresponding relations. As we have already mentioned, $RC(\mathbb{R})$ is isomorphic to $RC(\mathbb{J})$, i.e., to $RC(\mathbb{Z}_0^{\mathbb{N}^+})$ or, equivalently, to $RC(\omega^\omega)$. The last algebra, which is one of the collapsing algebras $RC(k^\omega)$ (where k is an infinite cardinal equipped with the discrete topology), has many abstract descriptions. The one, which is the most appropriate for our purposes, is the following: *a complete Boolean algebra C is isomorphic to the Boolean algebra $RC(k^\omega)$ iff it has a dense subset isomorphic to T^*, for the normal tree*

$$T = \biguplus\{k^n \mid n \in \mathbb{N}^+\}$$

(here

$$T^*$$

is the tree T with the opposite partial order and $k^n \cap k^m = \emptyset$ for $n \neq m$) (see, e.g., [77, 14.16(a),(b)]). (Recall that a partially ordered set $(T, \leq_T)$ is called a *tree* if for every $t \in T$, the set $pred(t)$ is well-ordered by $\leq_T$.) This shows that $RC(k^\omega)$ is isomorphic to the Boolean algebra $RC(T^*)$, where the ordered set T^* is endowed with the *left topology*, i.e., that one generated by the base $\{L_{T^*}(t) \mid t \in T\}$ (here

$$L_{T^*}(t) = \{t' \in T \mid t' \leq_{T^*} t\} = \{t' \in T \mid t \leq_T t'\},$$

for every $t \in T$) (see, e.g., [77, 4.11-4.16] and [53, 1.7.2]).

Let us add some details and introduce some notation.

Notation 4.4.2.5. For any $n \in \mathbb{N}^+$, we set

$$\underline{n} = \{1, \ldots, n\}.$$

We set

$$T_0 = \biguplus\{\mathbb{Z}_0^n \mid n \in \mathbb{N}^+\},$$

i.e., $\mathbb{Z}_0^n \cap \mathbb{Z}_0^m = \emptyset$ for $n \neq m$. Any element $t \in \mathbb{Z}_0^n$ is interpreted, as usual, as a function

$$t : \underline{n} \longrightarrow \mathbb{Z}_0.$$

Further, we let

$$\perp \subseteq t \text{ and } \perp \neq t, \text{ for any } t \in T_0;$$

if $n, n' \in \mathbb{N}^+$, $t \in \mathbb{Z}_0^n$ and $t' \in \mathbb{Z}_0^{n'}$, then we set

$$t \subseteq t' \text{ iff } t' \text{ is an extension of } t, \text{ i.e., iff } n \leq n' \text{ and } t(i) = t'(i) \text{ for any } i \in \underline{n}.$$

Then the ordered set $(T_0 \cup \{\perp\}, \subseteq)$ is a normal tree of height ω with $\mathbb{Z}_0^n$ as its nth level; thus we set

$$L_n = \mathbb{Z}_0^n.$$

We also put, for any $t, t' \in T_0 \cup \{\perp\}$,

$$t \leq t' \Leftrightarrow t' \subseteq t.$$

We set

$$T_0^* = (T_0 \cup \{\perp\}, \leq).$$

Let T_0^* be endowed with its left topology (i.e., let $(T_0 \cup \{\perp\}, \subseteq)$ be equipped with its right topology (which is defined analogously to the left topology (see [53, 1.7.2]))). Further, for any $t \in T_0 \cup \{\perp\}$, put

$$c_t = \{t' \in T_0 \mid t \text{ and } t' \text{ are } T_0^*\text{-compatible}\}.$$

(Recall that two elements x and y of a partially ordered set $(M, \preceq)$ are *compatible* if there is some $z \in M$ such that $z \preceq x$ and $z \preceq y$.) Then, as it is well known (see, e.g., [77, 4.13,4.16,the formula for $\mathrm{cl}(u_p)$ in the proof of 4.16]), the embedding e of the partially ordered set T_0^* into the Boolean algebra $RC(T_0^*)$ is given by the formula

$$e(t) = c_t, \ \forall t \in T_0 \cup \{\perp\}.$$

(Note that the map e is an embedding because T_0^* is a *separative partial order* (see, e.g., [77, 4.15,4.16,p.226]).) Also, let us recall that the left topology on $T_0 \cup \{\perp\}$ induced by the ordered set T_0^* is an *Alexandroff topology*, i.e., the union of arbitrarily many closed sets is a closed set (see, e.g., [53, 1.7.2]). Thus, the (finite or infinite) joins $\bigvee\{F_j \mid j \in J\}$ in $RC(T_0^*)$ are just the unions $\bigcup\{F_j \mid j \in J\}$.

Finally, for every $n \in \mathbb{N}^+ \setminus \{1\}$ and every $t \in L_n$ (i.e., $t : \underline{n} \longrightarrow \mathbb{Z}_0$), define

(4.50) $t_\lambda : \underline{n} \longrightarrow \mathbb{Z}_0$ by the formulas $(t_\lambda)_{| \ \underline{n-1}} = t_{| \ \underline{n-1}}$ and $t_\lambda(n) = (t(n))^-$;

let, for $t \in L_1$, $t_\lambda : \underline{1} \longrightarrow \mathbb{Z}_0$ be defined by $t_\lambda(1) = (t(1))^-$.

4.4.2.6. As we have already mentioned, the Boolean algebra $RC(\mathbb{Z}_0^{\mathbb{N}^+})$ is isomorphic to the Boolean algebra $RC(T_0^*)$ (see, e.g., [77, 14.16(a),(b),4.11-4.16]). We will recall the proof of this fact since we will use it later. For every $t \in T_0$, set

(4.51) $a_t = \{x \in \mathbb{Z}_0^{\mathbb{N}^+} \mid t \subseteq x\}$.

Note that if $t : \underline{n} \longrightarrow \mathbb{Z}_0$, where $n \in \mathbb{N}^+$, then

(4.52) $a_t = \displaystyle\bigcap_{i=1}^{n} \pi_i^{-1}(t(i))$

and thus a_t is a clopen subset of $\mathbb{Z}_0^{\mathbb{N}^+}$. Set

(4.53) $S = \{a_t \mid t \in T_0\} \cup \mathbb{Z}_0^{\mathbb{N}^+}$.

Then $S \subseteq CO(\mathbb{Z}_0^{\mathbb{N}^+}) \subseteq RC(\mathbb{Z}_0^{\mathbb{N}^+})$. Now it is easy to see that the set S is dense in $RC(\mathbb{Z}_0^{\mathbb{N}^+})$ and isomorphic to T_0^* (indeed, the map

(4.54) $s : T_0^* \longrightarrow S,$ where $s(\perp) = \mathbb{Z}_0^{\mathbb{N}^+}$ and $s(t) = a_t, \forall t \in T_0$

is an isomorphism). Therefore, $RC(\mathbb{Z}_0^{\mathbb{N}^+})$ is isomorphic to the Boolean algebra $RC(T_0^*)$.

We will now equip the Boolean algebra $RC(T_0^*)$ defined above with an LCA-structure $(RC(T_0^*), \theta, \mathbb{B}_T)$ and will prove that the obtained CLCA is LCA-isomorphic to the CLCA $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$. Recall that two elements x and y of a partially ordered set $(M, \preccurlyeq)$ are *comparable* if $x \preccurlyeq y$ or $y \preccurlyeq x$.

4.4.2.7. The construction of the triple $(RC(T_0^*), \theta, \mathbb{B}_T)$.

For every $k, n \in \mathbb{N}^+$ and for every $t \in L_k$ (recall that $L_k = \mathbb{Z}_0^k$), set

$$d_{tn} = \bigcup \{c_{t'} \mid (t' \in L_{k+1}) \ \& \ [(t_\lambda \subseteq t' \ \& \ t'(k+1) > n) \text{ or } (t \subseteq t' \ \& \ t'(k+1) < -n)]\}.$$

Note that the fact that the left topology on T_0^* is an Alexandroff topology implies that

(4.55) $d_{tn} =$

$$\bigvee \{c_{t'} \mid (t' \in L_{k+1}) \ \& \ [(t_\lambda \subseteq t' \ \text{ and } \ t'(k+1) > n) \text{ or } (t \subseteq t' \ \text{ and } \ t'(k+1) < -n)]\}.$$

Let

(4.56) $C_0 = \{c_t \mid t \in T_0\}$ and $C_1 = \{d_{tn} \mid t \in T_0, \ n \in \mathbb{N}^+\}$.

Denote by

$$\mathbb{B}_{T_0}$$

the ideal of $RC(T_0^*)$ generated by $C_0 \cup C_1$.

For every $k, k', n, n' \in \mathbb{N}^+$ and every $t \in L_k$, $t' \in L_{k'}$, set

(4.57) $c_t \theta c_{t'} \Leftrightarrow \begin{cases} t = t' \text{ or } t = t'_\lambda \text{ or } t' = t_\lambda, & \text{if } k = k' \\ t \text{ and } t' \text{ are comparable}, & \text{if } k \neq k', \end{cases}$

and

(4.58) $d_{tn} \theta d_{t'n'} \Leftrightarrow$

$$\begin{cases} (t' \subseteq t \text{ and } t(k'+1) < -n') \text{ or } (t'_\lambda \subseteq t \text{ and } t(k'+1) > n'), & \text{if } k > k'+1 \\ (t' \subseteq t \text{ and } t(k) \leq -n') \text{ or } (t'_\lambda \subseteq t \text{ and } t(k) > n'), & \text{if } k = k'+1 \\ t = t', & \text{if } k = k' \\ (t \subseteq t' \text{ and } t'(k') \leq -n) \text{ or } (t_\lambda \subseteq t' \text{ and } t'(k') > n), & \text{if } k = k'-1 \\ (t \subseteq t' \text{ and } t'(k+1) < -n) \text{ or } (t_\lambda \subseteq t' \text{ and } t'(k+1) > n), & \text{if } k < k'-1; \end{cases}$$

and also

(4.59) $d_{tn}\theta c_{t'} \Leftrightarrow c_{t'}\theta d_{tn} \Leftrightarrow$

$$\begin{cases} t' \subseteq t, & \text{if } k' < k \\ t' = t \text{ or } t' = t_\lambda, & \text{if } k' = k \\ (t_\lambda \subseteq t' \text{ and } t'(k') \geq n) \text{ or } (t \subseteq t' \text{ and } t'(k') \leq -n), & \text{if } k' = k+1 \\ (t_\lambda \subseteq t' \ \& \ t'(k+1) > n) \text{ or } (t \subseteq t' \ \& \ t'(k+1) < -n), & \text{if } k' > k+1. \end{cases}$$

Further, for every two elements c and d of $\mathbb{B}_{T_0}$, set

(4.60) $c(-\theta)d \Leftrightarrow (\exists k, l \in \mathbb{N}^+$ and $\exists c_1, \ldots, c_k, d_1, \ldots, d_l \in C_0 \cup C_1$ such that

$$c \subseteq \bigcup_{i=1}^{k} c_i, \ d \subseteq \bigcup_{j=1}^{l} d_j \text{ and } c_i(-\theta)d_j, \ \forall i = 1, \ldots, k \text{ and } \forall j = 1, \ldots, l).$$

Finally, for every two elements a and b of $RC(T_0^*)$, set

(4.61) $a\theta b \Leftrightarrow (\exists c, d \in \mathbb{B}_{T_0}$ such that $c \subseteq a, \ d \subseteq b$ and $c\theta d)$.

