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COMPUTABLE STRUCTURE THEORY : JUMP OF STRUCTURE, CODING AND DECODING

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4.5. MATHEMATICS
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Dedicated to my daughter Mariya Soskova,
to my grandson Edgar Evan Soskov Miller,
and to the bright memory of
Ivan Soskov,
Maria Velcheva,
Andrey Velchev

Contents

Chapter 1

Introduction

The mathematical analysis of the notion of *definability* is one of the principal objectives of Mathematical Logic. Our only access to the objects of mathematics is by explicitly referring to them. Some objects are intrinsically more complicated than others, and that difference is reflected in the means needed to define them. The goal of Computability Theory is to understand the definability for such objects and the relative definability between them.

Now we will be more specific. The first step is to study the most basic mathematical objects - the real numbers, or equivalently, the sets of natural numbers. For X and Y, sets of natural numbers, we intend "X is definable from Y" to mean that whether n is an element of X can be determined concretely once Y is given. For example, there could be an algorithm to determine whether $n \in X$ when given Y as data—-*Turing reducibility* $\leq_T$, or the algorithm could be used to enumerate instances of membership in X from instances of membership in Y—*enumeration reducibility* $\leq_e$. When we omit Y, the set X, defined this way, is *computable* or *computably enumerable,* respectively. Each reducibility $\leq$ has a natural representation as an ordered degree structure by identification of sets reducible to each other: $\mathcal{D}_T$ for the Turing degrees ordered by $\leq_T$, and its extension $\mathcal{D}_e$ the enumeration degrees ordered by $\leq_e$. Elements that stand higher up in this order are more difficult to describe than elements that are closer to the computable or c.e. elements.

The relation $\leq_T$ is a preorder on the subsets of the natural numbers and induces an equivalence relation: $A \equiv_T B$ if and only if $A \leq_T B$ and $B \leq_T A$. The equivalence class of a set A under this relation is the Turing degree of A, denoted by $d_T(A)$. The Turing degrees are ordered by $d_T(A) \leq_T d_T(B)$ if and only if $A \leq_T B$. The least upper bound of two degrees $d_T(A) \vee d_T(B)$

is $d_T(A \oplus B)$, where $A \oplus B = 2A \cup (2B+1)$ is the disjoint union of A and B, also known as join. Finally, relativizing the halting problem to any set induces a *jump operation*, which maps a degree $\mathbf{a}$ to a degree $\mathbf{a}'$, such that $\mathbf{a} <_T \mathbf{a}'$. Thus the structure of the Turing degrees $\mathcal{D}_T$ is an upper semi-lattice with jump operation.

Just like Turing reducibility, enumeration reducibility is a pre-order on the set of the natural numbers, it induces an equivalence relation $\equiv_e$ and a degree structure $\mathcal{D}_e$. The structure of the enumeration degrees is also an upper semi-lattice.

There is a strong connection between the relations that we defined: $A \leq_T B$ if and only if $A \oplus \overline{A}$ is c.e. in B if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$. This gives the natural embedding ι of the Turing degrees into the enumeration degrees: $\iota(d_T(A)) = d_e(A \oplus \overline{A})$. The embedding preserves the least element, the partial order and the jump operation. The image of the Turing degrees under the embedding ι, defines a substructure of the enumeration degrees, isomorphic to $\mathcal{D}_T$. An enumeration degree is total if it is an image of a Turing degree. In 1967 Rogers wrote his famous expository text [Rog67a], describing the state of the art of the field and marking the important open questions that stood open. Among them was the question of the first order definability of the total enumeration degrees. The full answer to Rogers' question is finally obtained through the collaboration of Cai, Ganchev, Lempp, Miller and M. Soskova, [CGL+16], confirming Ganchev and Soskova's conjecture: The total enumeration degrees are first order definable in $\mathcal{D}_e$.

