Sofia University "St. Kliment Ohridski"
Faculty of Mathematics and Informatics

SIMULTANEOUS APPROXIMATION BY THE BERNSTEIN OPERATOR

Borislav Radkov Draganov

Dissertation for Doctor of Sciences Degree in Mathematics

Sofia • 2023

Declaration of Authorship

The author hereby declares that the dissertation contains original results he obtained. All other researchers' results used are acknowledged as references.

Borislav Radkov Draganov

Contents

Introduction

S. N. Bernstein introduced in 1912 an approximation operator, which is now named after him, in order to give a simple proof of Weierstrass's theorem that every continuous function on a finite closed interval can be uniformly approximated by algebraic polynomials [9].

The Bernstein operators or polynomials are defined for $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}_+$ by

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}. \tag{0.1}$$

We have

$$\lim_{n\to\infty} B_n f(x) = f(x) \quad \text{uniformly on} \quad [0,1],$$

that is

$$\lim_{n\to\infty} \|B_n f - f\| = 0, \quad f \in C[0,1],$$

where $\|\circ\|$ stands for the supremum norm on the interval $[0,1]$.

Moreover, clearly

$$\|B_n f\| \le \|f\|, \quad f \in C[0,1],\ n \in \mathbb{N}_+.$$

Thus $\{B_n\}_{n=1}^{\infty}$ is a strong approximation process on $C[0,1]$ (see [13, Definition 12.0.1]).

Various estimates of the supremum norm of the error $B_n f(x) - f(x)$ were established. Some of the earliest ones were stated in terms of the so-called moduli of smoothness (or continuity). For example, Popoviciu [86] (or see [79, Theorem 1.6.1]) showed that

$$\|B_n f - f\| \le \frac{5}{4}\,\omega_1(f, n^{-1/2}).$$

Above $\omega_1(f,t)$ is the modulus of continuity of f, defined by

$$\omega_1(f,t) := \sup_{|x-y|\le t} |f(x)-f(y)|.$$

Since $B_n f$ interpolates f at the ends of the interval, one can expect that it approximates the function better in their neighbourhood. This is indeed so. The following estimate holds for $f \in AC^1_{loc}(0,1)$ such that $\varphi^2 f'' \in L_\infty[0,1]$, where $\varphi(x) := \sqrt{x(1-x)}$ (see e.g. [18, Chapter 10, § 7] or [23, Chapter 9])

$$\|B_n f - f\| \le \frac{c}{n}\,\|\varphi^2 f''\|, \quad n \in \mathbb{N}_+. \tag{0.2}$$

Here and henceforward c denotes absolute constants.

This estimate can be further generalized for any $f \in C[0,1]$ and $n \in \mathbb{N}_+$ in the form

$$\|B_n f - f\| \le c\,\omega_\varphi^2(f, n^{-1/2}), \tag{0.3}$$

where $\omega_\varphi^2(f,t)$ is the Ditzian-Totik modulus of smoothness of second order with varying step controlled by the weight $\varphi(x)$ in the sup-norm on $[0,1]$. It is defined by

$$\omega_\varphi^2(f,t) := \sup_{0<h\le t}\ \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|, \quad t>0.$$

Adell and G. Sangüesa [4] proved that (0.3) holds with $c=4$. Later on Gavrea, Gonska, Păltănea and Tachev [42] improved this estimate to $c=3$, then Păltănea [85, p. 96] to $c=5/2$ (or see [12, p. 183]).

It turns out that (0.2) and (0.3) cannot be improved. The converse to (0.3) is also valid—there holds (see [70] and [93])

$$\|B_n f - f\| \ge c\,\omega_\varphi^2(f, n^{-1/2}), \quad n \ge n_0, \tag{0.4}$$

where $n_0 \in \mathbb{N}_+$ is independent of f.

Earlier, Ditzian and Ivanov [22, Theorem 8.1] obtained a similar two-term converse inequality.

The last estimate implies that $B_n f$ cannot approximate f in the supremum norm on $[0,1]$ with a rate faster than $1/n$ unless $B_n f$ preserves f, that is, f is an algebraic polynomial of degree at most 1. This is known as saturation. It was first observed by Voronovskaya [99] (or see e.g. [18, Chapter 10, Theorem 3.1]). She proved that if $f \in C^2[0,1]$, then

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2}\, f''(x) \tag{0.5}$$

uniformly on $[0,1]$.

The Bernstein polynomial possesses another property. As it was established by Chlodowsky [17], Wigert [100] and Lorentz [78] (see e.g. [18, Chapter 10, Theorem 2.1], or [12, p. 232]), it not only approximates the function, but also its derivatives. More precisely, we have

$$\lim_{n\to\infty} (B_n f)^{(s)}(x) = f^{(s)}(x) \quad \text{uniformly on} \quad [0,1],$$

provided that $f \in C^s[0,1]$. That phenomenon is referred to as simultaneous approximation.

The main subject of the dissertation is to present estimates of the rate of this approximation. We prove both direct estimates and matching one- or two-term converse estimates, which show that the direct estimates are sharp. The estimates are given in the $\operatorname{ess\,sup}$-norm on $[0,1]$ with Jacobi weights. As a further application of those results we characterize the rate of the simultaneous approximation of the iterated Boolean sums of B_n and of two modifications of B_n, which are polynomials with integer coefficients. Finally, we investigate the rate of convergence in Voronovskaya's theorem (0.5).

The contents of the dissertation are organized as follows.

In Chapter 1 we collect the definitions and the basic properties of the standard K-functionals and moduli of smoothness that are used in problems of the type we consider. In later chapters, we introduce other K-functions as well. The latter are more straightforwardly related to the error of the approximation processes under considerations.

In Chapter 2 we establish inequalities between the weighted essential supremum norms of the derivatives of functions as well as between them and the norms of the values of the differential operators that are associated with the simultaneous approximation by the Bernstein operator and by its iterated Boolean sums. They enable us to relate the weighted supremum norm of the values of these differential operators to the weighted supremum norms of derivatives of the approximated function. They play a key role in the approach we adopt to investigating the rate of the simultaneous approximation of the aforementioned operators. The results presented in this chapter were published in [26, 27, 33].

In Chapter 3 we establish matching direct and two-term strong converse estimates of the rate of the simultaneous approximation by the Bernstein operator in the weighted essential supremum norm. We consider Jacobi weights as the range of the exponents is the broadest natural one. The obtained characterization of the rate of weighted simultaneous approximation by B_n is stated in terms of moduli of smoothness and K-functionals. Also, analogous results concerning the generalized Kantorovich operators are derived. The strong converse inequality is further improved to one that exactly matches the direct inequality for lower order derivatives and under additional restrictions on the weight exponents. Here we also obtain as auxiliary inequalities, but having merit of its own, Voronovskaya-, Bernstein-, and Zamansky-type inequalities. The material presented in this chapter was published in [27, 28].

In Chapter 4 most of the estimates of the previous chapter are extended to iterated Boolean sums of the Bernstein operator. They provide higher order approximation than B_n. Here we also apply results obtained in the previous chapter to derive a direct and a matching two-term converse inequality for the approximation by the iterated Boolean sums of the Bernstein operator itself. The results presented in this chapter were published in [25, 26, 27, 31, 32].

In Chapter 5 we deal with two modifications of the Bernstein polynomials, which provide approximation by algebraic polynomials with integer coefficients. We prove that they possess the property of simultaneous approximation as well and establish direct estimates of the error of that approximation in uniform norm by means of moduli of smoothness. These estimates are established under certain peculiar assumptions, but we show that they are also necessary. In addition, we prove a weak converse estimate for that approximation process. It is stated in terms of moduli of smoothness. In particular, it yields a big O-characterization of the rate of that approximation. We also show that the approximation process is saturated and identify its saturation rate and trivial class. The results presented in this chapter were published in [29, 30].

In Chapter 6 we characterize the rate of the convergence in the Voronovskaya's theorem (0.5). The characterization is given in terms of K-functionals and moduli of smoothness. The results presented in this chapter were published in [33], written jointly with I. Gadjev.

Acknowledgments. I am especially grateful to Professor Kamen Ivanov, Professor Dany Leviatan, Kiril Delev and the anonymous Referees for comments and corrections on the manuscripts. I am also indebted to Professor Gancho Tachev for providing me with papers and discussions on problems treated in the dissertation. I especially appreciate the support, encouragement and advice, which I have been receiving from Professor Kamen Ivanov throughout my research. Also, the support I received from Professor Geno Nikolov, Professor Nadejda Ribarska, Associate Professor Ivan Gadjev, Associate Professor Parvan Parvanov and Associate Professor Rumen Uluchev (in alphabetical order) has been very important for me.

Chapter 1

Basic K-functionals and moduli of smoothness

In this chapter we will formulate the definition of the classical moduli of smoothness and K-functionals that are extensively used in Approximation Theory as well as their weighted analogues that are most popular. In later chapters we will introduce other, K-functionals, which are more complicated, but more naturally related to the approximation processes we will study. We will characterize them by the former.

Generally, the K-functional between a normed space X with the norm $\| \circ \|_X$ and a semi-normed space $Y \subset X$ with the semi-norm $| \circ |_Y$ is defined for $f \in X$ and $t > 0$ by

$$K(f, t; X, Y) := \inf_{g \in Y} \{ \|f - g\|_X + t\, |g|_Y \}.$$

In Approximation Theory, the K-functionals are particularly useful to describe how well we can approximate elements in X by elements in Y.

The basic properties of the K-functional $K(f, t) := K(f, t; X, Y)$ are the following:

(a) As a function of $t \geq 0$, $K(f, t)$ is increasing, concave, continuous and bounded;

(b) $K(f, t)$ is sub-additive on t: $K(f, t_1 + t_2) \leq K(f, t_1) + K(f, t_2)$ and $K(f, t) \leq \dfrac{t}{s} K(f, s)$ if $0 < s < t$;

(c) $K(f + g, t) \leq K(f, t) + K(g, t)$.

As we indicated in the beginning, we will make use of various K-functionals to derive strong—exact or nearly exact, estimates of the weighted simultaneous approximation by the Bernstein operator and certain modifications of it. We will consider weighted (and, in particular, unweighted) K-functionals in weighted essential supremum (semi-)norms on the interval $[0, 1]$.

Before we define them, let us introduce some notations. We denote by $\|f\|$ the essential supremum norm of the function f on the interval $[0, 1]$. When the norm is taken on an interval $J \subseteq \mathbb{R}$, we will write $\|f\|_J$. As usual, $AC^k[a, b]$, $k \in \mathbb{N}_0$, stands for the set of all functions, which are k times differentiable and along with their derivatives up to order k are absolutely continuous on $[a, b]$; $AC^k_{loc}(0, 1)$ is the set of the functions, which are in $AC^k[a, b]$ for all $0 < a < b < 1$ (e.g. the monograph [76] contains the basic properties of these and related spaces). By c we will denote positive constants, not necessarily the same at each occurrence, which are independent of the functions involved in the inequalities but the weights.

The simplest K-functional we will use is the one between the spaces $X = L_\infty[0, 1]$ and $Y = W^m_\infty[0, 1]$. It is given by

$$K_m(f, t) := \inf_{g \in W^m_\infty[0,1]} \left\{ \|f - g\| + t\, \|g^{(m)}\| \right\}.$$

Here, as usual, $W^m_\infty[0, 1]$ is the Sobolev space

$$W^m_\infty[0, 1] := \{g \in AC^{m-1}[0, 1] : g^{(m)} \in L_\infty[0, 1]\}.$$

If $f \in C[0, 1]$, it is more natural to take the infimum on $C^m[0, 1]$ instead.

On the other hand, we are interested in function characteristics which are more directly related to the approximated function f. They are called moduli of smoothness. To define them, we first introduce the difference operator.

The forward difference of $f : [0, 1] \to \mathbb{R}$ with step $h > 0$ of order $m \in \mathbb{N}_+$ is given by

$$\overrightarrow{\Delta}^m_h f(x) = \begin{cases} \displaystyle\sum_{i=0}^{m} (-1)^i \binom{m}{i} f(x + (m - i)h), & x \in [0, 1 - mh], \\ 0, & x \in (1 - mh, 1]. \end{cases} \tag{1.1}$$

Similarly, the backward difference is given by

$$\overleftarrow{\Delta}^m_h f(x) = \begin{cases} \displaystyle\sum_{i=0}^{m} (-1)^i \binom{m}{i} f(x - ih), & x \in [mh, 1], \\ 0, & x \in [0, mh). \end{cases}$$

We will also make use of the symmetric difference, which is defined on $[0,1]$ by

$$\bar{\Delta}_h^m f(x) = \begin{cases} \sum_{i=0}^{m} (-1)^i \binom{m}{i} f\left(x + \left(\frac{m}{2} - i\right) h\right), & x \in \left[\frac{mh}{2}, 1 - \frac{mh}{2}\right], \\ 0, & \text{otherwise.} \end{cases}$$

The classical unweighted fixed-step modulus of smoothness of order m of $f \in L_\infty[0,1]$ is then defined for $t > 0$ by

$$\omega_m(f,t) := \sup_{0<h\le t} \|\overrightarrow{\Delta}_h^m f\|. \tag{1.2}$$

Similarly, we can use backward or symmetric differences.

Its basic properties are the following (see e.g. [18, Ch. 2,§§ 7-8]):

(a) $\omega_m(f+g,t) \le \omega_m(f,t) + \omega_m(g,t)$;

(b) $\omega_m(cf,t) = |c|\omega_m(f,t), \quad c \in \mathbb{R}$;

(c) $\omega_m(f,\lambda t) \le (\lambda+1)^m \omega_m(f,t), \quad \lambda > 0$;

(d) $\omega_m(f,t) \le t^m \|f^{(m)}\|, \quad f \in W_\infty^m[0,1]$;

(e) If $\omega_m(f,t) = O(t^m)$, then $f \in W_\infty^m[0,1]$;

(f) $\omega_m(f,t) = 0$ for all t if f is equal a.e. with regard to the Lebesgue measure to an algebraic polynomial of degree at most $m-1$; conversely, if $\omega_m(f,t) = o(t^m)$, then f is equal a.e. to an algebraic polynomial of degree at most $m-1$;

(g) $\lim_{t\to 0} \omega_m(f,t) = 0$ if and only if f is equal a.e. to a continuous function on $[0,1]$.

Johnen [64] (see also [65]) showed that $K_m(f,t^m)$ and $\omega_m(f,t)$ are equivalent, that is, there exists $c > 0$ such that for all $f \in L_\infty[0,1]$ and $t > 0$

$$c^{-1}\omega_m(f,t) \le K_m(f,t^m) \le c\,\omega_m(f,t). \tag{1.3}$$

We will denote the above relation by $K_m(f,t^m) \sim \omega_m(f,t)$. More generally, we say that $\Phi(f,t)$ and $\Psi(f,t)$ are equivalent and write $\Phi(f,t) \sim \Psi(f,t)$ if there exists a positive constant c such that $c^{-1}\Phi(f,t) \le \Psi(f,t) \le c\,\Phi(f,t)$ for all f and t under consideration.

We will study approximation processes whose approximation rate is better at the ends of the interval $[0,1]$. Thus, in order to get, precise error estimates, we will need K-functionals and moduli of smoothness, which take this into account; moreover, we will be interested in approximation in weighted L_∞-spaces of functions on $[0,1]$. The weight is given by (2.2), that is,

$$w(x) := w(\gamma_0,\gamma_1;x) := x^{\gamma_0}(1-x)^{\gamma_1}, \quad \gamma_0,\gamma_1 \ge 0.$$

We will make use of the K-functionals:

$$K_m(f,t)_w := \inf_{g\in AC_{loc}^{m-1}(0,1)} \left\{\|w(f-g)\| + t\|wg^{(m)}\|\right\} \tag{1.4}$$

and

$$K_{m,\varphi}(f,t)_w := \inf_{g\in AC_{loc}^{m-1}(0,1)} \left\{\|w(f-g)\| + t\|w\varphi^m g^{(m)}\|\right\}, \tag{1.5}$$

where $\varphi(x) := \sqrt{x(1-x)}$. For the unweighted case $w=1$ we set

$$K_{m,\varphi}(f,t) := K_{m,\varphi}(f,t)_1.$$

Similarly to (1.3), we have

$$K_m(f,t^m)_w \sim \omega_m(f,t)_w, \quad 0 < t \le 1, \tag{1.6}$$

where the weighted modulus of smoothness $\omega_m(f,t)_w$ is defined by

$$\omega_m(f,t)_w := \sup_{0<h\le t} \|w\overrightarrow{\Delta}_h^m f\|_{[0,3/4]} + \sup_{0<h\le t} \|w\overleftarrow{\Delta}_h^m f\|_{[1/4,1]}. \tag{1.7}$$

If $\gamma_1 = 0$, we can use the simpler form

$$\omega_m(f,t)_w := \sup_{0<h\le t} \|w\overrightarrow{\Delta}_h^m f\|,$$

and similarly in the case $\gamma_0 = 0$.

In the case $w=1$ we will rather use $\omega_m(f,t)$, that is, we set

$$\omega_m(f,t)_1 := \omega_m(f,t).$$

One generalization of the classical moduli, which is equivalent to the K-functional $K_{m,\varphi}(f,t^m)$, was introduced by Ditzian and Totik [23, (2.1.2)]. In the unweighted case $w = 1$ it is given by

$$\omega_\varphi^m(f,t) := \sup_{0<h\le t} \|\bar{\Delta}_{h\varphi}^m f\|.$$

Similarly, we can use forward or backward differences (see [23, Section 3.2]).

The generalization of that modulus of smoothness to the weighted case is more complicated. For $\gamma_0, \gamma_1 > 0$ it is defined by (see [23, Appendix B])

$$\begin{aligned}(1.8)\quad \omega_\varphi^m(f,t)_w := \sup_{0<h\le t} \|w\bar{\Delta}_{h\varphi}^m f\|_{[m^2t^2,1-m^2t^2]} &+ \sup_{0<h\le m^2t^2} \|w\overrightarrow{\Delta}_h^m f\|_{[0,12m^2t^2]} \\ &+ \sup_{0<h\le m^2t^2} \|w\overleftarrow{\Delta}_h^m f\|_{[1-12m^2t^2,1]},\end{aligned}$$

where $0 < t \le 1/(m\sqrt{2})$.

When either γ_0 or γ_1 are equal to 0, its definition is modified as in the case of $\omega_m(f,t)_w$.

We set

$$\omega_\varphi^m(f,t)_1 := \omega_\varphi^m(f,t).$$

The weighted Ditzian-Totik modulus of smoothness possesses similar properties like the classical one, which were often established for $0 < t \le t_0$ with $t_0 > 0$ independent of f (see [23, Chapters 4 and 6] and [24]). In particular, as it was shown in [23, Theorems 2.1.1 and 6.1.1], there exists t_0 such that

$$(1.9)\qquad K_{m,\varphi}(f,t^m)_w \sim \omega_\varphi^m(f,t)_w, \quad 0 < t \le t_0.$$

It was shown in [72, Theorem 2.7] that we can take $t_0 = 2/r$. A smaller value of t_0 was given in [18, Chapter 6, Theorem 6.2].

Earlier, weighted moduli of smoothness, which are equivalent to K-functionals such as $K_{m,\varphi}(f,t)$ with general weight φ, were introduced by Ivanov [56, 57, 58, 59, 60]. A modification of the Ditzian-Totik modulus of smoothness was considered by Dzyadyk, Kopotun, Leviatan and Shevchuk [38, 71, 72, 73, 74, 75]. K-functionals such as $K_{m,\varphi}(f,t)_w$ again with general φ were characterized by the classical moduli of smoothness but taken on certain linear transform of the function in [34, 36, 37]. All aforementioned results were established in L_p-spaces with $1 \le p \le \infty$ and spaces of continuous functions.

Chapter 2

Embedding inequalities

We will extensively use embedding inequalities, that is, inequalities for the norms of intermediate derivatives, in order to simplify estimates or show that certain integrals are well-defined. Such inequalities are typical for that setting; see e.g. [8, Lemmas 2, 3 and 4], [23, p. 135], [53, Lemma 2] and [54, pp. 127-128].

First, we recall the well-known inequality (e.g. [18, Chapter 2, Theorem 5.6])

$$\|f^{(j)}\|_J \leq c\left(\|f\|_J + \|f^{(m)}\|_J\right), \quad j = 0, \dots, m, \tag{2.1}$$

where J is an interval on the real line and $f \in W_\infty^m(J)$.

Next, we will establish a generalization of [23, p. 135, (a) and (b)] by means of an argument similar to the one used there. We set

$$w(x) := w(\gamma_0, \gamma_1; x) := x^{\gamma_0}(1-x)^{\gamma_1}, \quad \gamma_0, \gamma_1 \geq 0. \tag{2.2}$$

Proposition 2.1. *Let $j, m \in \mathbb{N}_0$ as $j < m$. Let $w_\mu := w(\gamma_{\mu,0}, \gamma_{\mu,1})$ be given by (2.2) with $\gamma_{\mu,0}, \gamma_{\mu,1} > 0$ for $\mu = 1, 2$ and let $\gamma_{2,\nu} \leq \gamma_{1,\nu} + m - j$ for $\nu = 0, 1$. Let also $g \in AC_{loc}^{m-1}(0,1)$ be such that $w_2 g^{(m)} \in L_\infty[0,1]$. Then*

$$\|w_1 g^{(j)}\| \leq c\left(\|g\|_{[1/4,3/4]} + \|w_2 g^{(m)}\|\right).$$

The value of the constant c is independent of g.

Proof. Let $x \in [0, 1/2]$. By Taylor's formula we have

$$g^{(j)}(x) = \sum_{i=0}^{m-j-1} \frac{g^{(i+j)}(1/2)}{i!} \left(x - \frac{1}{2}\right)^i + \frac{1}{(m-j-1)!} \int_{1/2}^{x} (x-u)^{m-j-1} g^{(m)}(u)\, du.$$

Consequently, for $x \in [0, 1/2]$ we have

$$x^{\gamma_{1,0}} |g^{(j)}(x)| \leq x^{\gamma_{1,0}} \sum_{i=0}^{m-j-1} \left| g^{(i+j)}\left(\frac{1}{2}\right)\right| + \sum_{k=0}^{m-j-1} \psi_k(x), \tag{2.3}$$

where we have set

$$\psi_k(x) := x^{k+\gamma_{1,0}} \int_x^{1/2} u^{m-j-k-1} |g^{(m)}(u)|\, du, \quad k = 0, \ldots, m-j-1.$$

Clearly, by (2.1), we have

$$\left| g^{(j)}\left(\frac{1}{2}\right)\right| \leq c \left(\|g\|_{[1/4,1/2]} + \|g^{(m)}\|_{[1/4,1/2]} \right), \quad j = 0, \ldots, m-1. \tag{2.4}$$

We set $\chi(x) := x$. We get for $x \in (0, 1/2]$

$$\begin{aligned} \psi_k(x) &\leq x^{k+\gamma_{1,0}} \int_x^{1/2} u^{-\gamma_{1,0}-k-1} du \, \|\chi^{m+\gamma_{1,0}-j} g^{(m)}\|_{[0,1/2]} \\ &\leq \frac{1}{\gamma_{1,0}} \|\chi^{\gamma_{2,0}} g^{(m)}\|_{[0,1/2]}, \quad k = 0, \ldots, m-j-1, \end{aligned} \tag{2.5}$$

as for the second estimate above we have used that $\gamma_{2,0} \leq \gamma_{1,0} + m - j$.

Now, (2.3)-(2.5) imply the inequality

$$\|\chi^{\gamma_{1,0}} g^{(j)}\|_{[0,1/2]} \leq c \left(\|g\|_{[1/4,1/2]} + \|\chi^{\gamma_{2,0}} g^{(m)}\|_{[0,1/2]} \right). \tag{2.6}$$

By symmetry, we get

$$\|(1-\chi)^{\gamma_{1,1}} g^{(j)}\|_{[1/2,1]} \leq c \left(\|g\|_{[1/2,3/4]} + \|(1-\chi)^{\gamma_{2,1}} g^{(m)}\|_{[1/2,1]} \right). \tag{2.7}$$

The last two estimates yield the assertion of the proposition. □

For an easier reference, we collect in the next propositions a number of particular corollaries of Proposition 2.1 and (2.1), which we will use later on.

Proposition 2.2. *Let* $m \in \mathbb{N}_+$, $w := w(\gamma_0, \gamma_1)$ *be given by* (2.2) *with* $\gamma_0, \gamma_1 \geq 0$, $\varphi(x) := \sqrt{x(1-x)}$ *and* $g \in C[0,1]$ *be such that* $g \in AC_{loc}^{m-1}(0,1)$.

(a) If $w\varphi^{2m} g^{(m)} \in L_\infty[0,1]$, *then*

$$\|w\varphi^{2j} g^{(j)}\| \leq c\left(\|wg\| + \|w\varphi^{2m} g^{(m)}\|\right), \quad j = 0, \dots, m.$$

(b) If $w\varphi^{m} g^{(m)} \in L_\infty[0,1]$, *then*

$$\|w\varphi^{j} g^{(j)}\| \leq c\left(\|wg\| + \|w\varphi^{m} g^{(m)}\|\right), \quad j = 0, \dots, m.$$

(c) If $wg^{(m)} \in L_\infty[0,1]$, *then*

$$\|wg^{(j)}\| \leq c\left(\|wg\| + \|wg^{(m)}\|\right), \quad j = 0, \dots, m.$$

The value of the constant c *is independent of* g.

Proof. We need to consider $j = 1, \dots, m-1$, $m \geq 2$.

Assertion (a) follows from Proposition 2.1 with $w_1 = w\varphi^{2j}$, $w_2 = w\varphi^{2m}$.

Likewise, to get (b) we apply Proposition 2.1 with $w_1 = w\varphi^j$ and $w_2 = w\varphi^m$.

Assertion (c) with $\gamma_0, \gamma_1 > 0$ follows from Proposition 2.1 with $w_1 = w_2 = w$, whereas for $\gamma_0 = \gamma_1 = 0$ it reduces to (2.1). In order to treat the cases when one of the weight exponents is positive and the other is 0, we just need to estimate $|x^{\gamma_0} g^{(j)}(x)|$ on $[0, 1/2]$, and $|(1-x)^{\gamma_1} g^{(j)}(x)|$ on $[1/2, 1]$ by Proposition 2.1, or (2.1) depending on whether $\gamma_i > 0$ or $\gamma_i = 0$. □

Proposition 2.3. *Let* $r \in \mathbb{N}_+$, $\varphi(x) := \sqrt{x(1-x)}$ *and* $g \in C[0,1]$ *be such that* $g \in AC_{loc}^{2r-1}(0,1)$ *and* $\varphi^{2r} g^{(2r)} \in L_\infty[0,1]$.

(a) If $r \geq 2$, *then* $\varphi^2 g'' \in L_\infty[0,1]$ *and*

$$\|\varphi^2 g^{(i)}\| \leq c\left(\|\varphi^2 g''\| + \|\varphi^{2r} g^{(2r)}\|\right), \quad i = 3, \dots, r+1. \tag{2.8}$$

(b) If $r \geq 3$, *then*

$$\|\varphi^{2i} g^{(i+r)}\| \leq c\left(\|\varphi^2 g''\| + \|\varphi^{2r} g^{(2r)}\|\right), \quad i = 2, \dots, r-1.$$

The value of the constant c is independent of g.

Proof. To prove (a), we first apply Proposition 2.2(b) with $w = 1$, $j = 2$ and $m = 2r$. Then we use Proposition 2.1 with g'' in place of g, $w_1 = \varphi^2$ and $w_2 = \varphi^{2r}$, $m = 2r - 2$ and $j = i - 2$ to get (2.8).

Similarly, (b) follows from Proposition 2.1 with g'' in place of g, $w_1 = \varphi^{2i}$, $w_2 = \varphi^{2r}$, $m = 2r - 2$ and $j = i + r - 2$. □

Proposition 2.4. *Let $\varphi(x) := \sqrt{x(1-x)}$ and $g \in C[0,1]$ be such that $g \in AC^5_{loc}(0,1)$ and $\varphi^6 g^{(6)} \in L_\infty[0,1]$. Then $\varphi^2 g^{(4)}, \varphi^4 g^{(5)} \in L_\infty(\varphi)[0,1]$ too, as moreover*

$$\|\varphi^2 g^{(4)}\| \le c\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right) \tag{2.9}$$

and

$$\|\varphi^4 g^{(5)}\| \le c\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right). \tag{2.10}$$

The value of the constant c is independent of g.

Proof. The assertion follows from Proposition 2.1 with $g^{(4)}$ in place of g, $w_1 = \varphi^{2(j+1)}$, $w_2 = \varphi^6$, $m = 2$ and $j = 0, 1$. □

We proceed to several embedding inequalities, which will enable us to transfer estimates in terms of the semi-norms $\|w\varphi^{2i} g^{(j)}\|$ to such in terms of the more complicated one $\|w(D^r g)^{(s)}\|$, where $Dg := \varphi^2 g''$ and $\varphi(x) := \sqrt{x(1-x)}$. Their proof is based on the following Taylor-type formulas.

Lemma 2.5. *Let $s \in \mathbb{N}_+$ and $g \in AC^{s+1}[0,1]$.*

(a) *If $s \ge 2$, then*

$$g^{(s)}(x) = \int_0^1 \mathcal{K}_s(x,u)\,(Dg)^{(s)}(u)\,du, \quad x \in [0,1],$$

where

$$\mathcal{K}_s(x,u) := -\frac{1}{s-1}\begin{cases} \left(\dfrac{u}{x}\right)^{s-1}, & u \le x, \\ \left(\dfrac{1-u}{1-x}\right)^{s-1}, & x \le u. \end{cases}$$

(b) If $s \geq 1$, then

$$g^{(s+1)}(x) = \int_0^1 \mathcal{L}_s(x,u)\,(Dg)^{(s)}(u)\,du, \quad x \in [0,1],$$

where

$$\mathcal{L}_s(x,u) := \begin{cases} \dfrac{u^{s-1}}{x^s}, & u \leq x, \\ -\dfrac{(1-u)^{s-1}}{(1-x)^s}, & x < u. \end{cases}$$

Proof. Assertion (a) is verified by integration by parts. More precisely, we expand $(Dg)^{(s)}(u)$ to get

$$(Dg)^{(s)}(u) = -s(s-1)g^{(s)}(u) + s(1-2u)g^{(s+1)}(u) + u(1-u)g^{(s+2)}(u). \tag{2.11}$$

Next, we evaluate the integral

$$\int_0^1 \mathcal{K}_s(x,u)\,\left[s(1-2u)g^{(s+1)}(u) + u(1-u)g^{(s+2)}(u)\right]\,du.$$

We get by integration by parts

$$\begin{aligned} &\int_0^x u^{s-1}\left[s(1-2u)g^{(s+1)}(u) + u(1-u)g^{(s+2)}(u)\right]\,du \\ &\quad = x^s(1-x)g^{(s+1)}(x) - (s-1)\int_0^x u^s g^{(s+1)}(u)\,du \\ &\quad = x^s(1-x)g^{(s+1)}(x) - (s-1)x^s g^{(s)}(x) + s(s-1)\int_0^x u^{s-1}g^{(s)}(u)\,du \end{aligned}$$

and

$$\begin{aligned} &\int_x^1 (1-u)^{s-1}\left[s(1-2u)g^{(s+1)}(u) + u(1-u)g^{(s+2)}(u)\right]\,du \\ &\quad = -x(1-x)^s g^{(s+1)}(x) + (s-1)\int_x^1 (1-u)^s g^{(s+1)}(u)\,du \\ &\quad = -x(1-x)^s g^{(s+1)}(x) - (s-1)(1-x)^s g^{(s)}(x) \\ &\qquad + s(s-1)\int_x^1 (1-u)^{s-1}g^{(s)}(u)\,du. \end{aligned}$$

Consequently,

$$\int_0^1 \mathcal{K}_s(x,u)\ \left[s(1-2u)g^{(s+1)}(u)+u(1-u)g^{(s+2)}(u)\right]\ du$$
$$= g^{(s)}(x) + s(s-1)\int_0^1 \mathcal{K}_s(x,u)\, g^{(s)}(u)\, du,$$

which, in view of (2.11), completes the proof of (a).

Assertion (b) for $s \geq 2$ is directly verified by differentiating the formula in (a). If $s = 1$, we just have

$$\frac{1}{x}\int_0^x (Dg)'(u)\, du = \frac{Dg(x)}{x} = (1-x)g''(x)$$

and

$$-\frac{1}{1-x}\int_x^1 (Dg)'(u)\, du = \frac{Dg(x)}{1-x} = xg''(x).$$

Hence (b) for $s = 1$ follows. □

Proposition 2.6. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \leq \gamma_0, \gamma_1 < s$. Set $j_s := 1$ if $s = 1$, and $j_s := 0$ otherwise. Then for all $g \in AC^{2r+s-1}[0,1]$ there hold*

$$\|wg^{(j+s)}\| \leq c\,\|w(D^r g)^{(s)}\|, \quad j = j_s, \dots, r, \tag{2.12}$$

and

$$\|w\varphi^{2r} g^{(2r+s)}\| \leq c\,\|w(D^r g)^{(s)}\|. \tag{2.13}$$

The value of the constant c is independent of g.

Proof. We will establish the assertions by induction on r. In order to verify them for $r = 1$ we apply Lemma 2.5 and estimate the integrals in the formulas in there.

We set

$$\Psi_1(x) := x^{-s+1}\int_0^x u^{s-1}(Dg)^{(s)}(u)\, du,$$
$$\Psi_2(x) := x^{-s}\int_0^x u^{s-1}(Dg)^{(s)}(u)\, du.$$

and, to recall, $\chi(x) := x$. Clearly,

$$\|w\Psi_1\| \leq \|w\Psi_2\|. \tag{2.14}$$

We will estimate the sup-norm of $w\Psi_2$ separately on the intervals $[0,1/2]$ and $[1/2,1]$.

For the estimate on $[0,1/2]$ we use that $s-\gamma_0 > 0$ to arrive for $x \in (0,1/2]$ at

$$\begin{aligned}|x^{\gamma_0}\Psi_2(x)| &\leq x^{\gamma_0-s}\int_0^x u^{s-\gamma_0-1}du\,\|\chi^{\gamma_0}(Dg)^{(s)}\|_{[0,1/2]}\\ &= \frac{1}{s-\gamma_0}\,\|\chi^{\gamma_0}(Dg)^{(s)}\|_{[0,1/2]}.\end{aligned}$$

Consequently,

$$\|\chi^{\gamma_0}\Psi_2\|_{[0,1/2]} \leq c\,\|w(Dg)^{(s)}\|_{[0,1/2]}. \tag{2.15}$$

Let $x \in [1/2,1)$. Since $\gamma_1 \geq 0$, then $(1-u)^{-\gamma_1} \leq (1-x)^{-\gamma_1}$ for $u \in [0,x]$. Therefore,

$$\begin{aligned}|(1-x)^{\gamma_1}\Psi_2(x)| &\leq (1-x)^{\gamma_1}\int_0^x u^{s-\gamma_0-1}(1-u)^{-\gamma_1}du\,\|w(Dg)^{(s)}\|\\ &\leq \int_0^x u^{s-\gamma_0-1}du\|w(Dg)^{(s)}\|\\ &= \frac{1}{s-\gamma_0}\,\|w(Dg)^{(s)}\|.\end{aligned}$$

Thus

$$\|(1-\chi)^{\gamma_1}\Psi_2\|_{[1/2,1]} \leq c\,\|\chi^{\gamma_0}(Dg)^{(s)}\|. \tag{2.16}$$

Inequalities (2.14)-(2.16) imply

$$\|w\Psi_1\| \leq \|w\Psi_2\| \leq c\,\|w(Dg)^{(s)}\|.$$

By symmetry, we get the analogue of the last estimates for the terms

$$(1-x)^{-s+i}\int_x^1 (1-u)^{s-1}(Dg)^{(s)}(u)\,du, \quad i=0,1.$$

Thus, we establish that

$$\left\| w \int_0^1 \mathcal{K}_s(\circ, u)\, (Dg)^{(s)}(u)\, du \right\| \le c\, \| w (Dg)^{(s)} \|, \quad s \ge 2,$$

and

$$\left\| w \int_0^1 \mathcal{L}_s(\circ, u)\, (Dg)^{(s)}(u)\, du \right\| \le c\, \| w (Dg)^{(s)} \|, \quad s \ge 1.$$

Now, we complete the proof of inequalities (2.12) for $r = 1$ by Lemma 2.5. Then (2.13) follows from (2.11). The proposition is established for $r = 1$.

We proceed by induction on r, so let us assume that (2.12)-(2.13) are valid for some r. Then applying (2.12) with Dg in place of g, we arrive at

$$\| w (Dg)^{(j+s)} \| \le c\, \| w (D^{r+1} g)^{(s)} \|, \quad j = j_s, \ldots, r. \tag{2.17}$$

On the other hand, by what we have already shown in the case $r = 1$, we have

$$\| w g^{(j'+j+s)} \| \le c\, \| w (Dg)^{(j+s)} \|, \quad j' = 0, 1. \tag{2.18}$$

Let us note that $j_{j+s} = 0$ because $j + s \ge 2$ for $j \ge j_s$.

Now, (2.17)-(2.18) yield

$$\| w g^{(j+s)} \| \le c\, \| w (D^{r+1} g)^{(s)} \|, \quad j = j_s, \ldots, r + 1.$$

Thus (2.12) is verified for $r + 1$ in place of r.

To complete the proof of (2.13), we need to show that

$$\| w \varphi^{2r+2} g^{(2r+s+2)} \| \le c\, \| w (D^{r+1} g)^{(s)} \|.$$

In view of (2.11) with $2r+s$ in place of s, that will follow from the inequalities

$$\| w \varphi^{2r} (Dg)^{(2r+s)} \| \le c\, \| w (D^{r+1} g)^{(s)} \| \tag{2.19}$$

and

$$\| w \varphi^{2r} g^{(j+2r+s)} \| \le c\, \| w (D^{r+1} g)^{(s)} \|, \quad j = 0, 1. \tag{2.20}$$

Inequality (2.19) follows from (2.13) with Dg in place of g. To establish (2.20) we first apply (2.12) with $r = 1$, $w\varphi^{2r}$ in place of w, and $2r + s$ in place of s and thus get

$$\| w \varphi^{2r} g^{(j+2r+s)} \| \le c\, \| w \varphi^{2r} (Dg)^{(2r+s)} \|, \quad j = 0, 1. \tag{2.21}$$

Inequalities (2.19) and (2.21) imply (2.20). $\square$

The last inequalities are due to Gonska and Zhou [53, (1), (2) and (4)]. We include them for an easier reference.

Proposition 2.7. *For $f \in C^{2r}[0,1]$ there hold:*

(a) $\|D^r f\| \le c\left(\|f\| + \|\varphi^{2r} f^{(2r)}\|\right);$

(b) $\|\varphi^{2r} f^{(2r)}\| \le c\,\|D^r f\|;$

(c) $\|D^j f\| \le c\,\|D^r f\|, \quad j = 1, \dots, r.$

The value of the constant c is independent of f.

Actually, Gonska and Zhou [53] stated the assertions above only for algebraic polynomials since that was what they needed, but the same considerations verify them for all functions in $C^{2r}[0,1]$.

Remark 2.8. There is an elegant Taylor-type formula through which the embedding inequalities in Proposition 2.7 can be verified.

Let $f \in AC^1_{loc}(0,1)$ be such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0} \varphi^2(x) f'(x) = \lim_{x\to 1} \varphi^2(x) f'(x) = 0.$$

Then

$$f(x) = \int_0^1 [xu - \min\{x,u\}] f''(u)\,du, \quad x \in [0,1]. \tag{2.22}$$

This formula is verified by integration by parts.

If $f \in AC^1_{loc}(0,1)$ is such that $\varphi^2 f'' \in L_\infty[0,1]$, then

$$f'(x) = f'\left(\frac{1}{2}\right) + \int_{1/2}^{x} f''(t)\,dt, \quad x \in (0,1/2];$$

hence

$$|xf'(x)| \le x\left|f'\left(\frac{1}{2}\right)\right| + 2x|\ln x|\,\|\varphi^2 f''\|, \quad x \in (0,1/2],$$

and we arrive at $\lim_{x\to 0} \varphi^2(x) f'(x) = 0$.

By symmetry, we get $\lim_{x\to 1} \varphi^2(x) f'(x) = 0$ as well.

Thus, if $f \in C[0,1]$ is such that $f(0) = f(1) = 0$, $f \in AC^1_{loc}(0,1)$ and $\varphi^2 f'' \in L_\infty[0,1]$, then formula (2.22) is applicable and yields

$$|f(x)| \le \int_0^1 \frac{\min\{x,u\} - xu}{\varphi^2(u)}\, du\, \|Df\|, \quad x \in [0,1].$$

Hence, taking into account that,

$$\int_0^1 \frac{\min\{x,u\} - xu}{\varphi^2(u)}\, du = -x\log x - (1-x)\log(1-x) \le \log 2, \quad x \in (0,1),$$

we arrive at the inequality

$$\|f\| \le \log 2\, \|Df\|.$$

Iterating it, we get Proposition 2.7(c) for $f \in C^{2r-2}[0,1]$ such that $f^{(2r-2)} \in AC^1_{loc}(0,1)$.

Formula (2.22) can be extended. Let $r \in \mathbb{N}_+$ and $f \in C^{2r-2}[0,1]$ be such that $f^{(2r-2)} \in AC^1_{loc}(0,1)$ and $f(0) = f(1) = 0$. Then

$$f(x) = \int_0^1 K_r(x,u) D^r f(u)\, du, \quad x \in [0,1], \tag{2.23}$$

where the kernel is defined by the recurrence relation

$$K_1(x,u) := \frac{xu - \min\{x,u\}}{\varphi^2(u)}, \quad K_{j+1}(x,u) := \int_0^1 K_j(x,v) K_1(v,u)\, dv.$$

This kernel possesses various properties. They include the symmetries

$$K_j(x,u) = K_j(u,x), \quad K_j(x,u) = K_j(1-x, 1-u)$$

and the relation

$$\varphi^2(x)\, \frac{\partial^2 K_{j+1}}{\partial x^2}(x,u) = K_j(x,u).$$

However, its explicit form is quite complicated even for $j = 2$. So it is easier to verify Proposition 2.7 (a) and (b) by the method used by H. Gonska and X.-l. Zhou rather than by (2.23).

Chapter 3

Weighted simultaneous approximation by the Bernstein operator

3.1 Background

As it follows from a result due to Voronovskaya [99], the Bernstein operator (0.1) cannot approximate a function with a rate faster than n^{-1} unless it preserves it. To recall, Voronovskaya's classic result states (see e.g. [18, Chapter 10, Theorems 3.1 and 5.1])

$$\lim_{n\to\infty} n\,(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x) \quad \text{uniformly on } [0,1] \tag{3.1}$$

for $f \in C^2[0,1]$.

This is known as saturation of an approximation process (see [13, Definition 12.0.2], or [18, p. 336]). Thus the sequence of approximating operators $\{B_n\}_{n=1}^{\infty}$ is saturated, as its saturation rate is n^{-1}.

Relation (3.1) shows that the differential operator which describes the rate of approximation of B_n (up to a constant multiple) is $Df(x) := \varphi^2(x) f''(x)$ with $\varphi(x) := \sqrt{x(1-x)}$. A quantitative description of this rate is given by (see (0.3)-(0.4))

$$\|B_n f - f\| \sim \omega_\varphi^2(f, n^{-1/2}), \quad n \ge n_0, \tag{3.2}$$

with some $n_0 \in \mathbb{N}_+$, which is independent of $f \in C[0,1]$.

It is known that the derivatives of the Bernstein polynomial of a smooth function approximate the corresponding derivatives of the function (see [18, Chapter 10, Theorem 2.1]). López-Moreno, Martínez-Moreno and Muñoz-Delgado [77] and Floater [40] extended (3.1), showing that for $f \in C^{s+2}[0,1]$ we have

$$(3.3) \qquad \lim_{n\to\infty} n\left((B_n f(x))^{(s)} - f^{(s)}(x)\right) = \frac{1}{2}(Df(x))^{(s)} \quad \text{uniformly on } [0,1].$$

Hence the differential operator that describes the simultaneous approximation by B_n is $(d/dx)^s D$. Results about the rate of convergence in (3.3) were established in [46, 47, 52].

The first quantitative result for the simultaneous approximation by means of B_n was given by Popoviciu [87] (or see [12, p. 232]). It states

$$\|(B_n f)^{(s)} - f^{(s)}\| \le \frac{3+2\sqrt{s}}{2}\,\omega_1\left(f^{(s)}, \frac{1}{\sqrt{n-s}}\right) + \frac{s(s-1)}{2n}\,\|f^{(s)}\|, \quad n > s.$$

Numerous improvements of this estimate have been established since then (see [6, 40, 44, 67, 69, 92] and [12, Section 4.6]). Approximation in Hölder norms was considered in [49, 50], and in the uniform and the Hausdorff metrics in [88, 89].

To the best of my knowledge, all but one estimate established previously (see Remark 3.6 below) use the classical fixed-step modulus of smoothness of first and second order. The estimates we will prove use the Ditzian-Totik modulus and take into account that the approximation is better near the ends of the interval, besides we consider approximation generally in weighted spaces. Moreover, we also establish matching converse inequalities, which show that the direct estimates are sharp. A point-wise direct inequality, which demonstrates that the approximation improves near the ends of the interval was established by Jiang [61] (or see [12, p. 237]), who proved for the first derivative that

$$|(B_n f(x))' - f'(x)| \le \frac{13}{4}\,\omega_2\left(f', \frac{2\varphi(x)}{\sqrt{n-1}}\right) + \omega_1(f', n^{-1}).$$

We consider simultaneous approximation by B_n with the Jacobi weights (2.2):

$$w(x) := w(\gamma_0, \gamma_1; x) := x^{\gamma_0}(1-x)^{\gamma_1}, \quad x \in [0,1],$$

where $\gamma_0, \gamma_1 \ge 0$.

To characterize the rate of the simultaneous approximation by B_n, we will use the K-functional

$$K_s^D(f,t)_w := \inf_{g\in C^{s+2}[0,1]} \left\{\|w(f-g^{(s)})\| + t\|w(Dg)^{(s)}\|\right\}.$$

However, we will not directly relate this K-functional to the norm of the error $\|w(B_nf)^{(s)} - f^{(s)})\|$. It is much easier to establish estimates in terms of the norms of the components into which $(Dg)^{(s)}$ expands and then making use of certain embedding inequalities we can get to estimates in terms of $\|w(Dg)^{(s)}\|$. That will allow us not only to avoid some technical difficulties, but also to derive characterizations of $\|w(B_nf - f)^{(s)}\|$ both by the more natural K-functional $K_s^D(f,t)_w$ and by the more useful ones $K_{2,\varphi}(f,t)_w$ and $K_1(f,t)_w$.

3.2 One elementary result

We begin with one direct estimate of the sup-norm of the error of simultaneous approximation by B_n, which is quite straightforward to get. We include it here because it is based on a neat representation of the derivatives of the Bernstein polynomial. Its shortcoming, however, is an additional factor, which appears with this derivative and depends on n.

In all our considerations, c denotes a positive constant, not necessarily the same at each occurrence, which is independent of the approximated function and the order of the approximation operator.

Theorem 3.1. *Let $s \in \mathbb{N}_+$. Then there exists $n_0 \in \mathbb{N}_+$ with $n_0 > s$ such that for $f \in C^s[0,1]$ and $n \geq n_0$ there holds*

$$\left\| \frac{n^s(n-s)!}{n!}(B_nf)^{(s)} - f^{(s)} \right\| \leq c\left(\omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1})\right).$$

The value of the constant c is independent of f and n.

Proof. It is known (see [83] or [18, Chapter 10, (2.3)], [23, p. 125]) that for $n \geq s$

$$(B_nf)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}(x). \tag{3.4}$$

We set

$$\widetilde{D}_{s,n}f(x) := n^s \overrightarrow{\Delta}^s_{1/n} f\left(\frac{n-s}{n}\,x\right), \quad x \in [0,1].$$

Then by (3.4)

$$\frac{n^s(n-s)!}{n!}\,(B_n f)^{(s)}(x) = B_{n-s}(\widetilde{D}_{s,n}f)(x), \quad x \in [0,1]. \tag{3.5}$$

Hence

$$\begin{aligned}\left\|\frac{n^s(n-s)!}{n!}\,(B_n f)^{(s)} - f^{(s)}\right\| &= \|B_{n-s}(\widetilde{D}_{s,n}f) - f^{(s)}\|\\ &\le \|B_{n-s}(\widetilde{D}_{s,n}f) - B_{n-s}(f^{(s)})\| + \|B_{n-s}(f^{(s)}) - f^{(s)}\|\\ &\le \|\widetilde{D}_{s,n}f - f^{(s)}\| + \|B_{n-s}(f^{(s)}) - f^{(s)}\|.\end{aligned}$$

By virtue of (0.3), we have

$$\|B_{n-s}(f^{(s)}) - f^{(s)}\| \le c\,\omega^2_\varphi(f^{(s)}, n^{-1/2}), \quad n \ge n_0, \tag{3.6}$$

where $n_0 \in \mathbb{N}_+$ is independent of f.

To estimate $\|\widetilde{D}_{s,n}f - f^{(s)}\|$ we use that the finite forward difference of order s of $f \in AC^{s-1}[a,b]$ can be represented in the integral form

$$\overrightarrow{\Delta}^s_h f(x) = h^{s-1}\int_0^{sh} M_s(u/h) f^{(s)}(x+u)\,du, \quad x \in [a, b-sh], \tag{3.7}$$

where M_s is the s-fold convolution of the characteristic function of $[0,1]$ with itself (see e.g. [18, p. 45]). Consequently,

$$\widetilde{D}_{s,n}f(x) = n\int_0^{s/n} M_s(nu) f^{(s)}\left(\frac{n-s}{n}\,x + u\right) du, \quad x \in [0,1],$$

and

$$\begin{aligned}&|\widetilde{D}_{s,n}f(x) - f^{(s)}(x)|\\ &\quad\le n\int_0^{s/n} M_s(nu)\left|f^{(s)}\left(\frac{n-s}{n}\,x+u\right) - f^{(s)}(x)\right| du\\ &\quad\le c\,\omega_1(f^{(s)}, n^{-1}), \quad x \in [0,1].\end{aligned} \tag{3.8}$$

Above we have used that

$$\int_0^s M_s(u)\,du = 1.$$

Now, (3.6) and (3.8) imply the assertion of the theorem. □

Remark 3.2. Based on Ditzian [20], Jiang and Xie [62] (or see [63, (16)]) gave a pointwise version of the estimate in Theorem 3.1.

