Distributions in Insurance Risk Models

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Chapter 1

Introduction

1.1 Risk Model

The foundation of the modern risk theory goes back to the works of Filip Lundberg and Harald Cramér. In 1903 Filip Lundberg proposed the Poisson process as a simple process in solving the problem of the first passage time. Lundberg's work has been extended by Harald Cramér in 1930 for modeling the ruin of an insurance company as a first passage time problem. The basic model is called a Cramér - Lundberg model or classical risk model. Insurance Risk Theory is a synonym of non - life insurance mathematics.

The standard model of an insurance company, called *risk process* $\{X(t), t \geq 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$, is given by

$$X(t) = ct - \sum_{k=1}^{N(t)} Z_k, \quad (\sum_{1}^{0} = 0). \tag{1.1}$$

Here c is a positive real constant representing the *gross risk premium* rate. The sequence $\{Z_k\}_{k=1}^{\infty}$ of mutually independent and identically distributed random variables with common distribution function F, $F(0) = 0$, and mean value μ is independent of the *counting process* $N(t)$, $t \geq 0$. The process $N(t)$ is interpreted as the number of claims to the company during the interval $[0, t]$.

In the classical risk model, the process $N(t)$ is a *stationary* Poisson counting process, see for instance Grandell [37]. In this case, the aggregate claim amount up to time t is given by the compound Poisson process $S(t) = \sum_{i=1}^{N(t)} Z_i$. If the number of claims $N(t)$ forms a renewal counting process, the model (1.1) is called a renewal risk model. There are many directions in which the classical risk model and the renewal model are generalized in order to become a reasonably realistic description. D. Dickson [25] studied a generalization of the renewal model, assuming that claims occur as an Erlang process and extended several classical results. Lin and Sendova [53] have analyzed the multiple thresholds in the compound Poisson risk model. References are given in Asmussen [5] and Rolski *et* al. [77]. Our interest is in the generalization of the counting process $N(t)$.

The basic stochastic elements of the general risk model are:

i) The times $0 \leq \sigma_1 \leq \sigma_2 \leq \ldots$, of claim arrivals. Suppose that $\sigma_0 = 0$. The random variables $T_n = \sigma_n - \sigma_{n-1},\ n = 1, 2, \ldots$, called *inter - occurrence* or *inter - arrival times* are nonnegative.

ii) $N(t) = \sup\{n : \sigma_n \leq t\},\ t \geq 0$ is the number of claims up to time t. The relations between the times $\{\sigma_0, \sigma_1, \ldots\}$ and the counting process $\{N(t),\ t \geq 0\}$ are given by

$$\{N(t) = n\} = \{\sigma_n \leq t < \sigma_{n+1}\},\ n = 0, 1, \ldots.$$

iii) The sequence $\{Z_n,\ n = 1, 2, \ldots\}$ of independent identically distributed random variables represents the amounts of the successful claims to the insurance company. Suppose that the sequence $\{Z_n\}$ is independent of the counting process $N(t)$.

The accumulated sum of claims up to time t is given by

$$S(t) = \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0.$$

The process $S = (S(t))_{t \geq 0}$ is defined by the sum $S_n = Z_1 + \ldots + Z_n$, where n is a realization of the random variable $N(t)$:

$$S(t) = Z_1 + \ldots + Z_{N(t)} = S_{N(t)}, \quad t \geq 0,$$

or a random sum of random variables. Suppose that $S(t) = 0$, if $N(t) = 0$.

The risk reserve of the insurance company with initial capital u is given by

$$U(t) = u + ct - S(t), \quad t \geq 0, \tag{1.2}$$

where $S(t)$ is the risk model, given in (1.1).

As we mentioned, our interest is related to the counting process. In these notes we define some compound counting processes and the corresponding risk models.

In Chapter 2 we define the family of Inflated - parameter generalized power series distribution (IGPSDs), as a compound Generalized Power series distributions with a geometric compounding distribution. The defined distributions are related to a homogeneous two - state Markov chain.

In Chapter 3 the Pólya - Aeppli distribution of order k is defined. It is a compound Poisson distribution with truncated geometric compounding distribution.

In Chapter 4 some discrete distributions related to a multi - state Markov chain are defined.

The corresponding I - stochastic processes are defined in Chapter 5. We give an alternative definition as a pure birth process.

In Chapter 6 the application of these processes in risk theory is given.

1.2 Counting probability distributions

During the last decade, a vast activity have been observed in generalizing of the classical discrete distributions. The main idea was to apply the extended versions for modeling different kinds of dependent count or frequency data structure in various fields (Econometrics, Insurance, Finance, Biometrics, etc.), see for example Bowers et *al.* [15], Johnson et *al.* [45], Rolski et *al.* [77], Winkelmann [86] and references therein.

In the general introduction of the monographs [15], [77], [86] is emphasized the need for richer classes of probability distributions when modeling count data. Since the probability distributions for counts are nonstandard, special

attention is paid here for more flexible distributions. They can be used as a building blocks for improved count data models with immediate application, for instance in insurance describing the number of claims.

In this section we suggest a short review of the classical counting distributions.

The random variables considered are assumed to be defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let N be a non-negative integer-valued random variable representing the number of claims to an insurance company for a fixed time period with probability mass function (PMF), given by

$$p_m = P(N = m), \quad m = 0, 1, \ldots.$$

Let $P_N(s)$ be the probability generating function (PGF) of the random variable N.

There are at least three particular cases that are applicable in insurance as a claim number distributions: the Poisson, Negative Binomial and Binomial distribution. We write

(i) $N \sim Po(\lambda)$ if N has a Poisson distribution with parameter $\lambda > 0$, i.e.

$$p_m = \frac{e^{-\lambda}\lambda^m}{m!}, \quad m = 0, 1, \ldots;$$

(ii) $N \sim NB_0(\pi, r)$ if N has a negative binomial (NB_0) distribution starting from zero with parameters $\pi \in (0, 1)$ and $r > 0$, i.e.

$$p_m = \binom{r+m-1}{m}\pi^r(1-\pi)^m, \quad m = 0, 1, \ldots.$$

In the special case $r = 1$, we obtain the geometric distribution, $Ge_0(\pi)$ on the non-negative integers.

(iii) $N \sim Bi(n, \pi)$ if N has a binomial distribution with parameters $\pi \in (0, 1)$ and $n \in \{1, 2, \ldots\}$, i.e.

$$p_m = \binom{n}{m}\pi^m(1-\pi)^{n-m}, \quad m = 0, 1, \ldots, n.$$

When $n = 1$, we obtain the Bernoulli random variable with parameter π;

(iv) $N \sim LS(\pi)$ if N has a logarithmic series distribution with parameter $\pi \in (0,1)$, i.e.

$$p_m = \frac{\pi^m}{-m\log(1-\pi)}, \quad m = 1, 2, \ldots.$$

The logarithmic series distribution is used rather rarely. It can be obtained as a limiting distribution of the truncated at zero NB distribution. Chatfield et *al.* [18] use the logarithmic series distribution to model the number of purchases in any time period, excluding zeros or non-buyers;

(v) $N \sim \delta_m$, if N is concentrated on the integer $m \in \{0, 1, \ldots\}$, i.e. degenerated at $N = m$ with

$$p_m = 1, \quad p_i = 0 \quad \text{for} \quad i \neq m.$$

The equality of mean and variance is a characterization of the Poisson distribution and can be referred to as *equi - dispersion*. Departures from equi - dispersion can be either as *over - dispersion* (variance is greater than mean) or *under - dispersion* (variance is less than the mean). The NB distribution is over-dispersed and the Binomial distribution is under-dispersed with respect to the Poisson distribution. The logarithmic series distribution displays over - dispersion for $0 < -[\log(1-\pi)]^{-1} < 1$ and under - dispersion for $-[\log(1-\pi)]^{-1} > 1$.

1.3 Generalized Power Series distributions

Many univariate discrete probability distributions, with a single parameter belong to the class of *generalized power series distributions* (GPSD) or to the classes of their generalizations, see Gupta [39], Consul [20].

Definition 1.1 *A discrete distribution with a parameter $\theta > 0$ and PGF given by*

$$\psi(s) = \frac{g(\theta s)}{g(\theta)}, \tag{1.3}$$

where $g(\theta)$ is a positive, finite and differentiable function, are called Generalized Power Series Distributions (GPSD). For any member of this family, the PMF of the corresponding random variable X can be written as

$$P(N = m) = \frac{a(m)\theta^m}{g(\theta)}, \quad m \in S, \qquad \theta > 0, \tag{1.4}$$

where S is any nonempty enumerable set of nonnegative integers, $a(m) \geq 0$ and $g(\theta) = \sum_{m \in S} a(m)\theta^m$.

The Poisson, Negative binomial, Binomial and logarithmic series distributions belong to this class, see Patil [70]. In the NB and Binomial cases, the corresponding additional parameters n and r are treated as nuisance parameters.

In the particular cases, the functions $a(m)$, $g(\theta)$ and the parameter θ, are given by the following expressions

$$\begin{aligned}
X \sim Po(\theta): &\quad a(m) = \tfrac{1}{m!}, &\quad g(\theta) = e^{\theta}, &\quad \theta = \lambda;\\
X \sim NB(r,\theta): &\quad a(m) = \tbinom{m+r-1}{m}, &\quad g(\theta) = (1-\theta)^{-r}, &\quad \theta = 1-\pi;\\
X \sim Bi(n,\theta): &\quad a(m) = \tbinom{n}{m}, &\quad g(\theta) = (1+\theta)^{n}, &\quad \theta = \tfrac{\pi}{1-\pi};\\
X \sim LS(\theta): &\quad a(m) = \tfrac{1}{m}, &\quad g(\theta) = -ln(1-\theta), &\quad \theta = 1-\pi.
\end{aligned}$$

1.4 The (a, b) class of discrete distributions

The following class of discrete distributions is introduced by Panjer [69] in 1981.

Suppose the PMF of the random variable N satisfies the recursion

$$P(N = m) = \left(a + \frac{b}{m}\right) P(N = m-1), \quad m = 1, 2, \ldots, \tag{1.5}$$

where a and b are constants. The family of distributions, satisfying (1.5) is called (a, b) class and (1.5) is known as Panjer recursion formula. The Poisson, negative binomial, binomial and logarithmic series distributions satisfy the Panjer recursion.

Sundt and Jewell [82] show that the only nondegenerate discrete distributions that satisfy the Panjer recursion formula are the Poisson, Negative binomial, binomial and logarithmic series distributions.

1.5 Zero - inflated distributions

The *zero - inflated distributions* play a crucial role in our exposition and we will briefly introduce and discuss them, see Johnson et *al.* [45], p. 312.

On occasion, the models for counts based on some discrete distribution may encounter lack - of - fit due to disproportionately large frequencies of zeros. Johnson et *al.* [45] describe a simple way of modifying a discrete distribution to handle extra zeros in the following way. Let ξ be an arbitrary nonnegative integer valued random variable defined by $P(\xi = m) = p_m,\ m = 0, 1, \ldots$ and let $G_\xi(s) = E(s^\xi)$ be its PGF. We will refer this discrete distribution as *original distribution*. An extra proportion of zeros, $\rho \in (0, 1)$, is *added* to the proportion of zeros from the distribution of the random variable ξ, while decreasing the remaining proportions in an appropriate way. The PMF of the corresponding *zero - inflated* random variable η can be written as

$$\begin{aligned} P(\eta = 0) &= \rho + (1 - \rho)p_0, \\ P(\eta = j) &= (1 - \rho)p_j, \quad j = 1, 2, \ldots . \end{aligned} \tag{1.6}$$

It has as a PGF

$$G_\eta(s) = \rho + (1 - \rho)G_\xi(s). \tag{1.7}$$

Zero - inflated models address the problem, that the data display a higher fraction of zeros, or non occurrences, than can be possibly explained through any fitted standard count model. The zero - inflated distributions are appropriate alternatives for modeling clustered samples, for example, when the population consists of two sub - populations, the first containing only zeros, while in the other, one observes counts from a discrete distribution.

If $\rho = 1$, than the corresponding zero - inflated distribution is the degenerated at zero one; if $\rho = 0$, nothing is changed in (1.7), i. e. $G_\eta(t) = G_\xi(t)$.

In general, the inflation parameter ρ may take *negative* values provided that $P(\eta = 0) \geq 0$, i. e., $\rho \geq -\frac{p_0}{1-p_0}$ and therefore $\rho \geq \max\{-1, -\frac{p_0}{1-p_0}\}$. This case corresponds to the opposite phenomena - excluding some proportions of zeros from the basic discrete distribution, if necessary.

Let us note, that the zero - inflated distributions are known as a ρ-modification also, which can be considered as a reverse truncated operation, see Rolski et *al.* [77] p. 35.

Chapter 2

I - Generalized Power Series Distributions

During the last decade, a vast activity have been observed in generalizing of the classical discrete distributions. The main idea was to apply the extended versions for modeling different kinds of dependent count or frequency data structure in various fields (Econometrics, Insurance, Finance, Biometrics, etc.), see for example Bowers et *al.* [15], Collett [19], Luceño [55], Rolski et *al.* [77], Winkelmann [86], and references therein.

It is known that many basic counting distributions are defined on a sequence of independent and identically distributed Bernoulli random variables. Many other distributions are defined by compounding and mixing. Another way of obtaining new discrete distributions is to define the counting distributions related to some Markov chain, see Viveros et al. (1994), [85] and Omey et al. (2008), [68]. Assuming some dependency in the sequence of Bernoulli variables, gives an additional parameter, and could be a more realistic model in practice, see Omey and Van Gulck (2006), [67].

The new family of counting distributions, given in this chapter is defined in [57] and [59], and [66]. Some interpretations are given in [51]. In order to obtain new distributions for modeling heterogeneity, we use at first the con-

struction related to two - state Markov chain. All the counting distributions are defined. Then we consider the second construction, by compounding the basic counting distributions. We use the geometric compounding distribution. The compound geometric distribution and compound negative binomial distribution coincide with the corresponding distributions, related to the Markov chain.

2.1 Geometric distribution related to Markov chain

In this section we develop a generalization of the usual geometric distribution by including an additional parameter $\rho \in [0, 1)$. Consider a sequence of binary random variables (ν_n), $n = 0, 1, \ldots$, where the states 1 and 0 appear with stationary probabilities $P(\nu_n = 1) = \pi$ and $P(\nu_n = 0) = q = 1 - \pi$. We call the state 1 "success" and the state 0 "failure". We assume that the sequence ν_n forms a Markov chain with transition probabilities

$$P(\nu_n = 1|\nu_{n-1} = 1) = 1 - q(1 - \rho),$$

$$P(\nu_n = 0|\nu_{n-1} = 0) = 1 - \pi(1 - \rho),$$

where $\rho \in [0, 1)$ and $n = 1, 2, \ldots$.

The transition matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 1 - \pi(1 - \rho) & \pi(1 - \rho) \\ \\ q(1 - \rho) & 1 - q(1 - \rho) \end{pmatrix}. \tag{2.1}$$

The parameter ρ in the defined Markov chain is a correlation coefficient $corr(\nu_{n-1}, \nu_n) = \rho, \quad n = 1, 2, \ldots$ with $\max\{-1, -\frac{1-\pi}{\pi}\} \leq \rho < 1$. For the n step transition matrix we find

$$\boldsymbol{P}^n = \begin{pmatrix} 1 - \pi(1 - \rho^n) & \pi(1 - \rho^n) \\ \\ q(1 - \rho^n) & 1 - q(1 - \rho^n) \end{pmatrix}.$$

Let the random variable X counts the number of failures in the Markov chain up to the first success. The PMF and PGF are given by

$$P(X=k)=\begin{cases}\pi, & k=0\\ (1-\pi)[1-\pi(1-\rho)]^{k-1}(1-\rho)\pi, & k=1,2,\ldots,\end{cases} \tag{2.2}$$

and

$$P_X(s)=\frac{\pi(1-\rho s)}{1-(1-\pi(1-\rho))s}. \tag{2.3}$$

It is easy to verify that the above equations define a proper probability distribution.

Definition 2.1 *The random variable X, defined by (2.2) and (2.3) is said to be a geometrically distributed related to the Markov chain. We use the notation $IGe(\pi,\rho)$.*

Remark 2.1 *If $\rho=0$, the defined geometric distribution coincides with the usual geometric distribution on the nonnegative integer values, with parameter π, i. e. $Ge_0(\pi)\equiv IGe(\pi,0)$.*

Remark 2.2 *The mean and the variance of the $IGe(\pi,\rho)$ distribution are given by*

$$E(X)=\frac{1-\pi}{\pi(1-\rho)} \quad \textit{and} \quad Var(X)=\frac{(1-\pi)(1+\pi\rho)}{\pi^2(1-\rho)^2}.$$

2.1.1 ρ - Type Lack of Memory Property

It is well known, from Galambos and Kotz [35], that the equation

$$P(U\geq b+x\,|\,U\geq b)=P(U\geq x),\quad x\geq 0,\quad b>0, \tag{2.4}$$

is true for a random variable U which is nonnegative and non - degenerate at zero, if and only if it has either the exponential or the geometric distribution.

Equation (2.4) is known as the *lack of memory* property for the random variable U or for its distribution function.

The following theorem gives the corrected form of the lack of memory property in the case of IGe distribution.

Theorem 2.1 *Let $X \sim IGe(\pi, \rho)$. Then for any $b > 0$, the conditional probability $P(X \geq b+x \,|\, X \geq b)$ has the following equivalent representations:*

$$(i) \quad [1-\pi(1-\rho)]^x, \quad x \geq 0;$$

$$(ii) \quad \tfrac{1-\pi(1-\rho)}{1-\pi} P(X \geq x), \quad x > 0;$$

$$(iii) \quad P(X \geq x) + \rho \tfrac{\pi}{1-\pi} P(X \geq x), \quad x > 0;$$

$$(iv) \quad P(X \geq x) + \rho\pi P(X \geq x \,|\, X > 0), \quad x \geq 0;$$

$$(v) \quad (1-\rho)P(X \geq x) + \rho P(X \geq x \,|\, X > 0), \quad x \geq 0;$$

$$(vi) \quad P(X \geq x+1), \quad x \geq 0.$$

Proof. For any fixed integer $b \geq 1$ from (2.2) we have

$$P(X \geq b) \quad = \quad \sum_{k=b}^{\infty} (1-\pi)\pi(1-\rho)[1-\pi(1-\rho)]^{k-1},$$

i. e.

$$P(X \geq b) \quad = \quad (1-\pi)[1-\pi(1-\rho)]^{b-1}.$$

Then for any $x \geq 0$

$$P(X \geq b+x \,|\, X \geq b) \quad = \quad \frac{P(X \geq b+x)}{P(X \geq b)} \quad = \quad [1-\pi(1-\rho)]^x$$

and the representation (i) is obtained.

By simple transformations from (i) one can obtain relations (ii) - (v). Representation (vi) follows from the definition of the r. v. X and (i).

□

The statement (v) of the Theorem 2.1 gives the reasons of the following extension of the usual lack - of - memory property.

Definition 2.2 *The random variable X has a $\rho-$ type lack - of - memory property if*

$$P(X \geq b + x \mid X \geq b) = (1 - \rho)P(X \geq x) + \rho P(X \geq x \mid X > 0),$$

for any $x > 0$ and $b > 0$.

2.2 Negative binomial distribution related to a Markov chain

Let r be a positive integer and $X_1, X_2, \ldots, X_r$ be independent identically distributed (i.i.d.) random variables having $IGe(\pi, \rho)$ distribution, given by (2.2).

Define the random variable $Y(r) = X_1 + \ldots + X_r$. Since X_i are i.i.d. random variables, when a success appears, the Markov chain starts again at the origin and the initial PMF is given by $P(\nu_n = 1) = \pi$, $P(\nu_n = 0) = 1-\pi$. Let the sequence (ν_n) denote a Markov chain with the following transition matrix $\mathbf{P}^*$:

$$\mathbf{P}^* = \begin{pmatrix} 1 - \pi(1 - \rho) & \pi(1 - \rho) \\ 1 - \pi & \pi \end{pmatrix}.$$

This Markov chain will be called an *interrupted Markov chain.* The r. v. $Y(r)$ counts the number of failures up to the rth success in this Markov chain.

The n step transition matrix is given by

$$\mathbf{P}^{*n} = \begin{pmatrix} 1 - \dfrac{\pi(1-\rho)}{1-\pi\rho}\left(1-(\pi\rho)^n\right) & \dfrac{\pi(1-\rho)}{1-\pi\rho}\left(1-(\pi\rho)^n\right) \\ \dfrac{1-\pi}{1-\pi\rho}\left(1-(\pi\rho)^n\right) & 1 - \dfrac{1-\pi}{1-\pi\rho}\left(1-(\pi\rho)^n\right) \end{pmatrix}.$$

Definition 2.3 *We say that the random variable $Y(r) = X_1 + \ldots + X_r$ has a negative binomial distribution related to the interrupted Markov chain with parameters $\pi \in (0,1)$, $\rho \in (\max\{-1, -\frac{1-\pi}{\pi}\}, 1)$ and $r \geq 1$. $Y(r)$ is called also an Inflated - parameter negative binomial distributed with the notation $Y(r) \sim INB(r, \pi, \rho)$.*

Since $Y(r)$ is a sum of r i.i.d. random variables, each having a PGF given by (2.3), the PGF of the random variable $Y(r)$ has the following form

$$P_{Y(r)}(s) \quad = \quad \left[\frac{\pi(1-\rho s)}{1-(1-\pi+\rho\pi)s}\right]^r. \tag{2.5}$$

The PMF of the inflated - parameter negative binomial distribution is given by the next proposition.

Proposition 2.1 *The PMF of the $INB(r, \pi, \rho)$ distributed random variable $Y(r)$ is given by the following relation*

$$P(Y(r) = y)$$

$$= \pi^r \sum_{y_1, y_2, \ldots} \binom{y_1 + y_2 + \cdots + r - 1}{y_1, y_2, \ldots, r-1} [(1-\pi)(1-\rho)]^{y_1+y_2+\cdots} \rho^{y_2+2y_3+\cdots}, \tag{2.6}$$

where $y = 0, 1, \ldots$ and the summation is over all nonnegative integers $y_1, y_2, y_3, \ldots$, such that $y_1 + 2y_2 + 3y_3 + \ldots = y$.

Proof. Using some combinatorial equations, from the PGF (2.5) we obtain

$$\begin{aligned} P_{Y(r)}(s) &= \left[\frac{\pi(1-s\rho)}{1-s(1-\pi+\rho\pi)}\right]^r = \frac{\pi^r}{\left[1-(1-\pi)(1-\rho)s\frac{1}{1-s\rho}\right]^r} \\ &= \pi^r \sum_{i=0}^{\infty} \binom{i+r-1}{i} \left[(1-\pi)(1-\rho)s(1+\rho s+(\rho s)^2+\ldots)\right]^i \\ &= \pi^r \sum_{i=0}^{\infty} \binom{i+r-1}{i} [(1-\pi)(1-\rho)s]^i \sum_{i_1,i_2,\ldots} \binom{i}{i_1,i_2,\ldots} (\rho s)^{i_2+2i_3+\ldots}, \end{aligned}$$

where the summation is over all nonnegative integers $i_1, i_2, i_3, \ldots$, such that $i_1 + i_2 + i_3 + \ldots = i$. Now, taking into account the equality

$$\binom{i+r-1}{i}\binom{i}{i_1,i_2,\ldots} = \binom{i+r-1}{i_1,i_2,\ldots,r-1}$$

we have

$$P_{Y(r)}(s) = \pi^r \sum_{i=0}^{\infty} [(1-\pi)(1-\rho)s]^i \sum_{i_1,i_2,\ldots} \binom{i+r-1}{i_1,i_2,\ldots,r-1} (\rho s)^{i_2+2i_3+\ldots}.$$

Substituting in the last expression $i_j = y_j,\ j \geq 1$ and $i = y - \sum_{j=1}^{\infty}(j-1)y_j$ we obtain

$$P_{Y(r)}(s) = \sum_{y=0}^{\infty} t^y P(Y=y),$$

where the probability $P(Y=y)$ is given by (2.6) for $y \geq 0$.

□

Remark 2.3 *In the case $\rho = 0$ in (2.6), we obtain the PMF of the usual NB distribution.*

Remark 2.4 *The probabilities of the first four values of the random variable*

$Y(r) \sim INB(r, \pi, \rho)$ are given by the following expressions

$$\begin{aligned}
P(Y(r)=0) &= \pi^r, \\
P(Y(r)=1) &= \pi^r r(1-\pi)(1-\rho), \\
P(Y(r)=2) &= \pi^r(1-\rho)(1-\pi)\left[\binom{r+1}{2}(1-\rho)(1-\pi)+r\rho\right], \\
P(Y(r)=3) &= \pi^r(1-\rho)(1-\pi) \\
&\quad \times \left[\binom{r+2}{3}(1-\rho)^2(1-\pi)^2 + r(r+1)(1-\rho)(1-\pi)\rho + r\rho^2\right],
\end{aligned}$$

derived from (2.6).

Remark 2.5 *The mean and the variance of the $INB(\pi, \rho, r)$ distribution are given by*

$$E(Y(r)) = \frac{r(1-\pi)}{\pi(1-\rho)} \quad \textit{and} \quad Var(Y(r)) = \frac{r(1-\pi)(1+\pi\rho)}{\pi^2(1-\rho)^2}.$$

2.3 Poisson limiting distribution

Here we will obtain the PGF and PMF of a new distribution, by finding the limits of the expressions (2.5) and (2.6) when

$$r \longrightarrow \infty \quad \text{and} \quad \pi \longrightarrow 1, \quad \text{such that} \quad r(1-\pi) = \lambda = const > 0 \tag{2.7}$$

The limiting PGF is given by the following proposition.

Proposition 2.2 *Under the limiting conditions (2.7) the following relation is true*

$$\lim_{r\to\infty}\lim_{\pi\to 1} P_{Y(r)}(s) = exp\left[\frac{\lambda(s-1)}{1-s\rho}\right], \tag{2.8}$$

where $P_{Y(r)}(s)$ is the PGF given by (2.5).

Proof. Taking logarithm on both sides of (2.5) we have

$$\ln P_{Y(r)}(s) = r\left\{\ln[1-(1-\pi+\rho\pi s)] - \ln[1-(1-\pi+\rho\pi)s]\right\}.$$

Using the Taylor expansion of the logarithmic function $\ln(1-x)$, after some simple transformations we obtain that

$$\ln P_{Y(r)}(s) = r(1-\pi)(s-1)\left\{1+\frac{1}{2}[2\rho\pi s+(1-\pi)(s+1)] + \frac{1}{3}[3(\rho\pi s)^2+3\rho\pi(1-\pi)s(s+1)+(1-\pi)^2(s^2+s+1)]+\ldots\right\}.$$

Now using the limiting conditions (2.7) we have

$$\lim_{r\to\infty}\lim_{\pi\to 1}\ln P_{Y(r)}(s) = \lambda(s-1)[1+\rho s+(\rho s)^2+\cdots] = \frac{\lambda(s-1)}{1-\rho s}.$$

Taking anti - logarithm in the last relation we obtain (2.8).

□

Remark 2.6 *In the case of $\rho = 0$, the limiting PGF given (2.8) coincides with the PGF of the usual Poisson distribution with parameter $\lambda > 0$.*

This motivate the following definition.

Definition 2.4 *The random variable Z defined by the PGF*

$$P_Z(s) = \exp\left[\frac{\lambda(s-1)}{1-\rho s}\right]. \tag{2.9}$$

has a Poisson distribution related to the Markov chain with parameters $\lambda > 0$ and $\rho \in [0,1)$. It is called also Inflated - parameter Poisson distribution ($Z \sim IPo(\lambda,\rho)$).

By analogy with the NB case, we will obtain the PMF of the $IPo(\lambda,\rho)$ distribution by the following proposition.

Proposition 2.3 *The PMF of the $IPo(\lambda,\rho)$ distributed random variable Z is given by the following relation*

$$P(Z=z) = \sum_{z_1,z_2,\ldots}\frac{e^{-\lambda}}{z_1!z_2!\ldots}[\lambda(1-\rho)]^{z_1+z_2+\cdots}\rho^{z_2+2z_3+\cdots}, \tag{2.10}$$

where $z = 0, 1, \ldots$ *and the summation is over all nonnegative integers* $z_1, z_2, z_3, \ldots$ *such that*

$$z_1 + 2z_2 + 3z_3 + \cdots = z.$$

Proof. From the PGF (2.9) we have

$$\begin{aligned} P_Z(s) &= \exp\left[\lambda(s-1)(1+\rho s+\rho^2 s^2+\ldots)\right] \\ &= \exp\left\{\lambda[-1+(1-\rho)s+\rho(1-\rho)s^2+\rho^2(1-\rho)s^3+\ldots]\right\}. \end{aligned}$$

Using the Taylor expansion of the exponential function $\exp(x)$, we obtain

$$P_Z(s) = \sum_{n=0}^{\infty} \frac{\lambda^n[-1+(1-\rho)s+\rho(1-\rho)s^2+\rho^2(1-\rho)s^3+\cdots]^n}{n!},$$

i. e.

$$\begin{aligned} &P_Z(s) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{n_0, n_1, \ldots} \binom{n}{n_0, n_1, \ldots} (-1)^{n_0} (1-\rho)^{n_1+n_2+\cdots} \rho^{n_2+2n_3+\cdots} s^{n_1+2n_2+3n_3+\cdots}, \end{aligned}$$

where the summation is over all nonnegative integers $n_0, n_1, n_2, \ldots$, such that $n_0 + n_1 + n_2 + \cdots = n$. Substituting in the last expression $n_i = z_i,\ i \geq 0$ and $n = z - \sum_{j=0}^{\infty}(j-1)z_j$ we obtain

$$P_Z(s) = \sum_{z=0}^{\infty} s^z \sum_{z_1, z_2, \ldots} \frac{[\lambda(1-\rho)]^{z_1+z_2+\cdots} \rho^{z_2+2z_3+\cdots}}{z_1! z_2! \ldots} \sum_{z_0=0}^{\infty} \frac{(-\lambda)^{z_0}}{z_0!},$$

where $z = 0, 1, \ldots$ and the summation is over all nonnegative integers $z_1, z_2, z_3, \ldots$ such that $z_1 + 2z_2 + 3z_3 + \cdots = z$. Therefore,

$$P_Z(s) = \sum_{z=0}^{\infty} s^z P(Z = z),$$

where the probability $P(Z = z)$ is given by (2.10) for $z \geq 0$.

□

Remark 2.7 *From (2.10) we obtain the probabilities of the first four values of $Z \sim IPo(\lambda, \rho)$, given by the following expressions*

$$\begin{aligned} P(Z=0) &= e^{-\lambda}, \\ P(Z=1) &= e^{-\lambda}\lambda(1-\rho), \\ P(Z=2) &= e^{-\lambda}\lambda(1-\rho)\left[\tfrac{\lambda(1-\rho)}{2!}+\rho\right], \\ P(Z=3) &= e^{-\lambda}\lambda(1-\rho)\left[\tfrac{\lambda^2(1-\rho)^2}{3!}+\lambda(1-\rho)\rho+\rho^2\right]. \end{aligned}$$

Remark 2.8 *We will show how to obtain the PMF (2.10) of the inflated - parameter Poisson random variable from the PMF (2.6) of the INB distribution by using the limiting conditions (2.7).*

It is easy to show that the following equality is true

$$\binom{y_1+y_2+\ldots s+r-1}{y_1, y_2, \ldots, r-1}$$

$$= \prod_{i=1}^{\infty} \frac{(r-1+y_i)(r-1+y_i-1)\cdots[r-1+y_i-(y_i-1)]}{y_i!} = \prod_{i=1}^{\infty} A_i.$$

Then (2.6) can be represented as

$$P(Y(r)=y) = \pi^r \sum_{y_1, y_2, \ldots} (1-\rho)^{y_1+y_2+\cdots}\rho^{y_2+2y_3+\cdots}A_1(1-\pi)^{y_1}\prod_{i=2}^{\infty} A_i(1-\pi)^{y_i},$$

Under the limiting conditions (2.7) the following two relations

$$\lim_{r\to\infty}\lim_{\pi\to 1}\pi^r = \lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r = e^{-\lambda} \quad \text{and} \quad \lim_{r\to\infty}(1-\pi)^{y_i} = \frac{\lambda^{y_i}}{r^{y_i}}, \quad i=1,2,\ldots$$

are valid. Then

$$\lim_{r\to\infty}\lim_{\pi\to 1} A_1\pi^r(1-\pi)^{y_1} = \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \quad \text{and} \quad \lim_{r\to\infty}\lim_{\pi\to 1} A_i = \frac{\lambda^{y_i}}{y_i!}, \quad i=2,3,\ldots$$

Therefore,

$$\lim_{r\to\infty}\lim_{\pi\to 1} P(Y(r)=y) = P(Z=z),$$

where $P(Z=z)$ is given by (2.10) for $z=0,1,\ldots$.

Remark 2.9 *The mean and the variance of the* $IPo(\pi, \lambda)$ *distribution are given by*

$$E(Z) = \frac{\lambda}{(1-\rho)} \quad \text{and} \quad Var(Z) = \frac{\lambda(1+\rho)}{(1-\rho)^2}.$$

Remark 2.10 *Let us note that the* $IPo(\lambda, \rho)$ *coincides with the Polya - Aeppli distribution, studied by Evans [30].*

2.4 I - parameter Binomial distribution

Until now, in a natural way we have defined the geometric distribution, negative binomial distribution and in the limiting case the Poisson distribution, related to Markov chain.

Our aim in this section is to obtain the same inflated - parameter Poisson distribution (given by (2.9) or (2.10), but starting from an appropriate defined inflated-parameter Binomial distribution (as in the classical theory). The construction gives an useful interpretation of the defined distributions.