Theorem 4.4.2.8. *The triple $(RC(T_0^*), \theta, \mathbb{B}_{T_0})$, constructed in 4.4.2.7, is a CLCA; it is LCA-isomorphic to the complete local contact algebra $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$. Thus, the triple $(RC(T_0^*), \theta, \mathbb{B}_{T_0})$ completely determines the real line $\mathbb{R}$ with its natural topology.*

Proof. In this proof, we will use the notation introduced in 4.4.2.2, 4.4.2.5, 4.4.2.6 and 4.4.2.7. As it follows from 4.4.2.6 and [77, the proof of 4.14], there is an isomorphism

$$h : RC(T_0^*) \longrightarrow RC(\mathbb{Z}_0^{\mathbb{N}^+})$$

defined by the formula

$$h(c) = \bigvee_{RC(\mathbb{Z}_0^{\mathbb{N}^+})} \{a_t \mid t \in T_0^*, c_t \subseteq c\},$$

for every $c \in RC(T_0^*)$. Thus, $h(c_t) = a_t = \bigcap_{i=1}^{k} \pi_i^{-1}(t(i))$ and c_t corresponds to $\bigwedge_{i=1}^{k} \varphi_i(t(i))$ (see 4.4.2.2), where $t \in L_k \subseteq T_0^*$ (i.e., $t : \underline{k} \longrightarrow \mathbb{Z}_0$). This implies that

$h(C_0) = B'_0 = \{a_t \mid t \in T_0\}$ and C_0 corresponds to $B_0 = \{\bigwedge_{i=1}^{k} \varphi_i(t(i)) \mid k \in \mathbb{N}^+, t \in L_k\}$ (see (4.56), (4.45), (4.32)). Note that t_λ corresponds to b_- (see (4.50) and (4.33)). Since h is a complete homomorphism, we get that $h(d_{tn}) = Q_{a_t n}$ and thus d_{tn} corresponds to $q_{a_t n}$, for every $k, n \in \mathbb{N}^+$ and every $t \in L_k$ (see (4.55), (4.46), (4.34)). Then $h(C_1) = B'_1$ and hence C_1 corresponds to B_1 (see (4.56), (4.49), (4.35)). Hence, $h(\mathbb{B}_{T_0}) = \mathbb{B}$ and therefore $\mathbb{B}_{T_0}$ corresponds to $\widetilde{\mathbb{B}}$ (see the line after (4.56), (4.44) and the paragraph after (4.49), the line after (4.35)). Having all these facts in mind, we obtain easily that the formula (4.57) follows from the formula (4.36), (4.58) from (4.37), (4.59) from (4.39), (4.60) from (4.40) and (4.61) from (4.41). This completes the proof of our theorem. □

Theorem 4.4.2.9. *A CLCA $(M, \mu, \mathbb{M})$ is LCA-isomorphic to the complete local contact algebra $(RC(\mathbb{R}), \rho_\mathbb{R}, CR(\mathbb{R}))$ iff there exists an embedding (between partially ordered sets)*

$$\zeta : T_0^* \longrightarrow M$$

such that the following three conditions are satisfied:

(a) $\zeta(T_0)$ is dense in M,

(b) the ideal $\mathbb{M}$ is generated by the set

$$Z = \zeta(T_0) \cup \{\widetilde{d_{tn}} \mid t \in T_0, n \in \mathbb{N}^+\},$$

where the elements $\widetilde{d_{tn}}$ are defined by the formula (4.55) in which d_{tn} is replaced by $\widetilde{d_{tn}}$ and c_t is replaced by $z_t = \zeta(t)$, for every $t \in T_0$,

(c) the formulas (4.57), (4.58), (4.59), (4.40), (4.41), in which θ and $\tilde{\sigma}$ are replaced by μ, c_t by z_t, d_{tn} by $\widetilde{d_{tn}}$, $\widetilde{\mathbb{B}}$ by $\mathbb{M}$, $B_0 \cup B_1$ by Z, and $\tilde{A}$ by M, take place.

Proof. It follows from Theorem 4.4.2.8 and [77, 4.14,14.16]. □

4.4.3 A Whiteheadian-type description of Tychonoff cubes, spheres and tori

Theorem 4.4.3.1. *For every $n \in \mathbb{N}^+$, the CNCA $(RC(\mathbb{S}^n), \rho_{\mathbb{S}^n})$ $(= \Psi^t(\mathbb{S}^n))$ is CA-isomorphic to the CNCA $(\tilde{A}_n, C_{(\tilde{\sigma}_n, \widetilde{\mathbb{B}}_n)})$ (see 4.4.2.4 for the LCA $(\tilde{A}_n, \tilde{\sigma}_n, \widetilde{\mathbb{B}}_n)$, and 1.2.3.4 for $C_{(\tilde{\sigma}_n, \widetilde{\mathbb{B}}_n)}$); thus, the CNCA $(\tilde{A}_n, C_{(\tilde{\sigma}_n, \widetilde{\mathbb{B}}_n)})$ completely determines the n-dimensional sphere $\mathbb{S}^n$ with its natural topology. Note that $\tilde{A}_n$ is isomorphic to $\tilde{A}$, for every $n \in \mathbb{N}^+$.*

Proof. As it follows from 0.4.2.2 and 1.2.3.4, if X is a locally compact Hausdorff space then the complete normal contact algebra $(RC(\alpha X), \rho_{\alpha X})$ is CA-isomorphic to the complete normal contact algebra $(RC(X), C_{(\rho_X, CR(X))})$. Now, since $\alpha\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n$, our result follows from Theorem 4.4.2.4. □

For every cardinal number τ, denote by $\mathbb{T}^\tau$ the space $(\mathbb{S}^1)^\tau$ (for finite τ, this is just the τ-dimensional torus).

Theorem 4.4.3.2. *For every cardinal number τ, the complete normal contact algebra $(RC(\mathbb{T}^\tau), \rho_{\mathbb{T}^\tau})$ $(= \Psi^t(\mathbb{T}^\tau))$ is CA-isomorphic to the* **DHC**-*sum of τ copies of the CNCA $(\tilde{A}, C_{\tilde{\sigma}, \widetilde{\mathbb{B}}})$ (see Theorem 4.4.3.1 for it); therefore, this* **DHC**-*sum completely determines the space $\mathbb{T}^\tau$.*

Proof. Since the CNCA $(RC(\mathbb{S}^1), \rho_{\mathbb{S}^1})$ is CA-isomorphic to the CNCA $(\tilde{A}, C_{(\tilde{\sigma}, \widetilde{\mathbb{B}})})$ (see Theorem 4.4.3.1), our result follows from Theorem 2.3.3.7. □

Using 2.6.3.3, we obtain the following result:

Theorem 4.4.3.3. *Let $(M, \mu, \mathbb{M})$ be a CLCA which is LCA-isomorphic to the CLCA $(RC(\mathbb{R}), \rho_{\mathbb{R}}, CR(\mathbb{R}))$ and $\zeta : T_0^* \longrightarrow M$ be the embedding described in Theorem 4.4.2.9. Then, for each $t \in T_0$, the CNCA $(M|\zeta(t), \mu')$, where μ' is the restriction of the relation μ to $M|\zeta(t)$, is NCA-isomorphic to the CNCA $(RC(\mathbb{I}), \rho_{\mathbb{I}})$.*

Proof. By (4.42), (4.52) and the beginning of the proof of Theorem 4.4.2.3, if $t \in T_0$, i.e., $t : \underline{n} \longrightarrow \mathbb{Z}_0$ for some $n \in \mathbb{N}^+$, then the element $\zeta(t)$ coresponds to the element $\Delta_{t(1)\dots t(n)}$ of $RC(\mathbb{I}')$ (see also the proofs of theorems 4.4.2.8 and 4.4.2.9). Since $\Delta_{t(1)\dots t(n)}$ is homeomorphic to $\mathbb{I}$, our assertion follows from Theorem 2.6.3.3. □

The last theorem shows, in particular, that the following assertion holds:

Theorem 4.4.3.4. *Let $(\tilde{A}, \tilde{\sigma}, \widetilde{\mathbb{B}})$ be the CLCA described in 4.4.2.2, $m \in \mathbb{N}^+$, $n_1, \dots, n_m \in \mathbb{Z}_0$, $a_j = \{n_j\}$ for $j = 1, \dots, m$, $u = \bigwedge_{j=1}^m \varphi_j(a_j)$ (see 4.4.2.2 for φ_j) and $B = \tilde{A}|u$. Then the CNCA $(B, \tilde{\sigma}')$, where $\tilde{\sigma}'$ is the restriction of the relation $\tilde{\sigma}$ to B, is NCA-isomorphic to the CNCA $(RC(\mathbb{I}), \rho_{\mathbb{I}})$. In particular, the CNCA $(RC(\mathbb{I}), \rho_{\mathbb{I}})$ is NCA-isomorphic to the CNCA $(\tilde{A}|\varphi_1(\{1\}), \tilde{\sigma}')$.*

A direct description of the CNCA $(RC(\mathbb{I}), \rho_{\mathbb{I}})$ is given below.

4.4.3.5. The construction of $(\tilde{A}, \tilde{\sigma}')$. We will use the notation from 4.4.2.2.

We will define a relation $\tilde{\sigma}'$ on the Boolean algebra $\tilde{A}$ constructed in 4.4.2.2.

For every $n \in \mathbb{N}^+$, set

$$u_n^{\uparrow} = \varphi_1(succ(n)) \text{ and } u_n^{\downarrow} = \varphi_1(pred(-n))$$

and let

$$B_2 = \{u_n^{\uparrow}, u_n^{\downarrow} \mid n \in \mathbb{N}^+\}.$$

For every $a, b \in B_0 \cup B_1 \cup B_2$, set

$$a\tilde{\sigma}'b \Leftrightarrow a\tilde{\sigma}b$$

(see 4.4.2.2 for the definition of the relation $\tilde{\sigma}$). For convenience of the reader, we will write down the corresponding formulae. For every $n, m \in \mathbb{N}^+$,

$$u_n^{\uparrow}\tilde{\sigma}'u_m^{\uparrow},\ u_n^{\downarrow}\tilde{\sigma}'u_m^{\downarrow} \ \text{ and } \ u_n^{\downarrow}(-\tilde{\sigma}')u_m^{\uparrow}.$$

Further, for every $n, r \in \mathbb{N}^+$ and every $b = \varphi_1(a_1) \wedge \ldots \wedge \varphi_k(a_k) \in B_0$, where $a_1 = \{m\}$,

(4.62) $b\tilde{\sigma}'u_n^{\uparrow} \Leftrightarrow \begin{cases} m \geq n, & \text{if } k = 1 \\ m > n, & \text{if } k > 1 \end{cases}, \quad b\tilde{\sigma}'u_n^{\downarrow} \Leftrightarrow \begin{cases} m \leq -n, & \text{if } k = 1 \\ m < -n, & \text{if } k > 1 \end{cases}$

and

(4.63) $q_{br}\tilde{\sigma}'u_n^{\uparrow} \Leftrightarrow m > n,\ q_{br}\tilde{\sigma}'u_n^{\downarrow} \Leftrightarrow \begin{cases} m \leq -n, & \text{if } k = 1 \\ m < -n, & \text{if } k > 1. \end{cases}$

Now, for every $c, d \in \tilde{A}$, set

(4.64) $c(-\tilde{\sigma}')d \Leftrightarrow (\exists k, l \in \mathbb{N}^+$ and $\exists c_1, \ldots, c_k, d_1, \ldots, d_l \in B_0 \cup B_1 \cup B_2$ such

$$\text{that } c \leq \bigvee_{i=1}^{k} c_i,\ d \leq \bigvee_{j=1}^{l} d_j \text{ and } c_i(-\tilde{\sigma}')d_j,\ \forall i = 1, \ldots, k \text{ and } \forall j = 1, \ldots, l).$$

Theorem 4.4.3.6. *The pair $(\tilde{A}, \tilde{\sigma}')$, constructed in 4.4.3.5, is a complete normal contact algebra; it is CA-isomorphic to the CNCA $(RC(\mathbb{I}), \rho_{\mathbb{I}})$. Thus, the pair $(\tilde{A}, \tilde{\sigma}')$ completely determines the closed interval $\mathbb{I}$ with its natural topology.*

Proof. The proof of this assertion is analogous to the proof of Theorem 4.4.2.3. We will use in it the notation introduced in 4.4.2.3, 4.4.2.2 and 4.4.3.5.

Clearly, $RC(\mathbb{R})$ is isomorphic to $RC(\mathbb{I})$ (by Lemma 0.4.2.2). Thus, $RC(\mathbb{I})$ is isomorphic to $RC(X)$, where $X = \mathbb{Z}_0^{\mathbb{N}^+}$ (see the proof of Theorem 4.4.2.3). We will now construct an NCA $(RC(X), \sigma')$ CA-isomorphic to $(RC(\mathbb{I}), \rho_{\mathbb{I}})$. Then, identifying $RC(X)$ with $\tilde{A}$, we will show that $\sigma' = \tilde{\sigma}'$.

For every two elements M and N of $RC(\mathbb{J}_2)$, set

$$M\rho_1 N \Leftrightarrow \mathrm{cl}_{\mathbb{I}}(M) \cap \mathrm{cl}_{\mathbb{I}}(N) \neq \emptyset.$$

Then, using Lemma 0.4.2.2, we get that the pair $(RC(\mathbb{J}_2), \rho_1)$ is CA-isomorphic to the NCA $(RC(\mathbb{I}), \rho_{\mathbb{I}})$. Now, for every two elements $F, G \in RC(X)$, we set

(4.65) $F\sigma' G \Leftrightarrow f(F)\rho_1 f(G)$,

where $f : X \longrightarrow \mathbb{J}_2$ is the homeomorphism constructed in the proof of Theorem 4.4.2.3. Obviously, $(RC(X), \sigma')$ is CA-isomorphic to $(RC(\mathbb{I}), \rho_{\mathbb{I}})$. In the rest of this proof, we will show that the definition of σ' given above agrees with the definition of $\tilde{\sigma}'$ given in 4.4.3.5.

Using the proof of Proposition 2.3.3.2, it is easy to see that the set

$$B'_2 = \{\pi_1^{-1}(succ(n)), \pi_1^{-1}(pred(-n)) \mid n \in \mathbb{N}^+\}$$

corresponds to the set B_2 introduced in 4.4.3.5. Now, the formula (4.42) implies that, for every $n \in \mathbb{N}^+$,

(4.66) $\mathrm{cl}_{\mathbb{I}}(f(\pi_1^{-1}(succ(n)))) = [1 - \dfrac{1}{2^{n+1}}, 1]$ and $\mathrm{cl}_{\mathbb{I}}(f(\pi_1^{-1}(pred(-n)))) = [0, \dfrac{1}{2^{n+1}}]$.