Understanding the fundamental building blocks of mathematical objects is only the first step in these investigations. The next step is to understand the objects of higher type: mathematical structures. We all know that in mathematics there are proofs that are more difficult than others, constructions that are more complicated than others, and objects that are harder to describe than others. Different fields in mathematics study different structures: in Algebra we study groups, rings, fields, in Topology and Analysis we study topological spaces, metric spaces, polish spaces, in Discrete Mathematics we study linear orderings, graphs and trees. In Computable Structure Theory we want to understand all of these structures from the point of view of logic and definability. Computable Structure Theory studies the interplay between complexity and structures. It is an area inside Computability Theory and Logic that is concerned with the computable aspects of mathematical objects and constructions. Our motivations come from questions of the following sort: Are there syntactical properties that explain why certain objects (like structures, relations, or isomorphisms) are easier or harder to compute or to describe?

In **Chapter 2.** of the thesis we introduce the basic notions and facts that we need. We begin with the properties of the structures of Turing and enumeration degrees with jumps and present the properties of degree spectra and enumeration degree spectra and their co-spectra. We introduce one of the general methods in Computable Structure Theory - the forcing method and genericity. We present the normal form of relatively intrinsically Σ^0_α relations in a given countable structure for a computable ordinal α, by computable infinitary Σ^c_α formulas.

Our work concentrates on the complexity of structures. By complexity, we mean descriptional or computational complexity, in the sense of how difficult it is to describe or compute a certain object. We want to measure the complexity of a structure, so we attach to every structure a set of degrees that describes it: *the degree spectrum of a structure*. Since computability theory is developed on the natural numbers we need to work with structures with countable domains, whose elements can be enumerated by natural numbers. Given a structure $\mathcal{A}$, *a presentation* (enumeration) of $\mathcal{A}$ is nothing more than an isomorphic or homomorphic copy of $\mathcal{A}$ whose domain is either the set of the natural numbers $\mathbb{N}$, or an initial segment of $\mathbb{N}$. The degree spectrum $DS(\mathcal{A})$ is the set of all Turing degrees of the atomic diagrams of the presentations of the structure $\mathcal{A}$, a notion, introduced by Richter [Ric81] and investigated by Knight [Kni86] and many others. Knight introduces the jump spectrum $DS_1(\mathcal{A})$ as the set of all jumps of elements of the spectrum of $\mathcal{A}$. Soskov [Sos04] initiates a generalization of the notion of spectrum, basing it on enumeration reducibility. The advantage is that these spectra are closed upwards relative to total enumeration degrees. He introduces also the co-spectrum $CS(\mathcal{A})$ of the structure $\mathcal{A}$ as the set of all enumeration degrees, which are lower bounds of elements of the spectrum and shows that every countable ideal of enumeration degrees is a co-spectrum of a structure. He shows a number of structural properties of spectra and co-spectra, such as the existence of minimal pairs and quasi-minimal degrees for the spectrum. The author [SS04, Sos05b, Sos05a, Sos07b, Sos06] generalizes the notion of spectrum relative to a finite sequence of structures - called joint and relative spectra, and shows that all the properties of spectra and co-spectra are preserved. Together with Soskov [Sos07a, SS07, SS09a], we show that every jump spectrum is a spectrum of a structure and we prove a jump inversion theorem for spectra.

One of the important methods used in computable structure theory is the method of forcing, introduced first by Cohen, in order to show the independence of the continuum hypothesis from Zermelo-Fraenkel set theory and later adapted to arithmetic by Feferman. It is very effective for the description of the structural properties of the Turing degrees. We show the standard construction of a generic set, which includes a sequence of initial segments of the desired set.We also give a proof of Friedberg's [Fri57] jump inversion theorem, which we will generalize for structures and will use this method in Chapter 3.