Let us define the random variable Q as follows:

$$P(Q=i) = \begin{cases} 1-\pi, & i=0, \\ \\ \pi\rho^{i-1}(1-\rho), & i=1,2,\ldots \end{cases} \tag{2.11}$$

The above equation define a proper probability distribution *on the set of the nonnegative integers.*

Definition 2.5 *The random variable* Q *defined by (2.11) is called inflated-parameter Bernoulli distributed with parameters* $\pi \in (0,1)$ *and* $\rho \in [0,1)$ *(*$Q \sim IBe(\pi, \rho)$*). It is known also as modified geometric distribution.*

Remark 2.11 *If* $\rho = 0$*, the defined inflated - parameter Bernoulli distribution* $IBe(\pi, 0)$ *coincides with the usual Bernoulli distributed r. v. with parameter* π*, taking values 0 and 1.*

From (2.11) we calculate the corresponding PGF $P_Q(s)$ given by the following expression

$$P_Q(s) \;=\; 1 - \frac{\pi(1-s)}{1-\rho s}. \tag{2.12}$$

Now, if we sum n i.i.d. $IBe(\pi, \rho)$ random variables we will obtain the random variable B with the following PGF

$$P_B(s) \;=\; \left[1 - \frac{\pi(1-s)}{1-\rho s}\right]^n. \tag{2.13}$$

Definition 2.6 *We call the random variable B defined by the PGF (2.13) inflated - parameter binomial distributed with parameters $\pi \in (0,1)$, $\rho \in [0,1)$ and n, and will denote this by $B \sim IBi(n, \pi, \rho)$.*

The next proposition represents the PMF of the $IBi(n, \pi, \rho)$ distributed random variable

Proposition 2.4 *The PMF of the $IBi(n, \pi, \rho)$ distributed random variable B is given by the following relation*

$$P(B=b) = (1-\pi)^n \sum_{b_1, b_2, \ldots} \binom{n}{n - b_1 - b_2 - \ldots, b_1, b_2, \ldots} \rho^b \left[\frac{\pi(1-\rho)}{(1-\pi)\rho}\right]^{b_1+b_2+\ldots}, \tag{2.14}$$

where $b = 0, 1, \ldots$ and the summation is over all nonnegative integers $b_1, b_2, b_3 \ldots$ such that

$$b_1 + 2b_2 + 3b_3 \ldots = b.$$

Proof. From the PGF (2.14) we consequently obtain

$$\begin{aligned} P_B(s) &= \left[1-\pi+\frac{\pi(1-\rho)s}{1-\rho s}\right]^n \\ &= (1-\pi)^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\pi(1-\rho)\rho s}{\rho(1-\pi)(1-\rho s)}\right]^k \\ &= (1-\pi)^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\pi(1-\rho)}{\rho(1-\pi)}(\rho s+\rho^2 s^2+\ldots)\right]^k \\ &= (1-\pi)^n \sum_{k=0}^{n} \binom{n}{k} \sum_{k_1,k_2,\ldots} \binom{k}{k_1,k_2,\ldots} \left[\frac{\pi(1-\rho)}{\rho(1-\pi)}\right]^{k_1+k_2+\ldots} (\rho s)^{k_1+2k_2+\ldots}, \end{aligned}$$

where the summation is over all nonnegative integers $k_1, k_2, \ldots$, such that $k_1 + k_2 + \cdots = k$. Substituting in the last expression $k_i = b_i$, $i \geq 1$ and $k = b - \sum_{j=1}^{\infty}(j-1)b_j$ we obtain

$$P_B(s) = \sum_{b=0}^{\infty} s^b P(B=b),$$

where the probability $P(B=b)$ is given by (2.14) for $b \geq 0$.

□

Remark 2.12 *Representation (2.14) has the following equivalent form:*

$$P(B=b) = \sum_{b_1,b_2,\ldots} \binom{n}{n_0,b_1,b_2,\ldots} (1-\pi)^{n_0} [\pi(1-\rho)]^{b_1+b_2+\ldots} \rho^{b_2+2b_3+\ldots}, \tag{2.15}$$

with $b = 0, 1, \ldots$ and summation over all nonnegative integers $b_1, b_2, b_3, \ldots$, such that $b_1 + 2b_2 + 3b_3 + \ldots = b$, under condition that $n_0 + b_1 + b_2 + \ldots = n$.

Remark 2.13 *Let us note that the defined $IBi(\pi, \rho, n)$ distributed random variable can take infinite number of non-negative values, since by construction the corresponding inflated - parameter Bernoulli distributed random variable can take all non-negative integer values.*

Remark 2.14 *The first four values of the probabilities of a random variable* $B \sim IBi(n, \pi, \rho)$ *are given by the following expressions*

$$\begin{aligned}
P(B=0) &= (1-\pi)^n, \\
P(B=1) &= (1-\pi)^{n-1}\pi(1-\rho)n, \\
P(B=2) &= (1-\pi)^{n-2}\pi(1-\rho)\left[\tbinom{n}{2}\pi(1-\rho) + n(1-\pi)\rho\right], \\
P(B=3) &= (1-\pi)^{n-3}\pi(1-\rho) \\
&\quad \times \left[\tbinom{n}{3}\pi^2(1-\rho)^2 + n(n-1)(1-\pi)\pi(1-\rho)\rho + n(1-\pi)^2\rho^2\right].
\end{aligned}$$

Now, by putting in (2.13)

$$n \longrightarrow \infty \quad \text{and} \quad \pi \longrightarrow 0, \quad \text{such that} \quad n\pi = \lambda = const > 0, \tag{2.16}$$

we would like to obtain in the limit the PGF (2.9) of the inflated - parameter Poisson distribution. Indeed,

$$\lim_{n\to\infty}\lim_{\pi\to 0} P_B(s) \;=\; \lim_{n\to\infty}\left[1 - \frac{\lambda(1-s)}{n(1-\rho s)}\right]^n \;=\; exp\left[-\frac{\lambda(1-s)}{1-\rho s}\right].$$

Remark 2.15 *The mean and the variance of the* $IBi(n, \pi, \rho)$ *distribution are given by*

$$E(B) = \frac{n\pi}{1-\rho} \quad and \quad Var(B) = \frac{n\pi(1-\pi+\rho)}{(1-\rho)^2}.$$

2.5 I - parameter Logarithmic series distribution

Let $Y(r) \sim INB(r, \pi, \rho)$ and its PGF $P_{Y(r)}(s)$ is given by (2.5). From (2.6) we find that $P(Y(r)=0) = \pi^r$. Then $P(Y(r)>0) = 1-\pi^r$. Now, let us consider the *truncated at zero* $INB(\pi, \rho, r)$ distributed r. v. Y_1. Its PMF is given by the following relation

$$P(Y_1 = y_1) = \frac{P(Y(r)=y_1)}{1-\pi^r}, \quad y_1 = 1, 2, \ldots$$

The corresponding PGF $P_{Y_1}(s)$ has the form

$$P_{Y_1}(s) = \frac{1}{1-\pi^r}\left[P_{Y(r)}(s) - \pi^r\right] = \frac{\pi^r}{1-\pi^r}\left\{\frac{(1-\rho s)^r - [1-s(1-\pi+\rho\pi)]^r}{[1-s(1-\pi+\rho\pi)]^r}\right\}.$$

Assuming $r \to 0$ in the last expression, after using the L'Hôpital's rule, we obtain the relation

$$\lim_{r\to 0} P_{Y_1}(s) \;=\; \ln\left[\frac{1-\rho s}{1-s(1-\pi+\rho\pi)}\right](-\ln\pi)^{-1}.$$

If we denote by L the random variable having the limiting PGF, then the following equality is fulfilled

$$P_L(s) \;=\; \ln\left[1+\frac{(1-\pi)(1-\rho)s}{1-s(1-\pi+\rho\pi)}\right](-\ln\pi)^{-1}.$$

After simple transformations, the PGF $P_L(s)$ can be given by the following equivalent representation

$$P_L(s) \;=\; \ln\left[1-\frac{(1-\pi)(1-\rho)s}{1-\rho s}\right]^{-1}(-\ln\pi)^{-1}. \tag{2.17}$$

Remark 2.16 *If we put $\rho = 0$ in the last expression, we derive the PGF of the usual logarithmic series distribution.*

So, we are ready to give the following definition.

Definition 2.7 *The random variable L defined by the PGF (2.17) is called inflated - parameter logarithmic series distribution with parameters $\pi \in (0,1)$ and $\rho \in [0,1)$. We denote this by $L \sim ILS(\pi,\rho)$.*

The following proposition represents the PMF of the defined inflated - parameter logarithmic series distribution.

Proposition 2.5 *The PMF of the $ILS(\pi,\rho)$ distributed random variable L is given by the following relation*

$$P(L=l) = \sum_{l_1,l_2,\ldots} \frac{(-1+l_1+l_2+\ldots)!}{(-\ln\pi)l_1!l_2!\ldots}[(1-\pi)(1-\rho)]^{l_1+l_2+\cdots}\rho^{l_2+2l_3+\cdots}, \tag{2.18}$$

where $l = 1, 2, \ldots$ *and the summation is over all nonnegative integers* $l_1, l_2, l_3, \ldots$ *such that*

$$l_1 + 2l_2 + 3l_3 + \ldots = l.$$

Proof. From (2.17), using the Taylor expansion of the logarithmic function $\ln(1-x)$, we have

$$\begin{aligned} P_L(s) &= \ln\left[1-(1-\pi)(1-\rho)s(1+\rho s+\rho^2 s^2+\ldots)\right](\ln\pi)^{-1} \\ &= -(\ln\pi)^{-1}\sum_{i=0}^{\infty}\frac{[s(1-\pi)(1-\rho)(1+\rho s+\rho^2 s^2+\ldots)]^{i+1}}{i+1}, \end{aligned}$$

and

$$\begin{aligned} P_L(s) &= -(\ln\pi)^{-1}\sum_{i=0}^{\infty}\frac{1}{i+1} \\ &\times \sum_{n_1,n_2,\ldots}\binom{i+1}{n_1,n_2,\ldots}[(1-\rho)(1-\pi)]^{n_1+n_2+\cdots}\rho^{n_2+2n_3+\cdots}s^{n_1+2n_2+3n_3+\cdots}. \end{aligned}$$

The summation is over all nonnegative integers $n_1, n_2, \ldots$, such that $n_1 + n_2 + \ldots = i+1$. Substituting $n_i = l_i,\ i \geq 1$ and $i+1 = l - \sum_{j=1}^{\infty}(j-1)l_j$ we obtain

$$\begin{aligned} P_L(s) &= -(\ln\pi)^{-1}\sum_{l=1}^{\infty}s^l \\ &\times \sum_{l_1,l_2,\ldots}\frac{(-1+l_1+l_2+\ldots)!}{l_1!l_2!\ldots}[(1-\rho)(1-\pi)]^{l_1+l_2+\cdots}\rho^{l_2+2l_3+\cdots}, \end{aligned}$$

i. e. we derived the PMF (2.17).

□

Proposition 2.6 *Let the random variable* $Y(r) \sim INB(r, \pi, \rho)$*. Then the following convergence is fulfilled*

$$\lim_{r\to 0} P(Y(r) = y \,|\, Y(r) \geq 1) \quad = \quad P(L = y),$$

where the random variable $L \sim ILS(\pi, \rho)$*.*

Proof. Since $P(Y(r) \geq 1) = 1 - \pi^r$, we have

$$\begin{aligned} P(Y(r) = y \mid Y(r) \geq 1) &= \frac{P(Y(r) = y, Y(r) \geq 1)}{P(Y(r) \geq 1)} \\ &= \frac{P(Y(r) = y, Y(r) \geq 1)}{1 - \pi^r}. \end{aligned}$$

From the last relation and (2.5) we have

$$\begin{aligned} & P(Y(r) = y | Y(r) \geq 1) \\ &= \frac{\pi^r}{1 - \pi^r} \sum_{y_1, y_2, \ldots} \binom{y_1 + y_2 + \ldots + r - 1}{y_1, y_2, \ldots, r - 1} [(1 - \pi)(1 - \rho)]^{y_1 + y_2 + \cdots} \rho^{y_2 + 2y_3 + \cdots} \\ &= \frac{r\pi^r}{1 - \pi^r} \sum_{y_1, y_2, \ldots} \frac{(y_1 + y_2 + \ldots + r - 1) \ldots (r + 2)(r + 1)}{y_1! y_2! \ldots} \\ &\times [(1 - \pi)(1 - \rho)]^{y_1 + y_2 + \cdots} \rho^{y_2 + 2y_3 + \cdots}. \end{aligned}$$

Notice that according to the L'Hôpital's rule $\lim_{r \to 0} \frac{r\pi^r}{1 - \pi^r} = -(\ln \pi)^{-1}$. Then it can be seen that $\lim_{r \to 0} P(Y(r) = y | Y(r) \geq 1)$ converges to the PMF $P(L = y), y = 1, 2, \ldots$ of the inflated - parameter logarithmic series distribution as given by (2.17).

□

Remark 2.17 *The univariate NB and logarithmic series distributions are related with the same limiting results stated by the last two propositions, see Qu et al. (1990), [75].*

Remark 2.18 *The probabilities of the first three values of the random variable $L \sim ILS(\pi, \rho)$, are given by the following expressions*

$$\begin{aligned} P(L = 1) &= -(\ln \pi)^{-1}(1 - \rho)(1 - \pi), \\ P(L = 2) &= -(\ln \pi)^{-1}(1 - \rho)(1 - \pi) \left[\tfrac{(1-\rho)(1-\pi)}{2} + \rho \right], \\ P(L = 3) &= -(\ln \pi)^{-1}(1 - \rho)(1 - \pi) \left[\tfrac{(1-\rho)^2(1-\pi)^2}{3} + (1 - \rho)(1 - \pi)\rho + \rho^2 \right]. \end{aligned}$$

Remark 2.19 *The mean and the variance of the random variable* $L \sim ILS(\pi, \rho)$ *are given by the following expressions*

$$E(L) = \frac{-(1-\pi)}{\pi(1-\rho)\ln\pi} \quad and \quad Var(L) = \frac{-(1-\pi)[\ln\pi(1+\pi\rho)+1-\pi]}{\pi^2(1-\rho)^2(\ln\pi)^2}.$$

2.6 The family of I - parameter GPSD

Having in hands the inflated - parameter discrete distributions, defined in the previous sections, it is natural to propose an "inflated-parameter" generalization of the GPSD. In this section we define the family of inflated-parameter GPSD. We give common representation of the PMF's and PGF's in the corresponding subsections. An over-dispersed property of the new family is discussed and a new constructive interpretation of the additional parameter ρ is given. We will assume hereafter that $\rho \in [0, 1)$.

2.6.1 Common representation of the PMFs

One can observe that the PMF's of the inflated-parameter binomial, Poisson, negative binomial and logarithmic series distributions given by (2.12), (2.10), (2.6) and (2.17) correspondingly, have similar representations according to the additional parameter ρ. Therefore, we can expect a common expression of the corresponding PMF's, as states the following proposition.

Proposition 2.7 *The PMF's given by (2.14), (2.10), (2.6) and (2.17), correspondingly, have the following representation*

$$P(N=k) = \frac{1}{g(\theta)} \sum_{k_1, k_2, \ldots} a(k)[\theta(1-\rho)]^{k_1+k_2+\cdots}\rho^{k_2+2k_3+\cdots}, \qquad (2.19)$$

with $k = 0, 1, 2, \ldots$, $\rho \in [0, 1)$, $\theta > 0$, *and the summation is on the set of all nonnegative integers* $k_1, k_2, \ldots$, *such that* $k_1 + 2k_2 + \ldots = k$. *If the random*

variable $X \sim ILS(\theta, \rho)$, *its values begin from 1 and the summation in (2.19) is over nonnegative integers, such that* $k_1 + 2k_2 + \ldots = k + 1$.

In the particular cases, the functions $a(k)$, $g(\theta)$ *and the parameter* θ, *are given by the following expressions*

$$N \sim IPo(\theta, \rho): \quad a(k) = \tfrac{1}{k_1!k_2!\ldots}, \quad g(\theta) = e^{\theta}, \quad \theta = \lambda,$$

$$N \sim INB(\theta, \rho, r): \quad a(k) = \tbinom{k_1+k_2+\ldots+r-1}{k_1,k_2,\ldots,r-1}, \quad g(\theta) = (1-\theta)^{-r}, \quad \theta = 1-\pi,$$

$$N \sim IBi(\theta, \rho, n): \quad a(k) = \tbinom{n}{n-k_1-k_2-\ldots,k_1,k_2,\ldots}, \quad g(\theta) = (1+\theta)^{n}, \quad \theta = \tfrac{\pi}{1-\pi},$$

$$N \sim ILS(\theta, \rho): \quad a(k) = \tfrac{(-1+k_1+k_2+\ldots)!}{k_1!k_2!\ldots}, \quad g(\theta) = -\ln(1-\theta), \quad \theta = 1-\pi.$$

Proof. Using simple transformations one can obtain the above relations from (2.10), (2.6), (2.14) and (2.18), respectively.

□

Definition 2.8 *The random variable* N *belongs to the family of Inflated - parameter GPSD with parameters* $\theta > 0$ *and* $\rho \in [0, 1)$ *if its PMF can be represented by (2.19).*

Remark 2.20 *Let us note that the defined family is different than the corresponding family studied by Gupta et al. [40].*

2.6.2 Explicit expressions for PMFs

The PMF's given by Proposition 2.7 can be represented by alternative explicit expressions, which are given by the following

Proposition 2.8 *The PMF's of the* $IPo(\theta,\rho), INB(r,\theta,\rho), IBi(n,\theta,\rho)$ *and* $ILS(\theta,\rho)$ *distributed random variables can be given by the following equivalent expressions*

(i) $$P(N=k)=\begin{cases} e^{-\lambda}, & k=0 \\ e^{-\lambda}\sum_{i=1}^{k}\binom{k-1}{i-1}\frac{[\lambda(1-\rho)]^i}{i!}\rho^{k-i}, & k=1,2,\ldots, \end{cases}$$

if N *has the Inflated - parameter Poisson distribution with parameters* $\lambda>0$ *and* ρ*, say* $N\sim IPo(\lambda,\rho)$*;*

(ii) $$P(N=k)=\begin{cases} \pi^r, & k=0 \\ \pi^r\sum_{i=1}^{k}\binom{k-1}{i-1}\binom{r+i-1}{i}[(1-\pi)(1-\rho)]^i\rho^{k-i}, & k=1,2,\ldots, \end{cases}$$

if N *has the Inflated - parameter Negative binomial distribution with parameters* $\pi\in(0,1)$*,* ρ *and* $r>0$*, say* $N\sim INB(\pi,\rho,r)$*;*

(iii) $$P(N=k)=\begin{cases} (1-\pi)^n, & k=0 \\ \sum_{i=1}^{min(k,n)}\binom{n}{i}\binom{k-1}{i-1}[\pi(1-\rho)]^i(1-\pi)^{n-i}\rho^{k-i}, & k=1,2,\ldots, \end{cases}$$

if N *has the Inflated - parameter Binomial distribution with parameters* $\pi\in(0,1)$*,* ρ *and* $n\in\{1,2,\ldots\}$*, say* $N\sim IBi(\pi,\rho,n)$*.*

When $n=1$*, we obtain the Inflated - parameter Bernoulli r. v. with parameters* π *and* ρ*;*

(iv) $$P(N=k)=(-ln\pi)^{-1}\sum_{i=1}^{k}\binom{k-1}{i-1}\frac{[(1-\pi)(1-\rho)]^i\rho^{k-i}}{i}, \quad k=1,2,\ldots,$$

if N *has the Inflated - parameter logarithmic-series distribution with parameters* $\pi\in(0,1)$ *and* ρ*, say* $N\sim ILS(\pi,\rho)$*.*

(v) $P(N=k) = \begin{cases} \pi, & k=0 \\ (1-\pi)[1-\pi(1-\rho)]^{k-1}(1-\rho)\pi, & k=1,2,\ldots, \end{cases}$

if N has the Inflated - parameter geometric distribution, say $N \sim IGe(\pi,\rho)$. The $IGe(\pi,\rho)$ distribution is in fact the $INB(\pi,\rho,1)$.

Proof. We will demonstrate how to obtain the PMF for the random variable $N \sim IPo(\lambda,\rho)$. The remaining expressions can be deduced in a similar way.

The starting point here is to use the following relation

$$\frac{1}{(1-y)^j} = \sum_{l=0}^{\infty} \binom{l+j-1}{l} y^l, \quad 0<y<1, \tag{2.20}$$

valid for any $j = 1, 2, \ldots$.

For the PGF (2.9) we have

$$P_N(s) \;=\; \exp\left\{\lambda[-1+(1-\rho)s+\rho(1-\rho)s^2+\rho^2(1-\rho)s^3+\ldots]\right\}.$$

Using the Taylor expansion of the exponential function $\exp(x)$, after some algebra we obtain

$$P_N(s) \;=\; e^{-\lambda}\left\{1+\sum_{j=1}^{\infty}\frac{[\lambda(1-\rho)s]^j}{j!}\frac{1}{(1-\rho s)^j}\right\}.$$

Applying (2.20) in the above expression we have

$$\sum_{m=0}^{\infty} P(N=m)s^m \;=\; e^{-\lambda}\left\{1+\sum_{j=1}^{\infty}\frac{[\lambda(1-\rho)s]^j}{j!}\sum_{l=0}^{\infty}\binom{l+j-1}{l}\rho^l s^l\right\}.$$

The PMF of the random variable $N \sim IPo(\lambda,\rho)$ given by the proposition is obtained by equating the coefficients of s^m on both sides of the last equality for fixed $m = 0, 1, 2, \ldots$.

□

2.7 The PGF of the family of IGPSDs

Let $P_Y(s)$ and $P_X(s)$ be PGF's of the non-negative integer valued random variables Y and X, respectively. Then the resulting function $P_Y(P_X(s))$, is

also a PGF, corresponding to the sum of independent identically distributed random variables having a PGF $P_X(s)$, where the number of summands is independent of the random variables and has a PGF $P_Y(s)$.

This situation describes the portfolio of insurance policies during a given length of time. Let N denote the number of claims and let X_i denote the amount of the i-th claim. Let $\{X_1, X_2, \ldots\}$ be a sequence of independent and identically distributed random variables with common PGF $P_X(s)$. Assume that $X_i,\ i = 1, 2, \ldots$ are also independent on N and consider the random sum

$$N = X_1 + X_2 + \ldots + X_Y,$$

with the convention that $N = 0$ if $Y = 0$. Then N equals aggregate claims, and the corresponding PGF $P_N(s)$ is just $P_Y(P_X(s))$, see for example Bowers et *al.* [15].

Now, if Y belongs to the family of GPSD with parameter θ defined by (2.1) and X is an arbitrary discrete distribution, then the resulting random sum S has a PGF given by the following expression

$$P_N(s) = \frac{g(\theta P_X(s))}{g(\theta)}, \tag{2.21}$$

where the possible choices of the function $g(\cdot)$ are given in the Proposition 2.7, see Hirano et *al.* [42] also.

After the inflated - parameter binomial, Poisson, NB and logarithmic series distributions have a common representation for their PMF's one can expect that they have the corresponding common representation of their PGF's. This is precised by the following statement, which gives the PGF of the inflated - parameter GPSD defined by Definition 2.8.

Proposition 2.9 *The PGF of the inflated - parameter GPSD is given by (2.21), where the functions $g(\theta)$ are given by Proposition 2.7 and*

$$P_X(s) = \frac{(1-\rho)s}{1-\rho s}. \tag{2.22}$$

Proof. Using simple transformations from the corresponding functions $g(\theta)$, given by Proposition 2.7, relations (2.21) and (2.22), one can get easy the PGF's (2.13), (2.9), (2.5) and (2.17), respectively.

□

In fact, Proposition 2.9 gives a constructive representation of the distributions belonging to the family of inflated-parameter GPSD. Indeed, (2.22) is the PGF of the geometric distribution with parameter $1-\rho$ and taking positive integer values. It gives also a new interpretation of the additional parameter ρ, different than the correlation coefficient for the Markov chain, as discussed earlier.

In terms of the collective risk model, this means that the aggregated claim S has inflated-parameter GPSD when the individual claims have geometric distribution with parameter $1-\rho$, i.e. $X_i \sim Ge_1(1-\rho)$, and number of claims N for a given length of time is a r.v. belonging to the usual family of GPSD with parameter θ.

Remark 2.21 *From (2.21) and (1.3) it is easy to establish, that the variance-mean ratio of the inflated - parameter GPSD is greater than the corresponding variance - mean ratio of the original GPSD. This means that the new family is over - dispersed relative to the family of GPSD, if the additional parameter* $\rho \in (0,1)$.

2.8 Compound Generalized Power series distributions

According to the Proposition 2.9, the family of IGPSDs can be defined as a Compound GPSDs.

Let us suppose that the random variable X has the shifted geometric distribution $Ge_1(1-\rho)$ with parameter $\rho \in [0,1)$, probability mass function

$$P(X=k) = \rho^{k-1}(1-\rho), \quad k = 1,2,3,\ldots \tag{2.23}$$

and the PGF given by (2.22).

Let the discrete random variable N with parameters θ and ρ has the following PGF

$$P_N(t) = \frac{g(\theta P_X(t))}{g(\theta)}, \tag{2.24}$$

where the *series function* $g(\theta)$ is a positive, finite and differentiable. We are ready to give the next definition.

Definition 2.9 *The random variable N belongs to the family of Inflated - parameter Power Series Distributions (IPSD), if it is defined on the non - negative integers and has the PGF given by (2.24).*

Definition 2.10 *The random variable N belongs to the family of Inflated - parameter Generalized Power Series Distributions (IGPSD), if it is defined on any subset of the non - negative integers and has the PGF given by (2.24).*

Remark 2.22 *It is clear that IPSD $\subset$ IGPSD.*

Example 2.1 *The first three distributions in the previous section belong to the family of IPSD. The corresponding series functions and parameters are the following (see [20])*

$$N \sim IPo(\theta, \rho): \qquad g(\theta) = e^{\theta}, \qquad \theta = \lambda,$$

$$N \sim INB(\theta, \rho, r): \quad g(\theta) = (1-\theta)^{-r}, \quad \theta = 1 - \pi,$$

$$N \sim IBi(\theta, \rho, n): \quad g(\theta) = (1+\theta)^{n}, \quad \theta = \tfrac{\pi}{1-\pi}.$$

Example 2.2 *The Inflated - parameter logarithmic - series distribution belongs to the family of IGPSD, but not to IPSD:*

$$N \sim ILS(\theta, \rho): \quad g(\theta) = -\ln(1-\theta), \quad \theta = 1 - \pi.$$

The definitions 2.9 and 2.10 give a constructive representation of the distributions belonging to the family of IGPSD. Let $X_1, X_2, \ldots$ be independent, identically geometric distributed random variables, having the probability mass function (2.23). The random variable Y belongs to the usual family of GPSD with parameter $\theta > 0$ and Y be independent of the above sequence. Then the random sum $X_1 + X_2 + \ldots + X_Y$ has a PGF given by (2.24).

Using the above interpretation we will give the next representation of the PMF of the IGPSD.

It is well known that the sum of m independent, geometrically distributed random variables is a negative binomial distributed random variable, i.e. if we denote $S_m = X_1 + \ldots + X_m$, then

$$P(S_m = k) = \binom{k-1}{m-1}(1-\rho)^m \rho^{k-m}, \quad k = m, m+1, \ldots \tag{2.25}$$

Let us note another useful interpretation of the model. If we suppose that any insurance policy produces two types of claims named "success" with probability $1-\rho$ and "failure" with probability ρ, then the r.v. S_m represents the total number of claims until the m-th successive claim appears. The parameter m in (2.25) is equal to the number of geometrically distributed random variables. We suppose that m is an outcome from the random variable Y independent of $\{X_1, X_2, \ldots\}$. Then S_m has the mixed Negative binomial distribution. If Y is discrete distributed random variable and $p_m = P(Y = m), \quad m \in S$, where S is any nonempty enumerable set of nonnegative integers, then

$$P(S_Y = k) = \sum_{m \in S} \binom{k-1}{m-1}(1-\rho)^m \rho^{k-m}\, p_m, \quad k = m, m+1, \ldots. \tag{2.26}$$

According to the above interpretation, by suitable choice of the random variable Y, (2.26) leads up to the PMF's of the IGPSD. This means that the Inflated - parameter distributions are mixed negative binomial distributions. Moreover the PMF's can be presented in terms of the Gaussian hypergeometric function and the Confluent hypergeometric function given in the Appendix. The PMF of the Inflated-parameter logarithmic-series distribution has an explicit expression. This is the statement of the following Proposition.

Proposition 2.10 *Let us suppose that the random variable Y belongs to the family of GPSD and is independent of the random variables $X_1, X_2, \ldots$. Then the random sum $N = X_1 + \ldots + X_Y$ has the following PMF:*

$$(i) \quad P(N=m) = \begin{cases} e^{-\lambda}, & m = 0 \\ \lambda(1-\rho)\rho^{m-1} e^{-\frac{\lambda}{\rho}} {}_1F_1[m+1; 2; \frac{\lambda(1-\rho)}{\rho}], & m = 1, 2, \ldots, \end{cases}$$

if $Y \sim Po(\lambda)$ *and* $\rho \neq 0$;

$$(ii) \quad P(N=m) = \begin{cases} \pi^r, & m = 0 \\ r(1-\rho)\rho^{m+r} \frac{1-\pi}{\pi} [\frac{\pi}{1-\pi(1-\rho)}]^{r+1} {}_2F_1[m+1; r+1; 2; \frac{(1-\pi)(1-\rho)}{1-\pi(1-\rho)}], & m = 1, 2, \ldots, \end{cases}$$

if $Y \sim NB(\pi, r)$;

$$(iii) \quad P(N=m) = \begin{cases} (1-\pi)^n, & m = 0 \\ \frac{n\pi(1-\rho)}{\rho^{n+1}(1-\pi)^{m+1}} (\rho - \pi)^{m+n} {}_2F_1[m+1; n+1; 2; \frac{\pi}{1-\pi} \frac{1-\rho}{\rho}], & m = 1, 2, \ldots, \end{cases}$$

if $Y \sim Bi(n, \pi)$ *and* $\rho \neq 0$;

$$(iv) \quad P(N=m) = (-\ln \pi)^{-1} \frac{1}{m} \left\{ [1 - \pi(1-\rho)]^m - \rho^k \right\}, \quad m = 1, 2, \ldots,$$

if $Y \sim LS(\pi)$.

The random variable N coincides with the corresponding IPo(θ, ρ), INB(θ, ρ, r), IBi(θ, ρ, n) and ILS(θ, ρ) distributed random variables, where θ is given by the examples.

Proof. Taking into account the above interpretation we have

$$P(S_Y = 0) = P(Y = 0).$$

Since Y belongs to the family of PSD we get

$$P(Y = 0) = [g(\theta)]^{-1}.$$

In this way we get the probabilities $P(N = 0)$ in the first three cases. Let us begin with the probabilities for $k = 1, 2, \ldots$.

(i) Let $Y \sim Po(\lambda)$, i. e. $P(Y = i) = \frac{e^{-\lambda}\lambda^i}{i!}$, $i = 0, 1, \ldots$. Then the PMF (2.26) becomes

$$\begin{aligned} P(N = m) &= \sum_{i=1}^{\infty} \binom{m-1}{i-1} (1-\rho)^i \rho^{m-i} \frac{\lambda^i}{i!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{i=1}^{\infty} \binom{m-1}{i-1} \frac{[\lambda(1-\rho)]^i \rho^{m-i}}{i!}, \quad i = 1, 2, \ldots \end{aligned} \tag{2.27}$$

The sum (2.27) stops for $i > m$ and presents the IPo(λ, ρ) distribution.

After some algebraic operations the PMF is presented by

$$P(N = m) = \lambda(1-\rho)\rho^{m-1} e^{-\lambda} {}_1F_1[-(m-1); 2; -\frac{\lambda(1-\rho)}{\rho}],$$

where ${}_1F_1[a; b; x]$ is the Confluent hypergeometric function, given by (A3). Using the Kummer's theorem (see the Appendix), one can have the statement (i).

(ii) Let $Y \sim NB(r, \pi)$, i.e. $P(Y = i) = \binom{r+i-1}{i} \pi^r (1-\pi)^i$, $i = 0, 1, \ldots$. In this case the PMF (2.26) becomes

$$\begin{aligned} P(N = m) &= \sum_{i=1}^{\infty} \binom{m-1}{i-1} (1-\rho)^i \rho^{m-i} \binom{r+i-1}{i} \pi^r (1-\pi)^i \\ &= \pi^r \sum_{i=1}^{\infty} \binom{m-1}{i-1} \binom{r+i-1}{i} [(1-\pi)(1-\rho)]^i \rho^{m-i}, \quad m = 1, 2, \ldots \end{aligned} \tag{2.28}$$

Again the sum stops for $m > k$ and this is just the PMF of the INB(r, π, ρ) distributed random variable. After some algebra the PMF (2.7) can be represented as

$$P(N = m) = r\pi^r \rho^{m-1}(1-\pi)(1-\rho) {}_2F_1[-(m-1), r+1; 2; -\frac{(1-\pi)(1-\rho)}{\rho}],$$

where ${}_2F_1[a, b; c; x]$ is the Gaussian hypergeometric function given by (A1). Using the second Euler transformation one can obtain the statement (ii) of the Proposition.

(iii) Let $Y \sim Bi(n,\pi)$, i.e. $P(Y=i) = \binom{n}{i}\pi^i(1-\pi)^{n-i}$, $i = 0,1,\ldots,n$. Then the PMF (2.26) becomes

$$\begin{aligned} P(N=m) &= \sum_{i=1}^{n}\binom{m-1}{m-1}(1-\rho)^m\rho^{k-m}\binom{n}{m}\pi^m(1-\pi)^{n-m} \\ &= \sum_{m=1}^{\infty}\binom{n}{m}\binom{k-1}{i-1}[\pi(1-\rho)]^i(1-\pi)^{n-i}\rho^{m-i}, \quad m=1,2,\ldots \end{aligned} \tag{2.29}$$

The sum in (2.29) stops for $i > min(n,m)$. After some relations similar to these in the INB-case the PMF (2.29) can be represented as

$$P(N=m) = n\pi(1-\rho)\rho^{m-1}(1-\pi)^{n-1}{}_2F_1[-(m-1),-(n-1);2;\frac{\pi(1-\rho)}{(1-\pi)\rho}].$$

The third Euler transformation in the last relation goes to the statement (iii) of the Proposition.

(iv) Let us suppose that Y has the logarithmic-series distribution, i.e. $P(Y=i) = -(\ln\pi)^{-1}\frac{(1-\pi)^i}{i}$, $i = 1,2,\ldots$. Then the PMF (2.26) becomes

$$P(N=m) = -(\ln\pi)^{-1}\sum_{i=1}^{\infty}\binom{m-1}{i-1}\frac{[(1-\rho)(1-\pi)]^i\rho^{k-i}}{i}. \tag{2.30}$$

The sum (2.30) stops for $m > k$ and coincides with the PMF of the ILSD. The representation (iv) of the Proposition follows easy from (2.30). This proves the Proposition.

□

The next proposition gives another one presentation of the PMF of INB and IBi distributions.

Proposition 2.11 *The INB and IBi distributions have the following probability mass functions:*

$$P(N=m) = \begin{cases} \pi^r, & m=0 \\ \pi^r\dfrac{r(1-\rho)q\rho^{m+r}}{\Gamma(m+1)\Gamma(1-m)}\displaystyle\int_0^1\left(\frac{t}{1-t}\right)^m[1-\pi(1-\rho)-q(1-\rho)t]^{-(r+1)}dt, & m=1,2,\ldots, \end{cases}$$

and

$$P(N=m) = \begin{cases} q^n, & m=0 \\ \dfrac{n\pi(1-\rho)\rho^{m-1}q^{n-1}}{\Gamma(m+1)\Gamma(1-m)} \displaystyle\int_0^1 t^m(1-t)^m \left(1-t\frac{\pi(1-\rho)}{q\rho}\right)^{n-1} dt, & m=1,2,\ldots, \end{cases}$$

where $q = 1-\pi$.

Proof. The proof follows from the integral representation formula (A2).

□

2.8.1 Properties of the IGPSDs

Property G1. If $X_1, X_2, \ldots, X_n$ are mutually independent IGPSD random variables with series function $g(\theta)$, then the sum $X_1 + \ldots + X_n$ is an IGPSD with series function $f(\theta) = g^n(\theta)$.

Property G2. Let $\{Y_1, Y_2, \ldots\}$ be a sequence of mutually independent and identically ILSD r. v's. Let N be a Poisson distributed r.v. with parameter λ, independent of the above sequence. Then the random sum $Z_N = Y_1 + \ldots + Y_N$ has the INBD.