Thus, for every $m, n \in \mathbb{N}^+$, $\mathrm{cl}_{\mathbb{I}}(f(\pi_1^{-1}(succ(n)))) \cap \mathrm{cl}_{\mathbb{I}}(f(\pi_1^{-1}(pred(-m)))) = \emptyset$. Also, for every $m, n \in \mathbb{N}^+$, we have that $f(\pi_1^{-1}(succ(n))) \cap f(\pi_1^{-1}(succ(m))) \neq \emptyset$ and $f(\pi_1^{-1}(pred(-n))) \cap f(\pi_1^{-1}(pred(-m))) \neq \emptyset$. Having in mind these formulae and the fact that $\mathrm{cl}_{\mathbb{I}}(f(F)) = \mathrm{cl}_{\mathbb{I}'}(f(F))$, for every $F \in B'_0 \cup B'_1$ (see the proof of Theorem 4.4.2.3 for the notation), we get that $G\sigma H \Leftrightarrow G\sigma' H$, for every $G, H \in B'_0 \cup B'_1 \cup B'_2$. This shows that $a\tilde{\sigma}' b \Leftrightarrow a\tilde{\sigma} b$, for every $a, b \in B_0 \cup B_1 \cup B_2$. Hence, the definitions of σ' and $\tilde{\sigma}'$ agree on $B'_0 \cup B'_1 \cup B'_2$ (or, equivalently, on $B_0 \cup B_1 \cup B_2$).

Further, using (4.66), we get that the family $\mathcal{B}_1 = \mathcal{B} \cup \{\mathrm{int}_{\mathbb{I}}(\mathrm{cl}_{\mathbb{I}}(f(F))) \mid F \in B'_2\}$ (see the proof of Theorem 4.4.2.3 for the notation and for the fact that $\mathcal{B}$ is a base of $\mathbb{I}'$) is a base of $\mathbb{I}$. Thus, by the regularity of $\mathbb{I}$, every two disjoint elements of $RC(\mathbb{I})$ can be separated by the finite unions of the elements of the family $\{\mathrm{cl}_{\mathbb{I}}(f(F)) \mid F \in B'_0 \cup B'_1 \cup B'_2\}$. This implies that the definitions of σ' and $\tilde{\sigma}'$ agree on $RC(X)$ (or, equivalently, on $\tilde{A}$). $\square$

Theorem 4.4.3.7. *For every cardinal number τ, the complete normal contact algebra $(RC(\mathbb{I}^\tau), \rho_{\mathbb{I}^\tau})$ $(= \Psi^t(\mathbb{I}^\tau))$ is CA-isomorphic to the* **DHC***-sum of τ copies of the CNCA $(\tilde{A}, \tilde{\sigma}')$ (see Theorem 4.4.3.6 for it); therefore, this* **DHC***-sum completely determines the space $\mathbb{I}^\tau$.*

Proof. It follows from Theorems 4.4.3.6 and 2.3.3.7. $\square$

Chapter 5

Some Isomorphism Theorems for Scott and Tarski consequence systems

5.1 Introduction

The notion of *Scott consequence system* (briefly, *S-system*) was introduced by D. Vakarelov in [113] in an analogy to a similar notion given by D. Scott in [100]. A standard example of an S-system is the set of all formulas of some formalized logical language with consequence relation $X \vdash Y$ between sets of formulas X and Y. A detailed study of such consequence relations in the context of propositional languages is given by Segerberg in [101] (see also [59]). The axioms of S-systems are abstract versions of some properties of the consequence relation $\vdash$ taken from logic. There are however many non-logical examples of S-systems and the main aim of this chapter is a study of some mathematical properties of this notion taken in its full generality. One such typical example is connected with the notion of a *property system* (briefly *P-system*), which is a kind of a very simple information system $P = (Ob, Pr, f)$, where Ob is a non-empty set of "objects", Pr is a set of "properties" and $f : Ob \longrightarrow P(Pr)$ is a function (called an *information function*), which assigns to each object x the set $f(x)$ of the "properties of x".

The structure of this chapter is the following. In the second section, we give some preliminary results. In the third section, we introduce the notion of an *S-morphism* between two S-systems, which enables us to define the category **SSyst** of all S-systems and all S-morphisms between them. The category **SSyst**, as well as its full subcategory **TSyst** of all *Tarski consequence systems*, are the main objects of our investigations in this chapter. In the fourth section, we prove some isomorphism theorems for these categories. With one of these theorems we extend the representation theory of S-systems in property systems, presented by D. Vakarelov in [113]. We also show that the categories **BoolAlg** (of all Boolean algebras and all Boolean homomorphisms) and **DLat** (of all distributive lattices and all lattice homomorphisms) are isomorphic to some reflective full subcategories of the category **SSyst**. Let us give a more detailed description of our isomorphism theorems. We define a category **TPS**, called *the category of topological property systems*, and we prove that the category **SSyst** is isomorphic to a full subcategory **TPSS** of **TPS**; then we show that the restriction of this isomorphism to the full subcategory **TSyst** of **SSyst** is in fact an isomorphism between the category **TSyst** and a subcategory $\mathbf{T}'$ of the category **Top** of all topological spaces and all continuous maps; the objects of the category $\mathbf{T}'$ are some hyperspaces.

Let us note that the connections of P-systems and S-systems with some notions of *informational relations* and some modal logics of information systems were studied in [114]. Other references on this subject can also be found in [114]. Let us also mention that in the book [10] of J. Barwise and J. Seligman, the P-systems and S-systems (presented there under the names of *classification systems* and *Gentzen systems*) play a crucial role in the definitions of the *"information flows"* and the *"logic of distributed systems"* which are basic notions in [10].

For all undefined here notions and notation, see [53], [75] and [1].

The results of this chapter were published in [39]. A generalization of a result of Iv. Prodanov [95], presented in [44], is used here as well.

5.2 Preliminaries

5.2.1 The definitions of S-systems, T-systems and P-systems

We will now give the precise definitions of some notions mentioned in the Introduction.

Definition 5.2.1.1. (see [101, 100, 113]) Let W be a non-empty set. By a *Scott consequence relation on* W we mean a binary relation $\vdash$ on $P(W)$ satisfying the following conditions for any $A, B, A', B' \in P(W)$ and $x \in W$:

(Refl) If $A \cap B \neq \emptyset$ then $A \vdash B$,

(Mono) If $A \vdash B$, $A \subseteq A'$ and $B \subseteq B'$ then $A' \vdash B'$,

(Cut) If $A \vdash (B \cup \{x\})$ and $(\{x\} \cup A) \vdash B$ then $A \vdash B$, (Fin) If $A \vdash B$ then there exist finite subsets $X \subseteq A$ and $Y \subseteq B$ such that $X \vdash Y$.

We say that $(W, \vdash)$ is a *Scott consequence system*, briefly, *S-system*, if W is a non-empty set and $\vdash$ is a Scott consequence relation on W.

We will denote by "$\nvdash$" the negation of "$\vdash$".

Definition 5.2.1.2. (see [59, 113]) Let $S = (W, \vdash)$ be an S-system. We say that $\vdash$ is a *Tarski consequence relation on* W and S is a *Tarski consequence system* (briefly, *T-system*), if the following condition is satisfied for any $A, B \in P(W)$:

(TFin) If $A \vdash B$ then there exist a finite set $X \subseteq A$ and an element $b \in B$ such that $X \vdash \{b\}$.

We will now recall some definitions and results from [115, 113]. They play a crucial role in our further investigations:

Definition 5.2.1.3. ([115]) By a *property system* (briefly, *P-system*) we mean any triple $P = (Ob, Pr, f)$, where Ob and Pr are sets, $Ob \neq \emptyset$ and $f \in \mathbf{Set}(Ob, P(Pr))$. The elements of Ob (resp. Pr; $f(x)$) are called *objects* (resp. *properties*; *properties of the object* x). A P-system $P = (Ob, Pr, f)$ is called a *set-theoretical P-system* if

$$Pr \subseteq P(Ob) \text{ and } f(x) = \{A \in Pr \mid x \in A\}$$

for any $x \in Ob$.

5.2.1.4. Let $(W, \vdash)$ be an S-system. A subset $p \subseteq W$ is called a *prime ideal of* $(W, \vdash)$ if for all finite subsets A and B of W such that $A \vdash B$,

$$A \cap p = \emptyset \text{ implies } B \setminus p \neq \emptyset.$$

A subset $q \subseteq W$ is called a *prime filter in* $(W, \vdash)$ if the set $W \setminus q$ is a prime ideal of $(W, \vdash)$. The set of all prime ideals (resp., prime filters) of $(W, \vdash)$ will be denoted by

$$PrI(W, \vdash) \text{ (resp., by } PrF(W, \vdash)).$$

Let us put

$$f(a) = \{p \in PrI(W, \vdash) \mid a \notin p\}$$

and

$$f'(a) = \{q \in PrF(W, \vdash) \mid a \in q\}$$

for all $a \in W$. Then the system

$$(W, PrI(W, \vdash), f)$$

is a P-system, called *the canonical P-system over* $(W, \vdash)$. It is denoted by

$$PS(W, \vdash).$$

The system

$$(W, PrF(W, \vdash), f')$$

is a set-theoretical P-system. It is called *the canonical set-theoretical P-system over* $(W, \vdash)$ and is denoted by

$$PSS(W, \vdash).$$

5.2.1.5. Let $W \neq \emptyset$ be a set, $L \in |\mathbf{DLat}|$ and $f \in \mathbf{Set}(W, L)$. Define a binary relation

$$\vdash_L$$

in $P(W)$ as follows. For any $A = \{a_i \in W \mid i = 1, \ldots, n\}$ and $B = \{b_j \in W \mid j = 1, \ldots, m\}$, put

$$A \vdash_L B \iff \bigwedge\{f(a_i) \mid i = 1, \ldots, n\} \leq \bigvee\{f(b_j) \mid j = 1, \ldots, m\}$$

(here n and m could be equal to zero as well). For arbitrary sets $A', B' \subseteq W$ let

$$A' \vdash_L B' \iff \text{ there exist finite subsets } A \subseteq A' \text{ and } B \subseteq B' \text{ such that } A \vdash_L B.$$

Then $(W, \vdash_L)$ is an S-system. In the special case of this construction when $W = L$ and $f = id$, the S-system $(L, \vdash_L)$ is denoted by

$$Sc(L).$$

One more special case will be used here. Let $P = (Ob, Pr, f)$ be a P-system. Put $L = (P(Pr), \cup, \cap, \emptyset, Pr)$ and $W = Ob$. By the definition of a P-system, we have that $f \in \mathbf{Set}(W, L)$. Hence, applying the above construction, we obtain the S-system $(W, \vdash_L)$. The relation $\vdash_L$ is denoted in this case by

$$\vdash_P .$$

The S-system

$$(Ob, \vdash_P)$$

is called *the canonical S-system over P* and is denoted by

$$Sc(P).$$

Proposition 5.2.1.6. ([113]) *Let $(W, \vdash)$ be an S-system. Then, for any $A, B \subseteq W$, the following conditions are equivalent:*

(a) $A \vdash B$;

(b) if p is a prime ideal of $(W, \vdash)$ and $A \cap p = \emptyset$ then $B \setminus p \neq \emptyset$;

(c) if p is a prime filter in $(W, \vdash)$ and $A \subseteq p$ then $B \cap p \neq \emptyset$.

Proposition 5.2.1.7. ([113]) *Let $(W, \vdash)$ be an S-system. Then:*

(1) For any $F \subseteq W$, the following conditions are equivalent:

(a) F is a prime filter (resp., prime ideal);

(b) $(\forall A \subseteq W)$ $((F \vdash A)$ implies $(F \cap A \neq \emptyset))$ (resp., $(\forall A \subseteq W)$ $(((W \setminus F) \vdash A)$ implies $(A \setminus F \neq \emptyset)))$;

(c) $F \not\vdash (W \setminus F)$ (resp., $(W \setminus F) \not\vdash F$).

(2) W is a prime filter (resp., prime ideal) iff $(\forall A \subseteq W)(A \not\vdash \emptyset)$ (resp., $(\forall A \subseteq W)(\emptyset \not\vdash A)$).

5.2.2 Coherent spaces and coherent maps

We now recall the definitions of coherent spaces and coherent maps (see, for example, [75]):

5.2.2.1. Let $(X, \mathcal{T})$ be a topological space. A closed subset F of X is called *irreducible* if the equality $F = F_1 \cup F_2$, where F_1 and F_2 are closed subsets of X, implies that $F = F_1$ or $F = F_2$. The space $(X, \mathcal{T})$ is called *sober* if it is a T_0-space and for every non-void irreducible subset F of X there exists a $x \in X$ such that $F = cl_X\{x\}$. The space $(X, \mathcal{T})$ is called *coherent* if it is a compact sober space, the family $KO(X, \mathcal{T})$ is closed under finite intersections and $KO(X, \mathcal{T})$ is a base for the topology $\mathcal{T}$. A continuous map $f : (X', \mathcal{T}') \longrightarrow (X'', \mathcal{T}'')$ is called *coherent* if $U'' \in KO(X'')$ implies that $f^{-1}(U'') \in KO(X')$.