Another way to characterize the complexity of a structure $\mathcal{A}$ is to analyze the definable sets in $\mathcal{A}$. This gives a finer measure as it may happen that two structures have the same degree spectra but greatly differ in their definability power and model theoretic properties. A relation R on a structure $\mathcal{A}$ is *relatively intrinsically* Σ^0_α for some computable ordinal α, if for every isomorphic copy (presentation) $(\mathcal{B};Q)$ of $(\mathcal{A};R)$ on the natural numbers, Q is Σ^0_α in the atomic diagram $D(\mathcal{B})$ of $\mathcal{B}$. Thus, these relations are exactly the ones that can be defined within α Turing jumps of the structure, independently of the presentation of the structure. In order to characterize syntactically the relatively intrinsically Σ^0_α relations in a structure, we need some infinite $L_{\omega_1,\omega}$ formulas – which allow countable disjunctions and conjunctions of formulas, all with finitely many free fixed variables (see [AK00]). The Σ^c_0 and Π^c_0 formulas are formulas without quantifiers. For $\alpha > 0$, a computable Σ^c_α formula $\varphi(\overline{x})$ is a c.e. disjunction of formulas of the form $\exists\overline{y}\psi(\overline{x},\overline{y})$, where $\psi(\overline{x},\overline{y})$ is a Π^c_β formula for some $\beta < \alpha$. Similarly, a computable Π^c_α formula is a c.e. conjunction of formulas of the form $\forall\overline{y}\psi(\overline{x},\overline{y})$, where $\psi(\overline{x},\overline{y})$ is a computable Σ^c_β formula, for some $\beta < \alpha$. Ash, Knight, Manasse and Slaman [AKMS89] and independently Chisholm [Chi90] prove that the relation R on a structure $\mathcal{A}$ is relatively intrinsically Σ^0_α iff R is definable by a Σ^c_α formula $\varphi(\overline{x},\overline{y})$ with finitely many parameters, i.e. there exists $\overline{b} \in |\mathcal{A}|^k$, $\overline{a} \in R \iff \mathcal{A} \models \varphi(\overline{a},\overline{b})$, for every $\overline{a}$.

Chapter 3. introduces a notion of jump of a structure and represents the proof of two Jump inversion theorems for structures, one — based on Marker's extensions and the other on forcing. Some applications of the Jump inversion theorems are presented.

The idea of the *jump of a structure* is first considered by Soskov and his student Baleva [Bal06] in the context of s-reducibility between structures, a reducibility based on relative search computability. Given a structure $\mathcal{A}$, the goal is to define a structure $\mathcal{A}'$ — the jump of $\mathcal{A}$, so that $\mathcal{A}'$ knows the sets definable by computable infinitary Σ^c_1 formulas in $\mathcal{A}$. Moreover the definable subsets of the domain of $\mathcal{A}$ by computable infinitary Σ^c_2 formulas are exactly those, that are definable by computable infinitary Σ^c_1 in $\mathcal{A}'$. This notion resurfaced in computable structure theory in the period 2002–2010 independently in our works with Soskov [Sos07a, SS07, SS09a], in papers of Montalbán [Mon09, Mon12, HM12] and in results of Stukachev [Stu09, Stu10]. With Soskov, we define the jump $\mathcal{A}'$ of the structure $\mathcal{A}$ by considering the Moschovakis extension of $\mathcal{A}$, together with a new predicate — an analogue of the Kleene's Halting set, which codes all the sets, definable by computable infinitary Σ^c_1 formulas with parameters. This changes the domain of the structure, but keeps the language finite, if the original is finite. Montalbán's approach is to keep the domain of the structure the same and to add a complete set of relations definable by computable infinitary Π^c_1 formulas. This can possibly make the language infinite, however Montalbán gives some examples of structures, such as linear orderings and Boolean algebras, where the complete set of relations is finite and natural. With Knight, Montalbán, Soskov, et.al. [Mon12] we give some additional examples of structures with finite complete Π^c_2 set of relations, and of others, which do not have finite complete Σ^c_1 set. Since the results remained unpublished, Antonio Montalbán included them in his paper [Mon12]. Morozov [Mor04] and later Puzarenko [Puz09] also define the jump, but for an admissible structure. Stukachev extends that definition to all structures in the terms of Σ-definability in hereditarily finite extension of the structure. Vatev [Vat13, Vat14, Vat15] extends the notion of jump of a structure to the α-th jump of a structure for arbitrary computable ordinal α.