Proof. The PGF of the ILSD is given by

$$P_Y(s) = -g^{-1}(\theta)\ln[1-\theta P_X(s)],$$

where $g(\theta) = -\ln(1-\theta)$.

Thus, the PGF of the Z_N is obtained as

$$\begin{aligned} P_{Z_N}(s) &= \exp\{\lambda[P_Y(s)-1]\} = \exp(-\lambda)\exp\{-\lambda g^{-1}(\theta)\ln(1-\theta P_X(s))\} \\ &= [\exp(g(\theta))]^{-\lambda g^{-1}(\theta)}[1-\theta P_X(s)]^{-\lambda g^{-1}(\theta)} \\ &= \left(\frac{1}{1-\theta}\right)^{-\lambda g^{-1}(\theta)}[1-\theta P_X(s)]^{-\lambda g^{-1}(\theta)}. \end{aligned}$$

The above formula gives the PGF of the INBD$(\theta, \rho, \lambda g^{-1}(\theta))$.

Property G3. Let us denote by $\mu_i, \quad i = 1, 2, \ldots$ the ordinary moments of the IGPSD. Then μ_i satisfy the following recurrent relation

$$\mu_{i+1} = \frac{\theta}{1-\rho}\frac{\partial \mu_i}{\partial \theta} + \mu_1 \mu_i + \frac{\rho}{1-\rho}\sum_{j=1}^{i}\binom{i}{j}\mu_{i-j+1}, \quad i = 1, 2, \ldots. \tag{2.31}$$

Proof. From the PGF it is easy to obtain

$$\frac{\partial p_i(\theta,\rho)}{\partial \theta} = [\frac{i}{\theta} - \frac{g'(\theta)}{g(\theta)}]p_i(\theta,\rho) - \frac{(i-1)\rho}{\theta}p_{i-1}(\theta,\rho), \quad i = 1, 2, \ldots.$$

and

$$\frac{\partial p_0(\theta,\rho)}{\partial \theta} = -\frac{g'(\theta)}{g(\theta)}p_0(\theta,\rho).$$

The relation (2.31) is obtained by usual algebraic operations, using the last relations and the definition of the PGF.

Property G4. If we denote by $m_i, \quad i = 1, 2, \ldots$ the central moments, then we have

$$\begin{aligned} m_{i+1} &= \frac{\theta}{1-\rho}\frac{\partial m_i}{\partial \theta} + i m_2 m_{i-1} \\ &+ \frac{\rho}{1-\rho}\sum_{j=1}^{i-1}\binom{i}{j}m_{i-j+1} + \frac{\rho}{1-\rho}EX\sum_{j=1}^{i-1}\binom{i}{j+1}m_{i-j-1} \end{aligned} \tag{2.32}$$

for every $i = 1, 2, \ldots$ and $\sum_1^0 = 0$.

The proof follows by the same arguments as in the Property G3.

□

Let $p_i = P(N = i), \quad i = 0, 1, \ldots$ be the probability mass function of the random variable N. It is natural to expect that the additional parameter ρ leads up to an extension of the Panjer - recursion, or a second - order difference equation. This is the statement of the following Property.

Property G5. The PMF of the IPSD satisfies the following recursions:

$$p_i = (a + \frac{b}{i})p_{i-1} + (1 - \frac{2}{i})\ c\ p_{i-2}, \tag{2.33}$$

for $i = 1, 2, \ldots$ and $p_{-1} = 0$.

In the particular cases, the constants a, b and c are given by the following expressions:

$N \sim IPo(\lambda, \rho)$:
$a = 2\rho, \qquad b = \lambda(1-\rho) - 2\rho, \qquad c = -\rho^2;$

$N \sim INB(\pi, \rho, r)$:
$a = (1-\rho)(1-\pi) + 2\rho, \quad b = (r-1)(1-\rho)(1-\pi) - 2\rho, \quad c = -\rho(1-\pi(1-\rho)).$

$N \sim IBi(\theta, \rho, n)$:
$a = 2\rho - \theta(1-\rho), \qquad b = (n+1)\theta(1-\rho) - 2\rho, \qquad c = -\rho(\rho - \theta(1-\rho));$

Proof. Differentiation in (2.24) leads to

$$(1-\rho t)^2 P'_N(t) = (1-\rho)\theta \frac{g'(\theta P_X(t))}{g(\theta P_X(t))} P_N(t), \tag{2.34}$$

where $P_N(s) = \sum_{k=0}^{\infty} p_i s^i$ and $P'_N(s) = \sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$.

Taking into account the series functions and parameters given in the Example 1 one can get the particular cases of (2.34). The relation (2.33) is obtained by equating the coefficients of s^i on both sides for fixed $i = 0, 1, 2, \ldots$.

□

Remark 2.23 *If $\rho = 0$ the IPSD becomes the usual PSD and the recursion (14) coincides with the Panjer - recursion (1.5). Such type of recursions are studied by many authors. Sundt et al. [81] study the distributions with PMF's satisfying k - order difference equation. Schröter [80] characterizes a family of distributions satisfying the second order difference equation. Note that the recursion (2.33) is an extension of the recursion studied by Schröter and will help us to estimate the tails of distributions, which is useful in risk theory.*

Having in hands the recursion (2.33) we are going to get the second - order recursion for the probability distribution function. Let us denote by

$F(k) = P(N \leq k)$ the distribution function of the random variable N, and by $\bar{F}(k) = 1 - F(k)$ the survivor function, or the tail distribution. Then the following property of the tail distribution is fulfilled.

Property G6. The tail distribution of the IPSD satisfies the following inequality:

$$\bar{F}(k) \leq (a+b)\bar{F}(k-1) - c\bar{F}(k-2), \quad k = 1, 2, \ldots, \tag{2.35}$$

where $\bar{F}(-1) = 1$ and for $k \geq 2$ the equation is fulfilled for the IGe(π, ρ).

The following is an analogy of the similar property related to the classical Poisson and Negative binomial distributions (Grandell [38], p. 2-4).

Property G7. The mixed IPo(λ, ρ) (or Pólya - Aeppli) distribution by Gamma mixing distribution is an Inflated - parameter Negative binomial distribution.

Proof. Let us suppose that for given λ, the random variable N has an IPo(λ, ρ) distribution. Let us consider λ as an outcome of a Gamma distributed random variable with parameter β and density function given by

$$f(\lambda) = \frac{\beta^r}{\Gamma(r)} e^{-\beta\lambda} \lambda^{r-1}, \quad \beta > 0, \quad \lambda \geq 0.$$

The unconditional distribution of N, given by

$$P(N = k) = \int_0^\infty P(N = k \mid \lambda) f(\lambda) d\lambda, \quad k = 0, 1, 2, \ldots$$

is just the Inflated - parameter Negative binomial distribution with parameter $\pi = \frac{\beta}{\beta+1}$ and PMF given by

$$P(N = m)$$

$$= \begin{cases} \left(\dfrac{\beta}{\beta+1}\right)^r, & m = 0 \\ \left(\dfrac{\beta}{\beta+1}\right)^r \displaystyle\sum_{i=1}^{m} \binom{m-1}{i-1} \binom{r+i-1}{i} \left[\left(1 - \dfrac{\beta}{\beta+1}\right)(1-\rho)\right]^i \rho^{m-i}, & m = 1, 2, \ldots. \end{cases}$$

□

Using the representation formula (2.33) we arrive at the following recursion for the ordinary moments.

Property G8. The ordinary moments of the IPSD satisfy the following recurrent relation:

$$\mu_k = \frac{1}{1-a-c}\sum_{j=1}^{k}\binom{k-1}{j-1}\left[\frac{k}{j}a+b+(\frac{k}{j}-1)2^j c\right]\mu_{k-j}, \quad k=1,2,\ldots.$$

This relation follows from Sundt et *al.* [81].

Among the elements studied in Risk Theory is the random sum $S_N = X_1 + \ldots + X_N$, $S_0 = 0$, called *compound*. It describes the aggregate claim amount in the risk model. We assume that $N, X_1, X_2, \ldots$ are mutually independent random variables and say that S_N has a *compound distribution*. If we suppose that the random variable N has the IPSD, then the relation (2.33) for the PMF is useful for deriving the compound IPSD. The corresponding formula in the case of discrete random variables is given below.

Property G9. Let suppose that $X_1, X_2, \ldots$ are independent identically distributed r.v's with common PMF $p(x)$, $x = 0, 1, 2\ldots$. Let us denote $f_S(x) = P(S_N = x), x = 0, 1, 2\ldots$. Then the following recursion formula is fulfilled

$$\begin{aligned} f_{S_N}(x) &= \frac{1}{1-ap(0)-cp^2(0)} \\ &\times\left[\sum_{i=1}^{x}(a+\frac{bi}{x})p(i)f_S(x-i) - c\sum_{i=1}^{x}p(i)\sum_{j=1}^{x-i}p(j)f_S(x-i-j)\right]. \end{aligned}$$

2.8.2 A Characterization of the I - Geometric distribution

In Theorem 2.1 it is shown that the IGe(π, ρ) distribution satisfies the ρ - type lack of memory property, i. e., if the r.v. X is IGe(π, ρ) distributed,

then

$$P(X \geq x+b) = \frac{1-\pi(1-\rho)}{1-\pi} P(X \geq x)P(X \geq b), \tag{2.36}$$

for all $x > 0$ and $b > 0$.

In the case of $\rho = 0$ the relation (2.36) coincides with the classical lack of memory property characterizing the usual geometric distribution.

In the next proposition we will show that the property (2.36) characterizes the IGe distribution.

Proposition 2.12 *Let X be a non - negative, integer - valued random variable satisfying (2.36) for any $\pi \in (0,1)$ and $\rho \in [0,1)$. If $P(X=0) = \pi$, then X is IGe(π, ρ) distributed random variable.*

Proof. Let us first write (2.36) for $b = 1$:

$$P(X \geq x+1) = [1-\pi(1-\rho)]P(X \geq x), \quad x > 0. \tag{2.37}$$

From (2.37) by induction we get the following formula for the tail distribution of X

$$P(X \geq x+1) = (1-\pi)[1-\pi(1-\rho)]^x, \quad x \geq 0. \tag{2.38}$$

The relation

$$P(X = x) = P(X \geq x) - P(X \geq x+1)$$

and (2.38) give the probability mass function of the IGe(π, ρ) distribution. □

Proposition 2.13 *Let X be a non - negative, integer - valued random variable satisfying*

$$P(X=0) = \pi \, and \;\; P(X=1) = (1-\pi)\pi(1-\rho), \tag{2.39}$$

where $\pi \in (0,1)$ and $\rho \in [0,1)$. Then $X \sim IGe(\pi, \rho)$ if and only if

$$(1-\pi)P(X > x+b) = P(X > x)P(X > b), \tag{2.40}$$

for every $x, b = 1, 2, \ldots$.

Proof. From (2.38) we have that

$$P(X > x) = (1 - \pi)(1 - \pi(1 - \rho))^x, \ x = 1, 2, \ldots,$$

and then (2.40) follows. Conversely, if (2.40) holds, then

$$\begin{aligned}(1 - \pi)P(X > x + 1) &= P(X > x)P(X > 1) \\ &= P(X > x)(1 - \pi - (1 - \pi)\pi(1 - \rho)) \\ &= (1 - \pi)uP(X > x),\end{aligned}$$

where $u = 1 - \pi(1 - \rho)$. It follows that

$$P(X > x + 1) = uP(X > x), \ x = 1, 2, \ldots,$$

and again (2.38) follows.

□

Theorem 2.2 *Suppose that X is a r. v. with $P(X \leq x) = 1 - \theta e^{-\lambda x}, \ \ x \geq 0, \ \ \lambda > 0, \ \ 0 < \theta < 1$. Let $|X| = k$ if and only if $k-1 < X \leq k, \ \ k \geq 0$. Then $|X|$ has an $IGe(\pi, \rho)$ distribution with $\pi = 1 - \theta$ and $\rho = (e^{-\lambda} - \theta)/(1 - \theta)$.*

Proof. For $k = 0$ we have

$$P(|X| = 0) = P(X \leq 0) = 1 - \theta.$$

For $k \geq 1$, we have

$$\begin{aligned}P(|X| = k) &= P(k - 1 < X \leq k) \\ &= \theta e^{-\lambda(k-1)} - \theta e^{-\lambda k} \\ &= \theta(e^{-\lambda})^{k-1}(1 - e^{-\lambda}).\end{aligned}$$

Comparing with (**??**), we have $\pi = 1 - \theta$ and $(1 - \pi)\pi(1 - \rho) = \theta(1 - e^{-\lambda})$. It follows that

$$1 - \rho = \frac{1 - e^{-\lambda}}{\pi},$$

and then also that

$$\rho = 1 - \frac{1-e^{-\lambda}}{\pi} = \frac{\pi - 1 + e^{-\lambda}}{\pi} = \frac{e^{-\lambda} - \theta}{1-\theta}.$$

Clearly we have $1 - \pi(1-\rho) = e^{-\lambda}$ as required.

□

Remark 2.24 *If $\theta = e^{-\lambda}$, then $P(X \leq x) = 1 - e^{-\lambda(1+x)}$ and $X + 1 \sim Exp(\lambda)$. The relation between an exponential distribution and the geometric distribution is well known, see Johnson et al. (2005), [45].*

Remark 2.25 *If $\theta = 1$, then $P(X \leq x) = 1 - e^{-\lambda x}$ and $X \sim EXP(\lambda)$, see Deng and Chhikara (1990), [23], for example. Now we find that*

$$P(|X| = 0) \quad = 0,$$

$$P(|X| = k) \quad = (e^{-\lambda})^{k-1}(1 - e^{-\lambda}),\ k \geq 1.$$

In the remainder of this section we will suppose that the random variable X has the IGe(π, ρ) distribution with PMF given in the Introduction or equivalently with the tail distribution (2.38). We will give the ρ - type characterization of the Inflated - parameter geometric distribution similar to this given by Zijlstra [88] for the geometric distribution.

Proposition 2.14 *Let X and Y be independent non - negative, integer - valued random variables and assume that X is non - degenerate at 0 and $P(X \geq Y) > 0$. Then X has $IGe(\pi, \rho)$ distribution for any $\pi \in (0,1)$ and $\rho \in [0,1)$ if and only if*

$$P(X \geq Y + i) = P(X \geq i)\left[P(Y = 0) + \frac{1 - \pi(1-\rho)}{1-\pi} P(X \geq Y)\mathbf{I}_{\{Y \neq 0\}}\right], \tag{2.41}$$

for $i = 1, 2, \ldots$.

Proof. Let us suppose that X has the IGe distribution and $P(X \geq Y) > 0$. Then by the relation (2.36), for every $i = 1, 2, \ldots$ we have

$$P(X \geq Y + i) = \sum_{j=0}^{\infty} P(X \geq j + i)P(Y = j)$$

$$= P(X \geq i)P(Y = 0) + \frac{1-\pi(1-\rho)}{1-\pi} P(X \geq i) \sum_{j=1}^{\infty} P(X \geq j)P(Y = j)$$

$$= P(X \geq i)P(Y = 0) + \frac{1-\pi(1-\rho)}{1-\pi} P(X \geq i)P(X \geq Y)\mathbf{I}_{\{Y \neq 0\}}.$$

This proves (2.41). We will now consider the integer - valued random variable X, non - negative, independent of Y. Let that (2.41) be true. Obviously

$$\sum_{j=1}^{\infty} P(X \geq j+i)P(Y = j) = \frac{1 - \pi(1-\rho)}{1-\pi} P(X \geq i) \sum_{j=1}^{\infty} P(X \geq j)P(Y = j),$$

for every $i = 1, 2 \ldots$. Rearranging the terms leads to

$$\sum_{j=1}^{\infty} \left[P(X \geq j + i) - \frac{1 - \pi(1-\rho)}{1-\pi} P(X \geq i)P(Y \geq j) \right] P(Y = j) = 0.$$

The expression in the brackets equals to zero for every $i = 1, 2, \ldots$ and $P(X = 0) = \pi$ is the characterization property of the IGe(π, ρ) distribution. □

Remark 2.26 *Let us note that for every $i = 1, 2, \ldots$. the relation (2.41) can be rewritten in the following equivalent form*

$$(1 - \pi)P(X \geq Y + i) = P(X \geq i)P(X > Y).$$

Proof. Follows immediately by combining (2.37 and (2.41). □

In the paper of Zijlstra [88], Theorem 2 it is shown that (2.42) is a characterization property of the geometric distribution. The following Proposition gives one counter - example.

Proposition 2.15 *Let X, Y and Z be independent non - negative integer - valued random variables. Suppose X is non - degenerate, Y has the IPo(λ, ρ)*

distribution, Z has the IPo(μ, ρ) distribution. If X has the IGe(π, ρ) distribution for some $\pi \in (0, 1)$, then

$$P(X \geq Y + Z) = P(X \geq Y)P(X \geq Z). \tag{2.42}$$

Proof. The proof follows directly from the fact that $Y + Z$ has the IPo$(\lambda + \mu, \rho)$ distribution and

$$P(X \geq Y) = \frac{1}{1 - \pi(1 - \rho)} \left\{ \pi\rho e^{-\lambda} + (1 - \pi) exp\left(-\frac{\lambda\pi(1 - \rho)}{1 - \rho[1 - \pi(1 - \rho)]} \right) \right\}.$$

□

2.8.3 Properties of the INBD

In our first result we show how the PMF of the INB distributed r. v. $Y(r)$ can be obtained recursively.

Lemma 2.1 *Let $p_r(k) = P(Y(r) = k)$ and $p_{r+1}(k) = P(Y(r + 1) = k)$. The following relation holds:*

$$\begin{aligned} p_{r+1}(0) &= \pi p_r(0); \\ p_{r+1}(k) &= \pi p_r(k) + (1 - \pi)\pi(1 - \rho) \textstyle\sum_{j=0}^{k-1} p_r(j) u^{k-j-1}, \quad k = 1, 2, \ldots, \end{aligned} \tag{2.43}$$

where $u = 1 - \pi(1 - \rho)$.

Proof. Using the PGF, we see that $Y(r + 1) \stackrel{d}{=} Y(r) + Y_1$. It follows that

$$p_{r+1}(k) = \sum_{j=0}^{k} p_r(j) P(Y_1 = k - j).$$

From here and using (2.2), it follows that $p_{r+1}(0) = p_r(0)\pi$ and for $k \geq 1$ that (2.43) holds.

□

Remark 2.27 *If $\rho = 0$, (2.43) can be simplified and we find $p_{r+1}(0) = \pi p_r(0)$ and*

$$p_{r+1}(k) = \pi \sum_{j=0}^{k} p_r(j)(1-\pi)^{k-j}, \ k = 1, 2, \ldots.$$

Remark 2.28 *Using the PGF of $Y(r)$ in (2.5), we also have the following alternative representation. We have*

$$\begin{aligned} \psi_{Y(r)}(t) &= \left(\frac{1-\rho t}{1-\rho}\right)^r \left(\frac{\pi(1-\rho)}{1-(1-\pi(1-\rho))t}\right)^r \\ &= \left(\frac{1}{1-\rho} + \frac{-\rho}{1-\rho}t\right)^r \left(\frac{\pi(1-\rho)}{1-(1-\pi(1-\rho))t}\right)^r. \end{aligned}$$

If $\rho < 0$, the first term is the PGF of a binomial r. v. $B(r) \sim Bi(r, \delta)$ with parameter $\delta = \frac{-\rho}{1-\rho}$. The second term is the PGF of an usual negative binomial r.v. $N^*(r)$ with parameter $\pi(1-\rho)$. We obtain that $Y(r) \stackrel{d}{=} B(r) + N^*(r)$. This representation can be useful for simulation purposes.

If $\rho > 0$, we have

$$\left(\frac{1-\rho}{1-\rho t}\right)^r \psi_{N(r)}(t) = \psi_{N^*(r)}(t).$$

The first term on the left hand side corresponds to the PGF of an usual negative binomial r.v. $B^*(r)$ with parameter ρ. Now we obtain that $B^*(r) + Y(r) \stackrel{d}{=} Y^*(r)$.

Since $Y(r)$ is the sum of i.i.d. random variables, we can apply the central limit theorem here.

Theorem 2.3 *As $r \to \infty$, we have*

$$\frac{Y(r) - r\mu}{\sqrt{r\sigma^2}} \stackrel{d}{\Longrightarrow} Z,$$

where μ and σ^2 are given in Remark 2.5 and where Z denotes a standard normal r. v.

2.9 Numerical example

In this section we will approximate the frequency data given in the column headed "Observed" of the Table 10.1 by using $IPo(\lambda, \rho)$ and $INB(\pi, \rho, r)$ distributions. The data, taken from Daykin et *al.* [22] p. 52, and related to the claims under UK comprehensive motor policies. The 421240 policies were classified according to the number of claims in the year 1968.

Let us denote by $\bar{X}_n$ and σ_n^2 the sampling mean and variance. Then the average number of claims per policy is $\bar{X}_n = 0.13174$ and $\sigma_n^2 = 0.13852$.

In the column headed "Poisson" of the Table 10.1 are given the corresponding expected values by using the usual Poisson distribution with a parameter $\lambda = \bar{X}_n = 0.13174$. The column of the Table 10.2 headed "NB" sets out the resulting NB approximation with parameters $\pi = 0.951$ and $r = 2.558$. The last rows show the corresponding values of the Pearon's χ^2.

The value of the chi-square in the Poisson case is too high, so the insufficiency of the Poisson law for the data is evident. The reason is that the sampling variance is greater than the sampling mean, whereas they should be almost equal in the case of Poisson distribution. The value of χ^2 in the NB case is 9.18 which gives probability 0.05 for 5 degrees of freedom, so that the representation is acceptable.

We will not discuss the Maximum Likelihood (ML) estimates of the parameters and their properties, but they can be calculated numerically. Here we give the corresponding results for comparison only. The ML estimates are obtained by a direct minimization approach of the log - likelihood following Mickey and Britt [56]. The minimization procedure is based on derivative - free algorithm for nonlinear least squares proposed by Ralston and Jennrich [76].

2.9.1 $IPo(\lambda, \rho)$ - case

The mean and variance of the $IPo(\lambda, \rho)$ distribution are given by Remark 2.9. Solving the corresponding system we obtain the following moment estimates

for the parameters ρ and λ

$$\hat{\rho} = \frac{\sigma_n^2 - \bar{X}_n}{\sigma_n^2 + \bar{X}_n} \quad \text{and} \quad \hat{\lambda} = \frac{2\bar{X}_n^{\,2}}{\sigma_n^2 + \bar{X}_n}.$$

The same estimates can be found, for example, in Johnson et *al.* [45], pp. 381 - 382, where they were reported for the Pólya - Aeppli distribution.

In our case we obtain the values

$$\hat{\rho} = 0.0251 \quad \text{and} \quad \hat{\lambda} = 0.12843.$$

The corresponding expected values are given in the column headed "IPo" of the Table 10.1.

Table 10.1. Poisson case

k	Observed	Poisson	IPo	IPo-ML
0	370412	369246.88	370469.93	370435.30
1	46545	48643.57	46385.30	46447.48
2	3935	3204.09	4068.21	4045.88
3	317	140.70	296.20	291.57
4	28	4.63	19.14	18.61
5	3	0.12	1.13	1.09
≥ 6	0	0.01	0.07	0.06
	Chi-square	667.52	13.60	13.61

The comparison of the expected values given in the columns headed "Poisson" and "IPo" shows that $IPo(\lambda, \rho)$ distribution fits the observed frequencies much better than the usual Poisson distribution, which has a shorter tail than the data. The value of χ^2 is 13.60 which gives probability 0.04 for 5 degrees of freedom, so the approximation by $IPo(\lambda, \rho)$ distribution is acceptable (observe that the use of the NB distribution is preferable if our criterion is the value of the χ^2 statistics).

Remark 2.29 *We obtained a positive value for the moment estimate of the parameter ρ. This can be interpreted in the following way: the observed number of zeros is more than it can be predicted by the usual Poisson distribution (as it can be seen from the Table 10.1).*

We obtain the following ML estimates

$$\hat{\rho}_{ML} = 0.02441 \quad \text{and} \quad \hat{\lambda}_{ML} = 0.12852$$

with $\chi^2 = 13.61$. One can see that the ML estimates of the parameters are close to the values of the corresponding moment estimates. The corresponding expected values are given in the last column of the Table 10.1.

2.9.2 $INB(\pi, \rho, r)$ - case

To estimate the parameters of the $INB(\pi, \rho, r)$ distribution we need additionally the third moment together with sampling mean and variance.

The mean and the variance of the r. v. $X \sim INB(\pi, \rho, r)$ are given by Remark 2.5. From the PGF (2.5), after some algebra we obtain the following relation for the third moment

$$E(X^3) = \frac{r(1-\pi)}{\pi}\left[1 + 4\rho + \rho^2 + \frac{3(r+1)(\rho+1)(1-\pi)}{\pi} + \frac{(r+1)(r+2)(1-\pi)^2}{\pi^2}\right].$$

The solution of the corresponding system gives the following procedure for calculation the moment estimates of the parameters.

Step 1. The moment estimate of the parameter ρ is a solution of the following quadratic equation

$$a\rho^2 + b\rho + c = 0,$$

where

$$a = \left(\bar{X}_n + \frac{\sigma_n^2}{\bar{X}_n}\right)\left(\bar{X}_n + \frac{2\sigma_n^2}{\bar{X}_n}\right) - \frac{m_3 - \sigma_n^2}{\bar{X}_n},$$

$$b = 2 - 2\left(\bar{X}_n + \frac{\sigma_n^2}{\bar{X}_n}\right)\left(\bar{X}_n + \frac{2\sigma_n^2}{\bar{X}_n}\right) + 2\frac{m_3}{\bar{X}_n},$$

and

$$c = \left(\bar{X}_n + \frac{\sigma_n^2}{\bar{X}_n}\right)\left(\bar{X}_n + \frac{2\sigma_n^2}{\bar{X}_n}\right) - \frac{m_3 + \sigma_n^2}{\bar{X}_n}$$

with m_3 being the third sample moment;

Step 2. The moment estimate for the parameter π is given by

$$\hat{\pi} = \left[\frac{\sigma_n^2}{\bar{X}_n}(1 - \hat{\rho}) - \hat{\rho}\right]^{-1},$$

where $\hat{\rho}$ is the result from Step 1;

Step 3. Finally, the moment estimate of the parameter r can be calculated by the following formula

$$\hat{r} = \frac{\bar{X}_n \hat{\pi}(1 - \hat{\rho})}{1 - \hat{\pi}},$$

where $\hat{\rho}$ and $\hat{\pi}$ are the calculated values from Step 1 and Step 2, correspondingly.

For the considered data we obtain the following moment estimates

$$\hat{\rho} = -.03869, \quad \hat{\pi} = 0.88428 \quad \text{and} \quad \hat{r} = 1.04564.$$

In the column headed "INB" of the Table 10.2 are given the corresponding estimated frequencies, when using the computed moment estimates of the parameters. One can see that the $INB(\pi, \rho, r)$ distribution fits the observed frequencies perfectly.

Table 10.2. NB case

k	Observed	NB	INB	INB-ML
0	370412	370459.94	370409.99	370412.37
1	46545	46413.30	46553.37	46545.63
2	3935	4043.97	3922.17	3928.82
3	317	300.92	325.21	324.23
4	28	20.48	26.85	26.58
5	3	1.32	2.21	2.17
≥ 6	0	0.09	0.20	0.19
	Chi-square	9.18	0.78	0.76

Remark 2.30 *We calculated a negative value for the parameter ρ. This is possible and can be interpreted in the following way: the observed number of zeros is less than predicted by the usual NB distribution. Let us note, that in the Poisson case we observed just the opposite situation.*

We obtain the following ML estimates for the parameters

$$\hat{\rho}_{ML} = -.03670, \quad \hat{\pi}_{ML} = 0.88748 \quad \text{and} \quad \hat{r}_{ML} = 1.07727,$$

with $\chi^2 = 0.76$. The corresponding estimated frequencies are given in the last column of the Table 10.2.

2.10 Comments

In this chapter we derive some properties of the IGPSD in the sense of application in Risk theory. It seams that, using these distributions as counting distributions in the risk model, the corresponding generalization will give good results. The advantage is in the interpretation of the additional parameter ρ. In general an additional parameter exhibits over - dispersion. The distributions are suitable for modelling dependent data, which naturally appear in insurance and finance. Moreover the inflation parameter ρ may take

negative values, i. e.,

$$\rho \geq max\{-1, -\frac{p_0}{1-p_0}\}.$$

If the observed number of zeros is more than it can be predicted by the usual GPSD, then the parameter ρ is positive. If the observed number of zeros is less than predicted by the GPSD, the parameter ρ is negative.

The distributions, given in this chapter are published in [51], [57] and [59]. Some additional results, related to the distributions and the Markov chain are derived in [66].

Aoyama et. *all* in [7] have defined a generalization of the shifted inverse trinomial distributions. A part of these distributions generalizes the IGPS distributions.

Chadjiconstantinidis S. and Pitselis G. (2009) in [16], give some recursion formulas for a class of Compound distributions. As examples they use the $IGe(\pi, \rho)$ and $ILS(\pi, \rho)$ distributions.

Vinogradov in [84], Remark 2.3 gives some basic properties of the $ILS(\pi, \rho)$ distribution.

Jazi and Alamatsaz in [44] compared the $ILS(\pi, \rho)$ distribution with its mixture with respect to several stochastic orderings.

Borges et al. (2011), [14], have applied the IGPS distributions to cutaneous melanoma data. The number of initiated cells related to the occurrence of a tumor is IGPS distributed and the parameter ρ is a measure of the association between the tumor cells. The defined model is called a correlated destructive generalized power series model (CDGPS model).

Chapter 3

Probability distributions of order k

The distributions of order k are related to a sequence of independent trials with success probability p. For a given fixed positive, integer number k, Feller [31] derived the PGF of the number of trials N until the first occurrence of $k-$th consecutive success

$$g_N(s) = \frac{(1-ps)p^k s^k}{1-s+(1-p)p^k s^{k+1}}. \tag{3.1}$$

Philippou A., Georghiou C. and Philippou G. [72] obtained the exact probability function corresponding to (3.1) and called it geometric distribution of order k, $(Ge_k(p))$. Philippou and Muwafi [71] show the relations between the geometric distribution of order k and the Fibonacci numbers. Since then many authors gave their contribution to the exact theory for the so called discrete distributions of order k which can be called already classical. Aki and Hirano in [4] derived the distribution of the number successes and failures until the first occurrence of k consecutive successes.

The negative binomial $(NB_k(r,p))$ distribution of order k is the distribution of the sum of r independent, identically $Ge_k(p)$ distributed random variables. Some properties of the negative binomial distribution of order k

are given by Philippou in [73]. The Poisson distribution of order k, $(Po_k(\lambda))$, is a limiting distribution of a sequence of shifted NB_k distributed random variables.

The binomial distribution of order k $(Bi_k(n,p))$ was derived by Hirano [43] and by Philippou and Makri [74]. They considered the number of success runs of length k $(N_n^{(k)})$ in n independent trials and derived the PMF. In this definition, not every run in successes of length k gives a contribution of 1 to $N_n^{(k)}$. According to this definition, $N_n^{(k)}$ counts the non overlapping success runs. Ling (1988) in [54] named this distribution Type I binomial distribution of order k. Ling defined the Type II binomial distribution of order k, which counts the overlapping success runs.

Aki, Kuboku and Hirano [2] introduced the logarithmic series distribution of order k $(LS_k(p))$ as a limiting distribution of a sequence of truncated at zero NB_k distributed random variables. A good references for the distributions of order k are given in the books of Balakrishnan and Koutras [13] and Fu and Lou [34]. Later it is proved that the discrete distributions of order k can be represented as Compound GPSDs, (see Charalambides [17] also). Let the random variable N has a compound distribution, i.e. $N = X_1 + \ldots + X_Y$, where X is the compounding random variable, Y is independent of X and $N = 0$ in $Y = 0$.

Definition 3.1 *If the compounding random variable X is a discrete distributed, truncated at 0 and from the right away from $k+1$, the random variable N has a distribution of order k.*

The $Ge_k(p)$, $NB_k(r,p)$ and the $LS_k(p)$ distributions belong to the family of Compound GPSDs by truncated geometric compounding distribution.

In this case we suppose that the random variable X has a PMF and PGF, given by

$$P(X = m) = \frac{1-\rho}{1-\rho^k}\rho^{m-1}, \quad m = 1, 2, \ldots k \tag{3.2}$$

and

$$P_X(s) = \frac{(1-\rho)s}{1-\rho^k}\frac{1-\rho^k s^k}{1-\rho s}, \tag{3.3}$$

where $k \geq 1$ is fixed integer number.

If $k \to \infty$, the truncated geometric distribution approaches to the $Ge_1(1-\rho)$ distribution, defined by (2.22) and (2.23).

In this way the $Ge_k(p)$, $NB_k(r,p)$ and $LS_k(p)$ distributions converge to the corresponding IGPSD.

The $Po_k(\lambda)$ distribution is obtained by discrete uniform compounding distribution with PMF

$$P(X = m) = \frac{1}{k}, \quad m = 1, 2, \ldots k$$

and PGF

$$P_X(s) = \frac{s}{k}\frac{1-s^k}{1-s}. \tag{3.4}$$

As we mentioned above, the motivation of this study is related to the risk model. In many cases the counting process in the defined model is a compound non homogeneous Poisson process. As usual, the compounding distribution is a discrete distribution, taking non negative integer values. The interpretation is the following. If the insurance policies are separated in independent groups, the number of groups has a Poisson distribution. We suppose that the groups are homogeneous, identically distributed and the number of policies in each of the groups has the compounding distribution. It is not realistic if the compounding distribution takes values over all non negative integers.

Here we define a compound Poisson distribution with compounding distribution taking values in $\{1, 2, ...k\}$.

3.1 Poisson distribution of order k

In this section we present some results obtained for the Poisson distribution of order k. Denote by Y the $Po_k(\lambda)$ distributed random variable. Then its PGF is given by

$$P_Y(s) = e^{-\lambda(k-(s+s^2+\ldots+s^k))}, \ |s| \leq 1. \tag{3.5}$$

The PMF is

$$P(Y=y)=\sum_{y_1,\ldots,y_k} e^{-k\lambda}\frac{\lambda^{y_1+\ldots+y_k}}{y_1!\ldots y_k!},\ y=0,1,\ldots,$$

where the summation is over all non - negative integers $y_1, y_2, \ldots, y_k$, such that $y_1+2y_2+\ldots+ky_k=y$. It follows from the definition that $Po_1(\lambda)\equiv Po(\lambda)$. Therefore $Po_k(\lambda)$ is a generalized Poisson distribution.

For the random variable $Y\sim Po_k(\lambda)$, we get the mean and the variance

$$E(Y)=\lambda\frac{k(k+1)}{2}\ \text{ and }\ Var(Y)=\lambda\frac{k(k+1)(2k+1)}{6}.$$

The Fisher index of dispersion is

$$FI(Y)=\frac{Var(Y)}{E(Y)}=1+\frac{2}{3}(k-1)>1$$

for $k>1$.