We denote by **CohSp** the category of all coherent spaces and all coherent maps between them.

5.2.2.2. Let $L = (L, \vee, \wedge, 0, 1) \in |\mathbf{DLat}|$. Recall that (see, for example, [75]):

(a) an ideal p of L is called a *prime ideal* if $1 \notin p$ and $(a \wedge b \in p) \Rightarrow (a \in p$ or $b \in p)$;

(b) the set of all prime ideals of L is denoted by

$$spec(L);$$

(c) the family

$$\mathcal{O} = \{O_I \mid I \text{ is an ideal of } L\},$$

where

$$\{O_I = \{p \in spec(L) \mid I \not\subseteq p\},$$

is a topology on the set $spec(L)$, called *the Stone topology*;

(d) the topological space

$$(spec(L), \mathcal{O})$$

is the *classical spectrum of the lattice* L; it is a coherent space.

By the famous Stone duality theorem for distributive lattices (see [109]), the categories **DLat** and **CohSp** are dual. Let's recall the descriptions of the duality functors

$$S_L^t : \mathbf{CohSp} \longrightarrow \mathbf{DLat} \text{ and } S_L^a : \mathbf{DLat} \longrightarrow \mathbf{CohSp}.$$

If X is a coherent space then

$$S_L^t(X) = (KO(X), \cup, \cap, \emptyset, X);$$

if $f \in \mathbf{CohSp}(X_1, X_2)$ then

$$S_L^t(f) : S_L^t(X_2) \longrightarrow S_L^t(X_1)$$

is defined by the formula

$$S_L^t(f)(U) = f^{-1}(U)$$

for every $U \in KO(X_2)$; if $L \in |\mathbf{DLat}|$ then

$$S_L^a(L) = (spec(L), \mathcal{O}),$$

where $(spec(L), \mathcal{O})$ is the classical spectrum of the lattice L; if $f \in \mathbf{DLat}(L_1, L_2)$ then

$$S_L^a(f) : S_L^a(L_2) \longrightarrow S_L^a(L_1)$$

is defined by the formula

$$S_L^a(f)(p) = f^{-1}(p)$$

for every $p \in spec(L_2)$.

5.3 The category of S-systems and S-morphisms

5.3.1 The categories SSyst and SDLat

Definition 5.3.1.1. Let $(W, \vdash)$ and $(W', \vdash')$ be two S-systems and $f \in \mathbf{Set}(W, W')$. The function f is called an *S-morphism* if

$$(A \vdash B) \Rightarrow (f(A) \vdash' f(B))$$

for any $A, B \in P(W)$. We denote by **SSyst** the category of all S-systems and all S-morphisms between them.

The following simple fact will be often used in this section:

Proposition 5.3.1.2. *Let $f : (W, \vdash) \longrightarrow (W', \vdash')$ be an S-morphism and $F' \subseteq W'$ be a prime filter (resp., prime ideal) in $(W', \vdash')$. Then $f^{-1}(F')$ is a prime filter (resp., prime ideal) in $(W, \vdash)$.*

Proof. Let $F' \subseteq W'$ be a prime filter in $(W', \vdash')$, $A \subseteq W$ and $f^{-1}(F') \vdash A$. Then $f(f^{-1}(F')) \vdash' f(A)$. Hence $F' \vdash' f(A)$. Thus, by 5.2.1.7(1), $F' \cap f(A) \neq \emptyset$. Then $A \cap f^{-1}(F') \neq \emptyset$. Therefore, by 5.2.1.7(1), $f^{-1}(F')$ is a prime filter in $(W, \vdash)$.

The corresponding statement for the prime ideals follows directly from the just proved one. □

Definition 5.3.1.3. We will denote by **SDLat** the full subcategory of the category **SSyst** whose objects are of the form $Sc(L)$, where $L \in |\mathbf{DLat}|$ (see 5.2.1.5 for the notation).

Proposition 5.3.1.4. *The category* **DLat** *is isomorphic to the subcategory* **SDLat** *of the category* **SSyst**.

Proof. We will prove that if $D_L : \mathbf{DLat} \longrightarrow \mathbf{SDLat}$ is defined on the objects by

$$D_L(L) = Sc(L)$$

and on the morphisms by

$$D_L(l) = l,$$

then D_L is an isomorphism (see 5.2.1.5 for the notation). We have that

$$a \leq b \iff \{a\} \vdash_L \{b\},$$

for every $a, b \in L$ (see 5.2.1.5 for $\vdash_L$). This shows that if $L, L' \in |\mathbf{DLat}|$ then $Sc(L) = Sc(L')$ is equivalent to $L = L'$. Let now $l \in \mathbf{DLat}(L, L')$. We will prove that $l \in \mathbf{SDLat}(Sc(L), Sc(L'))$. Let $A, B \subseteq L$ and $A \vdash_L B$. Then there exist finite subsets $A' = \{a_i \mid i = 1, \ldots, n\} \subseteq A$ and $B' = \{b_j \mid j = 1, \ldots, m\} \subseteq B$ such that $A' \vdash_L B'$. This means that $\bigwedge\{a_i \mid i = 1, \ldots, n\} \leq \bigvee\{b_j \mid j = 1, \ldots, m\}$. Thus we obtain that $\bigwedge\{l(a_i) \mid i = 1, \ldots, n\} \leq \bigvee\{l(b_j) \mid j = 1, \ldots, m\}$. Hence $l(A') \vdash_{L'} l(B')$ and this implies that $l(A) \vdash_{L'} l(B)$. Therefore l is an S-morphism, i.e., $l \in \mathbf{SDLat}(Sc(L), Sc(L'))$. Conversely, if $l \in \mathbf{SDLat}(Sc(L), Sc(L'))$, then $l \in \mathbf{DLat}(L, L')$. Indeed, if $a, b \in L$ and $a \leq b$ then $\{a\} \vdash_L \{b\}$ and hence $\{l(a)\} \vdash_{L'} \{l(b)\}$. Thus $l(a) \leq l(b)$. So, l is an order-preserving map. Further, let $a \vee b = c$ in L. Then $\{c\} \vdash_L \{a, b\}$. Therefore $\{l(c)\} \vdash_{L'} \{l(a), l(b)\}$. This implies that $l(a \vee b) = l(c) \leq l(a) \vee l(b)$. On the other hand, the inequalities $a \leq c$ and $b \leq c$ imply (since l is order-preserving) that $l(a) \vee l(b) \leq l(c) = l(a \vee b)$. So, $l(a \vee b) = l(a) \vee l(b)$. Analogously we prove that $l(a \wedge b) = l(a) \wedge l(b)$. Finally, since $\emptyset \vdash_L \{1_L\}$, we have that $\emptyset \vdash_{L'} \{l(1_L)\}$. Hence $1_{L'} = \bigwedge \emptyset \leq l(1_L)$. So, $l(1_L) = 1_{L'}$. Analogously, $\{0_L\} \vdash_L \emptyset$ implies that $l(0_L) = 0_{L'}$. Therefore, $l \in \mathbf{DLat}(L, L')$. All this shows that D_L is a functor. It is now easily seen that D_L is an isomorphism. $\square$

5.3.2 SDLat is a reflective subcategory of the category SSyst

We are now going to demonstrate that **SDLat** is a reflective subcategory of the category **SSyst**. Let's start with the following theorem which is a generalization of a result of Iv. Prodanov from [95]. We formulated and proved it in [44].

Theorem 5.3.2.1. *Let X be a set and $\mathcal{S} \subseteq P(X)$. Setting, for every $x \in X$,*

$$U_x^- = \{p \in \mathcal{S} \mid x \notin p\},$$

let $\mathcal{T}^-$ be the topology on $\mathcal{S}$ having as a subbase the family

$$\mathcal{P}^- = \{U_x^- \mid x \in X\}.$$

Suppose that $(\mathcal{S}, \mathcal{T}^-)$ is a coherent space and let

$$L = S_L^t(\mathcal{S}, \mathcal{T}^-)$$

(see 5.2.2.2 for the notation). Then $U_x^- \in L$ for every $x \in X$. Set

$$\varphi : X \longrightarrow L, \quad x \mapsto U_x^-.$$

Then:

(i) the set $\varphi(X)$ generates L;

(ii) $\varphi^{-1}(q) \in \mathcal{S}$ for every $q \in spec(L)$ (see 5.2.2.2 for the notation);

(iii) $\Phi' : spec(L) \longrightarrow \mathcal{S}$, $q \longrightarrow \varphi^{-1}(q)$, is a **CohSp***-isomorphism;*

(iv) if $L' \in |\mathbf{DLat}|$ and $\theta : X \longrightarrow L'$ is a function such that:

(1) *$\theta^{-1}(q) \in \mathcal{S}$ for every $q \in spec(L')$, and*

(2) *$\Theta : spec(L') \longrightarrow \mathcal{S}$, $q \longrightarrow \theta^{-1}(q)$, is a* **CohSp***-morphism,*

then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ with $l \circ \varphi = \theta$;

(v) $\varphi : X \longrightarrow L$ is an injection iff for any two different points x and y of X there exists a $p \in \mathcal{S}$ containing exactly one of them.

We also need the following result of Iv. Prodanov (see [44]):

Proposition 5.3.2.2. *Let X be a set, $\mathcal{S} \subseteq P(X)$ and $\mathcal{T}^-$ be the topology on $\mathcal{S}$ defined in 5.3.2.1. Then the following conditions are equivalent:*

(a) $(\mathcal{S}, \mathcal{T}^-)$ is a coherent space;

(b) $\mathcal{S}$ is a closed subset of the Cantor cube $\mathbf{2}^X$ (where $\mathcal{S}$ is identified with a subset of $\mathbf{2}^X$ in the following way: any $A \in \mathcal{S}$ is identified with its characteristic function $\chi_A : X \longrightarrow \mathbf{2}$ (i.e., $\chi_A(x) = 1$ iff $x \in A$)).

Proposition 5.3.2.3. *Let $(W, \vdash)$ be an S-system. Put $\mathcal{S} = PrI(W, \vdash)$ and define the topology $\mathcal{T}^-$ on $\mathcal{S}$ exactly as in 5.3.2.1. Then $(\mathcal{S}, \mathcal{T}^-)$ is a coherent space.*

Proof. Identifying $\mathcal{S}$ with a subset of $\mathbf{2}^W$ as in 5.3.2.2, we have to prove, according to 5.3.2.2, that $\mathcal{S}$ is a closed subset of the Cantor cube $\mathbf{2}^W$.

Let $\{p_\sigma, \sigma \in \Sigma\}$ be a net in $\mathcal{S}$ converging in $\mathbf{2}^W$ to a point $p \in \mathbf{2}^W$. This means that if $f_\sigma : W \longrightarrow \mathbf{2}$ and $f : W \longrightarrow \mathbf{2}$ are functions such that $f_\sigma^{-1}(1) = p_\sigma$ for every $\sigma \in \Sigma$ and $f^{-1}(1) = p$, then $\{f_\sigma, \sigma \in \Sigma\}$ converges to f in $\mathbf{2}^W$. We have to prove that $p \in \mathcal{S}$, i.e., that $f^{-1}(1) \in \mathcal{S}$.