3.3 A characterization of the rate of the weighted simultaneous approximation by the Bernstein operator

We will establish the following direct estimate of the rate of simultaneous approximation by the Bernstein operator.

Theorem 3.3. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_n f - f)^{(s)}\| \le c\, K_s^D(f^{(s)}, n^{-1})_w. \tag{3.9}$$

The value of the constant c is independent of f and n.

This estimate can be simplified. The K-functional $K_s^D(f,t)_w$ can be characterized by the simpler $K_1(f,t)_w$ and $K_{2,\varphi}(f,t)_w$ given in (1.4) and (1.5). We will show in Theorem 4.4 in the next chapter that if $0 < \gamma_0, \gamma_1 < s$, then for all $wf \in L_\infty[0,1]$ and $0 < t \le 1$ there holds

$$K_s^D(f,t)_w \sim \begin{cases} K_{2,\varphi}(f,t)_w + K_1(f,t)_w, & s = 1, \\ K_{2,\varphi}(f,t)_w + t\,\|wf\|, & s \ge 2. \end{cases} \tag{3.10}$$

The result in the case $w = 1$ is of a different form (Theorem 4.5):

$$K_s^D(f,t)_1 \sim \begin{cases} K_{2,\varphi}(f,t) + K_1(f,t), & s = 1, \\ K_{2,\varphi}(f,t) + K_1(f,t) + t\,\|f\|, & s \ge 2, \end{cases} \tag{3.11}$$

for all $f \in C[0,1]$ and $0 < t \le 1$. The characterization of $K_s^D(f,t)_w$ in the case when one of the γs is 0 and the other is positive is a "mixture" of (3.10) and (3.11).

Remark 3.4. Let us note that the assertion in (3.10) in the case $s = 1$ actually holds for all $0 \le \gamma_0, \gamma_1 < 1$, as it will be briefly shown in the proof of Theorem 4.4 in Section 4.7.

Further, we can take into account that $K_1(f,t)_w$ is equivalent to the weighted modulus of smoothness $\omega_1(f,t)_w$, and $K_{2,\varphi}(f,t^2)_w$ is equivalent to the weighted Ditzian-Totik modulus of smoothness $\omega_\varphi^2(f,t)_w$ (see (1.6) and (1.9)) to get the following Jackson-type estimates.

Theorem 3.5. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0,\gamma_1)$ be given by (2.2). Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_nf-f)^{(s)}\| \\ \le c \begin{cases} \omega_\varphi^2(f',n^{-1/2})_w + \omega_1(f',n^{-1})_w, & s=1,\ 0\le \gamma_0,\gamma_1<1,\\ \omega_\varphi^2(f^{(s)},n^{-1/2}) + \omega_1(f^{(s)},n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\|, & s\ge 2,\ \gamma_0=\gamma_1=0,\\ \omega_\varphi^2(f^{(s)},n^{-1/2})_w + \dfrac{1}{n}\,\|wf^{(s)}\|, & s\ge 2,\ 0<\gamma_0,\gamma_1<s. \end{cases}$$

The value of the constant c is independent of f and n.

Note that the direct estimates are stated for all $n \in \mathbb{N}_+$, whereas the equivalence between $K_{2,\varphi}(F,t^2)$ and $\omega_\varphi^2(F,t)$ was established for $t>0$ small enough (see (1.9)). We will give brief details how we can get rid of this limitation when we consider the generalization of the Bernstein operator given by its iterated Boolean sum in Theorems 4.7 and 4.8.

Remark 3.6. Jiang and Xie [62] (or see [12, Theorem 4.57]) proved a pointwise direct estimate, which implies the estimate in Theorem 3.5 in the case $s\ge 2,\ \gamma_0=\gamma_1=0$. It also follows from Theorem 3.1.

Remark 3.7. The range of γ_0 and γ_1 in Theorems 3.3 and 3.5 is the broadest possible, which allows direct estimates under natural assumptions on the functions (see Remark 3.15 below).

The direct estimates stated above are sharp—the following strong converse estimate holds.

Theorem 3.8. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0,\gamma_1)$ be given by (2.2) as $0 \le \gamma_0,\gamma_1 < s$. Then there exists $R \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $k,n \in \mathbb{N}_+$ with $k \ge Rn$ there holds*

$$K_s^D(f^{(s)},n^{-1})_w \le c\,\frac{k}{n}\left(\|w(B_nf-f)^{(s)}\| + \|w(B_kf-f)^{(s)}\|\right).$$

In particular,

$$K_s^D(f^{(s)}, n^{-1})_w \le c\left(\|w(B_nf - f)^{(s)}\| + \|w(B_{Rn}f - f)^{(s)}\|\right).$$

The value of the constant c *is independent of* f, n *and* k.

Remark 3.9. Let us note that the last estimate in Theorem 3.5 is, in general, not true in the case $s \ge 2$, $\gamma_0\gamma_1 = 0$. To avoid certain technical details we will show that for $\gamma_0 = \gamma_1 = 0$. Let $f^{(s)}(x) = x\log x$. Then $f^{(s)}, \varphi^2 f^{(s+2)} \in L_\infty[0,1]$ but $f^{(s+1)} \notin L_\infty[0,1]$. If the last estimate in Theorem 3.5 was true for $s \ge 2$, $\gamma_0 = \gamma_1 = 0$, then the last assertion of Theorem 3.8 and (3.11) would imply $K_1(f^{(s)}, n^{-1}) = O(n^{-1})$; hence $f^{(s+1)} \in L_\infty[0,1]$ (see Chapter 1 or [18, Chapter 2, Theorem 9.3 and Chapter 6, Theorem 2.4]), which is a contradiction.

Remark 3.10. We have stated Theorems 3.3, 3.5 and 3.8 under minimal assumptions on f. However, we have an approximation if and only if $\lim_{t\to 0}\omega_\varphi^2(f^{(s)}, t)_w = 0$ and, in addition, $\lim_{t\to 0}\omega_1(f^{(s)}, t)_w = 0$ in the cases $s = 1$, $0 \le \gamma_0, \gamma_1 < 1$ or $s \ge 2$, $\gamma_0 = \gamma_1 = 0$. In the case $w = 1$, we have $\lim_{t\to 0}\omega_1(g, t) = 0$ if and only if $g \in C[0,1]$; similarly $\lim_{t\to 0}\omega_\varphi^2(g, t) = 0$ if and only if $g \in C[0,1]$ (considering two functions which are equal a.e. with regard to the Lebesgue measure as identical); see [23, p. 37]. If $\gamma_0 > 0$, then we must have that $g(x)$ is continuous on $(0,1)$ and $\lim_{x\to 0} x^{\gamma_0} g(x) = 0$; if $\gamma_1 > 0$, then we must have $\lim_{x\to 1}(1-x)^{\gamma_1} g(x) = 0$ (see e.g. [35, p. 94]).

In the following two sections we will establish a number of auxiliary results we need to prove the theorems stated above. Some of this results are of independent importance since they describe the approximation rate for smooth functions. They include Jackson- and Voronovskaya-type inequalities. The proof of Theorems 3.3 and 3.8 are given in Section 3.6

3.4 Auxiliary identities for the Bernstein operator

In this section we will present several technical results for the Bernstein operator, which we will use.

Direct computation yields the following formulas for the derivatives of the polynomials $p_{n,k}$, $k = 0, \dots, n$ (see e.g. [18, Chapter 10, (2.1)]):

$$p'_{n,k}(x) = n[p_{n-1,k-1}(x) - p_{n-1,k}(x)] \tag{3.12}$$

and

$$p'_{n,k}(x) = \varphi^{-2}(x)(k - nx)p_{n,k}(x), \tag{3.13}$$

where we have set for convenience $p_{n,k} = 0$ if $k < 0$ or $k > n$.

For a sequence $\{a_k\}_{k\in\mathbb{Z}}$ we set $\Delta a_k := a_k - a_{k-1}$. Now, if we put $p_k(n, x) := p_{n,k}(x)$, then iterating (3.12) we get

$$p^{(s)}_{n,k}(x) = (-1)^s \frac{n!}{(n-s)!}\, \Delta^s p_k(n - s, x). \tag{3.14}$$

Similarly, using (3.13), it is verified by induction that (cf. [23, (9.4.8)])

$$p^{(s)}_{n,k}(x) = \varphi^{-2s}(x)\, p_{n,k}(x) \sum_{j=0}^{s} (k - nx)^j \sum_{0\le i\le (s-j)/2} q_{s,j,i}(x) \left(n\varphi^2(x)\right)^i,$$

where $q_{s,j,i}(x)$ are polynomials, whose coefficients are independent of n. Rearranging the summands, we get

$$p^{(s)}_{n,k}(x) = \varphi^{-2s}(x)\, p_{n,k}(x) \sum_{0\le i\le s/2} \left(n\varphi^2(x)\right)^i \sum_{j=0}^{s-2i} q_{s,j,i}(x)(k - nx)^j. \tag{3.15}$$

We will often use the quantities

$$T_{n,\ell}(x) := \sum_{k=0}^{n} (k - nx)^\ell p_{n,k}(x).$$

It is known (see [18, Chapter 10, Theorem 1.1]) that

$$T_{n,\ell}(x) = \sum_{1\le\rho\le\ell/2} t_{\ell,\rho}(x) \left(n\varphi^2(x)\right)^\rho, \quad \ell \in \mathbb{N}_+, \tag{3.16}$$

where $t_{\ell,\rho}(x)$ are polynomials, whose coefficients are independent of n.

In particular (see e.g. [18, p. 304] and [79, p. 14]),

$$\begin{aligned} &T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0, \quad T_{n,2}(x) = n\varphi^2(x), \\ &T_{n,3}(x) = (1 - 2x)n\varphi^2(x), \\ &T_{n,4}(x) = 3n^2\varphi^4(x) + n\varphi^2(x)(1 - 6\varphi^2(x)). \end{aligned} \tag{3.17}$$

Identity (3.16) implies (see also [23, Lemma 9.4.4]) that

$$0 \le T_{n,2m}(x) \le c \begin{cases} n\varphi^2(x), & n\varphi^2(x) \le 1, \\ (n\varphi^2(x))^m, & n\varphi^2(x) \ge 1. \end{cases} \tag{3.18}$$

Let $\alpha > 0$. We fix $m \in \mathbb{N}_+$ such that $2m/\alpha > 1$. Then Hölder's inequality, (3.18) and the identity $\sum_{k=0}^{n} p_{n,k}(x) \equiv 1$ imply

$$\begin{aligned} 0 \le \sum_{k=0}^{n} |k - nx|^\alpha p_{n,k}(x) &\le T_{n,2m}^{\alpha/(2m)}(x) \\ &\le c \begin{cases} 1, & n\varphi^2(x) \le 1, \\ (n\varphi^2(x))^{\alpha/2}, & n\varphi^2(x) \ge 1. \end{cases} \end{aligned} \tag{3.19}$$

We will need the analogue of $T_{n,\ell}$ associated with the differentiated Bernstein polynomial. We set

$$T_{s,n,\ell}(x) := \sum_{k=0}^{n} (k - nx)^\ell p_{n,k}^{(s)}(x).$$

The following formula, similar to (3.16), holds.

Lemma 3.11. *Let $\ell, n, s \in \mathbb{N}_+$. Then*

$$T_{s,n,\ell}(x) = \sum_{\rho=1}^{s} \tilde{t}_{s,\ell,\rho}(x)\, n^\rho + n^s \sum_{1 \le \rho \le (\ell - s)/2} t_{s,\ell,\rho}(x) \left(n\varphi^2(x)\right)^\rho,$$

where $t_{s,\ell,\rho}(x)$ and $\tilde{t}_{s,\ell,\rho}(x)$ are polynomials, whose coefficients are independent of n.

Above we follow the usual convention that an empty sum is considered to be equal to 0.

Proof of Lemma 3.11. Let $\ell \ge 2$. We apply (3.15). Then we sum on k, use (3.16) and finally reorder the summands to get

$$\begin{aligned} T_{s,n,\ell}(x) &= n^s \sum_{0 \le i \le s/2} \left(n\varphi^2(x)\right)^{i-s} \sum_{j=0}^{s-2i} q_{s,j,i}(x)\, T_{n,j+\ell}(x) \\ &= n^s \sum_{0 \le i \le s/2} \; \sum_{1 \le \rho \le (s+\ell-2i)/2} t_{s,i,\ell,\rho}(x) \left(n\varphi^2(x)\right)^{i+\rho-s}, \end{aligned}$$

where we have set

$$t_{s,i,\ell,\rho}(x) = \sum_{j=\max\{0,2\rho-\ell\}}^{s-2i} q_{s,j,i}(x)\, t_{j+\ell,\rho}(x).$$

Let us note that $t_{s,i,\ell,\rho}(x)$ are polynomials, whose coefficients are independent of n.

Consequently,

$$T_{s,n,\ell}(x) = n^s \sum_{1-s\le\rho\le(\ell-s)/2} t_{s,\ell,\rho}(x) \left(n\varphi^2(x)\right)^{\rho}$$

with some polynomials $t_{s,\ell,\rho}(x)$, whose coefficients are independent of n. To get the assertion of the lemma for $\ell \ge 2$, we need only take into account that the left-hand side of the last identity is a polynomial in x; hence so is $\varphi^{2\rho}(x)t_{s,\ell,\rho}(x)$ for each negative ρ. Here we also use that $t_{s,\ell,\rho}(x)$ are independent of n.

Minor changes in the above argument establish the lemma for $\ell = 1$ too. □

We proceed to several identities concerning the derivatives of the error of the Bernstein operators. We will use them to establish Jackson- and Voronovskaya-type estimates. We denote the set of the algebraic polynomials of degree at most j by π_j.

Lemma 3.12. *Let $s \in \mathbb{N}_+$, $f \in C[0,1]$, $f \in AC_{loc}^{s+1}(0,1)$ and $\varphi^{2s+2} f^{(s+2)} \in L[0,1]$. Then*

$$(3.20)\quad (B_n f(x) - f(x))^{(s)} = \frac{1}{n}\, A_{s,n} f^{(s)}(x) + \frac{1}{n}\, B_{s,n}(x) f^{(s+1)}(x) + \frac{1}{(s+1)!} \sum_{k=0}^{n} p_{n,k}^{(s)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{s+1} f^{(s+2)}(u)\, du, \quad x \in (0,1),$$

where

$$A_{s,n} = \sum_{\nu=0}^{s-2} a_{s,\nu}\, n^{-\nu}, \quad B_{s,n}(x) = \sum_{\nu=0}^{s-1} b_{s,\nu}(x)\, n^{-\nu},$$

and $a_{s,\nu}$ and $b_{s,\nu}(x)$ are respectively constants and linear functions, which are independent of n.

Above we again use the usual convention that an empty sum is zero. Note that the order of the derivatives on the right of (3.20) is at least $\max\{2, s\}$.

Proof of Lemma 3.12. Let us make two observations that will justify our usage of Taylor's expansions, integration by parts and induction on s below.

First, if $f \in AC_{loc}^{\sigma+1}(0,1)$ and $\varphi^{2\sigma+2} f^{(\sigma+2)} \in L[0,1]$ for some $\sigma \in \mathbb{N}_+$, then

$$\varphi^{2\sigma} f^{(\sigma+1)} \in L[0,1]. \tag{3.21}$$

That follows from Proposition 2.1 with $p = 1$, $g = f$, $j = \sigma + 1$, $m = \sigma + 2$, $w_1 = \varphi^{2\sigma}$ and $w_2 = \varphi^{2\sigma+2}$.

Further, using the representation

$$\begin{aligned} u^{\sigma+1} f^{(\sigma+1)}(u) = \frac{1}{2^{\sigma+1}} f^{(\sigma+1)}\left(\frac{1}{2}\right) - (\sigma+1) \int_u^{1/2} v^{\sigma} f^{(\sigma+1)}(v)\, dv \\ - \int_u^{1/2} v^{\sigma+1} f^{(\sigma+2)}(v)\, dv, \quad u \in (0,1), \end{aligned}$$

we deduce that $\lim_{u \to 0+0} u^{\sigma+1} f^{(\sigma+1)}(u)$ exists as a finite limit. Moreover, if we assume that it is not 0, then we will get that $u^{\sigma}|f^{(\sigma+1)}(u)| \geq C/u$ for $u \in (0, \delta)$ with some positive constants C and δ, which contradicts $\varphi^{2\sigma} f^{(\sigma+1)} \in L[0,1]$. Consequently,

$$\lim_{u \to 0+0} u^{\sigma+1} f^{(\sigma+1)}(u) = 0. \tag{3.22}$$

By symmetry, we get

$$\lim_{u \to 1-0} (1-u)^{\sigma+1} f^{(\sigma+1)}(u) = 0. \tag{3.23}$$

Let us proceed to the proof of the lemma. We will establish it by means of induction on s. To check it for $s = 1$ we note that by (3.21) with $\sigma = 1$ we have $\varphi^2 f'' \in L[0,1]$ and we can expand $f(t)$ at $x \in (0,1)$ by Taylor's formula to get

$$f(t) = f(x) + (t-x) f'(x) + \int_x^t (t-u) f''(u)\, du, \quad t \in [0,1].$$

Then we apply the operator B_n to both sides of the above identity, take into account that it preserves the linear functions and arrive at

$$B_n f(x) - f(x) = \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du. \tag{3.24}$$

We differentiate (3.24), integrate by parts as we take into account (3.22)-(3.23) with $\sigma = 1$ and use (3.13) and (3.17) to derive

$$\begin{aligned}(B_n f(x) - f(x))' &= -\frac{1}{n}\, T_{n,1}(x) f''(x) + \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du \\ &= \frac{\varphi^{-2}(x)}{2n^2}\, T_{n,3}(x) f''(x) \\ &\quad + \frac{1}{2} \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^2 f'''(u)\, du \\ &= \frac{1-2x}{2n}\, f''(x) + \frac{1}{2} \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^2 f'''(u)\, du.\end{aligned}$$

Thus the lemma is verified for $s = 1$.

Next, let us assume that the assertion of the lemma is true for some s and let $f \in C[0,1]$, $f \in AC_{loc}^{s+2}(0,1)$ and $\varphi^{2s+4} f^{(s+3)} \in L[0,1]$. Then by (3.21) with $\sigma = s+1$ we have $\varphi^{2s+2} f^{(s+2)} \in L[0,1]$. Therefore, by the induction hypothesis, formula (3.20) is valid for that s. We differentiate it and integrate by parts using (3.22)-(3.23) with $\sigma = s+1$. Thus we arrive at

$$\begin{aligned}(B_n f(x) - f(x))^{(s+1)} &= \frac{1}{n} \left(A_{s,n} + B'_{s,n}(x)\right) f^{(s+1)}(x) \\ &\quad + \frac{1}{n}\, B_{s,n}(x) f^{(s+2)}(x) \\ &\quad - \frac{1}{(s+1)!} \sum_{k=0}^{n} p_{n,k}^{(s)}(x) \left(\frac{k}{n} - x\right)^{s+1} f^{(s+2)}(x) \\ &\quad + \frac{1}{(s+2)!} \sum_{k=0}^{n} p_{n,k}^{(s+1)}(x) \left(\frac{k}{n} - x\right)^{s+2} f^{(s+2)}(x) \\ &\quad + \frac{1}{(s+2)!} \sum_{k=0}^{n} p_{n,k}^{(s+1)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{s+2} f^{(s+3)}(u)\, du.\end{aligned} \tag{3.25}$$

According to the induction hypothesis the expression $A_{s+1,n} = A_{s,n} + B'_{s,n}(x)$ is of the form $\sum_{\nu=0}^{s-1} a_{s+1,\nu}\, n^{-\nu}$ with some constants $a_{s+1,\nu}$, which are independent of n.

Let us denote by $B_{s+1,n}(x)$ the factor of $f^{(s+2)}(x)/n$ in the expansion (3.25). From the induction hypothesis and Lemma 3.11 with $\ell = s+1$ it follows that it is of the form

$$B_{s+1,n}(x) = \sum_{\nu=0}^{s} b_{s+1,\nu}(x)\, n^{-\nu}, \tag{3.26}$$

where $b_{s+1,\nu}(x)$ are polynomials, whose coefficients are independent of n. To show that they are of degree 1, we set in (3.25) $f(x) = x^{s+2}$. We get

$$(B_n f(x) - f(x))^{(s+1)} = \frac{A_{s+1,n}\,(s+2)!}{n}\, x + \frac{(s+2)!}{n}\, B_{s+1,n}(x).$$

Since $B_n f \in \pi_{s+2}$, we deduce that $B_{s+1,n} \in \pi_1$; hence $b_{s+1,\nu} \in \pi_1$ because their coefficients are independent of n.

This completes the proof of the lemma. □

Lemma 3.13. *Let* $s \in \mathbb{N}_+$, $f \in C[0,1]$, $f \in AC_{loc}^{s+2}(0,1)$ *and* $\varphi^{2s+4} f^{(s+3)} \in L[0,1]$. *Then*

$$\left(B_n f(x) - f(x) - \frac{1}{2n}\, Df(x)\right)^{(s)} = \frac{1}{n^2}\, \widetilde{A}_{s,n} f^{(s)}(x) + \frac{1}{n^2}\, \widetilde{B}_{s,n}(x) f^{(s+1)}(x)$$
$$+ \frac{1}{n^2}\, \widetilde{C}_{s,n}(x) f^{(s+2)}(x) + \frac{1}{(s+2)!} \sum_{k=0}^{n} p_{n,k}^{(s)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{s+2} f^{(s+3)}(u)\, du,$$
$$x \in (0,1),$$

where

$$\widetilde{A}_{s,n} = \sum_{\nu=0}^{s-3} \tilde{a}_{s,\nu}\, n^{-\nu}, \quad \widetilde{B}_{s,n}(x) = \sum_{\nu=0}^{s-2} \tilde{b}_{s,\nu}(x)\, n^{-\nu}, \quad \widetilde{C}_{s,n}(x) = \sum_{\nu=0}^{s-1} \tilde{c}_{s,\nu}(x)\, n^{-\nu}$$

and $\tilde{a}_{s,\nu}$, $\tilde{b}_{s,\nu}(x)$ *and* $\tilde{c}_{s,\nu}(x)$ *are polynomials of degree respectively* 0, 1 *and* 2, *whose coefficients are independent of* n.

Let us note that the order of the derivatives on the right of the formula in the lemma is at least $\max\{3, s\}$.

Proof of Lemma 3.13. We verify the lemma just similarly to the previous one.

To check it for $s = 1$ we apply (3.21) with $\sigma = 2$ and get $\varphi^4 f''' \in L[0,1]$. Then

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}\,(t-x)^2 f''(x) + \frac{1}{2}\int_x^t (t-u)^2 f'''(u)\, du, \quad t \in [0,1].$$

We apply the operator B_n to both sides of the above identity, take into account that it preserves the linear functions and also that $T_{n,2}(x) = n\varphi^2(x)$ (see (3.17)) and arrive at

$$B_n f(x) - f(x) - \frac{1}{2n}\, Df(x) = \frac{1}{2} \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^2 f^{(3)}(u)\, du. \tag{3.27}$$

We set

$$V_n f(x) := B_n f(x) - f(x) - \frac{1}{2n}\, Df(x). \tag{3.28}$$

We differentiate (3.27), integrate by parts, taking into account (3.22)-(3.23) with $\sigma = 2$, and apply (3.13). Thus we arrive at

$$\begin{aligned}
(V_n f)'(x) &= -\frac{1}{2n^2}\, T_{n,2}(x) f^{(3)}(x) + \frac{1}{2} \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^2 f^{(3)}(u)\, du \\
&= \frac{1}{6n^2} \left(\frac{\varphi^{-2}(x)}{n}\, T_{n,4}(x) - 3T_{n,2}(x) \right) f^{(3)}(x) \\
&\quad + \frac{1}{3!} \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^3 f^{(4)}(u)\, du.
\end{aligned}$$

To complete the proof for $s = 1$ we apply (3.17), which yields

$$\frac{\varphi^{-2}(x)}{n}\, T_{n,4}(x) - 3T_{n,2}(x) = 1 - 6\varphi^2(x).$$

Next, let us assume that the lemma is valid for some s. Let $f \in C[0,1]$, $f \in AC_{loc}^{s+3}(0,1)$ and $\varphi^{2s+6} f^{(s+4)} \in L[0,1]$. Then by (3.21) with $\sigma = s+2$ we have $\varphi^{2s+4} f^{(s+3)} \in L[0,1]$; hence the formula of the lemma is true for that s. We differentiate it and integrate by parts as we use (3.22)-(3.23) with $\sigma = s+2$. Thus we arrive at

$$\begin{aligned}
(V_n f)^{(s+1)} &= \frac{1}{n^2} \left(\widetilde{A}_{s,n} + \widetilde{B}'_{s,n}(x) \right) f^{(s+1)}(x) \\
&\quad + \frac{1}{n^2} \left(\widetilde{B}_{s,n}(x) + \widetilde{C}'_{s,n}(x) \right) f^{(s+2)}(x) \\
&\quad + \frac{1}{n^2} \left(\widetilde{C}_{s,n}(x) + \widetilde{D}_{s,n}(x) \right) f^{(s+3)}(x) \\
&\quad + \frac{1}{(s+3)!} \sum_{k=0}^{n} p_{n,k}^{(s+1)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{s+3} f^{(s+4)}(u)\, du,
\end{aligned} \tag{3.29}$$

where we have set

$$\widetilde{D}_{s,n}(x) = \frac{n^2}{(s+3)!}\left(\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\left(\frac{k}{n}-x\right)^{s+3}\right)'.$$

The induction hypothesis implies that the factors of $f^{(s+1)}(x)$ and $f^{(s+2)}(x)$ are of the stated form. To establish that for the factor of $f^{(s+3)}(x)$, we use Lemma 3.11 with $\ell = s+3$ to deduce that

$$\widetilde{D}_{s,n}(x) = \sum_{\nu=0}^{s} \tilde{d}_{s,\nu}(x)\, n^{-\nu}$$

with some polynomials $\tilde{d}_{s,\nu}$, whose coefficients do not depend on n. Consequently, if we set

$$\widetilde{C}_{s+1,n}(x) = \widetilde{C}_{s,n}(x) + \widetilde{D}_{s,n}(x),$$

then

$$\widetilde{C}_{s+1,n}(x) = \sum_{\nu=0}^{s} \tilde{c}_{s+1,\nu}(x)\, n^{-\nu}$$

with some polynomials $\tilde{c}_{s+1,\nu}$, whose coefficients do not depend on n. To prove that they are of degree 2, we set $f(x) := x^{s+3}$ in (3.29) and argue as in the proof of Lemma 3.12. □

3.5 Basic estimates for the simultaneous approximation by the Bernstein operator

In this section we will establish the basic inequalities which imply the characterization of the error of the simultaneous approximation by means of B_n. We use techniques, which have already become standard for this set of problems (see e.g. [23, Chapters 9 and 10]). To establish the converse estimate we apply the general method developed by Ditzian and Ivanov [22, Theorem 3.2]. These methods allow us to establish both, the direct and the converse estimate, by means of several other basic estimates concerning the approximation properties of the operator.

3.5.1 Boundedness

We begin with the following basic estimate concerning the boundedness of the weighted L_∞-norm of $(B_n f)^{(s)}$.

Proposition 3.14. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_n f)^{(s)}\| \le c\,\|wf^{(s)}\|.$$

The value of the constant c is independent of f and n.

Proof. The inequality is trivial for $n < s$. For $n \ge s$ we use (3.4), which states

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}(x),$$

and (3.7) in the form

$$\overrightarrow{\Delta}_h^s f(x) = h^s \int_0^s M_s(u) f^{(s)}(x+hu)\,du, \quad x \in [0, 1-sh],$$

to get by Hölder's inequality the estimate

$$\left|\overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right)\right| \le \frac{w_{n,k}}{n^s}\,\|wf^{(s)}\|_{[k/n,(k+s)/n]}, \quad k = 0, \dots, n-s, \tag{3.30}$$

where

$$w_{n,k} := n \int_{k/n}^{(k+s)/n} \left|\frac{M_s(nu-k)}{w(u)}\right| du. \tag{3.31}$$

Relations (3.4) and (3.30) yield

$$|w(x)(B_n f)^{(s)}(x)| \le c\,w(x) \sum_{k=0}^{n-s} w_{n,k}\, p_{n-s,k}(x)\,\|wf^{(s)}\|_{[k/n,(k+s)/n]}. \tag{3.32}$$

We will show that the right-hand side of (3.32) is bounded above by $c\,\|wf^{(s)}\|$. Due to symmetry it is sufficient to consider only the summands for $k =$

$0,\dots,[(n-s)/2]$ on the right-hand side of (3.32). Indeed, for $\bar{w}(x)=w(1-x)$, $\bar{f}(x)=f(1-x)$, and $\bar{w}_{n,k}$, defined by (3.31) with $\bar{w}$ in place of w, we have

$$\begin{aligned} w_{n,n-s-k} &= \bar{w}_{n,k}, \quad \|wf^{(s)}\| = \|\bar{w}\bar{f}^{(s)}\|, \\ \|wf^{(s)}\|_{[(n-s-k)/n,(n-k)/n]} &= \|\bar{w}\bar{f}^{(s)}\|_{[k/n,(k+s)/n]}, \end{aligned} \tag{3.33}$$

as for the first relation above we have taken into account that $M_s(s-u)=M_s(u)$. Consequently, with $y=1-x$ we have

$$\begin{aligned} &\sum_{(n-s)/2\le k\le n-s} w_{n,k}\, p_{n-s,k}(x)\, \|wf^{(s)}\|_{[k/n,(k+s)/n]} \\ &\qquad\qquad = \sum_{0\le k\le (n-s)/2} \bar{w}_{n,k}\, p_{n-s,k}(y)\, \|\bar{w}\bar{f}^{(s)}\|_{[k/n,(k+s)/n]}. \end{aligned} \tag{3.34}$$

Thus it is sufficient to consider only the summands for $k=0,\dots,[(n-s)/2]$ on the right-hand side of (3.32).

It is known that

$$0\le M_s(u)\le c[u(s-u)]^{s-1}, \quad 0\le u\le s.$$

Hence the assertion follows for $n=s$.

Let $n>s$. We have

$$\frac{M_s(nu)}{w(u)} \le c\, n^{\gamma_0}\, (nu)^{s-\gamma_0-1}, \quad u\in(0,s/n],$$

and

$$\frac{M_s(nu-k)}{w(u)} \le c\, n^{\gamma_0} k^{-\gamma_0}, \quad u\in[k/n,(k+s)/n], \quad 1\le k\le (n-s)/2;$$

hence, under the assumptions on γ_0, we get

$$w_{n,k}\le c\left(\frac{n}{k+1}\right)^{\gamma_0}, \quad 0\le k\le (n-s)/2. \tag{3.35}$$

Inequality (3.35) and Hölder's inequality imply

$$\begin{aligned} \sum_{k=0}^{[(n-s)/2]} w_{n,k}\, p_{n-s,k}(x) &\le c\sum_{k=0}^{n-s}\left(\frac{n}{k+1}\right)^{\gamma_0} p_{n-s,k}(x) \\ &\le c\left(\sum_{k=0}^{n-s}\left(\frac{n}{k+1}\right)^{s} p_{n-s,k}(x)\right)^{\gamma_0/s}. \end{aligned} \tag{3.36}$$

There holds (see [23, (10.2.4)])

$$\sum_{k=0}^{n}\left(\frac{n}{k+1}\right)^{s} p_{n,k}(x) \le c\,x^{-s}, \quad x\in(0,1]. \tag{3.37}$$

Consequently,

$$w(x)\sum_{k=0}^{[(n-s)/2]} w_{n,k}\,p_{n-s,k}(x) \le c, \quad x\in[0,1]. \tag{3.38}$$

Now, (3.32), (3.34) and (3.38) imply the assertion of the proposition for and $n > s$. □

Certain particular cases of the proposition have been established before—e.g. [23, (9.3.7)] contains it with $w = \varphi^{2\ell}$ and $s = 2\ell$.

Remark 3.15. The range of γ_0 and γ_1 in the proposition above cannot be generally expanded even for a fixed n unless we impose additional assumptions on f. This means that the range of the γs cannot be expanded in Theorems 3.3 and 3.5 either because each one of them implies the inequality in Proposition 3.14.

We will consider only the case $s = 1$. Analogous arguments can be used for $s \ge 2$.

If $\gamma_0 < 0$, then since $(B_n f)'(0) = n(f(1/n) - f(0))$ (see (3.4)), the function f has to satisfy quite restrictive and specific assumptions in a neighbourhood of 0.

Let $\gamma_0 \ge 1$. We consider the functions $f_m(x) := \ln(x + 1/m)$, $x \in [0,1]$, $m \in \mathbb{N}_+$. We have $f_m \in C^1[0,1]$ and $0 \le x^{\gamma_0} f_m'(x) < 1$, for all $x \in [0,1]$ and all m. On the other hand, since $f_m(x)$ is increasing, (3.4) yields for

$$\begin{aligned}(B_n f_m)'\left(\frac{1}{2}\right) &= n\sum_{k=0}^{n-1}\overrightarrow{\Delta}_{1/n} f_m\left(\frac{k}{n}\right) p_{n-1,k}\left(\frac{1}{2}\right)\\ &\ge n2^{1-n}\left[f_m\left(\frac{1}{n}\right) - f_m(0)\right]\\ &= n2^{1-n}\ln\left(\frac{m}{n}+1\right) \to +\infty \text{ as } m\to+\infty,\end{aligned}$$

for any fixed n. Therefore, the assertion of Proposition 3.14 is not valid for $\gamma_0 \ge s = 1$.

3.5.2 Jackson-type estimates

Now, we will establish Jackson-type estimates for the operators $(B_n f)^{(s)}$. We will use the following technical result.

Lemma 3.16. *Let $\alpha, \beta, \delta \in \mathbb{R}$ be such that $0 \le \alpha, \beta \le \delta$. Set $\gamma := \min\{\alpha, \beta\}$. Then for $x, t \in (0,1)$ and u between x and t there holds*

$$\frac{|t-u|^\delta}{u^\alpha(1-u)^\beta} \le 2^{|\gamma-1|} \frac{|t-x|^\delta}{x^\alpha(1-x)^\beta}.$$

Proof. For u between t and x such that $x, t \in (0,1)$ we have the inequalities:

$$\frac{|t-u|}{u} \le \frac{|t-x|}{x}, \qquad \frac{|t-u|}{1-u} \le \frac{|t-x|}{1-x}. \tag{3.39}$$

The first one is checked directly and the second one follows from it by symmetry.

Next, we will show that under the same conditions on x, t and u we have

$$\frac{|t-u|^\mu}{[u(1-u)]^\mu} \le 2^{|\mu-1|} \frac{|t-x|^\mu}{[x(1-x)]^\mu}, \quad \mu \ge 0. \tag{3.40}$$

To establish that we raise each of the inequalities in (3.39) to the power of μ and sum them up. Thus we arrive at

$$|t-u|^\mu \frac{u^\mu + (1-u)^\mu}{[u(1-u)]^\mu} \le |t-x|^\mu \frac{x^\mu + (1-x)^\mu}{[x(1-x)]^\mu}.$$

To get (3.40), it remains to observe that $\min\{1, 2^{1-\mu}\} \le x^\mu + (1-x)^\mu \le \max\{1, 2^{1-\mu}\}$ for $x \in [0,1]$.

Further, we set $\hat{\gamma} := \max\{\alpha, \beta\}$ and

$$\phi(x) := \begin{cases} x, & \alpha \ge \beta, \\ 1-x, & \beta > \alpha. \end{cases}$$

Now, to prove the lemma we need only multiply the inequalities:

$$\frac{|t-u|^\gamma}{[u(1-u)]^\gamma} \le 2^{|\gamma-1|} \frac{|t-x|^\gamma}{[x(1-x)]^\gamma}, \tag{3.41}$$

$$\left(\frac{|t-u|}{\phi(u)}\right)^{\hat{\gamma}-\gamma} \le \left(\frac{|t-x|}{\phi(x)}\right)^{\hat{\gamma}-\gamma} \tag{3.42}$$

and

$$|t-u|^{\delta-\hat{\gamma}} \le |t-x|^{\delta-\hat{\gamma}}. \tag{3.43}$$

Inequality (3.41) is (3.40) with $\mu = \gamma \ge 0$, (3.42) follows from (3.39) and $\gamma \le \hat{\gamma}$, and (3.43) from $\hat{\gamma} \le \delta$. □

Proposition 3.17. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2). Set $s' := \max\{2, s\}$. If $0 < \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+1}(0,1)$ and $wf^{(s')}, w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_n f - f)^{(s)}\| \le \frac{c}{n}\left(\|wf^{(s')}\| + \|w\varphi^2 f^{(s+2)}\|\right). \tag{3.44}$$

If $\gamma_0\gamma_1 = 0$ and still $0 \le \gamma_0, \gamma_1 \le s$, then

$$\|w(B_n f - f)^{(s)}\| \le \frac{c}{n}\left(\|wf^{(s')}\| + \|wf^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\|\right), \tag{3.45}$$

provided that $wf^{(s+1)} \in L_\infty[0,1]$ too.

The value of the constant c is independent of f and n.

Proof. First, let us note that if $\gamma_0, \gamma_1 > 0$, the assumption $w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$ implies $wf^{(s')} \in L_\infty[0,1]$. This follows from Proposition 2.1 with $w_1 = w$, $w_2 = w\varphi^2$, $j = s'$ and $m = s+2$.

The proof of the proposition is based on Lemma 3.12. Since $w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$, then $\varphi^{2s+2} f^{(s+2)} \in L_\infty[0,1]$; and hence the lemma is applicable.

We will prove that if $0 \le \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+1}(0,1)$ and $wf^{(s')}, wf^{(s+1)}, w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds

$$\|w(B_n f - f)^{(s)}\| \le \frac{c}{n}\left(\|wf^{(s')}\| + \|wf^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\|\right). \tag{3.46}$$

That contains, in particular, (3.45), and estimate (3.44) follows from (3.46) and the inequality

$$\|wf^{(s+1)}\| \le c\left(\|wf^{(s')}\| + \|w\varphi^2 f^{(s+2)}\|\right),$$

which is established by means of Proposition 2.1 with $g = f^{(s')}$, $j = s - s' + 1$, $m = s - s' + 2$, $w_1 = w$ and $w_2 = w\varphi^2$.

Let us set

$$R_{s,n}f(x) := \frac{1}{(s+1)!}\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^{s+1} f^{(s+2)}(u)\,du.$$

We will show that

$$\|wR_{s,n}f\| \le \frac{c}{n}\left(\|wf^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\|\right), \tag{3.47}$$

which verifies (3.46) in view of Lemma 3.12.

In order to simplify our argument, we will consider two cases for the domain of x.

Case 1. Let $n\varphi^2(x) \ge 1$. We make use of (3.15) and Lemma 3.16 with $\delta = s+1$, $\alpha = \gamma_0 + 1$ and $\beta = \gamma_1 + 1$ to get

$$\begin{aligned} &|w(x)\,R_{s,n}f(x)| \\ &\quad \le \frac{c}{n}\sum_{0\le i\le s/2}\left(n\varphi^2(x)\right)^{i-s-1}\sum_{j=0}^{s-2i}\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^{s+j+2}\|w\varphi^2 f^{(s+2)}\|. \end{aligned} \tag{3.48}$$

Further, we apply estimate (3.19) and get

$$\begin{aligned} &\sum_{0\le i\le s/2}\left(n\varphi^2(x)\right)^{i-s-1}\sum_{j=0}^{s-2i}\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^{s+j+2} \\ &\qquad \le c\sum_{0\le i\le s/2}\sum_{j=0}^{s-2i}\left(n\varphi^2(x)\right)^{(2i+j-s)/2} \le c, \end{aligned} \tag{3.49}$$

as at the last inequality we have taken into account that $n\varphi^2(x) \ge 1$ and $2i + j - s \le 0$.

Now, (3.48)-(3.49) imply

$$\|wR_{s,n}f\|_{I_n} \le \frac{c}{n}\,\|w\varphi^2 f^{(s+2)}\|, \tag{3.50}$$

where $I_n := \{x \in [0,1] : n\varphi^2(x) \ge 1\}$.

Case 2. Let $n\varphi^2(x) \le 1$. Due to symmetry, we may also assume that $x \le 1/2$. Therefore, $x \le 2/n$. By means of (3.14) and Abel's transform we derive for $n \ge s$ the relation (cf. (3.4))

$$R_{s,n}f(x) = \frac{1}{(s+1)!}\,\frac{n!}{(n-s)!}\sum_{k=0}^{n-s}\overrightarrow{\Delta}_{1/n}^{s} r_{s,x}\left(\frac{k}{n}\right)p_{n-s,k}(x),$$

where we have set

$$r_{s,x}(t) := \int_x^t (t-u)^{s+1} f^{(s+2)}(u)\, du.$$

Consequently,

$$(3.51) \qquad |w(x)R_{s,n}f(x)| \le c\,n^s \max_{i=0,\dots,s} \sum_{k=0}^{n-s} \left| w(x)\, r_{s,x}\left(\frac{k+i}{n}\right)\right| p_{n-s,k}(x).$$

Just as in Case 1 we estimate $r_{s,x}(t)$ by means of Lemma 3.16 and get

$$(3.52) \qquad \left| w(x)\, r_{s,x}\left(\frac{k+i}{n}\right)\right| \le c\,\varphi^{-2}(x) \left|\frac{k+i}{n} - x\right|^{s+2} \|w\varphi^2 f^{(s+2)}\|.$$

Next, we observe that for $k \ge 1$ and $i = 0,\dots,s$ we have $k+i+1 \ge 2 \ge nx$. Therefore for $n > s$ there holds

$$\begin{aligned}
\sum_{k=1}^{n-s} \left|\frac{k+i}{n} - x\right|^{s+2} p_{n-s,k}(x) &\le \frac{x}{n^{s+1}} \sum_{k=0}^{n-s-1} |k+i+1-nx|^{s+2} p_{n-s-1,k}(x) \\
&\le \frac{c\,x}{n^{s+1}} \left(1 + \sum_{k=1}^{n-s-1} (k+i+1-nx)^{s+2} p_{n-s-1,k}(x)\right) \\
&\le \frac{c\,x}{n^{s+1}} \left(1 + \sum_{k=1}^{n-s-1} (k+s+1-nx)^{s+2} p_{n-s-1,k}(x)\right).
\end{aligned}$$

Further, we use the binomial formula to represent $(k+s+1-nx)^{s+2}$ in the form

$$\begin{aligned}
(k+s+1-nx)^{s+2} &= \big([k-(n-s-1)x] + [(s+1)(1-x)]\big)^{s+2} \\
&= \sum_{j=0}^{s+2} \binom{s+2}{j} [k-(n-s-1)x]^j [(s+1)(1-x)]^{s-j+2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{k=1}^{n-s} \left|\frac{k+i}{n} - x\right|^{s+2} p_{n-s,k}(x) \\
&\qquad \le \frac{c\,x}{n^{s+1}} \left(1 + \sum_{j=0}^{s+2} \sum_{k=1}^{n-s-1} |k-(n-s-1)x|^j p_{n-s-1,k}(x)\right) \\
&\qquad \le \frac{c\,x}{n^{s+1}},
\end{aligned}$$

where at the last estimate, we applied (3.19). Consequently, by (3.52) we get

$$\sum_{k=1}^{n-s} \left| w(x)\, r_{s,x}\left(\frac{k+i}{n}\right)\right| \, p_{n-s,k}(x) \le \frac{c}{n^{s+1}}\, \|w\varphi^2 f^{(s+2)}\|, \quad i = 0,\dots,s;$$

hence we arrive at

$$\left\| \sum_{k=1}^{n-s} w\, r_{s,\circ}\left(\frac{k+i}{n}\right) p_{n-s,k} \right\|_{I_n'} \le \frac{c}{n^{s+1}}\, \|w\varphi^2 f^{(s+2)}\|, \quad i = 0,\dots,s, \tag{3.53}$$

where $I_n' := \{x \in [0,1/2] : n\varphi^2(x) \le 1\}$.

It remains to estimate the terms for $k = 0$ in (3.51). First, we observe that by (3.52) with $k = i = 0$ we have

$$|w(x) r_{s,x}(0)| \le c\, x^{s+1} |w(x)\varphi^2(x) f^{(s+2)}(x)| \le \frac{c}{n^{s+1}}\, \|w\varphi^2 f^{(s+2)}\|;$$

hence

$$\|w\, r_{s,\circ}(0)\|_{I_n'} \le \frac{c}{n^{s+1}}\, \|w\varphi^2 f^{(s+2)}\|. \tag{3.54}$$

To estimate $w(x) r_{s,x}(i/n)$ for $i = 1,\dots,s$, we expand $(i/n - u)^{s+1}$ by the binomial formula and get

$$\left| w(x) r_{s,x}\left(\frac{i}{n}\right)\right| \le \frac{c\, x^{\gamma_0}}{n^{s+1}} \sum_{j=0}^{s+1} \left| \int_x^{i/n} (nu)^j f^{(s+2)}(u)\, du \right|. \tag{3.55}$$

Further, taking into account that in the case under consideration we have $nx \le 2$, we get for $i = 2,\dots,s$ and $n \ge s$ but not $i = n = s$ the inequality

$$\left| w(x) r_{s,x}\left(\frac{i}{n}\right)\right| \le \frac{c}{n^{s+1}}\, x^{\gamma_0} \int_x^{s/(s+1)} |f^{(s+2)}(u)|\, du.$$

Consequently, if $\gamma_0 > 0$, then

$$\left\| w\, r_{s,\circ}\left(\frac{i}{n}\right)\right\|_{I_n'} \le \frac{c}{n^{s+1}}\, \|w\varphi^2 f^{(s+2)}\| \tag{3.56}$$

for $i = 2,\dots,s$ and $n \ge s$ but not $i = n = s$.

For $\gamma_0 > 0$, $i = 1$, $n \geq s$ but not $n = s = 1$ we split the interval I_n' into two intervals. On $[0, 1/n]$ (note that $n \geq 2$), the same considerations as above yield

$$\left\| w\, r_{s,\circ}\left(\frac{1}{n}\right) \right\|_{[0,1/n]} \leq \frac{c}{n^{s+1}} \, \|w\varphi^2 f^{(s+2)}\|. \tag{3.57}$$

Let us denote the right end of the interval I_n' by x_n. We have $x_n \leq 2/n$. Then for $x \in [1/n, x_n]$ there hold

$$\int_{1/n}^{x} |f^{(s+2)}(u)|\, du \leq c\, n^{\gamma_0+1} \int_{1/n}^{x} |w(u)\varphi^2(u) f^{(s+2)}(u)|\, du \leq c\, x^{-\gamma_0} \|w\varphi^2 f^{(s+2)}\|.$$

Consequently, we have for $x \in [1/n, x_n]$

$$x^{\gamma_0} \int_{1/n}^{x} |f^{(s+2)}(u)|\, du \leq c\, \|w\varphi^2 f^{(s+2)}\|.$$

Thus, in view of (3.55), we have established

$$\left\| w\, r_{s,\circ}\left(\frac{1}{n}\right) \right\|_{[1/n, x_n]} \leq \frac{c}{n^{s+1}} \, \|w\varphi^2 f^{(s+2)}\|. \tag{3.58}$$

Combining (3.57) and (3.58), we get

$$\left\| w\, r_{s,\circ}\left(\frac{1}{n}\right) \right\|_{I_n'} \leq \frac{c}{n^{s+1}} \, \|w\varphi^2 f^{(s+2)}\| \tag{3.59}$$

for $\gamma_0 > 0$.

For $\gamma_0 = 0$, $i = 1, \ldots, s$ and $n \geq s$ but not $i = n = s$ we apply (3.55) to derive

$$\begin{aligned} &\left| w(x) r_{s,x}\left(\frac{i}{n}\right) \right| \\ &\quad \leq \frac{c}{n^{s+1}} \left| \int_{x}^{i/n} f^{(s+2)}(u)\, du \right| + \frac{c}{n^{s+1}} \sum_{j=1}^{s+1} \left| \int_{x}^{i/n} (nu)^j |f^{(s+2)}(u)|\, du \right| \\ &\quad \leq \frac{c}{n^{s+1}} \left| \int_{x}^{i/n} f^{(s+2)}(u)\, du \right| + \frac{c}{n^{s}} \left| \int_{x}^{i/n} u |f^{(s+2)}(u)|\, du \right|. \end{aligned} \tag{3.60}$$

For the first term on the right above we have

$$
(3.61)\qquad \left|\int_x^{i/n} f^{(s+2)}(u)\,du\right| \le |f^{(s+1)}(x)| + \left|f^{(s+1)}\left(\frac{i}{n}\right)\right| \le 2\,\|f^{(s+1)}\|_{[0,s/(s+1)]} \le c\,\|wf^{(s+1)}\|.
$$

We estimate the second term on the right of (3.60) in the following way:

$$
(3.62)\qquad \left|\int_x^{i/n} u|f^{(s+2)}(u)|\,du\right| \le \frac{c}{n}\,\|\chi f^{(s+2)}\|_{[0,s/(s+1)]} \le \frac{c}{n}\,\|w\varphi^2 f^{(s+2)}\|.
$$

Combining (3.60)-(3.62) we deduce that

$$
(3.63)\qquad \left\|w\, r_{s,\circ}\left(\frac{i}{n}\right)\right\|_{I_n'} \le \frac{c}{n^{s+1}}\left(\|wf^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\|\right)
$$

for $\gamma_0 = 0$, $i = 1, \ldots, s$ and $n \ge s$ except $i = n = s$.