The proof of the following Lemma follows from the PGF in (3.5).

Lemma 3.1 *Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. Poisson of order k distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Then the r. v. $\sum_{j=1}^{n} Y_j$ is a Poisson of order k distributed with parameter $\lambda_1+\lambda_2+\ldots+\lambda_n$.*

Remark 3.1 *Let $Y_1, Y_2, \ldots, Y_n$ be independent identically Poisson distributed of order k random variables and set $\overline{Y}=\frac{1}{n}\sum_{j=1}^{n} Y_j$. The moment estimator $\widehat{\lambda}$ of λ is the unique admissible root of the equation*

$$\overline{Y}=\lambda\frac{k(k+1)}{2}.$$

3.2 Pólya - Aeppli distribution of order k

In this section we introduce the Pólya - Aeppli distribution of order k as a compound Poisson distribution with truncated geometric compounding distribution. It is defined in [63]. The PGF is given by

$$P_N(s)=e^{\lambda(P_X(s)-1)}, \tag{3.6}$$

where $P_X(s)$ is the PGF of the compounding distribution in (3.3).

Definition 3.2 *The probability distribution defined by the PGF (3.6) and compounding distribution, given by (3.2) and (3.3) is called a Pólya - Aeppli distribution of order k with parameters $\lambda > 0$ and $\rho \in [0, 1)$ $(PA_k(\lambda, \rho))$.*

The Pólya - Aeppli distribution of order k belongs to the family of Compound GPSD, compounded by the truncated geometric distribution. In the next theorem we give the probability mass function. Let us denote $Z = \frac{\lambda(1-\rho)}{1-\rho^k}$.

Theorem 3.1 *The probability mass function of the $PA_k(\lambda, \rho)$ distributed random variable is given by:*

$$p_0 = e^{-\lambda},$$

$$p_i = e^{-\lambda} \sum_{j=1}^{i} \binom{i-1}{j-1} \frac{Z^j}{j!} \rho^{i-j}, \quad i = 1, 2, \ldots, k$$

$$p_i = e^{-\lambda} [\sum_{j=1}^{i} \binom{i-1}{j-1} \frac{Z^j}{j!} \rho^{i-j}$$

$$- \sum_{n=1}^{l} (-1)^{n-1} \frac{(Z\rho^k)^n}{n!} \sum_{j=0}^{i-n(k+1)} \binom{i-n(k+1)+n-1}{j+n-1} \frac{Z^j}{j!} \rho^{i-n(k+1)-j}],$$

$$i = l(k+1) + m, \ m = 0, 1, \ldots k, \ l = 1, 2, \ldots \infty.$$

Proof: The PMF is obtained by equating the coefficients of t^i on both sides of the Taylor expansion of the PGF.

□

The probability mass function of the $PA_k(\lambda, \rho)$ distributed random variable, given in the Theorem 3.1 can be rewrite in detail as follows

$$p_0 = e^{-\lambda},$$

$$p_i = e^{-\lambda} \sum_{j=1}^{i} \binom{i-1}{j-1} \frac{Z^j}{j!} \rho^{i-j}, \quad i = 1, 2, \ldots, k$$

$$p_i = e^{-\lambda} [\sum_{j=1}^{i} \binom{i-1}{j-1} \frac{Z^j}{j!} \rho^{i-j} - Z\rho^k \sum_{j=0}^{i-k-1} \binom{i-k-1}{j} \frac{Z^j}{j!} \rho^{i-k-1-j}],$$

$$i = k+1, k+2, \ldots, 2k+1$$

$$p_i = e^{-\lambda} [\sum_{j=1}^{i} \binom{i-1}{j-1} \frac{Z^j}{j!} \rho^{i-j} - Z\rho^k \sum_{j=0}^{i-k-1} \binom{i-k-1}{j} \frac{Z^j}{j!} \rho^{i-k-1-j} +$$

$$+ \frac{(Z\rho^k)^2}{2!} \sum_{j=0}^{i-2k-2} \binom{i-2k-1}{j+1} \frac{Z^j}{j!} \rho^{i-2k-2-j}], \quad i = 2k+2, \ldots, 3k+2$$

$$p_i = e^{-\lambda} [\sum_{j=1}^{i} \binom{i-1}{j-1} \frac{Z^j}{j!} \rho^{i-j} - Z\rho^k \sum_{j=0}^{i-k-1} \binom{i-k-1}{j} \frac{Z^j}{j!} \rho^{i-k-1-j}$$

$$+ \frac{(Z\rho^k)^2}{2!} \sum_{j=0}^{i-2k-2} \binom{i-2k-1}{j+1} \frac{Z^j}{j!} \rho^{i-2k-2-j} -$$

$$- \frac{(Z\rho^k)^3}{3!} \sum_{j=0}^{i-3k-3} \binom{i-3k-1}{j+2} \frac{Z^j}{j!} \rho^{i-3k-3-j}], \quad i = 3k+3, \ldots, 4k+3$$

...

The following proposition gives an extension of the Panjer recursion formulas.

Proposition 3.1 *The PMF of the Pólya-Aepply distribution of order k sat-*

isfies the following recursions:

$$p_1 = Zp_0,$$

$$p_i = (2\rho + \tfrac{Z-2\rho}{i})p_{i-1} - (1 - \tfrac{2}{i})\rho^2 p_{i-2},\ i = 2, 3, \ldots k$$

$$p_i = (2\rho + \tfrac{Z-2\rho}{i})p_{i-1} - (1 - \tfrac{2}{i})\rho^2 p_{i-2} - \tfrac{k+1}{i} Z\rho^k p_{i-k-1} + \tfrac{k}{i} Z\rho^{k+1} p_{i-k-2},$$
$$i = k+1, k+2, \ldots$$

and $p_{-1} = 0$.

Proof. Differentiation in (3.6) leads to

$$(1-\rho s)^2 P_N'(s) = \frac{1-\rho}{1-\rho^k}[1 - (k+1)\rho^k s^k + k\rho^{k+1} s^{k+1}]P_N(s), \tag{3.7}$$

where $P_N(s) = \sum_{m=0}^{\infty} p_m s^m$ and $P_N'(s) = \sum_{m=0}^{\infty}(m+1)p_{m+1}s^m$. The recursions are obtained by equating the coefficients of t^m on both sides for fixed $m = 0, 1, 2, \ldots$.

□

Remark 3.2 *If $k \to \infty$, the Pólya - Aeppli distribution of order k, approaches to the usual Pólya - Aeppli distribution. If $\rho = 0$, it is a Poisson distribution.*

Remark 3.3 *The mean and the variance of the Pólya - Aeppli distribution of order k are given by*

$$EN = \frac{1 + \rho + \ldots + \rho^{k-2} + \rho^{k-1} - k\rho^k}{1-\rho^k}\lambda$$

and

$$Var(N) = \frac{1 + 3\rho + 5\rho^2 + 7\rho^3 + \ldots + (2k-3)\rho^{k-2} + (2k-1)\rho^{k-1} - k^2\rho^k}{1-\rho^k}\lambda.$$

It is easy to verify that the Fisher index of dispersion is

$$FI(N) = \frac{Var(N)}{E(N)} > 1,$$

and therefore, the Pólya - Aeppli distribution is overdispersed related to the Poisson distribution.

3.3 Comments

The PMF of the Pólya-Aepply distribution of order k, given in the Theorem 3.1 is published in [63]. The detailed probabilities are given in the Proceedings of the ASMDA - 2007 Conference.

Chapter 4

Distributions on Markov chain trials

We consider the time - homogeneous multi - state Markov chain $\{X_n, n \geq 0\}$ taking values $\{0, 1, 2, \ldots\}$, where the sequence $\{X_0, X_1, \ldots\}$ is determined by the distribution of the initial states

$$P(X_0 = g_1) = p_{g_1},\ 0 < p_0 < 1,\ p_{g_1} \geq 0,\ g_1 = 1, 2, \ldots,\ \sum_{g_1=0}^{\infty} p_{g_1} = 1,$$

and transition probabilities

$$P(X_{n+1} = g_2 \mid X_n = g_1) = p_{g_1 g_2},\quad g_1, g_2 = 0, 1, \ldots,\quad n = 0, 1, \ldots$$

where

$$0 < p_{00} < 1,\quad \sum_{g_2=0}^{\infty} p_{g_1 g_2} = 1,\quad g_1 = 0, 1, \ldots$$

and

$$p_{g_1 g_2} \geq 0,\quad g_1 = 1, 2, \ldots,\quad g_2 = 0, 1, \ldots.$$

We regard the value $''0''$ as success and the remaining values $''f''$ as failures, $f = 1, 2, \ldots$.

We adopt the following classical way of counting of a "success run of length k": a sequence of n outcomes of successes and failures contains as

many runs of length k as there are non - overlapping, uninterrupted sequence of exactly k zeros.

4.1 Geometric distribution of order k

Let $\mathbf{SK_0}$ be the event

$$\mathbf{SK_0} = \{\text{success run of length } \ k \ \text{ in the sequence} X_0, X_1, \ldots\}\,.$$

Let us define the random variables

$$S_{g_1} \ = \ \begin{cases} 1 & \text{if } X_0 = g_1, \\ 0 & \text{if } X_0 \neq g_1, \end{cases}$$

for $g_1 = 0, 1, \ldots$ and

$$S_{g_1 g_2} \ = \ \left\{ \begin{array}{c} \text{number of transitions of type } g_1 \longrightarrow g_2 \text{ in the} \\ \text{Markov chain until the event } \mathbf{SK_0} \text{ occurs} \end{array} \right\},$$

for $g_1, g_2 = 0, 1, \ldots$.

In this section we deal with the distribution of the random vector

$$\mathbf{S} = (S_0, S_1, \ldots, S_{00}, S_{01}, \ldots),$$

under the condition that the event $\mathbf{SK_0}$ occurs with probability 1. To avoid trivial complications we assume the chain to be irreducible.

Let us write $f_{g_1 g_2}(n)$ for the probability that starting from state g_1, the first passage to the state g_2 occurs at the nth step, $g_1, g_2 = 0, 1, \ldots$. The probability of ever visiting state g_2, starting from the state g_1, is $f_{g_1 g_2}$, where

$$f_{g_1 g_2} = \sum_{n=1}^{\infty} f_{g_1 g_2}(n).$$

Then the condition $P(\mathbf{SK_0}) = 1$ is equivalent to $f_{g_1 0} = 1$, which means that for each $g_1 = 1, 2, \ldots$ the sequences $\{f_{g_1 0}(n)\}_{n=1}^{\infty}$ determine a proper discrete distribution.

The following random variables are related to the defined Markov chain:

$$S = \begin{array}{c}\text{the total number of successes}\\ \text{until the event } \mathbf{SK_0} \text{ occurs,}\end{array}$$

$$F = \begin{array}{c}\text{the total number of failures}\\ \text{until the event } \mathbf{SK_0} \text{ occurs}\end{array}$$

and

$$N = S + F.$$

Definition 4.1 *The distribution of the random variable N is called a geometric distribution of order k for the homogeneous $\{0, 1, 2, \ldots\}-$ valued Markov chain.*

4.1.1 The joint PGF of the random vector S

Under the given notations the following basic lemma is true

Lemma 4.1 *The joint PGF of the random vector $\mathbf{S} = (S_0, S_1, \ldots, S_{00}, S_{01}, \ldots)$ is given by*

$$\begin{aligned} g_{\mathbf{S}}(t_0, t_1, \ldots, t_{00}, t_{01}, \ldots) &= E\{(t_0)^{S_0}(t_1)^{S_1}\ldots(t_{00})^{S_{00}}(t_{01})^{S_{01}}\ldots\} \\ &= \frac{\mathbb{A}_0}{1-\mathbb{A}_{00}}(p_{00}t_{00})^{k-1}, \end{aligned} \tag{4.1}$$

which converges at least for

$$\{(t_0, t_1, \ldots, t_{00}, t_{01}, \ldots) : |t_i| \leq 1 \quad \textit{and} \quad |t_{ij}| \leq 1, \quad i, j = 0, 1, \ldots\},$$

where

$$\mathbb{A}_0 = p_0 t_0 + \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} \mathbb{F}_{g_1 0}, \tag{4.2}$$

$$\mathbb{A}_{00} = \frac{1-(p_{00}t_{00})^{k-1}}{1-p_{00}t_{00}} \sum_{g_1=1}^{\infty} p_{0g_1} t_{0g_1} \mathbb{F}_{g_1 0}, \tag{4.3}$$

and

$$\begin{aligned} \mathbb{F}_{g_1 0} &= p_{g_1 0} t_{g_1 0} + \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} p_{g_2 0} t_{g_2 0} \\ &+ \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} \sum_{g_3=1}^{\infty} p_{g_2 g_3} t_{g_2 g_3} p_{g_3 0} t_{g_3 0} + \ldots, \end{aligned} \tag{4.4}$$

for $g_1 = 1, 2, \ldots$.

Proof. Let $X_0 = 0$ and the first success is followed by $k-1$ consecutive successes. The contribution to the joint PGF $g_{\mathbf{S}}(\cdot)$ is

$$\mathbb{S}_0 = p_0 t_0 (p_{00} t_{00})^{k-1}.$$

Let the initial trial is successful, i.e. $X_0 = 0$ and exactly $r \geq 1$ subsequences containing failures $f = 1, 2, \ldots$ are observed before the event $\mathbf{SK_0}$ occurs. Then the first success "0" is followed by no more than $k-2$ consecutive successes "0", before the first subsequence of failures. The second subsequence of failures in the event $\mathbf{SK_0}$ (if it exists) may be preceded by no more than $k-1$ consecutive successes "0", followed the first subsequence of failures. This process will be repeated till the last subsequence of failures in the event $\mathbf{SK_0}$, which is followed by exactly k consecutive successes "0". Then, the possible sequence of outcomes terminating in k consecutive successes "0" and containing exactly r subsequences of failures can be described as

$$0\underbrace{00\ldots0}_{i_1}\underbrace{ff\ldots f}_{j_1}\underbrace{00\ldots0}_{i_2}\underbrace{ff\ldots f}_{j_2}\ldots\underbrace{00\ldots0}_{i_r}\underbrace{ff\ldots f}_{j_r}\underbrace{00\ldots0}_{k},$$

where

$$0 \leq i_1 \leq k-2;\ 1 \leq i_a \leq k-1,\ a = 2, 3, \ldots, r;\ 1 \leq j_b < \infty,\ b = 1, 2, \ldots, r.$$

From the Markov property of the sequences follows that the net contribution $\mathbb{S}_r$ to the joint PGF $g_{\mathbf{S}}(\cdot)$ of all such sequences is

$$\mathbb{S}_r = p_0 t_0 (\mathbb{A}_{00})^r (p_{00} t_{00})^{k-1},$$

where $\mathbb{A}_{00}$ is given by (4.3);

Let $X_0 = f$ for $f = 1, 2, \ldots$. Then the distribution of the sequence just after the first failure is the same as the distribution after the first failure in the above case, i.e. the contribution to the joint PGF is

$$\mathbb{F}_r = \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} \mathbb{F}_{g_1 0} (\mathbb{A}_{00})^{r-1} (p_{00} t_{00})^{k-1}.$$

Adding all of these contributions gives

$$g_{\mathbf{S}}(t_0, t_1, \ldots, t_{00}, t_{01}, \ldots) = \mathbb{S}_0 + \sum_{r=1}^{\infty} (\mathbb{S}_r + \mathbb{F}_r).$$

To see that $g_{\mathbf{S}}(\cdot)$ is a PGF, it is enough to note that for each $g_1 = 1, 2, \ldots$ the relation

$$\mathbb{F}_{g_1 0}(\mathbf{1}) = \sum_{n=1}^{\infty} f_{g_1 g_2}(n) = 1$$

is fulfilled under the assumption that $P(\mathbf{SK_0}) = 1$, where $\mathbf{1} = (1, 1, \ldots, 1, \ldots)$ is the unit vector.

□

Remark 4.1 *The same result is given in [48] in a different way. The expression (4.1) is preferable to be used since it is simpler than the corresponding joint PGF derived in [48].*

4.1.2 Number of successes in $\mathbf{SK_0}$.

The total number of successes S may be represented as follows

$$S = S_0 + \sum_{g_1=0}^{\infty} S_{g_1 0}.$$

Theorem 4.1 *The distribution of S is given by the PGF*

$$g_S(t) = \frac{(1 - p_{00}t)(p_{00})^{k-1} t^k}{1 - t + (1 - p_{00})(p_{00})^{k-1} t^k}. \tag{4.5}$$

Proof. The relation (4.5) is obtained from (4.1) by substituting

$$t_0 = t_{g_1 0} = t, \quad g_1 = 0, 1, \ldots$$

and

$$t_{g_1} = 1, \ g_1 = 1, 2, \ldots, \quad t_{g_1 g_2} = 1, \quad g_1 = 0, 1, \ldots \text{ and } g_2 = 1, 2, \ldots.$$

□

Remark 4.2 *The presentation (4.5) is the PGF of the shifted geometric distribution of order $k-1$ and its support begins with k. The probability mass function of S is given by*

$$P(S = m) = \sum_{m_1, \ldots m_{k-1}} \binom{m_1 + \ldots + m_{k-1}}{m_1, \ldots, m_{k-1}} (p_{00})^m \left(\frac{q_{00}}{p_{00}}\right)^{m_1 + \ldots + m_{k-1}},$$

for $m = k, k+1, \ldots$, where $q_{00} = 1 - p_{00}$ and the summation is over all nonnegative integers $m_1, m_2, \ldots m_{k-1}$, such that

$$m_1 + 2m_2 + \ldots + (k-1)m_{k-1} = m - k + 1.$$

It is somewhat surprising to note from (4.5) that the distribution of S depends only on p_{00} and it is unaffected by the values p_{g_1} and $p_{g_1 g_1}$, $g_1 = 0, 1, \ldots$ and $g_2 = 1, 2, \ldots$. Hence we can treat a homogeneous multi-state Markov chain as the independent, identically distributed sequence. This conclusion was obtained by Aki and Hirano [4] for the $\{0, 1\}-$ valued homogeneous Markov chain (their Theorem 3.2, p.199).

4.1.3 Number of failures

The total number of failures F may be represented as follows

$$F = \sum_{g_1=1}^{\infty} S_{g_1} + \sum_{g_1=0}^{\infty} \sum_{g_2=1}^{\infty} S_{g_1 g_2}.$$

Theorem 4.2 *The distribution of F is given by the PGF*

$$g_F(t) = \left[p_0 + \sum_{g_1=1}^{\infty} p_{g_1}\mathbf{H}_{g_1}(t)\right] \sum_{n=0}^{\infty} \left(\frac{1-(p_{00})^{k-1}}{1-p_{00}} \sum_{g_1=1}^{\infty} p_{0g_1}\mathbf{H}_{g_1}(t)\right)^n (p_{00})^{k-1}, \tag{4.6}$$

where $\mathbf{H}_{g_1}(t)$ is the PGF of the discrete distribution $\{f_{g_1 0}(n)\}_{n=1}^{\infty}$.

Proof. The relation (4.6) is obtained from (4.1) by substituting

$$t_0 = t_{g_1 0} = 1, \quad g_1 = 0, 1, \ldots$$

and

$$t_{g_1} = t, \ g_1 = 1, 2, \ldots, \ t_{g_1 g_2} = t, \ g_1 = 0, 1, \ldots \quad \text{and} \quad g_2 = 1, 3, \ldots.$$

□

Corollary 4.1 *Let $\mathbf{H}_{g_1}$ be the distribution function of $\{f_{g_1 0}(n)\}_{n=1}^{\infty}$ and $\mathbf{H}$ the distribution function of the mixture*

$$\sum_{g_1=1}^{\infty} \frac{p_{0g_1}}{1-p_{00}} \mathbf{H}_{g_1}.$$

Then the conditional distribution of F

(i) given that $X_0 = g_1, \quad g_1 = 1, 2, \ldots$ is the composition

$$\mathbf{H}_{g_1} * \mathbf{F}_1,$$

where $\mathbf{F}_1$ is the distribution function of the mixture

$$\sum_{n=0}^{\infty} (p_{00})^{k-1}[1-(p_{00})^{k-1}]^n \mathbf{H}^n;$$

(ii) given that $X_0 = 0$, is the mixture

$$p_{00}^{k-1}\delta(0) + [1-(p_{00})^{k-1}]\mathbf{F}_2,$$

where $\delta(0)$ *means the unit mass at zero and* $\mathbf{F}_2$ *is the distribution function of the mixture*

$$\sum_{n=1}^{\infty}(p_{00})^{k-1}[1-(p_{00})^{k-1}]^{n-1}\mathbf{H}^n.$$

Proof. It follows from Theorem 4.2 by considering the conditional PGFs $g_{F|X_0=g_1}(t)$ and $g_{F|X_0=0}(t)$, $g_1 = 1, 2, \ldots$.

□

Corollary 4.2 *The PMF of the random variable* F *is given by*

$$P(F=m) = \begin{cases} p_0(p_{00})^{k-1}, & m=0 \\ \sum_{g_1=1}^{\infty} P_{g_1}(F=m), & m=1,2,\ldots, \end{cases}$$

where

$$P_{g_1}(F=m) = (p_{00})^{k-1}\left(p_0 + \frac{1-(p_{00})^{k-1}}{1-p_{00}}p_{0g_1}\right)f_{g_10}(m)$$

$$+\frac{1-(p_{00})^{k-1}}{1-p_{00}}p_{0g_1}\sum_{i=1}^{m-1}P(F=i)f_{g_10}(m-i).$$

Proof. The distribution of F is obtained by equating the coefficients of t^m on both sides of (4.6).

□

4.1.4 Number of trials

The total number of trials N can be represented as follows

$$N = S + F = \sum_{g_1=0}^{\infty} S_{g_1} + \sum_{g_1=0}^{\infty}\sum_{g_2=0}^{\infty} S_{g_1g_2}.$$

Theorem 4.3 *The distribution of N is given by the PGF*

$$g_N(t) = \frac{p_0 + \sum_{g_1=1}^{\infty} p_{g_1} \mathbf{H}_{g_1}(t)}{1 - \frac{1-(p_{00}t)^{k-1}}{1-p_{00}t} \sum_{g_1=1}^{\infty} p_{0g_1} t \mathbf{H}_{g_1}(t)} (p_{00})^{k-1} t^k, \tag{4.7}$$

where $\mathbf{H}_{g_1}(t)$ is the PGF of the discrete distribution $\{f_{g_1 0}(n)\}_{n=1}^{\infty}$.

Proof. The relation (4.7) is obtained from (4.1) by substituting

$$t_{g_1} = t_{g_1 g_2} = t, \quad g_1, g_2 = 0, 1, \ldots.$$

□

This leads to the second definition of the geometric distributions of order k.

Definition 4.2 *The distribution with PGF given by (4.7) is called a geometric distribution of order k related to homogeneous multi-state Markov chain.*

Corollary 4.3 *The PMF of the random variable N is given by*

$$P(N = k) = p_0 (p_{00})^{k-1}$$

$$P(N = k+1) = (p_{00})^{k-1} \sum_{g_1=1}^{\infty} p_{g_1} p_{g_1 0}$$

$$\begin{aligned} P(N = m) &= (p_{00})^{k-1} \sum_{g_1=1}^{\infty} p_{g_1} p_{f_1 0}(m-k) \\ &\quad + \sum_{i+n+j=m-1} P(N = i)(p_{00})^n \sum_{g_1=1}^{\infty} p_{0g_1} f_{g_1 0}(j), \quad m = k+2, k+3, \ldots, \end{aligned}$$

where the summation is over the nonnegative integers i, n and j such that $j \geq k$, $0 \leq n \leq k-2$ and $j \geq 1$.

Proof. The distribution of N is obtained by equating the coefficients of t^m on both sides in (4.7), $m \geq k$.

□

Remark 4.3 *For two - state Markov chain, the distribution of N and its PGF was outlined by Aki and Hirano [3] where it is understood that X_0 is not counted in measuring the length of the run (Corollary on p. 469).*

Corollary 4.4 *The expected number of trials in* $\mathbf{SK_0}$ *is given by*

$$EN = 1 + \sum_{g_1=1}^{\infty} p_{g_1} E(g_1) + \frac{(p_{00})^{1-k} - 1}{1 - p_{00}} \left(\sum_{g_1=1}^{\infty} p_{0g_1} E(g_1) + 1 \right), \tag{4.8}$$

where for $g_1 = 1, 2, \ldots$

$$E(g_1) = \sum_{i=1}^{\infty} i f_{g_1 0}(i).$$

Proof. The expression (4.8) is obtained after simplification of the relation

$$EN = \frac{d}{dt} g_N(t) \Big|_{t=1}.$$

□

Example 4.1 *(Random walk model). Let us consider the motion of a particle which moves in discrete jumps with certain probabilities from point to point. We imagine the particle starts at the point $x = i$ with probability p_i on the x - axis, $i = 1, 2, \ldots$. For each subsequent time it moves one unit to the right, one unit to the left or remains where it is with probabilities $0 < p_{i,i+1} < 1$, $0 < p_{i,i-1} < 1$ or p_{ii}, respectively, where*

$$p_{i,i-1} + p_{ii} + p_{i,i+1} = 1, \quad i = 1, 2, \ldots.$$

If it happens that the particle starts at the point $x = 0$ *with probability* $p_0 \in (0, 1)$, *it moves at the next time only one unit to the right or remains where it is with probabilities* p_{01} *or* $p_{00} \in (0, 1)$, *respectively where*

$$p_{00} + p_{01} = 1.$$

The same rule is fulfilled if in some time the particle is at the point $x = 0$ *on the* x *- axis.*

The considered random walk model can be described by a homogeneous Markov chain with the following transition probability matrix

$$\begin{pmatrix} p_{00} & p_{01} & 0 & 0 & 0 & \cdots \\ p_{10} & p_{11} & p_{12} & 0 & 0 & \cdots \\ 0 & p_{21} & p_{22} & p_{23} & 0 & \cdots \\ 0 & 0 & p_{32} & p_{33} & p_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Let us note that the PGF $g_N(t)$ *in (4.7) is unaffected by the zero-values of the above matrix. In particular, this means that* $\mathbf{H}_{g_1}(t) = \sum_{i=g_1}^{\infty} f_{g_1 0}(i) t^i$ *for* $g_1 = 1, 2, \ldots$.

Example 4.2 *(Urn model). Suppose that there are urns numbered by* $0, 1, 2, \ldots$. *Each urn contains balls numbered by* $0, 1, 2, \ldots$. *The content of each urn remains constant (each sampled ball is returned to the urn). Let* p_{g_1} *denote the probability that the initial drawing will be from the urn with number* g_1, $g_1 = 0, 1, 2, \ldots$, $\sum_{g_1=0}^{\infty} p_{g_1} = 1$. *Let* $p_{g_1 g_2}$ *be the probability that a ball bearing the number* g_2 *is drawn from the* g_1*th urn,* $\sum_{g_2=0}^{\infty} p_{g_1 g_2} = 1$. *Then the next ball will be drawn from the urn numbered by* g_2, $g_2 = 0, 1, 2, \ldots$.

The process continues until k consecutive times of drawing is made from the urn numbered by 0. If N denote the number of balls drawn, then N has an geometric distribution of order k for the homogeneous multi - state Markov chain, defined by Corollary 4.3.

4.1.5 The joint PGF $g_{S,F}(t,s)$

By substituting

$$t_0 = t_{g_1 0} = t, \quad g_1 = 0, 1, \ldots$$

and

$$t_{g_1} = s, \ g_1 = 1, 2, \ldots, \ t_{g_1 g_2} = s, \ g_1 = 0, 1, \ldots \text{ and } g_2 = 1, 2, \ldots$$

in (4.1) we get the following

Theorem 4.4 *The joint distribution (S, F) is defined by the PGF*

$$g_{S,F}(t,s) = \frac{\mathbb{A}_0(t,s)}{1 - \mathbb{A}_{00}(t,s)} (p_{00} t)^{k-1},$$

where $\mathbb{A}_0(t,s)$ and $\mathbb{A}_{00}(t,s)$ are obtained from (4.2) and (4.3).

4.2 Quotas on runs in a multi-state Markov chain

The waiting times to be discussed will arise by setting quotas on runs of successes and failures. Let c and d be fixed positive integers and let us consider the events

$$\mathbf{ST}(c,d) \quad = \quad \left\{ \begin{array}{c} \text{the trials are performed sequentially} \\ \text{until either } c \text{ consecutive successes} \\ \text{or } d \text{ consecutive failures are observed} \end{array} \right\},$$

$$\mathbf{ST}_1(c) = \left\{ \begin{array}{c} \text{the trials are performed sequentially} \\ \text{until } c \text{ consecutive successes are observed} \end{array} \right\},$$

$$\mathbf{ST}_2(d) = \left\{ \begin{array}{c} \text{the trials are performed sequentially} \\ \text{until } d \text{ consecutive failures are observed} \end{array} \right\},$$

and

$$\mathbf{LT}(c,d) = \left\{ \begin{array}{c} \text{the trials are performed sequentially upon} \\ \text{completion of both a run of } c \text{ successes} \\ \text{and a run of } d \text{ failures} \end{array} \right\}.$$

It is clear that the sooner occurring event $\mathbf{ST}(c,d)$ and the later occurring event $\mathbf{LT}(c,d)$ between $\mathbf{ST}_1(c)$ and $\mathbf{ST}_2(d)$ are determined by the relations

$$\mathbf{ST}(c,d) = \mathbf{ST}_1(c) \cup \mathbf{ST}_2(d) \tag{4.9}$$

and

$$\mathbf{LT}(c,d) = \mathbf{ST}_1(c) \cap \mathbf{ST}_2(d). \tag{4.10}$$

In this section we extend to the present homogeneous Markov chain the waiting time problems related with the events $\mathbf{ST}(c,d)$ and $\mathbf{LT}(c,d)$. These events are studied by Balasubramanian et al. [12] for Markov correlated Bernoulli trials, where the review of literature is given.

Similar results for two - state homogeneous Markov chain are obtained by Aki and Hirano ([3]) and Uchida and Aki [83]. The distribution theory of runs in different Markov fashions is developed by many authors. Some important references are Aki et *al.* [1], Balakrishnan [8], Doi and Yamamoto [27], Ebneshahrashoob and Sobel [29], Fu and Koutras [33], Koutras [52]. In Balakrishnan et al. [9], Balakrishnan and Chan [10] and Balakrishnan et al. [11] the start - up demonstration tests are discussed as an application of the distributions of order k. It is a demonstration of the starting - up reliability of some equipment.

4.2.1 Sooner waiting time problems

We start with the distribution of the random vector

$$\mathbf{S} = (S_0, S_1, \ldots, S_{00}, S_{01}, \ldots),$$

under the condition that the event $\mathbf{ST}(c,d)$ occurs with probability 1. Under the given notations the following basic lemma is true.

Lemma 4.2 *The joint probability generating function of the random vector* $\mathbf{S}$ *related with the event* $\mathbf{ST}(c,d)$ *is given by*

$$\begin{aligned} g_{\mathbf{S}}(t_0,t_1,\ldots,t_{00},t_{01},\ldots) &= E\{(t_0)^{S_0}(t_1)^{S_1}\ldots(t_{00})^{S_{00}}(t_{01})^{S_{01}}\ldots\} \\ &= \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} F_{g_1}^{(d)} + \frac{A_0 C_{00}}{1-A_{00}} \end{aligned} \tag{4.11}$$

which converges at least for

$$\{(t_0,t_1,\ldots,t_{00},t_{01},\ldots) : |t_i| \le 1 \quad and \quad |t_{ij}| \le 1, \quad i,j = 0,1,\ldots\}$$

and

$$A_0 = p_0 t_0 + \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} F_{g_1 0}^{(d-1)}, \tag{4.12}$$

$$A_{00} = \frac{1-(p_{00}t_{00})^{c-1}}{1-p_{00}t_{00}} \sum_{g_1=1}^{\infty} p_{0g_1} t_{0g_1} F_{g_1 0}^{(d-1)}, \tag{4.13}$$

$$C_{00} = (p_{00}t_{00})^{c-1} + \frac{1-(p_{00}t_{00})^{c-1}}{1-p_{00}t_{00}} \sum_{g_1=1}^{\infty} p_{0g_1} t_{0g_1} F_{g_1}^{(d)},$$

where for $g_1 = 1,2,\ldots$ *and* $d > 1$

$$F_{g_1}^{(d)} = \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} \cdots \sum_{g_d=1}^{\infty} p_{g_{d-1} g_d} t_{g_{d-1} g_d}, \tag{4.14}$$

$$\begin{aligned} F_{g_1 0}^{(d-1)} &= p_{g_1 0} t_{g_1 0} + \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} p_{g_2 0} t_{g_2 0} + \ldots \\ &+ \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} \cdots \sum_{g_{d-1}=1}^{\infty} p_{g_{d-2} g_{d-1}} t_{g_{d-2} g_{d-1}} p_{g_{d-1} 0} t_{g_{d-1} 0}. \end{aligned} \tag{4.15}$$

If $d = 1$ *the expressions (4.14) and (4.15) are equal to*

$$F_{g_1}^{(d)} = 1 \quad and \quad F_{g_1 0}^{(d-1)} = 0.$$

Proof. Let us consider at first the sequences of outcomes terminating in c consecutive successes "0". The possible disjoint events for $d > 1$ are as follows:

– $X_0 = 0$ and the first success is followed by $c-1$ consecutive successes. The contribution to the joint PGF $g_{\mathbf{S}}(\cdot)$ is

$$S^0 = p_0 t_0 (p_{00} t_{00})^{c-1};$$

– $X_0 = 0$ and exactly $r \geq 1$ subsequences containing no more than $d-1$ failures are observed.

The possible sequences of outcomes can be described as

$$0\underbrace{00\ldots0}_{i_1}\underbrace{f f\ldots f}_{j_1}\underbrace{00\ldots0}_{i_2}\underbrace{f f\ldots f}_{j_2}\cdots\underbrace{00\ldots0}_{i_r}\underbrace{f f\ldots f}_{j_r}\underbrace{00\ldots0}_{c}$$

where

$$0 \leq i_1 \leq c-2;\ 1 \leq i_a \leq c-1,\ a = 2,3,\ldots,r;\ 1 \leq j_b < d,\ b = 1,2,\ldots,r.$$

From the Markov property of the sequences follows that the net contribution S_r^1 to the joint PGF $g_{\mathbf{S}}(\cdot)$ of all such sequences is

$$S_r^1 = p_0 t_0 (A_{00})^r (p_{00} t_{00})^{c-1},$$

where A_{00} is given by (4.13);

– $X_0 = f,\ f = 1,2,\ldots$ and exactly $r \geq 1$ subsequences containing no more than $d-1$ failures are observed.

The contribution to the joint PGF in this case is

$$S_r^2 = \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} F_{g_1 0}^{(d-1)} (A_{00})^{r-1} (p_{00} t_{00})^{c-1}.$$

Adding all of these contributions gives

$$S = S^0 + \sum_{r=1}^{\infty} (S_r^1 + S_r^2)$$

and after simplifications we obtain

$$S = \frac{A_0}{1 - A_{00}} (p_{00} t_{00})^{c-1}.$$

Let us consider the sequences of outcomes terminating in d consecutive failures $f = 1, 2, \ldots$ when $d > 1$.