Let $A = \{a_i \mid i = 1, \ldots, n\}$ and $B = \{b_j \mid j = 1, \ldots, m\}$ be two finite subsets of W and $A \cap p = \emptyset$. We have to show that $B \setminus p \neq \emptyset$ (see 5.2.1.4). For every $i = 1, \ldots, n$ we have that $f(a_i) = 0$. Let $i \in \{1, \ldots, n\}$. Since the net $\{f_\sigma(a_i), \sigma \in \Sigma\}$ converges to $f(a_i)$, there exists a $\sigma_i \in \Sigma$ such that $f_\sigma(a_i) = 0$ for every $\sigma \geq \sigma_i$. Let $\sigma_0 = \sup\{\sigma_i \mid i = 1, \ldots, n\}$. Then, for every $\sigma \geq \sigma_0$ and for every $i = 1, \ldots, n$, we have that $f_\sigma(a_i) = 0$. Hence, for every $\sigma \geq \sigma_0$, we get that $A \cap p_\sigma = \emptyset$. Since p_σ is a prime ideal, we obtain that $B \setminus p_\sigma \neq \emptyset$ for every $\sigma \geq \sigma_0$. Consequently, for every $\sigma \geq \sigma_0$, there exists a $j(\sigma) \in \{1, \dots, m\}$ such that $b_{j(\sigma)} \notin p_\sigma$, i.e., $f_\sigma(b_{j(\sigma)}) = 0$. Defining a function $\alpha : \{\sigma \in \Sigma \mid \sigma \geq \sigma_0\} \longrightarrow B$ by the formula $\alpha(\sigma) = b_{j(\sigma)}$, for every $\sigma \geq \sigma_0$, we get that $f_\sigma(\alpha(\sigma)) = 0$ for every $\sigma \geq \sigma_0$. Obviously, there exists a $j' \in \{1, \dots, m\}$ such that the set $\Sigma' = \alpha^{-1}(b_{j'})$ is a cofinal subset of the directed set $(\Sigma, \leq)$. Then, for every $\sigma' \in \Sigma'$, we have that $f_{\sigma'}(b_{j'}) = f_{\sigma'}(\alpha(\sigma')) = 0$. Since $\{f_{\sigma'}(b_{j'}), \sigma' \in \Sigma'\}$ is a net finer than the net $\{f_\sigma(b_{j'}), \sigma \in \Sigma\}$ and the last one converges to $f(b_{j'})$, we obtain that the net $\{f_{\sigma'}(b_{j'}), \sigma' \in \Sigma'\}$ converges also to $f(b_{j'})$. Thus $f(b_{j'}) = 0$, i.e., $b_{j'} \in B \setminus p$. Therefore, we proved that $B \setminus p \neq \emptyset$. This implies that p is a prime ideal of $(W, \vdash)$, i.e., $p \in \mathcal{S}$. Hence, $\mathcal{S}$ is a closed subset of $\mathbf{2}^W$. Therefore, $(\mathcal{S}, \mathcal{T}^-)$ is a coherent space. $\square$

Theorem 5.3.2.4. *Let $(W, \vdash)$ be an S-system. Then there exists a distributive lattice $(L, \vee, \wedge)$ with 0 and 1, and a function $\varphi : W \longrightarrow L$ such that:*

(i) the set $\varphi(W)$ generates L;

(ii) for any two finite subsets A and B of W we have that $A \vdash B$ iff $\varphi(A) \vdash_L \varphi(B)$ (see 5.2.1.5 for the notation);

(iii) if $L' \in |\mathbf{DLat}|$ and $\theta : (W, \vdash) \longrightarrow Sc(L')$ is an S-morphism (see 5.2.1.5 for the notation) then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ such that $l \circ \varphi = \theta$.

(iv) $\varphi : W \longrightarrow L$ is an injection iff for any two different points x and y of W there exists a prime ideal p in $(W, \vdash)$ containing exactly one of them.

Proof. Put $\mathcal{S} = PrI(W, \vdash)$ and let $\mathcal{T}^-$ be the topology on $\mathcal{S}$ defined exactly as in 5.3.2.1. Then, by 5.3.2.3, we have that $(\mathcal{S}, \mathcal{T}^-)$ is a coherent space. Hence, setting

$$L = S_L^t(\mathcal{S}, \mathcal{T}^-)$$

and

$$\varphi : W \longrightarrow L, \quad x \mapsto U_x^- = \{p \in \mathcal{S} \mid x \notin p\}$$

(see 5.2.2.2 for the notation and 5.3.2.1 for φ), we obtain, applying Theorem 5.3.2.1, that the set $\varphi(W)$ generates L. Hence, condition (i) is fulfilled. It is obvious that 5.3.2.1(v) implies our condition (iv). So, let's prove (ii).

Let $A = \{a_i \mid i = 1, \dots, n\}$ and $B = \{b_j \mid j = 1, \dots, m\}$ be two finite subsets of W. Recall that $\varphi(A) \vdash_L \varphi(B)$ iff $\bigcap\{\varphi(a_i) \mid i = 1, \dots, n\} \subseteq \bigcup\{\varphi(b_j) \mid j = 1, \dots, m\}$. The following four cases are possible.

Case 1: $n \neq 0$ and $m \neq 0$.

Let $A \vdash B$ and $p \in \bigcap\{\varphi(a_i) \mid i = 1, \ldots, n\}$. Then $A \cap p = \emptyset$. Hence, by 5.2.1.4, $B \setminus p \neq \emptyset$. Therefore $p \in \bigcup\{\varphi(b_j) \mid j = 1, \ldots, m\}$. Thus $\varphi(A) \vdash_L \varphi(B)$. Conversely, let $\varphi(A) \vdash_L \varphi(B)$. Take a $p \in \mathcal{S}$ such that $A \cap p = \emptyset$. Then $p \in \bigcap\{\varphi(a_i) \mid i = 1, \ldots, n\}$. Thus $p \in \bigcup\{\varphi(b_j) \mid j = 1, \ldots, m\}$, i.e., there exists a $j \in \{1, \ldots, m\}$ such that $b_j \notin p$. Therefore $B \setminus p \neq \emptyset$. This shows, by 5.2.1.6, that $A \vdash B$.

Case 2: $n = 0$ and $m = 0$.

We have that $A = B = \emptyset$. Let $A \vdash B$. Then $\mathcal{S} = \emptyset$. Indeed, if $p \in \mathcal{S}$ then $A \cap p = \emptyset$ and $B \setminus p = \emptyset \setminus p = \emptyset$, which is a contradiction. Hence $\mathcal{S} = \emptyset$. Then $|L| = 1$, i.e., $0 = 1$. Therefore the inequality $1 \leq 0$ takes place. Thus $\bigwedge \emptyset = 1 \leq 0 = \bigvee \emptyset$. So, $\varphi(A) \vdash_L \varphi(B)$. Conversely, if $\varphi(A) \vdash_L \varphi(B)$ then $1 \leq 0$ and, hence, $|L| = 1$. This shows that $\mathcal{S} = \emptyset$. Now, 5.2.1.6 implies that $A \vdash B$.

Case 3: $n = 0$ and $m \neq 0$.

Let $A \vdash B$. We will prove that $\bigcup\{\varphi(b_j) \mid j = 1, \ldots, m\} = \mathcal{S}$. Suppose that there exists a $p \in \mathcal{S}$ such that $p \notin \bigcup\{\varphi(b_j) \mid j = 1, \ldots, m\}$. Then $B \subseteq p$. This is a contradiction because $A \cap p = \emptyset$. Hence $\bigcup\{\varphi(b_j) \mid j = 1, \ldots, m\} = \mathcal{S}$. Therefore $\varphi(A) \vdash_L \varphi(B)$. Conversely, if $\varphi(A) \vdash_L \varphi(B)$ then $\bigcup\{\varphi(b_j) \mid j = 1, \ldots, m\} = \mathcal{S}$. Let $p \in \mathcal{S}$. Then $A \cap p = \emptyset$ and $B \setminus p \neq \emptyset$. This shows, by 5.2.1.6, that $A \vdash B$.

Case 4: $n \neq 0$ and $m = 0$.

Let $A \vdash B$. We will prove that $\bigcap\{\varphi(a_i) \mid i = 1, \ldots, n\} = \emptyset$. Suppose that there exists a $p \in \bigcap\{\varphi(a_i) \mid i = 1, \ldots, n\}$. Then $A \cap p = \emptyset$. Hence, by 5.2.1.6, $B \setminus p \neq \emptyset$. This is a contradiction because $B \setminus p = \emptyset \setminus p = \emptyset$. Therefore $\bigcap\{\varphi(a_i) \mid i = 1, \ldots, n\} = \emptyset$. Thus $\varphi(A) \vdash_L \varphi(B)$. Conversely, let $\varphi(A) \vdash_L \varphi(B)$. Then $\bigcap\{\varphi(a_i) \mid i = 1, \ldots, n\} = \emptyset$. Let $p \in \mathcal{S}$. Suppose that $A \cap p = \emptyset$. Then $p \in \bigcap\{\varphi(a_i) \mid i = 1, \ldots, n\}$, which is a contradiction. Hence, for every $p \in \mathcal{S}$, we have that $A \cap p \neq \emptyset$. Now, 5.2.1.6 implies that $A \vdash B$. So, (ii) is proved.

We prove (iii) now. Let $\theta : W \longrightarrow L'$ be as in (iii). Obviously, it is enough to show that θ satisfies conditions (1) and (2) of 5.3.2.1(iv). In order to check condition (1) of 5.3.2.1(iv), let's take a $q \in \mathit{spec}(L')$. We have to prove that $p = \theta^{-1}(q) \in \mathcal{S}$.

Suppose that $p \notin \mathcal{S}$. Then there exist two finite subsets A and B of W such that $A \vdash B$, $A \cap p = \emptyset$ and $B \subseteq p$. Then $\theta(A) \cap q = \emptyset$ and $\theta(B) \subseteq q$. Let $A = \{a_i \mid i = 1, \ldots, n\}$ and $B = \{b_j \mid j = 1, \ldots, m\}$. Since $A \vdash B$, we have that $\bigwedge'\{\theta(a_i) \mid i = 1, \ldots, n\} \leq \bigvee'\{\theta(b_j) \mid j = 1, \ldots, m\}$. The equality $\theta(A) \cap q = \emptyset$ and the fact that q is a prime ideal imply that $\bigwedge'\{\theta(a_i) \mid i = 1, \ldots, n\} \notin q$. Hence $\bigvee'\{\theta(b_j) \mid j =$

$1, \dots, m\} \notin q$. But this is impossible, since $\theta(B)$ is a subset of q and, therefore, $\bigvee'\{\theta(b_j) \mid j = 1, \dots, m\} \in q$ (because q is an ideal). So, we got a contradiction. Hence $p = \theta^{-1}(q) \in \mathcal{S}$. Therefore, condition (1) of 5.3.2.1(iv) is fulfilled.

Now, we will show that condition (2) of 5.3.2.1(iv) is fulfilled, i.e., we will prove that the function

$$\Theta : spec(L') \longrightarrow \mathcal{S}, \quad q \mapsto \theta^{-1}(q),$$

is a **CohSp**-morphism. Let's show first that $\Theta : (spec(L'), \mathcal{O}') \longrightarrow (\mathcal{S}, \mathcal{T}^-)$ is a continuous map (here $\mathcal{O}'$ is the Stone topology on $spec(L')$ (see 5.2.2.2)). Recall that the family $\mathcal{P}^- = \{U_x^- \mid x \in W\}$, where $U_x^- = \{p \in \mathcal{S} \mid x \notin p\}$ for every $x \in W$, is a subbase of the topology $\mathcal{T}^-$ on $\mathcal{S}$. Hence, we have to prove that $\Theta^{-1}(U_x^-) \in \mathcal{O}'$ for every $x \in W$.

Let $x \in W$. Then

$$\Theta^{-1}(U_x^-) = \{q \in spec(L') \mid \Theta(q) \in U_x^-\} =$$

$$= \{q \in spec(L') \mid \theta^{-1}(q) \in U_x^-\} = \{q \in spec(L') \mid x \notin \theta^{-1}(q)\} =$$

$$= \{q \in spec(L') \mid \theta(x) \notin q\} = \{q \in spec(L') \mid I(\theta(x)) \not\subseteq q\} = O_{I(\theta(x))}$$

(see 5.2.2.2 for the notation), where $I(\theta(x)) = \{l \in L' \mid l \leq \theta(x)\}$. Since $I(\theta(x))$ is an ideal of L', we obtain that $\Theta^{-1}(U_x^-) \in \mathcal{O}'$. Therefore, Θ is a continuous map.

Let K be a compact open subset of $(\mathcal{S}, \mathcal{T}^-)$. Then, obviously, K is a finite union of elements of the family $\mathcal{B}^-$ of all finite intersections of the elements of $\mathcal{P}^-$. Hence, for showing that $\Theta^{-1}(K)$ is a compact subset of $spec(L')$, it is enough to show that $\Theta^{-1}(U_x^-)$ is a compact subset of $spec(L')$ for every $x \in W$. (Here we use the fact that the family $KO(spec(L'))$ of all compact open subsets of $spec(L')$ is closed under finite intersections. It is so because the space $spec(L')$ is coherent (see 5.2.2.2)). Let $x \in X$. As we have shown, $\Theta^{-1}(U_x^-) = O_{I(\theta(x))}$. Since $O_{I(\theta(x))}$ is a compact set (see [109]), the proof is completed. □

This theorem implies the following result:

Theorem 5.3.2.5. *The category* **DLat** *is isomorphic to a reflective full subcategory of the category* **SSyst** *of all S-systems and their morphisms.*

Proof. In 5.3.1.4, we proved that the category **DLat** is isomorphic to the full subcategory **SDLat** of the category **SSyst**. Let's show that **SDLat** is a reflective subcategory of **SSyst**. Take an S-system $(W, \vdash)$. Then, by 5.3.2.4, there exists an $L \in |\mathbf{DLat}|$ and a function $\varphi : W \longrightarrow L$ which, by 5.3.2.4(ii), is an S-morphism between $(W, \vdash)$

and $Sc(L)$. So $\varphi \in \mathbf{SSyst}(W, Sc(L))$. Now, using 5.3.2.4(iii) and the fact that $l \in \mathbf{DLat}(L, L')$ implies $l \in \mathbf{SSyst}(Sc(L), Sc(L'))$ (see the proof of 5.3.1.4), we get that φ is an **SDLat**-reflection arrow. Therefore, **SDLat** is a reflective subcategory of **SSyst**. $\square$

Since the category **BoolAlg** is a reflective full subcategory of the category **DLat** (see [88] or [75](Exercise 4.5)), we obtain immediately (using also 4G from [1]) the following corollary:

Corollary 5.3.2.6. *The category* **BoolAlg** *is isomorphic to a reflective full subcategory of the category* **SSyst** *of all S-systems and their morphisms.*

Theorem 5.3.2.4 implies also the following two results of D. Vakarelov [113]:

Corollary 5.3.2.7. ([113]) *Let $(W, \vdash)$ be an S-system, satisfying the following additional condition:*

(Antisymm) if $\{a\} \vdash \{b\}$ and $\{b\} \vdash \{a\}$ then $a = b$ ($a, b \in W$).