It remains to estimate the sup-norm of $w(x)r_{s,x}(i/n)$ on I_n' for $i = n = s$. It is enough to do so for the function $x^{\gamma_0} r_{s,x}(1)$ on $[0, 1/2]$. To this end, we split the integral in $r_{s,x}(1)$ by means of the intermediate point $1/2$ and consider the two quantities separately. For the first one we get for $x \in [0, 1/2]$ and $\gamma_0 > 0$

$$
(3.64)\qquad \begin{aligned} &\left|x^{\gamma_0}\int_x^{1/2} (1-u)^{s+1} f^{(s+2)}(u)\,du\right| \\ &\le x^{\gamma_0}\int_x^{1/2} u^{-\gamma_0-1}du\,\|\chi^{\gamma_0+1} f^{(s+2)}\|_{[0,1/2]} \\ &\le c\,\|w\varphi^2 f^{(s+2)}\|. \end{aligned}
$$

In the case $\gamma_0 = 0$ we apply the same considerations, by which we established (3.63), to arrive at

$$
(3.65)\qquad \left|\int_x^{1/2} (1-u)^{s+1} f^{(s+2)}(u)\,du\right| \le c\left(\|wf^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\|\right), \quad x \in [0, 1/2].
$$

For the other one we simply have for $x \in [0,1/2]$ and any $\gamma_0 \geq 0$

$$\begin{aligned} &\left| x^{\gamma_0} \int_{1/2}^{1} (1-u)^{s+1} f^{(s+2)}(u)\, du \right| \\ &\qquad \leq c \left| x^{\gamma_0} \int_{1/2}^{1} w(u)\varphi^2(u) f^{(s+2)}(u)\, du \right| \leq c\|w\varphi^2 f^{(s+2)}\|. \end{aligned} \tag{3.66}$$

Relations (3.64)-(3.66) show that

$$\|w\, r_{s,\circ}(1)\|_{[0,1/2]} \leq c \left(\|w f^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\| \right). \tag{3.67}$$

To summarize, (3.54), (3.56), (3.59), (3.63) and (3.67) yield

$$\left\| w\, r_{s,\circ}\left(\frac{i}{n}\right) \right\|_{I_n'} \leq \frac{c}{n^{s+1}} \left(\|w f^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\| \right) \tag{3.68}$$

for $i = 0, \ldots, s$, $n \geq s$ and a weight w satisfying the assumptions in assertion (3.46). Let us explicitly note that (3.63) and (3.65) are used only if $\gamma_0 = 0$. So the term $\|w f^{(s+1)}\|$ in (3.68) is redundant except when $\gamma_0 = 0$.

Now, (3.51), (3.53) and (3.68) imply

$$\|w R_{s,n} f\|_{I_n'} \leq \frac{c}{n} \left(\|w f^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\| \right). \tag{3.69}$$

Finally, estimates (3.50) and (3.69) yield (3.47). Thus (3.46) is verified. □

The upper estimate can be stated in a more concise form in terms of the differential operator $(d/dx)^s D$. This result follows directly from Proposition 2.6 and Proposition 3.17.

Corollary 3.18. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \leq \gamma_0, \gamma_1 < s$. Then for all $f \in AC^{s+1}[0,1]$ such that $w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_n f - f)^{(s)}\| \leq \frac{c}{n} \|w(Df)^{(s)}\|.$$

The value of the constant c is independent of f and n.

A very neat though generally less practical Jackson-type estimate of the error of simultaneous approximation by the Bernstein operator can be stated in terms of the differential operator D.

Proposition 3.19. *Let $s \in \mathbb{N}_+$. Then for all $f \in C^{2s+2}[0,1]$ and $n \in \mathbb{N}_+$ there holds*

$$\|D^s(B_n f - f)\| \le \frac{c}{n}\,\|D^{s+1}f\|.$$

The value of the constant c is independent of f and n.

Proof. The estimate is a direct corollary of (3.44) and Proposition 2.7. Indeed, applying consecutively Proposition 2.7(a), (3.44) with $w = \varphi^{2s}$, (4.6), and Proposition 2.7, (b) and (c), we get

$$\begin{aligned}\|D^s(B_n f - f)\| &\le c\left(\|B_n f - f\| + \|\varphi^{2s}(B_n f - f)^{(2s)}\|\right)\\ &\le \frac{c}{n}\left(\|Df\| + \|\varphi^{2s}f^{(2s)}\| + \|\varphi^{2s+2}f^{(2s+2)}\|\right)\\ &\le \frac{c}{n}\,\|D^{s+1}f\|.\end{aligned}$$

Thus the assertion of the proposition is verified. □

3.5.3 Voronovskaya-type estimates

We proceed to Voronovskaya-type estimates.

Proposition 3.20. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0,\gamma_1)$ be given by (2.2). Set $s'' := \max\{3, s\}$. If $0 < \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+3}(0,1)$ and $wf^{(s'')}, w\varphi^4 f^{(s+4)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\left\|w\left(B_n f - f - \frac{1}{2n}\,Df\right)^{(s)}\right\| \le \frac{c}{n^2}\left(\|wf^{(s'')}\| + \|w\varphi^4 f^{(s+4)}\|\right).$$

If $\gamma_0\gamma_1 = 0$ and still $0 \le \gamma_0, \gamma_1 \le s+1$, then

$$\left\|w\left(B_n f - f - \frac{1}{2n}\,Df\right)^{(s)}\right\| \le \frac{c}{n^2}\left(\|wf^{(s'')}\| + \|wf^{(s+2)}\| + \|w\varphi^4 f^{(s+4)}\|\right)$$

provided that $wf^{(s+2)} \in L_\infty[0,1]$ too.

The value of the constant c is independent of f and n.

Proof. First, let us note that if $\gamma_0, \gamma_1 > 0$, the assumption $w\varphi^4 f^{(s+4)} \in L_\infty[0,1]$ implies $wf^{(s'')} \in L_\infty[0,1]$. This follows from Proposition 2.1 with $w_1 = w$, $w_2 = w\varphi^4$, $j = s''$ and $m = s+4$.

The proof of the proposition is based on Lemma 3.13 and is similar to that of the previous proposition.

Using $\|w\varphi^4 f^{(s+4)}\| < \infty$, we get by Proposition 2.1 with $g = f$, $j = s+3$, $m = s+4$, $w_1 = \varphi^{2s+4}$ and $w_2 = w\varphi^4$ that $\varphi^{2s+4} f^{(s+3)} \in L_\infty[0,1]$ and we can apply Lemma 3.13.

We will prove that if $0 \le \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+3}(0,1)$ and $wf^{(s'')}, wf^{(s+2)}, w\varphi^4 f^{(s+4)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds

$$\left\| w \left(B_n f - f - \frac{1}{2n} \, Df \right)^{(s)} \right\| \le \frac{c}{n^2} \left(\|wf^{(s'')}\| + \|wf^{(s+2)}\| + \|w\varphi^4 f^{(s+4)}\| \right). \tag{3.70}$$

That establishes the second assertion of the proposition; the first one follows from (3.70) and

$$\|wf^{(s+2)}\| \le c \left(\|wf^{(s'')}\| + \|w\varphi^4 f^{(s+4)}\| \right),$$

which is established by Proposition 2.1 with $g = f^{(s'')}$, $j = s - s'' + 2$, $m = s - s'' + 4$, $w_1 = w$ and $w_2 = w\varphi^4$.

Let us set

$$\widetilde{R}_{s,n} f(x) := \frac{1}{(s+2)!} \sum_{k=0}^{n} p_{n,k}^{(s)}(x) \int_x^{k/n} \left(\frac{k}{n} - u \right)^{s+2} f^{(s+3)}(u) \, du.$$

We will show that

$$\|w\widetilde{R}_{s,n} f\| \le \frac{c}{n^2} \left(\|wf^{(s+2)}\| + \|w\varphi^2 f^{(s+3)}\| + \|w\varphi^4 f^{(s+4)}\| \right). \tag{3.71}$$

Then Lemma 3.13 implies

$$\|w(V_n f)^{(s)}\| \le \frac{c}{n^2} \left(\sum_{k=s''}^{s+2} \|wf^{(k)}\| + \|w\varphi^2 f^{(s+3)}\| + \|w\varphi^4 f^{(s+4)}\| \right), \tag{3.72}$$

where $V_n f(x)$ is defined in (3.28). By Proposition 2.2(c) with $g = f^{(s)}$, $j = 1$ and $m = 2$ we have for $s \geq 3$

$$\|wf^{(s+1)}\| \leq c\left(\|wf^{(s'')}\| + \|wf^{(s+2)}\|\right), \tag{3.73}$$

and by Proposition 2.2(a) with $g = f^{(s+2)}$, $j = 1$ and $m = 2$ we have

$$\|w\varphi^2 f^{(s+3)}\| \leq c\left(\|wf^{(s+2)}\| + \|w\varphi^4 f^{(s+4)}\|\right). \tag{3.74}$$

Now, estimate (3.70) follows from (3.72)-(3.74).

It remains to prove (3.71). We consider two cases for the domain of x.

Case 1. Let $n\varphi^2(x) \geq 1$. Since $w\varphi^4 f^{(s+4)} \in L_\infty[0,1]$, then $\varphi^{2s+6} f^{(s+4)} \in L[0,1]$; hence (3.22)-(3.23) are valid for $\sigma = s+2$. Using them we integrate by parts in $\widetilde{R}_{s,n} f$ and represent it in the form

$$\widetilde{R}_{s,n} f(x) = \widetilde{S}_{s,n} f(x) + \widetilde{R}'_{s,n} f(x),$$

where

$$\widetilde{S}_{s,n} f(x) := \frac{1}{(s+3)!} \sum_{k=0}^{n} p_{n,k}^{(s)}(x) \left(\frac{k}{n} - x\right)^{s+3} f^{(s+3)}(x)$$

and

$$\widetilde{R}'_{s,n} f(x) := \frac{1}{(s+3)!} \sum_{k=0}^{n} p_{n,k}^{(s)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{s+3} f^{(s+4)}(u)\, du.$$

We will show that

$$\left|\sum_{k=0}^{n} p_{n,k}^{(s)}(x) \left(\frac{k}{n} - x\right)^{s+3}\right| \leq \frac{c}{n^2}\, \varphi^2(x), \quad x \in I_n, \tag{3.75}$$

and

$$\|w\widetilde{R}'_{s,n} f\|_{I_n} \leq \frac{c}{n^2}\, \|w\varphi^4 f^{(s+4)}\|, \tag{3.76}$$

where $I_n := \{x \in [0,1] : n\varphi^2(x) \geq 1\}$. Then it will follow that

$$\|w\widetilde{R}_{s,n} f\|_{I_n} \leq \frac{c}{n^2}\left(\|w\varphi^2 f^{(s+3)}\| + \|w\varphi^4 f^{(s+4)}\|\right). \tag{3.77}$$

Estimate (3.75) follows directly from Lemma 3.11 with $\ell = s+3$ and from $n^{-1} \le \varphi^2(x)$.

We make use of (3.15) and Lemma 3.16 with $\delta = s+3$, $\alpha = \gamma_0 + 2$ and $\beta = \gamma_1 + 2$ to get

$$(3.78)\quad |w(x)\,\widetilde{R}'_{s,n} f(x)| \le \frac{c}{n^2} \sum_{0\le i\le s/2} \bigl(n\varphi^2(x)\bigr)^{i-s-2} \sum_{j=0}^{s-2i} \sum_{k=0}^{n} p_{n,k}(x)|k-nx|^{s+j+4} \|w\varphi^4 f^{(s+4)}\|.$$

Further, we apply estimate (3.19) and get

$$(3.79)\quad \sum_{0\le i\le s/2} \bigl(n\varphi^2(x)\bigr)^{i-s-2} \sum_{j=0}^{s-2i} \sum_{k=0}^{n} p_{n,k}(x)|k-nx|^{s+j+4} \le c \sum_{0\le i\le s/2} \sum_{j=0}^{s-2i} \bigl(n\varphi^2(x)\bigr)^{(2i+j-s)/2} \le c.$$

Now, (3.78) and (3.79) imply (3.76).

Case 2. Let $n\varphi^2(x) \le 1$ and, because of the symmetry, we may also assume that $x \le 1/2$. Just as in the proof of Proposition 3.17, case 2, we represent $\widetilde{R}_{s,n} f$ in the form

$$\widetilde{R}_{s,n} f(x) = \frac{1}{(s+2)!} \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}^{s}_{1/n} r_{s+1,x}\left(\frac{k}{n}\right) p_{n-s,k}(x)$$

and derive (cf. (3.51))

$$(3.80)\quad |w(x)\widetilde{R}_{s,n} f(x)| \le c\, n^s \max_{i=0,\dots,s} \sum_{k=0}^{n-s} \left| w(x)\, r_{s+1,x}\left(\frac{k+i}{n}\right)\right| p_{n-s,k}(x).$$

Just similarly to (3.53) and (3.68) we establish the following estimates

$$\left\| \sum_{k=1}^{n-s} w\, r_{s+1,\circ}\left(\frac{k+i}{n}\right) p_{n-s,k} \right\|_{I'_n} \le \frac{c}{n^{s+2}} \|w\varphi^2 f^{(s+3)}\|$$

and

$$\left\| w\, r_{s+1,\circ}\left(\frac{i}{n}\right) \right\|_{I'_n} \le \frac{c}{n^{s+2}} \bigl(\|w f^{(s+2)}\| + \|w\varphi^2 f^{(s+3)}\|\bigr)$$

for $i = 0, \dots, s$ and $n \geq s$. Actually the second estimate follows directly from (3.68).

Consequently,

$$\|w\widetilde{R}_{s,n}f\|_{I'_n} \leq \frac{c}{n^2}\left(\|wf^{(s+2)}\| + \|w\varphi^2 f^{(s+3)}\|\right). \tag{3.81}$$

Estimates (3.77) and (3.81) yield (3.71). Thus (3.70) is verified. □

Remark 3.21. In Proposition 3.20 we have assumed higher degree of smoothness than usual—$w\varphi^4 f^{(s+4)} \in L_\infty[0,1]$ rather than the weaker $w\varphi^3 f^{(s+3)} \in L_\infty[0,1]$. However, the latter assumption yields an order of $n^{-3/2}$ on the right in the corresponding Voronovskaya-type estimate. It still can be used to prove the converse inequality about simultaneous approximation by B_n, but the order of n^{-2} as in Proposition 3.20 seems more natural in this setting and is easier to work with (see [54, Lemma 2.1]).

Similarly to Corollary 3.18 we get by Proposition 2.6 and Proposition 3.20 the following Voronovskaya-type estimate.

Corollary 3.22. *Let $s \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \leq \gamma_0, \gamma_1 < s$. Then for all $f \in AC^{s+3}[0,1]$ such that $w\varphi^4 f^{(s+4)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\left\| w\left(B_n f - f - \frac{1}{2n}\, Df\right)^{(s)} \right\| \leq \frac{c}{n^2}\, \|w(D^2 f)^{(s)}\|.$$

The value of the constant c is independent of f and n.

3.5.4 Bernstein-type inequalities

The last several estimates, we will need, are traditionally regarded to as Bernstein-type inequalities.

Proposition 3.23. *Let $\ell, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \leq \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there hold:*

(a) $\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\| \leq c\, n^\ell \|wf^{(s)}\|$;

(b) $\|w(B_n f)^{(\ell+s)}\| \leq c\, n^\ell \|wf^{(s)}\|$.

The value of the constant c is independent of f and n.

Proof. Again we will consider two cases for the domain of x.

Case 1. Let $(n-s)\varphi^2(x) \geq 1$. Differentiating (3.4) we get

$$(B_n f)^{(2\ell+s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}^{(2\ell)}(x). \tag{3.82}$$

Next, we express $p_{n-s,k}^{(2\ell)}(x)$ by means of (3.15) and estimate $\left|\overrightarrow{\Delta}_{1/n}^{s} f(k/n)\right|$ by (3.30). Thus we arrive at

$$\begin{aligned} \left|w(x)\varphi^{2\ell}(x)(B_n f)^{(2\ell+s)}(x)\right| &\leq c\, n^{\ell} \sum_{i=0}^{\ell} \sum_{j=0}^{2(\ell-i)} \left(n\varphi^2(x)\right)^{i-\ell} \\ &\quad \times w(x) \sum_{k=0}^{n-s} p_{n-s,k}(x)\, w_{n,k}\, \|wf^{(s)}\|_{[k/n,(k+s)/n]} |k-(n-s)x|^j \\ &\leq c\, n^{\ell} \sum_{j=0}^{2\ell} \left(n\varphi^2(x)\right)^{-j/2} \\ &\quad \times w(x) \sum_{k=0}^{n-s} p_{n-s,k}(x)\, w_{p,n,k}\, \|wf^{(s)}\|_{[k/n,(k+s)/n]} |k-(n-s)x|^j, \end{aligned} \tag{3.83}$$

where at the last step we have used that $n\varphi^2(x) \geq 1$ and $i-\ell \leq -j/2$.

We have to estimate the weighted sup-norm of the right-hand side of the last inequality. Moreover, due to symmetry, we can restrict the range of summation on k to $\{0, \ldots, [(n-s)/2]\}$ (see (3.33)-(3.34) and note that $|k-(n-s)x| = |n-s-k-(n-s)y|$ with $y = 1-x$).

We apply Cauchy's inequality to derive

$$\begin{aligned} w(x) &\sum_{k=0}^{[(n-s)/2]} p_{n-s,k}(x)\, w_{n,k}\, \|wf^{(s)}\|_{[k/n,(k+s)/n]} |k-(n-s)x|^j \\ &\leq \left(w^2(x) \sum_{k=0}^{[(n-s)/2]} w_{n,k}^2\, p_{n-s,k}(x)\right)^{1/2} (T_{n-s,2j}(x))^{1/2}\, \|wf^{(s)}\|. \end{aligned} \tag{3.84}$$

Further, just as in (3.38) we see that

$$w^2(x)\sum_{k=0}^{[(n-s)/2]} w_{n,k}^2\, p_{n-s,k}(x) \le c, \quad x\in[0,1]. \tag{3.85}$$

Also, (3.18) yields

$$\big(n\varphi^2(x)\big)^{-j/2}\,(T_{n-s,2j}(x))^{1/2} \le c, \quad (n-s)\varphi^2(x)\ge 1. \tag{3.86}$$

Relations (3.83)-(3.86) imply

$$\|w\varphi^{2\ell}(B_nf)^{(2\ell+s)}\|_{I_{n-s}} \le c\,n^{\ell}\|wf^{(s)}\|, \tag{3.87}$$

where, to recall, $I_n := \{x\in[0,1] : n\varphi^2(x)\ge 1\}$.

Case 2. Let $(n-s)\varphi^2(x)\le 1$ and $n\ge 2\ell+s$. Differentiating ℓ times (3.4) with $\ell+s$ in place of s, we get

$$(B_nf)^{(2\ell+s)}(x) = \frac{n!}{(n-\ell-s)!}\sum_{k=0}^{n-\ell-s}\overrightarrow{\Delta}_{1/n}^{\ell+s}f\left(\frac{k}{n}\right)p_{n-\ell-s,k}^{(\ell)}(x).$$

Consequently,

$$\begin{aligned}|(B_nf)^{(2\ell+s)}(x)|& \\ &\le c\,n^{\ell}\max_{\nu=0,\dots,\ell}\frac{n!}{(n-s)!}\sum_{k=0}^{n-\ell-s}\left|\overrightarrow{\Delta}_{1/n}^{s}f\left(\frac{k+\nu}{n}\right)\right||p_{n-\ell-s,k}^{(\ell)}(x)|.\end{aligned} \tag{3.88}$$

Just as in Case 1 we estimate $\left|\overrightarrow{\Delta}_{1/n}^{s}f((k+\nu)/n)\right|$ by means of (3.30) and express $p_{n-\ell-s,k}^{(\ell)}(x)$ by means of (3.15). Thus for each $\nu=0,\dots,\ell$ we have

$$\begin{aligned}&\frac{n!}{(n-s)!}w(x)\varphi^{2\ell}(x)\sum_{k=0}^{n-\ell-s}\left|\overrightarrow{\Delta}_{1/n}^{s}f\left(\frac{k+\nu}{n}\right)\right||p_{n-\ell-s,k}^{(\ell)}(x)|\\
&\quad\le c\sum_{0\le i\le \ell/2}\sum_{j=0}^{\ell-2i}\big(n\varphi^2(x)\big)^{i}\,w(x)\sum_{k=0}^{n-\ell-s}p_{n-\ell-s,k}(x)\\
&\qquad\times w_{n,k+\nu}\,\|wf^{(s)}\|_{[(k+\nu)/n,(k+\nu+s)/n]}|k-(n-\ell-s)x|^{j}\\
&\quad\le c\sum_{j=0}^{\ell}w(x)\sum_{k=0}^{n-\ell-s}p_{n-\ell-s,k}(x)\\
&\qquad\times w_{n,k+\nu}\,\|wf^{(s)}\|_{[(k+\nu)/n,(k+\nu+s)/n]}|k-(n-\ell-s)x|^{j},\end{aligned} \tag{3.89}$$

where at the last estimate we have taken into account that $n\varphi^2(x) \le c$.

We proceed as in Case 1. Again due to symmetry it is sufficient to restrict the range of summation on k to $\{0, \dots, [(n-\ell-s)/2]\}$, as now we have with $\bar{k} = n - \ell - s - k$ and $\bar{\nu} = \ell - \nu$ (cf. (3.33)-(3.34)) the relations

$$\begin{aligned} &w_{n,\bar{k}+\nu} = \bar{w}_{n,k+\bar{\nu}}, \\ &\|wf^{(s)}\|_{[(\bar{k}+\nu)/n,(\bar{k}+\nu+s)/n]} = \|\bar{w}\bar{f}^{(s)}\|_{[(k+\bar{\nu})/n,(k+\bar{\nu}+s)/n]}. \end{aligned} \tag{3.90}$$

Let us note that we still have

$$w_{n,k+\nu} \le c\left(\frac{n}{k+1}\right)^{\gamma_0}, \quad 0 \le k \le (n-\ell-s)/2, \tag{3.91}$$

for $\nu = 0, \dots, \ell$. Consequently, there holds

$$w^2(x) \sum_{k=0}^{[(n-\ell-s)/2]} w_{n,k+\nu}^2 \, p_{n-\ell-s,k}(x) \le c, \quad x \in [0,1].$$

Also, (3.18) implies

$$T_{n-\ell-s,2j}(x) \le c, \quad (n-s)\varphi^2(x) \le 1.$$

Now, just similarly to Case 1, we derive from (3.88), (3.89), the symmetry on k, and the last two estimates above the inequality

$$\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\|_{I''_{n-s}} \le c\, n^{\ell} \|wf^{(s)}\|, \tag{3.92}$$

where $I''_n := \{x \in [0,1] : n\varphi^2(x) \le 1\}$.

Estimates (3.87) and (3.92) yield

$$\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\| \le c\, n^{\ell} \|wf^{(s)}\|.$$

To establish (b) we apply (3.4) with $\ell + s$ in place of s and (3.30). Thus we get

$$\begin{aligned} |(B_n f)^{(\ell+s)}(x)| &\le \frac{n!}{(n-\ell-s)!} \sum_{k=0}^{n-\ell-s} \left| \overrightarrow{\Delta}_{1/n}^{\ell+s} f\left(\frac{k}{n}\right) \right| p_{n-\ell-s,k}(x) \\ &\le c\, n^{\ell+s} \max_{\nu=0,\dots,\ell} \sum_{k=0}^{n-\ell-s} \left| \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k+\nu}{n}\right) \right| p_{n-\ell-s,k}(x) \\ &\le c\, n^{\ell} \max_{\nu=0,\dots,\ell} \sum_{k=0}^{n-\ell-s} w_{n,k+\nu}\, p_{n-\ell-s,k}(x)\, \|wf^{(s)}\|. \end{aligned}$$

To complete the proof we need only recall (3.91), (3.36)-(3.37) and use the symmetry on k, see (3.90). □

Further, we will state two analogues of the above Bernstein-type inequalities in terms of the differential operator D.

Corollary 3.24. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(DB_nf)^{(s)}\| \le c\,n\|wf^{(s)}\|.$$

The value of the constant c is independent of f and n.

Proof. We have

$$(Dg(x))^{(s)} = \varphi^2(x)g^{(s+2)}(x) + s(1-2x)g^{(s+1)}(x) - s(s-1)g^{(s)}(x).$$

Hence

$$\|w(Dg)^{(s)}\| \le c\big(\|wg^{(s')}\| + \|wg^{(s+1)}\| + \|w\varphi^2 g^{(s+2)}\|\big), \tag{3.93}$$

where $s' := \max\{2, s\}$.

Now, the assertion of the corollary follows from (3.93) with $g = B_nf$ and Propositions 3.14 and 3.23, (a) and (b), with $\ell = 1$. □

Corollary 3.25. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC^{s+1}[0,1]$ and $w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(D^2B_nf)^{(s)}\| \le c\,n\,\|w(Df)^{(s)}\|.$$

The value of the constant c is independent of f and n.

Proof. We iterate (3.93) to arrive with at

$$\begin{aligned}\|w(D^2g)^{(s)}\| &\le c\big(\|w(Dg)^{(s')}\| + \|w(Dg)^{(s+1)}\| + \|w\varphi^2(Dg)^{(s+2)}\|\big)\\ &\le c\big(\|wg^{(s')}\| + \|wg^{(s'+1)}\| + \|w\varphi^2 g^{(s'+2)}\|\big)\\ &\quad + c\big(\|wg^{(s+1)}\| + \|wg^{(s+2)}\| + \|w\varphi^2 g^{(s+3)}\|\big)\\ &\quad + c\big(\|w\varphi^2 g^{(s+2)}\| + \|w\varphi^2 g^{(s+3)}\| + \|w\varphi^4 g^{(s+4)}\|\big)\\ &\le c\big(\|wg^{(s')}\| + \|wg^{(s+1)}\| + \|wg^{(s+2)}\| + \|w\varphi^2 g^{(s+3)}\| + \|w\varphi^4 g^{(s+4)}\|\big).\end{aligned}$$

We apply this estimate with $g = B_n f$.

Proposition 3.14 implies

$$\|wg^{(s')}\| \le c\,\|wf^{(s')}\|$$

and

$$\|wg^{(s+1)}\| \le c\,\|wf^{(s+1)}\|;$$

Proposition 3.23(b) with $\ell = 1$ implies

$$\|wg^{(s+2)}\| \le c\,n\|wf^{(s+1)}\|;$$

Proposition 3.23(a) with $\ell = 1$ but $s+1$ in place of s implies

$$\|w\varphi^2 g^{(s+3)}\| \le c\,n\|wf^{(s+1)}\|;$$

Proposition 3.23(a) with $\ell = 1$ but $w\varphi^2$ in place of w and $s+2$ in place of s implies

$$\|w\varphi^4 g^{(s+4)}\| \le c\,n\|w\varphi^2 f^{(s+2)}\|.$$

We combine all the above estimates to get

$$\|w(D^2 B_n f)^{(s)}\| \le c\big(\|wf^{(s')}\| + n\|wf^{(s+1)}\| + n\|w\varphi^2 f^{(s+2)}\|\big).$$

Now, the assertion of the corollary follows from Proposition 2.6 with $r = 1$. □

3.6 Proof of the characterization of the rate of the simultaneous approximation

We are now able to give the proofs of the direct and converse estimates stated in Section 3.3.

Proof of Theorem 3.3. The estimate follows from Proposition 3.14 and Corollary 3.18 via a standard argument (see e.g. [22, Theorem 3.4]). Namely, for any $g \in C^{s+2}[0,1]$ we have

$$\begin{aligned}\|w(B_n f - f)^{(s)}\| &\le \|w(f^{(s)} - g^{(s)})\| + \|w(B_n g - g)^{(s)}\| \\ &\quad + \|w(B_n(f-g))^{(s)}\| \\ &\le c\left(\|w(f^{(s)} - g^{(s)})\| + \frac{1}{n}\|w(D^r g)^{(s)}\|\right).\end{aligned}$$

Taking an infimum on $g \in C^{s+2}[0,1]$, we arrive at (3.9). □

To prove the converse inequality in Theorem 3.8 we will apply the general method to prove such converse estimates given in [22].

Proof of Theorem 3.8. To establish the converse estimate we apply [22, Theorem 3.2] with the operator $Q_n = B_n$ on the space

$$X = \{f \in C[0,1] : f \in AC_{loc}^{s-1}(0,1), wf^{(s)} \in L_\infty[0,1]\}$$

with a semi-norm $\|f\|_X := \|wf^{(s)}\|$. Let us note that [22, Theorem 3.2] continues to hold for a semi-norm $\|\circ\|_X$ since in its proof the property that distinguishes a norm from a semi-norm is not used. Let also $Y = C^{s+2}[0,1]$ and $Z = C^{s+4}[0,1]$.

Proposition 3.14 implies that Q_n is a bounded operator on X, so that [22, (3.3)] holds.

By virtue of Corollary 3.22, we have for $\Phi(f) = \|w(D^2 f)^{(s)}\|$ and $f \in Z$

$$\left\| w\left(Q_n f - f - \frac{1}{2n}\, Df\right)^{(s)} \right\| \le \frac{c}{n^2}\,\Phi(f),$$

which shows that [22, (3.4)] is valid with $\lambda(n) = (2n)^{-1}$ and $\lambda_1(n) = c\,n^{-2}$, where the constant c is the one from Corollary 3.22.

Further, we set $g := B_n f$ for $f \in X$ and apply Corollary 3.25 to obtain

$$\Phi(Q_n^2 f) = \Phi(B_n g) \le c\,n\,\|w(Dg)^{(s)}\| = c\,n\,\|w(DB_n f)^{(s)}\|.$$

Hence [22, (3.5)] is established with $m = 2$ and $\ell = 1$.

Finally, Corollary 3.24 yields for $f \in X$

$$\|w(DQ_n f)^{(s)}\| \le c\,n\|wf^{(s)}\|,$$

which is [22, (3.6)].

Now, [22, Theorem 3.2] implies the assertion of the theorem. □

3.7 An improved converse estimate

We are able to prove a stronger converse estimate that the one given in Theorem 3.8 for small order derivatives and a narrower range of the weight exponents, but still including the unweighted case $w = 1$.

Theorem 3.26. *Let $s \in \mathbb{N}_+$ as $s \le 6$, and let $w := w(\gamma_0, \gamma_1)$ be given by (2.2) with $\gamma_0, \gamma_1 \in [0, s/2]$. Then there exists $n_0 \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ with $n \ge n_0$ there holds*

$$K_s^D(f^{(s)}, n^{-1})_w \le c\, \|w(B_n f - f)^{(s)}\|.$$

The value of the constant c is independent of f and n.

Remark 3.27. Using earlier results, it can be easily shown that, in the unweighted case, the converse inequality above holds for all n. We will demonstrate that after proving the theorem.

Remark 3.28. The proof of the theorem is based on a number of very technical results. In establishing just a small fragment of them (namely (3.140) for $j = 0$) we imposed an upper bound on s—all the other ones are verified for all positive integers s. Refinements of the calculations can yield the validity of the theorem for s larger than 6. However, it seems that settling the general case requires much effort or another approach.

Remark 3.29. The assumption $\gamma_0, \gamma_1 \in [0, s/2]$ in Theorem 3.26 is due to the method of proof we use. It is quite plausible that Theorem 3.26 remains valid for all $\gamma_0, \gamma_1 \in [0, s)$.

Theorem 3.26 holds for $s = 0$ (see [70, 93]). Its assertion for $s = 1$ and $w = 1$ has already been established in [54].

Combining Theorems 3.3 and 3.26, we verify that the error of the weighted simultaneous approximation by the Bernstein operator is equivalent to the K-functional $K_s^D(f^{(s)}, n^{-1})_w$. Thus the following characterization of the rate of the weighted simultaneous approximation by the Bernstein operator holds true.

Theorem 3.30. *Let $s \in \mathbb{N}_+$ as $s \le 6$, and let $w := w(\gamma_0, \gamma_1)$ be given by (2.2) with $\gamma_0, \gamma_1 \in [0, s/2]$. Then there exists $n_0 \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ with $n \ge n_0$ there holds*

$$\|w(B_n f - f)^{(s)}\| \sim K_s^D(f^{(s)}, n^{-1})_w.$$

Similarly, Theorems 3.5 and 3.26 along with (3.10)-(3.11) yield

Theorem 3.31. *Let $s \in \mathbb{N}_+$, as $s \le 6$, and $w := w(\gamma_0, \gamma_1)$ be given by* (2.2)*. Then there exists $n_0 \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ with $n \ge n_0$ there hold:*

$$\|w(B_n f - f)'\| \sim \omega_\varphi^2(f', n^{-1/2})_w + \omega_1(f', n^{-1})_w, \quad s = 1,\ 0 \le \gamma_0, \gamma_1 \le 1/2,$$

$$\|(B_n f - f)^{(s)}\| \sim \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + n^{-1}\|f^{(s)}\|, \\ 2 \le s \le 6,\ \gamma_0 = \gamma_1 = 0,$$

$$\|w(B_n f - f)^{(s)}\| \sim \omega_\varphi^2(f^{(s)}, n^{-1/2})_w + n^{-1}\|wf^{(s)}\|, \\ 2 \le s \le 6,\ 0 < \gamma_0, \gamma_1 \le s/2.$$

To compare, the characterization in the case $s = 0$ is of the form (see (3.2))

$$\|B_n f - f\| \sim \omega_\varphi^2(f, n^{-1/2}).$$

3.7.1 Strengthened Bernstein-type inequalities

To prove the converse inequality of Theorem 3.26, we again apply the method developed by Ditzian and Ivanov [22]. As we saw earlier in this chapter, it allows us to establish such converse estimates by means of several other basic estimates concerning the approximation properties of the operator. All but one of them were established in Section 3.5. What remains to be shown is that the more iterates of B_n we apply, the smaller constant we can take on the right-hand side of the Bernstein-type inequalities in Proposition 3.23 and Corollary 3.25.

As we established in Proposition 3.14, if $0 \le \gamma_0, \gamma_1 < s$, then

$$\|w(B_n f)^{(s)}\| \le c\,\|wf^{(s)}\| \tag{3.94}$$

for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$. We will need a stronger form of this estimate that gives an upper bound of the order by which the constant c can increase when we take iterates of the Bernstein operator.

Proposition 3.32. *Let $m, s \in \mathbb{N}_+$ as $m \ge 2$, and let $w := w(\gamma_0, \gamma_1)$ be given by* (2.2) *with $\gamma_0, \gamma_1 \in [0, s)$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ such that $n \ge m + s$ there holds*

$$\|w(B_n^m f)^{(s)}\| \le c \log m\,\|wf^{(s)}\|.$$

The value of the constant c is independent of f, n and m.

Proof. It is known that

$$(3.95)\quad \overrightarrow{\Delta}_h^s f(x) = \int_0^h \cdots \int_0^h f^{(s)}(x+u_1+\cdots+u_s)\,du_1\cdots du_s, \quad x\in[0,1-sh].$$

Note that, under the assumptions of the proposition, $f^{(s)}(x+u_1+\cdots+u_s)$ is a summable function of the variables $(u_1,\ldots,u_s)$ on the cube $[0,h]^s$ for each $x\in[0,1-sh]$.

Identities (3.4) and (3.95) yield the representation

$$(B_nf)^{(s)}(x) = \frac{n!}{(n-s)!} \times\sum_{k=0}^{n-s}\int_0^{1/n}\cdots\int_0^{1/n} f^{(s)}\left(\frac{k}{n}+u_1+\cdots+u_s\right)du_1\cdots du_s\,p_{n-s,k}(x),\ x\in[0,1].$$

Iterating it, we arrive at the formula

$$(3.96)\quad (B_n^mf)^{(s)}(x) = \frac{n!}{(n-s)!} \times\sum_{\bar{k}}\int_0^{1/n}\cdots\int_0^{1/n} f^{(s)}\left(\frac{k_1}{n}+u_1+\cdots+u_s\right)du_1\cdots du_s\,P_{n,s,\bar{k}}\,p_{n-s,k_m}(x),$$

where the summation is carried over $k_j=0,\ldots,n-s$, $j=1,\ldots,m$, and we have set $\bar{k}:=(k_1,\ldots,k_m)$,

(3.97)

$$P_{n,s,\bar{k}} := \prod_{j=1}^{m-1} p_{n,s,k_j}\left(\frac{k_{j+1}}{n}\right),$$

$$p_{n,i,k}(x) := \frac{n!}{(n-i)!}\int_0^{1/n}\cdots\int_0^{1/n} p_{n-i,k}(x+u_1+\cdots+u_i)\,du_1\cdots du_i.$$

Taking into account (3.95), we can write (3.96) in the form

$$(3.98)\quad (B_n^mf)^{(s)}(x) = \frac{n!}{(n-s)!}\sum_{\bar{k}}\overrightarrow{\Delta}_{1/n}^s f\left(\frac{k_1}{n}\right) P_{n,s,\bar{k}}\,p_{n-s,k_m}(x), \quad x\in[0,1].$$

As it follows from (3.30) and (3.35) and symmetry, there holds

$$\left|\overrightarrow{\Delta}^{s}_{1/n} f\left(\frac{k_1}{n}\right)\right| \le \frac{c}{n^s}\, w\left(\frac{k_1+1}{n}\right)^{-1} \|wf^{(s)}\|, \quad k_1 = 0, \ldots, n-s. \tag{3.99}$$

We will establish in (3.116) of Lemma 3.35 that

$$w(x)\sum_{\bar{k}} w\left(\frac{k_1+1}{n}\right)^{-1} P_{n,s,\bar{k}}\, p_{n-s,k_m}(x) \le c\,\log m, \quad x\in[0,1],$$

for $m \ge 2$ and $n \ge m+s$ with a constant c independent of m and n. Now, (3.98), (3.99) and the last estimate imply the assertion of the proposition. □

Next, we proceed to the Bernstein-type inequalities for the iterated Bernstein operator.

Proposition 3.33. *Let $m, s \in \mathbb{N}_+$ as $m \ge 2$, and let $w := w(\gamma_0, \gamma_1)$ be given by (2.2) with $\gamma_0, \gamma_1 \in [0, s/2]$. Then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ such that $n \ge m+s$ there hold:*

(a) $\|w\varphi(B_n^m f)^{(s+1)}\| \le c\sqrt{\frac{\log m}{m}}\,\sqrt{n}\,\|wf^{(s)}\|, \quad 2 \le s \le 9;$

(b) $\|w\varphi^2(B_n^m f)^{(s+2)}\| \le c\,\frac{\log m}{m}\,n\,\|wf^{(s)}\|, \quad 2 \le s \le 8;$

(c) $\|w(B_n^m f)^{(s+1)}\| \le c\sqrt{\frac{\log m}{m}}\,n\,\|wf^{(s)}\|, \quad 2 \le s \le 9.$

The value of the constant c is independent of f, n and m.

Proof. To prove assertion (a), we follow the argument in [70, pp. 318–320]. We differentiate (3.98) in x and apply the formula (see e.g. [18, Chapter 10, (2.1)])

$$p'_{n,k}(x) = n[p_{n-1,k-1}(x) - p_{n-1,k}(x)], \tag{3.100}$$

where we have set for convenience $p_{n,k} = 0$ if $k < 0$ or $k > n$. Then we use the Abel transform to derive $m-1$ different representations of $(B_n^m f)^{(s+1)}$. This is the key step in the considerations of Knoop and Zhou in [70, pp. 318–320].

Thus we arrive at the formula

$$(3.101)\quad (B_n^m f)^{(s+1)}(x) = \frac{1}{m-1}\,\frac{n!}{(n-s)!}\sum_{\bar{k}} \overrightarrow{\Delta}^{s}_{1/n} f\left(\frac{k_1}{n}\right) P_{n,s,\bar{k}} Q_{n,s,\bar{k}} p_{n-s-1,k_m}(x),$$

where the summation is carried over $k_j = 0, \dots, n-s$ and $j = 1, \dots, m$, $P_{n,s,\bar{k}}$ is given in (3.97), and we have set

$$Q_{n,s,\bar{k}} := \sum_{j=1}^{m-1} Q_{n,s,j,\bar{k}}, \quad Q_{n,s,m-1,\bar{k}} := \ell^*_{n,s,k_{m-1}}\left(\frac{k_m}{n}\right),$$

$$Q_{n,s,j,\bar{k}} := \ell^*_{n,s,k_j}\left(\frac{k_{j+1}}{n}\right) \ell_{n,s,k_{j+1}}\left(\frac{k_{j+2}}{n}\right) \cdots \ell_{n,s,k_{m-1}}\left(\frac{k_m}{n}\right), j = 1, \dots, m-2,$$

$$\ell^*_{n,s,k}(x) := \frac{(n-s)\int_0^{1/n} p'_{n,s,k}(x+u)\,du}{p_{n,s,k}(x)}, \quad \ell_{n,s,k}(x) := \frac{p_{n,s+1,k}(x)}{p_{n,s,k}(x)}.$$

Further, we apply Cauchy's inequality and (3.99) to derive from (3.101) the estimate

(3.102)

$$|w(x)\varphi(x)(B_n^m f)^{(s+1)}(x)| \le \frac{c}{m}\left(\varphi^2(x)\sum_{\bar{k}} P_{n,s,\bar{k}} Q^2_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x)\right)^{1/2} \times \|wf^{(s)}\| \left(w^2(x)\sum_{\bar{k}} w^{-2}\left(\frac{k_1+1}{n}\right) P_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x)\right)^{1/2}.$$

We will show in (3.127) of Lemma 3.36 below that

$$\varphi^2(x)\sum_{\bar{k}} P_{n,s,\bar{k}}\, Q^2_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\, mn, \quad x \in [0,1],$$

for $2 \le s \le 9$, $m \ge 2$, and $n \ge m+s$. Also, (3.117) of Lemma 3.35 with w^2 in place of w yields

$$w^2(x)\sum_{\bar{k}} w^{-2}\left(\frac{k_1+1}{n}\right) P_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\,\log m, \quad x \in [0,1],$$

for $m \geq 2$, and $n \geq m+s$. In view of these two inequalities, (3.102) implies assertion (a).

To prove (b) for even $m \geq 4$ we just apply (a) twice with $m/2$ in place of m, as the first time we take $w\varphi$ in place of w, and $s+1$ in place of s.

We reduce the case of odd $m \geq 5$ to the case of even m's greater than or equal to 4 by applying (3.94) with $B_n^{m-1}f$ in place of f, $s+2$ in place of s, and $w\varphi^2$ in place of w. Assertion (b) for $m = 2, 3$ follows directly from (3.94) and Proposition 3.23(a) with $\ell = 1$.

Assertion (c) is verified similarly to (a) as instead of (3.127) we use (3.128). □

Corollary 3.34. *Let $m, s \in \mathbb{N}_+$ as $s \leq 6$ and $m \geq 2$, and let $w := w(\gamma_0, \gamma_1)$ be given by* (2.2) *with $\gamma_0, \gamma_1 \in [0, s/2]$. Then for all $f \in C^{s+2}[0,1]$ and $n \in \mathbb{N}_+$ such that $n \geq m+s+2$ there holds*

$$\|w(D^2 B_n^m f)^{(s)}\| \leq c' \sqrt{\frac{\log m}{m}}\, n\|w(Df)^{(s)}\|.$$

The value of the constant c' is independent of f, n and m.

Proof. As we have shown in the beginning of the proof of Corollary 3.25,

$$\begin{aligned}(3.103)\quad &\|w(D^2 g)^{(s)}\| \\ &\leq c\big(\|wg^{(s')}\| + \|wg^{(s+1)}\| + \|wg^{(s+2)}\| + \|w\varphi^2 g^{(s+3)}\| + \|w\varphi^4 g^{(s+4)}\|\big).\end{aligned}$$

We will apply this estimate with $g = B_n^m f$ and show that each of the terms on the right above is estimated from above by $c\sqrt{\frac{\log m}{m}}\, n\|w(Df)^{(s)}\|$, where the constant c is independent of m, n and f.

By Proposition 2.6 with $r = 1$ we have

$$\|wf^{(s')}\| \leq c\,\|w(Df)^{(s)}\|, \tag{3.104}$$

$$\|wf^{(s+1)}\| \leq c\,\|w(Df)^{(s)}\| \tag{3.105}$$

and

$$\|w\varphi^2 f^{(s+2)}\| \leq c\,\|w(Df)^{(s)}\|. \tag{3.106}$$

Proposition 3.32 with s' in place of s and inequality (3.104) imply the estimates

$$\begin{aligned}\|w(B_n^m f)^{(s')}\| &\leq c\log m\|wf^{(s')}\| \leq c\log m\|w(Df)^{(s)}\| \\ &\leq c\sqrt{\frac{\log m}{m}}\, n\|w(Df)^{(s)}\|.\end{aligned} \tag{3.107}$$

Similarly, Proposition 3.32 with $s+1$ in place of s and (3.105) yield

$$\|w(B_n^m f)^{(s+1)}\| \le c\sqrt{\frac{\log m}{m}}\, n\|w(Df)^{(s)}\|. \tag{3.108}$$

Next, Proposition 3.33(c) with $s+1$ in place of s and (3.105) imply

$$\|w(B_n^m f)^{(s+2)}\| \le c\sqrt{\frac{\log m}{m}}\, n\,\|wf^{(s+1)}\| \le c\sqrt{\frac{\log m}{m}}\, n\,\|w(Df)^{(s)}\|. \tag{3.109}$$

Further, by Proposition 3.33(b) with $s+1$ in place of s, and (3.105), we get

$$\begin{aligned}\|w\varphi^2(B_n^m f)^{(s+3)}\| &\le c\,\frac{\log m}{m}\, n\,\|w\varphi^2 f^{(s+1)}\|\\ &\le c\sqrt{\frac{\log m}{m}}\, n\,\|w(Df)^{(s)}\|.\end{aligned} \tag{3.110}$$

Finally, again by means of Proposition 3.33(b) but with $s+2$ in place of s and $w\varphi^2$ in place of w, and (3.106) we arrive at

$$\begin{aligned}\|w\varphi^4(B_n^m f)^{(s+4)}\| &\le c\,\frac{\log m}{m}\, n\,\|w\varphi^2 f^{(s+2)}\|\\ &\le c\sqrt{\frac{\log m}{m}}\, n\,\|w(Df)^{(s)}\|.\end{aligned} \tag{3.111}$$

Estimates (3.103) and (3.107)-(3.111) imply the assertion of the corollary. □

3.7.2 Proof of the improved converse estimate

Below we will prove Theorem 3.26 and the assertion of Remark 3.27.

Proof of Theorem 3.26. We apply [22, Theorem 4.1] with the operator $Q_\alpha = B_n$ on the space

$$X = \{f \in C[0,1] : f \in AC_{loc}^{s-1}(0,1), wf^{(s)} \in L_\infty[0,1]\}$$

with the semi-norm $\|f\|_X := \|wf^{(s)}\|$. Let also $Y = C^{s+2}[0,1]$ and $Z = C^{s+4}[0,1]$.

Inequality (3.94) shows that B_n is a bounded operator on X, so that [22, (3.3)] holds.

Next, by virtue of Corollary 3.22 yields

$$\left\| w\left(B_n f - f - \frac{1}{2n}\, Df\right)^{(s)} \right\| \le \frac{c''}{n^2}\, \|w(D^2 f)^{(s)}\|, \quad f \in Z,$$

where c'' is a positive constant, which is independent of f and n. Thus [22, (3.4)] with $\Phi(f) := \|w(D^2 f)^{(s)}\|$, $\lambda(n) := 1/(2n)$ and $\lambda_1(n) := c''/n^2$ is valid.

Further, we apply Corollary 3.34 with $B_n f$ in place of f to obtain

$$\|w(D^2 B_n^{m+1} f)^{(s)}\| \le c' \sqrt{\frac{\log m}{m}}\, n \|w(DB_n f)^{(s)}\|, \quad f \in X.$$

Hence [22, (3.5)] is established with $m+1$ in place of m, $\ell = 1$, and

$$A = 2c'c'' \sqrt{\frac{\log m}{m}}.$$

We fix $m \ge 2$ so large that $A < 1$.

Finally, by Corollary 3.24 we have

$$\|w(DB_n f)^{(s)}\| \le c\, n \|w f^{(s)}\|, \quad f \in X,$$

which is [22, (3.6)] with $\ell = 1$.

Now, [22, Theorem 4.1] implies the converse estimate for $n \ge m+s+2$. □

Proof of Remark 3.27. To show that the converse estimate in Theorem 3.26 for $w = 1$ holds also for small n, we follow the considerations in [70, p. 317].

Let $n < m + s + 2$, where m is the positive integer fixed in the proof of Theorem 3.26. By Proposition 3.23(b) we have the estimates

$$\|(B_n f)^{(s+\ell)}\| \le c\, \|f^{(s)}\|, \quad \ell = 1, 2, \tag{3.112}$$

for $f \in C^s[0,1]$.

We readily deduce from (3.4) and (3.95) that for $f \in C^s[0,1]$ and $s \ge 2$ there holds

$$\|(B_n f)^{(s)}\| \le \frac{n-1}{n}\, \|f^{(s)}\|;$$

hence

$$\|(B_n^i f)^{(s)}\| \le \left(\frac{n-1}{n}\right)^i \|f^{(s)}\|. \tag{3.113}$$

Further, as is known [68, Theorem 1],

$$\lim_{i\to\infty} B_n^i f(x) = B_1 f(x), \quad x \in [0,1]. \tag{3.114}$$

We apply (3.94) and (3.112)-(3.114) to get for $j = 0, 1, 2$ and $s + j \geq 2$

$$\begin{aligned} \|(B_n f)^{(s+j)}\| &\leq \sum_{i=1}^{\infty} \left\| \left(B_n^i (f - B_n f)\right)^{(s+j)} \right\| \\ &\leq \sum_{i=1}^{\infty} \left(\frac{n-1}{n}\right)^{i-1} \left\| (B_n(f - B_n f))^{(s+j)} \right\| \\ &\leq c\, \|(f - B_n f)^{(s)}\|. \end{aligned} \tag{3.115}$$

By (3.93) with $w = 1$ we have

$$\|(DB_n f)^{(s)}\| \leq c \left(\|(B_n f)^{(s')}\| + \|(B_n f)^{(s+1)}\| + \|(B_n f)^{(s+2)}\| \right),$$

where $s' := \max\{s, 2\}$.

Now, (3.115) imply

$$\|(DB_n f)^{(s)}\| \leq c\, \|(f - B_n f)^{(s)}\|.$$

Consequently,

$$K_s^D(f^{(s)}, n^{-1})_1 \leq \|(f - B_n f)^{(s)}\| + \|(DB_n f)^{(s)}\| \leq c\, \|(f - B_n f)^{(s)}\|$$

for $n < m + s + 2$ as well. □

3.7.3 Auxiliary lemmas

Here we will provide proofs of the technical lemmas we used to verify Propositions 3.32 and 3.33.