In the classical computability theory, Friedberg [Fri57] shows a jump inversion theorem: if $X \geq_T \varnothing'$, then there is a set Y, such that $Y' \equiv_T X$. *Jump inversion for a structure* can be formulated in the following way: for every structure $\mathcal{A}$ which codes $\varnothing'$, i.e. $\mathbf{0}'$ is a lower bound of $DS(\mathcal{A})$, there is a structure $\mathcal{C}$, such that $\mathcal{C}'$ is equivalent to $\mathcal{A}$. A commonly used notion is to say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent if they have the same Turing-degree spectrum. In this case we say that $\mathcal{A}$ and $\mathcal{B}$ are Muchnik equivalent, and write $\mathcal{A} \equiv_w \mathcal{B}$ (w for weak, as it is weaker than the Medvedev equivalence), where $\mathcal{A} \leq_w \mathcal{B} \iff DS(\mathcal{B}) \subseteq DS(\mathcal{A})$. Another notion, often used in work of Russian mathematicians, is to say that $\mathcal{A}$ and $\mathcal{B}$ are Σ equivalent, if each one can be interpreted in the structure of hereditary finite sets over the other one in an effective way; in this case we write $\mathcal{A} \equiv_\Sigma \mathcal{B}$ (This notion is due to Ershov[Ers85]). We prove with Soskov, [Sos07a, SS07, SS09a], this jump inversion theorem for structures, using Marker extensions. Furthermore, we present a relativized version of the theorem to all structures. That is, if $\mathcal{A} \geq_w \mathcal{B}'$, then there is a structure $\mathcal{C} \geq_w \mathcal{B}$, such that $\mathcal{A} \equiv_w \mathcal{C}'$. Actually, our proof is in the terms of degree spectra, i.e. if $DS(\mathcal{A}) \subseteq DS(\mathcal{B}')$, then there exists a structure $\mathcal{C}$ with the property $DS_1(\mathcal{C}) = DS(\mathcal{A})$ and $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$. The jump inversion theorem was proved later by Stukachev [Stu09, Stu10] for the notion of Σ equivalence.

This result for any computable successor ordinal appears in some form in Goncharov, Harizanov, et.al. [GHK⁺05]. They only do it for graphs, but we know that any degree spectrum can be realized as the degree spectrum of a graph [HKSS02]. They proved the result above only as a tool to get other results to build a structure that is Δ_α-categorical but not relatively so. They do not mention the jump of a structure. Based on their method Vatev [Vat13, Vat14, Vat15] extends the jump inversion of a structure for arbitrary successor ordinal α. Vatev's approach relies also on the notion of *conservative extension*. This notion provides a finer way to compare the relative definability between two structures at arbitrary levels of the Σ^c_α-hierarchy. Soskov [Sos13b] gives an example that the jump inversion theorem does not hold for a limit ordinal.

Another way to formulate the jump inversion on structures is: for every structure $\mathcal{A}$, if $Y \subseteq \mathbb{N}$ computes a copy of the jump $\mathcal{A}'$, then there is $X \subseteq \mathbb{N}$ such that $X' \equiv_T Y$ and X computes a copy of $\mathcal{A}$. Montalbán [Mon09, Mon12, Mon] calls this the second jump inversion theorem. In other words: the jump spectrum of $\mathcal{A}$ is the spectrum of $\mathcal{A}'$, i.e. $DS_1(\mathcal{A}) = DS(\mathcal{A}')$. We prove this result with Soskov [SS07, SS09a] and independently later Montalbán [Mon09] also proved it in another terms.

In **Chapter 4.** we present some general sufficient conditions for a structure to admit strong jump inversion.

Downey and Jockusch [DJ94] show that for every Boolean algebra $\mathcal{A}$, if X' computes a copy of $\mathcal{A}$ with added predicate $atom(x)$, then X computes a copy of $\mathcal{A}$. Montalbán [Mon09] proved that a Boolean algebra $\mathcal{A}$ with added predicate $atom(x)$ is in some sense equivalent to $\mathcal{A}'$. This suggests the following strengthening of the jump inversion.

The structure $\mathcal{A}$ *admits strong jump inversion*, if whenever X' computes a copy of $\mathcal{A}'$, then X computes a copy of $\mathcal{A}$. This is equivalent to: for all X, if $\mathcal{A}$ has a copy that is low over X, then it has a copy that is computable in X. Here, when we say that $\mathcal{C}$ is low over X, we mean that $D(\mathcal{C})' \leq_T X'$.