Then there exists a distributive lattice $(L, \vee, \wedge)$ with 0 and 1, and an injection $\varphi : W \longrightarrow L$ such that:

(i) the set $\varphi(W)$ generates L;

(ii) for any two finite subsets $A = \{a_i \mid i = 1, \ldots, n\}$ and $B = \{b_j \mid j = 1, \ldots, m\}$ of W we have that $A \vdash B$ iff $\bigwedge\{\varphi(a_i) \mid i = 1, \ldots, n\} \leq \bigvee\{\varphi(b_j) \mid j = 1, \ldots, m\}$.

Proof. By 5.3.2.4, there exist a distributive lattice L and a function $\varphi : W \longrightarrow L$ which satisfy conditions (i) and (ii). We have only to show that the function φ is an injection.

Let $a, b \in W$ and $a \neq b$. Suppose that every prime ideal p in $(W, \vdash)$ contains (resp. doesn't contain) both a and b. Then, by the definition of φ (see the proof of 5.3.2.4), we get that $\varphi(a) = 0 = \varphi(b)$ (resp. $\varphi(a) = 1 = \varphi(b)$). Hence, by (ii), we obtain, in both cases, that $\{a\} \vdash \{b\}$ and $\{b\} \vdash \{a\}$. Now, condition (Antisymm) implies that $a = b$, which is a contradiction. Therefore, there exists a prime ideal p in $(W, \vdash)$ containing exactly one of the points a and b. We thus get, using 5.3.2.4(iv), that φ is an injection. $\square$

Corollary 5.3.2.8. (Vakarelov's Representation Theorem for S-systems in P-systems) *Let $(W, \vdash)$ be an S-system. Then $(W, \vdash) = Sc(PS(W, \vdash)) = Sc(PSS(W, \vdash))$ (see 5.2.1.4 and 5.2.1.5 for the notation).*

Proof. Let $\mathcal{S} = PrI(W, \vdash)$. The definition of the function φ from the proof of 5.3.2.4 and Definition 5.2.1.4 show that $PS(W, \vdash) = (W, \mathcal{S}, \varphi)$. Put $P = (W, \mathcal{S}, \varphi)$. Then, using the notation of 5.2.1.5, we obtain, by 5.3.2.4(ii), that $\vdash_P = \vdash$. Therefore,

$$(W, \vdash) = Sc(P) = Sc(PS(W, \vdash)).$$

Put $P' = PSS(W, \vdash)$. It is easy to see, using only the definitions of the relevant notions and notation, that $\vdash_P = \vdash_{P'}$. Thus $Sc(PS(W, \vdash)) = Sc(PSS(W, \vdash))$. $\square$

In the next section we will extend this representation theorem to an isomorphism theorem (see 5.4.1.4).

5.4 Some Isomorphism Theorems

5.4.1 The categories SPS, TPS and TPSS

Definition 5.4.1.1. (a) Let (W, V, f) and (W', V', f') be two set-theoretical P-systems and $\varphi \in \mathbf{Set}(W, W')$. The function φ is called a *P-morphism* if $\varphi^{-1}(V') \subseteq V$. We denote by **SPS** the category of all set-theoretical P-systems and P-morphisms.

(b) We denote by **TPS** the category whose objects are all pairs $(X, \mathcal{P})$, where X is a non-empty set and $\mathcal{P} \subseteq P(X)$, and, for any $(X, \mathcal{P}), (X', \mathcal{P}') \in |\mathbf{TPS}|$, the set $\mathbf{TPS}((X, \mathcal{P}), (X', \mathcal{P}'))$ consists of all $f \in \mathbf{Set}(X, X')$ such that $f^{-1}(\mathcal{P}') \subseteq \mathcal{P}$. The objects of the category **TPS** are called *topological property systems*.

Remark 5.4.1.2. (a) The full subcategory $\mathbf{Top}'$ of **Top**, consisting of all non-empty topological spaces, is a full subcategory of **TPS**.

(b) **SPS** and **TPS** are isomorphic categories.

For proving (b), define two functors

$$H^a : \mathbf{SPS} \longrightarrow \mathbf{TPS} \ \text{ and } \ H^t : \mathbf{TPS} \longrightarrow \mathbf{SPS}$$

by

$$H^a(W, V, f) = (W, V)$$

(on the objects of **SPS**),

$$H^a\varphi = \varphi$$

(on the morphisms of **SPS**),

$$H^t(X, \mathcal{P}) = (X, \mathcal{P}, f),$$

$$\text{where } f \in \mathbf{Set}(X, P(\mathcal{P})) \text{ is defined by } f(x) = \{A \in \mathcal{P} \mid x \in A\}$$

(on the objects of **TPS**) and

$$H^t(\varphi) = \varphi$$

(on the morphisms of **TPS**). Then it is easy to see that $H^a \circ H^t = Id_{\mathbf{TPS}}$ and $H^t \circ H^a = Id_{\mathbf{SPS}}$. Hence, **SPS** and **TPS** are isomorphic categories.

Definition 5.4.1.3. We denote by **TPSS** the full subcategory of **TPS** whose objects are all $(X, \mathcal{P}) \in |\mathbf{TPS}|$ which satisfy the following condition:

(TPSS) If $V \subseteq X$ is such that for any two finite sets $F \subseteq V$ and $G \subseteq X \setminus V$ there exists a $U \in \mathcal{P}$ with $F \subseteq U$ and $U \cap G = \emptyset$, then $V \in \mathcal{P}$.

Theorem 5.4.1.4. *The categories* **SSyst** *and* **TPSS** *are isomorphic.*

Proof. The proof will consist of several steps.

Step 1. In this step we will define two functors

$$T^a : \mathbf{SSyst} \longrightarrow \mathbf{TPS} \ \text{ and } \ T^t : \mathbf{TPS} \longrightarrow \mathbf{SSyst}.$$

For any $(W, \vdash) \in |\mathbf{SSyst}|$, put

$$T^a(W, \vdash) = (W, PrF(W, \vdash))$$

(see 5.2.1.4 for the notation) and let

$$T^a(\varphi) = \varphi$$

on the morphisms of **SSyst**. It is easily seen, using 5.3.1.2, that T^a is a functor.

For any $(X, \mathcal{P}) \in |\mathbf{TPS}|$, put

$$T^t(X, \mathcal{P}) = (X, \vdash_{\mathcal{P}}),$$

where the binary relation $\vdash_{\mathcal{P}}$ in $P(X)$ is defined as follows: if A and B are two finite subsets of X then

$$A \vdash_{\mathcal{P}} B \iff [(\forall U \in \mathcal{P})((A \subseteq U) \rightarrow (U \cap B \neq \emptyset))];$$

if A and B are two arbitrary subsets of X then

$$A \vdash_{\mathcal{P}} B \iff (\text{ there exist finite subsets } A' \subseteq A \text{ and } B' \subseteq B \text{ such that } A' \vdash_{\mathcal{P}} B').$$

Put

$$T^t(\varphi) = \varphi$$

on the morphisms of **TPS**.

Let's show that T^t is a functor from the category **TPS** to the category **SSyst**. Take a $(X, \mathcal{P}) \in |\mathbf{TPS}|$. Define $f : X \longrightarrow P(\mathcal{P})$ putting

$$f(x) = \{U \in \mathcal{P} \mid x \in U\},$$

for every $x \in X$. Then $P = (X, \mathcal{P}, f)$ is a (set-theoretical) P-system such that $T^t(X, \mathcal{P}) = Sc(P)$ (see 5.2.1.5 for $Sc(P)$). For proving this, take two finite subsets $A = \{a_i \in X \mid i = 1, \ldots, n\}$ and $B = \{b_j \in X \mid j = 1, \ldots, m\}$ of X. Let $A \vdash_P B$. This means that $\bigcap\{f(a_i) \mid i = 1, \ldots, n\} \subseteq \bigcup\{f(b_j) \mid j = 1, \ldots, m\}$. Let $U \in \mathcal{P}$ and $A \subseteq U$. Then $U \in \bigcap\{f(a_i) \mid i = 1, \ldots, n\}$. Hence $U \in \bigcup\{f(b_j) \mid j = 1, \ldots, m\}$. Thus $B \cap U \neq \emptyset$. So, we have proved that $A \vdash_{\mathcal{P}} B$. Conversely, let $A \vdash_{\mathcal{P}} B$. Take a $U \in \bigcap\{f(a_i) \mid i = 1, \ldots, n\}$. Then $A \subseteq U$. Now, the definition of the relation $\vdash_{\mathcal{P}}$ implies that $U \cap B \neq \emptyset$. Thus $U \in \bigcup\{f(b_j) \mid j = 1, \ldots, m\}$. So, $\bigcap\{f(a_i) \mid i = 1, \ldots, n\} \subseteq \bigcup\{f(b_j) \mid j = 1, \ldots, m\}$, i.e., $A \vdash_P B$. Therefore the relations $\vdash_P$ and $\vdash_{\mathcal{P}}$ coincide on the finite subsets of X. Then, by their definitions, they coincide on arbitrary subsets of X. So, $T^t(X, \mathcal{P}) = Sc(P)$. Since, by 5.2.1.5, $Sc(P)$ is an S-system, we get that the images of the objects of the category **TPS** by T^t are objects of the category **SSyst**. Let's show that the images of the morphisms of the category **TPS** by T^t are morphisms of the category **SSyst**. Indeed, let

$$\varphi \in \mathbf{TPS}((X, \mathcal{P}), (X', \mathcal{P}')).$$

Take two finite subsets A and B of X such that $A \vdash_{\mathcal{P}} B$. We have to prove that $\varphi(A) \vdash_{\mathcal{P}'} \varphi(B)$. Let $U' \in \mathcal{P}'$ be such that $\varphi(A) \subseteq U'$. Then $\varphi^{-1}(U') \in \mathcal{P}$ and $A \subseteq \varphi^{-1}(U')$. Since $A \vdash_{\mathcal{P}} B$, we obtain that $B \cap \varphi^{-1}(U') \neq \emptyset$. Thus $U' \cap \varphi(B) \neq \emptyset$. So, $\varphi(A) \vdash_{\mathcal{P}'} \varphi(B)$. Therefore φ is an S-morphism. It is now easily seen that T^t is a functor from the category **TPS** to the category **SSyst**.

Step 2. We will prove that the functor $T^t \circ T^a$ coincides with the identity functor $Id_{\mathbf{SSyst}}$ of the category **SSyst**.

Let $(W, \vdash) \in |\mathbf{SSyst}|$. Then $(T^t \circ T^a)(W, \vdash) = T^t(W, \mathcal{P}_W) = (W, \vdash_{\mathcal{P}_W})$ (using the notation of *Step 1* and denoting by $\mathcal{P}_W$ the family $PrF(W, \vdash)$). For any two finite subsets A and B of W, we have, by 5.2.1.6, that $A \vdash_{\mathcal{P}_W} B$ iff $A \vdash B$. This implies that the same is valid for arbitrary subsets of W. So, $(T^t \circ T^a)(W, \vdash) = (W, \vdash)$, i.e., $T^t \circ T^a$

and $Id_{\mathbf{SSyst}}$ coincide on the objects of **SSyst**. Since they, obviously, coincide on the morphisms of **SSyst**, the equality $T^t \circ T^a = Id_{\mathbf{SSyst}}$ is proved.

Step 3. We will prove that:

(a) if $\mathbf{C} = (T^a \circ T^t)(\mathbf{TPS})$ then $\mathbf{C} = (T^a \circ T^t)(\mathbf{C})$ and $C = (T^a \circ T^t)(C)$ for every $C \in |\mathbf{C}|$;

(b) the subcategory $\mathbf{C}$ of **TPS** is isomorphic to **SSyst**.

Using *Step 2*, we obtain that $T^a(\mathbf{SSyst}) = (T^a \circ T^t \circ T^a)(\mathbf{SSyst}) \subseteq (T^a \circ T^t)(\mathbf{TPS}) = \mathbf{C}$. Hence the functor $T^a : \mathbf{SSyst} \longrightarrow \mathbf{TPS}$ can be regarded also as a functor from **SSyst** to $\mathbf{C}$. Denote this functor by T^a_1, i.e.,

$$T^a_1 : \mathbf{SSyst} \longrightarrow \mathbf{C}.$$

Denote by T^t_1 the restriction of the functor T^t to the subcategory $\mathbf{C}$ of the category **TPS**, i.e.,

$$T^t_1 : \mathbf{C} \longrightarrow \mathbf{SSyst}.$$

Then, by *Step 2*, $T^t_1 \circ T^a_1 = Id_{\mathbf{SSyst}}$. Further, we will show that $T^a_1 \circ T^t_1 = Id_{\mathbf{C}}$. This is obviously true on the morphisms of $\mathbf{C}$. Let $C \in |\mathbf{C}|$. Then $C = (T^a \circ T^t)(D)$ for some $D \in |\mathbf{TPS}|$. Using again *Step 2*, we obtain that $(T^a_1 \circ T^t_1)(C) = (T^a \circ T^t \circ T^a \circ T^t)(D) = (T^a \circ T^t)(D) = C$. So, $T^a_1 \circ T^t_1 = Id_{\mathbf{C}}$. Hence, $\mathbf{C} = (T^a \circ T^t)(\mathbf{C})$ and $\mathbf{C}$ is isomorphic to **SSyst**.