Lemma 3.35. *Let $m, n, s \in \mathbb{N}_+$, $n \geq m + s$, $m \geq 2$ and $w := w(\gamma_0, \gamma_1)$ be given by* (2.2) *with $0 \leq \gamma_0, \gamma_1 \leq s$. Then*

$$w(x) \sum_{\bar{k}} w\left(\frac{k_1 + 1}{n}\right)^{-1} P_{n,s,\bar{k}}\, p_{n-s,k_m}(x) \leq c \log m, \quad x \in [0,1], \tag{3.116}$$

and

$$(3.117)\quad w(x)\sum_{\bar{k}} w\left(\frac{k_1+1}{n}\right)^{-1} P_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\,\log m, \quad x\in[0,1],$$

where the summation is carried over $k_j = 0,\dots,n-s$ *and* $j=1,\dots,m$. *The value of the constant* c *is independent of* m, n *and* x.

Proof. We follow the considerations of Knoop and Zhou [70] (see the proof of Lemma 3.1 there). Throughout c denotes a constant whose value is independent of m,n and x in the specified ranges.

By means of the inequalities:

$$\frac{1}{2}\left(x^{-\gamma_0}+(1-x)^{-\gamma_1}\right) \le w(x)^{-1} \le 2^{s-1}\left(x^{-\gamma_0}+(1-x)^{-\gamma_1}\right), \quad x\in(0,1),$$

Hölder's inequality and the relations

$$\sum_{\bar{k}} P_{n,s,\bar{k}} p_{n-s-r,k_m}(x) \equiv \left(\frac{n!}{(n-s)!\,n^s}\right)^{m-1} \le 1, \quad r=0,1,$$

we reduce the assertion of the lemma to the estimates

$$\sum_{\bar{k}}(k_1+1)^{-s}P_{n,s,\bar{k}}\, p_{n-s-r,k_m}(x) \le c\,\log m\,(nx)^{-s}, \quad x\in(0,1),$$

and

$$\sum_{\bar{k}}(n-k_1)^{-s}P_{n,s,\bar{k}}\, p_{n-s-r,k_m}(x) \le c\,\log m\,(n(1-x))^{-s}, \quad x\in(0,1),$$

where $r=0,1$.

We set for $\tau\in[0,1]$

$$F_{n,0}(\tau) := 1-\tau, \quad F_{n,j}(\tau) := 1-e^{-\frac{n-s}{n}F_{n,j-1}(\tau)}, \quad j=1,2,\dots .$$

Just as in [70, pp. 322–324] we show that

$$\sum_{\bar{k}}(k_1+1)^{-s}P_{n,s,\bar{k}}\, p_{n-s-r,k_m}(x)$$
$$\le \left(\frac{n}{n-s}\right)^{s(m-1)} \int_0^1\cdots\int_0^1 \frac{F^s_{n,m-1}(\tau_1\cdots\tau_s)}{F^s_{n,0}(\tau_1\cdots\tau_s)} e^{-(n-s-r)F_{n,m-1}(\tau_1\cdots\tau_s)x} d\tau_1\cdots d\tau_s$$

and

$$\sum_{\bar{k}} (n-k_1)^{-s} P_{n,s,\bar{k}}\, p_{n-s-r,k_m}(x)$$
$$\leq \left(\frac{n}{n-s}\right)^{s(m-1)} \int_0^1 \cdots \int_0^1 \frac{F^s_{n,m-1}(\tau_1 \cdots \tau_s)}{F^s_{n,0}(\tau_1 \cdots \tau_s)} e^{-(n-s-r)F_{n,m-1}(\tau_1 \cdots \tau_s)(1-x)} d\tau_1 \cdots d\tau_s.$$

Since

$$\left(\frac{n}{n-s}\right)^m \leq e^s, \quad n \geq m+s, \tag{3.118}$$

to complete the proof of the lemma it is sufficient to show

$$\int_0^1 \cdots \int_0^1 \frac{F^s_{n,m-1}(\tau_1 \cdots \tau_s)}{F^s_{n,0}(\tau_1 \cdots \tau_s)} e^{-(n-s-1)F_{n,m-1}(\tau_1 \cdots \tau_s)x} d\tau_1 \cdots \tau_s \leq c \log m\,(nx)^{-s} \tag{3.119}$$

for all $n \geq m+s$, $m \geq 2$ and $x \in (0,1]$.

Using that $y^s e^{-y} \leq c$, $y \geq 0$, we get

$$F^s_{n,m-1}(\tau) e^{-(n-s-1)F_{n,m-1}(\tau)x} \leq c(nx)^{-s}, \quad x \in (0,1],\ \tau \in [0,1]. \tag{3.120}$$

Also, we clearly have $F_{n,0}(\tau) \geq 1/2$ for $\tau \in [0,1/2]$. Therefore, if $\mathcal{D} \subset [0,1]^s$ is a parallelepiped with at least one side of the form $[0,1/2]$, then

$$\int_{\mathcal{D}} \frac{F^s_{n,m-1}(\tau_1 \cdots \tau_s)}{F^s_{n,0}(\tau_1 \cdots \tau_s)} e^{-(n-s-1)F_{n,m-1}(\tau_1 \cdots \tau_s)x}\, d\tau_1 \cdots d\tau_s \leq c(nx)^{-s} \tag{3.121}$$

for all $n \geq m+s$, $m \geq 2$ and $x \in (0,1]$.

In order to estimate the integral on the cube $[1/2,1]^s$, we set

$$F_{n,m-1}(\tau, x) := \frac{F^s_{n,m-1}(\tau)}{F^s_{n,0}(\tau)} e^{-(n-s-1)F_{n,m-1}(\tau)x},$$

make the change of the variables, defined by the formulae $\sigma_j = \tau_1 \cdots \tau_j$,

$j = 1, \ldots, s$, and arrange the order of integration from σ_1 to σ_s to get

$$\int_{1/2}^{1} \cdots \int_{1/2}^{1} F_{n,m-1}(\tau_1 \cdots \tau_s, x)\, d\tau_1 \cdots d\tau_s$$
$$\leq \int_{2^{-s}}^{1} \left[F_{n,m-1}(\sigma_s, x) \int_{\sigma_s}^{1} \left(\cdots \left(\frac{1}{\sigma_3} \int_{\sigma_3}^{1} \left(\frac{1}{\sigma_2} \int_{\sigma_2}^{1} \frac{1}{\sigma_1}\, d\sigma_1 \right) d\sigma_2 \right) \cdots \right) d\sigma_{s-1} \right] d\sigma_s$$
$$\leq c \int_{2^{-s}}^{1} \left[F_n(\sigma_s, x) \int_{\sigma_s}^{1} \left(\cdots \left(\int_{\sigma_3}^{1} \left(\int_{\sigma_2}^{1} d\sigma_1 \right) d\sigma_2 \right) \cdots \right) d\sigma_{s-1} \right] d\sigma_s$$
$$\leq c \int_{2^{-s}}^{1} F_{n,m-1}(\sigma, x)(1-\sigma)^{s-1} d\sigma.$$

We make the change of the variable $\sigma = 1 - t$ and set $G_{n,j}(t) := F_{n,j}(1-t)$. Thus we arrive at

$$\int_{1/2}^{1} \cdots \int_{1/2}^{1} \frac{F_{n,m-1}^{s}(\tau_1 \cdots \tau_s)}{F_{n,0}^{s}(\tau_1 \cdots \tau_s)}\, e^{-(n-s-1)F_{n,m-1}(\tau_1 \cdots \tau_s)x}\, d\tau_1 \cdots d\tau_s \leq c \int_{0}^{1} t^{-1} G_{n,m-1}^{s}(t)\, e^{-(n-s-1)G_{n,m-1}(t)x}\, dt. \tag{3.122}$$

By means of induction on m we show that (cf. [70, (4.7)])

$$\left(\frac{n-s}{n} \right)^{m} \left(t - \frac{m}{2}\, t^2 \right) \leq G_{n,m-1}(t) \leq t, \quad t \in [0,1]. \tag{3.123}$$

We split the integral on the right-hand side of (3.122) by means of the intermediate point $1/m$. For the one between 0 and $1/m$ we apply (3.118) and (3.123) to get

$$\int_{0}^{1/m} t^{-1} G_{n,m-1}^{s}(t)\, e^{-(n-s-1)G_{n,m-1}(t)x}\, dt \leq \int_{0}^{1} t^{s-1} e^{-cnxt}\, dt \leq c(nx)^{-s}, \tag{3.124}$$

as the last estimate is verified by integration by parts.

For the other integral we again use (3.120) to derive

$$\int_{1/m}^{1} t^{-1} G_{n,m-1}^{s}(t)\, e^{-(n-s-1)G_{n,m-1}(t)x}\, dt \leq c(nx)^{-s} \int_{1/m}^{1} \frac{dt}{t} = c \log m\, (nx)^{-s}. \tag{3.125}$$

Estimates (3.122), (3.124) and (3.125) yield

$$\int_{1/2}^{1} \cdots \int_{1/2}^{1} \frac{F^s_{n,m-1}(\tau)}{F^s_{n,0}(\tau)}\, e^{-(n-s-1)F_{n,m-1}(\tau)x}\, d\tau_1 \cdots d\tau_s \le c\,\log m\,(nx)^{-s} \tag{3.126}$$

for all $n \ge m+s$, $m \ge 2$ and $x \in (0,1]$.

Now, (3.121) and (3.126) imply (3.119). □

Lemma 3.36. *Let $m, n, s \in \mathbb{N}_+$ as $2 \le s \le 9$, $m \ge 2$, and $n \ge m+s$. Then*

$$\varphi^2(x) \sum_{\bar{k}} P_{n,s,\bar{k}}\, Q^2_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\,mn, \quad x \in [0,1], \tag{3.127}$$

and

$$\sum_{\bar{k}} P_{n,s,\bar{k}}\, Q^2_{n,s,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\,mn^2, \quad x \in [0,1], \tag{3.128}$$

where the summation is carried over $k_j = 0, \dots, n-s$ and $j = 1, \dots, m$. The value of the constant c is independent of m, n and x.

Remark 3.37. The proof of the lemma is reduced to several simpler inequalities. All but one of them is verified for all $s \ge 2$ (see Remark 3.40).

Proof of Lemma 3.36. Both estimates are verified just like [70, Lemma 3.2], where the case $s = 2$ was considered. We will indicate the modifications we need to make. Often that amounts only to replacing $n-2$ with $n-s$.

To establish (3.127) it is enough to verify (see [70, p. 328]) that

$$\sum_{\bar{k}} P_{n,s,\bar{k}}\, Q^2_{n,s,j,\bar{k}}\, p_{n-s-1,k_m}(x) \le c\,n\,\varphi^{-2}(x), \quad x \in (0,1), \quad j = 1, \dots, m-1.$$

It follows from the estimates:

$$\sum_{k=0}^{n-s} p_{n,s,k}\left(\frac{j}{n}\right) \ell^*_{n,s,k}\left(\frac{j}{n}\right)^2 \le c\,n\,\varphi^{-2}\left(\frac{j+1}{n-s+1}\right) = \frac{c\,n(n-s+1)^2}{(j+1)(n-s-j)} \tag{3.129}$$

and

$$\sum_{k=0}^{n-s-1} \frac{p_{n,s+1,k}^2\left(\frac{j}{n}\right)}{(k+1)(n-s-k)p_{n,s,k}\left(\frac{j}{n}\right)} \le \frac{1+\frac{c}{n}}{(j+1)(n-s-j)}, \tag{3.130}$$

for $j = 0, \ldots, n-s-1$, and also

$$\sum_{k=0}^{n-s-1} \frac{p_{n-s-1,k}(x)}{(k+1)(n-s-k)} \le \frac{c}{n^2}\,\varphi^{-2}(x), \quad x \in (0,1). \tag{3.131}$$

Inequalities (3.129) and (3.130) are established in Lemmas 3.38 and 3.39 below, and (3.131) directly follows from [70, (4.21)] with $n-2$ replaced with $n-s$.

Similarly, (3.128) follows from (3.129) and (3.130) and the trivial inequality

$$\sum_{k=0}^{n-s-1} \frac{p_{n-s-1,k}(x)}{(k+1)(n-s-k)} \le \frac{c}{n}, \quad x \in [0,1].$$

□

Lemma 3.38. *Let $n, s \in \mathbb{N}_+$, as $n \ge s+2$. Then*

$$\sum_{k=0}^{n-s} p_{n,s,k}\left(\frac{j}{n}\right) \ell^*_{n,s,k}\left(\frac{j}{n}\right)^2 \le c\,n\,\varphi^{-2}\left(\frac{j+1}{n-s+1}\right) \tag{3.132}$$

for $j = 0, \ldots, n-s-1$. The value of the constant c is independent of n.

Proof. We estimate each of the summands on the left-hand side as we consider two cases: $j = 0, n-s-1$ and $1 \le j \le n-s-2$.

For $j = 0$ we apply (3.100) to derive

$$\begin{aligned} &n^{s+1}\int_0^{1/n}\cdots\int_0^{1/n} |p'_{n-s,k}(u_1+\cdots+u_{s+1})|\,du_1\cdots du_{s+1} \\ &\quad\le (n-s)\left[\binom{n-s-1}{k-1}\left(\frac{s+1}{n}\right)^{k-1} + \binom{n-s-1}{k}\left(\frac{s+1}{n}\right)^{k}\right] \end{aligned} \tag{3.133}$$

for $k = 0, \ldots, n-s$, as we set for convenience $\binom{\alpha}{-1} = 0$.

Using that

$$(u_1+\cdots+u_s)^k \ge u_1^k$$

and

$$(1-u_1-\cdots-u_s)^{n-s-k}\ge\left(1-\frac{s}{n}\right)^n\ge c,$$

we estimate the denominators of the terms on the left-hand side of (3.132) by

$$n^s\int_0^{1/n}\cdots\int_0^{1/n}p_{n-s,k}(u_1+\cdots+u_s)\,du_1\cdots du_s\ge\frac{c}{k+1}\binom{n-s}{k}\frac{1}{n^k}. \tag{3.134}$$

Estimates (3.133) and (3.134) yield (cf. [70, p. 326])

$$\begin{aligned}\sum_{k=0}^{n-s}p_{n,s,k}\left(\frac{j}{n}\right)\ell^*_{n,s,k}\left(\frac{j}{n}\right)^2\le c\,n^2\Bigg\{&\sum_{k=1}^{n-s}\frac{(k+1)\binom{n-s-1}{k-1}^2}{\binom{n-s}{k}}\left[\frac{(s+1)^2}{n}\right]^{k-2}\\&+\sum_{k=0}^{n-s-1}\frac{(k+1)\binom{n-s-1}{k}^2}{\binom{n-s}{k}}\left[\frac{(s+1)^2}{n}\right]^{k}\Bigg\}.\end{aligned}$$

To complete the proof of the lemma for $j=0$, it remains to show that the two sums on the right above are bounded on n. For the first one we have

$$\begin{aligned}\sum_{k=1}^{n-s}&\frac{(k+1)\binom{n-s-1}{k-1}^2}{\binom{n-s}{k}}\left[\frac{(s+1)^2}{n}\right]^{k-2}\\&\le c\left(1+\sum_{k=3}^{n-s}\binom{n-s-3}{k-3}\left[\frac{(s+1)^2}{n}\right]^{k-3}\right)\\&=c\left(1+\sum_{k=0}^{n-s-3}\binom{n-s-3}{k}\left[\frac{(s+1)^2}{n}\right]^{k}\right)\\&\le c\left(1+\frac{(s+1)^2}{n}\right)^{n-s-3}\le c\,e^{(s+1)^2}.\end{aligned}$$

The other sum is treated in a similar way.

Next, we reduce the case $j=n-s-1$ to $j=0$. More precisely, we make the change of the variables $v_i=1/n-u_i$, $i=1,\dots,s+1$, and apply (3.135) to arrive at

$$\int_0^{1/n} \cdots \int_0^{1/n} p'_{n-s,k}\left(\frac{n-s-1}{n} + u_1 + \cdots + u_{s+1}\right) du_1 \cdots du_{s+1}$$
$$= \int_0^{1/n} \cdots \int_0^{1/n} p'_{n-s,k}(1 - v_1 - \cdots - v_{s+1})\, dv_1 \cdots dv_{s+1}$$
$$= -\int_0^{1/n} \cdots \int_0^{1/n} p'_{n-s,n-s-k}(v_1 + \cdots + v_{s+1})\, dv_1 \cdots dv_{s+1};$$

similarly, using the same change of the variables and the inequality

$$\left(\frac{1 - v_1 - \cdots - v_s - \frac{1}{n}}{1 - v_1 - \cdots - v_s}\right)^k \geq \left(1 - \frac{1}{n-s}\right)^{n-s} \geq c,$$

we deduce

$$p_{n,s,k}\left(\frac{n-s-1}{n}\right)$$
$$= \frac{n!}{(n-s)!} \int_0^{1/n} \cdots \int_0^{1/n} p_{n-s,n-s-k}\left(v_1 + \cdots + v_s + \frac{1}{n}\right) dv_1 \cdots dv_s$$
$$\geq c\, p_{n,s,n-s-k}(0).$$

Consequently,

$$p_{n,s,k}\left(\frac{n-s-1}{n}\right) \ell^*_{n,s,k}\left(\frac{n-s-1}{n}\right)^2 \leq c\, p_{n,s,n-s-k}(0) \ell^*_{n,s,n-s-k}(0)^2.$$

It only remains to observe that

$$\varphi^2\left(\frac{n-s}{n-s+1}\right) = \varphi^2\left(\frac{1}{n-s+1}\right)$$

to derive the assertion of the lemma for $j = n-s-1$ from the one for $j = 0$.

Let $1 \leq j \leq n-s-2$. Set $U := j/n + u_1 + \cdots + u_{s+1}$.

As we have already noted (see e.g. [18, Chapter 10, (2.1)]),

$$p'_{n,k}(x) = \varphi^{-2}(x)(k - nx)p_{n,k}(x). \tag{3.135}$$

By means of that identity and Cauchy's inequality, we get

$$p_{n,s,k}\left(\frac{j}{n}\right)\ell^*_{s,n,k}\left(\frac{j}{n}\right)^2 \le c\,n^{s+3}\int_0^{1/n}\cdots\int_0^{1/n}\varphi^{-4}(U)$$
$$\times\left(\frac{k}{n-s}-U\right)^2 \frac{p^2_{n-s,k}\left(\frac{j}{n}+u_1+\cdots+u_{s+1}\right)}{p_{n-s,k}\left(\frac{j}{n}+u_1+\cdots+u_s\right)}\,du_1\cdots du_{s+1}.$$

Further, we set

$$A := \frac{\left(\frac{j}{n}+u_1+\cdots+u_{s+1}\right)^2}{\frac{j}{n}+u_1+\cdots+u_s},\quad B := \frac{\left(1-\frac{j}{n}-u_1-\cdots-u_{s+1}\right)^2}{1-\frac{j}{n}-u_1-\cdots-u_s}. \tag{3.136}$$

There hold

$$\varphi^2(U) \ge c\,\varphi^2\left(\frac{j+1}{n-s+1}\right) \ge \frac{c}{n}, \tag{3.137}$$

$$A+B = 1+\frac{u^2_{s+1}}{\varphi^2\left(\frac{j}{n}+u_1+\cdots+u_s\right)} \le 1+\frac{c}{n} \tag{3.138}$$

and

$$\left(\frac{k}{n-s}-U\right)^2 \le 2\left(\frac{k}{n-s}-A\right)^2+\frac{c}{n^2}$$

for $0 \le u_i \le 1/n$, $i = 1,\ldots,s+1$.

Consequently, if we denote the sum at the left-hand side of (3.132) by S, we get

$$S \le \frac{c\,n^{s+3}}{\varphi^4\left(\frac{j+1}{n-s+1}\right)} \times\left[\int_0^{1/n}\cdots\int_0^{1/n}\left(\sum_{k=0}^{n-s}\left(\frac{k}{n-s}-A\right)^2\binom{n-s}{k}A^kB^{n-s-k} + \frac{1}{n^2}(A+B)^{n-s}\right)du_1\cdots du_{s+1}\right]. \tag{3.139}$$

Using (3.137) and (3.138), we readily get

$$\frac{c\,n^{s+1}}{\varphi^4\left(\frac{j+1}{n-s+1}\right)}\int_0^{1/n}\cdots\int_0^{1/n}(A+B)^{n-s}\,du_1\cdots du_{s+1} \le c\,n\,\varphi^{-2}\left(\frac{j+1}{n-s+1}\right).$$

So, to complete the proof of (3.132) for $1 \le j \le n-s-2$, it remains to estimate the first multiple integral on the right of (3.139). To this end, we apply the identity (cf. [70, (4.18)])

$$\sum_{k=0}^{n-s} \left(\frac{k}{n-s} - A \right)^2 \binom{n-s}{k} A^k B^{n-s-k} = (A+B)^{n-s-2} \left(A^2 (A+B-1)^2 + \frac{AB}{n-s} \right),$$

inequality (3.138) and the estimate (cf. [70, (4.19)])

$$A^2(A+B-1)^2 + \frac{AB}{n-s} \le \frac{c}{n} \varphi^2 \left(\frac{j+1}{n-s+1} \right).$$

The latter follows from the inequalities

$$A \le c\,\frac{j+1}{n}, \quad B \le c\,\frac{n-s-j}{n},$$

(3.137) and (3.138). □

Lemma 3.39. *Let $n, s \in \mathbb{N}_+$ as $2 \le s \le 9$ and $n \ge s+2$. Then*

$$\sum_{k=0}^{n-s-1} \frac{p_{n,s+1,k}^2 \left(\frac{j}{n}\right)}{(k+1)(n-s-k) p_{n,s,k}\left(\frac{j}{n}\right)} \le \frac{1+\frac{c}{n}}{(j+1)(n-s-j)} \tag{3.140}$$

for $j = 0, \dots, n-s-1$. The value of the constant c is independent of n.

Remark 3.40. The assertion of the lemma for $j = 1, \dots, n-s-1$ is verified for any positive integer $s \ge 2$ in the proof below.

Proof of Lemma 3.39. The assertion of the lemma was verified for $s = 2$ in [70, (4.20)]. So, we can assume that $s \ge 3$.

First, let $j = 0$. In order to estimate the denominators of the terms on the left-hand side of (3.140), we expand $(u_1 + \cdots + u_s)^k$ by the binomial formula to get

$$(u_1 + \cdots + u_s)^k = \sum_{i=0}^{k} \binom{k}{i} (u_1 + \cdots + u_{s-1})^{k-i} u_s^i,$$

apply the trivial estimate

$$(1-u_1-\cdots-u_s)^{n-s-k} \geq \left(1-u_1-\cdots-u_{s-1}-\frac{1}{n}\right)^{n-s-k}$$

for $u_s \in [0,1/n]$ and integrate on $u_s \in [0,1/n]$. Thus we get

$$\begin{aligned} p_{n,s,k}(0) \geq \frac{n!}{(n-s)!}\frac{1}{n(k+1)}\int_0^{1/n}\cdots\int_0^{1/n}\binom{n-s}{k} \\ \times \sum_{i=0}^{k}\binom{k}{i}(u_1+\cdots+u_{s-1})^{k-i}\left(\frac{1}{n}\right)^i \\ \times\left(1-u_1-\cdots-u_{s-1}-\frac{1}{n}\right)^{n-s-k} du_1\dots du_{s-1}. \end{aligned}$$

We apply the binomial formula once again and arrive at

$$\begin{aligned} & p_{n,s,k}(0) \\ & \geq \frac{(n-1)!}{(k+1)(n-s)!}\int_0^{1/n}\cdots\int_0^{1/n} p_{n-s,k}\left(u_1+\cdots+u_{s-1}+\frac{1}{n}\right)du_1\cdots du_{s-1}. \end{aligned}$$

Further, we use Cauchy's inequality to get the estimate

$$\begin{aligned} \text{(3.141)}\quad & \frac{p^2_{n,s+1,k}(0)}{(k+1)(n-s-k)\,p_{n,s,k}(0)} \\ & \leq n^{s+1}\int_0^{1/n}\cdots\int_0^{1/n}\frac{p^2_{n-s-1,k}(u_1+\cdots+u_{s+1})}{(n-s-k)\,p_{n-s,k}\left(u_1+\cdots+u_{s-1}+\frac{1}{n}\right)}\,du_1\cdots du_{s+1}. \end{aligned}$$

We set

$$\widetilde{A} := \frac{(u_1+\cdots+u_{s+1})^2}{u_1+\cdots+u_{s-1}+\frac{1}{n}}, \quad \widetilde{B} := \frac{(1-u_1-\cdots-u_{s+1})^2}{1-u_1-\cdots-u_{s-1}-\frac{1}{n}}.$$

Then (3.141) yields

$$\begin{aligned} & \frac{p^2_{n,s+1,k}(0)}{(k+1)(n-s-k)\,p_{n,s,k}(0)} \\ & \qquad \leq \frac{n^{s+2}}{(n-s)^2}\int_0^{1/n}\cdots\int_0^{1/n}\binom{n-s-1}{k}\widetilde{A}^k\widetilde{B}^{n-s-1-k}\,du_1\cdots du_{s+1} \end{aligned}$$

and, consequently, for $n \geq s+3$ we have

$$(3.142)\qquad \sum_{k=2}^{n-s-1} \frac{p_{n,s+1,k}^2(0)}{(k+1)(n-s-k)\,p_{n,s,k}(0)} \leq \frac{n^{s+2}}{(n-s)^2}$$
$$\times \int_0^{1/n} \cdots \int_0^{1/n} [(\widetilde{A}+\widetilde{B})^{n-s-1} - \widetilde{B}^{n-s-1} - (n-s-1)\widetilde{A}\widetilde{B}^{n-s-2}]\, du_1 \cdots du_{s+1}.$$

By means of the inequality $1+x \leq e^x$, we get

$$\begin{aligned}\widetilde{A}+\widetilde{B} &= 1 + \frac{\left(u_s + u_{s+1} - \frac{1}{n}\right)^2}{\left(u_1 + \cdots + u_{s-1} + \frac{1}{n}\right)\left(1 - u_1 - \cdots - u_{s-1} - \frac{1}{n}\right)} \\ &\leq 1 + \frac{\left(u_s + u_{s+1} - \frac{1}{n}\right)^2}{\left(u_1 + \cdots + u_{s-1} + \frac{1}{n}\right)\left(1 - \frac{s}{n}\right)} \\ &\leq e^{\frac{(nu_s+nu_{s+1}-1)^2}{(n-s)(nu_1+\cdots+nu_{s-1}+1)}}.\end{aligned}$$

Therefore

$$(3.143)\qquad (\widetilde{A}+\widetilde{B})^{n-s-1} \leq e^{\frac{(nu_s+nu_{s+1}-1)^2}{nu_1+\cdots+nu_{s-1}+1}}.$$

Similarly, by means of the inequality $1+x \geq (1-x^2)e^x$, $x \in [-1,1]$, we establish

$$\begin{aligned}\widetilde{B} &\geq 1 + \frac{1}{n} - u_1 - \cdots - u_{s-1} - 2u_s - 2u_{s+1} \\ &\geq \left(1 - \left(\frac{s+2}{n}\right)^2\right) e^{\frac{1}{n} - u_1 - \cdots - u_{s-1} - 2u_s - 2u_{s+1}};\end{aligned}$$

hence, using Bernoulli's inequality $(1+x)^n \geq 1+nx$ for $x \geq -1$, and $e^x \geq 1+x$, we derive

$$(3.144)\qquad \widetilde{B}^{n-s-j} \geq \left(1 - \frac{c}{n}\right) e^{1-nu_1-\cdots-nu_{s-1}-2nu_s-2nu_{s+1}}, \quad j = 1,2.$$

We apply estimates (3.142)-(3.144), make the change of the variables $t_i = nu_i$, $i = 1,\ldots,s+1$, and use the representation

$$\frac{(t_1+\cdots+t_{s+1})^2}{t_1+\cdots+t_{s-1}+1} = -1 + t_1 + \cdots + t_{s-1} + 2t_s + 2t_{s+1} + \frac{(t_s+t_{s+1}-1)^2}{t_1+\cdots+t_{s-1}+1}$$

to obtain

$$\sum_{k=2}^{n-s-1} \frac{p^2_{n,s+1,k}(0)}{(k+1)(n-s-k)\,p_{n,s,k}(0)} \le \frac{1}{n-s}\left(1+\frac{c}{n}\right)(I'_s - I''_s), \tag{3.145}$$

where we have set

$$I'_s := \int_0^1 \cdots \int_0^1 e^{\frac{(t_s+t_{s+1}-1)^2}{t_1+\cdots+t_{s-1}+1}} dt_1 \cdots dt_{s+1},$$
$$I''_s := \int_0^1 \cdots \int_0^1 (t_1+\cdots+t_{s-1}+2t_s+2t_{s+1}) e^{1-t_1-\cdots-t_{s-1}-2t_s-2t_{s+1}} dt_1 \cdots dt_{s+1}.$$

We estimate the first integral by means of the inequality

$$e^x \le 1 + x + \frac{e\,x^2}{2}, \quad x \in [0,1],$$

and direct computations. Thus we get

$$\begin{array}{llll} I'_3 \le 1.11327, & I'_4 \le 1.08629, & I'_5 \le 1.06929, & I'_6 \le 1.05773, \\ I'_7 \le 1.0494, & I'_8 \le 1.04314, & I'_9 \le 1.03827. & \end{array} \tag{3.146}$$

We evaluate I''_s and get

$$I''_s = \frac{e}{4}\left(1-e^{-1}\right)^s \left(1+e^{-1}\right)\left[(s-1)(1-2e^{-1})(1+e^{-1}) + 2(1-3e^{-2})\right]; \tag{3.147}$$

hence

$$\begin{array}{llll} I''_3 \ge 0.44866; & I''_4 \ge 0.33725; & I''_5 \ge 0.24709; & I''_6 \ge 0.17762, \\ I''_7 \ge 0.12583, & I''_8 \ge 0.0881, & I''_9 \ge 0.0611. & \end{array} \tag{3.148}$$

We will now estimate the first two terms in the sum in (3.140). We use the inequalities $(1-x^2)e^x \le 1+x \le e^x$, $x \in [-1,1]$, to derive

$$\frac{p^2_{n,s+1,0}(0)}{(n-s)p_{n,s,0}(0)} = \frac{n!}{(n-s-1)!} \frac{\left(\int_0^{1/n} \cdots \int_0^{1/n} (1-u_1-\cdots-u_{s+1})^{n-s-1}\, du_1 \cdots du_{s+1}\right)^2}{\int_0^{1/n} \cdots \int_0^{1/n} (1-u_1-\cdots-u_s)^{n-s}\, du_1 \cdots du_s}$$

$$\le \frac{n!}{(n-s-1)!}\left(1+\frac{c}{n}\right)\frac{\left(\int_0^{1/n}\cdots\int_0^{1/n} e^{-(n-s-1)(u_1+\cdots+u_{s+1})}\,du_1\cdots du_{s+1}\right)^2}{\int_0^{1/n}\cdots\int_0^{1/n} e^{-(n-s)(u_1+\cdots+u_s)}\,du_1\cdots du_s}$$
$$\le \frac{n!}{(n-s-1)!}\left(1+\frac{c}{n}\right)\frac{\left(\int_0^{1/n}\cdots\int_0^{1/n} e^{-n(u_1+\cdots+u_{s+1})}\,du_1\cdots du_{s+1}\right)^2}{\int_0^{1/n}\cdots\int_0^{1/n} e^{-n(u_1+\cdots+u_s)}\,du_1\cdots du_s}$$
$$\le \frac{1}{n-s}\left(1+\frac{c}{n}\right)\frac{\left(\int_0^{1}\cdots\int_0^{1} e^{-(t_1+\cdots+t_{s+1})}\,dt_1\cdots dt_{s+1}\right)^2}{\int_0^{1}\cdots\int_0^{1} e^{-(t_1+\cdots+t_s)}\,dt_1\cdots dt_s}.$$

Consequently,

$$\frac{p^2_{n,s+1,0}(0)}{(n-s)p_{n,s,0}(0)} \le \frac{1}{n-s}\left(1+\frac{c}{n}\right)\left(1-e^{-1}\right)^{s+2}. \tag{3.149}$$

Similarly, we derive

$$\frac{p^2_{n,s+1,1}(0)}{2(n-s-1)p_{n,s,1}(0)} \tag{3.150}$$
$$\le \frac{1}{n-s}\left(1+\frac{c}{n}\right)\frac{1}{2}\frac{\left(\int_0^{1}\cdots\int_0^{1}(t_1+\cdots+t_{s+1})\,e^{-(t_1+\cdots+t_{s+1})}\,dt_1\cdots dt_{s+1}\right)^2}{\int_0^{1}\cdots\int_0^{1}(t_1+\cdots+t_s)\,e^{-(t_1+\cdots+t_s)}\,dt_1\cdots dt_s}.$$

We have

$$\int_0^1\cdots\int_0^1 (t_1+\cdots+t_s)\,e^{-(t_1+\cdots+t_s)}\,dt_1\cdots dt_s = s(1-e^{-1})^{s-1}(1-2e^{-1}).$$

Consequently,

$$\frac{p^2_{n,s+1,1}(0)}{2(n-s-1)p_{n,s,1}(0)} \le \frac{1}{n-s}\left(1+\frac{c}{n}\right)\frac{(s+1)^2}{2s}(1-e^{-1})^{s+1}(1-2e^{-1}). \tag{3.151}$$

For

$$J_s := \left(1-e^{-1}\right)^{s+2} + \frac{(s+1)^2}{2s}(1-e^{-1})^{s+1}(1-2e^{-1})$$

we have

$$\begin{aligned} &J_3 \le 0.21343, \quad J_4 \le 0.14714, \quad J_5 \le 0.10102, \quad J_6 \le 0.06901,\\ &J_7 \le 0.04691, \quad J_8 \le 0.03175, \quad J_9 \le 0.0214. \end{aligned} \tag{3.152}$$

By (3.145), (3.149) and (3.150) we have

$$\sum_{k=0}^{n-s-1} \frac{p_{n,s+1,k}^2\left(\frac{j}{n}\right)}{(k+1)(n-s-k)p_{n,s,k}\left(\frac{j}{n}\right)} \leq \frac{1}{n-s}\left(1+\frac{c}{n}\right)(J_s + I_s' - I_s'').$$

Inequalities (3.146), (3.148) and (3.152) imply

$$J_s + I_s' - I_s'' \leq 1, \quad s = 3, 4, \dots, 9.$$

Thus the lemma is established for $j = 0$.

Let $s \geq 2$. For $1 \leq j \leq n-s-1$ we get by means of Cauchy's inequality

$$\frac{p_{n,s+1,k}^2\left(\frac{j}{n}\right)}{p_{n,s,k}\left(\frac{j}{n}\right)} \leq n^{s+1}\int_0^{1/n}\cdots\int_0^{1/n} \frac{p_{n-s-1,k}^2\left(\frac{j}{n}+u_1+\cdots+u_{s+1}\right)}{p_{n-s,k}\left(\frac{j}{n}+u_1+\cdots+u_s\right)}\,du_1\cdots du_{s+1}.$$

Therefore,

$$\frac{p_{n,s+1,k}^2\left(\frac{j}{n}\right)}{(k+1)(n-s-k)p_{n,s,k}\left(\frac{j}{n}\right)} \leq \frac{n^{s+1}}{(n-s)^2}\int_0^{1/n}\cdots\int_0^{1/n} \frac{\binom{n-s}{k+1}A^{k+1}B^{n-s-k-1}}{A\left(1-\frac{j}{n}-u_1-\cdots-u_s\right)}\,du_1\cdots du_{s+1}$$

with A and B defined in (3.136). We sum up these inequalities for $k = 0, \dots, n-s-1$ and apply the binomial formula. Thus we get the estimate (3.153)

$$S \leq \frac{n^{s+1}}{(n-s)^2}\int_0^{1/n}\cdots\int_0^{1/n} \frac{(A+B)^{n-s}}{A\left(1-\frac{j}{n}-u_1-\cdots-u_s\right)}\,du_1\cdots du_{s+1},$$

where S denotes the sum on the left of estimate (3.140)

Further, we again use (3.138) and the inequality $1 + x \leq e^x$ to deduce

$$(A+B)^{n-s} \leq e^{\frac{nu_{s+1}^2}{\varphi^2(j/n+u_1+\cdots+u_s)}};$$

and hence

$$(A+B)^{n-s} \leq \left(1+\frac{c}{n}\right)\begin{cases} e^{\frac{n^2u_{s+1}^2}{\xi}}, & 1 \leq j \leq (n-s)/2, \\ e^{\frac{n^2u_{s+1}^2}{n-\xi}}, & (n-s)/2 \leq j \leq n-s-1, \end{cases}$$

where $\xi = j+nu_1+\cdots+nu_s$. We apply that estimate in (3.153) and make the change of the variables $t_i = nu_i$, $i = 1,\ldots,s+1$. Thus, for $1 \le j \le (n-s)/2$, we arrive at

$$\text{(3.154)}\quad S \le \left(1+\frac{c}{n}\right)\frac{1}{n-s-j} \times \int_0^1 \cdots \int_0^1 \frac{j+t_1+\cdots+t_s}{(j+t_1+\cdots+t_{s+1})^2}\, e^{\frac{t_{s+1}^2}{j+t_1+\cdots+t_s}}\, dt_1\cdots dt_{s+1}.$$

Using that the function $T(T+t)^{-2}e^{t^2/T}$ is decreasing on T in $[1,\infty)$ for any fixed $t \in [0,1]$, we deduce that

$$\text{(3.155)}\quad \frac{j+t_1+\cdots+t_s}{(j+t_1+\cdots+t_{s+1})^2}\, e^{\frac{t_{s+1}^2}{j+t_1+\cdots+t_s}} \le \frac{j+t_1+t_2}{(j+t_1+t_2+t_{s+1})^2}\, e^{\frac{t_{s+1}^2}{j+t_1+t_2}}$$

for all $t_i \in [0,1]$, $i = 1,\ldots,s+1$.

Combining (3.154), (3.155) and [70, (4.10)], we verify (3.140) for $1 \le j \le (n-s)/2$ and $s \ge 2$.

Similarly, for $(n-s)/2 \le j \le n-s-1$ we have

$$\text{(3.156)}\quad S \le \left(1+\frac{c}{n}\right)\frac{1}{j+1} \times \int_0^1 \cdots \int_0^1 \frac{1}{n-j-t_1-\cdots-t_s}\, e^{\frac{t_{s+1}^2}{n-j-t_1-\cdots-t_s}}\, dt_1\cdots dt_{s+1}.$$

Above we used that the function $T/(T+t)^2$ is decreasing on T in $[1,\infty)$ for any fixed $t \in [0,1]$ to derive

$$\frac{j+t_1+\cdots+t_s}{(j+t_1+\cdots+t_{s+1})^2} \le \frac{j}{(j+t_{s+1})^2} \le \frac{1}{j+1}\left(1+\frac{c}{n}\right).$$

Next, we make the change of the variables $v_i = 1-t_i$, $i = 1,\ldots,s$, in the integral in (3.156). Thus we arrive at

$$S \le \left(1+\frac{c}{n}\right)\frac{1}{j+1} \times \int_0^1 \cdots \int_0^1 \frac{1}{n-s-j+v_1+\cdots+v_s}\, e^{\frac{t_{s+1}^2}{n-s-j+v_1+\cdots+v_s}}\, dv_1\cdots du_s\, dt_{s+1}.$$

Now, (3.140) for $(n-s)/2 \le j \le n-s-1$ and $s \ge 2$ follows from the fact that the function $T^{-1}e^{t^2/T}$ is decreasing on T in $[1,\infty)$ for any fixed $t \in [0,1]$ and [70, (4.11)]. $\square$

3.8 Simultaneous approximation by the Kantorovich operator in weighted L_∞-spaces

Results about the simultaneous approximation by the Bernstein operator can be easily transferred to the Kantorovich operator. The Kantorovich operators or polynomials are defined for $f \in L[0,1]$ and $x \in [0,1]$ by

$$K_n f(x) := \sum_{k=0}^{n} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt\, p_{n,k}(x), \ p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

They are related to the Bernstein polynomials as follows

$$K_n f(x) = (B_{n+1} F(x))', \quad F(x) := \int_0^x f(t)\,dt. \tag{3.157}$$

More generally, we set for $f \in L[0,1]$ and $m \in \mathbb{N}_+$ (see [89])

$$K_n^{\langle m \rangle} f(x) := (B_{n+m} F_m(x))^{(m)},$$

where

$$F_m(x) := \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f(t)\,dt.$$

The operator $K_n^{\langle m \rangle}$ is referred to as the generalized Kantorovich operator of order m. That generalization of the Kantorovich polynomials or similar modifications of related operators were studied in [14, 15, 46, 47, 52, 55].

Using that B_n is degree reducing w.r.t. the algebraic polynomials (see e.g. [18, p. 306]), it can be verified by induction on j that

$$\left(K_n^{\langle m \rangle}\right)^j f = \left(B_{n+m}^j F_m\right)^{(m)}. \tag{3.158}$$

All that enables us to transfer all the above results about simultaneous approximation by B_n to $K_n^{\langle m \rangle}$. Theorems 3.3 and 3.8 with $s+m$ in place of s, F_m in place of f, and $n+m$ in place of n yield the following characterization of the rate of the simultaneous approximation by $K_n^{\langle m \rangle}$.

Theorem 3.41. *Let $m \in \mathbb{N}_+$, $s \in \mathbb{N}_0$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s+m$. Then for all $f \in L_\infty[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(K_n^{\langle m \rangle} f - f)^{(s)}\| \le c\, K_{s+m}^D(f^{(s)}, n^{-1})_w.$$

Conversely, there exists $R \in \mathbb{N}_+$ *such that for all* $f \in L_\infty[0,1]$ *with* $f \in AC_{loc}^{s-1}(0,1)$ *and* $wf^{(s)} \in L_\infty[0,1]$*, and all* $\ell, n \in \mathbb{N}_+$ *with* $\ell \ge Rn$ *there holds*

$$K_{s+m}^D(f^{(s)}, n^{-1})_w \le c \left(\frac{\ell}{n}\right)^r \left(\|w(K_n^{\langle m\rangle} f - f)^{(s)}\| + \|w(K_\ell^{\langle m\rangle} f - f)^{(s)}\|\right).$$

In particular,

$$K_{s+m}^D(f^{(s)}, n^{-r})_w \le c \left(\|w(K_n^{\langle m\rangle} f - f)^{(s)}\| + \|w(K_{Rn}^{\langle m\rangle} f - f)^{(s)}\|\right).$$

The value of the constant c *is independent of* f*,* n *and* ℓ*.*

In the statement of the last theorem the condition $f \in AC_{loc}^{s-1}(0,1)$ is to be ignored for $s = 0$.

Remark 3.42. As it is clear from the last theorem, the higher the order of the generalized Kantorovich operator is, the broader the space of functions it approximates is. More precisely, let us denote by $\mathcal{W}_m^s$ the set of functions, for which Theorem 3.41 is established, i.e. $\mathcal{W}_m^s$ is the set of all $f \in L_\infty[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$ for some Jacobi weight $w := w(\gamma_0, \gamma_1)$ with $0 \le \gamma_0, \gamma_1 < s + m$. Then we have $\mathcal{W}_m^s \subset \mathcal{W}_{m+1}^s$. Or, to put it otherwise, given an $s \in \mathbb{N}_0$ and a function $f \in AC_{loc}^{s-1}(0,1)$ such that $wf^{(s)} \in L_\infty[0,1]$ for some Jacobi weight $w := w(\gamma_0, \gamma_1)$, then $(K_n^{\langle m\rangle} f)^{(s)}$ approximates $f^{(s)}$ in L_∞ with a weight w, provided that we take m large enough, namely, $m > \max\{\gamma_0, \gamma_1\} - s$.

Remark 3.43. We can enlarge the domain of $K_n^{\langle m\rangle}$ if we replace F_m in its definition by

$$f_m(x) := \frac{1}{(m-1)!} \int_{1/2}^{x} (x-t)^{m-1} f(t)\, dt.$$

If $\tilde{w} f \in L_\infty[0,1]$, where $\tilde{w} := w(\tilde{\gamma}_0, \tilde{\gamma}_1)$ with $\tilde{\gamma}_0, \tilde{\gamma}_1 < m$, then $f_m \in C[0,1]$ (as we established in the proof of Lemma 4.24). Theorem 3.41 holds for this modification of $K_n^{\langle m\rangle}$ as the condition $f \in L_\infty[0,1]$ is replaced with $\tilde{w} f \in L_\infty[0,1]$, $\tilde{\gamma}_0, \tilde{\gamma}_1 < m$.

Theorem 3.30 with $s = 1$, F in place of f, and $n+1$ in place of n implies the following characterization of the error of approximation of the Kantorovich operator (i.e. $m = 1$) in weighted L_∞-spaces.

Theorem 3.44. *Let $w := w(\gamma_0, \gamma_1)$ be given by (2.2) with $\gamma_0, \gamma_1 \in [0, 1/2]$. Then there exists $n_0 \in \mathbb{N}_+$ such that for all $f \in L[0, 1]$ with $wf \in L_\infty[0, 1]$, and all $n \in \mathbb{N}_+$ with $n \geq n_0$ there holds*

$$\|w(K_n f - f)\| \sim K_1^D(f, n^{-1})_w.$$

The direct estimate for the Kantorovich operator in the case $w = 1$ and $s = 0$ is due to Berens and Xu [8, Theorem 6]. There a weak converse inequality was established as well. The corresponding one-term strong converse inequality and the characterization of the K-functional by the Ditzian-Totik modulus were proved by Gonska and Zhou [54]. Mache [81] established the direct estimate for the Kantorovich operator and a weak converse one in the case $w = \varphi^{2\ell}$ and $s = 2\ell$, $\ell \in \mathbb{N}_+$. All those results were obtained in the L_p-norm, $1 \leq p \leq \infty$.

Chapter 4

Weighted simultaneous approximation by iterated Boolean sums of Bernstein operators $\mathcal{B}_{r,n}$

4.1 Background

One way to increase the approximation rate of the Bernstein operator B_n, defined in (0.1), is to form the following linear combination of its iterates

$$\mathcal{B}_{r,n} := I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}_+$. Clearly, $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$ is a bounded linear operator.

If P and Q are operators on a linear space X, then their Boolean sum $P \oplus Q$ is defined by

$$P \oplus Q := P + Q - PQ.$$

Then we have (see [53])

$$\mathcal{B}_{r,n} = \underbrace{B_n \oplus \cdots \oplus B_n}_{r \text{ times}};$$

hence we can refer to $\mathcal{B}_{r,n}$ as iterated Boolean sums of B_n.

In [84] it was shown that the saturation order of $\mathcal{B}_{r,n}$ is n^{-r}.

An important and nice characterization of the error of $\mathcal{B}_{r,n}$ was given by Gonska ans Zhou [53]. They established the following upper estimate

$$\|\mathcal{B}_{r,n}f - f\| \le c\left(\omega_\varphi^{2r}(f, n^{-1/2}) + \frac{1}{n^r}\|f\|\right), \quad f \in C[0,1],\ n \in \mathbb{N}_+. \tag{4.1}$$

A Stechkin-type converse inequality was also proved. That enabled them to deduce the trivial class of the operator and a big O equivalence characterization of the error.

Since B_n preserve the algebraic polynomials of degree at most 1, replacing in (4.1) f with $f - p_1$, where p_1 is the polynomial of degree 1 of best approximation of f in the uniform norm on $[0,1]$, we immediately arrive at

$$\|\mathcal{B}_{r,n}f - f\| \le c\left(\omega_\varphi^{2r}(f, n^{-1/2}) + \frac{1}{n^r}E_1(f)\right), \quad f \in C[0,1],\ n \in \mathbb{N}_+, \tag{4.2}$$

where $E_1(f)$ denotes the best approximation of f by algebraic polynomials of degree 1 in the uniform norm on $[0,1]$.

Later on Ding and Cao [19] characterized the error of the multivariate generalization of $\mathcal{B}_{r,n}$ on the simplex. In the univariate case, the direct inequality they proved is of the form

$$\|\mathcal{B}_{r,n}f - f\| \le c\, K_{r,0}^D(f, n^{-r}), \quad f \in C[0,1],\ n \in \mathbb{N}_+, \tag{4.3}$$

where

$$K_{r,0}^D(f,t) := \inf_{g \in C^{2r}[0,1]} \{\|f-g\| + t\|D^r g\|\}$$

with, to recall, $Dg := \varphi^2 g''$ and $\varphi(x) := \sqrt{x(1-x)}$.

They also proved a strong converse inequality of type D (in the terminology introduced in [22]), that is

$$K_{r,0}^D(f, n^{-r}) \le c \max_{k \ge n} \|\mathcal{B}_{r,k}f - f\|, \quad f \in C[0,1],\ n \in \mathbb{N}_+. \tag{4.4}$$

However, as we will show in Theorem 4.23 below,

$$K_{r,0}^D(f,t) \sim K_{2r,\varphi}(f,t) + tE_1(f), \quad 0 < t \le 1.$$

Therefore, taking also into account (1.9), we see that the function characteristics on the right side of (4.2) and (4.3) are equivalent.

In addition, we will establish in this theorem that

$$K_{r,0}^{D}(f,t) \sim K_{2r,\varphi}(f,t) + K_{2,\varphi}(f,t), \quad 0 < t \le 1.$$

Then, taking into account (1.9) (along with [72, Theorem 2.7]), we arrive at the following relation between $K_{r,0}^{D}(f,t)$ and Ditzian-Totik moduli:

$$K_{r,0}^{D}(f,n^{-r}) \sim \omega_{\varphi}^{2r}(f,n^{-1/2}) + \omega_{\varphi}^{2}(f,n^{-r/2}), \quad f \in C[0,1],\ n \ge r^2.$$

When we apply it in (4.3), we get the direct estimate

$$\|\mathcal{B}_{r,n}f - f\| \le c\left(\omega_{\varphi}^{2r}(f,n^{-1/2}) + \omega_{\varphi}^{2}(f,n^{-r/2})\right), \quad f \in C[0,1],\ n \ge r^2.$$

Quite recently, Cheng and Zhou [16] derived another converse inequality from the Stechkin-type converse inequality in [53]. It is similar to (4.4), though weaker than it.

A historical overview of the study of $\mathcal{B}_{r,n}$ and the motivation to regard them as iterated Boolean sums can be found in [53].