The result of Downey and Jockusch shows that every Boolean algebra admits strong jump inversion. Lerman and Schmerl [LS79] prove that for every $\aleph_0$-categorical theory T, if $T \cap \Sigma_2$ is c.e., then every model of T admits strong jump inversion. Some equivalence structures and some abelian p-groups admit strong jump inversion. Recently, D. Marker and R. Miller [MM17] show that all countable models of the theory of differentially closed fields of characteristic 0 (DCF_0) admit strong jump inversion.

Not all structures admit strong jump inversion. Jockush and Soare [JS91] show that there are low linear orders without computable copies, and hence they do not admit strong jump inversion. There exist low complete extensions of Peano arithmetic T, for which there exists a model $\mathcal{A}$ whose complete diagram is computable in T, but since $\mathcal{A}$ is nonstandard, it does not possess a computable copy. The problem here is the following: find sufficient conditions for a structure to admit strong jump inversion. In particular, study some classes of linear orderings, which admit strong jump inversion. With Calvert, Frolov, et.al., [CFH+18], we establish a general result with sufficient conditions on a structure $\mathcal{A}$, which guarantee strong jump inversion of $\mathcal{A}$. They are expressed in terms of saturation and enumeration properties of sets of types having formulas of low arithmetic complexity: as computable enumeration R of the B_1-types, where these are made up of formulas that are Boolean combinations of existential formulas, effective type completion, and R-labeling of $\mathcal{A}$. When a structure $\mathcal{A}$ admits strong jump inversion, and $\mathcal{A}$ is low relative to an oracle X, we also consider the complexity of the isomorphisms between $\mathcal{A}$ and its X-computable copies.

Our general result applies to structures from some familiar classes, including certain classes of linear orderings and trees. While we do not get the result of Downey and Jockusch for arbitrary Boolean algebras, we do get a result for Boolean algebras with no 1-atom, with some extra information on the complexity of the isomorphism. Such an isomorphism can be chosen to be Δ^0_3 relative to X. This is interesting, because Knight and Stob established in 2000 that any low Boolean algebra has a computable copy and a corresponding Δ^0_4 isomorphism, and this bound has been proven to be sharp. We apply also our general conditions on the models of first order theory T such that $T \cap \Sigma_2$ is computably enumerable and for each tuple of variables $\overline{x}$, there are only finitely many B_1-types in variables $\overline{x}$ consistent with T. Our general result includes the result of Marker and Miller. As a side result, we get that the saturated model of DCF_0 has a decidable copy.

Chapter 5. is devoted to some uniform methods for coding and decoding of one class of structures to another.

We consider also classes of structures from the viewpoint of computability theory. By classes of structures we mean classes like the one of fields, groups, linear orderings, graphs, etc. Our general objective is to consider global properties of the classes and derive properties about their individual structures. In Model Theory, the relevant issues include ones about transferring model-theoretic phenomena from structures of one class such as graphs, where certain properties are easy to arrange, to others such as groups where they are less obvious (as in Mekler, [Mek81]). In Descriptive Set Theory, the analysis frequently centers around the issue of completeness of various properties at different levels of a hierarchy with respect to Borel reducibilities (as, for example, in Friedman and Stanley, [FS89], Camerlo and Gao, [CG01], Hjorth and Kechris [HK96]). There are various ways of mapping structures from one class into another. For each of these reducibilities we have classes, that are on top in the sense, that all other classes can be reduced to it. Starting by Borel reducibility, and moving on to effective reducibility as Turing-computable reducibility, sometimes we have a uniform method for coding each member of one class in some member of the other. We are interested in cases where there is a uniform effective procedure for decoding, and in cases where the decoding is highly non-effective. We consider also a stronger notion of reducibility, introduced by Montalbán [Mon14, Mon], based on the idea of effective interpretability between structures. It captures the idea of effective decoding. When a structure $\mathcal{A}$ from one class is coded effectively and uniformly in a structure $\mathcal{B}$ from another class, possibly in another signature, the question is: can we decode effectively and uniformly the structure $\mathcal{B}$ from $\mathcal{A}$? The uniformity we receive by finding some formulas, that the coded structure should satisfy, and this is true for all structures in the first class. One part of the uniform effective interpretability is the *Medvedev reduction*, i.e. there is a Turing operator, for a copy of $\mathcal{B}$ it gives a copy of $\mathcal{A}$, and the second part is two isomorphic structures from one class to be coded into isomorphic structures from the other class. R. Miller proposed a notion of effective interpretability based on computable functor—a pair of Turing operators, the first one gives the Medvedev reduction and the second the preserving the isomorphism, between copies. Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] prove that these the two notions of effective uniform interpretability coincide. Harrison-Trainor, R. Miller, and Montalbán [HTMM18] show similar result for Borel functors and infinitary interpretations.