Step 4. We will prove that the subcategories **TPSS** and $\mathbf{C}$ (see *Step 3* for $\mathbf{C}$) of the category **TPS** coincide.

Let $(X, \mathcal{P}) \in |\mathbf{C}|$. We will show that $(X, \mathcal{P}) \in |\mathbf{TPSS}|$. Let $V \subseteq X$ be such that for any two finite sets $F \subseteq V$ and $G \subseteq X \setminus V$ there exists a $U \in \mathcal{P}$ with $F \subseteq U$ and $U \cap G = \emptyset$. By *Step 3*, we have that $(X, \mathcal{P}) = (T^a \circ T^t)(X, \mathcal{P})$. Hence $\mathcal{P} = PrF(X, \vdash_{\mathcal{P}})$ (see *Step 1* for the notation). So, we have to prove that V is a prime filter in $(X, \vdash_{\mathcal{P}})$. By 5.2.1.7(1), it is enough to show that $V \not\vdash_{\mathcal{P}} (X \setminus V)$. Let A be a finite subset of V and B be a finite subset of $X \setminus V$. Then, by our hypothesis, there exists a $U \in \mathcal{P}$ with $A \subseteq U$ and $U \cap B = \emptyset$. This means that $A \not\vdash_{\mathcal{P}} B$ (see the definition of $\vdash_{\mathcal{P}}$ in *Step 1*). Thus $V \not\vdash_{\mathcal{P}} (X \setminus V)$. So, we have proved that $(X, \mathcal{P}) \in |\mathbf{TPSS}|$.

Let $(X, \mathcal{P}) \in |\mathbf{TPSS}|$. We will show that $(X, \mathcal{P}) \in |\mathbf{C}|$ by proving that $(X, \mathcal{P}) = (T^a \circ T^t)(X, \mathcal{P})$. Since $(T^a \circ T^t)(X, \mathcal{P}) = (X, PrF(X, \vdash_{\mathcal{P}}))$, we have to prove that $\mathcal{P} = PrF(X, \vdash_{\mathcal{P}})$. Let $V \in PrF(X, \vdash_{\mathcal{P}})$ and let F be a finite subset of V. Since, by 5.2.1.7(1), $V \not\vdash_{\mathcal{P}} (X \setminus V)$, we obtain that $F \not\vdash_{\mathcal{P}} G$, for every finite subset G of $X \setminus V$. Hence, by the definition of the relation $\vdash_{\mathcal{P}}$ (see *Step 1*), for every finite subset G of

$X \setminus V$ there exists an element U of $\mathcal{P}$ such that $F \subseteq U$ and $U \cap G = \emptyset$. Since $(X, \mathcal{P})$ satisfies condition (TPSS) from 5.4.1.3, we obtain that $V \in \mathcal{P}$. So, we have proved that $PrF(X, \vdash_{\mathcal{P}}) \subseteq \mathcal{P}$. Conversely, let $V \in \mathcal{P}$. Then the definition of the relation $\vdash_{\mathcal{P}}$ implies that if F is a finite subset of V and G is a finite subset of $X \setminus V$ then $F \not\vdash_{\mathcal{P}} G$. Hence $V \not\vdash_{\mathcal{P}} (X \setminus V)$. Thus, by 5.2.1.7(1), V is a prime filter in $(X, \vdash_{\mathcal{P}})$. So, $\mathcal{P} \subseteq PrF(X, \vdash_{\mathcal{P}})$. Hence, $\mathcal{P} = PrF(X, \vdash_{\mathcal{P}})$. Therefore $(X, \mathcal{P}) \in |\mathbf{C}|$. So, the subcategories **TPSS** and **C** of the category **TPS** coincide.

Now, we complete the proof of our theorem combining the results obtained in *Step 3* and *Step 4*. □

Let's remark that in *Step 4* of the proof of 5.4.1.4 we obtained, in fact, the following result:

Proposition 5.4.1.5. *Let $(X, \mathcal{P}) \in |\mathbf{TPS}|$ and $\mathcal{P}' = PrF(X, \vdash_{\mathcal{P}})$ (see Step 1 in the proof of 5.4.1.4 for the notation). Then a subset V of X belongs to $\mathcal{P}'$ iff for any two finite sets $F \subseteq V$ and $G \subseteq X \setminus V$ there exists an element U of $\mathcal{P}$ such that $F \subseteq U$ and $U \cap G = \emptyset$. In particular, $\mathcal{P} \subseteq \mathcal{P}'$.*

As a special case of this proposition, we obtain immediately the following corollary:

Corollary 5.4.1.6. *Let $(X, \mathcal{P}) \in |\mathbf{TPS}|$ and $\mathcal{P}' = PrF(X, \vdash_{\mathcal{P}})$ (see Step 1 in the proof of 5.4.1.4 for the notation). Then*

(a) $X \in \mathcal{P}'$ iff $\mathcal{P}$ is an ω-cover of X (i.e., for every finite subset F of X there exists an element U of $\mathcal{P}$ containing F);

(b) $\emptyset \in \mathcal{P}'$ iff for every finite subset F of X there exists an element U of $\mathcal{P}$ such that $U \cap F = \emptyset$.

We have also the following result:

Proposition 5.4.1.7. *Let $(X, \mathcal{P}) \in |\mathbf{TPS}|$, $\mathcal{P}' = PrF(X, \vdash_{\mathcal{P}})$ (see Step 1 in the proof of 5.4.1.4 for the notation) and let $\mathcal{P}$ be closed under finite intersections. Then $\mathcal{P}'$ is closed under arbitrary intersections.*

Proof. First of all, using the fact that $\mathcal{P}$ is closed under finite intersections, we will prove that for a subset V of X the following conditions are equivalent:

(1) $V \in \mathcal{P}'$;

(2) if F is a finite subset of V then $\bigcap\{U \in \mathcal{P} \mid F \subseteq U\} \subseteq V$.

The implication (1) $\Rightarrow$ (2) follows immediately from 5.4.1.5. Let's show that (2) $\Rightarrow$ (1). Let F be a finite subset of V and $G = \{g_i \mid i = 1, \ldots, n\} \subseteq X \setminus V$. Then, by (2), for every $i \in \{1, \ldots, n\}$ there exists an element U_i of $\mathcal{P}$ such that $F \subseteq U_i$ and $g_i \notin U_i$. Putting $U = \bigcap\{U_i \mid i = 1, \ldots, n\}$, we obtain that $U \in \mathcal{P}$, $F \subseteq U$ and $U \cap G = \emptyset$. Hence, by 5.4.1.5, $V \in \mathcal{P}'$. So, the conditions (1) and (2) are equivalent.

Let A be a set and, for every $\alpha \in A$, V_α be an element of $\mathcal{P}'$. Put $V = \bigcap\{V_\alpha \mid \alpha \in A\}$. We will check that V satisfies (2). Take a finite subset F of V. Then $F \subseteq V_\alpha$, for every $\alpha \in A$. Since $V_\alpha \in \mathcal{P}'$, we have, by (2), that $\bigcap\{U \in \mathcal{P} \mid F \subseteq U\} \subseteq V_\alpha$. Hence $\bigcap\{U \in \mathcal{P} \mid F \subseteq U\} \subseteq \bigcap\{V_\alpha \mid \alpha \in A\} = V$. So, condition (2) is fulfilled. Thus, $V \in \mathcal{P}'$. $\square$

5.4.2 T-systems and hyperspaces

Let's now concentrate on T-systems.

Proposition 5.4.2.1. *Let $(W, \vdash)$ be a T-system and $\mathcal{P} = PrF(W, \vdash)$ (see 5.2.1.4 for the notation). Then:*

(a) for a $V \subseteq W$, we have that $V \in \mathcal{P}$ iff $\bigcap\{U \in \mathcal{P} \mid F \subseteq U\} \subseteq V$ for every finite subset F of V;

(b) $\mathcal{P}$ is closed under arbitrary intersections.

Proof. (a) Let V be a subset of W. Using consecutively 5.2.1.7(1), (TFin) (see 5.2.1.2) and 5.2.1.6, we obtain that $(V \in \mathcal{P}) \iff (V \not\vdash (W \setminus V)) \iff (\forall\, w \in (W \setminus V)$ and $\forall$ finite subset F of V we have that $F \not\vdash w) \iff (\forall\, w \in (W \setminus V)$ and $\forall$ finite subset F of V $\exists\, U \in \mathcal{P}$ such that $F \subseteq U$ and $w \notin U) \iff (\bigcap\{U \in \mathcal{P} \mid F \subseteq U\} \subseteq V$, for every finite subset F of $V)$.

(b) Let A be a set and, for every $\alpha \in A$, V_α be an element of $\mathcal{P}$. Put $V = \bigcap\{V_\alpha \mid \alpha \in A\}$. Let F be a finite subset of V. Then $F \subseteq V_\alpha$, for every $\alpha \in A$. By a) , we obtain that $\bigcap\{U \in \mathcal{P} \mid F \subseteq U\} \subseteq V_\alpha\}$, for every $\alpha \in A$. Hence $\bigcap\{U \in \mathcal{P} \mid F \subseteq U\} \subseteq \bigcap\{V_\alpha \mid \alpha \in A\} = V$. Now, a) implies that $V \in \mathcal{P}$. $\square$

Corollary 5.4.2.2. *Let $(W, \vdash)$ be a T-system and $\mathcal{P} = PrF(W, \vdash)$ (see 5.2.1.4 for this notation). Then, for a $V \subseteq W$, we have that $V \in \mathcal{P}$ iff for every finite subset F of V there exists an element U of $\mathcal{P}$ such that $F \subseteq U \subseteq V$.*

Proof. It follows directly from a) and b) of 5.4.2.1. □

Definition 5.4.2.3. ([3]) A topological space X is called an *Alexandroff space* if the intersection of any family of open subsets of X is an open subset of X.

Remark 5.4.2.4. The Alexandroff spaces were introduced by P.S. Alexandroff in [3] under the name of *discrete spaces*. They are now known as *Alexandroff spaces* or, shortly, *A-spaces* (see [74]). Note that the term "A-space" or "Alexandroff space" is used in the literature with another meaning as well (see, for example, [68]).

Definition 5.4.2.5. Let X be a set, $\mathcal{M} \subseteq P(X)$ and $\mathcal{O}$ be a topology on the set $\mathcal{M}$. We say that $\mathcal{O}$ is a *topology of Tychonoff type on* $\mathcal{M}$ if the family $\mathcal{O} \cap \{A^+_{\mathcal{M}} \mid A \subseteq X\}$, where

$$A^+_{\mathcal{M}} = \{M \in \mathcal{M} \mid M \subseteq A\},$$

is an open base of $\mathcal{O}$. In what follows, we will denote by

$$\mathcal{P}_{\mathcal{O},\mathcal{M}}$$

the family $\{A \subseteq X \mid A^+_{\mathcal{M}} \in \mathcal{O}\}$. When $\mathcal{M} = Fin(X)$ (see 0.1.2.1 for this notation), we will write simply

$$\mathcal{P}_{\mathcal{O}}$$

instead of $\mathcal{P}_{\mathcal{O},Fin(X)}$.

Remark 5.4.2.6. The above definition was given in [39]. After the publication of [39], we learned that a particular case of such a topology (namely, when X is a topological space and $\mathcal{M}$ is the family of all closed subsets of X) was introduced earlier by M. Choban in his remarkable paper [17]. A detailed investigation of the topologies of Tychonoff type on arbitrary families $\mathcal{M}$ (as they were introduced above) was done later in [36].

Definition 5.4.2.7. We denote by $\mathbf{T}'$ the category whose objects are all Alexandroff spaces of the form $(Fin(X), \mathcal{O})$, where X is a non-empty set and $\mathcal{O}$ is a topology of Tychonoff type on $Fin(X)$, and, for any two objects $(Fin(X), \mathcal{O})$ and $(Fin(X'), \mathcal{O}')$ of $\mathbf{T}'$, the set $\mathbf{T}'((Fin(X), \mathcal{O}), (Fin(X'), \mathcal{O}'))$ consists of all $f \in \mathbf{Set}(X, X')$ for which the map $f_{Fin} : Fin(X) \longrightarrow Fin(X')$, defined by $f_{Fin}(F) = f(F)$ for any $F \in Fin(X)$, is a continuous map between $(Fin(X), \mathcal{O})$ and $(Fin(X'), \mathcal{O}')$.