4.2 A characterization of the rate of approximation by $\mathcal{B}_{r,n}$

We will first demonstrate that the result of Gonska and Zhou (4.1) can be derived from the direct estimates on simultaneous approximation by B_n, presented in the previous chapter. In my opinion, such an approach is more elementary and more straightforward (though not shorter) than the one used by H. Gonska and X.-l. Zhou. It is more elementary because essentially it uses only Taylor's formula and simple integral estimates, whereas highly non-trivial results on best approximation by algebraic polynomials were applied in [53]. Besides that it is more straightforward because it is independent of the close relation between best algebraic approximation and approximation by the Bernstein polynomials, as in both cases the weight $\varphi(x)$ plays an important role. However, it should be noted, the method used by H. Gonska and X.-l. Zhou enabled them to prove also an important converse inequality. The approach of Ding and Cao [19] was similar to that of H. Gonska and X-l. Zhou.

Clearly, since $\|B_n f\| \le \|f\|$ for all $f \in C[0,1]$ and $n \in \mathbb{N}_+$, then

$$\|\mathcal{B}_{r,n}f\| \le c\,\|f\|, \quad f \in C[0,1],\ n \in \mathbb{N}_+. \tag{4.5}$$

The following Jackson-type estimate holds.

Theorem 4.1. *Let $r \in \mathbb{N}_+$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r-1}(0,1)$ and $\varphi^{2r} f^{(2r)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|\mathcal{B}_{r,n} f - f\| \le \frac{c}{n^r} \left(\|f\| + \|\varphi^{2r} f^{(2r)}\| \right).$$

The value of the constant c is independent of f and n.

Proof. Let us set $F_\rho := (B_n f - f)^\rho$. As is known

$$\|B_n g - g\| \le \frac{c}{n} \|\varphi^2 g''\| \tag{4.6}$$

for any $g \in AC_{loc}^1(0,1)$ and $n \in \mathbb{N}_+$.

Therefore,

$$\|F_r\| \le \frac{c}{n} \|\varphi^2 F''_{r-1}\|.$$

Next, if $r \ge 2$, we estimate the norm on the right above by (3.44) with $w = \varphi^2$ and $s = 2$ and thus arrive at

$$\|F_r\| \le \frac{c}{n^2} \Big(\|\varphi^2 F_{r-2}^{(2)}\| + \|\varphi^4 F_{r-2}^{(4)}\| \Big).$$

If $r \ge 3$, we proceed in a similar fashion, i.e. we estimate above each of the two terms on the right by means of (3.44). Note that at each such step:

(i) The power of n increases by one,

(ii) The number of iterates of $B_n - I$ decreases by one,

(iii) The range of the index ℓ of the terms $\|\varphi^{2\ell} F_\rho^{(2\ell)}\|$ increases by one.

The inequality between the power of φ^2 and the order of the derivative in (3.44) is always satisfied.

Thus we arrive at the upper estimate

$$\|\mathcal{B}_{r,n} f - f\| \le \frac{c}{n^r} \sum_{k=1}^{r} \|\varphi^{2k} f^{(2k)}\|. \tag{4.7}$$

To complete the proof, we need only apply Proposition 2.2(b) with $w = 1$ and $m = 2r$. □

By the standard argument used in the proof of Theorem 3.3, we derive from (4.5) and Theorem 4.1 the estimate

$$\|\mathcal{B}_{r,n}f - f\| \le c\left(K_{2r,\varphi}(f, n^{-r}) + \frac{1}{n^r}\|f\|\right), \quad f \in C[0,1],\ n \in \mathbb{N}_+. \tag{4.8}$$

Relations (4.8) and (1.9) imply (4.1) for $n \ge n_0$ with some fixed $n_0 \in \mathbb{N}_+$. For $n \le n_0$ it trivially follows from (4.5).

Estimates we will further establish concerning the simultaneous approximation by $\mathcal{B}_{r,n}$ can be used to verify a two-term strong converse inequality that matches the direct one in (4.3) and improves the aforementioned earlier converse inequalities.

Theorem 4.2. *Let $r \in \mathbb{N}_+$. Then there exists $R \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ and $k, n \in \mathbb{N}_+$ with $k \ge Rn$ there holds*

$$K^D_{r,0}(f, n^{-r}) \le c\left(\frac{k}{n}\right)^r (\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,k}f - f\|).$$

In particular,

$$K^D_{r,0}(f, n^{-r}) \le c\,(\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,Rn}f - f\|).$$

The value of the constant c is independent of f, n and k.

4.3 A characterization of the rate of the weighted simultaneous approximation by $\mathcal{B}_{r,n}$

Taking into consideration relation (3.3), we arrive at the hypothesis that the differential operator related to the rate of the simultaneous approximation by $\mathcal{B}_{r,n}$ is $(d/dx)^s D^r$ and the saturation order is n^{-r}. Thus, to characterize this rate, we will use the K-functional

$$K^D_{r,s}(f,t)_w := \inf_{g \in C^{2r+s}[0,1]} \left\{\|w(f - g^{(s)})\| + t\|w(D^r g)^{(s)}\|\right\}.$$

We will establish the following direct estimate of the error of the simultaneous approximation by $\mathcal{B}_{r,n}$.

Theorem 4.3. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le c\, K_{r,s}^D(f^{(s)}, n^{-r})_w.$$

The value of the constant c is independent of f and n.

The estimate in Theorem 4.3 can be simplified. The involved K-functional $K_{r,s}^D(f,t)_w$ can be characterized by the simpler ones $K_{2r,\varphi}(f,t)_w$ and $K_m(f,t)_w$. In the last section of this chapter, we will show that the following characterization of $K_{r,s}(f,t)_w$ holds.

Theorem 4.4. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) with $0 < \gamma_0, \gamma_1 < s$. Then for all $wf \in L_\infty[0,1]$ and $0 < t \le 1$ there holds*

$$K_{r,s}^D(f,t)_w \sim \begin{cases} K_{2r,\varphi}(f,t)_w + K_1(f,t)_w, & s = 1, \\ K_{2r,\varphi}(f,t)_w + t\,\|wf\|, & s \ge 2. \end{cases}$$

The result in the case $w = 1$ is of a different form.

Theorem 4.5. *Let $r, s \in \mathbb{N}_+$. Then for all $f \in C[0,1]$ and $0 < t \le 1$ there holds*

$$K_{r,s}^D(f,t)_1 \sim \begin{cases} K_{2r,\varphi}(f,t) + K_r(f,t) + K_1(f,t), & s = 1, \\ K_{2r,\varphi}(f,t) + K_r(f,t) + t\,\|f\|, & s \ge 2. \end{cases}$$

Remark 4.6. The middle term on the right-hand side in the characterization in Theorem 4.5 *cannot* be omitted. Indeed, if $f(x) = x^r \log x$, then $f, \varphi^{2r} f^{(2r)} \in L_\infty[0,1]$ and $f' \in L_\infty[0,1]$ (the latter in the case $r \ge 2$), but $f^{(r)} \notin L_\infty[0,1]$.

Further, we can take into account that $K_r(f,t^r)_w \sim \omega_r(f,t)_w$ (see (1.3) and (1.6)) and $K_{2r,\varphi}(f,t^{2r})_w \sim \omega_\varphi^{2r}(f,t)_w$ (see (1.9)) and deduce from Theorems 4.3-4.5 the following Jackson-type estimates.

Theorem 4.7. *Let $r, s \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 < \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2})_w + \omega_1(f', n^{-r})_w, & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2})_w + \dfrac{1}{n^r}\,\|wf^{(s)}\|, & s \ge 2. \end{cases}$$

The value of the constant c is independent of f and n.

Theorem 4.8. *Let $r, s \in \mathbb{N}_+$. Then for all $f \in C^s[0,1]$ and $n \in \mathbb{N}_+$ there holds*

$$\|(\mathcal{B}_{r,n}f - f)^{(s)}\| \leq c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2}) + \omega_r(f', n^{-1}) + \omega_1(f', n^{-r}), & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2}) + \omega_r(f^{(s)}, n^{-1}) + \dfrac{1}{n^r}\|f^{(s)}\|, & s \geq 2. \end{cases}$$

The value of the constant c is independent of f and n.

Similar estimates can be stated in terms of the differential operator D. They are given in the next theorem. We state this direct estimate only for the unweighted case. We set

$$\widehat{K}_{r,s}(F, t) := \inf_{g \in C^{2(r+s)}[0,1]} \{\|F - D^s g\| + t\|D^{r+s}g\|\}.$$

Theorem 4.9. *Let $r, s \in \mathbb{N}_+$. Then for all $f \in C^{2s}[0,1]$ and $n \in \mathbb{N}_+$ there holds*

$$\|D^s(\mathcal{B}_{r,n}f - f)\| \leq c\,\widehat{K}_{r,s}(D^s f, n^{-r}).$$

The value of the constant c is independent of f and n.

The direct estimates above are sharp. We will verify a strong converse inequality that matches the direct one in Theorem 4.3.

Theorem 4.10. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \leq \gamma_0, \gamma_1 < s$. Then there exists $R \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $k, n \in \mathbb{N}_+$ with $k \geq Rn$ there holds*

$$K_{r,s}(f^{(s)}, n^{-r})_w \leq c \left(\frac{k}{n}\right)^r \left(\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| + \|w(\mathcal{B}_{r,k}f - f)^{(s)}\|\right).$$

In particular,

$$K_{r,s}(f^{(s)}, n^{-r})_w \leq c \left(\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| + \|w(\mathcal{B}_{r,Rn}f - f)^{(s)}\|\right).$$

The value of the constant c is independent of f, n and k.

To establish the results stated above, we adopt the same approach as in treating the simultaneous approximation by B_n. Since the differential operator associated with the simultaneous approximation by $\mathcal{B}_{r,n}$, that is $(d/dx)^s D^r$, is rather involved, we do not directly aim at establishing estimates by it. It is much easier to prove estimates in terms of the norms of the components into which $(D^r g)^{(s)}$ expands. They are of the form $q\varphi^{2i}g^{(j)}$, where q is an algebraic polynomial, which can be ignored, and $i, j \in \mathbb{N}_0$. Due to the validity of certain embedding inequalities their number can be reduced to two or three and the sum of their weighted L_∞-norms is equivalent to the norm of $(D^r g)^{(s)}$. That allows us not only to get round the technical difficulties of dealing with $(d/dx)^s D^r$, but also to derive almost simultaneously both characterizations of $\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|$: the more natural one by $K^D_{r,s}(f,t)_w$ and the more useful one by $K_{2r,\varphi}(f,t)_w$ and $K_m(f,t)_w$.

In the next section we will extend the basic inequalities for B_n in the previous chapter to $\mathcal{B}_{r,n}$. They will enable us to prove the converse estimate for the approximation rate of $\mathcal{B}_{r,n}$ and the direct and converse estimates for the simultaneous approximation by it. The proofs are then given in Sections 4.5 and 4.6. The proof of Theorems 4.4 and 4.5 are given in Section 4.7.

4.4 Basic estimates for the simultaneous approximation by $\mathcal{B}_{r,n}$

We will extend the estimates obtained in Section 3.5 for the Bernstein operator to its iterated Boolean sum.

We begin with the following basic estimates concerning the boundedness of the weighted L_∞-norm of $(\mathcal{B}_{r,n}f)^{(s)}$.

Proposition 4.11. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{B}_{r,n}f)^{(s)}\| \le c\,\|wf^{(s)}\|.$$

The value of the constant c is independent of f and n.

Proof. The assertion follows from Proposition 3.14 by iteration. □

4.4.1 A Jackson-type estimate

The following Jackson-type estimate of the error of $\mathcal{B}_{r,n}$ holds for smooth functions.

Proposition 4.12. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2). Set $s' := \max\{2, s\}$. If $0 < \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s-1}(0,1)$ and $wf^{(s')}, w\varphi^{2r} f^{(2r+s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \frac{c}{n^r}\left(\|wf^{(s')}\| + \|w\varphi^{2r} f^{(2r+s)}\|\right).$$

If $\gamma_0\gamma_1 = 0$ and still $0 \le \gamma_0, \gamma_1 \le s$, then

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \frac{c}{n^r}\left(\|wf^{(s')}\| + \|wf^{(r+s)}\| + \|w\varphi^{2r} f^{(2r+s)}\|\right)$$

provided that $wf^{(r+s)} \in L_\infty[0,1]$ too.

The value of the constant c is independent of f and n.

Proof. Actually, if $\gamma_0, \gamma_1 > 0$, the assumption $w\varphi^{2r} f^{(2r+s)} \in L_\infty[0,1]$ implies $wf^{(s')} \in L_\infty[0,1]$. This follows from Proposition 2.1 with $w_1 = w$, $w_2 = w\varphi^{2r}$, $j = s'$ and $m = 2r + s$.

We will prove that if $0 \le \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s-1}(0,1)$ and $wf^{(s')}, wf^{(r+s)}, w\varphi^{2r} f^{(2r+s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \frac{c}{n^r}\left(\|wf^{(s')}\| + \|wf^{(r+s)}\| + \|w\varphi^{2r} f^{(2r+s)}\|\right). \tag{4.9}$$

That already contains the second assertion of the corollary; to get the first one we apply

$$\|wf^{(r+s)}\| \le c\left(\|wf^{(s')}\| + \|w\varphi^{2r} f^{(2r+s)}\|\right), \tag{4.10}$$

which follows from Proposition 2.1 with $g = f^{(s')}$, $j = r+s-s'$, $m = 2r+s-s'$, $w_1 = w$ and $w_2 = w\varphi^{2r}$.

To establish (4.9) for $s \ge 2$ we use Proposition 3.17 to derive by induction on r the estimate

$$\|w[(B_n - I)^r f]^{(s)}\| \le \frac{c}{n^r} \sum_{i=0}^{r} \sum_{j=2i}^{i+r} \|w\varphi^{2i} f^{(j+s)}\|. \tag{4.11}$$

In order to estimate above the terms with $i = 0$ on the right side of the last relation, we apply Proposition 2.2(c) with $g = f^{(s)}$ and $m = r$ to get for $j = 0, \ldots, r$

$$\|wf^{(j+s)}\| \leq c \left(\|wf^{(s)}\| + \|wf^{(r+s)}\|\right), \tag{4.12}$$

whereas to estimate above the terms with $i > 0$, we apply Proposition 2.1 with $g = f^{(s)}$, $m = 2r$, $w_1 = w\varphi^{2i}$ and $w_2 = w\varphi^{2r}$ to get for $j = 2i, \ldots i + r$

$$\|w\varphi^{2i} f^{(j+s)}\| \leq c \left(\|wf^{(s)}\| + \|w\varphi^{2r} f^{(2r+s)}\|\right). \tag{4.13}$$

Now, (4.9) for $s \geq 2$ follows from (4.11)-(4.13).

To prove (4.9) for $s = 1$ we first observe that Proposition 3.17 and what we have already established yield

$$\begin{aligned}\|w(\mathcal{B}_{r,n}f - f)'\| &\leq \frac{c}{n} \left(\|w(\mathcal{B}_{r-1,n}f - f)''\| + \|w\varphi^2(\mathcal{B}_{r-1,n}f - f)'''\|\right) \\ &\leq \frac{c}{n^r}\Big(\|wf''\| + \|wf^{(r+1)}\| + \|w\varphi^{2r-2} f^{(2r)}\| \\ &\qquad + \|w\varphi^2 f'''\| + \|w\varphi^2 f^{(r+2)}\| + \|w\varphi^{2r} f^{(2r+1)}\|\Big).\end{aligned}$$

Next, to complete the proof in this case, we use that

$$\|w\varphi^{2j} f^{(r+j+1)}\| \leq c \left(\|wf^{(r+1)}\| + \|w\varphi^{2r} f^{(2r+1)}\|\right), \quad j = 1, r-1,$$

and

$$\|w\varphi^2 f'''\| \leq c \left(\|wf''\| + \|w\varphi^{2r} f^{(2r+1)}\|\right),$$

which follow from Proposition 2.1 respectively with $g = f^{(r+1)}$, $m = r$, $w_1 = w\varphi^{2j}$, $w_2 = w\varphi^{2r}$ (or see Proposition 2.2(a)) and $g = f''$, $j = 1$, $m = 2r - 1$, $w_1 = w\varphi^2$, $w_2 = w\varphi^{2r}$. □

The upper estimate can be stated in a more concise form in terms of the differential operator $(d/dx)^s D^r$. This result follows directly from Proposition 2.6 and Proposition 4.12.

Corollary 4.13. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \leq \gamma_0, \gamma_1 < s$. Then for all $f \in AC^{2r+s-1}[0,1]$ such that $w\varphi^{2r} f^{(2r+s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \leq \frac{c}{n^r} \|w(D^r f)^{(s)}\|.$$

The value of the constant c is independent of f and n.

4.4.2 Voronovskaya-type estimates

Now, we will extend the Voronovskaya-type estimates for the simultaneous approximation by B_n to $\mathcal{B}_{r,n}$.

Proposition 4.14. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2). Set $s'' := \max\{3, s\}$. If $0 < \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s+1}(0,1)$ and $wf^{(s'')}, w\varphi^{2r+2} f^{(2r+s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\left\| w \left(\mathcal{B}_{r,n} f - f - \frac{(-1)^{r-1}}{(2n)^r} D^r f \right)^{(s)} \right\| \le \frac{c}{n^{r+1}} \left(\|wf^{(s'')}\| + \|w\varphi^{2r+2} f^{(2r+s+2)}\| \right).$$

If $\gamma_0\gamma_1 = 0$ and still $0 \le \gamma_0, \gamma_1 \le s+1$, then

$$\left\| w \left(\mathcal{B}_{r,n} f - f - \frac{(-1)^{r-1}}{(2n)^r} D^r f \right)^{(s)} \right\| \le \frac{c}{n^{r+1}} \left(\|wf^{(s'')}\| + \|wf^{(r+s+1)}\| + \|w\varphi^{2r+2} f^{(2r+s+2)}\| \right).$$

provided that $wf^{(r+s+1)} \in L_\infty[0,1]$ too.

The value of the constant c is independent of f and n.

Proof. Actually, if $\gamma_0, \gamma_1 > 0$, the assumption $w\varphi^{2r+2} f^{(2r+s+2)} \in L_\infty[0,1]$ implies $wf^{(s'')} \in L_\infty[0,1]$. This follows from Proposition 2.1 with $w_1 = w$, $w_2 = w\varphi^{2r+2}$, $j = s''$ and $m = 2r+s+2$.

We will establish that if $0 \le \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s+1}(0,1)$ and $wf^{(s'')}, wf^{(r+s+1)}, w\varphi^{2r+2} f^{(2r+s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds

$$\left\| w \left(\mathcal{B}_{r,n} f - f - \frac{(-1)^{r-1}}{(2n)^r} D^r f \right)^{(s)} \right\| \le \frac{c}{n^{r+1}} \left(\|wf^{(s'')}\| + \|wf^{(r+s+1)}\| + \|w\varphi^{2r+2} f^{(2r+s+2)}\| \right). \tag{4.14}$$

That verifies the second estimate in the corollary; to get the first one we also use the inequality

$$\|wf^{(r+s+1)}\| \le c \left(\|wf^{(s'')}\| + \|w\varphi^{2r+2} f^{(2r+s+2)}\| \right),$$

which follows from Proposition 2.1 with $g = f^{(s'')}$, $j = r+s-s''+1$, $m = 2r+s-s''+2$, $w_1 = w$ and $w_2 = w\varphi^{2r+2}$.

So, let us proceed to the proof of (4.14). We set

$$V_{r,n}f := \mathcal{B}_{r,n}f - f - \frac{(-1)^{r-1}}{(2n)^r}\, D^r f.$$

For $s \geq 3$ we establish by induction on r that

$$\|w(V_{r,n}f)^{(s)}\| \leq \frac{c}{n^{r+1}} \sum_{i=0}^{r+1} \sum_{j=2i}^{i+r+1} \|w\varphi^{2i} f^{(j+s)}\|. \tag{4.15}$$

To this end, we use the relation

$$\|w(V_{r+1,n}f)^{(s)}\| \leq \|w(V_{1,n}F_{r,n})^{(s)}\| + \frac{1}{n}\,\|w(DV_{r,n}f)^{(s)}\|, \tag{4.16}$$

where $F_{r,n} := (B_n - I)^r f$, as we estimate $\|w(V_{1,n}F_{r,n})^{(s)}\|$ by means of Proposition 3.20 and (4.11), and the term $\|w(DV_{r,n}f)^{(s)}\|$ by (see (2.11))

$$\begin{aligned} \|w(DV_{r,n}f)^{(s)}\| \\ \leq c\left(\|w(V_{r,n}f)^{(s)}\| + \|w(V_{r,n}f)^{(s+1)}\| + \|w\varphi^2(V_{r,n}f)^{(s+2)}\|\right) \end{aligned} \tag{4.17}$$

and the induction hypothesis.

Next, we estimate above the terms of (4.15) with $i = 0$ by means of Proposition 2.2(c) with $g = f^{(s)}$ and $m = r+1$ to get for $j = 0, \ldots, r+1$

$$\|wf^{(j+s)}\| \leq c\left(\|wf^{(s)}\| + \|wf^{(r+s+1)}\|\right). \tag{4.18}$$

For the terms with $i > 0$, we apply Proposition 2.1 with $g = f^{(s)}$, $m = 2r+2$, $w_1 = w\varphi^{2i}$ and $w_2 = w\varphi^{2r+2}$ to get for $j = 2i, \ldots, i+r+1$

$$\|w\varphi^{2i} f^{(j+s)}\| \leq c\left(\|wf^{(s)}\| + \|w\varphi^{2r+2} f^{(2r+s+2)}\|\right). \tag{4.19}$$

Now, estimate (4.14) for $s \geq 3$ follows from (4.15)-(4.19).

The proof in the case $s = 2$ is similar. We verify by induction on r that

$$\|w(V_{r,n}f)''\| \leq \frac{c}{n^{r+1}} \sum_{i=0}^{r+1} \sum_{j=\max\{1,2i\}}^{i+r+1} \|w\varphi^{2i} f^{(j+2)}\|,$$

as besides (4.16), (4.17), Proposition 3.20 and (4.11) we also use (4.15). Then we complete the proof by means of Proposition 2.1 just similarly as in the case $s \geq 3$.

Finally, in the case $s = 1$, we apply (4.16) with $s = 1$ and $r - 1$ in place of r, Propositions 3.20 and 4.12, the trivial estimate

$$\|w(DV_{r-1,n}f)'\| \leq c\left(\|w(V_{r-1,n}f)''\| + \|w\varphi^2(V_{r-1,n}f)'''\|\right)$$

and what we have already established to deduce

$$\begin{aligned}\|w(V_{r,n}f)'\| \leq \frac{c}{n^{r+1}}\Big(&\|wf^{(3)}\| + \|wf^{(r+2)}\| + \|w\varphi^4 f^{(5)}\| + \|w\varphi^2 f^{(r+3)}\| \\ &+ \|w\varphi^{2r-2}f^{(2r+1)}\| + \|w\varphi^{2r}f^{(2r+2)}\| + \|w\varphi^{2r+2}f^{(2r+3)}\|\Big).\end{aligned}$$

To complete the proof of (4.14) for $s = 1$ we need only take into account the inequalities

$$\|w\varphi^{2j}f^{(r+j+2)}\| \leq c\left(\|wf^{(r+2)}\| + \|w\varphi^{2r+2}f^{(2r+3)}\|\right), \quad j = 1, r-1, r,$$

and

$$\|w\varphi^4 f^{(5)}\| \leq c\left(\|wf^{(3)}\| + \|w\varphi^{2r+2}f^{(2r+3)}\|\right),$$

which follow from Proposition 2.1 respectively with $g = f^{(r+2)}$, $m = r + 1$, $w_1 = w\varphi^{2j}$, $w_2 = w\varphi^{2r+2}$ and $g = f^{(3)}$, $j = 2$, $m = 2r$, $w_1 = w\varphi^4$, $w_2 = w\varphi^{2r+2}$. $\square$

Similarly to Corollary 3.18 we get by Propositions 2.6 and 4.14 the following Voronovskaya-type estimate.

Corollary 4.15. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \leq \gamma_0, \gamma_1 < s$. Then for all $f \in AC^{2r+s+1}[0,1]$ such that $w\varphi^{2r+2}f^{(2r+s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\left\|w\left(\mathcal{B}_{r,n}f - f - \frac{(-1)^{r-1}}{(2n)^r}\,D^r f\right)^{(s)}\right\| \leq \frac{c}{n^{r+1}}\,\|w(D^{r+1}f)^{(s)}\|.$$

The value of the constant c is independent of f and n.

4.4.3 Bernstein-type inequalities

The last several estimates, we will need, are several Bernstein-type inequalities.

Proposition 4.16. *Let $\ell, r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there hold:*

(a) $\|w\varphi^{2\ell}(\mathcal{B}_{r,n}f)^{(2\ell+s)}\| \le c\, n^\ell \|wf^{(s)}\|$;

(b) $\|w(\mathcal{B}_{r,n}f)^{(\ell+s)}\| \le c\, n^\ell \|wf^{(s)}\|$;

(c) $\|w\varphi^{2\ell}(\mathcal{B}_{r,n}f)^{(2\ell+s)}\| \le c\, n^\ell K_{2\ell,\varphi}(f^{(s)}, n^{-\ell})_w$.

The value of the constant c is independent of f and n.

Proof. Assertion (a) and (b) follow from Propositions 3.23 and 3.14 since $\mathcal{B}_{r,n}$ is a linear combination of iterates of B_n.

Finally, to prove (c) we apply (a) and Proposition 3.14 to derive for any $g \in AC_{loc}^{2\ell+s-1}(0,1)$ the estimate

$$\|w\varphi^{2\ell}(\mathcal{B}_{r,n}f)^{(2\ell+s)}\| \le c\, n^\ell \left(\|w(f^{(s)} - g^{(s)})\| + n^{-\ell}\|w\varphi^{2\ell}g^{(2\ell+s)}\|\right).$$

Taking an infimum on g we get (c). □

Similarly to the Bernstein operator, $\mathcal{B}_{r,n}$ satisfies analogues of the above Bernstein-type inequalities in terms of the differential operator $(D^r g)^s$. They directly follow from them and the embedding inequalities in Chapter 2.

Corollary 4.17. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(D^r\mathcal{B}_{r,n}f)^{(s)}\| \le c\, n^r \|wf^{(s)}\|.$$

The value of the constant c is independent of f and n.

Proof. It can be established by induction on r that (cf. [53, p. 24])

$$D^r g = \varphi^2 \sum_{i=2}^{r+1} q_{r,i-2}\, g^{(i)} + \sum_{i=2}^{r} \varphi^{2i}\, \tilde{q}_{r,r-i}\, g^{(i+r)},$$

where $q_{r,j}, \tilde{q}_{r,j} \in \pi_j$. Hence we derive that

$$(D^r g)^{(s)} = \sum_{i=s'}^{r+s} \hat{q}_{r,s,i}\, g^{(i)} + \sum_{i=1}^{r} \varphi^{2i}\, \hat{q}_{r,s,r+s+i}\, g^{(i+r+s)}, \tag{4.20}$$

where $s' := \max\{2, s\}$ and $\hat{q}_{r,s,j}$ are polynomials. Hence

$$\|w(D^r g)^{(s)}\| \le c \sum_{i=s'}^{r+s} \|wg^{(i)}\| + \sum_{i=1}^{r} \|w\varphi^{2i} g^{(i+r+s)}\|.$$

The embedding inequality Proposition 2.2(c) yields for $i = s', \ldots, r+s$

$$\begin{aligned}\|wg^{(i)}\| &\le c\left(\|wg^{(s')}\| + \|wg^{(r+s)}\|\right)\\ &\le c\left(\|wg^{(s)}\| + \|wg^{(r+s)}\|\right).\end{aligned}$$

Similarly, by means of Proposition 2.2(a) we get for $i = 1, \ldots, r$

$$\|w\varphi^{2i} g^{(i+r+s)}\| \le c\left(\|wg^{(r+s)}\| + \|w\varphi^{2r} g^{(2r+s)}\|\right).$$

Consequently,

$$\|w(D^r g)^{(s)}\| \le c\left(\|wg^{(s')}\| + \|wg^{(r+s)}\| + \|w\varphi^{2r} g^{(2r+s)}\|\right) \tag{4.21}$$

$$\le c\left(\|wg^{(s)}\| + \|wg^{(r+s)}\| + \|w\varphi^{2r} g^{(2r+s)}\|\right) \tag{4.22}$$

and the middle term can be omitted except when $\gamma_0\gamma_1 = 0$.

Now, the assertion of the corollary follows from (4.22) with $g = \mathcal{B}_{r,n}f$ and Propositions 4.11, 4.16, (a) and (b), with $\ell = r$. □

Corollary 4.18. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC^{2r+s-1}[0,1]$ and $w\varphi^{2r} f^{(2r+s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(D^{r+1}\mathcal{B}_{r,n} f)^{(s)}\| \le c\, n\, \|w(D^r f)^{(s)}\|.$$

The value of the constant c is independent of f and n.

Proof. Just as in the previous proof, we apply (4.21) with $r+1$ in place of r and $g = \mathcal{B}_{r,n}f$, Proposition 4.11, Proposition 4.16(a) with $\ell = 1$, $w\varphi^{2r}$ in place of w, and $2r+s$ in place of s, and Proposition 4.16(b) with $\ell=1$ and $r+s$ in place of s to derive the estimate

$$\|w(D^{r+1}\mathcal{B}_{r,n}f)^{(s)}\| \le c\left(\|wf^{(s')}\| + n\,\|wf^{(r+s)}\| + n\,\|w\varphi^{2r}f^{(2r+s)}\|\right).$$

In fact, the term $\|w(\mathcal{B}_{r,n}f)^{(r+s+1)}\|$ and hence $n\,\|wf^{(r+s)}\|$ appear only in the case $\gamma_0\gamma_1=0$.

Now, the assertion of the corollary follows from Proposition 2.6. □

4.5 Proof of the converse estimate for the approximation by $\mathcal{B}_{r,n}$

We establish Theorem 4.2 again by means of the method introduced by Ditzian and Ivanov [22]. To this end, we need a Voronovskaya-type inequality and several Bernstein-type inequalities, which relate the approximation operator $\mathcal{B}_{r,n}$ to the differential operator D^r.

We begin with two Voronovskaya-type estimates (cf. [53, Lemma 4]).

Proposition 4.19. *Let $r\in\mathbb{N}_+$. Then for all $g\in C[0,1]$ such that $g\in AC_{loc}^{2r+1}(0,1)$ and $\varphi^{2r+2}g^{(2r+2)}\in L_\infty[0,1]$, and all $n\in\mathbb{N}_+$ there holds*

$$\left\|\mathcal{B}_{r,n}g - g - \frac{(-1)^{r-1}}{(2n)^r}\,D^r g\right\| \le \frac{c}{n^{r+1}}\left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2}g^{(2r+2)}\|\right).$$

The value of the constant c is independent of f and n.

Proof. We note that by virtue of Proposition 2.3(a) we have $\varphi^2 g'', \varphi^2 g^{(3)}\in L_\infty[0,1]$ too.

First, we establish the assertion for $r=1$. Applying Taylor's formula, we have for $x\in(0,1)$

$$\begin{aligned} g\left(\frac{k}{n}\right) &= g(x) + \left(\frac{k}{n}-x\right)g'(x) + \frac{1}{2}\left(\frac{k}{n}-x\right)^2 g''(x) \\ &\quad + \frac{1}{6}\left(\frac{k}{n}-x\right)^3 g^{(3)}(x) + \frac{1}{6}\int_x^{k/n}\left(\frac{k}{n}-v\right)^3 g^{(4)}(v)\,dv. \end{aligned}$$

Multiplying both sides by $p_{n,k}(x)$, summing with respect to k and using the identities (3.17) we obtain

$$\begin{aligned}
&\left|B_n g(x) - g(x) - \frac{1}{2n}\,\varphi^2(x)g''(x)\right| \\
&= \left|\frac{(1-2x)\varphi^2(x)}{6n^2} g^{(3)}(x) + \frac{1}{6}\sum_{k=0}^{n} p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^3 g^{(4)}(v)\,dv\right| \\
&\le \frac{1}{6n^2}\|\varphi^2 g^{(3)}\| + \frac{1}{6}\|\varphi^4 g^{(4)}\|\left|\sum_{k=0}^{n} p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^3 \varphi^{-4}(v)\,dv\right|
\end{aligned}$$

We will show that

$$R_n(x) := \left|\sum_{k=0}^{n} p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^3 \varphi^{-4}(v)\,dv\right| \le \frac{c}{n^2}.$$

Obviously, it is enough to prove it for $0 < x \le 1/2$. We consider two cases.

Case 1. $1/n \le x \le 1/2$.

Then $\varphi^2(x) \ge 1/2n$ and by using (for v between x and k/n) the inequality [23, p. 141]

$$\frac{\left|\frac{k}{n}-v\right|}{\varphi^2(v)} \le \frac{\left|\frac{k}{n}-x\right|}{\varphi^2(x)}$$

and (3.17), we obtain

$$\begin{aligned}
R_n(x) &\le \sum_{k=0}^{n} p_{n,k}(x)\frac{\left(\frac{k}{n}-x\right)^2}{\varphi^4(x)}\left|\int_x^{k/n}\left(\frac{k}{n}-v\right)\,dv\right| \\
&= \frac{\varphi^{-4}(x)}{2}\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}-x\right)^4 \\
&= \frac{\varphi^{-4}(x)}{2}\left[\frac{3\varphi^4(x)}{n^2} + \frac{(1-6\varphi^2(x))\varphi^2(x)}{n^3}\right] \le \frac{c}{n^2}.
\end{aligned}$$

Case 2. $0 < x \le 1/n$.

Analogously to [22, Lemma 8.3], we will estimate the terms in the sum of $R_n(x)$ separately for $k = 0, 1$ and $k \ge 2$. We have for $k = 0$

$$\begin{aligned}
p_{n,0}(x)\int_0^x v^3\varphi^{-4}(v)\,dv &= (1-x)^n\int_0^x \frac{v^3\,dv}{\big(v(1-v)\big)^2} \\
&\le (1-x)^{n-2}\int_0^x v\,dv = \frac{x^2(1-x)^{n-2}}{2} \le \frac{c}{n^2}.
\end{aligned}$$

For $k = 1$ and $n \geq 2$ we have

$$p_{n,1}(x) \int_x^{1/n} \left(\frac{1}{n} - v\right)^3 \varphi^{-4}(v)\, dv = nx(1-x)^{n-1} \int_x^{1/n} \frac{\left(\frac{1}{n} - v\right)^3 dv}{\big(v(1-v)\big)^2}$$
$$\leq nx(1-x)^{n-1} \left(1 - \frac{1}{n}\right)^{-2} \int_x^{1/n} \frac{\left(\frac{1}{n}\right)^3 dv}{v^2} \leq \frac{c}{n^2}.$$

Trivially, for $n = k = 1$ we have

$$p_{1,1}(x) \int_x^1 (1-v)^3 \varphi^{-4}(v)\, dv = x \int_x^1 \frac{(1-v)^3 dv}{\big(v(1-v)\big)^2} \leq x \int_x^1 \frac{dv}{v^2} \leq 1.$$

For $k \geq 2$ and $n \geq 3$ we have

$$\left| \sum_{k=2}^n p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 \varphi^{-4}(v)\, dv \right|$$
$$\leq cx^{-2} \sum_{k=2}^n p_{n,k}(x) \left(\frac{k}{n} - x\right)^4 \leq cx^{-2} \sum_{k=2}^n p_{n,k}(x) \left(\frac{k}{n}\right)^4$$
$$= cx^{-2} \sum_{k=0}^{n-2} \frac{n!}{(k+2)!(n-k-2)!}\, x^{k+2}(1-x)^{n-k-2} \left(\frac{k+2}{n}\right)^4$$
$$\leq c \sum_{k=0}^{n-2} p_{n-2,k}(x) \left(\frac{k}{n-2}\right)^2 = c\left(x^2 + \frac{\varphi^2(x)}{n-2}\right) \leq \frac{c}{n^2},$$

where at the last but one estimate we have taken into account (3.17). The case $n = k = 2$ is again trivial. The proof is complete.

Let $r \geq 2$. We set $J_{r,n}g := (I - B_n)^r g$ and

$$V_{r,n}g := \mathcal{B}_{r,n}g - g - \frac{(-1)^{r-1}}{(2n)^r}\, D^r g.$$

We use the relation

$$V_{r,n}g = V_{1,n}J_{r-1,n}g - \frac{1}{2n} DV_{r-1,n}g.$$

It implies

$$\|V_{r,n}g\| \leq \|V_{1,n}J_{r-1,n}g\| + \frac{1}{n}\, \|\varphi^2(V_{r-1,n}g)''\|. \tag{4.23}$$

By virtue of Proposition 4.19 with $r = 1$,

$$\|V_{1,n}J_{r-1,n}g\| \le \frac{c}{n^2}\Big(\|\varphi^2(J_{r-1,n}g)^{(3)}\| + \|\varphi^4(J_{r-1,n}g)^{(4)}\|\Big). \tag{4.24}$$

Further, we estimate the first term on the right above by means of Proposition 4.12 with $r-1$ in place of r, $s = 3$ and $w = \varphi^2$. Thus we get

$$\|\varphi^2(J_{r-1,n}g)^{(3)}\| \le \frac{c}{n^{r-1}}\left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r}g^{(2r+1)}\|\right). \tag{4.25}$$

Similarly, again by Proposition 4.12 with and $r-1$ in place of r, but $s = 4$ and $w = \varphi^4$ we have for the other term

$$\|\varphi^4(J_{r-1,n}g)^{(4)}\| \le \frac{c}{n^{r-1}}\left(\|\varphi^4 g^{(4)}\| + \|\varphi^{2r+2}g^{(2r+2)}\|\right). \tag{4.26}$$

Next, by virtue of Proposition 2.1 with $j = 1$, $m = 2r-1$, $w_1 = \varphi^4$, $w_2 = \varphi^{2r+2}$ and $g^{(3)}$ in place of g, we get

$$\|\varphi^4 g^{(4)}\| \le c\left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2}g^{(2r+2)}\|\right). \tag{4.27}$$

Likewise, by means of the same proposition with $m = 2r-1$, $w_2 = \varphi^{2r+2}$ and $g^{(3)}$ in place of g, but with $j = 2r-2$ and $w_1 = \varphi^{2r}$, we get

$$\|\varphi^{2r}g^{(2r+1)}\| \le c\left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2}g^{(2r+2)}\|\right). \tag{4.28}$$

Combining, (4.24)-(4.28), we get

$$\|V_{1,n}J_{r-1,n}g\| \le \frac{c}{n^{r+1}}\Big(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2}g^{(2r+2)}\|\Big). \tag{4.29}$$

It remains to estimate the second term on the right side of (4.23). To this end, we apply Proposition 4.14 with $r-1$ in place of r, $s = 2$, and $w = \varphi^2$ and get

$$\|\varphi^2(V_{r-1,n}g)''\| \le \frac{c}{n^r}\left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2}g^{(2r+2)}\|\right). \tag{4.30}$$

Now, (4.23), (4.29) and (4.30) imply the assertion for $r \ge 2$. □

Corollary 4.20. *Let $r \in \mathbb{N}_+$. Then for all $g \in C^{2r+2}[0,1]$ and all $n \in \mathbb{N}_+$ there holds*

$$\left\|\mathcal{B}_{r,n}g - g - \frac{(-1)^{r-1}}{(2n)^r}\,D^r g\right\| \le \frac{c}{n^{r+1}}\,\|D^{r+1}g\|.$$

The value of the constant c is independent of f and n.

Proof. The estimate follows from the previous proposition and several embedding inequalities.

We apply Proposition 2.7(b) with $r+1$ in place of r to get

$$\|\varphi^{2r+2}g^{(2r+2)}\| \le c\,\|D^{r+1}g\|. \tag{4.31}$$

Also, by virtue of Proposition 2.3(a) with $i=3$ and $r+1$ in place of r, we have

$$\|\varphi^2 g^{(3)}\| \le c\left(\|\varphi^2 g''\| + \|\varphi^{2r+2}g^{(2r+2)}\|\right).$$

Taking into account (4.31) and Proposition 2.7(c) with $j=1$ and $r+1$ in place of r, we arrive at

$$\|\varphi^2 g^{(3)}\| \le c\,\|D^{r+1}g\|. \tag{4.32}$$

Now, Proposition 4.20 follows from Proposition 4.19, (4.31) and (4.32). □

Next we will establish several Bernstein-type inequalities.

Proposition 4.21. *Let $r\in\mathbb{N}_+$. Then for all $f\in C[0,1]$ and $n\in\mathbb{N}_+$ there holds*

$$\|D^r\mathcal{B}_{r,n}f\| \le c\,n^r\|f\|.$$

The value of the constant c is independent of f and n.

Proof. Let $g\in C^{2r}[0,1]$. It is established by induction on r that (cf. [53, p. 24])

$$D^r g = \varphi^2\sum_{i=2}^{r+1} q_{r,i-2}\,g^{(i)} + \sum_{i=2}^{r}\varphi^{2i}\,\tilde{q}_{r,r-i}\,g^{(i+r)},$$

where $q_{r,j}$ and $\tilde{q}_{r,j}$ are algebraic polynomials of degree at most j.

Therefore

$$\|D^r g\| \le c\left(\sum_{i=2}^{r+1}\|\varphi^2 g^{(i)}\| + \sum_{i=2}^{r}\|\varphi^{2i}g^{(i+r)}\|\right). \tag{4.33}$$

Let $r\ge 2$. By virtue of Proposition 2.3(a), we have

$$\|\varphi^2 g^{(i)}\| \le c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r}g^{(2r)}\|\right),\quad i=3,\dots,r+1. \tag{4.34}$$

Also, this trivially holds for $i=2$, $r=1$.

Let $r \geq 3$. By virtue of Proposition 2.3(b), we have

$$\|\varphi^{2i} g^{(i+r)}\| \leq c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\|\right), \quad i = 2, \dots, r-1. \tag{4.35}$$

The above estimate trivially holds for $i = r$, $r \geq 2$, as well.

Inequalities (4.33)-(4.35) yield

$$\|D^r g\| \leq c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\|\right), \quad r \in \mathbb{N}_+. \tag{4.36}$$

With $g = \mathcal{B}_{r,n} f$ we get

$$\|D^r \mathcal{B}_{r,n} f\| \leq c\left(\|\varphi^2 (\mathcal{B}_{r,n} f)^{(2)}\| + \|\varphi^{2r} (\mathcal{B}_{r,n} f)^{(2r)}\|\right). \tag{4.37}$$

Then we take into account that the operator $\mathcal{B}_{r,n}$ is a linear combination of iterates of B_n and also that (see Proposition 3.14 with $w = \varphi^{2\ell}$ and $s = 2\ell$)

$$\|\varphi^{2\ell} (B_n g)^{(2\ell)}\| \leq c\,\|\varphi^{2\ell} g^{(2\ell)}\|, \quad g \in C^{2\ell}[0,1], \tag{4.38}$$

to derive from (4.37) the estimate

$$\|D^r \mathcal{B}_{r,n} f\| \leq c\left(\|\varphi^2 (B_n f)^{(2)}\| + \|\varphi^{2r} (B_n f)^{(2r)}\|\right).$$

Now, the assertion of the proposition follows from

$$\|\varphi^{2\ell} (B_n f)^{(2\ell)}\| \leq c\, n^\ell \|f\|, \quad \ell \in \mathbb{N}_+,$$

which was established in [23, Theorem 9.4.1]. □

Proposition 4.22. *Let $r \in \mathbb{N}_+$. Then for all $g \in C^{2r}[0,1]$ and $n \in \mathbb{N}_+$ there holds*

$$\|D^{r+1} \mathcal{B}_{r,n} g\| \leq c\, n \|D^r g\|.$$

The value of the constant c is independent of f and n.

Proof. We make use of (4.36) with $r+1$ in place of r and $\mathcal{B}_{r,n} g$ in place of g, then apply (4.38), Proposition 3.23(a) with $w = \varphi^{2r}$, $\ell = 1$, $s = 2r$, and, finally, Proposition 2.7(c) with $j = 1$, to arrive at

$$\begin{aligned}
\|D^{r+1} \mathcal{B}_{r,n} g\| &\leq c\left(\|\varphi^2 (\mathcal{B}_{r,n} g)^{(2)}\| + \|\varphi^{2r+2} (\mathcal{B}_{r,n} g)^{(2r+2)}\|\right) \\
&\leq c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r+2} (B_n g)^{(2r+2)}\|\right) \\
&\leq c\left(\|\varphi^2 g^{(2)}\| + n\,\|\varphi^{2r} g^{(2r)}\|\right) \\
&\leq c\, n \|D^r g\|.
\end{aligned}$$

Thus the proposition is verified. □

Equipped with the estimates established in the previous section, we are now ready to verify Theorem 4.2.

Proof of Theorem 4.2. We apply [22, Theorem 3.2] with the operator $Q_n = \mathcal{B}_{r,n}$ and the spaces $X = C[0,1]$ (with the uniform norm on $[0,1]$), $Y = C^{2r}[0,1]$ and $Z = C^{2r+2}[0,1]$.

As is known,

$$\|B_n f\| \le \|f\|.$$

Therefore, since $\mathcal{B}_{r,n}$ is linear combination of iterates of B_n, we have

$$\|\mathcal{B}_{r,n} f\| \le c\,\|f\|, \quad f \in C[0,1],\ n \in \mathbb{N}_+.$$

Thus [22, (3.3)] is satisfied.

By virtue of the Voronovskaya-type inequality Proposition 4.20, we have [22, (3.4)] with $(-1)^{r-1}D^r$ in place of D, $\Phi(f) = \|D^{r+1}f\|$, $\lambda(n) = (2n)^{-r}$ and $\lambda_1(\alpha) = c\,n^{-r-1}$, where the constant c is the one in Proposition 4.20.

Next, Proposition 4.22 with $g = \mathcal{B}_{r,n}f$ implies [22, (3.5)] with $\ell = 1$ and $m = 2$.

Finally, Proposition 4.21 yields [22, (3.6)]. □

Let us note that [22, Theorems 10.4 and 10.5] are not applicable because condition (c) there is not satisfied.

4.6 Proof of the characterization of the rate of the simultaneous approximation by $\mathcal{B}_{r,n}$

We are ready now to verify the theorems in Section 4.3.

Proof of Theorem 4.3. The estimate follows from Proposition 4.11 and Corollary 4.13 via a standard argument (see e.g. [22, Theorem 3.4]). For any $g \in C^{2r+s}[0,1]$ we have

$$\begin{aligned}\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| &\le \|w(f^{(s)} - g^{(s)})\| + \|w(\mathcal{B}_{r,n}g - g)^{(s)}\| \\ &\quad + \|w(\mathcal{B}_{r,n}(f-g))^{(s)}\| \\ &\le c\left(\|w(f^{(s)} - g^{(s)})\| + \frac{1}{n^r}\|w(D^r g)^{(s)}\|\right).\end{aligned}$$

Taking an infimum on $g \in C^{2r+s}[0,1]$, we arrive at the estimate stated in the theorem. □

Proof of Theorems 4.7 and 4.8. By virtue of (1.6) and (1.9), the assertions of the corollaries follow for $n \geq n_0$ with some $n_0 \in \mathbb{N}_+$ from Theorems 4.3-4.5. For $n < n_0$ we apply Proposition 4.11 to get

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \leq \frac{c}{n^r}\, \|wf^{(s)}\|, \tag{4.39}$$

which completes the proof for $s \geq 2$. For $s = 1$ we use that $\mathcal{B}_{r,n}f$ preserves the linear functions to deduce from (4.39) the estimate

$$\|w(\mathcal{B}_{r,n}f - f)'\| \leq \frac{c}{n^r}\, E_0(f')_w, \quad n < n_0.$$

Then we apply (4.54) below with f' in place of f and the relation $K_1(f', t)_w \leq c\,\omega_1(f', t)_w$, $0 < t \leq 1$. □

Proof of Theorem 4.9. We proceed as in the proof of Theorem 4.3. We need to show that

$$\|D^s\mathcal{B}_{r,n}f\| \leq c\,\|D^s f\|, \quad n \in \mathbb{N}_+. \tag{4.40}$$

To this end, we apply consecutively Propositions 2.7(a), 4.11 with $w = \varphi^{2s}$ and Proposition 2.7(b) to derive

$$\begin{aligned}\|D^s\mathcal{B}_{r,n}f\| &\leq c\,(\|\mathcal{B}_{r,n}f\| + \|\varphi^{2s}(\mathcal{B}_{r,n}f)^{(2s)}\|)\\ &\leq c\,(\|f\| + \|\varphi^{2s}f^{(2s)}\|)\\ &\leq c\,(\|f\| + \|D^s f\|);\end{aligned}$$

hence

$$\|D^s\mathcal{B}_{r,n}f\| \leq c\,(E_1(f) + \|D^s f\|).$$

To complete the proof of (4.40), we need only to take into account (see (4.6))

$$E_1(g) \leq \|B_1 g - g\| \leq c\,\|\varphi^2 g''\|, \quad g \in AC^1_{loc}(0,1),$$

and apply Proposition 2.7(c). □

We proceed to the proof of the converse inequality.

Proof of Theorem 4.10. Just as in the proof of Theorem 3.8, we apply [22, Theorem 3.2] with the operator $Q_n = \mathcal{B}_{r,n}$ on the space

$$X = \{f \in C[0,1] : f \in AC^{s-1}_{loc}(0,1), wf^{(s)} \in L_p[0,1]\}$$

with a semi-norm $\|f\|_X := \|wf^{(s)}\|$. Let also $Y = C^{2r+s}[0,1]$ and $Z = C^{2r+s+2}[0,1]$.

Proposition 4.11 implies that Q_n is a bounded operator on X, so that [22, (3.3)] holds.

By virtue of Corollary 4.15, we have for $\Phi(f) := \|w(D^{r+1}f)^{(s)}\|$ and $f \in Z$

$$\left\| w\left(Q_n f - f - \frac{(-1)^{r-1}}{(2n)^r}\, D^r f\right)^{(s)} \right\| \leq \frac{c}{n^{r+1}}\,\Phi(f),$$

which shows that [22, (3.4)] is valid with $(-1)^{r-1}D^r$ in place of D, $\lambda(n) := (2n)^{-r}$ and $\lambda_1(n) := c\,n^{-r-1}$, where the constant c is the one from Corollary 4.15.