Historically the first well-known notion of Borel reducibility, introduced by Friedman and Stanley [FS89], in order to get a classification of some classes of structures. The effective version is the Turing-computable reducibility [CCKM04, KMVB07], introduced by Julia Knight and her students. The class of undirected graphs and the class of linear orderings both lie on top under Turing computable embeddings. The standard Turing computable embeddings of directed graphs (or structures for an arbitrary computable relational language) in undirected graphs come with uniform effective interpretations. The question is: does the class of linear orderings lie also on the top of uniform effective interpretability? With Knight and Vatev [KSV19], we give examples of graphs that are not Medvedev reducible to any linear ordering, or to the jump of any linear ordering. We observe that any graph can be coded in the second jump of a linear ordering, so we have a Medvedev reduction. For the known Turing computable embedding of graphs in linear orderings, due to Friedman and Stanley, we show that there is no uniform effective interpretation, defined even by $L_{\omega_1\omega}$ formulas. Our conjecture is that there is no effective uniform way for coding graphs in linear orders with uniform effective decoding. In support of this Montalbán and Harrison-Trainor [HT20] independently prove that for each computable ordinal α there is a structure $\mathcal{A}$ with no Δ^0_α copy, but the Friedman and Stanley's embedding $L(\mathcal{A})$ has a computable copy. We relativize: if $\mathcal{A}$ is interpreted in $L(\mathcal{A})$ using Σ^X_α formulas, then any copy of $L(\mathcal{A})$ will $\Delta^{0,X}_\alpha$-computes a copy of $\mathcal{A}$.

Our second result here is positive. With Alvir, Calvert, et.al. [ACG+20], we consider an effective uniform interpretation of fields in some 2-step nilpotent groups. We improve on and generalize a 1960 result of Mal'tsev. For a field F, we denote by $H(F)$ the Heisenberg group with entries in F. Mal'tsev [Mal60] showed that there is a copy of F defined in $H(F)$, using existential formulas with an arbitrary non-commuting pair (u, v) as parameters. We show that F is interpreted in $H(F)$ using computable Σ^c_1-formulas with no parameters. We give two proofs. The first is an existence proof, relying on a result of Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] based on a computable functor. This proof allows the possibility that the elements of F are represented by tuples in $H(F)$ of no fixed arity. The second proof is direct, giving explicit finitary existential formulas that define the interpretation, with elements of F represented by triples in $H(F)$. Looking at what was used to arrive at this parameter-free interpretation of F in $H(F)$, we give general conditions sufficient to eliminate parameters from interpretations.

For an algebraically closed field C of characteristic 0, let $SL_2(C)$ be a special linear group of 2×2 matrices over C with determinant 1. Clearly, $SL_2(C)$ is defined in C without parameters. With Alvir, Knight, R. Miller, [AKMS] we define an interpretation of the field C in $SL_2(C)$ using finitary existential formulas with two parameters. There are old model theoretic results, due to Poizat [Poi01], that give uniform definability of a copy of C in $SL_2(C)$ using elementary first order formulas without parameters. So, we have, not necessarily an *effective* interpretation without parameters, but one that is defined by elementary first order formulas. We do not know the complexity of the formulas.

In **Chapter 6.** we consider some model theoretical properties of cohesive powers of linear orders.