Let $X_0 = f$ and the first failure is followed by $d-1$ consecutive failures. The contribution to the joint PGF in this case is

$$F_1 = \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} F_{g_1}^{(d)}.$$

Next, the contribution to the joint PGF, when the initial trial is successful followed by no more than $c-1$ consecutive successes preceded the sequence of d consecutive failures, is given by

$$F_2 = p_0 t_0 \frac{1-(p_{00}t_{00})^{c-1}}{1-p_{00}t_{00}} \sum_{g_1=1}^{\infty} p_{0g_1} t_{0g_1} F_{g_1}^{(d)}.$$

The corresponding contributions F_r^1 and F_r^2 from the sequences ending with d consecutive failures, containing exactly $r \geq 1$ subsequences of failures and beginning with a success or a failure are

$$F_r^1 = p_0 t_0 (A_{00})^r \frac{1-(p_{00}t_{00})^{c-1}}{1-p_{00}t_{00}} \sum_{g_1=1}^{\infty} p_{0g_1} t_{0g_1} F_{g_1}^{(d)}$$

and

$$F_r^2 = \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} F_{g_10}^{(d-1)} (A_{00})^{r-1} \frac{1-(p_{00}t_{00})^{c-1}}{1-p_{00}t_{00}} \sum_{g_1=1}^{\infty} p_{0g_1} t_{0g_1} F_{g_1}^{(d)},$$

where $A_{00}, F_{g_1}^{(d)}$ and $F_{g_10}^{(d-1)}$ are given by (4.13), (4.14) and (4.15). Adding of all of these contributions gives

$$F = F_1 + F_2 + \sum_{r=1}^{\infty} (F_r^1 + F_r^2),$$

i.e.

$$F = \sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} F_{g_1}^{(d)} + \frac{A_0}{1-A_{00}} \left[C_{00} - (p_{00}t_{00})^{c-1} \right].$$

The joint PGF $g_{\mathbf{S}}(\cdot)$ given by (4.11) is obtained by

$$g_{\mathbf{S}}(t_0, t_1, \ldots, t_{00}, t_{01}, \ldots) = S + F.$$

At the end, if $d = 1$, the joint PGF is given by the following expression

$$\sum_{g_1=1}^{\infty} p_{g_1} t_{g_1} + p_0 t_0 \left[(p_{00} t_{00})^{c-1} + \frac{1 - (p_{00} t_{00})^{c-1}}{1 - p_{00} t_{00}} \sum_{g_1=1}^{\infty} p_{0g_1} t_{0g_1} \right],$$

i.e. one can substitute $F_{g_1}^{(d)} = 1$ and $F_{g_1 0}^{(d-1)} = 0$ in (4.11).

To see that $g_{\mathbf{S}}(\cdot)$ is a PGF, let us put in (4.11) all arguments equal to 1 and $\mathbf{1} = (1, 1, \ldots, 1, 1, \ldots)$ be the corresponding unit vector. Then the following relations

$$F_{g_1 0}^{(d-1)}(\mathbf{1}) = \sum_{i=1}^{d-1} f_{g_1 0}(i) = 1 - F_{g_1}^{(d)}(\mathbf{1})$$

and

$$1 - A_{00}(\mathbf{1}) = C_{00}(\mathbf{1}),$$

are true. Here $f_{g_1 0}(i)$ means the probability that first visit to state 0 from some failure state occurs in exactly i steps, $i = 1, 2, \ldots, d-1$.

□

Remark 4.4 *Lemma 4.2 is true under the condition that* $P(\mathbf{ST}(c, d)) = 1$, *which is always fulfilled for irreducible Markov chains with persistent set of states.*

If the states of the chain are transient, for the realization of the event $\mathbf{ST}(c, d)$ with probability 1, it is necessary all possible sequences of outcomes between the first and the last observed successive state $''0''$ in the event $\mathbf{ST}(c, d)$ to belong to the same state, say $\mathbf{P}_0$, of transient states.

Let $X_0 = 0$ and the sequences of outcomes terminate in d consecutive transient failure states. Beginning from some of the last d failure states it is possible the elements of the following subsequence to belong to the upper (according $\mathbf{P}_0$) sets of states.

Let the chain starts with a failure state. Then some subsequence of consecutive failure states, followed the initial state of the chain and preceding the first successive state, may belong to the lower (according $\mathbf{P}_0$) set of transient states.

Remark 4.5 *If the homogeneous Markov chain has two states from (4.11) one can obtain the corresponding joint PGF given by Balasubramanian et al. ([12]) in their Proposition 1. Lemma 4.2 generalizes the PGF given in Theorem 1 by Aki and Hirano ([3]), where it is understood that X_0 is not counted in measuring the length of the run in two - state Markov chain.*

Remark 4.6 *By substituting in (4.11)*

$$t_0 = t_{g_1 0} = s, \quad g_1 = 0, 1, \ldots$$

and

$$t_{g_1} = t, \ g_1 = 1, 2, \ldots; \quad t_{g_1 g_2} = t, \quad g_1 = 0, 1, \ldots \quad \text{and} \quad g_2 = 1, 2, \ldots$$

we obtain the joint PGF

$$g_{S,F}(s,t) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} P(X = x, F = y) s^x t^y \tag{4.16}$$

of the random vector (S, F), where S and F are the numbers of successes and failures in the event $\mathbf{ST}(c, d)$, respectively. From (4.16) it is possible to calculate the $Cov(S, F)$.

Let us denote by

$$a_1(j) = \sum_{g_1=1}^{\infty} p_{g_1} f_{g_1 0}(j), \quad j = 1, 2, \ldots, d-1,$$

$$a_0(j) = \sum_{g_1=1}^{\infty} p_{0 g_1} f_{g_1 0}(j), \quad j = 1, 2, \ldots, d-1,$$

$$a_1(d) = \sum_{g_1=1}^{\infty} p_{g_1} F_{g_1}^{(d)}(\mathbf{1})$$

and

$$a_0(d) = \sum_{g_1=1}^{\infty} p_{0g_1} F_{g_1}^{(d)}(\mathbf{1}),$$

where $f_{g_1 0}(j)$ *are the probabilities defined in the proof of Theorem 4.2 and* $\mathbf{1}$ *means the corresponding unit vector.*

From (4.16) one can obtain the following exact join distribution of the random vector (S, F) :

$$\begin{aligned}
P(S=x, F=y) &= 0 && \text{for} \quad x<c \quad \text{and} \quad y<d;\\
P(S=c, F=0) &= p_0(p_{00})^{c-1};\\
P(S=0, F=d) &= a_1(d);\\
P(S=c, F=d) &= 0;\\
P(S=c+i, F=0) &= 0 && \text{for} \quad i=1,2,\ldots;\\
P(S=0, F=d+j) &= 0 && \text{for} \quad j=1,2,\ldots;\\
P(S=c, F=j) &= a_1(j)(p_{00})^{c-1} && \text{for} \quad j=1,2,\ldots,d-1;\\
P(S=i, F=d) &= p_0(p_{00})^{i-1}a_0(d) && \text{for} \quad i=1,2,\ldots,c-1,
\end{aligned}$$

$$\begin{aligned}
P(S=i, F=d+j) &= (p_{00})^{i-1}[a_1(j)a_0(d) - a_0(j)a_1(d)]\\
&+ \sum_{m=1}^{i}\sum_{n=1}^{j}(p_{00})^{m-1}a_0(n)P(S=i-m, F=d+j-n)\\
&\quad \text{for} \quad i=1,2,\ldots,c-1 \quad \text{and} \quad j=1,2,\ldots,d-1.
\end{aligned}$$

The remaining joint probabilities for $x=c,\ y=d+1, d+2, \ldots, 2d-1$ *and* $x \geq c+1,\ y \geq 2d$, *are given recursively by the following relation*

$$P(S=x, F=y) = \sum_{m=1}^{c-1}\sum_{n=1}^{d-1}(p_{00})^{m-1}a_0(n)P(S=x-m, F=y-n).$$

From (4.16) can be obtained the marginal PGF's $g_S(s)$, $g_F(t)$ and $g_N(u)$ of the total number of successes S, the total number of failures F and the total number of trials N in the event $\mathbf{ST}(c,d)$, respectively, as well as the corresponding probability mass functions and expected waiting times.

4.2.2 Exact Distribution of the number of successes

By substituting in (4.11)

$$t_0 = t_{g_1 0} = s, \quad g_1 = 0, 1, \ldots$$

and

$$t_{g_1} = 1,\ g_1 = 1, 2, \ldots; \quad t_{g_1 g_2} = 1,\ g_1 = 0, 1, \ldots \quad \text{and} \quad g_2 = 1, 2, \ldots$$

we obtain the PGF of the number of successes S in the event $\mathbf{ST}(c, d)$. The corresponding expression is given by

$$g_S(s) = a_1(d) + \frac{\left[p_0 + \sum_{j=1}^{d} a_1(j)\right]\left[(p_{00})^{c-1}s^c + sa_0(d)\frac{1-(p_{00}s)^{c-1}}{1-p_{00}s}\right]}{1 - \sum_{m=1}^{c-1}(p_{00})^{m-1}s^m \sum_{j=1}^{d-1} a_0(j)}, \quad (4.17)$$

where $a_0(d), a_1(d), a_0(j)$ and $a_1(j)$ are as defined in Remark 4.6. Now, for the exact distribution of the number of successes we obtain the following formulas:

$$\begin{aligned}
P(S=0) &= a_1(d); \\
P(S=i) &= \left[p_0 + \sum_{j=1}^{d-1} a_1(j)\right](p_{00})^{i-1}a_0(d) \\
&+ \sum_{m=1}^{i-1} P(S=i-m)(p_{00})^{m-1}\sum_{j=1}^{d-1} a_0(j) \quad \text{for} \quad i = 1, 2, \ldots, c-1; \\
P(S=c) &= \left[p_0 + \sum_{j=1}^{d-1} a_1(j)\right](p_{00})^{c-1}.
\end{aligned}$$

The remaining probabilities can be calculated recursively

$$P(S=x) = \sum_{m=1}^{c-1}\sum_{j=1}^{d-1}(p_{00})^{m-1}a_0(j)P(S=x-m) \quad \text{for} \quad x > c.$$

Example 4.3 *Let us consider a Markov chain having 3 states: a successive one and two types of failure states (1 and 2), with transition probabilities given by the matrix*

$$\begin{pmatrix} p_{00} & p_{01} & 0 \\ p_{10} & 0 & p_{12} \\ p_{20} & 0 & p_{22} \end{pmatrix}$$

and $p_0 + p_1 = 1$.

This matrix corresponds to the single corrective action model for a start-up demonstration test under the Markovian dependence structure studied by Balakrishnan et al. [11]. We slightly simplify the model accepting the same probability

$$p_{00} = P(\textit{success} \mid \textit{previous trial is successive}).$$

In a single corrective action model, one corrective action of the equipment is allowed (failure of type 1), resulting in a change in the probability of success just after a failure, until specified number of consecutive start-ups, say c, are observed. This means that just after the first failure the system changes his behavior. So, it is impossible the to begin with a failure of the second type. Because of this $p_2 = 0$.

Under these assumptions, one can obtain from (4.17), with $d \to \infty$, *the PGF given in Theorem 2.1 in [11].*

Let us substitute $d = 3$. *This means that under the corrective action model, the start-ups are terminated upon the c - th consecutive successive state of the equipment or just after the third observed consecutive failure.*

In this case, we have the following simple expressions

$$\begin{aligned}
a_0(1) &= p_{01}p_{10};\\
a_0(2) &= p_{01}p_{12}p_{20};\\
a_0(3) &= p_{01}p_{12}p_{22};\\
a_1(1) &= p_1p_{10};\\
a_1(2) &= p_1p_{12}p_{20};\\
a_1(3) &= p_1p_{12}p_{22},
\end{aligned}$$

which are used in calculating the probabilities $P(S = x),\ x = 0, 1, \ldots.$

Remark 4.7 *Letting* $d \to \infty$ *in the event* $\mathbf{ST}(c, d)$ *yields the event* $\mathbf{SK}_0$. *Then from (4.16) we obtain the joint PGF* $g(c, \infty)$ *until a run of* c *consecutive successes in a multi - state Markov chain is observed, given by*

$$g(c, \infty) = \frac{\mathbb{A}_0}{1 - \mathbb{A}_{00}}(p_{00}t_{00})^{c-1}, \tag{4.18}$$

where $\mathbb{A}_0$ *and* $\mathbb{A}_{00}$ *are given by (4.2) and (4.3).*

Let us note that in this limiting case, the relation (4.18) is true only for irreducible Markov chain with persistent set of states.

4.2.3 Exact distribution of the number of failures

By analogy with the previous subsection, let us substitute

$$t_0 = t_{g_1 0} = 1, \quad g_1 = 0, 1, \ldots$$

and

$$t_{g_1} = t,\ g_1 = 1, 2, \ldots, \quad t_{g_1 g_2} = t,\ \ g_1 = 0, 1, \ldots \quad \text{and} \quad g_2 = 1, 2, \ldots$$

As a result, from (4.11) one can obtain the PGF $g_F(t)$ of the total number F of failures in the event $\mathbf{ST}(c, d)$

$$g_F(t) = a_1(d)t^d + \frac{\left[p_0 + \sum_{j=1}^{d-1} a_1(j)t^j\right]\left[(p_{00})^{c-1} + a_0(d)\frac{1-(p_{00})^{c-1}}{1-p_{00}}t^d\right]}{1 - \frac{1-(p_{00})^{c-1}}{1-p_{00}}\sum_{j=1}^{d-1} a_0(j)t^j},$$

where the expressions for $a_0(d), a_1(d), a_0(j)$ and $a_1(j)$ are given in Remark 4.6. The probability mass function of the total number of failures is given

by the following relations

$$\begin{aligned}
P(F=0) &= p_0(p_{00})^{c-1};\\
P(F=j) &= \left[a_1(j)+p_0\frac{1-(p_{00})^{c-1}}{1-p_{00}}a_0(j)\right](p_{00})^{c-1} \quad \text{for} \quad j=1,2,\ldots,d-1;\\
P(F=d) &= a_1(d)+p_0\frac{1-(p_{00})^{c-1}}{1-p_{00}}a_0(d);\\
P(F=d+j) &= \frac{1-(p_{00})^{c-1}}{1-p_{00}}\left[a_1(j)a_0(d)+p_0\frac{1-(p_{00})^{c-1}}{1-p_{00}}a_0(j)a_0(d)\right.\\
&+ \left.\sum_{k=1,\ k\neq j}^{d-1} P(F=d+j-k)a_0(k)\right] \quad \text{for} \quad j=1,2,\ldots,d-1.
\end{aligned}$$

The remaining probabilities can be calculated recursively

$$P(F=y)=\frac{1-(p_{00})^{c-1}}{1-p_{00}}\sum_{j=1}^{d-1}a_0(j)P(F=y-j) \quad \text{for} \quad y\geq 2d.$$

Example 4.4 *Let us consider a random walk model over the non-negative integers determined by the following transition matrix*

$$\begin{pmatrix}
p_{00} & p_{01} & 0 & 0 & 0 & \cdots\\
p_{10} & p_{11} & p_{12} & 0 & 0 & \cdots\\
0 & p_{21} & p_{22} & p_{23} & 0 & \cdots\\
0 & 0 & p_{32} & p_{33} & p_{34} & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

and $p_0+p_1+\ldots=1$. *If we substitute* $d=3$, *we have the following expressions*

$$\begin{aligned}
a_0(1) &= p_{01}p_{10};\\
a_0(2) &= p_{01}p_{11}p_{10};\\
a_0(3) &= p_{01}[p_{11}(p_{11}+p_{12})+p_{12}];\\
a_1(1) &= p_1p_{10};\\
a_1(2) &= p_1p_{11}p_{10};\\
a_1(3) &= 1+p_1[p_{11}(p_{11}+p_{12})+p_{12}-1]+p_2p_{21}(p_{11}+p_{12}-1)-p_0
\end{aligned}$$

and one can calculate the probabilities $P(F = y),\ y \geq 0$.

Remark 4.8 *Letting* $c \to \infty$ *in the event* $\mathbf{ST}(c,d)$ *yields the event* $\mathbf{ST}_2(d)$. *From (4.16) we obtain the joint PGF* $g(\infty,d)$ *until a run of* d *consecutive failures in a multi - state Markov chain is observed, given by*

$$g(\infty,d) = \sum_{g_1=1}^{\infty} \left[p_{g_1} t_{g_1} + \frac{p_0 t_0 + \sum_{l=1}^{\infty} p_l t_1 F_{l0}^{(d-1)}}{1 - p_{00} t_{00} \sum_{l=1}^{\infty} p_{0l} t_{0l} F_{l0}^{(d-1)}} p_{0g_1} t_{0g_1} \right] F_{g_1}^{(d)}, \quad (4.19)$$

where $F_{g_1}^{(d)}$ *and* $F_{g_1 0}^{(d-1)}$ *are the expressions (4.14) and (4.15), respectively.*

As in the previous case, (4.19) is true only for irreducible Markov chains with persistent set of states.

4.2.4 Later waiting time problems

The latter case in this section arises when stopping the experiment upon completion of both a run of c successes and a run of d failures in a multi state Markov chain, i.e. we deal with the event $\mathbf{LT}(c,d)$ and associated random variables

$$S_{g_1 g_2}^{1} = \left\{ \begin{array}{c} \text{number of transitions of type } g_1 \longrightarrow g_2 \text{ in the} \\ \text{Markov chain until the event } \mathbf{LT}(c,d) \text{ occurs} \end{array} \right\},$$

for $g_1, g_2 = 0, 1, \ldots$ under condition that $P(\mathbf{LT}(c,d)) = 1$.

The joint PGF $g_{\mathbf{LT}}(c,d)$ related with the random vector $\mathbf{S}$ can be obtained from the results derived in Section 2 because of duality noticed by Balasubramanian et *al.* [12] and based on the relations (4.9) and (4.10).

If we denote by $g_{\mathbf{ST}}(c,d)$ the joint PGF given by (4.11), then the joint PGF $g_{\mathbf{LT}}(c,d)$ is determined by

$$g_{\mathbf{LT}}(c,d) = g(c,\infty) + g(\infty,d) - g_{\mathbf{ST}}(c,d), \quad (4.20)$$

where $g(c,\infty)$ and $g(\infty,d)$ are expressions (4.18) and (4.19). Similar relationships hold for any marginal PGF's.

Remark 4.9 *From (4.11) one can obtain the corresponding joint PGF, given by Aki and Hirano [3] in their Theorem 2 where it is understood that X_0 is not counted in measuring the length of the run in two-state Markov chain.*

Remark 4.10 *The condition $P(\mathbf{LT}(c,d)) = 1$ implies that the joint PGF determined by (4.20) have to be modified as follows*

$$g_{\mathbf{LT}}(c,d,c_1,d_1) = g(c,d_1) + g(c_1,d) - g_{\mathbf{ST}}(c,d),$$

where $c \leq c_1 < \infty$ and $d \leq d_1 < \infty$ with c_1 and d_1 being the maximal possible length of runs of successes and failures within the set $\mathbf{P}_0$, defined by Remark 4.4.

4.3 Run and frequency Quotas in a multi-state Markov cnain

For Markov correlated Bernoulli trials Balasubramanian et *al.* [12] studied the waiting time problems related with the events

$$\mathbf{RF}(c,d) = \left\{ \begin{array}{c} \text{the trials are performed sequentially} \\ \text{until either } c \text{ consecutive successes} \\ \text{or } d \text{ failures in total are observed} \end{array} \right\}$$

and

$$\mathbf{FR}(c,d) = \left\{ \begin{array}{c} \text{the trials are performed sequentially} \\ \text{until either } c \text{ successes in total} \\ \text{or } d \text{ consecutive failures are observed} \end{array} \right\}$$

where c and d are fixed positive integers.

Some waiting time problems related with quotas on runs of successes and failures are studied by Aki et *al.* [1], Balakrishnan et *al.* [11], Doi and Yamamoto [27], Fu and Koutras [33], Koutras [52].

Let us define the random variables

$$S_{g_1} = \begin{cases} 1 & \text{if } X_0 = g_1, \\ 0 & \text{if } X_0 \neq g_1, \end{cases}$$

for $g_1 = 0, 1, \ldots,$

$$S^1_{g_1 g_2} = \left\{ \begin{array}{c} \text{number of transitions of type } g_1 \longrightarrow g_2 \\ \text{in the Markov chain until the event} \\ \mathbf{RF}(c,d) \text{ occurs} \end{array} \right\}$$

and

$$S^2_{g_1 g_2} = \left\{ \begin{array}{c} \text{number of transitions of type } g_1 \longrightarrow g_2 \\ \text{in the Markov chain until the event} \\ \mathbf{FR}(c,d) \text{ occurs} \end{array} \right\},$$

for $g_1, g_2 = 0, 1, \ldots.$

Here we are interested on joint probability generating functions (PGF) related with the random vectors

$$\mathbf{S}^i = (S_0, S_1, \ldots, S^i_{00}, S^i_{01}, \ldots), \quad i = 1, 2,$$

under condition that the events $\mathbf{FR}(c,d)$ and $\mathbf{RF}(c,d)$ occur with probability 1. Interpretation of the results is given in terms of start - up demonstration tests.

4.3.1 Run Quota on successes and frequency Quota on failures

In this section we will study the waiting time problems, related with the event $\mathbf{RF}(c,d)$. The basic result is given by the following lemma.

Lemma 4.3 *The joint PGF of the random vector*

$$\mathbf{S}^1 = (S_0, S_1, \ldots, S^1_{00}, S^1_{01}, \ldots)$$

related with the event $\mathbf{RF}(c,d)$ *is given by*

$$g_{\mathbf{S}^1}(t_0,t_1,\ldots,t_{00},t_{01},\ldots) = g_{\mathbf{S}^1}(c,d) = E\{(t_0)^{S_0}(t_1)^{S_1}\ldots(t_{00})^{S^1_{00}}(t_{01})^{S^1_{01}}\ldots\}$$

$$= p_0t_0(p_{00}t_{00})^{c-1} + \textstyle\sum_{g_1=1}^{\infty} A_{g_1}B_{g_1}, \tag{4.21}$$

which converges at least for

$$\{(t_0,t_1,\ldots,t_{00},t_{01},\ldots):\ |t_i|\le 1 \ \ and \ \ |t_{ij}|\le 1, \quad i,j=0,1,\ldots\}$$

where for $d>1$

$$A_{g_1} = p_0t_0\frac{1-(p_{00}t_{00})^{c-1}}{1-p_{00}t_{00}}p_{0g_1}t_{0g_1} + p_{g_1}t_{g_1}, \tag{4.22}$$

$$B_{g_1} = B^{(d)}_{g_1} + B^{(d-1)}_{g_10}(p_{00}t_{00})^{c-1}, \tag{4.23}$$

$$B^{(d)}_{g_1} = \sum_{g_2=1}^{\infty} B_{g_1g_2}\sum_{g_3=1}^{\infty} B_{g_2g_3}\ldots\sum_{g_d=1}^{\infty} B_{g_{d-1}g_d}, \tag{4.24}$$

$$\begin{aligned} B^{(d-1)}_{g_10} &= p_{g_10}t_{g_10} + \sum_{g_2=1}^{\infty} B_{g_1g_2}p_{g_20}t_{g_20} + \ldots \\ &+ \sum_{g_2=1}^{\infty} B_{g_1g_2}\sum_{g_3=1}^{\infty} B_{g_2g_3}\ldots\sum_{g_{d-1}=1}^{\infty} B_{g_{d-2}g_{d-1}}p_{g_{d-1}0}t_{g_{d-1}0}, \end{aligned} \tag{4.25}$$

for $g_1=1,2,\ldots$ *and*

$$B_{g_ig_j} = p_{g_i0}t_{g_i0}\frac{1-(p_{00}t_{00})^{c-1}}{1-p_{00}t_{00}}p_{0g_j}t_{0g_j} + p_{g_ig_j}t_{g_ig_j} \tag{4.26}$$

for $i,j=1,2,\ldots.$

If $d=1$, $B^{(d)}_{g_1}=1$ *and* $B^{(d-1)}_{g_10}=0$.

Proof. The joint PGF given by (4.21) is obtained in a usual way by considering all possible sequences of outcomes beginning with a success or failure and terminating with c consecutive successes or until d failures in total are observed.

A_{g_1} given by (4.22) is the contribution to the joint PGF $g_{\mathbf{S}^1}(c,d)$ until the Markov chain moves to the first failure state $g_1 = 1, 2, \ldots$;

$B_{g_i g_j}$ given by (4.26) is the corresponding contribution when the Markov chain moves from some failure state g_i to the next failure state g_j, $i, j = 1, 2, \ldots$. Then B_{g_1} given by (4.23) is the contribution to the joint PGF (4.21) after the first observed failure state in the event $\mathbf{RF}(c,d)$. Really, $B_{g_1}^{(d)}$ given by (4.24), is the contribution to the joint PGF, when $d-1$ failure states in total are observed after the first failure state $g_1 = 1, 2, \ldots$ in the possible sequences of outcomes. By analogy, the term $B_{g_1 0}^{(d-1)}$ given by (4.25) represents the contribution to the $g_{\mathbf{S}^1}(c,d)$ when the possible subsequences of outcomes follow the first observed failure and precede the terminating sequence of $c-1$ consecutive successes.

To see that $g_{\mathbf{S}^1}(c,d)$ is a PGF, it is enough to check the validity of the following equations

$$B_{g_1}^{(d)}(\mathbf{1}) = 1 - (p_{00})^{c-1} B_{g_1 0}^{(d-1)}(\mathbf{1}) \tag{4.27}$$

for $g_1 = 1, 2, \ldots$, where $\mathbf{1}$ means the corresponding unit vector. This completes the proof.

□

Remark 4.11 *The marginal PGF's of the total number of successes, the total number of failures and the total number of trials in the event* $\mathbf{RF}(c,d)$ *may be obtained by evaluating* $g_{\mathbf{S}^1}(c,d)$ *at the appropriate values.*

In particular, the distribution of the total number of failures

$$F = \sum_{g_1=1}^{\infty} S_{g_1} + \sum_{g_1=0}^{\infty} \sum_{g_2=1}^{\infty} S_{g_1 g_2}^{1}$$

is given by the following relations

$$\begin{aligned}
P(F=0) &= p_0(p_{00})^{c-1};\\
P(F=1) &= \sum_{g_1=1}^{\infty} A_{g_1}(\mathbf{1})p_{g_10}(p_{00})^{c-1};\\
P(F=m) &= \sum_{g_1=1}^{\infty} A_{g_1}(\mathbf{1}) \sum_{g_2=1}^{\infty} B_{g_1g_2}(\mathbf{1}) \dots \sum_{g_m=1}^{\infty} B_{g_{m-1}g_m}(\mathbf{1})p_{g_m0}(p_{00})^{c-1},\\
&\quad \textit{for} \quad m=2,\dots,d-1;\\
P(F=d) &= \sum_{g_1=1}^{\infty} A_{g_1}(\mathbf{1})B_{g_1}^{(d)}(\mathbf{1}).
\end{aligned}$$

Example 4.5 *Let us consider a Markov chain given in the Example 4.3. The equipment intervention takes place at the time of failure immediately following successful finish of the test (failure of type 1), resulting in a change in the probability of success just after that failure.*

We use the notation

$$p_{00} = P(\textit{success} \mid \textit{previous trial is successive})$$

and treat each failure following a failure outcome as a failure of type 2. Denote

$$p_{22} = P(\textit{failure} \mid \textit{previous trial is failure}).$$

Usually, in a multiple corrective action models trials are performed until specified number of consecutive start - ups, say c, are observed.

Let us substitute $d = 3$. In our case, this means that the start - ups are terminated upon the c - th consecutive successive state of the equipment or just after the third observed consecutive failure.

Now, from (4.22) one can obtain

$$A_1(\mathbf{1}) = 1 - p_0(p_{00})^{c-1}$$

and the possible transitions (4.26) are given by

$$\begin{aligned} B_{11}(\mathbf{1}) &= p_{10}[1-(p_{00})^{c-1}]; \\ B_{12}(\mathbf{1}) &= p_{12}; \\ B_{21}(\mathbf{1}) &= p_{20}[1-(p_{00})^{c-1}]; \\ B_{22}(\mathbf{1}) &= p_{22}. \end{aligned}$$

Then, the exact distribution of the number of failures from Remark 4.11 is given by the following relations

$$\begin{aligned} P(F=0) &= p_0(p_{00})^{c-1}; \\ P(F=1) &= A_1(\mathbf{1})p_{10}(p_{00})^{c-1}; \\ P(F=2) &= A_1(\mathbf{1})[B_{11}(\mathbf{1})p_{10}+B_{12}(\mathbf{1})p_{20}](p_{00})^{c-1}; \\ P(F=3) &= A_1(\mathbf{1})[B_{11}^2(\mathbf{1})+B_{11}(\mathbf{1})B_{12}(\mathbf{1})+B_{12}(\mathbf{1})B_{21}(\mathbf{1})+B_{12}(\mathbf{1})B_{22}(\mathbf{1})]. \end{aligned}$$

Remark 4.12 *Three particular cases of (4.21) are of interest:*

– first, if the homogeneous Markov chain has two states one can obtain the corresponding joint PGF given by Balasubramanian et al. [12] in their Proposition 2;

– second, letting $d \longrightarrow \infty$ in (4.21) yields the following joint PGF

$$g_1(c,\infty) = \left[p_0t_0 + \sum_{g_1=1}^{\infty} A_{g_1}B_{g_10}^{(\infty)}\right](p_{00}t_{00})^{c-1} + \sum_{g_1=1}^{\infty} A_{g_1}B_{g_1}^{(\infty)}, \qquad (4.28)$$

where $B_{g_1}^{(\infty)}$ and $B_{g_10}^{(\infty)}$ are derived from (4.24) and (4.25) respectively, and A_{g_1} is given by (4.22). Let us note that in this limiting case, the relation (4.27) holds true, i. e. it is fulfilled

$$B_{g_1}^{(\infty)}(\mathbf{1}) = 1-(p_{00})^{c-1}B_{g_10}^{(\infty)}(\mathbf{1});$$

– *third, letting* $c \longrightarrow \infty$ *in (4.21) yields the following joint PGF until* d *failures in total are observed in a multi - state Markov chain*

$$g_1(\infty, d) = \sum_{g_1=1}^{\infty} A_{g_1}^1 B_{g_1}^{1(d)}, \tag{4.29}$$

where

$$A_{g_1}^1 = p_0 t_0 \frac{p_{0g_1} t_{0g_1}}{1 - p_{00} t_{00}} + p_{g_1} t_{g_1},$$

$$B_{g_1}^{1(d)} = \sum_{g_1=1}^{\infty} B_{g_1 g_2}^1 \sum_{g_2=1}^{\infty} B_{g_2 g_3}^1 \ldots \sum_{g_d=1}^{\infty} B_{g_{d-1} g_d}^1,$$

with

$$B_{g_i g_j}^1 = p_{g_i 0} t_{g_i 0} \frac{p_{0g_j} t_{0g_j}}{1 - p_{00} t_{00}} + p_{g_i g_j} t_{g_i g_j},$$

for $i, j = 1, 2, \ldots$.

Remark 4.13 *The PGF of the total number of trials in the event* $\mathbf{RF}(\infty, d)$*, obtained by equating all arguments in (4.29), is the negative binomial analogue under multi-state homogeneous Markov chain.*

Remark 4.14 *Because of duality notified by Balasubramanian et al. [12], the joint PGF* $g_{1L}(c, d)$ *related with the later occurring event*

$$\mathbf{RF}_L(c, d) \quad = \quad \left\{ \begin{array}{c} \text{the trials are performed sequentially} \\ \text{until both } c \text{ consecutive successes} \\ \text{and } d \text{ failures in total are observed} \end{array} \right\}$$

has the representation

$$g_{1L}(c, d) = g_1(c, \infty) + g_1(\infty, d) - g_{\mathbf{S}^1}(c, d),$$

where $g_{\mathbf{S}^1}(c, d)$*,* $g_1(c, \infty)$ *and* $g_1(\infty, d)$ *are the joint PGF's given by (4.21), (4.28) and (4.29), correspondingly. Let us underline, that the last relation is valid for Markov chain with persistent states only.*

4.3.2 Run Quota on failures and frequency Quota on successes

Under the notations given in the Introduction the following basic lemma concerning the event $\mathbf{FR}(c,d)$ is true.

Lemma 4.4 *The joint PGF of the random vector*

$$\mathbf{S}^2 = (S_0, S_1, \ldots, S_{00}^2, S_{01}^2, \ldots)$$

related with the event $\mathbf{FR}(c,d)$ *is given by*

$$g_{\mathbf{S}^2}(t_0, t_1, \quad \ldots, t_{00}, t_{01}, \ldots) = g_{\mathbf{S}^2}(c,d) = E\{(t_0)^{S_0}(t_1)^{S_1} \ldots (t_{00})^{S_{00}^2}(t_{01})^{S_{01}^2} \ldots\}$$

$$= \quad A_0(B_0)^{c-1} + \sum_{g_1=1}^{\infty} \left[A_0 \tfrac{1-(B_0)^{c-1}}{1-B_0} p_{0g_1} t_{0g_1} + p_{g_1} t_{g_1} \right] F_{g_1}^{(d)}, \tag{4.30}$$

which converges at least for

$$\{(t_0, t_1, \ldots, t_{00}, t_{01}, \ldots) : \ |t_i| \leq 1 \quad and \quad |t_{ij}| \leq 1, \quad i,j = 0,1,\ldots\},$$

where for $d > 1$

$$A_0 = p_0 t_0 + \sum_{l=1}^{\infty} p_l t_l F_{l0}^{(d-1)}, \tag{4.31}$$

$$B_0 = p_{00} t_{00} + \sum_{l=1}^{\infty} p_{0l} t_{0l} F_{l0}^{(d-1)}, \tag{4.32}$$

and

$$F_{g_1}^{(d)} = \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} \cdots \sum_{g_d=1}^{\infty} p_{g_{d-1} g_d} t_{g_{d-1} g_d}, \tag{4.33}$$

$$\begin{aligned} F_{g_1 0}^{(d-1)} &= p_{g_1 0} t_{10} + \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} p_{g_2 0} t_{g_2 0} + \cdots \\ &+ \sum_{g_2=1}^{\infty} p_{g_1 g_2} t_{g_1 g_2} \cdots \sum_{g_{d-1}=1}^{\infty} p_{g_{d-2} g_{d-1}} t_{g_{d-2} g_{d-1}} p_{g_{d-1} 0} t_{g_{d-1} 0}, \end{aligned} \tag{4.34}$$

for $g_1 = 1, 2, \ldots$.

If $d = 1$, $F_{g_1}^{(d)} = 1$ *and* $F_{g_1 0}^{(d-1)} = 0$.

Proof. The joint PGF (4.30) can be obtained by symmetry with the joint PGF (4.21).

The expression (4.33) for $F_{g_1}^{(d)}$ represents the contribution to the joint PGF when $d-1$ consecutive failure states follow the failure state g_1, $g_1 = 1, 2, \ldots$. In this case all possible outcomes terminate with d consecutive failures. By analogy, relation (4.34) for $F_{g_1 0}^{(d)}$ is the contribution to the joint PGF when the Markov chain moves from some failure state $g_1 = 1, 2, \ldots$ to a successive one, with preceded subsequences containing no more than $d-2$ consecutive failures.