Further, we set $g := \mathcal{B}_{r,n} f$ for $f \in X$ and apply Corollary 4.18 to obtain

$$\Phi(Q_n^2 f) = \Phi(\mathcal{B}_{r,n} g) \leq c\,n\,\|w(D^r g)^{(s)}\| = c\,n\,\|w(D^r \mathcal{B}_{r,n} f)^{(s)}\|.$$

Hence [22, (3.5)] is established with $m = 2$ and $\ell = 1$.

Finally, Corollary 4.17 yields for $f \in X$

$$\|w(D^r Q_n f)^{(s)}\| \leq c\,n^r \|wf^{(s)}\|,$$

which is [22, (3.6)].

Now, [22, Theorem 3.2] implies the converse estimate in Theorem 4.3. □

Let us explicitly note that the characterization of the weighted simultaneous approximation by $\mathcal{B}_{r,n}$ in terms of the K-functionals $K_{2r,\varphi}(f,t)_w$ and $K_j(f,t)_w$ can be directly derived from Propositions 4.11, 4.12, 4.14, 4.16, (a) and (b), by means of [22, Theorems 3.2 and 3.4].

4.7 Relations between K-functionals

In this section, first we will show that the direct estimates (4.2) and (4.3) are equivalent. More precisely, we will establish that the quantities on the right hand-side of (4.3) and (4.8) with $\|f\|$ replaced by $E_1(f)$ are equivalent. In addition, we will verify a characterization of $K^D_{r,0}(f,t)$ by means K-functionals of the form $K_{m,\varphi}(f,t)$.

Theorem 4.23. *Let $r \in \mathbb{N}_+$. For all $f \in C[0,1]$ and $0 < t \leq 1$ we have*

$$K^D_{r,0}(f,t) \sim K_{2r,\varphi}(f,t) + t\,E_1(f) \tag{4.41}$$

$$\sim K_{2r,\varphi}(f,t) + K_{2,\varphi}(f,t). \tag{4.42}$$

Proof. The first assertion will follow from the inequalities:

$$K_{2r,\varphi}(f,t) \le c\,K_{r,0}^{D}(f,t),$$
$$t\,E_1(f) \le c\,K_{r,0}^{D}(f,t)$$

and

$$K_{r,0}^{D}(f,t) \le c\,(K_{2r,\varphi}(f,t) + t\,E_1(f))\,.$$

The first one follows directly from Proposition 2.7(b). The second one follows from the estimate

$$\begin{aligned} t\,E_1(f) &\le E_1(f-g) + t\,E_1(g) \le c\,(\|f-g\| + t\,\|Dg\|) \\ &\le c\,(\|f-g\| + t\,\|D^r g\|)\,, \quad g \in C^{2r}[0,1], \quad 0 < t \le 1, \end{aligned}$$

where at the second step we have taken into account the estimate

$$E_1(g) \le \|B_1 g - g\| \le c\,\|\varphi^2 g''\|, \quad g \in AC_{loc}^1(0,1),$$

and at the third Proposition 2.7(c).

In order to verify the third inequality, we apply Proposition 2.7(a) to get for any $g \in C^{2r}[0,1]$ and $t \le 1$ that

$$\begin{aligned} t\,\|D^r g\| &\le c\,t\left(\|\varphi^{2r} g^{(2r)}\| + E_1(g)\right) \\ &\le c\left(\|f-g\| + t\,\|\varphi^{2r} g^{(2r)}\| + t\,E_1(f)\right). \end{aligned}$$

Consequently,

$$K_{r,0}^{D}(f,t) \le c\left(\inf_{g\in C^{2r}[0,1]} \left\{\|f-g\| + t\|\varphi^{2r} g^{(2r)}\|\right\} + t\,E_1(f)\right).$$

It remains to observe that

$$\inf_{g\in C^{2r}[0,1]} \left\{\|f-g\| + t\|\varphi^{2r} g^{(2r)}\|\right\} \le c\,K_{2r,\varphi}(f,t). \tag{4.43}$$

To justify the latter, we recall that the Steklov-type function used in [23, Chapter 2] to establish the inequality

$$K_{2r,\varphi}(f,t) \le c\,\omega_{\varphi}^{2r}(f,t)$$

belongs to $C^{2r}[0,1]$ (or see [18, Chapter 6, § 6], where a spline was used). Therefore the K-functional on the left hand-side of (4.43) is estimated above by $\omega_\varphi^{2r}(f,t)$ (at least for $t \le t_0$); and hence, in view of (1.9), by $K_{2r,\varphi}(f,t)$.

We proceed to the proof of the second assertion of the theorem:

$$K_{2r,\varphi}(f,t) + tE_1(f) \sim K_{2r,\varphi}(f,t) + K_{2,\varphi}(f,t), \quad 0 < t \le 1. \tag{4.44}$$

Trivially, for any $g \in C[0,1]$ such that $g \in AC^1_{loc}(0,1)$ and $\varphi^2 g'' \in L_\infty[0,1]$, and any $t \in (0,1]$ we have the estimates

$$tE_1(f) \le \|f-g\| + t\,\|g - B_1 g\| \le \|f-g\| + ct\|\varphi^2 g''\|;$$

hence

$$tE_1(f) \le c\,K_{2,\varphi}(f,t), \quad 0 < t \le 1.$$

Above we used the inequality

$$\|g - B_1 g\| \le \|\varphi^2 g''\|,$$

which is directly established by Taylor's formula (see e.g. [22, p. 87]).

To complete the proof of (4.44), it remains to show that

$$K_{2,\varphi}(f,t) \le c\left(K_{2r,\varphi}(f,t) + tE_1(f)\right), \quad 0 < t \le 1. \tag{4.45}$$

Let $g \in C[0,1]$ be such that $g \in AC^{2r-1}_{loc}(0,1)$ and $\varphi^{2r} g^{(2r)} \in L_\infty[0,1]$. Then, by e.g. Proposition 2.2 (b) with $w=1$, $j=1$ and $m=2r$, we deduce that $\varphi^2 g'' \in L_\infty[0,1]$ too, as, moreover,

$$\|\varphi^2 g''\| \le c\left(\|g\| + \|\varphi^{2r} g^{(2r)}\|\right).$$

Consequently, we have for $t \in (0,1]$

$$\begin{aligned} K_{2,\varphi}(f,t) &\le \|f-g\| + t\|\varphi^2 g''\| \\ &\le c\left(\|f-g\| + t\|\varphi^{2r} g^{(2r)}\|\right) + ct\|f\|. \end{aligned}$$

Taking the infimum on g, we arrive at

$$K_{2,\varphi}(f,t) \le c\left(K_{2r,\varphi}(f,t) + t\|f\|\right).$$

Finally, we replace f with $f - p_1$, where p_1 is the algebraic polynomial of degree 1 of best approximation in $C[0,1]$ to f, to get (4.45). □

Next, we will verify the assertions of Theorems 4.4 and 4.5 as well as of Remark 3.4. First, we will present a couple of auxiliary inequalities between K-functionals.

It is known that in the case $w = 1$ in the definition of $K_{m,\varphi}(f,t)$ the infimum can be equivalently taken on $C^m[0,1]$. That is evident from the proof of [23, Theorem 2.1.1] (see also [21, p. 110]). That equivalence probably holds for any Jacobi weight w. For our purposes weaker relations will suffice. They are given in the lemma below. Using them one can derive the above mentioned equivalence under the conditions of the lemma, but we will not establish that here.

Lemma 4.24. *Let $r, s \in \mathbb{N}_+$, and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) with $0 < \gamma_0, \gamma_1 < s$. Then for all $wf \in L_\infty[0,1]$ and $0 < t \le 1$ there holds*

$$(4.46)\qquad \inf_{g \in C^{2r+s}[0,1]} \left\{ \|w(f - g^{(s)})\| + t\, \|w\varphi^{2r} g^{(2r+s)}\| \right\} \le c\,(K_{2r,\varphi}(f,t)_w + t\,\|wf\|)\,, \quad s \ge 2,$$

and

$$(4.47)\qquad \inf_{g \in C^{2r+1}[0,1]} \left\{ \|w(f - g')\| + t\, \|w\varphi^{2r} g^{(2r+1)}\| \right\} \le c\,(K_{2r,\varphi}(f,t)_w + K_1(f,t)_w)\,.$$

The value of the constant c is independent of f.

Proof. For a given function f such that $wf \in L_\infty[0,1]$ with $\gamma_0, \gamma_1 < s$, $s \in \mathbb{N}_+$, we set

$$f_s(x) := \frac{1}{(s-1)!} \int_{1/2}^{x} (x-u)^{s-1} f(u)\, du, \quad x \in [0,1].$$

Clearly, $\varphi^{2s-2} f \in L[0,1]$. Hence $f_s(x)$ is well-defined and finite at $x = 0$ and $x = 1$; moreover, $f_s \in C[0,1]$. The continuity of $f_s(x)$ at every interior point for any s as well as at $x = 0, 1$ for $s = 1$ is clear. To see that $f_s(x)$ is continuous at $x = 0, 1$ for $s \ge 2$ we can apply Lebesgue's dominated convergence theorem.

Now, we are ready to verify the inequalities in the lemma. We set $n := [t^{-1/r}] + 1$ and $g_t := \mathcal{B}_{r,n} f_s$. In view of the above remarks, g_t is well defined and clearly $g_t \in C^{2r+s}[0,1]$. To verify (4.46) and (4.47) it is enough to show that

$$\|w(f - g_t^{(s)})\| \le c\left(K_{2r,\varphi}(f,t)_w + t\,\|wf\|\right), \quad s \ge 2, \tag{4.48}$$

$$\|w(f - g_t')\| \le c\left(K_{2r,\varphi}(f,t)_w + K_1(f,t)_w\right) \tag{4.49}$$

and

$$t\,\|w\varphi^{2r} g_t^{(2r+s)}\| \le c\,K_{2r,\varphi}(f,t)_w, \quad s \ge 1. \tag{4.50}$$

Let $G \in AC_{loc}^{2r-1}(0,1)$ with $wG, w\varphi^{2r}G^{(2r)} \in L_\infty[0,1]$ be arbitrarily fixed. Then $G_s \in C[0,1]$. Let $s \ge 2$. Applying Propositions 4.11 and 4.12 and the trivial estimate

$$\|wG\| \le \|w(f-G)\| + \|wf\|, \tag{4.51}$$

we get

$$\begin{aligned}
\|w(f - g_t^{(s)})\| &\le \|w(f-G)\| + \|w(\mathcal{B}_{r,n}G_s - G_s)^{(s)}\| \\
&\qquad + \|w(\mathcal{B}_{r,n}(f_s - G_s))^{(s)}\| \\
&\le c\,\|w(f-G)\| + \frac{c}{n^r}\left(\|wG_s^{(s)}\| + \|w\varphi^{2r}G_s^{(2r+s)}\|\right) \\
&\le c\left(\|w(f-G)\| + t\,\|w\varphi^{2r}G^{(2r)}\| + t\,\|wf\|\right).
\end{aligned}$$

We take an infimum on G and arrive at (4.48).

For $s = 1$ by means of a similar argument we arrive at

$$\|w(f - g_t')\| \le c\left(\|w(f-G)\| + t\,\|w\varphi^{2r}G^{(2r)}\| + t\,\|wG'\|\right). \tag{4.52}$$

Next, we estimate the last term on the right above by means of Proposition 2.1 with $j = 1$, $m = 2r$, $w_1 = w$ and $w_2 = w\varphi^{2r}$. Thus we get

$$\|wG'\| \le c\left(\|wG\| + \|w\varphi^{2r}G^{(2r)}\|\right).$$

Consequently, for an arbitrary real α we have

$$\|wG'\| \le c\left(\|w(G-\alpha)\| + \|w\varphi^{2r}G^{(2r)}\|\right).$$

Setting $E_0(f)_w := \inf_{\alpha\in\mathbb{R}} \|w(f-\alpha)\|$, we arrive at the estimate

$$t\,\|wG'\| \le c\left(\|w(f-G)\| + t\,\|w\varphi^{2r}G^{(2r)}\|\right) + ct\,E_0(f)_w. \tag{4.53}$$

For $wf \in L_\infty[0,1]$, where $\gamma_0, \gamma_1 \geq 0$, and $0 < t \leq 1$ we have

$$t\, E_0(f)_w \leq c\, K_1(f,t)_w. \tag{4.54}$$

That easily follows from the estimate

$$E_0(g)_{w,p} \leq \|w(g - g(1/2))\| = \left\| w \int_{1/2}^{\circ} g'(t)\, dt \right\| \leq c\, \|wg'\|,$$

where $g \in AC_{loc}(0,1)$.

Combining (4.52)-(4.54) we arrive at (4.49).

Finally, to verify (4.50) we apply Proposition 4.16(c) with $\ell = r$ to get

$$t\, \|w\varphi^{2r} g_t^{(2r+s)}\| \leq c\, tn^r K_{2r,\varphi}(f_s^{(s)}, n^{-r})_w \leq c\, K_{2r,\varphi}(f,t)_w.$$

□

Let us proceed now to the proof of Theorems 4.4 and 4.5.

Proof of Theorems 4.4 and 4.5. Let $0 \leq \gamma_0, \gamma_1 < s$ and $g \in C^{2r+s}[0,1]$. Proposition 2.6 yields

$$\|wf\| \leq \|w(f - g^{(s)})\| + c\, \|w(D^r g)^{(s)}\|, \quad s \geq 2, \tag{4.55}$$

$$\|wg^{(j+s)}\| \leq c\, \|w(D^r g)^{(s)}\|, \quad j = 1, r, \quad s \geq 1, \tag{4.56}$$

and

$$\|w\varphi^{2r} g^{(2r+s)}\| \leq c\, \|w(D^r g)^{(s)}\|, \quad s \geq 1. \tag{4.57}$$

Taking an infimum on $g \in C^{2r+s}[0,1]$ in (4.55) we get for $0 < t \leq 1$

$$t\, \|wf\| \leq c\, K_{r,s}(f,t)_w, \quad s \geq 2.$$

Next, since $g^{(s)} \in AC_{loc}^{j-1}(0,1)$ for $j = 1, r$, we derive from (4.56) that

$$K_j(f,t)_w \leq c \left(\|w(f - g^{(s)})\| + t\, \|w(D^r g)^{(s)}\| \right), \quad s \geq 1.$$

Taking an infimum on $g \in C^{2r+s}[0,1]$ we arrive at

$$K_j(f,t)_w \leq c\, K_{r,s}(f,t)_w, \quad j = 1, r, \quad s \geq 1.$$

Just similarly, using that $g^{(s)} \in AC_{loc}^{2r-1}(0,1)$ and (4.57), we establish that

$$K_{2r,\varphi}(f,t)_w \le c\, K_{r,s}(f,t)_w, \quad s \ge 1.$$

Thus we have shown that $K_{r,s}(f,t)_w$ estimates above the quantities on the right-hand side of the relations in Theorems 4.4 and 4.5.

Let us proceed to the reverse inequalities. Let $0 < \gamma_0, \gamma_1 < s$. Let $g \in C^{2r+s}[0,1]$. By (4.21) (see also (4.10)), we have

$$\|w(D^r g)^{(s)}\| \le c \left(\|w g^{(s')}\| + \|w\varphi^{2r} g^{(2r+s)}\| \right), \tag{4.58}$$

where $s' := \max\{2, s\}$. Hence, using (4.51) with $g^{(s)}$ in place of G, we get for $s \ge 2$ the estimate

$$\|w(D^r g)^{(s)}\| \le c \left(\|w(f - g^{(s)})\| + \|w\varphi^{2r} g^{(2r+s)}\| + \|wf\| \right).$$

Consequently, for $s \ge 2$ we have

$$\begin{aligned} K_{r,s}(f,t)_w &\le c \left(\inf_{g \in C^{2r+s}[0,1]} \left\{ \|w(f - g^{(s)})\| + t\, \|w\varphi^{2r} g^{(2r+s)}\| \right\} + t\, \|wf\| \right) \\ &\le c \left(K_{2r,\varphi}(f,t)_w + t\, \|wf\| \right). \end{aligned}$$

Here we have taken into account (4.46).

Similarly, relations (4.58) with $s = 1$, (4.53) with g' in place of G, (4.54) and (4.47) yield

$$K_{r,1}(f,t)_w \le c \left(K_{2r,\varphi}(f,t)_w + K_1(f,t)_w \right).$$

This completes the proof of Theorem 4.4.

To establish the upper estimate of $K_{r,s}(f,t)_1$ in Theorem 4.5 we use the quasi-interpolant $Q(f) := Q_T(f)$ constructed in the proof of [18, Chapter 6, Theorem 6.2] but with $2r$ in place of r and for the interval $[0,1]$ instead of $[-1,1]$. It has the properties (see [18, p. 191]):

$$\|f - Q(f)\| \le c\, \omega_\varphi^{2r}(f,t)$$

and

$$t^{2r} \|\varphi^{2r} Q(f)^{(2r)}\| \le c\, \omega_\varphi^{2r}(f,t),$$

where $t = 1/m$, $m \in \mathbb{N}_+$ and $m \geq m_0$ with some fixed $m_0 \in \mathbb{N}_+$. Likewise, by means of [18, Chapter 5, Proposition 4.6 and Chapter 6, Theorem 4.2] we get

$$t^{2r}\|Q(f)^{(j)}\| \leq c\, t^{2(r-j)}\omega_j(f,t^2), \quad j = 1, r,$$

for $t = 1/m$, $m \in \mathbb{N}_+$ and $m \geq m_0$. Hence, taking into account the inequalities $\omega_j(f,t) \leq c\, K_j(f,t^j)$ and $\omega_\varphi^{2r}(f,t) \leq c\, K_{2r,\varphi}(f,t^{2r})$, we arrive at

$$\begin{aligned} &\|f - Q(f)\| \leq c\, K_{2r,\varphi}(f,t^{2r}), \\ &t^{2r}\|\varphi^{2r}Q(f)^{(2r)}\| \leq c\, K_{2r,\varphi}(f,t^{2r}), \\ &t^{2r}\|Q(f)^{(j)}\| \leq c\, K_j(f,t^{2r}), \quad j = 1, r, \end{aligned} \tag{4.59}$$

for $t = 1/m$, $m \in \mathbb{N}_+$ and $m \geq m_0$.

Now, the upper estimate of $K_{r,s}(f,t)_1$ for all $t \in (0,1]$ follows from (4.21) with $w = 1$ and $g^{(s)} = Q(f)$, (4.51) with $Q(f)$ in place of G, or [18, Chapter 5, Theorem 4.4] (if $s \geq 2$), and the basic property of the K-functionals

$$K(f,t_1) \leq \max\left\{1, \frac{t_1}{t_2}\right\} K(f,t_2), \tag{4.60}$$

where $K(f,t)$ stands for any of the considered here K-functionals.

Let us briefly show the validity of Remark 3.4. In view of what already has been established, it is enough to demonstrate that

$$K_{1,1}(f,t)_w \leq c\,(K_{2,\varphi}(f,t)_w + K_1(f,t)_w)$$

for $\gamma_0 > 0$ and $\gamma_1 = 0$. To this end we will apply a well-known patching technique (see e.g. [18, p. 176]). By (2.6), 3/4 instead of 1/2, g' in place of g, $\gamma_{1,0} = \gamma_0, \gamma_{2,0} = \gamma_0 + 1$, $j = 1$ and $m = 2$, we get

$$\|\chi^{\gamma_0}g''\|_{[0,3/4]} \leq c\left(\|\chi^{\gamma_0}g'\|_{[0,3/4]} + \|\chi^{\gamma_0+1}g'''\|_{[0,3/4]}\right).$$

Then, just as above, we derive that

$$\begin{aligned} t\,&\|\chi^{\gamma_0}g''\|_{[0,3/4]} \\ &\leq c\left(\|\chi^{\gamma_0}(f-g')\|_{[0,3/4]} + t\,\|\chi^{\gamma_0+1}g'''\|_{[0,3/4]}\right) + c\,K_1(f,t)_w. \end{aligned}$$

Using the last inequality and (4.47) with $r = 1$, we deduce that there exists $\tilde{g}_t \in C^3[0,3/4]$ such that

$$\begin{aligned} &\|\chi^{\gamma_0}(f-\tilde{g}_t')\|_{[0,3/4]} + t^2\|\chi^{\gamma_0}\tilde{g}_t''\|_{[0,3/4]} + t^2\|\chi^{\gamma_0+1}\tilde{g}_t'''\|_{[0,3/4]} \\ &\qquad\leq c\left(K_{2,\varphi}(f,t^2)_w + K_1(f,t^2)_w\right). \end{aligned} \tag{4.61}$$

Further, let $\widetilde{Q}(f) := \widetilde{Q}_{\widetilde{T}}(f)$ be the quasi-interpolant considered above for $r = 1$ and modified for the interval $[1/5, 1]$. We set $\tilde{\varphi}(x) := \sqrt{(x-1/5)(1-x)}$ and denote by $K(f,t)_J$ the modification of the K-functional $K(f,t)$, in which the sup-norm is taken on the interval J instead of $[0,1]$. Then (cf. (4.59)) we have

$$\begin{aligned}
\|f - \widetilde{Q}(f)\|_{[1/4,1]} &\le \|f - \widetilde{Q}(f)\|_{[1/5,1]} \\
&\le c\,K_{2,\tilde{\varphi}}(f,t^2)_{[1/5,1]} \le c\,K_{2,\varphi}(f,t^2)_w, \\
t^2\|\varphi^2\widetilde{Q}(f)''\|_{[1/4,1]} &\le c\,t^2\|\tilde{\varphi}^2\widetilde{Q}(f)''\|_{[1/5,1]} \\
&\le c\,K_{2,\tilde{\varphi}}(f,t^2)_{[1/5,1]} \le c\,K_{2,\varphi}(f,t^2)_{w,}, \\
t^2\|\widetilde{Q}(f)'\|_{[1/4,1]} &\le t^2\|\widetilde{Q}(f)'\|_{[1/5,1]} \\
&\le c\,K_1(f,t^2)_{[1/5,1]} \le c\,K_1(f,t^2)_w
\end{aligned} \tag{4.62}$$

for $t = 1/m$, $m \in \mathbb{N}_+$ and $m \ge m_0$ with some fixed $m_0 \in \mathbb{N}_+$.

Let the function $g_t \in C^3[0,1]$ be such that $g_t' = (1-\psi)\tilde{g}_t' + \psi\widetilde{Q}(f)$, where $\psi \in C^\infty(\mathbb{R})$, $\psi(x) = 0$ for $x \le 1/4$ and $\psi(x) = 1$ for $x \ge 3/4$. It can be shown by (4.21) with $r = s = 1$, (4.61) and (4.62) that (see [18, p. 176])

$$\|w(f - g_t')\| + t^2\|w(Dg_t)'\| \le c\left(K_{2,\varphi}(f,t^2)_w + K_1(f,t^2)_w\right)$$

for $t = 1/m$, $m \in \mathbb{N}_+$ and $m \ge m_0$. In view of property (4.60) that completes the proof of Remark 3.4. □

4.8 Simultaneous approximation by the iterated Boolean sums of the Kantorovich operator in weighted L_∞-spaces

As in the last section of the previous chapter, the results about the simultaneous approximation by the iterated Boolean sums of the Bernstein operator can be easily transferred to their analogue with the Kantorovich operator.

To recall, the Kantorovich polynomials are defined for $f \in L[0,1]$ and $x \in [0,1]$ by

$$K_n f(x) := \sum_{k=0}^{n}(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt\,p_{n,k}(x),\ p_{n,k}(x) := \binom{n}{k}x^k(1-x)^{n-k}.$$

More generally, we will consider again the generalized Kantorovich operator of order $m \in \mathbb{N}_+$, which is defined for $f \in L[0,1]$ by

$$K_n^{\langle m \rangle} f(x) := \left(B_{n+m} F_m(x)\right)^{(m)},$$

where

$$F_m(x) := \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f(t)\, dt.$$

Further, we set

$$\mathcal{K}_{r,n}^{\langle m \rangle} := I - (I - K_n^{\langle m \rangle})^r.$$

Relation (3.158) yields

$$\mathcal{K}_{r,n}^{\langle m \rangle} f = \left(\mathcal{B}_{r,n+m} F_m\right)^{(m)}.$$

This enables us to transfer all the above results about simultaneous approximation by $\mathcal{B}_{r,n}$ to $\mathcal{K}_{r,n}$. In particular, we get the following characterization of the rate of the simultaneous approximation by $\mathcal{K}_{r,n}^{\langle m \rangle}$.

Theorem 4.25. *Let $m, r \in \mathbb{N}_+$, $s \in \mathbb{N}_0$ and $w := w(\gamma_0, \gamma_1)$ be given by (2.2) as $0 \le \gamma_0, \gamma_1 < s + m$. Then for all $f \in L_\infty[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{K}_{r,n}^{\langle m \rangle} f - f)^{(s)}\| \le c\, K_{r,s+m}^D(f^{(s)}, n^{-r})_w.$$

Conversely, there exists $R \in \mathbb{N}_+$ such that for all $f \in L[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $\ell, n \in \mathbb{N}_+$ with $\ell \ge Rn$ there holds

$$K_{r,s+m}^D(f^{(s)}, n^{-r})_w \le c \left(\frac{k}{n}\right)^r \left(\|w(\mathcal{K}_{r,n}^{\langle m \rangle} f - f)^{(s)}\| + \|w(\mathcal{K}_{r,\ell}^{\langle m \rangle} f - f)^{(s)}\|\right).$$

In particular,

$$K_{r,s+m}^D(f^{(s)}, n^{-r})_w \le c \left(\|w(\mathcal{K}_{r,n}^{\langle m \rangle} f - f)^{(s)}\| + \|w(\mathcal{K}_{r,Rn}^{\langle m \rangle} f - f)^{(s)}\|\right).$$

The value of the constant c is independent of f, n and ℓ.

In the statement of the last theorem the condition $f \in AC_{loc}^{s-1}(0,1)$ is to be ignored for $s = 0$.

The observations made in Remarks 3.42 and 3.43 hold for $\mathcal{K}_{r,n}^{\langle m \rangle}$ as well.

Chapter 5

Simultaneous approximation by Bernstein polynomials with integer coefficients

5.1 Background

There is an extensive literature on the approximation of functions by polynomials with integer coefficients. A quite helpful introduction to the subject is the monograph [39] and also [80, Chapter 2, § 4]. In particular, the extension of the classical results on simultaneous approximation by algebraic polynomials with real coefficients to the integer case is due to Gelfond [43] and Trigub [94, 95, 96, 97, 98].

Bernstein [10] posed the problem of determining to what extent the requirement on the coefficients of the algebraic polynomials to be integers affects the order of the best algebraic approximation in the uniform norm. To solve this problem Kantorovich [66] (or e.g. [39, pp. 3–4], or [80, Chapter 2, Theorem 4.1]) introduced an integer modification of B_n. It is given by

$$\widetilde{B}_n(f)(x) := \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k}.$$

Above $[\alpha]$ denotes the largest integer that is less than or equal to the real α. L. Kantorovich showed that if $f \in C[0,1]$ is such that $f(0), f(1) \in \mathbb{Z}$, then

$$\lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0.$$

Clearly, the conditions $f(0), f(1) \in \mathbb{Z}$ are also necessary in order to have $\lim_{n\to\infty} \widetilde{B}_n(f)(0) = f(0)$ and $\lim_{n\to\infty} \widetilde{B}_n(f)(1) = f(1)$, respectively.

Following L. Kantorovich and applying (0.3), we get a direct estimate of the error of $\widetilde{B}_n$ for $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$. For $x \in [0,1]$ and $n \in \mathbb{N}_+$ we have

$$\begin{aligned} |\widetilde{B}_n(f)(x) - f(x)| &\le |B_n f(x) - f(x)| \\ &\quad + \sum_{k=1}^{n-1} \left(f\left(\frac{k}{n}\right)\binom{n}{k} - \left[f\left(\frac{k}{n}\right)\binom{n}{k} \right] \right) x^k(1-x)^{n-k} \\ &\le \|B_n f - f\| + \sum_{k=1}^{n-1} x^k(1-x)^{n-k} \\ &\le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\sum_{k=1}^{n-1} p_{n,k}(x) \\ &\le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}. \end{aligned} \tag{5.1}$$

We will show that the simultaneous approximation by $\widetilde{B}_n(f)$ satisfies a similar estimate. Before stating that result, let us note that another integer modification of $B_n f$ possesses actually better properties regarding simultaneous approximation. In it, instead of the integer part $[\alpha]$ we use the nearest integer $\langle\alpha\rangle$ to the real α. More precisely, if $\alpha \in \mathbb{R}$ is not the arithmetic mean of two consecutive integers, we set $\langle\alpha\rangle$ to be the integer at which the minimum $\min_{m\in\mathbb{Z}} |\alpha - m|$ is attained. When α is right in the middle between two consecutive integers, we can define $\langle\alpha\rangle$ to be either of them even without following a given rule. The results we will prove are valid regardless of our choice in this case.

We will denote that integer modification of the Bernstein polynomial by $\widehat{B}_n(f)$, that is, we set

$$\widehat{B}_n(f)(x) := \sum_{k=0}^{n} \left\langle f\left(\frac{k}{n}\right)\binom{n}{k} \right\rangle x^k(1-x)^{n-k}$$

for $f \in C[0,1]$ and $x \in [0,1]$.

An argument similar to (5.1) yields

$$\|\widehat{B}_n(f) - f\| \le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{2n} \tag{5.2}$$

for all $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$ and all $n \in \mathbb{N}_+$.

Combining (5.1) and (5.2) with (3.2), we arrive at the characterization

$$\begin{aligned} c^{-1}\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) &\le \|\widetilde{B}_n(f) - f\| + \frac{1}{n} \\ &\le c\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} c^{-1}\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) &\le \|\widehat{B}_n(f) - f\| + \frac{1}{n} \\ &\le c\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) \end{aligned} \tag{5.4}$$

valid for all $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$ and all $n \ge n_0$ with some n_0, which is independent of f.

Consequently, if $0 < \alpha \le 1$, then

$$\|\widetilde{B}_n(f) - f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \omega_\varphi^2(f, h) = O(h^{2\alpha}) \tag{5.5}$$

and

$$\|\widehat{B}_n(f) - f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \omega_\varphi^2(f, h) = O(h^{2\alpha}), \tag{5.6}$$

provided that $f(0), f(1) \in \mathbb{Z}$; we assume $f \in C[0,1]$.

In addition, we will prove in the last section of the chapter that the approximation generated by $\widetilde{B}_n$ and $\widehat{B}_n$ in the uniform norm on $[0,1]$ is saturated with the saturation rate of $1/n$ and if $\|\widetilde{B}_n(f) - f\| = o(1/n)$ or $\|\widehat{B}_n(f) - f\| = o(1/n)$, then, similarly to the Bernstein operator, we have that $\widetilde{B}_n(f) = \widehat{B}_n(f) = f$ and f is a polynomial of the type $px + q$, where $p, q \in \mathbb{Z}$.

Let us explicitly note that for any fixed $n \ge 2$ the operator $\widetilde{B}_n : C[0,1] \to C[0,1]$ is not bounded in the sense that there does *not* exist a constant M such that

$$\|\widetilde{B}_n f\| \le M\,\|f\| \quad \forall\, f \in C[0,1].$$

Therefore we cannot drop the quantity $1/n$ on the right-hand side of the estimate (5.1), or replace it with $c\,\|f\|\,n^{-1}$. That operator is not continuous either. On the other hand, $\widehat{B}_n$ is bounded but not continuous. Both operators are not linear. To emphasize the latter we write $\widetilde{B}_n(f)$ and $\widehat{B}_n(f)$, not $\widetilde{B}_n f$ and $\widehat{B}_n f$.

5.2 A characterization of the rate of the simultaneous approximation by Bernstein polynomials with integer coefficients

We established in Theorem 3.5, that for all $f \in C^s[0,1]$ and $n \in \mathbb{N}_+$ there holds

$$(5.7)\quad \|(B_n f)^{(s)} - f^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}), & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\|, & s \ge 2, \end{cases}$$

as, moreover, these estimates cannot be improved since a matching strong converse inequality is also valid (see Theorem 3.8 and (3.11)).

We will verify that the integer forms of the Bernstein polynomials $\widetilde{B}_n$ and $\widehat{B}_n$ satisfy similar direct inequalities. They are stated in the following two theorems.

Theorem 5.1. *Let $s \in \mathbb{N}_+$. Let $f \in C^s[0,1]$ be such that*

$$f(0), f(1), f'(0), f'(1) \in \mathbb{Z} \text{ and } f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 2, \dots, s.$$

Let also there exist $n_0 \in \mathbb{N}_+$, $n_0 \ge s$, such that

$$(5.8)\quad f\left(\frac{k}{n}\right) \ge f(0) + \frac{k}{n}\, f'(0), \quad k = 1, \dots, s,\ n \ge n_0,$$

$$(5.9)\quad f\left(\frac{k}{n}\right) \ge f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \dots, n-1,\ n \ge n_0.$$

Then for $n \ge n_0$ there holds

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \ge 2. \end{cases}$$

The value of the constant c is independent of f and n.

Remark 5.2. Certainly, it suffices to assume instead of the cumbersome (5.8)-(5.9) that there exists $\delta \in (0,1)$ such that

$$\begin{aligned} f(x) &\geq f(0) + x\,f'(0), \quad x \in [0,\delta], \\ f(x) &\geq f(1) - (1-x)f'(1), \quad x \in [1-\delta, 1]. \end{aligned}$$

However, it turns out that the conditions (5.8)-(5.9) are also necessary unlike the ones above (see Theorem 5.12).

Remark 5.3. An analogous result holds for the integer form of the Bernstein operator defined by means of the ceiling function instead of the integer part. Then we assume that the reverse inequalities for $f(k/n)$ hold, that is,

$$\begin{aligned} f\left(\frac{k}{n}\right) &\leq f(0) + \frac{k}{n}\,f'(0), \quad k = 1,\dots,s,\ n \geq n_0, \\ f\left(\frac{k}{n}\right) &\leq f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s,\dots,n-1,\ n \geq n_0. \end{aligned}$$

The proof is quite similar and we will omit it.

The estimates of the rate of convergence for $\widehat{B}_n$ are valid under *weaker* assumptions.

Theorem 5.4. *Let $s \in \mathbb{N}_+$. Let $f \in C^s[0,1]$ be such that*

$$f(0), f(1), f'(0), f'(1) \in \mathbb{Z} \text{ and } f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 2,\dots,s.$$

Then

$$\begin{aligned} &\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| \\ &\qquad \leq c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \geq 2. \end{cases} \end{aligned}$$

The value of the constant c is independent of f and n.

We will show that the assumptions made in Theorems 5.1 and 5.4 are necessary in order to have uniform simultaneous approximation. Concerning the difference between the set of conditions on the derivatives for $s = 1$ and $s \geq 2$, let us note that $\widetilde{B}_n$ and $\widehat{B}_n$ preserve the polynomials of the form

$p\,x+q$, where $p,q \in \mathbb{Z}$ (that is verified just as for the Bernstein operator). Therefore it is not surprising that there are not any restrictions on the values of the function and its first derivative at the endpoints except that they must be integers. However, the requirement that the derivatives of order 2 and higher must be equal to 0 at the endpoints is quite unexpected. Technically, it is related to the fact that $\left(\frac{k}{n}\right)^s \binom{n}{k} \in \mathbb{Z}$ for all k and n iff $s=0$ or $s=1$.

I am aware of only one result concerning approximation of the derivatives of smooth functions by means of integer forms of the Bernstein polynomials. Martinez [82] considered this problem but the coefficients are replaced by their integer part *after* differentiating the Bernstein polynomial of the function.

Another result in this chapter is an analogue of the equivalence relations (5.5)-(5.6) for the simultaneous approximation by the operators $\widetilde{B}_n$ and $\widehat{B}_n$. We will establish the following weak converse relations that complement the direct estimates in Theorems 5.1 and 5.4.

Theorem 5.5. *Let $s \in \mathbb{N}_+$ and $0 < \alpha < 1$. Let $f \in C^s[0,1]$, $f(0), f(1) \in \mathbb{Z}$, and*

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}) \quad \text{or} \quad \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}).$$

Then

$$\omega_\varphi^2(f^{(s)},h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)},h) = O(h^{\alpha}).$$

Combining this theorem with Theorems 5.1 and 5.4, we get the following two big O equivalence relations.

Corollary 5.6. *Let $s \in \mathbb{N}_+$ and $0 < \alpha < 1$. Let $f \in C^s[0,1]$ be such that $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2,\dots,s$. Let also there exist $n_0 \in \mathbb{N}_+$, $n_0 \ge s$, such that*

$$f\left(\frac{k}{n}\right) \ge f(0) + \frac{k}{n}\, f'(0), \quad k = 1,\dots,s,\ n \ge n_0,$$
$$f\left(\frac{k}{n}\right) \ge f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s,\dots,n-1,\ n \ge n_0.$$

Then

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}) \iff \omega_\varphi^2(f^{(s)},h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)},h) = O(h^{\alpha}).$$

Corollary 5.7. *Let* $s \in \mathbb{N}_+$ *and* $0 < \alpha < 1$. *Let* $f \in C^s[0,1]$ *be such that* $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ *and* $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$. *Then*

$$\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}) \iff \omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \textit{and} \quad \omega_1(f^{(s)}, h) = O(h^\alpha).$$

In the next section, we will establish the direct estimates stated in Theorems 5.1 and 5.4. Then we will show in Section 5.4 that the assumptions on the approximated function near the ends of the interval and its derivatives are also necessary in order to have simultaneous approximation. In Section 5.5 we will prove the converse relations in Theorem 5.5. In Section 5.6, we consider the saturation of the approximation by $\widetilde{B}_n$ and $\widehat{B}_n$. In the last section we present a Kantorovich-type integral modification of $\widehat{B}_n$.

5.3 Proof of the direct estimates

The integer modifications of the Bernstein polynomials $\widetilde{B}_n$ and $\widehat{B}_n$ are not linear. That is why the simplest way to estimate their rate of approximation is to consider their deviation from the linear operator B_n (see (5.1)). We will apply the same approach to estimate their rate of simultaneous approximation.

For $n \in \mathbb{N}_+$ and $k = 0, \dots, n$, we set

$$\tilde{b}_n(k) := \tilde{b}_n^f(k) := \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] \binom{n}{k}^{-1}$$

and

$$\hat{b}_n(k) := \hat{b}_n^f(k) := \left\langle f\left(\frac{k}{n}\right) \binom{n}{k} \right\rangle \binom{n}{k}^{-1}.$$

Then the operators $\widetilde{B}_n$ and $\widehat{B}_n$ can be written respectively in the form

$$\widehat{B}_n(f)(x) = \sum_{k=0}^{n} \tilde{b}_n(k)\, p_{n,k}(x)$$

and

$$\widehat{B}_n(f)(x) = \sum_{k=0}^{n} \hat{b}_n(k)\, p_{n,k}(x).$$

We will use the forward finite difference operator $\overrightarrow{\Delta}_h$ with step h. If $h = 1$, we will omit the subscript, writing $\overrightarrow{\Delta} := \overrightarrow{\Delta}_1$. Thus

$$\overrightarrow{\Delta}^s \tilde{b}_n(k) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \tilde{b}_n(k+s-i), \quad k = 0, \dots, n-s; \tag{5.10}$$

and analogously for $\hat{b}_n$.

Similarly to (3.4), for $n \geq s$ we have

$$(\widetilde{B}_n(f))^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}^s \tilde{b}_n(k)\, p_{n-s,k}(x), \quad x \in [0,1], \tag{5.11}$$

and

$$(\widehat{B}_n(f))^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}^s \hat{b}_n(k)\, p_{n-s,k}(x), \quad x \in [0,1]. \tag{5.12}$$

We proceed to the results that relate $\widetilde{B}_n$ and $\widehat{B}_n$ to B_n.

Theorem 5.8. *Let $s \in \mathbb{N}_+$. Let $f \in C^s[0,1]$ be such that*

$$f(0), f(1), f'(0), f'(1) \in \mathbb{Z} \text{ and } f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 2, \dots, s.$$

Let also there exist $n_0 \in \mathbb{N}_+$, $n_0 \geq s$, such that

$$f\left(\frac{k}{n}\right) \geq f(0) + \frac{k}{n} f'(0), \quad k = 1, \dots, s,\ n \geq n_0,$$
$$f\left(\frac{k}{n}\right) \geq f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \dots, n-1,\ n \geq n_0.$$

Then

$$\|(B_n f)^{(s)} - (\widetilde{B}_n(f))^{(s)}\| \leq c \left(\omega_1(f^{(s)}, n^{-1}) + \frac{1}{n}\right), \quad n \geq n_0.$$

The value of the constant c is independent of f and n.

Theorem 5.9. *Let $s \in \mathbb{N}_+$. Let $f \in C^s[0,1]$ be such that*

$$f(0), f(1), f'(0), f'(1) \in \mathbb{Z} \text{ and } f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 2, \dots, s.$$

Then

$$\|(B_n f)^{(s)} - (\widehat{B}_n(f))^{(s)}\| \leq c \left(\omega_1(f^{(s)}, n^{-1}) + \frac{1}{n}\right).$$

The value of the constant c is independent of f and n.

Now, Theorems 5.1 and 5.4 follow directly from (5.7) and Theorems 5.8 and 5.9, respectively.

Let us establish Theorems 5.8 and 5.9.

Proof of Theorem 5.8. Let $n \ge n_0$. We make use of (3.4), (5.11), and the identities $\sum_{j=0}^{s} \binom{s}{j} = 2^s$ and $\sum_{k=0}^{n-s} p_{n-s,k}(x) \equiv 1$ to get

$$\begin{aligned}(5.13)\quad &\left|(B_n f)^{(s)}(x) - (\widetilde{B}_n(f))^{(s)}(x)\right| \\ &\qquad \le 2^s\, n^s \max_{0\le k\le n} \left(f\left(\frac{k}{n}\right) - \tilde{b}_n(k)\right), \quad x \in [0,1].\end{aligned}$$

Note that $f(k/n) - \tilde{b}_n(k) \ge 0$, $k = 0, \dots, n$, because $[\alpha] \le \alpha$.

We will estimate $f(k/n) - \tilde{b}_n(k)$ separately for $k \le s$, $s+1 \le k \le n-s-1$, and $k \ge n-s$. For the middle part, we simply use that if $n \ge 2s+2$, then

$$\begin{aligned}(5.14)\quad f\left(\frac{k}{n}\right) - \tilde{b}_n(k) &= \left(f\left(\frac{k}{n}\right)\binom{n}{k} - \left[f\left(\frac{k}{n}\right)\binom{n}{k}\right]\right)\binom{n}{k}^{-1} \\ &\le \binom{n}{s+1}^{-1} \le \frac{c}{n^{s+1}}, \quad k = s+1, \dots, n-s-1.\end{aligned}$$

Next, we will show that

$$(5.15)\qquad f\left(\frac{k}{n}\right) - \tilde{b}_n(k) \le \frac{c}{n^s}\, \omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s.$$

We apply Taylor's formula, as we take into consideration that $f^{(i)}(0) = 0$ for $i = 2, \dots, s$, to arrive at

$$\begin{aligned}(5.16)\quad f\left(\frac{k}{n}\right) &= f(0) + \frac{k}{n}\, f'(0) \\ &\quad + \frac{1}{(s-1)!}\int_0^{k/n} \left(\frac{k}{n} - t\right)^{s-1} \left(f^{(s)}(t) - f^{(s)}(0)\right) dt.\end{aligned}$$

That implies

$$\begin{aligned}(5.17)\quad f\left(\frac{k}{n}\right) - \left(f(0) + \frac{k}{n}\, f'(0)\right) &\le \frac{1}{s!}\left(\frac{k}{n}\right)^s \omega_1\left(f^{(s)}, \frac{k}{n}\right) \\ &\le \frac{c}{n^s}\, \omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s.\end{aligned}$$

At the second estimate, we have taken into account the well-known property of the modulus of continuity

$$\omega_1(F, rt) \le r\omega_1(F, t),$$

where $r \in \mathbb{N}_+$.

On the other hand, (5.8) and

$$f(0)\binom{n}{k} + f'(0)\,\frac{k}{n}\binom{n}{k} \in \mathbb{Z}, \tag{5.18}$$

imply

$$\left[f\left(\frac{k}{n}\right)\binom{n}{k}\right] \ge f(0)\binom{n}{k} + f'(0)\,\frac{k}{n}\binom{n}{k}, \quad k = 0, \dots, s.$$

Consequently,

$$\tilde{b}_n(k) \ge f(0) + \frac{k}{n}\,f'(0), \quad k = 0, \dots, s. \tag{5.19}$$

Estimates (5.17) and (5.19) imply (5.15).

Finally, we observe that, by symmetry, (5.15) yields

$$f\left(\frac{k}{n}\right) - \tilde{b}_n(k) \le \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = n-s, \dots, n. \tag{5.20}$$

More precisely, with $\bar{f}(x) := f(1-x)$ and

$$\bar{\tilde{b}}_n(k) := \left[\bar{f}\left(\frac{k}{n}\right)\binom{n}{k}\right]\binom{n}{k}^{-1}$$

we have

$$\begin{aligned} \bar{f}\left(\frac{k}{n}\right) &= f\left(\frac{n-k}{n}\right), \\ \bar{\tilde{b}}_n(k) &= \tilde{b}_n(n-k), \\ \omega_1(\bar{f}^{(s)}, t) &= \omega_1(f^{(s)}, t). \end{aligned} \tag{5.21}$$

Note also that $\bar{f} \in C^s[0,1]$, $\bar{f}(0), \bar{f}'(0) \in \mathbb{Z}$, $\bar{f}^{(i)}(0) = 0$, $i = 2, \dots, s$, and for $n \ge n_0$ and $k = 1, \dots, s$ we have by (5.9)

$$\bar{f}\left(\frac{k}{n}\right) = f\left(\frac{n-k}{n}\right) \ge f(1) - \frac{k}{n}f'(1) = \bar{f}(0) + \frac{k}{n}\bar{f}'(0).$$

So, $\bar{f}$ satisfies the condition (5.8) and, by virtue of (5.15), we have

$$\bar{f}\left(\frac{k}{n}\right) - \bar{\bar{b}}_n(k) \leq \frac{c}{n^s}\,\omega_1(\bar{f}^{(s)}, n^{-1}), \quad k = 0, \dots, s.$$

As we take into account (5.21), we get (5.20).

Inequalities (5.13)-(5.15) and (5.20) imply the assertion of the theorem. □

We will use the following elementary lemma in the proof the theorem about $\widehat{B}_n$.

Lemma 5.10. *Let $m \in \mathbb{Z}$ and $\alpha, \omega \in \mathbb{R}$. If $|\alpha - m| \leq \omega$, then $|\langle\alpha\rangle - m| \leq 2\omega$.*

Proof. If $\omega < 1/2$, then $\langle\alpha\rangle = m$. If, on the other hand, $\omega \geq 1/2$, then

$$|\langle\alpha\rangle - m| \leq |\langle\alpha\rangle - \alpha| + |\alpha - m| \leq \frac{1}{2} + \omega \leq 2\omega.$$

□

Proof of Theorem 5.9. We proceed similarly to the proof of the previous theorem. Since the assertion is trivial for $n < s$, we assume that $n \geq s$. We make use of (3.4) and (5.12) to get

$$(5.22) \quad \left|(B_n f)^{(s)}(x) - (\widehat{B}_n(f))^{(s)}(x)\right| \leq 2^s\, n^s \max_{0 \leq k \leq n} \left| f\left(\frac{k}{n}\right) - \hat{b}_n(k)\right|, \quad x \in [0,1].$$

Again we estimate separately the terms $|f(k/n) - \hat{b}_n(k)|$ for $k \leq s$, $s+1 \leq k \leq n-s-1$, and $k \geq n-s$. For the middle part, we have similarly to (5.14)

$$(5.23) \quad \left| f\left(\frac{k}{n}\right) - \hat{b}_n(k)\right| \leq \frac{c}{n^{s+1}}, \quad k = s+1, \dots, n-s-1,\ n \geq 2s+2.$$

Next, we will show that

$$(5.24) \quad \left| f\left(\frac{k}{n}\right) - \hat{b}_n(k)\right| \leq \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s.$$

By virtue of (5.16), we have

$$\left| f\left(\frac{k}{n}\right) - \left(f(0) + \frac{k}{n}\, f'(0)\right)\right| \le \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \tag{5.25}$$

That implies

$$\left| f\left(\frac{k}{n}\right)\binom{n}{k} - \left(f(0)\binom{n}{k} + f'(0)\,\frac{k}{n}\binom{n}{k}\right)\right| \le \frac{c}{n^s}\binom{n}{k}\omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \tag{5.26}$$

We apply Lemma 5.10 with

$$\begin{aligned} \alpha &= f\left(\frac{k}{n}\right)\binom{n}{k}, \\ m &= f(0)\binom{n}{k} + f'(0)\,\frac{k}{n}\binom{n}{k} \in \mathbb{Z}, \\ \omega &= \frac{c}{n^s}\binom{n}{k}\omega_1(f^{(s)}, n^{-1}), \end{aligned}$$

where the constant c is the one on the right-hand side of (5.26).

Thus we arrive at

$$\left|\left\langle f\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle - \left(f(0)\binom{n}{k} + f'(0)\,\frac{k}{n}\binom{n}{k}\right)\right| \le \frac{c}{n^s}\binom{n}{k}\omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s,$$

and, consequently,

$$\left|\hat{b}_n(k) - \left(f(0) + \frac{k}{n}\, f'(0)\right)\right| \le \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \tag{5.27}$$

Estimates (5.25) and (5.27) yield (5.24).

Finally, we derive

$$\left| f\left(\frac{k}{n}\right) - \hat{b}_n(k)\right| \le \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = n - s, \dots, n. \tag{5.28}$$

from (5.24) by symmetry just as in the proof of (5.20) with $\bar{\tilde{b}}_n(k)$ replaced with

$$\bar{\hat{b}}_n(k) := \left\langle \bar{f}\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle \binom{n}{k}^{-1}.$$

Inequalities (5.22)-(5.24) and (5.28) imply the assertion of the theorem. □

5.4 Optimality of the assumptions

We will establish the necessity of the assumptions made in Theorems 5.1 and 5.4. We begin with the operator $\widehat{B}_n$ since stronger results are valid for it.