Skolem's 1934 construction [Sko34] of a countable non-standard model of arithmetic was the first, using this technique. He considered the arithmetical cohesiveness of a set of natural numbers C, i.e. for every arithmetical $A \subseteq \mathbb{N}$, either $C \subseteq^* A$ or $C \subseteq^* \overline{A}$, one then showed that this structure is elementarily equivalent to $(\mathbb{N}; +, \cdot, <)$. Here, $C \subseteq^* A \iff C \setminus A$ is finite, i.e. an inclusion of sets up to finitely many elements. He define an equivalence relation $=_C$ on the arithmetical functions f: $\mathbb{N} \to \mathbb{N}$ by $f =_C g$ if and only if $C \subseteq^* \{n : f(n) = g(n)\}$. The elements of the structure are the classes of equivalence of this relation and the $+, \cdot, <$ are defined appropriately, e.g. $[f] < [g] \iff C \subseteq^* \{n : f(n) < g(n)\}$. The structure is countable because there are only countably many arithmetical functions, and it has non-standard elements, such as the element represented by the identity function. Cohesive powers of computable structures can be viewed as effective ultrapowers over effectively indecomposable sets called cohesive sets, where cohesive sets play the role of ultrafilters. The ultraproduct construction is a very powerful and widely used in Model theory. An infinite set $C \subseteq \mathbb{N}$ is *cohesive* (*r-cohesive*) if for every c.e. (computable) set W, either $W \cap C$ or $\overline{W} \cap C$ is finite. For computable functions f and g and a r-cohesive set C, Feferman, Scott, and Tennenbaum (see [FST59]) proved that the structure $\mathcal{R}/=_C$, with domain the set of recursive functions modulo $=_C$, is a model only of a fragment of arithmetic.

The effective version of cohesive powers of computable structures, based on partial computable functions has been introduced by Dimitrov, [Dim09], in relation to the study of automorphisms of the lattice $\mathcal{L}^*(V_\infty)$ of effective vector spaces. Cohesive powers on the field of rational numbers were used in [Dim08, DH16] to characterize certain principal filters and interesting orbits of $\mathcal{L}^*(V_\infty)$. Their isomorphism types and automorphisms were further studied in [DHMM14].

With Dimitrov, Harizanov, Morozov, Shafer and Vatev [DHM+19, DHM+20] we consider some properties of cohesive powers of linear orders. We show that if $\mathcal{A}$ is a computable structure that is ultrahomogeneous in a uniformly computable way, then $\mathcal{A}$ is isomorphic to its cohesive powers. We investigate the isomorphism types of cohesive powers $\Pi_C \mathcal{L}$ for familiar computable linear orders $\mathcal{L}$. The goal of this investigations is to show that the presentation of a computable structure matters for the isomorphism type of its cohesive power. If $\mathcal{L}$ is a computable copy of ω that is computably isomorphic to the standard presentation of ω, then every cohesive power of $\mathcal{L}$ has order-type $\omega + \zeta\eta$. There is a computable copy $\mathcal{L}$ of ω that is not computably isomorphic to the standard presentation of ω, but every cohesive power of $\mathcal{L}$ has order-type $\omega + \zeta\eta$. However, there are computable copies of ω, necessarily not computably isomorphic to ω, having cohesive powers not elementarily equivalent to $\omega + \zeta\eta$. For example, we show that there is a computable copy of ω with a cohesive power of order-type $\omega + \eta$. Our most general result is that if $X \subseteq \mathbb{N} \setminus \{0\}$ is either a Σ^0_2 set or a Π^0_2 set, thought of as a set of finite order-types, then there is a computable copy of ω with a cohesive power of order-type $\omega + \sigma(X \cup \{\omega + \zeta\eta + \omega^*\})$, where $\sigma(X \cup \{\omega + \zeta\eta + \omega^*\})$ denotes the shuffle sum of the order-types in X and the order-type $\omega + \zeta\eta + \omega^*$. Furthermore, if X is finite and non-empty, then there is a computable copy of ω with a cohesive power of order-type $\omega + \sigma(X)$.