Then A_0 given by (4.31) is the contribution to the joint PGF $g_{\mathbf{S}^2}(c, d)$ until the Markov chain moves to the first successive state and B_0 given by (4.32) is the corresponding contribution when the Markov chain moves from the successive state to the next successive state in the possible sequences of outcomes.

The relation (4.30) is a PGF, since the following relations

$$F_{g_1 0}^{(d-1)}(\mathbf{1}) + F_{g_1}^{(d)}(\mathbf{1}) = 1 \tag{4.35}$$

and

$$1 - B_0(\mathbf{1}) = \sum_{g_1=1}^{\infty} p_{0 g_1} F_{g_1}^{(d)}(\mathbf{1})$$

are true for $g_1 = 1, 2, \ldots$, where $\mathbf{1}$ means the corresponding unit vector. This completes the proof.

□

Remark 4.15 *The total number of successes* X *in the event* $\mathbf{FR}(c, d)$ *may be represented by the following equation*

$$X = S_0 + \sum_{g_1=0}^{\infty} S_{g_1 0}^2.$$

Now, evaluating $g_{\mathbf{S}^2}(c,d)$ *at the appropriate values we obtain the following exact distribution*

$$\begin{aligned} P(X=0) &= a_1(d); \\ P(X=1) &= A_0(\mathbf{1})a_0(d); \\ P(X=x) &= A_0(\mathbf{1})(B_0(\mathbf{1}))^{x-1}a_0(d), \quad \textit{for} \quad x=2,3,\ldots,c-1; \\ P(X=c) &= A_0(\mathbf{1})(B_0(\mathbf{1}))^{c-1}, \end{aligned}$$

where $\mathbf{1}$ *means the corresponding unit vector,*

$$a_1(d)=\sum_{g_1=1}^{\infty} p_{g_1} F_{g_1}^{(d)}(\mathbf{1}) \quad \textit{and} \quad a_0(d)=\sum_{g_1=1}^{\infty} p_{0g_1} F_{g_1}^{(d)}(\mathbf{1}).$$

Remark 4.16 *Let us consider a Markov chain with a matrix of transition probabilities from Example 4.3. We will give the corresponding formulas when* d *takes values 2, 3 and 4.*

If $d=2$, *from (4.31) and (4.32) we have*

$$A_0(\mathbf{1})=p_0+p_1p_{10} \quad \textit{and} \quad B_0(\mathbf{1})=p_{00}+p_{01}p_{10}.$$

The exact distribution of the number of successes from Remark 4.15 can be obtained with

$$a_1(2)=p_1p_{12} \quad \textit{and} \quad a_0(2)=p_{01}p_{12}.$$

If $d=3$, *the corresponding relations are given by*

$$\begin{aligned} A_0(\mathbf{1}) &= p_0+p_1(p_{10}+p_{12}p_{20}); \\ B_0(\mathbf{1}) &= p_{00}+p_{01}(p_{10}+p_{12}p_{20}); \\ a_1(3) &= p_1p_{12}p_{22}^2; \\ a_0(3) &= p_{01}p_{12}p_{22}^2. \end{aligned}$$

Finally, if $d = 4$ *we have*

$$\begin{aligned} A_0(\mathbf{1}) &= p_0 + p_1(p_{10} + p_{12}p_{20} + p_{12}p_{22}p_{20}); \\ B_0(\mathbf{1}) &= p_{00} + p_{01}(p_{10} + p_{12}p_{20} + p_{12}p_{22}p_{20}); \\ a_1(4) &= p_1 p_{12} p_{22}; \\ a_0(4) &= p_{01} p_{12} p_{22}. \end{aligned}$$

Remark 4.17 *Let us consider the following particular cases of (4.30):*

– *first, let* $d \longrightarrow \infty$. *For* $g_1 = 1, 2, \ldots,$ *(4.34) gives* $F_{g_1 0}^{(\infty)}(\mathbf{1}) = 1$, *since our assumption for persistent states of the Markov chain. Then from (4.35) we have*

$$\lim_{d \longrightarrow \infty} F_{g_1}^{(d)}(\mathbf{1}) = 0$$

and therefore from (4.33) it is fulfilled

$$\lim_{d \longrightarrow \infty} F_{g_1}^{(d)} = 0,$$

for $g_1 = 1, 2, \ldots$. *This limiting case yields the following joint PGF*

$$g_2(c, \infty) = A_0(\infty)(B_0(\infty))^{c-1}, \tag{4.36}$$

where $A_0(\infty)$ *and* $B_0(\infty)$ *are derived from (4.31) and (4.32) respectively, by putting* $d \longrightarrow \infty$.

– *second, letting* $c \longrightarrow \infty$ *in (4.30) yields the following joint PGF until* d *consecutive failures are observed in a multi - state Markov chain*

$$g_2(\infty, d) = \sum_{g_1=1}^{\infty} \left[\frac{A_0}{1 - B_0} p_{0g_1} t_{0g_1} + p_{g_1} t_{g_1} \right] F_{g_1}^{(d)}. \tag{4.37}$$

Remark 4.18 *By symmetry, the joint PGF* $g_{2L}(c,d)$ *related with the later occurring event*

$$\mathbf{FR}_L(c,d) \quad = \quad \left\{ \begin{array}{c} \textit{the trials are performed sequentially} \\ \textit{until both } c \textit{ successes in total and} \\ d \textit{ consecutive failures are observed} \end{array} \right\}$$

has the representation

$$g_{2L}(c,d) = g_2(c,\infty) + g_2(\infty,d) - g_{\mathbf{S}^2}(c,d),$$

where $g_{\mathbf{S}^2}(c,d)$, $g_2(c,\infty)$ *and* $g_2(\infty,d)$ *are the joint PGF's given by (4.30), (4.36) and (4.37), correspondingly.*

4.3.3 Comments

In this chapter the geometric distribution of order k related to multi - state Markov chain is defined. The main result is Lemma 4.1. As a Corollary of the lemma, the PGFs of the number of successes, the number of failures and the number of trials (geometric distribution of order k) are given in Theorem 4.1, Theorem 4.2 and Theorem 4.3. The joint distribution of the number of successes and the number of failures is given in the Theorem 4.4. The result is published in [47] and [48]. The application in quality control is mentioned in [87].

The sooner and later waiting time problems discussed by Balasubramanian et *al.* [12] and Aki and Hirano [3] have been generalized to the multi - state Markov chain, published in [49]. The main result is in Lemma 4.2. In this case the PGFs of the number of successes and the number of failures are given.

The obtained results can be applied for evaluating the system reliability (start - up demonstration tests), molecular biology (modeling the structure of DNA), for description of some random walk models, etc.

The waiting time problems related with run and frequency quotas have been generalized to the multi - state Markov chain. The basic results stated by Lemma 4.3 and Lemma 4.4 are published in [50].

Presented results can also be extended for non - overlapping way of counting and for more complex sooner and latter waiting time problems considered by Uchida and Aki [83] and Balakrishnan et *al.* [11].

Chapter 5

I - Stochastic processes

The family of IGPSDs, defined in Chapter 2 is a generalization of the the family of GPSD, or the classical discrete distributions. Having in hand the new family of distributions we ask the natural question: What will be the corresponding generalization of the counting stochastic process.

5.1 The Pólya - Aeppli process

The $IPo(\lambda, \rho)$ distribution is a generalization of the classical $Po(\lambda)$ distribution. As we mentioned in Remark 2.10 the $IPo(\lambda, \rho)$ distribution coincides with the Pólya-Aeppli distribution. It was derived by Anscombe [6] in 1950, who stated that it was given earlier by A. Aeppli in a thesis in 1923 and then studied subsequently by G. Pólya in 1930. For this reason, Anscombe called this distribution a Pólya - Aeppli distribution. In this section we will define a generalization of the Poisson process and will call it a Pólya - Aeppli process.

Let us consider the sequence $T_1, T_2, \ldots$ of non - negative, mutually independent random variables and the corresponding renewal process

$$S_n = \sum_{i=1}^{n} T_i, \quad n = 1, 2 \ldots, \quad S_0 = 0.$$

The process S_n can be interpreted as a sequence of *renewal* epochs. T_1 is the time until the first renewal epoch and $\{T_i\}_{i\geq 2}$ are the inter-arrival times.

Let $N(t) = \sup\{n \geq 0,\ S_n \leq t\},\ t \geq 0$ be the number of renewals occurring up to time t. The distribution of $N(t)$ is related to that of S_n, and for any $t \geq 0$ and $n \geq 0$, the following probability relation holds:

$$P(N(t) = n) = P(S_n \leq t) - P(S_{n+1} \leq t\}, \quad n = 0, 1, 2, \ldots. \tag{5.1}$$

We will suppose that $N(t)$ is described by the $IPo(\lambda t, \rho)$ distribution (or Pólya - Aeppli distribution), with mean function $\frac{\lambda}{1-\rho}t$, i.e.

$$P(N(t) = n) \ = \ \begin{cases} e^{-\lambda t}, & n = 0 \\ e^{-\lambda t} \displaystyle\sum_{i=1}^{n} \binom{n-1}{i-1} \frac{[\lambda(1-\rho)t]^i}{i!} \rho^{n-i}, & n = 1, 2, \ldots. \end{cases} \tag{5.2}$$

Let us denote by $LS_X(s) = \int_0^\infty e^{-sx} dF_X(x)$ the Laplace - Stieltjes transform (LST) of any random variable X with distribution function $F_X(x)$. Let $P_n(t) = P(N(t) = n)$.

For the next considerations we need the following result.

Lemma 5.1 *The LST of $P_n(t)$ is given by*

$$LS_{P_n(t)}(s) \ = \int_0^\infty e^{-st} dP_n(t)$$

$$= \begin{cases} -\dfrac{\lambda}{s+\lambda}, & n = 0 \\ (1-\rho)\dfrac{\lambda}{s+\lambda}\dfrac{s}{s+\lambda}\left[\rho + (1-\rho)\dfrac{\lambda}{s+\lambda}\right]^{n-1}, & n = 1, 2, \ldots. \end{cases}$$

Now we will show that the renewal process is characterized by the fact that T_1 is exponentially distributed and $\{T_2, T_3, \ldots\}$ are identically distributed. Moreover, T_2 is zero with probability ρ, and with probability $1 - \rho$ exponentially distributed with parameter λ. This means that the probability density functions and the mean values are the following:

$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \quad ET_1 = \frac{1}{\lambda} \tag{5.3}$$

and

$$f_{T_2}(t) = \rho\delta_0(t) + (1-\rho)\lambda e^{-\lambda t}, \quad t \geq 0, \quad ET_2 = \frac{1-\rho}{\lambda}, \tag{5.4}$$

where

$$\delta_0(t) = \begin{cases} 1, & \text{if } t = 0 \\ 0, & \text{otherwise,} \end{cases}$$

The process S_n is called a *delayed renewal process* with a *delay* T_1.

Theorem 5.1 *There exists exactly one renewal process such that the number of renewals up to time t has the Pólya - Aeppli distribution (5.2). In this case the time until the first renewal epoch T_1 is exponentially distributed with parameter λ. The inter-arrival times $T_2, T_3, \ldots$ are equal to zero with probability ρ and with probability $1-\rho$ exponentially distributed with parameter λ.*

Proof. To prove the theorem it suffices to show that the LST of the random variable S_n is equal to

$$LS_{S_n}(s) = \frac{\lambda}{s+\lambda}[\rho + (1-\rho)\frac{\lambda}{s+\lambda}]^{n-1}. \tag{5.5}$$

We will prove it by induction using the relations (5.1). For $n = 0$ (5.1) becomes

$$P(N(t) = 0) = 1 - P(T_1 \leq t) = 1 - F_{T_1}(t), \tag{5.6}$$

where $F_{T_1}(t)$ is the distribution function of T_1. On the other hand, from (5.2) it follows that

$$P(N(t) = 0) = e^{-\lambda t}. \tag{5.7}$$

Combining (5.6) and (5.7) gives that $F_{T_1}(t) = 1 - e^{-\lambda t}$, i.e. the r.v. T_1 is exponentially distributed with parameter λ and LST $\frac{\lambda}{s+\lambda}$.

Now from (5.1), for $n = 1$, we get

$$P(N(t) = 1) = P(S_1 \leq t) - P(S_2 \leq t).$$

Taking the LST leads to

$$(1-\rho)\frac{\lambda}{s+\lambda}\frac{s}{s+\lambda} = LS_{S_1}(s) - LS_{S_2}(s).$$

After some algebra we arrive at

$$LS_{T_1+T_2}(s) = \frac{\lambda}{s+\lambda}[\rho + (1-\rho)\frac{\lambda}{s+\lambda}],$$

which means that T_2 is independent of T_1. Moreover, T_2 is exponentially distributed random variable with parameter λ and mass at zero equal to ρ. The probability density function of T_2 is given by (5.4).

Let suppose now that for any $n \geq 2$, the LST of S_n is given by (5.5). Taking the LST in (5.1) we get

$$\begin{aligned} LS_{S_{n+1}}(s) &= \int_0^\infty e^{-st} dP(S_{n+1} \leq t) \\ &= \int_0^\infty e^{-st} dP(S_n \leq t) - \int_0^\infty e^{-st} dP_n(t). \end{aligned}$$

Applying Lemma 5.1 one can show that the LST of the renewal process S_{n+1} is equal to

$$LS_{S_{n+1}}(s) = \frac{\lambda}{s+\lambda}\left[\rho + (1-\rho)\frac{\lambda}{s+\lambda}\right]^n.$$

This proves the theorem.

□

Remark 5.1 *In the case of $\rho = 0$, the LST (5.5) becomes the LST of Gamma(n, λ) (or Erlang (n)) distributed random variable. This case coincides with the usual homogeneous Poisson process.*

Remark 5.2 *Let us note that the probability distribution function of T_2 is given by*

$$F_{T_2}(t) = 1 - (1-\rho)e^{-\lambda t}, \quad t \geq 0.$$

That family of distributions has a jump at zero. That is, $P(T_2 = 0) = \rho$.

Remark 5.3 *It is easy to see that the exponential distribution function of the delay, $F_{T_1}(t)$, and the distribution function $F_{T_2}(t)$, satisfy the following relation:*

$$F_{T_1}(t) = \frac{1}{ET_2} \int_0^t [1 - F_{T_2}(u)]du. \tag{5.8}$$

In this case the delayed renewal counting process is the only stationary renewal process, see [21]. From the renewal theory, it is known, that under the condition (5.8), the delayed renewal counting process has stationary increments, see for instance Rolski et al. [77], Theorem 6.1.8.

We proved the theorem using the LST and basic relation (5.1). The converse theorem is also true.

Theorem 5.2 *Let suppose that the inter-arrival times $\{T_i\}_{i\geq 2}$ of the stationary renewal process are equal to zero with probability ρ and with probability $1-\rho$ exponentially distributed with parameter λ. Then the number of renewals up to time t, has the Pólya - Aeppli distribution.*

Now we can define the inflated - parameter Poisson process, or the Pólya - Aeppli process.

Definition 5.1 *A counting process $\{N(t), t \geq 0\}$ is said to be a Pólya - Aeppli process if*

(a) it starts at zero, $N(0) = 0$;

(b) it has independent, stationary increments;

(c) for each $t > 0$, $N(t)$ is Pólya - Aeppli distributed.

The Pólya - Aeppli process is a time homogeneous process, such that i claims can arrive simultaneously with probability ρ^{i-1}, for $i = 1, 2, \ldots$. In the case of $\rho = 0$ it becomes a homogeneous Poisson process. So, we have a homogeneous process with an additional parameter ρ.

Remark 5.4 *In the definition of the Pólya - Aeppli process, the parameter ρ is interpreted as a probability and we assume that $\rho \in [0,1)$. At the same time we started with the interpretation of ρ as a correlation coefficient in a Markov chain. Let us suppose that the correlation coefficient is $r \in (-1,1)$. In this case the parameter ρ is equal to $\rho = |r|$ and the given definitions and properties are fulfilled.*

5.1.1 Moments

The mean value and the variance of Pólya - Aeppli process are given by

$$EN(t) = \frac{\lambda}{1-\rho}t \quad \text{and} \quad Var(N(t)) = \frac{\lambda(1+\rho)}{1-\rho}t.$$

The index of dispersion is

$$\frac{Var(N(t))}{EN(t)} = \frac{1+\rho}{1-\rho} > 1,$$

i.e. the Pólya - Aeppli process is overdispersed relative the Poisson process.

5.2 I - Pólya process

In this section we will obtain the Inflated - parameter negative binomial process, called also I - Pólya process by mixing the Pólya - Aeppli process. Suppose that the number of items in the time interval $[0,t]$ has a conditional Pólya - Aeppli distribution, i.e.

$$P(N_t = m \mid \lambda) = \begin{cases} e^{-\lambda t}, & m = 0, \\ e^{-\lambda t}\sum_{i=1}^{m}\binom{m-1}{i-1}\frac{[\lambda(1-\rho)t]^i}{i!}\rho^{m-i}, & m = 1,2,\ldots, \end{cases} \tag{5.9}$$

where $\lambda > 0$ and $\rho \in [0,1)$. The mean number of arrivals is $E(N_t \mid \lambda) = \frac{\lambda}{1-\rho}t$.

Suppose that λ is the outcome of a random variable Λ. Following the well known terminology [45] the resulting process is a mixed Pólya - Aeppli process. The probability distribution Λ is called mixing distribution. The

PMF (5.9) is interpreted as the conditional distribution of N_t, given the outcome $\Lambda = \lambda$.

Let the distribution of the mixing random variable Λ be gamma one with parameters β and r. Its probability density function is given by

$$\frac{\beta^r}{\Gamma(r)}\lambda^{r-1}e^{-\beta\lambda}, \quad \beta > 0, \quad \lambda > 0,$$

where $\Gamma(r)$ is the Gamma function, r is called the shape parameter and β the scale parameter.

In this case N_t has the following probability mass function:

$$P(N_t = m) = \begin{cases} \left(\frac{\beta}{\beta+t}\right)^r, & m = 0 \\ \left(\frac{\beta}{\beta+t}\right)^r \sum_{i=1}^{m} \binom{m-1}{i-1}\binom{r+i-1}{i}[(1-\rho)\frac{t}{\beta+t}]^i\rho^{m-i}, & m = 1, 2, \ldots. \end{cases} \tag{5.10}$$

This is just the Inflated - parameter negative binomial distribution with parameters $\frac{\beta}{\beta+t}$, ρ and r, INB$(\frac{\beta}{\beta+t}, \rho, r)$. In the case of $\rho = 0$ (5.10) coincides with the usual negative binomial distribution with parameters $\frac{\beta}{\beta+t}$ and r. If the number of events up to time t is negative binomial distributed, the counting process is known as Pólya process, see [37], p. 4. This motivated the following definition.

Definition 5.2 *The counting process, $\{N(t),\ t \geq 0\}$ is said to be an Inflated - parameter Pólya process or I - Pólya process, if it starts at zero, $N(0) = 0$ and for each $t > 0$, the distribution of $N(t)$ is given by (5.10).*

5.2.1 Properties of the I - Pólya process

Denote $S_n = T_1 + T_2 + \ldots + T_n$, $n = 1, 2, \ldots$, the waiting time until the nth event. The basic properties of the I - Pólya process are given by the next theorem.

Theorem 5.3 *Let $N(t)$ has the INB$(\frac{\beta}{\beta+t}, \rho, r)$distribution (5.10). Then*

(i) The time until the first epoch T_1 is Pareto distributed. The inter-arrival times $T_2, T_3, \ldots$ are zero with probability ρ and with probability $1-\rho$ Pareto distributed.

(ii) The waiting time until the nth event has the following probability density function

$$f_{S_n}(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} (1-\rho)^i \rho^{n-1-i} \frac{r}{\beta} \binom{r+i}{i} \left(\frac{\beta}{\beta+t}\right)^{r+1} \left(\frac{t}{\beta+t}\right)^i. \quad (5.11)$$

Proof. (i) According to the definition of the Pólya - Aeppli process the conditional distribution of T_1 is $\exp(\lambda)$. The unconditional distribution is given by the probability distribution function

$$F_{T_1}(t) = \int_0^\infty (1-e^{-\lambda t}) \frac{\beta^r}{\Gamma(r)} \lambda^{r-1} e^{-\beta\lambda} d\lambda = 1 - \left(\frac{\beta}{\beta+t}\right)^r, \quad t \geq 0 \quad (5.12)$$

and the probability density function

$$f_{T_1}(t) = \frac{r}{\beta} \left(\frac{\beta}{\beta+t}\right)^{r+1}, \quad t \geq 0.$$

We write $T_1 \sim P(\beta, r)$ for Pareto distribution with parameters β and r.

Similarly for the distribution of T_2 we obtain

$$F_{T_2}(t) = 1 - (1-\rho) \left(\frac{\beta}{\beta+t}\right)^r, \quad t \geq 0 \quad (5.13)$$

and

$$f_{T_2}(t) = \rho\delta(0) + (1-\rho) \frac{r}{\beta} \left(\frac{\beta}{\beta+t}\right)^{r+1}, \quad t \geq 0$$

where $\delta(x)$ is the delta function. T_2 is Pareto distributed with mass at zero equal to ρ.

(ii) We will prove (5.11) by induction, using the following relation

$$P(S_{n+1} \leq t) = P(S_n \leq t) - P(N(t) = n), \quad n = 0, 1, 2, \ldots. \quad (5.14)$$

For $n = 1$ (5.14) becomes

$$P(S_2 \leq t) = P(S_1 \leq t) - P(N(t) = 1). \quad (5.15)$$

According to (5.10)

$$P(N(t)=1)=r(1-\rho)\left(\frac{\beta}{\beta+t}\right)^{r}\frac{t}{\beta+t}.$$

The random variable S_1 coincides with the random variable T_1. Differentiating (5.14) and rearranging the terms leads to

$$f_{S_2}(t)=\rho\frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^{r+1}+(1-\rho)\frac{r(r+1)}{\beta}\left(\frac{\beta}{\beta+t}\right)^{r}\left(\frac{t}{\beta+t}\right).$$

Suppose now that (5.11) is true. Differentiating (5.14) and applying (5.10) and (5.11) we find that

$$f_{S_{n+1}}(t)=\sum_{i=0}^{n}\binom{n}{i}(1-\rho)^{i}\rho^{n-i}\frac{r}{\beta}\binom{r+i}{i}\left(\frac{\beta}{\beta+t}\right)^{r+1}\left(\frac{t}{\beta+t}\right)^{i}.$$

□

5.2.2 Moments

The mean value and the variance of I - Pólya process are given by

$$EN(t)=\frac{r}{(1-\rho)}\frac{t}{\beta}$$

and

$$Var(N(t))=\frac{r}{(1-\rho)}\frac{t}{\beta}\left[\frac{(1+\rho)\beta+t}{(1-\rho)\beta}\right]=EN(t)\left[\frac{1+\rho}{1-\rho}+\frac{t}{(1-\rho)\beta}\right]$$

The index of dispersion

$$\frac{Var(N(t))}{EN(t)}>\frac{1+\rho}{1-\rho}.$$

This means that the I - Pólya process is overdispersed relative the Pólya - Aeppli process.

5.3 I - Binomial process

Suppose that $N(t) = \sum_{i=1}^{N_1(t)} Y_i$, where $Y_1, Y_2, \ldots$ are independent identically distributed random variables, independent of $N_1(t)$. Let for $\alpha > 0,\ t < \alpha$ and $n \geq 1,\ N_1(t)$ has a binomial distribution with parameters n and $\frac{t}{\alpha}$ $(N_1(t) \sim Bi(n, \frac{t}{\alpha}))$. Let Y denotes the compounding random variable. Here we suppose that for $\rho \in [0, 1),\ Y$ is geometrically distributed with success probability $1 - \rho$ $(Y \sim Ge_1(1 - \rho))$, and has a probability mass function

$$P(Y = i) = (1 - \rho)\rho^{i-1}, \quad i = 1, 2, \ldots.$$

The compound binomial process $N(t)$ has Inflated - parameter binomial distribution and is called Inflated - parameter binomial process or I-Binomial process. The distribution is given by

$$P(N_t = m) = \begin{cases} \left(1 - \dfrac{t}{\alpha}\right)^n, & m = 0 \\ \displaystyle\sum_{i=1}^{m \wedge n} \binom{n}{i} \binom{m-1}{i-1} \left[(1 - \rho)\frac{t}{\alpha}\right]^i \left(1 - \frac{t}{\alpha}\right)^{n-i} \rho^{m-i}, & m = 1, 2, \ldots. \end{cases} \tag{5.16}$$

The I-Binomial process is not a simple process and the probability that two or more claims can arrive simultaneously is greater than zero.

This leads to the following definition

Definition 5.3 *The counting process* $\{N(t),\ t \geq 0\}$ *is said to be I - Binomial process, if it starts at zero,* $N(0) = 0$ *and for each* $t \in (0, \alpha)$*, the distribution of* $N(t)$ *is given by (5.16).*

5.3.1 Properties of the I-Binomial process

Denote $S_m = T_1 + T_2 + \ldots + T_m$, $m = 1, 2, \ldots$, the waiting time until the mth event. One of the basic properties of the I-Binomial process is given in the next theorem.

Theorem 5.4 *Let $N(t)$ has the IBi($\frac{t}{\alpha}, \rho, n$)distribution (5.16). Then the waiting time until the mth event for $m \geq 1$ and $t < \alpha$ has the following probability density function*

$$f_{S_m}(t) = \frac{n}{\alpha} \sum_{i=0}^{(m-1)\wedge(n-1)} \binom{n-1}{i} \binom{m-1}{i} \left[(1-\rho)\frac{t}{\alpha}\right]^i \left(1-\frac{t}{\alpha}\right)^{n-i-1} \rho^{m-i-1}. \tag{5.17}$$

Proof. We will prove (5.17) by induction, using the following relation

$$P(S_{m+1} \leq t) = P(S_m \leq t) - P(N(t) = m), \quad m = 0, 1, 2, \ldots. \tag{5.18}$$

For $m = 1$ (5.18) becomes

$$P(S_2 \leq t) = P(S_1 \leq t) - P(N(t) = 1). \tag{5.19}$$

According (5.16)

$$P(N(t) = 1) = n(1-\rho)\frac{t}{\alpha}\left(1-\frac{t}{\alpha}\right)^{n-1}. \tag{5.20}$$

The random variable S_1 coincides with the random variable T_1. From (5.18) for $m = 0$ we get

$$P(S_1 \leq t) = 1 - P(N(t) = 0) = 1 - \left(1-\frac{t}{\alpha}\right)^n. \tag{5.21}$$

Substituting (5.20) and (5.21) in (5.19) we obtain

$$P(S_2 \leq t) = 1 - \left[1 - (1-(1-\rho)n)\frac{t}{\alpha}\right]\left(1-\frac{t}{\alpha}\right)^{n-1}. \tag{5.22}$$

Differentiating (5.22) we get the pdf of $S_2 + T_1 + T_2$

$$f_{S_2}(t) = \rho\frac{n}{\alpha}\left(1-\frac{t}{\alpha}\right)^{n-1} + (1-\rho)\frac{n(n-1)}{\alpha}\frac{t}{\alpha}\left(1-\frac{t}{\alpha}\right)^{n-2}, \; t < \alpha.$$

Let suppose now that (5.17) is true. Differentiating the relation

$$F_{S_{m+1}}(t) = F_{S_m}(t) - P(N(t) = m)$$

and applying (5.16) and (5.17) we find that

$$f_{S_{m+1}}(t) = \frac{n}{\alpha} \sum_{i=0}^{m\wedge(n-1)} \binom{m}{i} \binom{n-1}{i} \left[(1-\rho)\frac{t}{\alpha}\right]^i \left(1-\frac{t}{\alpha}\right)^{n-i-1} \rho^{m-i}.$$

This completes the proof.

□

Corollary 5.1 *The time until the first claim T_1 has the following density function*

$$f_{T_1}(t) = \frac{n}{\alpha}\left(1-\frac{t}{\alpha}\right)^{n-1}, \quad t < \alpha. \tag{5.23}$$

Proof. Differentiating (5.21), leads to (5.23).

□

5.3.2 Moments

The mean value and the variance of I - Binomial process are given by

$$EN(t) = \frac{nt}{(1-\rho)\alpha}$$

and

$$Var(N(t)) = \frac{n}{(1-\rho)^2}\left[1-\frac{t}{\alpha}+\rho\right]\frac{t}{\alpha} = EN(t)\left[\frac{1+\rho}{1-\rho} - \frac{t}{(1-\rho)\alpha}\right].$$

If the index of dispersion

$$\frac{Var(N(t))}{EN(t)} = \frac{1+\rho}{1-\rho} - \frac{t}{(1-\rho)\alpha} > 1,$$

then the variance is larger than the mean and $\rho > \frac{t}{2\alpha}$. In this case the I-Binomial process is overdispersed. If $\rho < \frac{t}{2\alpha}$, the process is underdispersed.

5.4 Inflated - parameter modification of the pure birth process

In this section we give a new interpretation of the defined processes.

To begin, let $N(t)$ represents the state of the system at time $t \geq 0$. It is assumed that the process has state space $\mathcal{N}$, the non-negative integers. Let $\lambda_i(t) > 0,\ i = 1, 2, \ldots$ be any given functions of t, called intensity of (transition) frequencies, and $\rho \in (0, 1)$.

We assume the following postulates:

$$P(N(t+h) = n \mid N(t) = m) = \begin{cases} 1 - (1-\rho)\sum_{i=1}^{\infty} \lambda_i(t)\rho^{i-1}h + o(h), & n = m, \\ (1-\rho)\rho^{i-1}\lambda_i(t)h + o(h), & n = m+i, \\ & i = 1, 2, \ldots, \end{cases}$$

for every $m = 0, 1, \ldots$, where $o(h) \to 0$ as $h \to 0$.

Let $P_m(t) = P(N(t) = m),\ m = 0, 1, 2, \ldots$. Then the above postulates yield the following Kolmogorov forward equations:

$$P_0'(t) = -(1-\rho)\sum_{i=1}^{\infty} \lambda_i(t)\rho^{i-1}P_0(t),$$

$$P_m'(t) = -(1-\rho)\sum_{i=1}^{\infty} \lambda_i(t)\rho^{i-1}P_m(t) + (1-\rho)\sum_{i=1}^{m} \lambda_i(t)\rho^{i-1}P_{m-i}(t), \tag{5.24}$$

for $m = 1, 2, \ldots$, with the conditions

$$P_0(0) = 1 \quad \text{and} \quad P_m(0) = 0, \quad m = 1, 2, \ldots. \tag{5.25}$$

We call the process defined by (5.24) and (5.25) an Inflated - parameter modification of the pure birth process (or I - modification).

In the next we consider particular cases, useful for risk theory and queueing theory.

The Pólya-Aeppli process. Let us suppose that the functions $\lambda_i(t) = \lambda$ are constants for every $i = 1, 2 \ldots$. In this case the Kolmogorov equations (5.24) are given by

$$\begin{aligned} P_0'(t) &= -\lambda P_0(t), \\ P_m'(t) &= -\lambda P_m(t) + (1-\rho)\lambda \sum_{i=1}^{m} \rho^{i-1}P_{m-i}(t), \quad m = 1, 2, \ldots. \end{aligned}$$

Integrating the above system of differential equations with the conditions (5.25) we obtain the distribution (5.2), i. e.

$$P(N(t)=m)=\begin{cases} e^{-\lambda t}, & m=0 \\ e^{-\lambda t}\sum_{i=1}^{m}\binom{m-1}{i-1}\frac{[(1-\rho)\lambda t]^i}{i!}\rho^{m-i}, & m=1,2,\ldots. \end{cases}$$

The I-Pólya process. Let the intensity functions $\lambda_i(t),\ i=1,2,\ldots$ depend on t in the following way

$$\lambda_i(t)\rho^{i-1}=\frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^2\left(1-(1-\rho)\frac{\beta}{\beta+t}\right)^{i-1},\quad t\geq 0, \tag{5.26}$$

where $\beta>0$ is a parameter, and $\lambda_i(t)\rho^{i-1}\to 0$, as $t\to\infty$.

The Kolmogorov equations (5.24) become

$$\begin{aligned} P_0'(t) &= -\frac{r}{\beta+t}P_0(t), \\ P_m'(t) &= -\frac{r}{\beta+t}P_m(t) \\ &+ (1-\rho)\frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^2\sum_{i=1}^{m}\left(1-(1-\rho)\frac{\beta}{\beta+t}\right)^{i-1}P_{m-i}(t), m=1,2,\ldots. \end{aligned}$$

By integrating we obtain the explicit solutions

$$P(N(t)=m)=\begin{cases} \left(\frac{\beta}{\beta+t}\right)^r, & m=0 \\ \left(\frac{\beta}{\beta+t}\right)^r\sum_{i=1}^{m}\binom{m-1}{i-1}\binom{r+i-1}{i}[(1-\rho)\frac{t}{\beta+t}]^i\rho^{m-i}, & m=1,2,\ldots, \end{cases}$$

which can be easily verified by insertion into the system. This is the $INB(\frac{\beta}{\beta+t},\rho,r)$ distribution (5.10), where $\pi=\frac{\beta}{\beta+t}$.

The Inflated-parameter binomial process. Let the functions $\lambda_i(t),\ i=1,2,\ldots$ depend on t in the following way

$$\lambda_i(t)\rho^{i-1}=\frac{n}{\alpha}\left(\frac{\alpha}{\alpha-t}\right)^2\left(1-(1-\rho)\frac{\alpha}{\alpha-t}\right)^{i-1}, \tag{5.27}$$

where $\alpha>1$ is a parameter and $t<\alpha$. In this case $\lambda_i(t)\rho^{i-1}\to\infty$ as $t\to\alpha$.

The Kolmogorov differential equations in this case are the following

$$\begin{aligned} P_0'(t) &= -\frac{n}{\alpha-t}P_0(t), \\ P_m'(t) &= -\frac{n}{\alpha-t}P_m(t) \\ &+ (1-\rho)\frac{n}{\alpha}\left(\frac{\alpha}{\alpha-t}\right)^2\sum_{i=1}^{m}\left(1-(1-\rho)\frac{\alpha}{\alpha-t}\right)^{i-1}P_{m-i}(t), m=1,2,\ldots. \end{aligned}$$

Integrating and applying the conditions (5.25) we obtain the $IBi(\frac{t}{\alpha}, \rho, n)$ distribution (5.16) for $\pi = \frac{t}{\alpha}$.