First of all, let us note that if

$$\lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0, \tag{5.29}$$

then $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}$ for $i = 0, \ldots, s$. Indeed, as is known, for any $g \in C^s[0,1]$ we have (see (2.1))

$$\|g^{(i)}\| \le c\big(\|g\| + \|g^{(s)}\|\big), \quad i = 1, \ldots, s-1.$$

Therefore (5.29) implies

$$\lim_{n\to\infty} \|(\widehat{B}_n(f))^{(i)} - f^{(i)}\| = 0, \quad i = 0, \ldots, s; \tag{5.30}$$

hence $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}$ for $i = 0, \ldots, s$. A similar result holds for $\widetilde{B}_n$.

Theorem 5.11. *Let $s \in \mathbb{N}_+$, $s \ge 2$, and $f \in C^s[0,1]$. If*

$$\lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0, \tag{5.31}$$

then $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \ldots, s$.

Proof. It is sufficient to establish the theorem at the point $x = 0$; for $x = 1$ it follows by symmetry. We use induction on s.

Let $s = 2$. Relation (5.30), in particular, yields

$$\lim_{n\to\infty} (\widehat{B}_n(f))'(0) = f'(0),$$

that is (see (5.12) with $s = 1$),

$$\lim_{n\to\infty} n\overrightarrow{\Delta}\hat{b}_n(0) = f'(0). \tag{5.32}$$

Since $n\overrightarrow{\Delta}\hat{b}_n(0) \in \mathbb{Z}$ for all n, (5.32) implies

$$n\overrightarrow{\Delta}\hat{b}_n(0) = f'(0) \quad \text{for } n \text{ large enough;}$$

hence

$$\hat{b}_n(1) = \hat{b}_n(0) + \frac{1}{n}\, f'(0) = f(0) + \frac{1}{n}\, f'(0). \tag{5.33}$$

Similarly, from $\lim_{n\to\infty}(\widehat{B}_n(f))''(0) = f''(0)$ we derive

$$n(n-1)\overrightarrow{\Delta}^2\hat{b}_n(0) = f''(0) \quad \text{for } n \text{ large enough.} \tag{5.34}$$

By Taylor's formula, we have

$$f\left(\frac{2}{n}\right) = f(0)+\frac{2}{n}\, f'(0)+\frac{2}{n^2}\, f''(0)+\int_0^{2/n}\left(\frac{2}{n}-t\right)(f''(t)-f''(0))\,dt. \tag{5.35}$$

Next, we proceed similarly to the proof of Theorem 5.9. We multiply both sides of the above identity by $\binom{n}{2}$ and rearrange the terms to get

$$\begin{aligned} f\left(\frac{2}{n}\right)\binom{n}{2} - \left(f(0)\binom{n}{2} + (n-1)f'(0) + f''(0)\right) \\ = -\frac{1}{n}\, f''(0) + \binom{n}{2}\int_0^{2/n}\left(\frac{2}{n}-t\right)(f''(t)-f''(0))\,dt. \end{aligned} \tag{5.36}$$

Consequently,

$$\begin{aligned} \left| f\left(\frac{2}{n}\right)\binom{n}{2} - \left(f(0)\binom{n}{2} + (n-1)f'(0) + f''(0)\right)\right| \\ \le \frac{1}{n}\,|f''(0)| + \omega_1\left(f'', \frac{2}{n}\right), \end{aligned}$$

which shows that for large n we have

$$\left\langle f\left(\frac{2}{n}\right)\binom{n}{2}\right\rangle = f(0)\binom{n}{2} + (n-1)f'(0) + f''(0).$$

Therefore

$$\hat{b}_n(2) = f(0) + \frac{2}{n} f'(0) + \frac{2}{n(n-1)} f''(0) \quad \text{for } n \text{ large enough.} \tag{5.37}$$

Now, fixing some n large enough, we deduce from (5.33)-(5.37) that

$$\begin{aligned} f''(0) &= n(n-1)(\hat{b}_n(2) - 2\hat{b}_n(1) + \hat{b}_n(0)) \\ &= n(n-1)\left(f(0) + \frac{2}{n} f'(0) + \frac{2}{n(n-1)} f''(0) - 2\left(f(0) + \frac{1}{n} f'(0)\right) + f(0)\right) \\ &= 2f''(0); \end{aligned}$$

hence $f''(0) = 0$.

Let the assertion of the theorem hold for some $s-1$, $s \geq 3$. We will prove that then it holds for s too.

As we noted in the beginning of the section, (5.31) implies

$$\lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s-1)} - f^{(s-1)}\| = 0.$$

Hence, by virtue of the induction hypothesis, we have $f^{(i)}(0) = 0$ for $i = 2, \dots, s-1$.

By Taylor's formula we have

$$\begin{aligned} f\left(\frac{k}{n}\right) &= f(0) + \frac{k}{n} f'(0) + \left(\frac{k}{n}\right)^s \frac{f^{(s)}(0)}{s!} \\ &\quad + \frac{1}{(s-1)!} \int_0^{k/n} \left(\frac{k}{n} - t\right)^{s-1} (f^{(s)}(t) - f^{(s)}(0))\, dt. \end{aligned} \tag{5.38}$$

We multiply both sides by $\binom{n}{k}$. For $1 \leq k < s$ we derive the inequality

$$\begin{aligned} &\left| f\left(\frac{k}{n}\right)\binom{n}{k} - \left(f(0)\binom{n}{k} + f'(0)\frac{k}{n}\binom{n}{k}\right) \right| \\ &\qquad \leq \binom{n}{k}\left(\frac{k}{n}\right)^s \frac{1}{s!}\left(|f^{(s)}(0)| + \omega_1\left(f^{(s)}, \frac{k}{n}\right)\right) \\ &\qquad \leq \frac{c}{n}\left(|f^{(s)}(0)| + \omega_1(f^{(s)}, n^{-1})\right). \end{aligned}$$

Consequently, for large n we have

$$\left\langle f\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle = f(0)\binom{n}{k} + f'(0)\frac{k}{n}\binom{n}{k};$$

hence

$$\hat{b}_n(k) = f(0) + \frac{k}{n} f'(0) \quad \text{for } 0 \le k < s \text{ and large } n. \tag{5.39}$$

In order to calculate $\hat{b}_n(s)$, we observe that

$$\lim_{n\to\infty} \binom{n}{s} \left(\frac{s}{n}\right)^s = \frac{s^s}{s!}.$$

We proceed just as in this case $s = 2$: we multiple both sides of (5.38) by $\binom{n}{s}$ and rearrange the terms to arrive at

$$\begin{aligned} &\left| f\left(\frac{s}{n}\right)\binom{n}{s} - \left(f(0)\binom{n}{s} + f'(0)\frac{s}{n}\binom{n}{s} + \frac{s^s}{(s!)^2} f^{(s)}(0)\right)\right| \\ &\qquad \le \left(\frac{s^s}{s!} - \binom{n}{s}\left(\frac{s}{n}\right)^s\right)\frac{1}{s!}\,|f^{(s)}(0)| + \frac{1}{s!}\binom{n}{s}\left(\frac{s}{n}\right)^s \omega_1\left(f^{(s)}, \frac{s}{n}\right) \\ &\qquad \le \frac{c}{n}|f^{(s)}(0)| + c\,\omega_1(f^{(s)}, n^{-1}). \end{aligned}$$

Consequently, for large n

$$\left\langle f\left(\frac{s}{n}\right)\binom{n}{s}\right\rangle = f(0)\binom{n}{s} + f'(0)\frac{s}{n}\binom{n}{s} + \left\langle \frac{s^s}{(s!)^2} f^{(s)}(0)\right\rangle + r_{s,n},$$

where $r_{s,n} \in \{-1, 0, 1\}$. Consequently,

$$\hat{b}_n(s) = f(0) + \frac{s}{n} f'(0) + \left(\left\langle \frac{s^s}{(s!)^2} f^{(s)}(0)\right\rangle + r_{s,n}\right)\binom{n}{s}^{-1}. \tag{5.40}$$

Relations (5.39) and (5.40) yield

$$\overrightarrow{\Delta}^s \hat{b}_n(0) = \left(\left\langle \frac{s^s}{(s!)^2} f^{(s)}(0)\right\rangle + r_{s,n}\right)\binom{n}{s}^{-1}. \tag{5.41}$$

On the other hand, since $\lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0$, and, in particular, $\lim_{n\to\infty}(\widehat{B}_n(f))^{(s)}(0) = f^{(s)}(0)$, we have that

$$\lim_{n\to\infty} \frac{n!}{(n-s)!}\overrightarrow{\Delta}^s \hat{b}_n(0) = f^{(s)}(0).$$

Taking into account that

$$\frac{n!}{(n-s)!}\overrightarrow{\Delta}^s\hat{b}_n(0)\in\mathbb{Z}\quad\forall n,$$

we deduce that for large n there holds

$$\frac{n!}{(n-s)!}\overrightarrow{\Delta}^s\hat{b}_n(0)=f^{(s)}(0).$$

That, in combination with (5.41), yields

$$s!\left(\left\langle\frac{s^s}{(s!)^2}f^{(s)}(0)\right\rangle+r_{s,n}\right)=f^{(s)}(0)\quad\text{for } n \text{ large enough.}\tag{5.42}$$

First of all, this relation implies that the integer $f^{(s)}(0)$ is divisible by $s!$, i.e. $f^{(s)}(0)=s!\,m_s$ with some $m_s\in\mathbb{Z}$. Secondly, it implies that $r_{s,n}$ has one and the same value for large n; denote it by r_s. Thus (5.42) can be reduced to

$$\left\langle\frac{s^s}{s!}m_s\right\rangle+r_s=m_s.$$

Consequently,

$$|m_s|\left(\frac{s^s}{s!}-1\right)\le\frac{3}{2}.$$

It remains to take into account that $s^s/s!$ increases on s; hence $s^s/s!\ge 9/2$ for $s\ge 3$, and then $|m_s|\le 3/7$, which is possible only if $m_s=0$. Thus $f^{(s)}(0)=0$. □

Necessary conditions for the simultaneous approximation by means of $\widetilde{B}_n$ are given in the following theorem.

Theorem 5.12. *Let $s\in\mathbb{N}_+$ and $f\in C^s[0,1]$. If*

$$\lim_{n\to\infty}\|\widetilde{B}_n(f)-f\|=0\quad\textit{and}\quad\lim_{n\to\infty}\|(\widetilde{B}_n(f))^{(s)}-f^{(s)}\|=0,\tag{5.43}$$

then $f^{(i)}(0)=f^{(i)}(1)=0$, $i=2,\dots,s$, and there exists $n_0\in\mathbb{N}_+$, $n_0\ge s$, such that

$$\begin{aligned}f\left(\frac{k}{n}\right)&\ge f(0)+\frac{k}{n}f'(0),\quad k=1,\dots,s,\ n\ge n_0,\\ f\left(\frac{k}{n}\right)&\ge f(1)-\left(1-\frac{k}{n}\right)f'(1),\quad k=n-s,\dots,n-1,\ n\ge n_0.\end{aligned}\tag{5.44}$$

Proof. It is sufficient to establish the theorem at the point $x = 0$; for $x = 1$ it follows by symmetry.

We argue as in the proof of the preceding theorem. However, here more efforts are required.

Using induction on s, we will prove that $f^{(i)}(0) = 0$, $i = 2, \ldots, s$ and

$$\tilde{b}_n(k) = f(0) + \frac{k}{n} f'(0), \quad k = 1, \ldots, s, \ n \geq n_0. \tag{5.45}$$

with some n_0. The latter implies directly the inequalities (5.44) because

$$f\left(\frac{k}{n}\right) \geq \left[f\left(\frac{k}{n}\right)\binom{n}{k}\right]\binom{n}{k}^{-1} = f(0) + \frac{k}{n} f'(0), \quad k = 1, \ldots, s, \ n \geq n_0.$$

As in the proof of Theorem 5.11, we deduce from

$$\lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

that there exists $n_0 \in \mathbb{N}_+$, $n_0 \geq s$, such that

$$\frac{n!}{(n-i)!} \overrightarrow{\Delta}^i \tilde{b}_n(0) = f^{(i)}(0), \quad i = 1, \ldots, s, \ n \geq n_0. \tag{5.46}$$

That directly yields (5.45) for $s = 1$ and the assertion of the theorem is verified for $s = 1$.

In order to complete the proof for larger s, we use that if $f \in C^s[0,1]$ and $\lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0$, then

$$\lim_{n\to\infty} \|(B_n f)^{(s)} - (\widetilde{B}_n(f))^{(s)}\| = 0;$$

hence

$$\lim_{n\to\infty} \left((B_n f)^{(s)}\left(\frac{y}{n}\right) - (\widetilde{B}_n(f))^{(s)}\left(\frac{y}{n}\right)\right) = 0, \quad y \in [0,1]. \tag{5.47}$$

By (3.4) and (5.11), after reordering the terms, we arrive at the identity

$$\begin{aligned} &(B_n f)^{(s)}(x) - (\widetilde{B}_n(f))^{(s)}(x) \\ &= \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \sum_{j=k}^{k+s} (-1)^{s+j-k} \binom{s}{j-k} \left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j)\right) p_{n-s,k}(x). \end{aligned} \tag{5.48}$$

We observe that, by virtue of (5.14), for $n \ge 3s+2$ and $x \in [0,1]$ there holds (cf. (5.13))

$$\left| \sum_{k=s+1}^{n-2s-1} \sum_{j=k}^{k+s} (-1)^{s+j-k} \binom{s}{j-k} \left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j) \right) p_{n-s,k}(x) \right| \le \frac{c}{n^{s+1}} \tag{5.49}$$

and

$$\left| \sum_{k=1}^{s} \sum_{j=s+1}^{k+s} (-1)^{s+j-k} \binom{s}{j-k} \left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j) \right) p_{n-s,k}(x) \right| \le \frac{c}{n^{s+1}}. \tag{5.50}$$

Next, we observe that if $n \ge 4s+1$, then $p_{n-s,k}(y/n) \le c\,n^{-s-1}$ for all $y \in [0,1]$ and $k = n-2s, \dots, n-s$. Indeed, since in this case $(n-s)/2 \le n-2s$, then for $k = n-2s, \dots, n-s$ there holds

$$\binom{n-s}{k} \le \binom{n-s}{n-2s} = \binom{n-s}{s} \le c\,n^s.$$

Next, we take into account that for $n \ge 4s+1$ and $k \ge n-2s$ we have $k \ge 2s+1$; hence

$$\frac{y^k}{n^k} \le \frac{1}{n^{2s+1}}, \quad y \in [0,1].$$

These two relations along with the trivial estimate $(1-y/n)^{n-s-k} \le 1$ imply that $p_{n-s,k}(y/n) \le c\,n^{-s-1}$ for all $y \in [0,1]$ and $k = n-2s, \dots, n-s$, $n \ge 4s+1$.

Further, taking also into account that $0 \le f(j/n) - \tilde{b}_n(j) \le 1$ and arguing as in (5.14), we arrive at

$$\left| \sum_{k=n-2s}^{n-s} \sum_{j=k}^{k+s} (-1)^{s+j-k} \binom{s}{j-k} \left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j) \right) p_{n-s,k}\left(\frac{y}{n}\right) \right| \le \frac{c}{n^{s+1}}, \quad y \in [0,1]. \tag{5.51}$$

We subtract (5.49) and (5.50) with $x = y/n$, and (5.51) from (5.48) with $x = y/n$, reorder the terms and take into account (5.47) and $\tilde{b}_n(0) = f(0)$.

Thus, for $y \in [0,1]$, we deduce that

$$\lim_{n\to\infty} \frac{n!}{(n-s)!} \sum_{j=1}^{s} (-1)^{s-j} \left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j) \right) \times \sum_{k=0}^{j} (-1)^k \binom{s}{j-k} p_{n-s,k}\left(\frac{y}{n}\right) = 0. \tag{5.52}$$

We will evaluate that limit in another way. Clearly,

$$\lim_{n\to\infty} \sum_{k=0}^{j} (-1)^k \binom{s}{j-k} p_{n-s,k}\left(\frac{y}{n}\right) = \frac{1}{e^y} \sum_{k=0}^{j} (-1)^k \frac{y^k}{k!} \binom{s}{j-k}. \tag{5.53}$$

We proceed by induction on s. Relations (5.43) imply

$$\lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s-1)} - f^{(s-1)}\| = 0.$$

Therefore, by virtue of the induction hypothesis, we have that $f^{(i)}(0) = 0$, $i = 2, \ldots, s-1$, $s \geq 2$, and

$$\tilde{b}_n(j) = f(0) + \frac{j}{n} f'(0), \quad j = 1, \ldots, s-1, \; n \geq n_0. \tag{5.54}$$

Then Taylor's formula yields

$$f\left(\frac{j}{n}\right) = f(0) + \frac{j}{n} f'(0) + \frac{j^s}{n^s} \frac{f^{(s)}(0)}{s!} + o(n^{-s}), \quad j = 1, \ldots, s.$$

The relations (5.46) with $i = s$ and (5.54) imply

$$\tilde{b}_n(s) = f(0) + \frac{s}{n} f'(0) + \frac{(n-s)!}{n!} f^{(s)}(0), \quad n \geq n_0. \tag{5.55}$$

Therefore

$$\frac{n!}{(n-s)!} \left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j) \right) = \frac{j^s}{s!} f^{(s)}(0) + o(1), \quad j = 1, \ldots, s-1,$$

and

$$\frac{n!}{(n-s)!} \left(f\left(\frac{s}{n}\right) - \tilde{b}_n(s) \right) = \left(\frac{s^s}{s!} - 1 \right) f^{(s)}(0) + o(1).$$

Now, if we substitute the last two relations into (5.52) and take into account (5.53), we arrive at

$$f^{(s)}(0)\sum_{k=0}^{s}\frac{(-1)^k y^k}{k!}\left(\binom{s}{k}-\sum_{j=k}^{s}(-1)^{s-j}\frac{j^s}{s!}\binom{s}{j-k}\right)=0,\quad y\in[0,1]$$

(actually the summand for $k=0$ is 0). Consequently, the coefficient of y^s is equal to zero, that is,

$$\frac{(-1)^s f^{(s)}(0)}{s!}\left(1-\frac{s^s}{s!}\right)=0.$$

Therefore $f^{(s)}(0)=0$ and then, by virtue of (5.55), $\tilde{b}_n(s)=f(0)+\frac{s}{n}f'(0)$. □

5.5 Proof of the converse estimates

Let $s\in\mathbb{N}_+$ and $f\in C^s[0,1]$. As it follows from Theorem 3.8 with $w=1$, (3.11) and [23, Theorem 2.1.1], there hold the following strong converse inequalities:

$$\omega_\varphi^2(f^{(s)},n^{-1/2})\le c\left(\|(B_nf)^{(s)}-f^{(s)}\|+\|(B_{Rn}f)^{(s)}-f^{(s)}\|\right)\tag{5.56}$$

and

$$\omega_1(f^{(s)},n^{-1})\le c\left(\|(B_nf)^{(s)}-f^{(s)}\|+\|(B_{Rn}f)^{(s)}-f^{(s)}\|\right)\tag{5.57}$$

for $n\ge n_0$ with some positive integers R and n_0, which are independent of f and n. It was shown in Theorem 3.26 (see also (3.11)) that the two estimates above still hold true without the second term on the right-hand side for $s\le 6$.

The operators $\widehat{B}_n$ and $\widetilde{B}_n$ are not linear. We will use the following property to compensate that. It also incorporates a Bernstein-type inequality.

Lemma 5.13. *Let $s\in\mathbb{N}_+$, $f\in C^s[0,1]$ and $g\in C^{s+1}[0,1]$. Let $f(0),f(1),f'(0),f'(1)\in\mathbb{Z}$ and $f^{(i)}(0)=f^{(i)}(1)=0$, $i=2,\dots,s$. Then*

$$\|(\widehat{B}_n(f))^{(s+1)}-(B_ng)^{(s+1)}\|\le c\,n\left(\|f^{(s)}-g^{(s)}\|+\frac{1}{n}\|g^{(s+1)}\|+\frac{1}{n}\right),\ n\in\mathbb{N}_+.$$

If also there exists $n_0 \in \mathbb{N}_+$, $n_0 \geq s$, such that for $n \geq n_0$ there hold

$$f\left(\frac{k}{n}\right) \geq f(0) + \frac{k}{n}\, f'(0), \quad k = 1, \dots, s,$$
$$f\left(\frac{k}{n}\right) \geq f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \dots, n-1,$$

then

$$\|(\widetilde{B}_n(f))^{(s+1)} - (B_n g)^{(s+1)}\| \leq c\, n \left(\|f^{(s)} - g^{(s)}\| + \frac{1}{n}\, \|g^{(s+1)}\| + \frac{1}{n}\right), \ n \geq n_0.$$

The value of the constant c is independent of f, g, and n.

Proof. We will consider in detail only the operator $\widehat{B}_n$ and indicate, in due course, the minor changes for $\widetilde{B}_n$.

We assume that $n \geq s+1$ since otherwise the assertion is trivial. We apply (3.4) and (5.12) (or (5.11) for $\widetilde{B}_n$) with $s+1$ in place of s, and the identities $\sum_{j=0}^{s+1} \binom{s+1}{j} = 2^{s+1}$ and $\sum_{k=0}^{n-s-1} p_{n-s-1,k}(x) \equiv 1$ to deduce for $x \in [0,1]$ that

$$\begin{aligned}
&|(\widehat{B}_n(f))^{(s+1)}(x) - (B_n g)^{(s+1)}(x)| \\
&\quad \leq n^{s+1} \sum_{k=0}^{n-s-1} \left| \overrightarrow{\Delta}^{s+1} \hat{b}_n^f(k) - \overrightarrow{\Delta}_{1/n}^{s+1} g\left(\frac{k}{n}\right)\right| p_{n-s-1,k}(x) \\
&\quad \leq n^{s+1} \sum_{k=0}^{n-s-1} \left| \overrightarrow{\Delta}^{s+1} \hat{b}_n^f(k) - \overrightarrow{\Delta}_{1/n}^{s+1} f\left(\frac{k}{n}\right)\right| p_{n-s-1,k}(x) \\
&\qquad + n^{s+1} \sum_{k=0}^{n-s-1} \left| \overrightarrow{\Delta}_{1/n}^{s+1} (f-g)\left(\frac{k}{n}\right)\right| p_{n-s-1,k}(x) \\
&\quad \leq (2n)^{s+1} \max_{k=0,\dots,n} \left| f\left(\frac{k}{n}\right) - \hat{b}_n^f(k)\right| + n^{s+1} \|\overrightarrow{\Delta}_{1/n}^{s+1}(f-g)\|_{[0,1-(s+1)/n]}.
\end{aligned}$$

By virtue of (5.23), (5.24) and (5.28) (for $\widetilde{B}_n$ we use (5.14), (5.15) and (5.20) instead) and basic properties of the modulus of continuity, we arrive at

$$\begin{aligned}
\left| f\left(\frac{k}{n}\right) - \hat{b}_n^f(k)\right| &\leq \frac{c}{n^s}\left(\omega_1(f^{(s)}, n^{-1}) + \frac{1}{n}\right) \\
&\leq \frac{c}{n^s}\left(\omega_1(f^{(s)} - g^{(s)}, n^{-1}) + \omega_1(g^{(s)}, n^{-1}) + \frac{1}{n}\right) \\
&\leq \frac{c}{n^s}\left(\|f^{(s)} - g^{(s)}\| + \frac{1}{n}\, \|g^{(s+1)}\| + \frac{1}{n}\right), \quad k = 0, \dots, n.
\end{aligned}$$

To complete the proof it remains to recall that (see e.g. [18, p. 45])

$$\|\overrightarrow{\Delta}_{1/n}^{s+1}(f-g)\|_{[0,1-(s+1)/n]} \le 2\,\|\overrightarrow{\Delta}_{1/n}^{s}(f-g)\|_{[0,1-s/n]} \le \frac{2}{n^s}\,\|f^{(s)}-g^{(s)}\|.$$

□

Now, we are ready to give the proof of the weak converse estimate.

Proof of Theorem 5.5. We will consider in detail only the operator $\widehat{B}_n$. Just the same arguments, but based on the corresponding properties of $\widetilde{B}_n$, yield the assertion for it.

Let $\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| \le C_f\, n^{-\alpha}$ for $n \ge n_f$ with some constants $C_f > 0$ and $n_f \in \mathbb{N}_+$ that may depend on f. Henceforward we will denote by C_f positive constants, which may depend on f, but not on n and h, δ, and g to be specified below.

We have $\lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0$. Since $f(0), f(1) \in \mathbb{Z}$, we have $\lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0$ too. Now, Theorem 5.11 implies that $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$. For $\widetilde{B}_n$ we apply Theorem 5.12 instead. Note also that for both operators we have $f'(0), f'(1) \in \mathbb{Z}$.

Then Theorem 5.9 (or Theorem 5.8 for $\widetilde{B}_n$), (5.56) and the monotonicity of the modulus of continuity on its second argument imply

$$\begin{aligned}
\omega_\varphi^2(f^{(s)}, n^{-1/2}) &\le c\left(\|(B_n f)^{(s)} - f^{(s)}\| + \|(B_{Rn} f)^{(s)} - f^{(s)}\|\right)\\
&\le c\Big(\|(B_n f)^{(s)} - (\widehat{B}_n(f))^{(s)}\| + \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\|\Big)\\
&\quad + c\Big(\|(B_{Rn} f)^{(s)} - (\widehat{B}_{Rn}(f))^{(s)}\| + \|(\widehat{B}_{Rn}(f))^{(s)} - f^{(s)}\|\Big)\\
&\le C_f\left(\omega_1(f^{(s)}, n^{-1}) + n^{-\alpha}\right).
\end{aligned}$$

Thus, to complete the proof, it suffices to show that

$$\omega_1(f^{(s)}, h) = O(h^\alpha) \tag{5.58}$$

and take into account the monotonicity of $\omega_\varphi^2(f^{(s)}, h)$ on h.

We consider the K-functional

$$K(f^{(s)}, t) := \inf_{g\in C^{s+1}[0,1]} \{\|f^{(s)} - g^{(s)}\| + t\,\|g^{(s+1)}\|\}.$$

As is known (see e.g. [18, Chapter 6, Theorem 2.4 and its proof], or (1.3)),

$$\omega_1(f^{(s)}, t) \le 2K(f^{(s)}, t);$$

hence, to establish (5.58), it is sufficient to show

$$K(f^{(s)}, h) = O(h^\alpha). \tag{5.59}$$

To this end, we will apply a standard argument based on the Berens-Lorentz Lemma (see [7], or e.g. [18, Chapter 10, Lemma 5.2]).

Let $0 < h \le \delta \le 1/n_f$. Set $n := [1/\delta]$. For any $g \in C^{s+1}[0,1]$, we have

$$\begin{aligned} K(f^{(s)}, h) &\le \|f^{(s)} - (\widehat{B}_n(f))^{(s)}\| + h\,\|(\widehat{B}_n(f))^{(s+1)}\| \\ &\le C_f\, n^{-\alpha} + h\,\|(\widehat{B}_n(f))^{(s+1)} - (B_n g)^{(s+1)}\| + h\,\|(B_n g)^{(s+1)}\| \\ &\le C_f\,\delta^\alpha + c\,\frac{h}{\delta}\left(\|f^{(s)} - g^{(s)}\| + \delta\,\|g^{(s+1)}\| + \delta\right), \end{aligned}$$

where, at the last step, we estimated the second term by Lemma 5.13, and the third by Proposition 3.14 with $s+1$ in place of s, and $w = 1$. The value of the constant c above is independent of f, g, h, and δ, and C_f is a positive constant, which may depend on f, but not on g, h, and δ.

We take the infimum on $g \in C^{s+1}[0,1]$ and thus arrive at

$$K(f^{(s)}, h) + h \le C_f\,\delta^\alpha + c\,\frac{h}{\delta}\left(K(f^{(s)}, \delta) + \delta\right).$$

Now, the Berens-Lorentz Lemma with $\phi(x) := K(f^{(s)}, x^2) + x^2$ and 2α in place of α (in the notations of [18, Chapter 10, Lemma 5.2]) implies (5.59). □

5.6 Saturation

We will show that the approximation processes given by $(\widetilde{B}_n(f))^{(s)} \to f^{(s)}$ and $(\widehat{B}_n(f))^{(s)} \to f^{(s)}$ in the uniform norm are saturated with the saturation rate of $1/n$ and the trivial class consists of the polynomials of the form $px+q$ with $p, q \in \mathbb{Z}$. Note that these processes are neither linear, nor positive.

Theorem 5.14. *Let $s \in \mathbb{N}_0$ and $f \in C^s[0,1]$ be such that $f(0), f(1) \in \mathbb{Z}$. If*

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = o(1/n) \quad \text{or} \quad \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = o(1/n),$$

then $f(x) = px + q$ with some $p, q \in \mathbb{Z}$ and thus $\widetilde{B}_n(f) = \widehat{B}_n(f) = f$ for all n.

Proof. We consider $\widehat{B}_n$. The argument for $\widetilde{B}_n$ is just the same.

First of all, let us note that if $f(x) = px + q$ with $p, q \in \mathbb{Z}$, then

$$\left(p\,\frac{k}{n} + q\right)\binom{n}{k} \in \mathbb{Z}, \quad k = 0, \dots, n;$$

hence $\widehat{B}_n(f) = B_n f$. As is known, B_n preserves the linear functions. Therefore $\widehat{B}_n(f) = f$ for all n.

We consider the case $s = 0$. Let $\delta \in (0, 1/2)$ be fixed. For $x \in [\delta, 1-\delta]$ we have

$$\begin{aligned}
|B_n f(x) - \widehat{B}_n(f)(x)| &\le \sum_{k=1}^{n-1} \left| f\left(\frac{k}{n}\right)\binom{n}{k} - \left\langle f\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle \right| x^k (1-x)^{n-k} \\
&\le \frac{1}{2}\sum_{k=1}^{n-1} x^k (1-x)^{n-k} \le \frac{1}{2}\sum_{k=1}^{n-1} (1-\delta)^k (1-\delta)^{n-k} \\
&= \frac{n-1}{2}\,(1-\delta)^n.
\end{aligned}$$

Consequently,

$$\|B_n f - f\|_{[\delta, 1-\delta]} = o(1/n). \tag{5.60}$$

Further, by virtue of (5.6) with $\alpha = 1$ and $\|\widehat{B}_n(f) - f\| = o(1/n)$, we get $\omega_\varphi^2(f, h) = O(h^2)$.

By virtue of [23, Theorem 4.2.1(b)], we have for any $f \in C[0,1]$

$$\begin{aligned}
\omega_\varphi^2(f, t) = O(t^2) \\
\iff f \in AC[0,1],\ f' \in AC_{loc}(0,1),\ \varphi^2 f'' \in L_\infty[0,1].
\end{aligned} \tag{5.61}$$

Therefore $f \in W_\infty^2[\delta, 1-\delta]$.

Now, Voronovskaya's classical result (0.5) and (5.60) yield that $f''(x) = 0$ a.e. in $[\delta, 1-\delta]$. Since δ was arbitrarily fixed in $(0, 1/2)$, we arrive at $f''(x) = 0$ a.e. in $[0,1]$. Consequently, $f(x)$ is a linear function. It assumes integral values at 0 and 1; hence $f(x) = px + q$ with some $p, q \in \mathbb{Z}$.

Let $s \in \mathbb{N}_+$. As before, using the inequality (2.1)

$$\|g^{(i)}\| \le c\big(\|g\| + \|g^{(s)}\|\big), \quad i = 1, \dots, s-1,$$

where $g \in C^s[0,1]$, we deduce from

$$\lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

that

$$\lim_{n\to\infty} \|(\widehat{B}_n(f))^{(i)} - f^{(i)}\| = 0, \quad i = 1, \dots, s-1.$$

In particular, we have $\lim_{n\to\infty}(\widehat{B}_n(f))^{(i)}(0) = f^{(i)}(0)$, $i = 0, \dots, s-1$. Since $(\widehat{B}_n(f))^{(i)}(0) \in \mathbb{Z}$, we deduce that for all n large enough we have $(\widehat{B}_n(f))^{(i)}(0) = f^{(i)}(0)$, $i = 0, \dots, s-1$.

Consequently,

$$\widehat{B}_n(f)(x) - f(x) = \frac{1}{(s-1)!} \int_0^x (x-u)^{s-1} \Big((\widehat{B}_n(f))^{(s)}(u) - f^{(s)}(u) \Big)\, du;$$

hence

$$\|\widehat{B}_n(f) - f\| = o(1/n),$$

which reduces the assertion to the case $s = 0$. □

Now, Theorem 5.14 with $s = 0$, (5.3), (5.4) and (5.61) yield the following assertion about the saturation class of the integer forms $\widetilde{B}_n$ and $\widehat{B}_n$ of the Bernstein operator.

Corollary 5.15. *The approximation processes, generated by the operators $\widetilde{B}_n$ and $\widehat{B}_n$, are saturated with the saturation rate of $1/n$. Their saturation class consists of those functions $f \in AC[0,1]$ such that $f(0), f(1) \in \mathbb{Z}$, $f' \in AC_{loc}(0,1)$ and $\varphi^2 f'' \in L_\infty[0,1]$.*

I was not able to identify the saturation class of the approximation processes $(\widetilde{B}_n(f))^{(s)} \to f^{(s)}$ and $(\widehat{B}_n(f))^{(s)} \to f^{(s)}$ with $s \geq 1$. In the proof of Theorem 5.14 we observed that $(\widetilde{B}_n(f))^{(s)}(x)$ and $(\widetilde{B}_n(f))^{(s)}(x)$ interpolate $f^{(s)}(x)$ at 0 and 1 for large n, depending on f. Therefore the description of the saturation class of these approximation processes might not involve the classical modulus of continuity of $f^{(s)}$ as in Corollaries 5.6 and 5.7. However, under an additional assumption, it is quite straightforward to establish the following converse result.

Proposition 5.16. *Let* $s \in \mathbb{N}_+$. *Let* $f \in C^s[0,1]$, $f(0), f(1) \in \mathbb{Z}$, *and* $f^{(s)}(x)$ *be absolutely continuous with an essentially bounded derivative in some neighbourhoods of* 0 *and* 1. *If*

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-1}) \quad \textit{or} \quad \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-1}),$$

then

$$\omega_\varphi^2(f^{(s)}, h) = O(h^2) \quad \textit{and} \quad \omega_1(f^{(s)}, h) = O(h);$$

hence $f^{(s)} \in AC[0,1]$, $f^{(s+1)} \in AC_{loc}(0,1)$ *and* $f^{(s+1)}, \varphi^2 f^{(s+2)} \in L_\infty[0,1]$.

Proof. We will consider only the operator $\widehat{B}_n$. The proof for $\widetilde{B}_n$ is quite similar.

As in the proof of Theorem 5.5 we first deduce that $f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \ldots, s$. Then we observe that the considerations in the proof of Theorem 5.9 actually imply

$$\|(B_n f)^{(s)} - (\widehat{B}_n(f))^{(s)}\| \le c \left(\omega_1(f^{(s)}, n^{-1})_{[0,s/n]} + \omega_1(f^{(s)}, n^{-1})_{[1-s/n,1]} + \frac{1}{n} \right),$$

where we have set for the interval $J \subset [0,1]$

$$\omega_1(F,t)_J := \sup_{\substack{|x-y|\le t \\ x,y\in J}} |F(x) - F(y)|.$$

We have $f^{(s)} \in W_\infty^1[0, s/n]$ and $f^{(s)} \in W_\infty^1[1 - s/n, 1]$ for all n large enough; hence

$$\|(B_n f)^{(s)} - (\widehat{B}_n(f))^{(s)}\| = O(n^{-1}).$$

Consequently,

$$\|(B_n f)^{(s)} - f^{(s)}\| = O(n^{-1}).$$

By virtue of (5.56)-(5.57), this implies

$$\omega_\varphi^2(f^{(s)}, t) = O(t^2) \quad \text{and} \quad \omega_1(f^{(s)}, t) = O(t).$$

Basic properties of the moduli (see (5.61) and [18, Chapter 2, Theorem 9.3]) yield the second assertion of the proposition. □

5.7 Simultaneous approximation by a Kantorovich-type modification of the Bernstein polynomials with integer coefficients

Following the relation between the Kantorovich polynomials and the Bernstein polynomials given in (3.157), we define

$$\widehat{K}_n(f)(x) := \Big(\widehat{B}_{n+1}(F)(x)\Big)', \quad F(x) := \int_0^x f(t)\,dt,$$

where $f \in L[0,1]$ and $x \in [0,1]$.

Then we have

$$\begin{aligned}\widehat{K}_n(f)(x) = \sum_{k=0}^{n} \Bigg(&(k+1) \left\langle \int_0^{\frac{k+1}{n+1}} f(t)\,dt \binom{n+1}{k+1} \right\rangle \\ &- (n-k+1) \left\langle \int_0^{\frac{k}{n+1}} f(t)\,dt \binom{n+1}{k} \right\rangle \Bigg) x^k (1-x)^{n-k}.\end{aligned}$$

Now, Theorem 5.4 with F in place of f and $s+1$ in place of s implies the following direct estimate of the rate of simultaneous approximation by $\widehat{K}_n$.

Theorem 5.17. *Let $s \in \mathbb{N}_0$. Let $f \in C^s[0,1]$ be such that*

$$\int_0^1 f(t)\,dt \in \mathbb{Z}, \quad f(0), f(1) \in \mathbb{Z},$$
$$f^{(i)}(0) = f^{(i)}(1) = 0, \; i = 1, \dots, s.$$

Then

$$\begin{aligned}&\|(\widehat{K}_n(f))^{(s)} - f^{(s)}\| \\ &\qquad \le c \begin{cases} \omega_\varphi^2(f, n^{-1/2}) + \omega_1(f, n^{-1}) + \dfrac{1}{n}, & s = 0, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \ge 1. \end{cases}\end{aligned}$$

The value of the constant c is independent of f and n.

Clearly, the only advantage of $\widehat{K}_n$ to $\widehat{B}_n$ could be that it is defined by integrals of f rather than its values, which is useful in case the former are more readily available than the latter.

Chapter 6

Direct and converse Voronovskaya estimates for the Bernstein operator

6.1 Background

Our goal is to estimate the rate of the convergence in the Voronovskaya's theorem [99] (or see e.g. [18, p. 307], or [79, p. 22]), which states that if $f \in C^2[0,1]$, then

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x)$$

uniformly on $[0,1]$.

We introduce the linear operator

$$D_n f(x) := n(B_n f(x) - f(x))$$

and we will refer to it as the Voronovskaya operator.

We will consider it on the Sobolev-type function spaces

$$W_\infty^m(\varphi)[0,1] := \{f \in C[0,1] : f \in AC_{loc}^{m-1}(0,1), \varphi^m f^{(m)} \in L_\infty[0,1]\},$$

where $\varphi(x) := \sqrt{x(1-x)}$. Let us note that, by virtue of Proposition 2.2(b), we have $W_\infty^{m+1}(\varphi)[0,1] \subset W_\infty^m(\varphi)[0,1]$.

For $f \in W_\infty^2(\varphi)[0,1]$ we set $\mathcal{D} f(x) := \frac{\varphi^2(x)}{2} f''(x)$.

It is known that (see [22, Lemma 8.3])

$$\left\| B_n f - f - \frac{1}{2n}\,\varphi^2 f'' \right\| \le \frac{c}{n^{3/2}} \|\varphi^3 f^{(3)}\|, \quad f \in W^3_\infty(\varphi)[0,1], \tag{6.1}$$

which can be written in the form

$$\|D_n f - \mathcal{D} f\| \le \frac{c}{n^{1/2}} \|\varphi^3 f^{(3)}\|, \quad f \in W^3_\infty(\varphi)[0,1].$$

We will show, assuming a higher degree of smoothness, that

$$\left\| B_n f - f - \frac{1}{2n}\,\varphi^2 f'' \right\| \le \frac{c}{n^2} \left(\|\varphi^2 f^{(3)}\| + \|\varphi^4 f^{(4)}\| \right), \quad f \in W^4_\infty(\varphi)[0,1],$$

that is,

$$\|D_n f - \mathcal{D} f\| \le \frac{c}{n} \left(\|\varphi^2 f^{(3)}\| + \|\varphi^4 f^{(4)}\| \right).$$

Let us note that if $f \in W^4_\infty(\varphi)[0,1]$, then $\varphi^2 f^{(3)} \in L_\infty[0,1]$ (see Proposition 2.3(a) with $r = 2$ and $i = 3$). That slightly improves the estimate

$$\left\| B_n f - f - \frac{1}{2n}\,\varphi^2 f'' \right\| \le \frac{c}{n^2} \left(\|f^{(3)}\| + \|f^{(4)}\| \right), \quad f \in C^4[0,1],$$

established in [51] (see also [48]).

6.2 A characterization of the rate of approximation of the Voronovskaya operator

To state our main results we will use the K-functionals $K_{2,\varphi}(F,t)_w$ defined in (1.5) and

$$\widetilde{K}(F,t) := \inf_{g \in W^4_\infty(\varphi)[0,1]} \left\{ \|F - \mathcal{D} g\| + t \left(\|\varphi^2 g^{(3)}\| + \|\varphi^4 g^{(4)}\| \right) \right\}.$$

We will establish the following characterization of the rate of approximation of $\mathcal{D} f$ by means of $D_n f$.

Theorem 6.1. *For all $f \in W^2_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}_+$ there holds*

$$\|D_n f - \mathcal{D} f\| \le c\,\widetilde{K}(\mathcal{D} f, n^{-1}) \le c \left(K_{2,\varphi}(f'', n^{-1})_{\varphi^2} + \frac{1}{n} \|\varphi^2 f''\| \right). \tag{6.2}$$

Conversely, for all $f \in W^2_\infty(\varphi)[0,1]$ *and all* $k, n \in \mathbb{N}_+$ *there holds*

$$(6.3)\quad K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \le 2\,\|D_k f - \mathcal{D} f\| + c\frac{k}{n}\,K_{2,\varphi}(f'', k^{-1})_{\varphi^2} + \frac{c}{n}\,\|\varphi^2 f''\|.$$

The value of the constant c *is independent of* f*,* n *and* k*.*

The above two estimates can also be written in the form

$$(6.4)\quad \left\| B_n f - f - \frac{1}{2n}\,\varphi^2 f'' \right\| \le \frac{c}{n}\,\widetilde{K}(\mathcal{D} f, n^{-1}) \le \frac{c}{n}\,K_{2,\varphi}(f'', n^{-1})_{\varphi^2} + \frac{c}{n^2}\,\|\varphi^2 f''\|$$

and

$$(6.5)\quad \frac{c}{k}\,K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \le 2\left\| B_k f - f - \frac{1}{2k}\,\varphi^2 f'' \right\| + \frac{c}{n}\,K_{2,\varphi}(f'', k^{-1})_{\varphi^2} + \frac{c}{nk}\,\|\varphi^2 f''\|.$$

We will refer to (6.2) and (6.4) as *direct Voronovskaya inequalities*, and to (6.3) and (6.5) as *weak converse Voronovskaya inequalities.*[1]

Similar direct point-wise estimates were established in [45, Theorem 3.2] and [91, Theorem 2] ([45] contains an overview of other related results). The assumptions on the functions made there are more restrictive. However, the first of these results is very general and both give explicit values to the absolute constant.

Remark 6.2. Since the quantities $D_n f - \mathcal{D} f$ and $K_{2,\varphi}(f'', t)_{\varphi^2}$ are invariant to translations of f by a quadratic polynomial, the relations above directly imply the following slight improvement:

$$\|D_n f - \mathcal{D} f\| \le c\left(K_{2,\varphi}(f'', n^{-1})_{\varphi^2} + \frac{1}{n}\,E_0(f'')_{\varphi^2} \right)$$

and

$$K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \le 2\,\|D_k f - \mathcal{D} f\| + c\left(\frac{k}{n}\,K_{2,\varphi}(f'', k^{-1})_{\varphi^2} + \frac{1}{n}\,E_0(f'')_{\varphi^2} \right),$$

where $E_0(F)_{\varphi^2} := \inf_{\alpha\in\mathbb{R}} \|\varphi^2(F - \alpha)\|$.

[1] The term "inverse Voronovskaya theorem" is also used for a different type of results (see [1, 5]).

From Theorem 6.1 we will derive the following equivalence relation.

Corollary 6.3. *Let* $f \in W^2_\infty(\varphi)[0,1]$ *and* $0 < \alpha < 1$. *Then*

$$\|D_n f - \mathcal{D} f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad K_{2,\varphi}(f'', t)_{\varphi^2} = O(t^\alpha).$$

Bernstein [11] proved that if $f \in C^{2r}[0,1]$, then

$$\lim_{n\to\infty} n^r \left(B_n f(x) - f(x) - \sum_{i=1}^{2r} B_n\left((\circ - x)^i\right)(x)\, \frac{f^{(i)}(x)}{i!} \right) = 0$$

uniformly on $[0,1]$ (see also [90]). A quantitative estimate of this convergence for positive linear operators on $C[0,1]$ was established by Gonska [45] (see also [2, 3, 41]).

Setting $r = 2$ above we have for $f \in C^4[0,1]$ (see (3.17))

$$\lim_{n\to\infty} n(D_n f(x) - \mathcal{D} f(x)) = D' f(x) \tag{6.6}$$

uniformly on $[0,1]$, where

$$D' f(x) := \frac{(1-2x)\varphi^2(x)}{3!}\, f^{(3)}(x) + \frac{3\,\varphi^4(x)}{4!}\, f^{(4)}(x).$$

This shows that the operator D_n is saturated, as its saturation order is n^{-1} and its trivial class is the set of the algebraic polynomials of degree at most 2.

We will establish the following quantitative estimate of the convergence in (6.6).

Theorem 6.4. *For all* $f \in W^4_\infty(\varphi)[0,1]$ *and all* $n \in \mathbb{N}_+$ *there holds*

$$\left\| D_n f - \mathcal{D} f - \frac{1}{n}\, D' f \right\| \le \frac{c}{n}\, K_{2,\varphi^2}(f^{(4)}, n^{-1})_{\varphi^4} + \frac{c}{n^2}\, \|\varphi^4 f^{(4)}\|.$$

The value of the constant c *is independent of* f *and* n.

Instead of $K_{2,\varphi}(F,t)_{\varphi^r}$ one can use the weighted Ditzian-Totik modulus of smoothness $\omega^2_\varphi(F,t)_{\varphi^r}$ (see (1.8) and (1.9)). In fact, the weighted Ditzian-Totik main-part modulus of smoothness allows us to restate the characterization in Corollary 6.3 in a simpler form. Corollary 6.3 and [23, (6.2.6) and (6.2.10)] yield

Corollary 6.5. *Let $f \in W^2_\infty(\varphi)[0,1]$ and $0 < \alpha < 1$. Then*

$$\|D_n f - \mathcal{D} f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \|\varphi^2 \bar{\Delta}^2_{h\varphi} f''\|_{[2h^2, 1-2h^2]} = O(h^{2\alpha}).$$

In the next section we recall several pertinent properties of the Bernstein operator and establish Jackson, Bernstein, and Voronovskaya-type inequalities concerning D_n. Then in Section 6.4 we present proofs of Theorems 6.1 and 6.4 and of Corollary 6.3.

6.3 Basic properties of the Voronovskaya operator

First, we note that the operator D_n is bounded in the following sense.

Proposition 6.6. *For all $f \in W^2_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}_+$ there holds*

$$\|D_n f\| \le 2\, \|\mathcal{D} f\|.$$

Proof. As is known (see e.g. [22, p. 87]),

$$\|B_n f - f\| \le \frac{1}{n}\, \|\varphi^2 f''\|.$$

Hence the assertion immediately follows. □

Next, we will establish a Jackson-type estimate.

Proposition 6.7. *For all $g \in W^4_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}_+$ there holds*

$$\|D_n g - \mathcal{D} g\| \le \frac{c}{n} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^4 g^{(4)}\| \right).$$

The value of the constant c is independent of g and n.

Proof. First, we note that by virtue of Proposition 2.3(a) with $r = 2$ and $i = 3$ we have $\varphi^2 g'', \varphi^2 g^{(3)} \in L_\infty[0,1]$ too.

Applying Taylor's formula, we have for $x \in (0,1)$

$$g\left(\frac{k}{n}\right) = g(x) + \left(\frac{k}{n} - x\right) g'(x) + \frac{1}{2}\left(\frac{k}{n} - x\right)^2 g''(x) + \frac{1}{6}\left(\frac{k}{n} - x\right)^3 g^{(3)}(x) + \frac{1}{6}\int_x^{k/n} \left(\frac{k}{n} - v\right)^3 g^{(4)}(v)\, dv.$$

Multiplying both sides by $p_{n,k}(x)$, summing with respect to k and using identities (3.17) we obtain

$$|D_n g(x) - \mathcal{D}g(x)| = \left| n(B_n g(x) - g(x)) - \frac{1}{2}\,\varphi^2(x) g''(x) \right|$$
$$= \left| \frac{(1-2x)\varphi^2(x)}{6n} g^{(3)}(x) + \frac{n}{6} \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 g^{(4)}(v)\,dv \right|$$
$$\le \frac{1}{6n} \|\varphi^2 g^{(3)}\| + \frac{n}{6} \|\varphi^4 g^{(4)}\| \left| \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 \varphi^{-4}(v)\,dv \right|$$

We will show that

$$R_n(x) := \left| \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 \varphi^{-4}(v)\,dv \right| \le \frac{c}{n^2}.$$

Obviously, it is enough to prove it for $0 < x \le 1/2$. We consider two cases.

Case 1. $1/n \le x \le 1/2$.

Then $\varphi^2(x) \ge 1/2n$ and by using (for v between x and k/n) the inequality [23, p. 141]

$$\frac{\left|\frac{k}{n} - v\right|}{\varphi^2(v)} \le \frac{\left|\frac{k}{n} - x\right|}{\varphi^2(x)}$$

and (3.17), we obtain

$$R_n(x) \le \sum_{k=0}^{n} p_{n,k}(x) \frac{\left(\frac{k}{n} - x\right)^2}{\varphi^4(x)} \left| \int_x^{k/n} \left(\frac{k}{n} - v\right)\,dv \right|$$
$$= \frac{\varphi^{-4}(x)}{2} \sum_{k=0}^{n} p_{n,k}(x) \left(\frac{k}{n} - x\right)^4$$
$$= \frac{\varphi^{-4}(x)}{2} \left[\frac{3\varphi^4(x)}{n^2} + \frac{(1-6\varphi^2(x))\varphi^2(x)}{n^3} \right] \le \frac{c}{n^2}.$$

Case 2. $0 < x \le 1/n$.