Proposition 5.4.2.8. *Let $f \in \mathbf{Set}(X, X')$ and $\mathcal{O}$ (resp., $\mathcal{O}'$) be a topology of Tychonoff type on Fin(X) (resp., on $Fin(X')$). Then the following are equivalent:*

(a) $f_{Fin} \in \mathbf{Top}((Fin(X), \mathcal{O}), (Fin(X'), \mathcal{O}'))$ (see 5.4.2.7 for the definition of f_{Fin});

(b) $f \in \mathbf{TPS}((X, \mathcal{P}_{\mathcal{O}}), (X', \mathcal{P}_{\mathcal{O}'}))$ (see 5.4.2.5 for the notation).

Proof. Let's first remark that if $A \subseteq X'$ then

(5.1) $f^{-1}_{Fin}(A^+) = (f^{-1}(A))^+$

(here and below we write, for short,

$$A^+$$

instead of $A^+_{Fin(X)}$ (see 5.4.2.5 for the notation $A^+_{Fin(X)}$). Indeed, if F is a finite subset of X then we have: $(F \in f^{-1}_{Fin}(A^+)) \iff (f_{Fin}(F) \in A^+) \iff (f(F) \subseteq A) \iff (F \subseteq f^{-1}(A)) \iff (F \in (f^{-1}(A))^+)$. So, $f^{-1}_{Fin}(A^+) = (f^{-1}(A))^+$.

$(a) \Rightarrow (b)$. Take an $A \in \mathcal{P}_{\mathcal{O}'}$. Then $A^+ \in \mathcal{O}'$ and hence $f^{-1}_{Fin}(A^+) \in \mathcal{O}$. Thus, by (**), $(f^{-1}(A))^+ \in \mathcal{O}$. This implies that $f^{-1}(A) \in \mathcal{P}_{\mathcal{O}}$. So, $f^{-1}(\mathcal{P}_{\mathcal{O}'}) \subseteq \mathcal{P}_{\mathcal{O}}$.

$(b) \Rightarrow (a)$. Since, by the definition of Tychonoff type topology (see 5.4.2.5), the family

$$\mathcal{P}^+_{\mathcal{O}'} = \{A^+ \mid A \in \mathcal{P}_{\mathcal{O}'}\} = \mathcal{O}' \cap \{A^+ \mid A \subseteq X'\}$$

is an open base of the topology $\mathcal{O}'$, it is enough to show that $f^{-1}_{Fin}(A^+) \in \mathcal{O}$ for every $A \in \mathcal{P}_{\mathcal{O}'}$. So, let $A \in \mathcal{P}_{\mathcal{O}'}$. Then, by (5.1), $f^{-1}_{Fin}(A^+) = (f^{-1}(A))^+$. Since $f^{-1}(A) \in \mathcal{P}_{\mathcal{O}}$, we obtain that $(f^{-1}(A))^+ \in \mathcal{O}$. Hence, f_{Fin} is a continuous function. $\square$

Notation 5.4.2.9. We denote by **TSyst** the full subcategory of **SSyst** whose objects are all Tarski consequence systems.

Theorem 5.4.2.10. *The categories* **TSyst** *and* $\mathbf{T}'$ *are isomorphic.*

Proof. If $(W, \vdash) \in |\mathbf{SSyst}|$ then we will write

$$\mathcal{P}_W$$

instead of $PrF(W, \vdash)$; if X is a set then U^+ will stand for $U^+_{Fin(X)}$ (see 5.2.1.4 and 5.4.2.5 for the notation).

The proof of the theorem will be carry out in several steps.

Step 1. In this step we define a functor $T'' : \mathbf{TSyst} \longrightarrow \mathbf{T}'$.

For every $(W, \vdash) \in |\mathbf{TSyst}|$, put

$$T''(W, \vdash) = (Fin(W), \mathcal{O}),$$

where $\mathcal{O}$ is the topology on the set $Fin(W)$ having as a base the family

$$\mathcal{P}_W^+ = \{U^+ \mid U \in \mathcal{P}_W\}.$$

If $f \in \mathbf{TSyst}((W, \vdash), (W', \vdash'))$ then put

$$T''(f) = f.$$

Let's show that T'' is a functor from the category $\mathbf{TSyst}$ to the category $\mathbf{T}'$. Take a $(W, \vdash) \in |\mathbf{TSyst}|$. Then, by 5.2.1.7(2) and (TFin) (see 5.2.1.2), we obtain that $W \in \mathcal{P}_W$. Further, the family $\mathcal{P}_W$ is closed under arbitrary intersections (by 5.4.2.1(b)). Since, obviously,

(5.2) $(\bigcap\{U_\alpha \mid \alpha \in A\})^+ = \bigcap\{U_\alpha^+ \mid \alpha \in A\}, \quad$ for every set A,

we get that $\mathcal{P}_W^+$ is closed under arbitrary intersections. So, the family $\mathcal{P}_W^+$ can be taken as a base of a topology $\mathcal{O}$ on $Fin(W)$. Evidently, $(Fin(W), \mathcal{O})$ is an Alexandroff space. Since $\mathcal{O} \cap \{A^+ \mid A \subseteq W\} \supseteq \mathcal{P}_W^+$, we obtain that $\mathcal{O}$ is a topology of Tychonoff type on $Fin(W)$. Thus, $T''(W, \vdash) \in |\mathbf{T}'|$.

Let's prove that

(5.3) $\mathcal{O} \cap \{A^+ \mid A \subseteq W\} = \mathcal{P}_W^+$.

We will use this equality in Step 3 below. Obviously, it suffices to demonstrate that $\mathcal{O} \cap \{A^+ \mid A \subseteq W\} \subseteq \mathcal{P}_W^+$. Let $A \subseteq W$ and $A^+ \in \mathcal{O}$. Take an $F \in A^+$. Since $A^+ \in \mathcal{O}$ and $\mathcal{P}_W^+$ is a base of $\mathcal{O}$, there exists an element $U \in \mathcal{P}_W$ such that $F \in U^+ \subseteq A^+$. This implies that $F \subseteq U \subseteq A$. Now, by 5.4.2.2, we obtain that $A \in \mathcal{P}_W$. Hence, $A^+ \in \mathcal{P}_W^+$. So, the equality (5.3) is proved.

Take now a morphism $f \in \mathbf{TSyst}((W, \vdash), (W', \vdash'))$. We have to show that

$$T''(f) \in \mathbf{T}'(T''(W, \vdash), T''(W', \vdash')).$$

By 5.4.2.8, it is enough to prove that $f^{-1}(\mathcal{P}_{W'}) \subseteq \mathcal{P}_W$. Since this follows directly from 5.3.1.2, we get that $T''(f) \in \mathbf{T}'(T''(W, \vdash), T''(W', \vdash'))$. It is now easily seen that T'' is a functor from the category $\mathbf{TSyst}$ to the category $\mathbf{T}'$.

Step 2. We will define a functor

$$S'' : \mathbf{T}' \longrightarrow \mathbf{TSyst}.$$

If $(Fin(X), \mathcal{O}) \in |\mathbf{T}'|$ then we put

$$S''(Fin(X), \mathcal{O}) = T^t(X, \mathcal{P}_{\mathcal{O}}),$$

where T^t is the functor defined in *Step 1* of the proof of 5.4.1.4 (see also 5.4.2.5, 5.4.2.7 and 0.1.2.1 for the notation). (Hence, $S''(Fin(X), \mathcal{O}) = (X, \vdash_{\mathcal{P}_{\mathcal{O}}})$.)

If $f \in \mathbf{T}'((Fin(X), \mathcal{O}), (Fin(X'), \mathcal{O}'))$ then we put

$$S''(f) = f.$$

We will show that S'' is a functor from the category **T**' to the category **TSyst**. Let $(Fin(X), \mathcal{O}) \in |\mathbf{T}'|$. Then $(X, \mathcal{P}_{\mathcal{O}}) \in \mathbf{TPS}$ and, by the proof of 5.4.1.4, $T^t(X, \mathcal{P}_{\mathcal{O}})$ is an S-system. Hence $S''(Fin(X), \mathcal{O})$ is an S-system and we have to prove only that it satisfies condition (TFin) from 5.2.1.2. So, let A and B be two finite subsets of X and $A \vdash_{\mathcal{P}_{\mathcal{O}}} B$. Then, by the definition of $\vdash_{\mathcal{P}_{\mathcal{O}}}$ (see *Step 1* of the proof of 5.4.1.4 for it),

$$(\forall U \in \mathcal{P}_{\mathcal{O}})((A \subseteq U) \Rightarrow (U \cap B \neq \emptyset)).$$

Since the family $\mathcal{O}$ is closed under arbitrary intersections, the equality (5.2) implies that $\mathcal{P}_{\mathcal{O}}$ is also closed under arbitrary intersections. Hence

$$U_0 = \bigcap \{U \in \mathcal{P}_{\mathcal{O}} \mid A \subseteq U\}$$

is an element of $\mathcal{P}_{\mathcal{O}}$. Thus $U_0 \cap B \neq \emptyset$. Let $b \in U_0 \cap B$. Then, for every $U \in \mathcal{P}_{\mathcal{O}}$ such that $A \subseteq U$, we have that $b \in U_0 \cap B \subseteq U \cap B$. So $b \in U$, for every $U \in \mathcal{P}_{\mathcal{O}}$ such that $A \subseteq U$. This means that $A \vdash_{\mathcal{P}_{\mathcal{O}}} \{b\}$. Hence, condition (TFin) from 5.2.1.2 is fulfilled. Thus, $S''(Fin(X), \mathcal{O})$ is a T-system. Let now $f \in \mathbf{T}'((Fin(X), \mathcal{O}), (Fin(X'), \mathcal{O}'))$. Then, by 5.4.2.8, $f \in \mathbf{TPS}((X, \mathcal{P}_{\mathcal{O}}), (X', \mathcal{P}_{\mathcal{O}'}))$. So, $S''(f) = T^t(f)$. Since, by the proof of 5.4.1.4, $T^t(f)$ is an **SSyst**-morphism, we obtain that $S''(f)$ is a **TSyst**-morphism. It is now easily seen that S'' is a functor from the category **T**' to the category **TSyst**.

Step 3. We will prove that $S'' \circ T'' = Id_{\mathbf{TSyst}}$.

Let $(W, \vdash) \in |\mathbf{TSyst}|$. Then $(S'' \circ T'')(W, \vdash) = S''(Fin(W), \mathcal{O})$, where $\mathcal{O}$ has as a base the family $\mathcal{P}_W^+$ (see *Step 1* here). Hence $(S'' \circ T'')(W, \vdash) = (W, \vdash_{\mathcal{P}_{\mathcal{O}}})$. Since, by (5.3), $\mathcal{P}_{\mathcal{O}} = \mathcal{P}_W$, we obtain, as in *Step 2* of the proof of 5.4.1.4, that $\vdash$ coincides with

$\vdash_{\mathcal{P}_{\mathcal{O}}}$. So, $(S'' \circ T'')(W, \vdash) = Id_{\mathbf{TSyst}}(W, \vdash)$. Since the corresponding equality for the morphisms is obvious, we conclude that $S'' \circ T'' = Id_{\mathbf{TSyst}}$.

Step 4. We will prove that $T'' \circ S'' = Id_{\mathbf{T}'}$.

Let $(Fin(X), \mathcal{O}) \in |\mathbf{T}'|$. Then

$$(T'' \circ S'')(Fin(X), \mathcal{O}) = T''(X, \vdash_{\mathcal{P}_{\mathcal{O}}}) = (Fin(X), \mathcal{O}'),$$

where the topology $\mathcal{O}'$ on $Fin(X)$ has as a base the family $\mathcal{P}_X^+$ (recall that $\mathcal{P}_X = PrF(X, \vdash_{\mathcal{P}_{\mathcal{O}}})$). Using 5.4.1.5 and the fact that $\mathcal{P}_{\mathcal{O}}$ is closed under arbitrary intersections (since $\mathcal{O}$ has this property), we obtain: $(V \in \mathcal{P}_X) \iff (V \subseteq X$ and for any two finite sets $F \subseteq V$ and $G \subseteq X \setminus V$ there exists an element U of $\mathcal{P}_{\mathcal{O}}$ such that $F \subseteq U$ and $U \cap G = \emptyset) \iff (V \subseteq X$ and for every finite subset F of V there exists an element U of $\mathcal{P}_{\mathcal{O}}$ such that $F \subseteq U \subseteq V) \iff (V \subseteq X$ and for every $F \in V^+$ there exists an element U of $\mathcal{P}_{\mathcal{O}}$ such that $F \in U^+ \subseteq V^+) \iff (V^+ \in \mathcal{O}) \iff (V \in \mathcal{P}_{\mathcal{O}})$. Hence, $\mathcal{P}_X = \mathcal{P}_{\mathcal{O}}$. This implies that $\mathcal{O} = \mathcal{O}'$. Thus $(T'' \circ S'')(Fin(X), \mathcal{O}) = Id_{\mathbf{T}'}(Fin(X), \mathcal{O})$. Since the corresponding equality for the morphisms is obvious, we get $T'' \circ S'' = Id_{\mathbf{T}'}$.

All this shows that the categories **TSyst** and $\mathbf{T}'$ are isomorphic. $\square$

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