5.5 Generalized Delaporte distribution.

In this section we introduce a generalization of one very popular counting distribution. The Delaporte distribution, (see [78] and [38]), is a mixed Poisson distribution with shifted Gamma mixing distribution. Here we also suppose that the mixing distribution is a shifted Gamma distribution with probability density function given by

$$\frac{\beta^r}{\Gamma(r)}(\lambda-\alpha)^{r-1}e^{-\beta(\lambda-\alpha)}, \quad \beta > 0, \quad \lambda \geq \alpha.$$

The parameter α can be interpreted as a basic risk, [38]. We suppose that N_t is a Pólya - Aeppli process. The unconditional probability mass function in this case is the following:

$$P(N_t = m) = \int_\alpha^\infty P(N_t = m \mid \lambda)\frac{\beta^r}{\Gamma(r)}(\lambda-\alpha)^{r-1}e^{-\beta(\lambda-\alpha)}d\lambda. \tag{5.28}$$

Calculating the integral in (5.28) gives the following probability mass function

$$P(N_t = m)$$
$$= \begin{cases} e^{-\alpha t}\left(\frac{\beta}{\beta+t}\right)^r, & m = 0 \\ e^{-\alpha t}\left(\frac{\beta}{\beta+t}\right)^r \left[r(1-\rho)\frac{t}{\beta+t} + \alpha(1-\rho)t\right], & m = 1 \\ e^{-\alpha t}\left(\frac{\beta}{\beta+t}\right)^r \left[\sum_{i=1}^{m}\binom{m-1}{i-1}\binom{r+i-1}{i}\left((1-\rho)\frac{t}{\beta+t}\right)^i \rho^{m-i}\right. & \\ + \sum_{i=1}^{m}\binom{m-1}{i-1}\frac{[\alpha(1-\rho)t]^i}{i!}\rho^{m-i} + \sum_{i=1}^{m-1}\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{[\alpha(1-\rho)t]^j}{j!}\rho^{i-j} & \\ \left. \times \sum_{k=1}^{m-i}\binom{m-i-1}{k-1}\binom{r+k-1}{k}\left((1-\rho)\frac{t}{\beta+t}\right)^k \rho^{m-i-k}\right], & m = 2, 3, \ldots. \end{cases}$$

The random variable N_t in this case is a sum of an IPo$(\alpha t, \rho)$ variable and independent INB$(\frac{\beta}{\beta+t}, \rho, r)$ variable.

In the case $\rho = 0$ the distribution coincides with the Delaporte distribution, see [38]. So we say that the above random variable N_t has a generalized Delaporte distribution (I - Delaporte distribution).

Taking $\rho = 0$ in the above processes we obtain the homogeneous Poisson process, Pólya process and binomial process, respectively. The relationships between I - processes are similar as between their classical analogues. In all the cases the parametrization is chosen to ensure constant intensity ($\frac{d}{dt}N(t) = const$). We say that the Pólya - Aeppli process is a stationary renewal process. The I - Pólya and I - binomial can be interpreted as renewal processes with dependency.

5.6 Stochastic processes of order k

5.6.1 Pólya - Aeppli process of order k

Let $N(t)$ represents the state of the system at time $t \geq 0$. It is assumed that the process has state space $\mathcal{N}$, the non-negative integers. Let $\lambda > 0$ be any real number and $\rho \in [0, 1)$.

Suppose that $N(t)$ has a $PA_k(\lambda t, \rho)$ distribution. The PGF of $N(t)$ is given by

$$h(u, t) = e^{\lambda t[P_X(u)-1]}, \tag{5.29}$$

where $P_X(u)$ is the PDF of the truncated geometric distribution

$$P_X(t) = \frac{(1-\rho)t}{1-\rho^k}\frac{1-\rho^k t^k}{1-\rho t}. \tag{5.30}$$

Definition 5.4 *A compound Poisson process with truncated geometric compounding distribution is called Pólya - Aeppli process of order k.*

The second definition of the process is as a pure birth process. The properties of the defined process are specified by the following postulates:

$$P(N(t+h)=n \mid N(t)=m) = \begin{cases} 1-\lambda h+o(h), & n=m, \\ \frac{1-\rho}{1-\rho^k}\rho^{i-1}\lambda h+o(h), & n=m+i, \\ & i=1,2,\ldots,k \end{cases}$$

for every $m=0,1,\ldots$, where $o(h)\to 0$ as $h\to 0$. Note that the postulates imply that for $i=k+1,k+2,\ldots$, $P(N(t+h)=m+i \mid N(t)=m)=o(h)$. According to the postulates, the probability that $2,3,\ldots k$ claims to the insurance company arrive simultaneously is positive.

Let $P_m(t)=P(N(t)=m)$, $m=0,1,2,\ldots$. Then the above postulates yield the following Kolmogorov forward equations:

$$\begin{aligned} &P_0'(t)=-\lambda P_0(t), \\ &P_m'(t)=-\lambda P_m(t)+\tfrac{1-\rho}{1-\rho^k}\lambda\textstyle\sum_{j=1}^{m\wedge k}\rho^{j-1}P_{m-j}(t), \quad m=1,2,\ldots, \end{aligned} \tag{5.31}$$

with the conditions

$$P_0(0)=1 \quad \text{and} \quad P_m(0)=0, \quad m=1,2,\ldots. \tag{5.32}$$

Let

$$h(u,t)=\sum_{m=0}^{\infty}u^m P_m(t)$$

be the PGF of the process $N(t)$. Multiplying the mth equation of (5.31) by u^m and summing for all $m=0,1,2,\ldots$ we get the following differential equation

$$\frac{\partial h(u,t)}{\partial t}=-\lambda[1-P_X(u)]h(u,t). \tag{5.33}$$

The solution of (5.33) with the initial condition

$$P_0(0)=1$$

is

$$h(u,t)=e^{-\lambda t[1-P_X(u)]},$$

which is the PGF of the $PA_k(\lambda t,\rho)$ distribution, given by (5.29) and (5.30).

Definition 5.5 *The birth process defined by (5.31) and (5.32) is called a Pólya - Aeppli process of order k.*

Remark 5.5 *In the case of $k \to \infty$, the Pólya - Aeppli process of order k, coincides with the Pólya - Aeppli process. If $\rho = 0$, it is a homogeneous Poisson process.*

5.7 Comments

The Pólya - Aeppli process with characterization and the properties are given in [61]. The I - stochastic processes, defined in this chapter are published in [58]. The I - Pólya process with applications in Risk theory is given also in [65]. In [28], Dragieva described the Pólya - Aeppli process as an input process in a queueing system. The Pólya - Aeppli process of order k is defined in [63].

Chapter 6

Risk Models

Consider the standard model of an insurance company, called *risk process* $\{X(t), t \geq 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$, and given by the equation (1.1)), i.e.

$$X(t) = ct - \sum_{i=1}^{N(t)} Z_i, \quad (\sum_{1}^{0} = 0).$$

The risk process $X(t)$ represents the profit of the risky business in $(0, t]$.

The random variables $\{Z_i\}_{i=1}^{\infty}$ are mutually independent, identically distributed with common distribution function F, $F(0) = 0$, and mean value μ. In this chapter we consider several examples of the counting process $N(t), \quad t \geq 0$. A common used measure of the risk related to an insurance company is the ruin probability. Denote by

$$\tau(u) = \inf\{t > 0, u + X(t) \leq 0\}$$

the time to ruin of a company having initial capital u. We let $\tau = \infty$, if for all $t > 0$ $u + X(t) > 0$.

The ruin probability $\Psi(u)$ in the infinite horizon case for the insurance company with initial capital u is defined by

Definition 6.1

$$\Psi(u) = P(\tau(u) < \infty).$$

Sometimes it is more convenient to use the probability of non ruin $\Phi(u) = 1 - \Psi(u)$.

From the definition it follows that $\Psi(u) = 1$ for $u < 0$. We suppose that $u \geq 0$.

The relative *safety loading* coefficient θ of the insurance company is defined by

$$\theta = \frac{EX(t)}{E(\sum_{i=1}^{N(t)} Z_i)}. \tag{6.1}$$

The risk process X(t) has a positive safety loading, if $\theta > 0$. In this case X(t) has a trend to $+\infty$ and we say that there is a *net profit condition,* (NPC).

6.1 The Pólya - Aeppli risk model

Suppose that the counting process $N(t)$ in the risk process is the Pólya - Aeppli process independent of the claim sizes $\{Z_i\}_{i=1}^{\infty}$. We will call this process a Pólya - Aeppli risk process.

The relative *safety loading* θ is defined by

$$\theta = \frac{c(1-\rho) - \lambda\mu}{\lambda\mu} = \frac{c(1-\rho)}{\lambda\mu} - 1, \tag{6.2}$$

and in the case of *positive* safety loading $\theta > 0$, $c > \frac{\lambda\mu}{1-\rho}$.

The occurrence of the claims in the risk process (1.1) is described by a delayed renewal counting process. We will study the ruin probability in two cases following the renewal arguments, described by Feller [32] and Grandell [37]. The main in the two cases is to derive the integral equations of ruin probability.

6.1.1 The ordinary case

We suppose that the first claim has occurred and the subsequent claims occur as an ordinary renewal process. The inter - occurrence times $T_k,\ k = 1, 2, \ldots$

are exponentially distributed with mass at zero equal to ρ and probability density function, given by

$$f_{T_2}(t) = \rho\delta_0(t) + (1-\rho)\lambda e^{-\lambda t}, \quad t \geq 0, \quad ET_2 = \frac{1-\rho}{\lambda}.$$

The claim sizes $Z_1, Z_2, \ldots$ are independent and identically distributed random variables with common distribution function $F(x)$ with $F(0) = 0$ and mean value μ. Let

$$F_I(x) = \frac{1}{\mu}\int_0^x [1 - F(z)]dz$$

be the *integrated tail distribution*. Let us define the function

$$H(z) = \rho F(z) + \frac{\lambda\mu}{c} F_I(z) \tag{6.3}$$

and note that

$$H(\infty) = \rho F(\infty) + \frac{\lambda}{c}\int_0^\infty [1 - F(z)]dz = \rho + \frac{\lambda\mu}{c} < 1.$$

If we denote by $\Phi^0(u)$ and $\Psi^0(u)$ the non-ruin and ruin probabilities, respectively, in the ordinary case, then the following result holds.

Proposition 6.1 *The non-ruin function $\Phi^0(u)$ satisfies the integral equation*

$$\Phi^0(u) = \Phi^0(0) + \int_0^u \Phi^0(u-z)dH(z), \quad u \geq 0, \tag{6.4}$$

where $H(z)$ is defined by (6.3).

Proof. Suppose that the first claim occurs at epoch s. For no ruin to occur according to the renewal argument we get

$$\Phi^0(t) = \int_{0-}^\infty [\rho\delta_0 + (1-\rho)\lambda e^{-\lambda s}] \int_0^{t+cs} \Phi^0(t+cs-z)dF(z)ds$$

$$= \rho\int_{0-}^t \Phi^0(t-z)dF(z) + (1-\rho)\int_{0-}^\infty \lambda e^{-\lambda s}\int_0^{t+cs} \Phi^0(t+cs-z)dF(z)ds, \quad t \geq 0.$$

Change of variables and differentiation leads to

$$\Phi^{0\prime}(t) = \rho\Phi^0(0)F'(t) + \rho\int_{0-}^t \Phi^{0\prime}(t-z)dF(z) + \frac{\lambda}{c}[\Phi^0(t) - \frac{\lambda}{c}\int_0^t \Phi^0(t-z)dF(z)], \tag{6.5}$$

where $\Phi^{0\prime}(t)$ is the derivative of $\Phi^0(t)$. Integrating (6.5) in t over $[0,u]$ and performing integration by parts one gets

$$\Phi^0(u) = \Phi^0(0) + \rho \int_{0-}^{u} \Phi^0(u-z)dF(z) + \frac{\lambda}{c}\int_0^u \Phi^0(u-z)[1-F(z)]dz,$$

which is just equation (6.4).

□

Corollary 6.1 *The ruin probability $\Psi^0(u)$ satisfies the following integral equation:*

$$\Psi^0(u) = H(\infty) - H(u) + \int_0^u \Psi^0(u-z)dH(z), \quad u \geq 0. \tag{6.6}$$

Proof. Equation (6.6) follows directly from (6.4).

□

Since $H(\infty) < 1$, equations (6.4) and (6.6) are defective renewal equations.

Recalling that $H(\infty) = \rho + \frac{\lambda\mu}{c}$ and $\Phi^0(\infty) = 1$, in the case of positive safety loading we conclude that

$$\Phi^0(0) = 1 - H(\infty) = (1-\rho)\left[1 - \frac{\lambda\mu}{c(1-\rho)}\right],$$

and

$$\Psi^0(0) = 1 - \Phi^0(0) = H(\infty).$$

Define $L_{\Phi^0}(s) = \int_0^\infty e^{-sx}\Phi^0(x)dx$ to be the Laplace transform (LT) of $\Phi^0(u)$. Taking the LT of (6.4) we get

$$L_{\Phi^0}(s) = \frac{\Phi^0(0)}{s[1 - LS_H(s)]}, \tag{6.7}$$

where $LS_H(s)$ is the LST of $H(u)$. Using the standard properties of the transforms and their inversions leads to

$$\Phi^0(u) = (1 - H(\infty))\sum_{n=0}^{\infty} H^{*n}(u), \quad u \geq 0,$$

where $H^{*n}(u)$ means the n-th convolution of $H(u)$ with itself. The same result can be derived by using the fact that renewal equation (6.4) has an unique solution, see for instance [**?**], Lemma 6.1.2.

Now let us define

$$H_1(u) = \frac{H(u)}{H(\infty)} = \frac{H(u)}{\rho + \frac{\lambda\mu}{c}},$$

which is a proper probability distribution. For the non-ruin probability we have

$$\Phi^0(u) = [1 - H(\infty)] \sum_{n=0}^{\infty} [H(\infty)]^n H_1^{*n}(u), u \geq 0, \tag{6.8}$$

and for the ruin probability

$$\Psi^0(u) = [1 - H(\infty)] \sum_{n=1}^{\infty} [H(\infty)]^n \overline{H}_1^{*n}(u), \quad u \geq 0,$$

where $\overline{H}_1(u) = 1 - H_1(u)$.

In the above formula we recognize a version of the Pollaczeck - Khinchin formula (or Beekman convolution formula), see [77].

According to the definition of the relative safety loading (6.2), the following relations hold:

$$1 - H(\infty) = (1 - \rho)\frac{\theta}{1 + \theta} \quad \text{and} \quad H(\infty) = \frac{1 + \theta\rho}{1 + \theta}.$$

So

$$\Psi^0(u) = (1 - \rho)\frac{\theta}{1 + \theta} \sum_{n=1}^{\infty} \left(\frac{1 + \theta\rho}{1 + \theta}\right)^n \overline{H}_1^{*n}(u), \quad u \geq 0. \tag{6.9}$$

In the case of $\rho = 0$, $H_1(u) = H(u) = G(u)$ and (6.9) coincides with the ruin probability of the classical risk model.

Example 6.1 *Let us consider the case of exponentially distributed claim sizes, i.e.* $F(u) = 1 - e^{-\frac{u}{\mu}}, \quad u \geq 0, \quad \mu > 0$. *In this case, the integrated tail distribution* $F_I(u)$ *is exponential also and* $H(u) = (\rho + \frac{\lambda\mu}{c})[1 - e^{-\frac{u}{\mu}}]$. *So,* $H_1(u) = 1 - e^{-\frac{u}{\mu}}$.

Taking into account that the n-th convolution of $H_1(u)$ is an Erlang $(n, \frac{1}{\mu})$ distribution with distribution function

$$H_1^{*n}(u) = 1 - e^{-\frac{u}{\mu}} \sum_{j=0}^{n-1} \frac{\left(\frac{u}{\mu}\right)^j}{j!},$$

relation (6.8) leads to

$$\Phi^0(u) = 1 - H(\infty) \exp\left\{-\frac{1 - H(\infty)}{\mu} u\right\}.$$

The ruin probability in terms of the relative safety loading is given by

$$\Psi^0(u) = \frac{1 + \theta\rho}{1 + \theta} \exp\left\{-\frac{1 - \rho}{\mu} \frac{\theta}{1 + \theta} u\right\}. \tag{6.10}$$

If $\rho = 0$, the result coincides with the example of Grandell [37], p. 5-6 related to the ruin probability for the classical risk model. The example is to be continued.

6.1.2 The stationary case

According to the arguments, described by Grandell [37], if $\Phi^0(u)$ and $\Psi^0(u)$ are non - ruin and ruin probability, respectively, in the ordinary case, then in the stationary case we have the following result:

Proposition 6.2 *The non-ruin probability $\Phi(u)$ and the ruin probability $\Psi(u)$ in the stationary case satisfy the integral representations*

$$\Phi(u) = \Phi(0) + \frac{\lambda}{c(1 - \rho)} \int_0^u \Phi^0(u - z)(1 - F(z))dz \tag{6.11}$$

and

$$\Psi(u) = \frac{\lambda}{c(1 - \rho)} \left[\int_u^\infty (1 - F(z))dz + \int_0^u \Psi^0(u - z)(1 - F(z))dz\right]. \tag{6.12}$$

Since $\Phi(\infty) = \Phi^0(\infty) = 1$ when $c > \frac{\lambda\mu}{1-\rho}$, we have

$$\Phi(0) = \frac{1 - H(\infty)}{1 - \rho} = 1 - \frac{\lambda\mu}{c(1-\rho)}.$$

Taking the Laplace transform of (6.11) and applying (6.7) we have

$$L_\Phi(s) = \frac{\Phi(0)}{s} + \frac{\lambda\mu}{c(1-\rho)} LS_{F_I}(s) \frac{\Phi^0(0)}{s[1 - LS_H(s)]}.$$

Again, the standard properties of the transforms lead to

$$LS_\Phi(s) = \frac{1 - H(\infty)}{1 - \rho} + \frac{H(\infty) - \rho}{1 - \rho} LS_{F_I}(s)[1 - H\infty)] \sum_{n=0}^{\infty} [LS_H(s)]^n.$$

So, the ruin probability in the stationary case is given by

$$\Psi(u) = \frac{H(\infty) - \rho}{1 - \rho} \left[\overline{F}_I(u) + F_I(u) * [1 - H(\infty)] \sum_{n=1}^{\infty} [H(\infty)]^n \overline{H}_1^{*n}(u) \right], \tag{6.13}$$

where $\overline{H}_1(u) = 1 - H_1(u)$ and $\overline{F}_I(u) = 1 - F_I(u)$.

In terms of the relative safety loading the ruin probability is given by

$$\Psi(u) = \frac{1}{1+\theta} \overline{F}_I(u) + \frac{1}{1+\theta} F_I(u) \frac{(1-\rho)\theta}{1+\theta} \sum_{n=1}^{\infty} \left(\frac{1+\theta\rho}{1+\theta} \right)^n \overline{H}_1^{*n}(u).$$

Example 6.2 *Again, consider the case in which the claim amount distribution is exponential with mean value μ. Applying the argument of Example 6.1 to the ruin probability (6.13) yields*

$$\Psi(u) = \frac{H(\infty) - \rho}{1 - \rho} \exp\left\{ -\frac{1 - H(\infty)}{\mu} u \right\},$$

and in terms of the relative safety loading,

$$\Psi(u) = \frac{1}{1+\theta} \exp\left\{ -\frac{1-\rho}{\mu} \frac{\theta}{1+\theta} u \right\}. \tag{6.14}$$

In the case of $\rho = 0$, (6.14) coincides with that of example 6 of Grandell [37], p. 69.

6.1.3 The Cramér - Lundberg approximation

6.1.3.1 The ordinary case

Let us return to the defective integral equation (6.6) for the ruin probability in the ordinary case. Assume that there exists a constant $R > 0$ such that

$$\int_0^\infty e^{Rz} dH(z) = 1, \tag{6.15}$$

where $H(z)$ is given by (6.3), and denote $h(R) = \int_0^\infty e^{Rz} dF(z) - 1$. The equation (6.15) is known as a **Cramér condition.** For any functions $f_1(x)$ and $f_2(x)$, we write $f_1(x) \sim f_2(x)$ for $x \to \infty$, if $\lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = 1$.

Theorem 6.1 *Let, for the Pólya-Aeppli risk model, Cramér condition (6.15) holds and $h'(R) < \infty$. Then*

$$\Psi^0(u) \sim \frac{\mu\theta A(\mu,\theta,R,\rho)}{A^2(\mu,\theta,R,\rho)h'(R) - \mu(1+\theta)} e^{-Ru}, \tag{6.16}$$

where $A(\mu,\theta,R,\rho) = \frac{1-[1-\mu(1+\theta)R]\rho}{1-\rho}$.

Proof. Multiplying (6.6) by e^{Ru} yields

$$e^{Ru}\Psi^0(u) = e^{Ru}(H(\infty) - H(u)) + \int_0^u e^{R(u-z)}\Psi^0(u-z)e^{Rz}dH(z). \tag{6.17}$$

It follows from the definition of R that the integral equation (6.17) is a renewal equation.

The mean value of the probability distribution, given by

$$G(t) = \int_0^t e^{Rz} dH(z)$$

is

$$\int_0^\infty z e^{Rz} dH(z) = \frac{\lambda}{cR}\left(1 + \frac{c}{\lambda}R\rho\right)h'(R) - \frac{1-\rho}{R(1+\frac{c}{\lambda}R\rho)}.$$

Since

$$\int_0^\infty e^{Ru}(H(\infty) - H(u))dz = \frac{1-H(\infty)}{R} = \frac{1-(\rho+\frac{\lambda\mu}{c})}{R},$$

by the key renewal theorem, we have

$$\Psi^0(u) \sim \frac{\left(1+\frac{c}{\lambda}R\rho\right)\left((1-\rho)\frac{c}{\lambda}-\mu\right)}{\left(1+\frac{c}{\lambda}R\rho\right)^2 h'(R)-(1-\rho)\frac{c}{\lambda}}e^{-Ru}.$$

Taking into account that $\frac{c}{\lambda}=\frac{\mu(1+\theta)}{1-\rho}$ we get the approximation (6.16) in terms of the relative safety loading.

□

Definition 6.2 *(6.16) is called a* **Cramér - Lundberg approximation**. *The constant R, the non negative solution of the equation (6.15) is called a* **Lundberg exponent** *or* **adjustment coefficient.**

If $\rho=0$, $A(\mu,\theta,R,0)=1$ and (6.16) coincides with the Cramér - Lundberg approximation for the classical risk model ([37], p. 7).

Example 6.3 *If we take $F(x)=1-\exp(-\frac{x}{\mu})$, then $h(R)=\frac{\mu R}{1-\mu R}$. The constant R is a positive solution of the equation*

$$(1+\frac{c}{\lambda}R\rho)\ \frac{\mu R}{1-\mu R}=(1-\rho)\frac{c}{\lambda}R, \tag{6.18}$$

i.e.

$$R=\frac{1-\rho}{\mu}\left(1-\frac{\lambda\mu}{c(1-\rho)}\right)=\frac{1-\rho}{\mu}\frac{\theta}{1+\theta},$$

and $A(\mu,\theta,R,\rho)=1+\theta\rho$.

So, the Cramér - Lundberg approximation is exact when the claims are exponentially distributed and given by

$$\Psi^0(u)\sim\frac{1+\theta\rho}{1+\theta}e^{-Ru}.$$

6.1.3.2 The stationary case

Let us write integral representation (6.12) for the ruin probability in the stationary case in the following equivalent form:

$$\Psi(u) = \frac{\lambda\mu}{c(1-\rho)}\left[\bar{F}_I(u) - \int_0^u \Psi^0(u-z)d\bar{F}_I(z)\right].$$

Taking into account the Cramér - Lundberg approximation in the ordinary case we have

$$\Psi(u) \sim \frac{(1-\rho)\frac{c}{\lambda} - \mu}{\left(1+\frac{c}{\lambda}R\rho\right)^2 h'(R) - (1-\rho)\frac{c}{\lambda}} e^{-Ru},$$

and in terms of the relative safety loading

$$\Psi(u) \sim \frac{\mu\theta}{A^2(\mu,\theta,R,\rho)h'(R) - \mu(1+\theta)} e^{-Ru}. \tag{6.19}$$

In the case of $\rho = 0$, asymptotic relation (6.19) coincides with the Cramér - Lundberg approximation in the classical risk model.

Example 6.4 *Again, in the case of exponentially distributed claim sizes, the Cramér - Lundberg approximation is exact and is given by*

$$\Psi(u) \sim \frac{1}{1+\theta} e^{-Ru},$$

where R is a positive solution of (6.18).

6.1.4 Comparison of ruins

The value of the Lundberg exponent is a measure of the dangerousness of the risk business. Relative to this measure, the ordinary and the stationary cases are equally dangerous, (see [37], p. 70).

We will compare the Pólya - Aeppli risk model with the corresponding classical model. According to the definition given by De Vylder and Goovaerts [24], corresponding risk models are models with the same claim-size distribution, the same expected number of claims in any time interval

$[0, t]$, the same security loading, and the same initial risk reserve. The classical risk model corresponding to the Pólya - Aeppli risk model has a Poisson counting process with intensity $\frac{\lambda}{1-\rho}$ and a relative safety loading given by (6.2). The inter-arrival times are exponentially distributed with parameter $\frac{\lambda}{1-\rho}$. Now we need the following lemma [37]:

Lemma 6.1 *Let T^1 and T^2 be two random variables, representing the inter-arrival times of two risk models. Let R_1 and R_2 be the corresponding Lundberg exponents. If $LS_{T^1}(s) \leq LS_{T^2}(s)$ for all $s > 0$, then $R_1 \geq R_2$.*

Let T^{cl} and T be two random variables representing the inter-arrival times of the corresponding classical risk model and the Pólya - Aeppli risk model. Then

$$LS_{T^{cl}}(s) = \frac{\lambda/(1-\rho)}{s + \lambda/(1-\rho)} \quad \text{and} \quad LS_T(s) = \rho + (1-\rho)\frac{\lambda}{s+\lambda}.$$

It is easy to see that $LS_{T^{cl}}(s) \leq LS_T(s),\ s > 0$. Applying Lemma 6.1 it follows that $R^{cl} \geq R$, where R^{cl} is the Lundberg exponent for the classical model. This means that the Pólya - Aeppli risk model is more dangerous than the corresponding classical model.

The comparison of the exact ruin probabilities depends on the claim-size distribution. We can compare analytically the particular cases of exponentially distributed claim-sizes. At first, let us compare the ruin probability of stationary case (6.14) with the ruin probability of the corresponding classical risk model given by

$$\Psi_{cl}(u) = \frac{1}{1+\theta}\ \exp\left\{-\frac{1}{\mu}\ \frac{\theta}{1+\theta}u\right\}.$$

It is easy to see that for all $u \geq 0$

$$\Psi(u) \geq \Psi_{cl}(u).$$

On the other hand, the comparison between (6.10) and (6.14) states that for all $u \geq 0$

$$\Psi^0(u) \geq \Psi(u).$$

So, in the case of exponentially distributed claim-sizes the most dangerous is the ordinary case and the less dangerous is the classical risk model.

Comparing the ruin probabilities it is natural to mention the difference between the Pólya - Aeppli risk model and the classical model in the case of $\rho = 0$. It suffices to compare the ruin probability in the stationary case (6.14) and the corresponding ruin probability

$$\Psi_{\{\rho=0\}}(u) = \frac{1}{1+\theta_{\{\rho=0\}}} \exp\left\{-\frac{1}{\mu}\,\frac{\theta_{\{\rho=0\}}}{1+\theta_{\{\rho=0\}}}\,u\right\},$$

in the case of $\rho = 0$. From (6.2) it follows that

$$\theta_{\{\rho=0\}} \geq \theta.$$

Then for all $u \geq 0$ we have

$$\Psi_{\{\rho=0\}}(u) \leq \Psi(u).$$

In this case again the Pólya - Aeppli risk model is more dangerous than the classical risk model.

It is useful to analyze the differences between the ruin probabilities even in the particular cases. The distributions of the inter-arrival times of the corresponding models have the same expected values. In the Pólya - Aeppli risk model we have $P(T_2 = 0) = \rho > 0$, i.e. the probability that the claims arrive simultaneously is not equal to zero. This can cause the ruin and the ruin probability is greater.

6.1.5 Martingales for the Pólya - Aeppli risk model

Let us denote by $(\mathcal{F}_t^X)$ the natural filtration generated by any stochastic process $X(t)$. $(\mathcal{F}_t^X)$ is the smallest complete filtration to which $X(t)$ is adapted.

Let us denote by $LS_Z(r) = \int_0^\infty e^{-rx} dF(x)$ the Laplace-Stieltjes transform (LS-transform) of any random variable Z with distribution function $F(x)$.

Lemma 6.2 *For the Pólya - Aeppli risk model*

$$Ee^{-rX(t)} = e^{g(r)t},$$

where

$$g(r) = \frac{1}{1 - \rho LS_Z(-r)}[\rho cr LS_Z(-r) + \lambda(LS_Z(-r) - 1) - cr].$$

Proof: Let us consider the random sum from the standard risk model

$$S_t = \sum_{i=1}^{N(t)} Z_i,$$

where $N(t)$ is a Pólya - Aeppli process, independent of $Z_i, i = 1, 2, \ldots$. S_t is a compound Pólya-Aeppli process and the LS- transform is given by

$$LS_{S_t}(r) = P_{N(t)}(LS_Z(r)) = e^{-\lambda t \frac{1-LS(r)}{1-\rho LS(r)}}.$$

For the LS-transform of $X(t)$ we have the following

$$\begin{aligned} LS_{X(t)}(r) &= Ee^{-rX(t)} = Ee^{-r[ct-S_t]} = e^{-rct}Ee^{rS_t} \\ &= e^{-rct}P_{N(t)}(LS_Z(-r)) = e^{-rct}e^{-\lambda t \frac{1-LS_Z(-r)}{1-\rho LS_Z(-r)}} = e^{g(r)t}. \end{aligned}$$

□

From the martingale theory we get the following

Lemma 6.3 *For all $r \in R$ the process*

$$M_t = e^{-rX(t)-g(r)t}, \ t \geq 0$$

is an $\mathcal{F}_t^X$-martingale, provided that $LS_Z(-r) < \infty$.

6.1.6 Martingale approach to the Pólya-Aeppli risk model

The martingale approach to the insurance risk model is introduced at first by H. Gerber, see [36]. Using the martingale properties of M_t, we will give some useful inequalities for the ruin probability.

Proposition 6.3 *Let $r > 0$. For the ruin probabilities of the Pólya - Aeppli risk model we have the following results*

i) $\Psi(u,t) \leq e^{-ru} \sup_{0\leq s\leq t} e^{g(r)s}, 0 \leq t < \infty$

ii) $\Psi(u) \leq e^{-ru} \sup_{s\geq 0} e^{g(r)s}$.

iii) If the Lundberg exponent R exists, then R is the unique strictly positive solution of

$$\rho cr LS_Z(-r) + \lambda(LS_Z(-r) - 1) - cr = 0 \tag{6.20}$$

and

$$\Psi(u) \leq e^{-Ru}. \tag{6.21}$$

Proof.

i) For any $t_0 < \infty$, the martingale stopping time theorem yields the following

$$1 = M_0 = EM_{t_0\wedge\tau} = E[M_{t_0\wedge\tau}, \tau \leq t] + E]M_{t_0\wedge\tau}, \tau > t] \geq$$

$$\geq E[M_{t_0\wedge\tau}, \tau \leq t] = E[e^{-rX(\tau)-g(r)\tau}|\tau \leq t]P(\tau \leq t),$$

from which

$$P(\tau \leq t) = \frac{e^{-ru}}{E[e^{-g(r)\tau}|\tau \leq t]}.$$

The statement i) follows from the above relation.

ii) follows immediately from i) when $t \to \infty$.

iii) The Cramér condition (6.15) becomes

$$\rho \int_0^\infty e^{rx} dF(x) + \frac{\lambda}{c} \int_0^\infty e^{rx}(1 - F(x))dx = 1.$$

Using

$$\begin{aligned} \int_0^\infty e^{rx}(1 - F(x))dx &= \int_0^\infty \int_x^\infty e^{rx} dF(y)dx \\ &= \int_0^\infty \int_0^y e^{rx} dx dF(y) = \tfrac{1}{r}[LS_Z(-r) - 1], \end{aligned}$$

it can be written as

$$\rho LS_Z(-r) + \frac{\lambda}{cr}(LS_Z(-r) - 1) = 1.$$

This is equivalent to the equation (6.20).

Let us denote

$$f(r) = \rho cr LS_Z(-r) + \lambda(LS_Z(-r) - 1) - cr.$$

So R is a positive solution of $f(r) = 0$. Because $f(0) = 0, f'(0) = \lambda\mu - (1-\rho)c < 0$ and $f''(r) = (\rho cr + \lambda)LS_Z''(-r) + 2\rho c LS_Z'(-r) > 0$ there is at most one strictly positive solution.

□

Remark 6.1 *The equation (6.20) is equivalent to* $g(r) = 0$.

Remark 6.2 *The above inequalities are well known for the classical risk model. In the case of* $\rho = 0$ *the Pólya - Aeppli risk model becomes the classical risk model. The Cramér condition (6.15) and the function* $g(r)$ *are the same, see [79].*

6.1.7 Reinsurance

Let us write the surplus process in terms of the safety loading coefficient

$$U(t) = u + \frac{\lambda\mu}{1-\rho}(1+\theta)t - \sum_{i=1}^{N(t)} Z_i. \tag{6.22}$$

Suppose the insurer chooses proportional reinsurance with retention level $b \in [0,1]$. The premium rate for the reinsurance is

$$(1+\eta)(1-b)\frac{\lambda\mu}{1-\rho},$$

where $\eta > 0$ is the relative safety loading, defined by the reinsurance company. We consider the case $\eta > \theta$. The premium rate for the insurer is

$$\frac{\lambda\mu}{1-\rho}[(1+\theta) - (1+\eta)(1-b)] = \frac{\lambda\mu}{1-\rho}[b(1+\eta) - (\eta-\theta)],$$

and the surplus process becomes

$$U(t,b) = u + \frac{\lambda\mu}{1-\rho}[b(1+\eta) - (\eta-\theta)]t - \sum_{i=1}^{N(t)} bZ_i. \tag{6.23}$$

In order that the net profit condition is fulfilled we need

$$\frac{\frac{\lambda\mu}{1-\rho}[b(1+\eta) - (\eta-\theta)]}{\frac{\lambda\mu}{1-\rho}b} > 1,$$

i.e.

$$b > 1 - \frac{\theta}{\eta}.$$

Let $M_Z(r)$ be the moment generating function of the individual claim amount distribution evaluated at r. Then the adjustment coefficient $R(b)$ under proportional reinsurance is the unique positive solution of the equation

$$\frac{\lambda\mu}{1-\rho}\rho[b(1+\eta)-(\eta-\theta)]rM_Z(br)+\lambda[M_Z(br)-1]-\frac{\lambda\mu}{1-\rho}[b(1+\eta)-(\eta-\theta)]r = 0. \tag{6.24}$$

Let $\Psi(u,b)$ denote the probability of ultimate ruin when the proportional reinsurance is chosen. Then

$$\Psi(u,b) = P(U(t,b) < 0 \text{ for some } \ t > 0)$$

Our objective is to find the retention level that minimizes $\Psi(u,b)$. According the Lundberg inequality (6.21), the retention level will be optimal, if the corresponding Lundberg exponent R is maximal. We know that there is a unique $b \in [0,1]$ where the maximum is attained. If the maximizer $b > 1$, it follows from the uni-modality that the optimal b is 1, i.e. no reinsurance is chosen.

The next result gives the optimal retention level b and maximal adjustment coefficient $R(b)$. Similar result is obtained by Hald and Schmidli [41] for the classical risk model.