Analogously to [22, Lemma 8.3], we will estimate the terms in the sum of $R_n(x)$ separately for $k = 0, 1$ and $k \ge 2$. We have for $k = 0$

$$p_{n,0}(x) \int_0^x v^3 \varphi^{-4}(v)\,dv = (1-x)^n \int_0^x \frac{v^3\,dv}{\big(v(1-v)\big)^2}$$
$$\le (1-x)^{n-2} \int_0^x v\,dv = \frac{x^2(1-x)^{n-2}}{2} \le \frac{c}{n^2}.$$

For $k = 1$ and $n \geq 2$ we have

$$p_{n,1}(x) \int_x^{1/n} \left(\frac{1}{n} - v\right)^3 \varphi^{-4}(v)\, dv = nx(1-x)^{n-1} \int_x^{1/n} \frac{\left(\frac{1}{n} - v\right)^3 dv}{\big(v(1-v)\big)^2}$$
$$\leq nx(1-x)^{n-1} \left(1 - \frac{1}{n}\right)^{-2} \int_x^{1/n} \frac{\left(\frac{1}{n}\right)^3 dv}{v^2} \leq \frac{c}{n^2}.$$

Trivially, for $n = k = 1$ we have

$$p_{1,1}(x) \int_x^1 (1-v)^3 \varphi^{-4}(v)\, dv = x \int_x^1 \frac{(1-v)^3 dv}{\big(v(1-v)\big)^2}$$
$$\leq x \int_x^1 \frac{dv}{v^2} \leq 1.$$

For $k \geq 2$ and $n \geq 3$ we have

$$\left| \sum_{k=2}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - v\right)^3 \varphi^{-4}(v)\, dv \right|$$
$$\leq cx^{-2} \sum_{k=2}^{n} p_{n,k}(x) \left(\frac{k}{n} - x\right)^4 \leq cx^{-2} \sum_{k=2}^{n} p_{n,k}(x) \left(\frac{k}{n}\right)^4$$
$$= cx^{-2} \sum_{k=0}^{n-2} \frac{n!}{(k+2)!(n-k-2)!}\, x^{k+2}(1-x)^{n-k-2} \left(\frac{k+2}{n}\right)^4$$
$$\leq c \sum_{k=0}^{n-2} p_{n-2,k}(x) \left(\frac{k}{n-2}\right)^2 = c\left(x^2 + \frac{\varphi^2(x)}{n-2}\right) \leq \frac{c}{n^2},$$

where at the last but one estimate we have taken into account (3.17). The case $n = k = 2$ is again trivial. The proof is complete. □

To prove the weak converse inequality in Theorem 6.1, we will use the operator A_n defined for $f \in W^2_\infty(\varphi)[0,1]$ by

$$A_n f(x) := 2n \int_{1/2}^{x} (x-t)\varphi^{-2}(t)[B_n f(t) - f(t)]\, dt, \quad x \in [0,1].$$

It is easy to see that A_n is well-defined. In fact, we have that $A_n f \in C[0,1]$ for any $f \in W^2_\infty(\varphi)[0,1]$. That follows from the next lemma.

Lemma 6.8. *If* $f \in W^2_\infty(\varphi)[0,1]$*, then* $\varphi^{-2}.(B_n f - f) \in L[0,1]$.

Proof. Clearly, $\varphi^{-2}(x)[B_n f(x) - f(x)]$ is continuous on $(0,1)$. To complete the proof of the lemma, we will show that it is dominated by a summable function on $[0,1]$. To this end, we expand $f(t)$ at $x \in (0,1)$ by Taylor's formula to get

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u) f''(u)\, du, \quad t \in [0,1],$$

and apply B_n with respect to t to both sides of this identity to arrive at

$$B_n f(x) = f(x) + \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du.$$

Here we have used that $T_{n,0}(x) = 1$ and $T_{n,1}(x) = 0$ (see (3.17)).

Consequently,

$$B_n f(x) - f(x) = \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du. \tag{6.7}$$

For $k = 1, \dots, n-1$, $n \geq 2$, we have $\varphi^{-2} p_{n,k} \in C[0,1]$ and

$$\begin{aligned}\left|\int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du\right| &\leq \left|\int_x^{k/n} \frac{du}{u(1-u)}\right| \|\varphi^2 f''\| \\ &\leq [\ln n - \ln x - \ln(1-x)]\, \|\varphi^2 f''\|.\end{aligned}$$

For $k = 0$ and $k = n$, we have, respectively,

$$\left|\frac{p_{n,0}(x)}{\varphi^2(x)} \int_0^x u f''(u)\, du\right| \leq -\frac{\ln(1-x)}{x}\, \|\varphi^2 f''\|$$

and

$$\left|\frac{p_{n,n}(x)}{\varphi^2(x)} \int_x^1 (1-u) f''(u)\, du\right| \leq -\frac{\ln x}{1-x}\, \|\varphi^2 f''\|.$$

Hence the assertion of the lemma follows. □

To verify the converse inequality we need two inequalities concerning the derivatives of $A_n f$. The first one is a Bernstein-type inequality.

Proposition 6.9. *For all* $f \in W^2_\infty(\varphi)[0,1]$ *and* $n \in \mathbb{N}_+$ *we have* $A_n f \in AC^3_{loc}(0,1)$ *and*

$$\|\varphi^4 (A_n f)^{(4)}\| \le cn\, \|\varphi^2 f''\|.$$

The value of the constant c *is independent of* f *and* n.

Proof. Clearly, if $f \in W^2_\infty(\varphi)[0,1]$, then $A_n f \in AC^3_{loc}(0,1)$. To establish the inequality, we first evaluate the fourth derivative of $A_n f(x)$ for $x \in (0,1)$. Using the representation (6.7), we get

$$(A_n f)^{(3)}(x) = 2n \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du - \frac{2f''(x)}{\varphi^2(x)}\, T_{n,1}(x)$$

a.e. in $(0,1)$. Taking into account that $T_{n,1}(x) = 0$, $f''(x)$ is finite almost everywhere, and that the functions $(A_n f)^{(3)}(x)$ and the sum on the right above are continuous on $(0,1)$, we deduce that

(6.8)
$$(A_n f)^{(3)}(x) = 2n \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du, \quad x \in (0,1).$$

We differentiate once more and arrive at

$$(6.9)\quad (A_n f)^{(4)}(x) = 2n \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)'' \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du \\ - 2n f''(x) \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \left(\frac{k}{n} - x\right) \quad \text{a.e. in } (0,1).$$

Let us consider the second sum on the right above. We calculate the derivative

$$\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' = \frac{2x-1}{\varphi^4(x)}\, p_{n,k}(x) + \frac{p'_{n,k}(x)}{\varphi^2(x)},$$

and apply identities (3.13), $T_{n,1}(x) = 0$ and $T_{n,2}(x) = n\varphi^2(x)$ (see (3.17)), to arrive at

$$\sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \left(\frac{k}{n} - x\right) = \frac{1}{\varphi^2(x)}.$$

Consequently, by (6.9) we get

$$\varphi^4(x)(A_nf)^{(4)}(x) = 2n\varphi^4(x)\sum_{k=0}^{n}\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\int_x^{k/n}\left(\frac{k}{n}-u\right)f''(u)\,du - 2n\varphi^2(x)f''(x).$$

Thus to complete the proof of the proposition, it remains to show that

$$\left|\varphi^4(x)\sum_{k=0}^{n}\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\int_x^{k/n}\left(\frac{k}{n}-u\right)f''(u)\,du\right| \le c\,\|\varphi^2 f''\| \tag{6.10}$$

for $x \in (0,1)$. We use that (see e.g. Lemma 3.16)

$$\left|\int_x^{k/n}\left(\frac{k}{n}-u\right)f''(u)\,du\right| \le \frac{c}{\varphi^2(x)}\left(\frac{k}{n}-x\right)^2\|\varphi^2 f''\|.$$

So, to verify (6.10), it suffices to establish the estimate

$$\varphi^2(x)\sum_{k=0}^{n}\left|\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\right|\left(\frac{k}{n}-x\right)^2 \le c, \quad x \in (0,1). \tag{6.11}$$

First, let $n\varphi^2(x) \ge 1$. By means of (3.13) we represent the second derivative of $\varphi^{-2}(x)p_{n,k}(x)$ in the form

$$\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)'' = \left(\frac{2-n}{\varphi^4(x)} + \frac{2(1-2x)^2}{\varphi^6(x)}\right)p_{n,k}(x) - 3\frac{1-2x}{\varphi^6(x)}(k-nx)p_{n,k}(x) + \frac{1}{\varphi^6(x)}(k-nx)^2 p_{n,k}(x).$$

Consequently,

$$\varphi^2(x)\sum_{k=0}^{n}\left|\left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)''\right|\left(\frac{k}{n}-x\right)^2 \le \frac{1}{n^2}\left(\frac{n+2}{\varphi^2(x)} + \frac{2}{\varphi^4(x)}\right)T_{n,2}(x) + \frac{3}{n^2\varphi^4(x)}\sum_{k=0}^{n}|k-nx|^3 p_{n,k}(x) + \frac{1}{n^2\varphi^4(x)}T_{n,4}(x)$$

and (6.11) for $n\varphi^2(x) \ge 1$ follows from (3.19).

Now, let $n\varphi^2(x) \le 1$. Since

$$[x^{k-1}(1-x)^{n-k-1}]'' = (k-1)(k-2)x^{k-3}(1-x)^{n-k-1}$$
$$-2(k-1)(n-k-1)x^{k-2}(1-x)^{n-k-2}+(n-k-1)(n-k-2)x^{k-1}(1-x)^{n-k-3},$$

in order to verify (6.11), it is enough to show for $x \in (0,1)$ that

$$\sum_{k=0}^{n} \binom{n}{k} (k-1)(k-2)x^{k-3}(1-x)^{n-k-1}(k-nx)^2 \le \frac{cn^2}{\varphi^2(x)},$$
$$\sum_{k=0}^{n} \binom{n}{k} (k-1)(n-k-1)x^{k-2}(1-x)^{n-k-2}(k-nx)^2 \le \frac{cn^2}{\varphi^2(x)},$$
$$\sum_{k=0}^{n} \binom{n}{k} (n-k-1)(n-k-2)x^{k-1}(1-x)^{n-k-3}(k-nx)^2 \le \frac{cn^2}{\varphi^2(x)}.$$

We will prove the first of these inequalities. The proof of the other two is quite similar. We directly see that the terms for $k=0$ and $k=n$ are estimated above by $cn^2\varphi^{-2}(x)$. Hence it remains to show

$$\text{(6.12)} \qquad \sum_{k=3}^{n-1} \binom{n}{k} (k-1)(k-2)x^{k-3}(1-x)^{n-k-1}(k-nx)^2 \le cn^3,$$

where $n \ge 4$.

We change the summation index and use that $n/[\ell(n-\ell)] \le c$ for $\ell = 1, \ldots, n-1$, to deduce

$$\begin{aligned}
&\sum_{k=3}^{n-1} \binom{n}{k} (k-1)(k-2)x^{k-3}(1-x)^{n-k-1}(k-nx)^2 \\
&\quad = \sum_{k=0}^{n-4} \binom{n}{k+3} (k+1)(k+2)x^{k}(1-x)^{n-4-k}(k+3-nx)^2 \\
&\quad \le cn^3 \sum_{k=0}^{n-4} p_{n-4,k}(x)(k+3-nx)^2 \\
&\quad = cn^3 \sum_{k=0}^{n-4} p_{n-4,k}(x)\big[(k-(n-4)x)+(3-4x)\big]^2 \\
&\quad = cn^3\big[T_{n-4,2}(x)+2(3-4x)T_{n-4,1}(x)+(3-4x)^2T_{n-4,0}(x)\big] \le cn^3,
\end{aligned}$$

where at the last step we have taken into account (3.17). Thus (6.12) is verified.

This completes the proof of (6.11) for $n\varphi^2(x) \le 1$, and the proof of the proposition. □

Next, we will estimate $\|\varphi^4(A_n g)^{(4)}\|$ using higher order derivatives. In preparation, we establish the following auxiliary result.

Lemma 6.10. *If $f \in C^3[0,1]$, then the second derivative of $\varphi^{-2}(x)[B_n f(x) - f(x)]$ is continuous and bounded on $(0,1)$.*

Proof. Clearly, $\varphi^{-2}(x)[B_n f(x) - f(x)]$ is twice continuously differentiable on $(0,1)$. By (6.8) we have

$$\left(\frac{B_n f(x) - f(x)}{\varphi^2(x)}\right)' = \sum_{k=0}^{n} \left(\frac{p_{n,k}(x)}{\varphi^2(x)}\right)' \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\, du, \quad x \in (0,1).$$

The summands on the right above for $k = 1, \ldots, n-1$, $n \ge 2$, are in $C^1[0,1]$. Also, it is clear that the first derivatives of the terms with $k = 0$ and $k = n$ are continuous on $(0,1)$. It remains to show that they are bounded on $(0,1)$. We will demonstrate this only for $k = 0$; the case of $k = n$ is treated in a similar way.

For $k = 0$ we have

$$\left(\frac{p_{n,0}(x)}{\varphi^2(x)}\right)' \int_0^x u f''(u)\, du = -(n-1)\frac{(1-x)^{n-2}}{x} \int_0^x u f''(u)\, du \\ - \frac{(1-x)^{n-1}}{x^2} \int_0^x u f''(u)\, du$$

Set

$$F_1(x) := \frac{1}{x} \int_0^x u f''(u)\, du, \quad F_2(x) := \frac{1}{x^2} \int_0^x u f''(u)\, du.$$

For the derivative of $F_1(x)$ we have

$$F_1'(x) = f''(x) - \frac{1}{x^2} \int_0^x u f''(u)\, du$$

and since

$$\left|\frac{1}{x^2} \int_0^x u f''(u)\, du\right| \le \frac{1}{2}\, \|f''\|, \quad x \in (0,1),$$

then $F_1'(x)$ is bounded on $(0,1)$.

To show this for $F_2(x)$, we integrate by parts and get

$$F_2(x) = \frac{1}{2x^2}\int_0^x f''(u)d(u^2) = \frac{1}{2}\,f''(x) - \frac{1}{2x^2}\int_0^x u^2 f^{(3)}(u)\,du.$$

So, it remains to show that the derivative of the function

$$F_3(x) := \frac{1}{x^2}\int_0^x u^2 f^{(3)}(u)\,du$$

is bounded on $(0,1)$. This is verified again straightforwardly since we have

$$F_3'(x) = f^{(3)}(x) - \frac{2}{x^3}\int_0^x u^2 f^{(3)}(u)\,du$$

and

$$\left|\frac{1}{x^3}\int_0^x u^2 f^{(3)}(u)\,du\right| \le \frac{1}{3}\,\|f^{(3)}\|, \quad x \in (0,1).$$

□

Proposition 6.11. *For all $g \in C^4[0,1]$ and all $n \in \mathbb{N}_+$ we have*

$$\|\varphi^4 (A_n g)^{(4)}\| \le c\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right).$$

The value of the constant c is independent of g and n.

Proof. By virtue of Lemma 6.10 we have $A_n g \in AC^3[0,1]$. To establish the inequality, we apply (2.13) with $r = 1$, $s = 2$, $w = \varphi^2$, and $A_n g$ in place of g, and get (note that $D = 2\mathcal{D}$)

$$\|\varphi^4 (A_n g)^{(4)}\| \le c\,\|\mathcal{D}^2 (A_n g)\| = c\,\|\varphi^2 (D_n g)''\|.$$

Finally, we get by means of (3.44) with $s = 2$ and $w = \varphi^2$, the inequality

$$\|\varphi^2 (D_n g)''\| = n\,\|\varphi^2 (B_n g - g)''\| \le c\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right).$$

□

We proceed to a Voronovskaya-type estimate for the operator D_n.

Proposition 6.12. *For all $g \in W^6_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}_+$ there holds*

$$\left\| D_n g - \mathcal{D}g - \frac{1}{n}\, D' g \right\| \le \frac{c}{n^2} \left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\| \right).$$

The value of the constant c is independent of g and n.

Proof. First, we note that by virtue of Propositions 2.3(a) (with $r = 2$ and $i = 3$) and 2.4 we have $\varphi^2 g'', \varphi^2 g^{(3)}, \varphi^2 g^{(4)}, \varphi^4 g^{(5)} \in L_\infty[0,1]$ too.

Again we consider two cases.

Let $n\varphi^2(x) \ge 1$. We expand $g(t)$ at $x \in (0,1)$ by Taylor's formula to get for $t \in [0,1]$

$$g(t) = \sum_{i=0}^{5} g^{(i)}(x) \frac{(t-x)^i}{i!} + \frac{1}{5!} \int_x^t (t-u)^5 g^{(6)}(u)\, du.$$

We apply B_n to both sides of the above identity, take into account (3.17), multiply by n, and rearrange the terms to get

$$(6.13) \quad D_n g(x) - \mathcal{D}g(x) - \frac{1}{n}\, D' g(x) = \frac{1 - 6\varphi^2(x)}{4! n^2}\, \varphi^2(x) g^{(4)}(x) + \frac{1}{5! n^4}\, T_{n,5}(x) g^{(5)}(x) + \frac{n}{5!} \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u \right)^5 g^{(6)}(u)\, du.$$

By (3.16) with $\ell = 5$ and (2.10) we have

$$(6.14) \quad \left| T_{n,5}(x) g^{(5)}(x) \right| \le c (n\varphi^2(x))^2 \left| g^{(5)}(x) \right| \le c n^2 \left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\| \right).$$

Further, we use that (see e.g. Lemma 3.16)

$$\left| \int_x^{k/n} \left(\frac{k}{n} - u \right)^5 g^{(6)}(u)\, du \right| \le \frac{c}{\varphi^6(x)} \left(\frac{k}{n} - x \right)^6 \|\varphi^6 g^{(6)}\|.$$

Therefore, taking into account (3.19) with $m = 3$, we have

$$(6.15) \quad \left| \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u \right)^5 g^{(6)}(u)\, du \right| \le \frac{c}{n^6 \varphi^6(x)} T_{n,6}(x)\, \|\varphi^6 g^{(6)}\| \le \frac{c}{n^3}\, \|\varphi^6 g^{(6)}\|.$$

Combining (6.13), (2.9), (6.14) and (6.15), we arrive at the inequality

$$\left|D_n g(x) - \mathcal{D}g(x) - \frac{1}{n}\, D' g(x)\right| \le \frac{c}{n^2}\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right).$$

for $n\varphi^2(x) \ge 1$.

Now, let $n\varphi^2(x) \le 1$. Using symmetry, we can also assume that $0 < x \le 1/2$. Then $x \le 2/n$. In this case we start with the expansion

$$g(t) = \sum_{i=0}^{4} g^{(i)}(x)\frac{(t-x)^i}{i!} + \frac{1}{4!}\int_x^t (t-u)^4 g^{(5)}(u)\,du, \quad t \in [0,1].$$

We apply B_n to both sides of the above identity, take into account (3.17), multiply by n, and rearrange the terms to get

$$\text{(6.16)}\quad D_n g(x) - \mathcal{D}g(x) - \frac{1}{n}\, D' g(x) = \frac{1-6\varphi^2(x)}{4!n^2}\,\varphi^2(x) g^{(4)}(x) + \frac{n}{4!}\sum_{k=0}^{n} p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^4 g^{(5)}(u)\,du.$$

In order to estimate the sum on the right above, we consider separately the terms with $k = 0$ and $k = 1$.

For $k = 0$ we have, bearing in mind that $0 < x \le 1/2$ and $x \le 2/n$,

$$\left|(1-x)^n \int_0^x u^4 g^{(5)}(u)\,du\right| \le \int_0^x \frac{u^2}{(1-u)^2}\,du\,\|\varphi^4 g^{(5)}\| \le \frac{c}{n^3}\,\|\varphi^4 g^{(5)}\|.$$

Similarly, for $k = 1$ and $n \ge 2$ we have

$$\begin{aligned}\left|nx(1-x)^{n-1}\int_x^{1/n}\left(\frac{1}{n}-u\right)^4 g^{(5)}(u)\,du\right| &\le \frac{cx}{n^3}\left|\int_x^{1/n}\frac{du}{u^2(1-u)^2}\right|\|\varphi^4 g^{(5)}\| \\ &\le \frac{cx}{n^3}\left|\int_x^{1/n}\frac{du}{u^2}\right|\|\varphi^4 g^{(5)}\| \le \frac{c}{n^3}\,\|\varphi^4 g^{(5)}\|.\end{aligned}$$

Straightforward calculations yield for $n = k = 1$ the estimate

$$\left| x \int_x^1 (1-u)^4 g^{(5)}(u)\, du \right| \le x \int_x^1 \frac{(1-u)^2}{u^2}\, du\, \|\varphi^4 g^{(5)}\| \\ \le x \int_x^1 \frac{du}{u^2}\, \|\varphi^4 g^{(5)}\| \le \|\varphi^4 g^{(5)}\|.$$

Thus we have established that

$$\left| p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^4 g^{(5)}(u)\, du \right| \\ \le \frac{c}{n^3}\, \|\varphi^4 g^{(5)}\|, \quad k = 0, 1,\ n \in \mathbb{N}_+. \tag{6.17}$$

In order to estimate the remaining part of the sum on the right of (6.16), we take into account that (see e.g. Lemma 3.16)

$$\left| \int_x^{k/n} \left(\frac{k}{n} - u\right)^4 g^{(5)}(u)\, du \right| \le \frac{c}{\varphi^4(x)} \left| \frac{k}{n} - x \right|^5 \|\varphi^4 g^{(5)}\|.$$

Hence, for $n \ge 2$, we have

$$\left| \sum_{k=2}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^4 g^{(5)}(u)\, du \right| \\ \le \frac{c}{n^5 \varphi^4(x)} \sum_{k=2}^{n} p_{n,k}(x) |k - nx|^5 \|\varphi^4 g^{(5)}\|.$$

We estimate the sum on the right. We have

$$\sum_{k=2}^{n} p_{n,k}(x) |k - nx|^5 \\ = \sum_{k=0}^{n-2} \frac{n!}{(k+2)!(n-k-2)!}\, x^{k+2} (1-x)^{n-k-2} |k + 2 - nx|^5 \\ \le n^2 x^2 \sum_{k=0}^{n-2} p_{n-2,k}(x)\, |(k - (n-2)x) + 2(1-x)|^5 \\ \le c\, n^2 x^2 \sum_{k=0}^{n-2} p_{n-2,k}(x) \left[|k - (n-2)x|^5 + 1 \right] \\ \le c\, n^2 x^2,$$

as at the last step we have taken into account (3.19).

Thus we have established that

$$\left|\sum_{k=2}^{n} p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^4 g^{(5)}(u)\,du\right| \le \frac{c}{n^3}\,\|\varphi^4 g^{(5)}\|. \tag{6.18}$$

Combining (6.16), (6.17), (6.18), (2.9) and (2.10), we arrive at

$$\left|D_n g(x)-\mathcal{D}g(x)-\frac{1}{n}\,D'g(x)\right| \le \frac{c}{n^2}\left(\|\varphi^4 g^{(4)}\|+\|\varphi^6 g^{(6)}\|\right).$$

for $n\varphi^2(x)\le 1$.

The proof of the proposition is completed. □

6.4 Proof of the characterization

Proof of Theorem 6.1. The direct inequality

$$\|D_n f-\mathcal{D}f\| \le c\,\widetilde{K}(\mathcal{D}f,n^{-1}) \tag{6.19}$$

follows from Propositions 6.6 and 6.7 and Proposition 2.3(a) by means of a standard technique. For any $g\in W^4_\infty(\varphi)[0,1]$ we have

$$\begin{aligned}\|D_n f-\mathcal{D}f\| &\le \|D_n(f-g)\|+\|D_n g-\mathcal{D}g\|+\|\mathcal{D}(f-g)\|\\ &\le c\left[\|\mathcal{D}f-\mathcal{D}g\|+\frac{1}{n}\left(\|\varphi^2 g^{(3)}\|+\|\varphi^4 g^{(4)}\|\right)\right].\end{aligned}$$

We take the infimum on g and arrive at (6.19).

Next, we observe that Proposition 2.3(a) with $r=2$ and $i=3$ directly implies for $g\in C^4[0,1]$ and $0<t\le 1$

$$\begin{aligned}\widetilde{K}(\mathcal{D}f,t) &\le c\left[\|\varphi^2(f''-g'')\|+t\left(\|\varphi^2 g''\|+\|\varphi^4 g^{(4)}\|\right)\right]\\ &\le c\left(\|\varphi^2(f''-g'')\|+t\,\|\varphi^4 g^{(4)}\|\right)+ct\,\|\varphi^2 f''\|.\end{aligned}$$

Taking the infimum on g, we get

$$\begin{aligned}\widetilde{K}(\mathcal{D}f,t) &\le c\inf_{g\in C^4[0,1]}\left\{\|\varphi^2(f''-g'')\|+t\,\|\varphi^4 g^{(4)}\|\right\}+ct\,\|\varphi^2 f''\|\\ &\le c\left(K_{2,\varphi}(f'',t)_{\varphi^2}+t\,\|\varphi^2 f''\|\right),\end{aligned}$$

as, to get the last estimate, we have applied Lemma 4.24 with $r = 1$, $s = 2$ and $w = \varphi^2$.

That completes the proof of (6.2). We proceed to the weak converse one given in (6.3).

For any $g \in C^4[0,1]$ we have

$$\begin{aligned} K_{2,\varphi}(f'', n^{-1})_{\varphi^2} &\le \|\varphi^2[f'' - (A_k f)'']\| + \frac{1}{n}\|\varphi^4 (A_k f)^{(4)}\| \\ &\le 2\|D_k f - \mathcal{D} f\| + \frac{1}{n}\left(\|\varphi^4 (A_k(f-g))^{(4)}\| + \|\varphi^4 (A_k g)^{(4)}\|\right). \end{aligned} \tag{6.20}$$

We estimate the second term by Proposition 6.9 to get

$$\|\varphi^4 (A_k(f-g))^{(4)}\| \le ck\|\varphi^2(f'' - g'')\|. \tag{6.21}$$

For the third term, by means of Proposition 6.11, we derive the estimate

$$\begin{aligned} \|\varphi^4 (A_k g)^{(4)}\| &\le c\left(\|\varphi^2 g''\| + \|\varphi^4 g^{(4)}\|\right) \\ &\le c\left(\|\varphi^2(f'' - g'')\| + \|\varphi^4 g^{(4)}\| + \|\varphi^2 f''\|\right). \end{aligned} \tag{6.22}$$

Now, combining (6.20)-(6.22), we arrive at

$$\begin{aligned} K_{2,\varphi}(f'', n^{-1})_{\varphi^2} &\le 2\|D_k f - \mathcal{D} f\| \\ &\qquad + c\frac{k}{n}\left(\|\varphi^2(f'' - g'')\| + \frac{1}{k}\|\varphi^4 g^{(4)}\|\right) + \frac{c}{n}\|\varphi^2 f''\|. \end{aligned}$$

Consequently,

$$\begin{aligned} K_{2,\varphi}(f'', n^{-1})_{\varphi^2} &\le 2\|D_k f - \mathcal{D} f\| \\ &\qquad + c\left(\frac{k}{n}\inf_{g \in C^4[0,1]}\left\{\|\varphi^2(f'' - g'')\| + \frac{1}{k}\|\varphi^4 g^{(4)}\|\right\} + \frac{1}{n}\|\varphi^2 f''\|\right). \end{aligned}$$

Now, (6.3) follows from Lemma 4.24 with $r = 1$, $s = 2$ and $w = \varphi^2$. □

Proof of Corollary 6.3. If $K_{2,\varphi}(f'', t)_{\varphi^2} = O(t^\alpha)$ for some $\alpha \in (0,1]$, then (6.2) implies $\|D_n f - \mathcal{D} f\| = O(n^{-\alpha})$.

To establish the converse implication, we use a standard method based on the Berens-Lorentz Lemma (see [7], or e.g. [18, Chapter 10, Lemma 5.2]).

Let $\|D_n f - \mathcal{D} f\| = O(n^{-\alpha})$ for some $\alpha \in (0,1)$. Then (6.3) implies

$$K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \le C_f k^{-\alpha} + c\frac{k}{n} K_{2,\varphi}(f'', k^{-1})_{\varphi^2} + \frac{c}{n} \|\varphi^2 f''\|, \tag{6.23}$$

where C_f is a positive constant that generally may depend on f, but not on k or n.

Let $0 < s \le t \le 1$. Set $n :=]1/s[$ and $k :=]1/t[$, where $]\gamma[$ denotes the smallest integer not less than the positive real γ. Then

$$1 \le sn \le 2, \quad 1 \le tk \le 2. \tag{6.24}$$

Using (6.23), (6.24) and the subadditivity of the K-functional on its second argument, that is,

$$K_{2,\varphi}(F, \delta_1)_{\varphi^2} \le \max\left\{1, \frac{\delta_1}{\delta_2}\right\} K_{2,\varphi}(F, \delta_2)_{\varphi^2},$$

we arrive at the estimate

$$\begin{aligned} K_{2,\varphi}(f'', s)_{\varphi^2} &\le 2K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \\ &\le C_f t^\alpha + c\frac{s}{t} K_{2,\varphi}(f'', t)_{\varphi^2} + cs\,\|\varphi^2 f''\|, \end{aligned}$$

where, to recall, the constant c is independent of f, s and t, and the constant C_f may depend on f, but is independent of s and t.

Consequently,

$$K_{2,\varphi}(f'', s)_{\varphi^2} + s\,\|\varphi^2 f''\| \le C_f t^\alpha + c\frac{s}{t}\left(K_{2,\varphi}(f'', t)_{\varphi^2} + t\,\|\varphi^2 f''\|\right). \tag{6.25}$$

We set $\phi(y) := K_{2,\varphi}(f'', y^2)_{\varphi^2} + y^2\|\varphi^2 f''\|$. Let $0 < x \le y \le 1$. We put $s := x^2$ and $t := y^2$ in (6.25) and get

$$\phi(x) \le C_f\left(y^{2\alpha} + \frac{x^2}{y^2}\,\phi(y)\right), \quad 0 < x \le y \le 1, \tag{6.26}$$

with some constant C_f, which may depend on f, but is independent of x and y.

Now, the Berens-Lorentz Lemma yields

$$\phi(y) \le C_f y^{2\alpha}, \quad 0 \le y \le 1,$$

with some constant C_f, which may depend on f, but is independent of y.

Consequently,

$$K_{2,\varphi}(f'', t)_{\varphi^2} = O(t^\alpha).$$

□

Proof of Theorem 6.4. We proceed similarly to the proof of the direct inequality in Theorem 6.1.

Let $g \in C^6[0,1]$ and $h(x) := \alpha x^3/6$ with $\alpha := (f-g)^{(3)}(1/2)$. We have, by virtue of Propositions 6.7 and 6.12,

$$\begin{aligned}\left\|D_n f - \mathcal{D} f - \frac{1}{n}\, D' f\right\| &\le \|D_n(f-g-h) - \mathcal{D}(f-g-h)\| \\ &+ \frac{1}{n}\,\|D'(f-g-h)\| + \left\|D_n(g+h) - \mathcal{D}(g+h) - \frac{1}{n}\, D'(g+h)\right\| \\ &\le \frac{c}{n}\left(\|\varphi^2[(f-g)^{(3)}-\alpha]\| + \|\varphi^4 (f-g)^{(4)}\|\right) \\ &\quad + \frac{c}{n^2}\left(\|\varphi^4 g^{(4)}\| + \|\varphi^6 g^{(6)}\|\right). \end{aligned} \tag{6.27}$$

Trivially, we have

$$\|\varphi^4 g^{(4)}\| \le \|\varphi^4 (f-g)^{(4)}\| + \|\varphi^4 f^{(4)}\|. \tag{6.28}$$

Also, for $F \in AC_{loc}(0,1)$ such that $\varphi^4 F' \in L_\infty[0,1]$, and $x \in (0,1)$ there holds

$$\begin{aligned}|\varphi^2(x)(F(x)-F(1/2))| &= \left|\varphi^2(x)\int_{1/2}^x F'(u)\,du\right| \\ &\le \left|\varphi^2(x)\int_{1/2}^x \frac{du}{\varphi^4(u)}\right|\,\|\varphi^4 F'\| \le c\,\|\varphi^4 F'\|. \end{aligned} \tag{6.29}$$

The estimates (6.27), (6.28), and (6.29) with $F = (f-g)^{(3)}$ imply

$$\left\|D_n f - \mathcal{D} f - \frac{1}{n}\, D' f\right\| \le \frac{c}{n}\left(\|\varphi^4 (f^{(4)} - g^{(4)})\| + \frac{1}{n}\,\|\varphi^6 g^{(6)}\|\right) + \frac{c}{n^2}\,\|\varphi^4 f^{(4)}\|.$$

We complete the proof as we take the infimum on $g \in C^6[0,1]$ in the relation above and apply Lemma 4.24 with $r = 1$, $s = 4$ and $w = \varphi^4$. □

Bibliography

[1] U. Abel, M. Ivan, R. Păltănea, Geometric series of positive linear operators and the inverse Voronovskaya theorem on a compact interval, *J. Approx. Theory* **184** (2014), 163–175.

[2] J. A. Adell, J. Bustamante, J. M. Quesada, Estimates for the moments of Bernstein polynomials, *J. Math. Anal. Appl.* **432** (2015), 114–128.

[3] J. A. Adell, J. Bustamante, J. M. Quesada, Sharp upper and lower bounds for the moments of Bernstein polynomials, *Appl. Math. Comput.* **265** (2015), 723–732.

[4] J. A. Adell, G. Sangüesa, Upper constant in direct inequalities for Bernstein-type operators, *J. Approx. Theory* **109** (2001), 229–241.

[5] F. Altomare, S. Diomede, Asymptotic formulae for positive linear operators: direct and converse results, *Jaen J. Approx.* **2** (2010), 255–287.

[6] C. Badea, I. Badea, H. H. Gonska, Improved estimates on simultaneous approximation by Bernstein operators, *Rev. Anal. Numér. Théor. Approx.* 22 (1993), 1–21.

[7] H. Berens, G. G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972), 693–708.

[8] H. Berens, Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights, In: "Approximation Theory and Functional Analysis" (C. K. Chui, Ed.), 1991, 25–46.

[9] S. N. Bernstein, Démonstration du théorème de Weierstrass, fondeé sur le calcul des probabilités, *Commun. Soc. Math. Kharkow* (2), **13** (1912-13), 1–2.

[10] S. N. Bernstein, Some remarks on polynomials of least deviation with integer coefficients, *Dokl. Akad. Nauk SSSR* **16** (1930), 411–415 (in Russian).

[11] S. N. Bernstein, Complément à l'article de E. Voronovskaya "Détermination de la forme asymptotique de l'approximation des fonctions par les polynômes de M. Bernstein", *Dokl. Acad. Nauk SSSR A* **4** (1932), 86–92.

[12] J. Bustamante, Bernstein Operators and Their Properties, Birkhäuser, Cham, 2017.

[13] P.L. Butzer, R.J. Nessel, Fourier Analysis and Approximation, Birkhäser Verlag, Basel, 1971.

[14] J. de la Cal, A. M. Valle, A generalization of Bernstein-Kantorovič operators, *J. Math. Anal. Appl.* **252** (2000), 750–766.

[15] J.-D. Cao, On Sikkema-Kantorovič polynomials of order k, *Approx. Theory Appl.* **5** (1989), 99–109.

[16] L. Cheng, X. Zhou, Strong inequalities for the iterated Boolean sums of Bernstein operators, *Stud. Univ. Babeş-Bolyai Math.* **64** (2019), No. 3, 299–304.

[17] I. Chlodowsky, Erster Mathematikerkongress der USSR, Charkow, 1930.

[18] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.

[19] C. Ding, F. Cao, K-functionals and multivariate Bernstein polynomials, *J. Approx. Theory* **155** (2008), 125–135.

[20] Z. Ditzian, Direct estimates for Bernstein polynomials, *J. Approx. Theory* **79** (1994), 165–166.

[21] Z. Ditzian, Polynomial approximation and $\omega_\varphi^r(f,t)$ twenty years later, *Surveys Approx. Theory* **3** (2007), 106–151.

[22] Z. Ditzian, K. G. Ivanov, Strong converse inequalities, *J. Anal. Math.* **61** (1993), 61–111.

[23] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer-Verlag, New York, 1987.

[24] Z. Ditzian, V. Totik, K-functionals and weighted moduli of smoothness, *J. Approx. Theory* **63** (1990), 3–29.

[25] B. R. Draganov, Upper estimates of the approximation rate of combinations of iterates of the Bernstein operator, *Annuaire Univ. Sofia Fac. Math. Inform.* **101** (2013), 95–104.

[26] B. R. Draganov, On simultaneous approximation by iterated Boolean sums of Bernstein operators, *Results Math.* **66** (2014), 21–41.

[27] B. R. Draganov, Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums, *J. Approx. Theory* **200** (2015), 92–135.

[28] B. R. Draganov, An exact strong converse inequality for the weighted simultaneous approximation by the Bernstein operator, In: "Constructive Theory of Functions, Sozopol 2016" (K. Ivanov, G. Nikolov, R. Uluchev, Eds.), pp. 75–97, Marin Drinov Academic Publishing House, Sofia, 2018.

[29] B. R. Draganov, Simultaneous approximation by Bernstein polynomials with integer coefficients, *J. Approx. Theory* **237** (2019), 1–16.

[30] B. R. Draganov, Converse estimates for the simultaneous approximation by Bernstein polynomials with integer coefficients, In: "Constructive Theory of Functions, Sozopol 2019" (B. Draganov, K. Ivanov, G. Nikolov, R. Uluchev, Eds.), pp. 39–51, Marin Drinov Academic Publishing House, Sofia, 2020.

[31] B. R. Draganov, Corrigendum to "Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums" [J. Approx. Theory 200 (2015) 92–135], *J. Approx. Theory* **252** (2020), 105321.

[32] B. R. Draganov, A strong converse inequality for the iterated Boolean sums of the Bernstein operator, *Stud. Univ. Babeş-Bolyai Math.* **67** (2022), 591–598.

[33] B. R. Draganov, I. Gadjev, Direct and converse Voronovskaya estimates for the Bernstein operator, *Results Math.* **73**:11 (2018).

[34] B. R. Draganov, K. G. Ivanov, A new characterization of weighted Peetre K-functionals, *Constr. Approx.* **21** (2005), 113–148.

[35] B. R. Draganov, K. G. Ivanov, Characterizations of weighted K-functionals and their application, In: "Constructive Theory of Functions, Varna 2005" (B. Bojanov, Ed.), pp. 88-97, Marin Drinov Academic Publishing House, Sofia, 2006.

[36] B. R. Draganov, K. G. Ivanov, A new characterization of weighted Peetre K-functionals (II), *Serdica Math. J.* **33** (2007), 59–124.

[37] B. R. Draganov, K. G. Ivanov, A new characterization of weighted Peetre K-functionals (III), In: "Constructive Theory of Functions, Sozopol 2016" (K. Ivanov, G. Nikolov, R. Uluchev, Eds.), pp. 99–141, Marin Drinov Academic Publishing House, Sofia, 2018.

[38] V. K. Dzyadyk, I. A. Shevchuk, Theory of Uniform Approximation of Functions by Polynomials, Walter de Gruyter, Berlin, 2008.

[39] Le Baron O. Ferguson, Approximation by Polynomials with Integral Coefficients, Mathematical Surveys Vol. 17, American Mathematical Society, Providence, Rhode Island, 1980.

[40] M. S. Floater, On the convergence of derivatives of Bernstein approximation, *J. Approx. Theory* **134** (2005), 130–135.

[41] I. Gavrea, M. Ivan, The Bernstein Voronovskaja-type theorem for positive linear approximation operators, *J. Approx. Theory* **192** (2015), 291–296.

[42] I. Gavrea, H. H. Gonska, R. Păltănea, G. Tachev, General estimates for the Ditzian-Totik modulus, *East J. Approx.* **2** (2003), 175–194.

[43] A. O. Gelfond, On uniform approximation by polynomials with rational integral coefficients, *Uspekhi Mat. Nauk* **10** (1955), 41–65 (in Russian).

[44] H. Gonska, Quantitative Korovkin-type theorems on simultaneous approximation, *Math. Z.* **186** (1984), 419–433.

[45] H. H. Gonska, On the degree of approximation in Voronovskaja's theorem, *Stud. Univ. Babeş-Bolyai Math.* **52** (2007), 3, 103–115.

[46] H. Gonska, M. Heilmann, I. Raşa, Asymptotic behaviour of differentiated Bernstein polynomials revisited, *Gen. Math.* **18** (2010), 45–53.

[47] H. Gonska, M. Heilmann, I. Raşa, Kantorovich operators of order k, *Numer. Funct. Anal. Optim.* **32** (2011), 717–738.

[48] H. Gonska, R. Păltănea, General Voronovskaya and asymptotic theorems in simultaneous approximation, *Mediterr. J. Math.* **7** (2010), 37–49.

[49] H. Gonska, J. Prestin, G. Tachev, A new estimate for Hölder approximation by Bernstein operators, *Appl. Math. Lett.* **26** (2013), 43–45.

[50] H. H. Gonska, J. Prestin, G. Tachev, D. X. Zhou, Simultaneous approximation by Bernstein operator in Hölder norms, *Math. Nachr.* **286** (2013), 349–359.

[51] H. H. Gonska, I. Raşa, The limiting semigroup of the Bernstein iterates: degree of convergence, *Acta Math. Hung.* **111** (2006), 119–130.

[52] H. Gonska, I. Raşa, Asymptotic behaviour of differentiated Bernstein polynomials, *Mat. Vesnik* **61** (2009), 53–60.

[53] H. Gonska, X.-l. Zhou, Approximation theorems for the iterated Boolean sums of Bernstein operators, *J. Comput. Appl. Math.* **53** (1994), 21–31.

[54] H. Gonska, X.-l. Zhou, The strong converse inequality for Bernstein-Kantorovich operators, *Comput. Math. Appl.* **30** (1995), 103–128.

[55] M. Heilmann, I. Raşa, k-th order Kantorovich type modification of the operators U_n^ρ, *J. Appl. Funct. Anal.* **9** (2014), 320–334.

[56] K. G. Ivanov, Direct and converse theorems for the best algebraic approximation in $C[-1,1]$ and $L_p[-1,1]$, *C. R. Acad. Bulgare Sci.* **33** (1980), 1309–1312.

[57] K. G. Ivanov, Some characterizations of the best algebraic approximation in $L_p[-1,1]$ $(1 \leq p \leq \infty)$, *C. R. Acad. Bulgare Sci.* **34** (1981), 1229–1232.

[58] K. G. Ivanov, On a new characteristics of functions, I, *Serdica* **8** (1982), 262–279.

[59] K. G. Ivanov, A constructive characteristic of the best algebraic approximation in $L_p[-1,1]$ $(1 \leq p \leq \infty)$, In: Proc. Internat. Conf. "Constructive Theory of Functions" (B. Sendov, B. Boyanov, D. Vačov, R. Maleev, S. Markov, T. Boyanov, Eds.), Varna, Bulgaria, June 1981, pp. 357–367, Bulgarian Acad. Sci., Sofia, 1983.

[60] K. G. Ivanov, A characterization of weighted Peetre K-functionals, *J. Approx. Theory* **56** (1989), 185–211.

[61] G. Jiang, On Bernstein operators and its compositions, *J. Zhaoqing Univ.* **26**(5) (2005), 5-7 (in Chinese).

[62] H. B. Jiang, L. S. Xie, Simultaneous approximation by Bernstein operators, *Pure Appl. Math.* (Xi'an) **4**:22 (2006), 471–476 (in Chinese).

[63] X. J. Jiang, L. S. Xie, Simultaneous approximation by Bernstein-Sikkema operators, *Anal. Theory Appl.* **24**:3 (2008), 237–246.

[64] H. Johnen, Inequalities connected with moduli of smoothness, *Mat. Vesnik* **3** (1972), 289–305.

[65] H. Johnen, K. Scherer, On equivalence of K-functional and moduli of continuity and some applications, In: Lecture Notes in Math. vol. 571, pp. 119–140, Springer, Berlin, New York, 1976.

[66] L. V. Kantorovich, Some remarks on the approximation of functions by means of polynomials with integer coefficients, *Izv. Akad. Nauk SSSR, Ser. Mat.* **9** (1931), 1163–1168 (in Russian).

[67] D. P. Kacsó, Simultaneous approximation by almost convex operators, *Rend. Circ. Mat. Palermo* (2) **68** (2002), 523–538.

[68] R. P. Kelisky, T. J. Rivlin, Iterates of Bernstein polynomials, *Pacific J. Math.* **21** (1967), 511–520.

[69] H. Knoop, P. Pottinger, Ein satz vom Korovkin-typ für C^k-raüme, *Math. Z.* **148** (1976), 23–32.

[70] H.-B. Knoop, X.-L. Zhou, The lower estimate for linear positive operators (II), *Results Math.* **25** (1994), 315–330.

[71] K. Kopotun, D. Leviatan, I. A. Shevchuk, New moduli of smoothness, *Publ. Math. Inst. (Beograd)* **96** (110) (2014), 169–180.

[72] K. Kopotun, D. Leviatan, I. A. Shevchuk, New moduli of smoothness: weighted DT moduli revisited and applied, *Constr. Approx.* **42** (2015), 129–159.

[73] K. Kopotun, D. Leviatan and I. A. Shevchuk, On the moduli of smoothness with Jacobi weights, *Ukrainian Math. J.* **70** (2018), 3, 437–466.

[74] K. Kopotun, D. Leviatan and I. A. Shevchuk, On weighted approximation with Jacobi weights, *J. Approx. Theory* **237** (2019), 96–112.

[75] K. Kopotun, D. Leviatan and I. A. Shevchuk, On some properties of moduli of smoothness with Jacobi weights, In: “Topics in Classical and Modern Analysis. Applied and Numerical Harmonic Analysis” (M. Abell, E. Iacob, A. Stokolos, S. Taylor, S. Tikhonov, J. Zhu, Eds.), pp. 19–31, Birkhäuser, Cham, 2019.

[76] G. Leoni, A First Course in Sobolev Spaces, AMS, Providence, Rhode Island, 2009.

[77] A. J. López-Moreno, J. Martínez-Moreno, F. J. Muñoz-Delgado, Asymptotic expression of derivatives of Bernstein type operators, *Rend. Circ. Mat. Palermo. Ser. II* **68** (2002), 615–624.

[78] G. G. Lorentz, Zur theorie der polynome von S. Bernstein, *Mat. Sb (N.S.)* **2** (1937), 543–556.

[79] G. G. Lorentz, Bernstein Polynomials, Chelsea Publishing Company, New York, Second Edition, 1986.

[80] G. G. Lorentz, M. v.Golitschek, Y. Makovoz, Constructive Approximation, Advanced Problems, Springer-Verlag, Berlin, 1996.

[81] D. H. Mache, Equivalence theorem on weighted simultaneous L_p-approximation by the method of Kantorovič operators, *J. Approx. Theory* **78** (1994), 321–350.

[82] F. L. Martinez, Some properties of two-dimensional Bernstein polynomials, *J. Approx. Theory* **59** (1989), 300–306.

[83] R. Martini, On the approximation of functions together with their derivatives by certain linear positive operators, *Indag. Math.* **31** (1969), 473–481.

[84] C. Micchelli, The saturation class and iterates of the Bernstein polynomials, *J. Approx. Theory* **8** (1973), 1–18.

[85] R. Păltănea, Approximation Theory Using Positive Linear Operators, Birkhäuser, Boston, 2004.

[86] T. Popoviciu, Sur l'approximation des fonctions convexes d'ordre supérieur, *Mathematica (Cluj)* **10** (1935), 49–54.

[87] T. Popoviciu, Despre cea mai ba aproximare a functiilor continue prin polinoame, Monografii matematice, Sec. Mat. A Univ. Din Cluj, fasc. III, 1937.

[88] Bl. Sendov, Convergence of the derivatives of linear operators, In: Proc. Internat. Conf. "Constructive Theory of Functions" (B. Sendov, D. Vačov, Eds.), Blagoevgrad, Bulgaria, May 30-June 6, 1977, pp. 131–143, Bulgarian Acad. Sci., Sofia, 1980 (in Russian).

[89] Bl. Sendov, V. A. Popov, Convergence of the derivatives of linear positive operators, *Bulgar. Akad. Nauk Izv. Mat. Inst.* **11** (1970), 107–115 (in Bulgarian).

[90] G. T. Tachev, The complete asymptotic expansion for Bernstein operators, *J. Math. Anal. Appl.* **385** (2012), 1179–1183.

[91] G. Tachev, New estimates in Voronovskaja's theorem, *Numer. Algorithms* **59** (2012), 119–129.

[92] G. Tachev, A Modified pointwise estimate on simultaneous approximation by Bernstein polynomials, In: Mathematical Analysis, Approximation Theory and Their Applications (Rassias, M. Themistocles and V. Gupta, Eds.), 2016, Springer, 631–637.

[93] V. Totik, Approximation by Bernstein polynomials, *Amer. J. Math.* **116** (1994), 995–1018.

[94] R. M. Trigub, Approximation of functions by polynomials with integer coefficients, *Dokl. Akad. Nauk SSSR* **140** (1961), 773–775 (in Russian).

[95] R. M. Trigub, Approximation of functions by polynomials with integer coefficients, *Izv. Akad. Nauk SSSR, Ser. Mat.* **26** (1962), 261–280 (in Russian).

[96] R. M. Trigub, On the approximation of functions by polynomials with positive coefficients, *East J. Approx.* **4**(3) (1998), 379–389.

[97] R. M. Trigub, On the approximation of smooth functions and constants by polynomials with integer and natural coefficients, *Mat. Zametki* **70**(1) (2001), 123–136 (in Russian).

[98] R. M. Trigub, On the best approximation of constants by polynomials with integer coefficients, *J. Math. Sci. (N.Y.)*, 2023.

[99] E. Voronovskaya, Détermnation de la forme asymptotice l'approximation des fonctions par les polynomes de M. Bernstein, *Dokl. Akad. Nauk SSSR* **4** (1932), 79–85.

[100] S. Wigert, Sur l'approximation par polynômes des fonctions continues, *Ark. Math. Astr. Fys.* **22 B** (1932), 1–4.