Lemma 6.4 *The solution of equation (6.24) is given by*

$$R(b(r)) = \frac{(1+\eta)[1-\rho M_Z(r)]\mu r - (1-\rho)[1-M_Z(r)]}{(\eta-\theta)[1-\rho M_Z(r)]\mu}, \tag{6.25}$$

where $b \to r(b)$ is invertible.

Proof. Assume that $r(b) = bR((b))$, where $R(b)$ will be the maximal value of the adjustment coefficient and $r(b)$ is invertible. If we consider the function $r \to b(r)$, it follows that

$$b(r) = \frac{(\eta - \theta)[1 - \rho M_Z(r)]\mu r}{(1+\eta)[1 - \rho M_Z(r)]\mu r - (1-\rho)[1 - M_Z(r)]}. \tag{6.26}$$

Now $R(b(r)) = \frac{r}{b(r)}$ in detail is given by (6.25).

□

Theorem 6.2 *Assume that $M_Z(r) < \infty$. Suppose there is a unique solution r to*

$$(1-\rho)^2 M_Z'(r) - (1+\eta)\mu[1 - \rho M_Z(r)]^2 = 0. \tag{6.27}$$

Then $r > 0$, the maximal value of $R(b(r))$ and the retention level $b(r)$ are given by (6.25) and (6.26).

Proof. The necessary condition for maximizing the value of the adjustment coefficient is given by equation (6.27).

Because $R'(b(0)) = \frac{\eta}{\eta - \theta} > 0$, the function $R(b(r))$ is strictly increasing in 0. The second derivative in zero $R''(b(0)) = -\frac{(1-\rho)^2}{(\eta-\theta)\mu}\frac{(1-\rho)EZ^2 + 2\rho\mu^2}{(1-\rho)^3} < 0$ shows that $R(b(r))$ is strictly concave. Consequently, the function $R(b(r))$ has an unique maximum in r, which is the solution of (6.27). The retention level is given by (6.26).

□

Remark 6.3 *Note that the value of the adjustment coefficient does not depend on c but on the relative safety loadings only.*

Example 6.5 *Suppose that the claim sizes are exponentially distributed with parameter $\mu > 0$, i.e.*

$$F(z) = 1 - e^{-\frac{z}{\mu}}.$$

The moment generating function is given by

$$M_Z(r) = \frac{1}{1-\mu r},$$

and

$$M_Z'(r) = \frac{\mu}{(1-\mu r)^2}.$$

In this case the adjustment coefficient relative to retention level can be obtain as a solution of the equation (6.27) and is given by

$$r(b) = \frac{(1-\rho)[\theta - \eta(1-b)]}{\mu[1+\theta-(1+\eta)(1-b)]}.$$

If $\rho = 0$ and $\mu = 1$, the corresponding formula for the classical model is obtained by Dickson and Waters [26].

The equation (6.27) and its solution are given by

$$(1-\rho)^2 \frac{\mu}{(1-\mu r)^2} - (1+\eta)\mu \left[1 - \rho \frac{1}{1-\mu r}\right]^2 = 0$$

and

$$r = \frac{1-\rho}{\mu}\left(1 - (1+\eta)^{-\frac{1}{2}}\right).$$

From (6.26) we find

$$b = \frac{(\eta - \theta)}{\sqrt{1+\eta}(1+\sqrt{1+\eta})}$$

and the adjustment coefficient is given by

$$R(b) = \frac{(1-\rho)(\sqrt{1+\eta}-1)^2}{\mu(\eta-\theta)}.$$

In this example there are closed form expressions for b and $R(b)$, if $b \leq 1$ and the retention level does not depend from ρ.

Analyzing the optimal retention levels for the Pólya - Aeppli risk model and the classical model gives the possibility to compare the ruin probabilities. In the case of proportional reinsurance and $\rho = 0$, the retention level $b(r)$, given by (6.26) coincides with the retention level for the classical risk model, say b_0, obtained by Hald and Schmidli [41]). It is easy to verify that $b(r) < b_0$. For the maximal values $R(b_0)$ and $R(b(r))$ is fulfilled

$$R(b_0) \geq R(b(r)),$$

which means that the Pólya - Aeppli risk model with proportional reinsurance is more dangerous than the classical risk model.

6.2 Compound Birth process

In this section we consider again the modified birth process. Let $\{p_i,\ i = 1, 2, \ldots\}$ be the distribution of some compounding random variable with values in $\{1, 2, \ldots\}$.

The transition probabilities of the counting process $N(t)$, for every $m = 0, 1, \ldots$ are defined by the following postulates:

$$P(N(t+h) = n \mid N(t) = m) = \begin{cases} 1 - \sum_{i=1}^{\infty} p_i \lambda_i(t) h + o(h), & n = m, \\ p_i \lambda_i(t) h + o(h), & n = m + i, \\ & i = 1, 2, \ldots, \end{cases}$$

where $o(h) \to 0$ as $h \to 0$ and $\lambda_i(t),\ i = 1, 2, \ldots$ are intensity functions.

Let $A(t) = \sum_{i=1}^{\infty} p_i \lambda_i(t)$. If $P_m(t) = P(N(t) = m),\ m = 0, 1, 2, \ldots$, the above postulates yield the following Kolmogorov forward equations:

$$\begin{aligned} P_0'(t) &= -A(t) P_0(t), \\ P_m'(t) &= -A(t) P_m(t) + \sum_{i=1}^{m} p_i \lambda_i(t) P_{m-i}(t),\ m = 1, 2, \ldots. \end{aligned} \tag{6.28}$$

This leads to the following definition

Definition 6.3 *The counting process $N(t)$, defined by the differential equations (6.28) with initial conditions*

$$P_0(0) = 1 \quad \textit{and} \quad P_m(0) = 0, \quad m = 1, 2, \ldots$$

is called a compound birth process.

The compound birth process extends the well known Pólya - Aeppli, I-Polya and I-Binomial processes.

6.2.1 Probability generating function

Multiplying the mth equation of (6.28) by u^m and summing for all $m = 0, 1, 2, \ldots$ we get the following differential equation for the probability generating function

$$\frac{\partial P_{N(t)}(s)}{\partial t} = -\sum_{i=1}^{\infty} p_i \lambda_i(t)(1 - s^i) P_{N(t)}(s). \tag{6.29}$$

The solution of (6.29) with the initial condition $P_{N(t)}(0) = 1$ is given by

$$P_{N(t)}(s) = \exp\left(-\sum_{i=1}^{\infty} p_i (1 - s^i) \int_0^t \lambda_i(u) du \right). \tag{6.30}$$

6.2.2 Application to Risk Theory

We consider the risk model in which the counting process $N(t)$ is a compound birth process. The relative safety loading θ is given by

$$\theta = \frac{ct}{\mu \sum_{i=1}^{\infty} i p_i \int_0^t \lambda_i(s)} - 1,$$

and in the case of positive safety loading $\theta > 0$, $c > \frac{\mu}{t} \sum_{i=1}^{\infty} i p_i \int_0^t \lambda_i(s) ds$.

In the following we use the notation of [46]. Let $G(u, y)$ be the joint probability distribution of the time to ruin τ and the deficit at the time of ruin D, i.e.

$$G(u,y) = P(\tau \leq \infty, D \leq y). \tag{6.31}$$

and

$$\lim_{y \longrightarrow \infty} G(u,y) = \Psi(u). \tag{6.32}$$

Using the postulates we have

$$G(u,y) = (1 - A(t)h)\, G(u+ch, y) +$$

$$+ \sum_{i=1}^{\infty} p_i \lambda_i(t) h \left[\int_0^{u+ch} G(u+ch-x, y) dF^{\star i}(x) + F^{\star i}(u+ch+y) - F^{\star i}(u+ch) \right]$$

$$+ o(h),$$

where $F^{\star i}(x), \;\; i = 1, 2, \ldots$ is the distribution function of $Z_1 + Z_2 + \ldots + Z_i$.

Rearranging the terms leads to

$$\frac{G(u+ch, y) - G(u,y)}{ch} = \frac{A(t)}{c} G(u+ch, y) -$$

$$- \frac{1}{c} \sum_{i=1}^{\infty} p_i \lambda_i(t) \left[\int_0^{u+ch} G(u+ch-x, y) dF^{\star i}(x) + F^{\star i}(u+ch+y) - F^{\star i}(u+ch) \right]$$

$$+ \frac{o(h)}{h},$$

Let

$$H(x) = \sum_{i=1}^{\infty} p_i \lambda_i(t) F^{*i}(x)$$

be the defective probability distribution function of the claims with

$$H(0) = 0 \;\; \text{and} \;\; H(\infty) = A(t).$$

By letting $h \to 0$ we obtain the following differential equation

$$\frac{\partial G(u,y)}{\partial u} = \frac{A(t)}{c} G(u,y) - \frac{1}{c} \left[\int_0^u G(u-x, y) dH(x) + [H(u+y) - H(u)] \right]. \tag{6.33}$$

In terms of the proper probability distribution function $H_1(x) = \frac{H(x)}{A(t)}$ the equation (6.33) is given by

$$\frac{\partial G(u,y)}{\partial u} = \frac{A(t)}{c}\left[G(u,y) - \int_0^u G(u-x,y)dH_1(x) + [H_1(u+y) - H_1(u)]\right]. \tag{6.34}$$

Theorem 6.3 *The function $G(0,y)$ is given by*

$$G(0,y) = \frac{A(t)}{c}\int_0^y [1 - H_1(u)]du. \tag{6.35}$$

Proof. Integrating (6.34) from 0 to ∞ with $G(\infty,y) = 0$ leads to

$$-G(0,y) =$$

$$= \frac{A(t)}{c}\left[\int_0^\infty G(u,y)du - \int_0^\infty \int_0^u G(u-x,y)dH_1(x)du - \int_0^\infty (H_1(u+y) - H_1(u))du\right]$$

The change of variables in the double integral and simple calculations yield

$$G(0,y) = \frac{A(t)}{c}\int_0^\infty [H_1(u+y) - H_1(u)]du$$

and (6.35).

□

Theorem 6.4 *The probability of ruin $\Psi(u)$ satisfies the equation*

$$\frac{\partial \Psi(u)}{\partial u} = \frac{A(t)}{c}\left[\Psi(u) - \int_0^u \Psi(u-x)dH_1(x) + [1 - H_1(u)]\right], \quad u \geq 0. \tag{6.36}$$

Proof. The result follows from (6.34) and (6.32).

□

Theorem 6.5 *The ruin probability with no initial capital satisfies*

$$\Psi(0) = \frac{\mu}{c}[p_1\lambda_1(t) + 2p_2\lambda_2(t) + 3p_3\lambda_3(t) + \ldots]. \tag{6.37}$$

Proof. According (6.32) and (6.35)

$$\Psi(0) = \lim_{y\to\infty} G(0,y) = \frac{A(t)}{c}\int_0^\infty [1 - H_1(u)]du.$$

Let X be a random variable with distribution function $H_1(x)$. By the definition of $H_1(x)$ and $EZ = \mu$ we obtain

$$EX = \frac{\mu}{A(t)}[p_1\lambda_1(t) + 2p_2\lambda_2(t) + 3p_3\lambda_3(t) + \ldots].$$

Using the fact that $EX = \int_0^\infty [1 - H_1(x)]dx$ we obtain (6.37).

□

6.2.3 Examples

Consider the case of geometric compounding distribution with parameter $\rho \in [0,1)$, i.e.

$$p_i = (1-\rho)\rho^{i-1}, \quad i = 1,2,\ldots.$$

A special choice of the intensity functions leads to the counting processes defined in [58] with distributions of [59]. The good properties of the defined processes are explained by the lack of memory property of the geometric distribution.

6.2.4 Pólya - Aeppli process

In the case of constant intensity functions:

$$\lambda_i(t) = \lambda, \quad i = 1,2,\ldots$$

the function $A(t)$ is also a constant:

$$A(t) = \lambda.$$

The solution of the equations (6.28) is given by the Pólya - Aeppli distribution, given by (5.2).

6.2.5 I - Pólya process

Consider for some $r \geq 1$ and $\beta > 0$ and $\rho > 0$, the intensity functions:

$$\lambda_i(t) = \frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^{i+1}\left[1+\frac{t}{\beta\rho}\right]^{i-1}, \quad \lambda_i(0) = \frac{r}{\beta}, \quad i = 1, 2, \ldots.$$

The function $A(t)$ is given by

$$A(t) = \frac{r}{\beta+t}.$$

In this case the solution of the equations (6.28) gives the Inflated - parameter Negative binomial distribution with parameters $\frac{\beta}{\beta+t}$, ρ and r, given by (5.10), i.e. $N(t) \sim INB\left(\frac{\beta}{\beta+t}, \rho, r\right)$.

The distribution function of the claims is given by

$$H_1(x) = (1-\rho)\frac{\beta}{\beta+t}\sum_{i=1}^{\infty}\left[1-(1-\rho)\frac{\beta}{\beta+t}\right]^{i-1}F^{*i}(x).$$

6.2.6 I - Binomial process

In this case, for some $\alpha > 0$, $n \geq 1$ and $\rho > 0$, the intensity functions are given by

$$\lambda_i(t) = \frac{n}{\alpha}\left(\frac{\alpha}{\alpha-t}\right)^{i+1}\left[1-\frac{t}{\alpha\rho}\right]^{i-1}, \; t < \alpha, \quad i = 1, 2, \ldots$$

and

$$A(t) = \frac{n}{\alpha-t}.$$

The solution of the equations (6.28) is given by the Inflated-parameter binomial distribution with parameters $\frac{t}{\alpha}$, ρ and n, defined in (5.16), i.e.

$$N(t) \sim IBi\left(\frac{t}{\alpha}, \rho, n\right).$$

For the I-Binomial process the distribution function of the claims is

$$H_1(x) = (1-\rho)\frac{\alpha}{\alpha-t}\sum_{i=1}^{\infty}\left[1-(1-\rho)\frac{\alpha}{\alpha-t}\right]^{i-1}F^{*i}(x).$$

6.2.7 Exponentially distributed claims

Let us consider the case of exponentially distributed claim sizes, i.e. $F(u) = 1 - e^{-\frac{u}{\mu}},\ u \geq 0,\ \mu > 0$. In this case, the probability of ruin for the defined risk model is

$$\Psi(u) = \frac{1}{1+\theta} e^{-\frac{A(t)}{c}\theta u},$$

where the function $A(t)$ and the relative safety loading coefficient θ are defined by the corresponding counting processes.

6.3 Comments

The Pólya - Aeppli process and the problem of ruin probability is given in [61]. The compound birth process together with particular cases are defined in [64]. In [60] and [62], the martingale approach to the Pólya - Aeppli risk model and the reinsurance problem are discussed. The I - Pólya risk model and ruin probability in detail is given in [65].

Chapter 7

Concluding remarks

The main results in these notes could be summaries as follows.

In Chapter 2, a new family of discrete distributions is defined. Using two different approaches, we define at first the geometric distribution and negative binomial distribution, related to a Markov chain. The additional parameter ρ has a natural interpretation in terms of both a zero - inflated proportion and a correlation coefficient. The defined distributions can be presented as a compound geometric and compound negative binomial distributions with a geometric compounding distribution. Then the compound Poisson and compound logarithmic series distributions as limiting distributions are defined. We derive some properties, recursion formulas, explicit formulas for PMFs, etc.

In Chapter 3, we started to enlarge the family of compound GPSDs with a truncated compounding distribution (k point distribution). The Poisson distribution of order k is mentioned. The Pólya - Aeppli distribution of order k is defined and some properties, recursion formulas and PMF are derived.

In Chapter 4 the geometric distribution of order k related to a multi - state Markov chain is defined. The PGFs of the number of successes, the number of failures and the number of trials up to the first run of k consecutive successes are given. We discuss the joint distribution of the number of successes and

the number of failures. Then the sooner and later waiting time problems are discussed.

In Chapter 5 the corresponding stochastic processes are defined. The lack of memory property of the geometric distribution leads to some good properties of the defined processes. The Pólya - Aeppli process is a stationary renewal process.

In Chapter 6 the defined processes are described as counting processes in risk models. We show in detail the Pólya - Aeppli risk model. A differential equation and Pollaczeck - Khinchin formula for ruin probability is derived. We discuss the Cramér - Lundberg approximation, martingale approximation, the particular case of exponentially distributed claims, and compare the model with the classical risk model. The definition of the compound birth process gives the joint distribution of the time to ruin and the deficit at ruin. The reinsurance problem for the Pólya - Aeppli risk model is also discussed.

Appendix

Gaussian hypergeometric function

The Gaussian Hypergeometric function has the form

$$ {}_2F_1[a,b;c;x] = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{x^j}{j!}, \quad c \neq 0, -1, -2, \ldots, \tag{A1} $$

where $(a)_j = a(a+1)\ldots(a+j-1)$ is the Pochhammer's symbol, see [45]. We have interested only in the case where a, b, c and x are real. If a is a non-positive integer, then $(a)_j$ is zero for $j > -a$, and series terminates.

Properties:

1. ${}_2F_1[b,a;c;x] = {}_2F_1[a,b;c;x]$;

2. The Gaussian Hypergeometric function satisfies the second-order linear differential equation

$$ x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - aby = 0; $$

3. The Euler transformations are

$$ \begin{aligned} {}_2F_1[a,b;c;x] &= (1-x)^{-a}\, {}_2F_1[a,c-b;c;\tfrac{x}{x-1}] \\ &= (1-x)^{-b}\, {}_2F_1[c-a,b;c;\tfrac{x}{x-1}] \\ &= (1-x)^{c-a-b}\, {}_2F_1[c-a,c-b;c;x]. \end{aligned} $$

4. Euler's integral representation is

$$ {}_2F_1[a,b;c;x] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du, \tag{A2} $$

where $c > a > 0$.

Confluent Hypergeometric function

The Confluent Hypergeometric function (or Kummer's series) has the form

$$ {}_1F_1[a;c;x] = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j} \frac{x^j}{j!}, \quad c \neq 0, -1, -2, \ldots. \tag{A3} $$

Again $(a)_j$ is the Pochhammer's symbol and if a is a non-positive integer, the series terminates, see [45].

The Confluent Hypergeometric function satisfies Kummer's differential equation

$$ x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0. $$

From the properties we need the Kummer's first theorem, which yields the transformation

$$ {}_1F_1[a;c;x] = e^x {}_1F_1[c-a;c;-x]. $$

The integral representation is

$$ {}_1F_1[a;c;x] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} e^{xu} du, \tag{A4} $$

where $c > a > 0$.

Incomplete Gamma functions

The incomplete Gamma function $\gamma(a,x)$ is defined by

$$ \gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt, \quad x > 0. \tag{A5} $$

Its compliment $\Gamma(a,x)$ is given by

$$ \Gamma(a,x) = \int_x^{\infty} t^{a-1} e^{-t} dt, \quad x > 0 \tag{A6} $$

and

$$ \gamma(a,x) + \Gamma(a,x) = \Gamma(a), $$

where $\Gamma(a)$ is a Gamma function.

Bibliography

[1] Aki S., Balakrishnan N. and Mohanty S. G. (1996). Sooner and later waiting time problems for success and failure runs in higher order Markov dependent trials, *Ann. Inst. Statist. Math.* **48**, 773 - 787. 4.2, 4.3

[2] Aki S., Kuboku H. and Hirano K. (1984). On discrete distributions of order k, *Ann. Inst. Statist. Math.,* **36,** Part A, 431 - 440. 3

[3] Aki S. and Hirano K. (1993). Discrete distributions related to succession events in a two - state Markov chain, *Statistical Sciences and Data Analysis: Proceedings of the Third Pacisfic Area Statistical Conference* (eds. K. Matusira, M.L. Puri and T. Hayakawa), 467 - 474, VSP International Science Publishers, Zeist. 4.3, 4.2, 4.5, 4.9, 4.3.3

[4] Aki S. and Hirano K. (1994). Number of failures and successes until the first consecutive k successes, *Ann. Inst. Statist. Math.,* **46,** 193 - 202. 3, 4.1.2

[5] Asmussen S. (2000). *Ruin Probabilities*, World Scientific Publishing Co. Pte Ltd. 1.1

[6] Anscombe F. J. (1950). Sampling theory of the negative binomial and logarithmic seriesdistributions, *Biometrika,* **37**, 358 - 382. 5.1

[7] Aoyama K., Shimizu K. and Ong S. H. (2008). A first - passage time random walk distribution with five transition probabilities: a general-

ization of the shifted inverse trinomial, *Ann. Inst. Stat. Math.,* **60**, 1 - 20. 2.10

[8] Balakrishnan N. (1997). Joint distributions of numbers of success - runs and failures until the first consecutive k successes in a binary sequence, *Ann. Inst. Statist. Math.* **49**, 519 - 529. 4.2

[9] Balakrishnan N., Balasubramanian K. and Viveros R. (1995). Probability Models and Inference for Start - Up Demonstration Tests Under Correlation and Corrective Action, *Naval Research Ligistics,* **42**, 1271 - 1276. 4.2

[10] Balakrishnan N. and Chan P. S. (2000). Start - up demonstration tests with rejection of units upon observing d failures, *Ann. Inst. Statist. Math.* **52**, 184 - 196. 4.2

[11] Balakrishnan N., Mohanty S. G. and Aki S. (1997). Start - up demonstration tests under Markov dependence model with corrective actions, *Ann. Inst. Stat. Math.,* **49**, 155 - 169. 4.2, 4.3, 4.3, 4.3.3

[12] Balasubramanian K., Viveros R. and Balakrishnan N. (1993). Sooner and later waiting time problems for Markovian Bernoulli trials, *Statist. Probab. Lett.*, **18**, 153 - 161. 4.2, 4.5, 4.2.4, 4.3, 4.12, 4.14, 4.3.3

[13] Balakrishnan N. and Koutras M. V. (2002). *Runs and Scans with Applications*, Wiley Series in Probability and Statistics. 3

[14] Borges P., Rodrigues J. and Balakrishnan N. (2012). Correlated destructive generalized power series cure rate models and associated inference with application to a cutaneous melanoma data, *Computational Statistics and Data Analysis,* **56**, 1703 - 1713. 2.10

[15] Bowers N. L., Gerber H. U., Hickman J. C., Jones D. A. and Nesbitt C. J. (1997). *Actuarial Mathematics*, 2nd ed. Schaumburg, Ill.: Society of Actuaries. 1.2, 2, 2.7

[16] Chadjiconstantinidis S. and Pitselis G. (2009). Further Improved Recursions for a Class of Compound Poisson distributions, *Insurance: Mathematics and Economics,* **44**, 278 286. 2.10

[17] Charalambides Ch. (1986). On discrete distributions of order k, *Ann. Inst. Statist. Math.* **38**, Part A, 557 - 568. 3

[18] Chatfield C., Ehrenberg A. S. C. and Goodhart G. J. (1966). Progress on a simplified model of stationary purchasing behavior (with discussion), *J. of the Royal Statistical Society* **A 129**, 317 - 367. 1.2

[19] Collett, D. (1991). *Modelling Binary Data,* Chapman and Hall: London. 2

[20] Consul P. C. (1990). New class of location - parameter discrete probability distributions and their characterizations, *Commun. Statist. - Theory and Methods,* **19**, 4653 - 4666. 1.3, 2.1

[21] Daley D.J. and Vere - Jones D. (2003). *An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods*, 2nd edition, Springer. 5.3

[22] Daykin C. D., Pentikäinen T. and Pesonen M. (1994). *Practical Risk Theory for Actuaries*, Chapman & Hall. 2.9

[23] Deng L - Y. and Chhikara R. S. (1990). On the characterization of the exponential distribution by the independence of its integer and fractional parts, *Statistica Neerlandica,* **44**, 83 - 85. 2.25

[24] De Vylder F. E. and Goovaerts M. J. (1998). Discussion to Dickson D. C. M. On a Class of Renewal Risk Processes, *North American Actuarial Journal*, **2**, 60 - 68; 68 - 70. 6.1.4

[25] Dickson D. C. M. (1998). On a Class of Renewal Risk Processes, *North American Actuarial Journal*, **2**, 60 - 68, 72 - 73. 1.1

[26] Dickson D. C. M. and Watres H. R. (1996). Reinsurance and ruin, *Insurance: Mathematics & Economics,* **19**, 61 - 80. 6.5

[27] Doi M. and Yamamoto E. (1998). On the joint distribution of runs in a sequence of multi-state trials, *Statist. Probab. Lett.*, **39**, 133 - 141. 4.2, 4.3

[28] Dragieva V. (2011). Queueing system with Pólya - Aeppli input process, *Ann. UACEG,* Sofia (accepted). 5.7

[29] Ebneshahrashoob M. and Sobel M. (1995). Dirichlet analysis for inverse multinomial with both quota and quota-free cells, *Computational Statistics & Data Analysis*, **19**, 293 - 307. 4.2

[30] Evans D.A. (1953). Experimental evidence concerning contagious distributions in ecology, *Biometrika* **4**, 186 - 211. 2.10

[31] Feller W. (1971).*An Introduction to Probability Theory and its Applications,* I, 2nd ed., J. Wiley & Sons, New York. 3

[32] Feller W. (1971). *An Introduction to Probability Theory and its Applications*, II, 2nd ed., J. Wiley & Sons, New York. 6.1

[33] Fu J. C. and Koutras M. V. (1996). Distribution theory of runs: A Markov chain approach, *J. Am. Statist. Assoc.*, **89**, 1050 - 1058. 4.2, 4.3

[34] Fu J. C. and Lou W. Y. W. (2003). *Distribution Theory of Runs and Patterns and its Applications. A Finite Markov Chain Imbedding Approach,* World Scientific Publishing Co. 3

[35] Galambos J. and Kotz S. (1978). *Characterization of Probability Distributions.* Lecture Notes in Math. 675. Springer - Verlag, New York. 2.1.1

[36] Gerber H. U. (1979). *An Introduction to Mathematical Risk Theory*, Richard D. Irwin, Homewood. 6.1.6

[37] Grandell, J. (1991). *Aspects of Risk Theory,* Springer - Verlag, New York. 1.1, 5.2, 6.1, 6.1, 6.1.2, 6.2, 6.1.3.1, 6.1.4

[38] Grandell, J. (1997). *Mixed Poisson Processes,* Chapman & Hall, London. 2.8.1, 5.5, 5.5

[39] Gupta, R. C. (1974). Modified power series distributions and some of its applications, *Sankhyã: Series B* **35**, 288 - 298. 1.3

[40] Gupta L. P., Gupta R. C. and Thripati R. C. (1995). Inflated modified power series distributions with applications, *Communications in Statistics: Theory and Methods* **24**, 2355 - 2374. 2.20

[41] Hald M. and Schmidli H. (2004). On the maximisation of the adjustment coefficient under proportional reinsurance, *ASTIN Bulletin,* **34**, 75 - 83. 6.1.7, 6.1.7

[42] Hirano K., Kuboki H., Aki S. and Kyribayashi K. (1984). *Figures of Probability Density Functions - Discrete Univariate Case*, Computer Science Monographs, Tokyo. 2.7

[43] Hirano K. (1986). Some properties of the distributions of order k, in *Fibonacci Numbers and Their Applications, A.N. Philippou et al.(eds)*, 43 - 53. 3

[44] Jazi M.A. and Alamatsaz M.H. (2010). Ordering Comparison of Logarithmic Series Random Variables with their Mixtures, *Communications in Statistics - Theory and Methods,* **39**, 3255 - 3263. 2.10

[45] Johnson N. L., Kemp A. W. and Kotz S. (2005). *Univariate Discrete Distributions*, Third edition, John Wiley & Sons, Hoboken, New Jersey. 1.2, 1.5, 2.24, 2.9.1, 5.2, (document)

[46] Klugman S. A., Panjer H. and Willmot G. (1998) *Loss Models. From Data to Decisions*, John Wiley & Sons, Inc. 6.2.2

[47] **Kolev N. and Minkova L. D. (1995). On joint distribution of successes and failures related to success runs of length K in homogeneous Markov chain,** *Compt. Randue Bulg. Acad. Sci.,* **48, Vol.9, 19 - 22.** 4.3.3

[48] **Kolev N. and Minkova L. D. (1997). Discrete distributions related to success runs of length K in a multi - state Markov chain,** *Commun. Statist. - Theory and Methods,* **26, 1031 - 1049.** 4.1, 4.3.3

[49] **Kolev N. and Minkova L. D. (1999). Quotas on runs of successes and failures in a multi - state Markov chain,** *Commun. Statist. - Theory and Methods,* **28, 2235 - 2248.** 4.3.3

[50] **Kolev N. and Minkova L. D. (1999). Run and frequency quotas in a multi-state Markov chain,** *Commun. Statist. - Theory and Methods,* **28, 2223 - 2233.** 4.3.3

[51] **Kolev N., Minkova L. D. and Neytchev P. (2000). Inflated-Parameter Family of Generalized Power Series Distributions and Their Application in Analysis of Overdispersed Insurance Data,** *ARCH Research Clearing House,* **2, 295 - 320.** 2, 2.10

[52] Koutras, M. V. (1997). Waiting time distributions associated with runs of fixed length in two - state Markov chains, *Ann. Inst. Stat. Math.,* **49**, 123 - 139. 4.2, 4.3

[53] Lin X. S. and Sendova K. (2008). The compound Poisson risk model with multiple thresholds, *Insurance: Mathematics & Economics,* **42**, 617 - 627. 1.1

[54] Ling K. D. (1988). On binomial distributions of order k, *Stat. Probab. Lett.,* **6**, 247 - 250. 3

[55] Luceño, A. (1995). A family of partially correlated Poisson models for over - dispersion, *Computational Statistics & Data Analysis* **20**, 511-520. 2

[56] Mickey M. R. and Britt P. M. (1974). Obtaining linear scores from preference pairs, *Commun. Statist. - Theory and Methods,* **3**, 501 - 511. 2.9

[57] **Minkova L. (2001). A Family of Compound Discrete Distributions,** *Compt. Randue Bulg. Acad. Sci.,* **54(2), 9 - 12.** 2, 2.10

[58] **Minkova L. D. (2001). Inflated - parameter modification of the pure birth process,** *Compt. Randue Bulg. Acad. Sci.,* **54(11), 17 - 22.** 5.7, 6.2.3

[59] **Minkova L. (2002). A generalization of the classical discrete distributions,** *Commun. Statist. - Theory and Methods,* **31, 871 - 888.** 2, 2.10, 6.2.3

[60] **Minkova L. D. (2004). A modified model of risk business,** *Pliska Stud. Math. Bulgar.,* **16, 129 - 135.** 6.3

[61] **Minkova L. D. (2004). The Pólya - Aeppli process and ruin problems,** *J. Appl. Math. Stoch. Analysis,* **3, 221 - 234.** 5.7, 6.3

[62] **Minkova L. D. (2009). Compound Compound Poisson Risk Model,** *Serdica Math. J.,* **35, 301 - 310.** 6.3

[63] **Minkova L. D. (2010). The Polya - Aeppli distribution of order K,** *Commun. Statist. - Theory and Methods,* **39, 408 - 415.** 3.2, 3.3, 5.7

[64] **Minkova L. D.(2010). Compound Birth processes in Risk Models,** *Proceedings of the 6th Conference in Actuarial Science & Finance on Samos,* **June 2 - 6, 2010.** 6.3

[65] **Minkova L. D. (2011). I - Pólya Process and Applications,** *Commun. Statist. - Theory and Methods,* **40, 2847 - 2855.** 5.7, 6.3

[66] **Minkova L. D. and Omey E. (2011). A new Markov Binomial distribution,** *HUB RESEARCH PAPERS,* **2011/24, Economics & Management, November 2011.** 2, 2.10

[67] Omey E. and Van Gulck S. (2006). Markovian Black and Scholes, *Publications de l'institut mathèmatique,* **79**, 65 - 72. 2

[68] Omey E., Santos J. and Van Gulck S. (2008). A Markov - binomial distribution, *Applicable Analysis and Discrete Mathematics,* **2**, 38 - 50. 2

[69] Panjer H. (1981). Recursive evaluation of a family of compound discrete distributions, *ASTIN Bull.,* **12**, 22 - 26. 1.4

[70] Patil G. P. (1962). Certain properties of the generalized power series distribution. *Ann. Inst. Statist. Math.,* **14**, 179 - 182. 1.3

[71] Philippou A. N. and Muwafi A. A. (1982). Waiting for the kth consecutive success and the Fibonacci sequence of order k, *The Fibonacci Quarterly,* **20**, 28 - 32. 3

[72] Philippou A. N., Georghiou C. and Philippou G. N. (1983). A Generalized Geometric Distribution and some of its Properties, *Stat. Probab. Lett.,* **1**, 171 - 175. 3

[73] Philippou A. N. (1984). The Negative Binomial Distribution of order k and Some of Its Properties, *Biometrical J.,* **26**, 789 - 794. 3

[74] Philipou A. N. and Makri F. S. (1986). Successes, runs and longest runs, *Stat. Probab. Lett.,* **4**, 211 - 215. 3

[75] Qu, Y., Beck, G. J. and Williams, G. W. (1990). Polya - Eggenberhger distribution: parameter estimation and hypothesis tests, *Biometrical J.,* **32**, 229 - 242. 2.17

[76] Ralston M.L. and Jennrich R.I. (1978). Derivative - free algorithm for nonlinear least squares, *Technometrics*, **20**, 7 - 14. 2.9

[77] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999). *Stochastic Processes for Insurance and Finance*, John Wiley & Sons, Chichester. 1.1, 1.2, 1.5, 2, 5.3, 6.1.1

[78] Ruohonen M. (1988). On a model for the claim number process, *ASTIN Bull.,* **18**, 57 - 68. 5.5

[79] Schmidli H. (1996). Martingales and Insurance Risk, in: *8th International Summer School on Probability Theory and Mathematical Statistics,* 155 - 188. 6.2

[80] Schröter, K. J. (1990). On a family of counting distributions and recursions for related compound distributions, *Scand. Actuarial J.,* 161 - 175. 2.23

[81] Sundt, B., Dhaene, J. and De Pril, N. (1998). Some Results on Moments and Cumulants, *Scand. Actuarial J.,* **1**, 24 - 40. 2.23, 2.8.1

[82] Sundt B. and Jewell (1981). Further results on recursive evaluation of compound distributions, *ASTIN Bull.,* **12**, 27 - 39. 1.4

[83] Uchida M. and Aki S. (1995). Sooner and later waiting time problem in a two - state Markov chain, *Ann. Inst. Stat. Math.,* **47**, 415 - 433. 4.2, 4.3.3

[84] Vinogradov V. (2007). On Structutural and Asymptotic Properties of Some Classes of Distributions, *Acta Appl. Math.,* **97**, 335 - 351. 2.10

[85] Viveros R., Balasubramanian K. and Balakrishnan N. (1994). Binomial and Negative Binomial Analogues Under Correlated Bernoulli Trials, *The American Statistician,* **48**, 234 - 247. 2

[86] Winkelmann, R. (1997). *Econometric Analysis of Count Data*, Springer-Verlag, Berlin. 1.2, 2

[87] Yasui S., Ojima Y. and Suzuki T. (2006). Generalization of the Run Rules for the Shewhart Control Charts, in: *Frontiers in Statistical Control*, 207 - 219, Physica Verlag HD. 4.3.3

[88] Zijlstra, M. (1983). Characterizations of the geometric distribution by distributional properties, *J. Appl. Prob.,* **20**, 843 - 850. 2.8.2, 2.8.2