In **Chapter 7.** our last result is in the degree theory, more specifically in enumeration degrees. With Andrews, Ganchev, et.al. [AGK+19] we investigate the properties of a substructure of the enumeration degrees: the cototal degrees. A set $A \subseteq \mathbb{N}$ is *cototal* if it is enumeration reducible to its complement, $\overline{A}$. The *skip* of A is the uniform upper bound of the complements of all sets enumeration reducible to A. These are closely connected: A has cototal degree if and only if it is enumeration reducible to its skip. We study cototality and related properties, using the skip operator as a tool in our investigation. We give many examples of classes of enumeration degrees that either guarantee or prohibit cototality. Our study of cototality is motivated by two examples of cototal sets that were pointed out to us by Jeandel [Jea15]. He shows that the set of non-identity words in a finitely generated simple group is cototal. Jeandel also gives an example from symbolic dynamics: The set of words that appear in a minimal subshift is cototal.

The complement of a graph of a total function is cototal and these degrees that contain such set we call graph-cototal. An enumeration degree is weakly-cototal if it contains a set A such that $\overline{A}$ has total enumeration degree. We have

$$\text{graph-cototal} \implies \text{cototal} \implies \text{weakly cototal}.$$

We show that these three properties are distinct. The harder separation is to construct a cototal degree that is not graph-cototal, where we use an infinite-injury argument of $\mathbf{0}'''$ relative to $\mathbf{0}'$. Case [Cas69] is conjecturing that, in our terms, if $\overline{A}$ has weakly cototal degree, then it has total degree. Gutteridge [Gut71, Chapter II] disproved this conjecture by constructing a quasiminimal graph-cototal degree. In particular, quasiminimal degrees are nontotal. Sanchis [San78], apparently unaware of Case's conjecture, gave an explicit construction of a cototal set that is not total. Sorbi [Sor88] constructed also a quasiminimal cototal degree. The name "cototal" was essentially first used, in an abstract of Pankratov from 2000 [Pan00]. The graph-cototal sets and degrees are further studied by Solon, Pankratov's advisor. In [Sol06], he used "co-total" to refer to what we call "graph-cototal".

We explain Jeandel's examples in more detail, and we give several other examples of cototal sets and degrees. We show that every Σ^0_2-set is cototal, in fact, graph-cototal. We show that the complement of a maximal independent subset of a computable graph is cototal, and that every cototal degree contains the complement of a maximal independent subset of $\omega^{<\omega}$. Ethan McCarthy [McC18] proves that the same is true of complements of maximal antichains in $\omega^{<\omega}$. We show that joins of nontrivial K-pairs are cototal. A pairs of sets $\{A, B\}$ form a $\mathcal{K}$*-pair* if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. A $\mathcal{K}$-pair is *nontrivial* if neither of its components is c.e. $\mathcal{K}$-pairs are introduced by Kalimullin [Kal03]. He shows that they are first-order definable in the structure of the enumeration degrees and used them to give a first-order definition of the enumeration jump. Cai, Ganchev, Lempp, Miller, and M. Soskova [CGL+16] used $\mathcal{K}$-pairs to define the class of total enumeration degrees. The structure of continuos degrees is introduced by Miller [Mil04] in order to capture the complexity of elements of computable metric spaces, such as $\mathcal{C}[0,1]$ and $[0,1]^\omega$, and is motivated by a question of Pour-El and Lempp from computable analysis. We show that the natural embedding of the continuous degrees into the enumeration degrees maps into the cototal degrees. Finally, we note that Harris [Har10] proved that sets with a good approximation have cototal degree.

Cototality is closely related to the other main subject: the skip operator. We define the *skip* of $A \subseteq \mathbb{N}$ to be $A^\lozenge = \overline{K_A}$. In fact a set A has cototal degree if and only if $A \leq_e A^\lozenge$. In some ways, the skip is analogous to the jump operator in the Turing degrees. For example, a standard diagonalization argument shows that $A^\lozenge \nleq_e A$. We restate the well-known fact that $A \leq_e B$ if and only if $A^\lozenge \leq_1 B^\lozenge$, mirroring the jump in the Turing degrees. We prove a skip inversion theorem, as analogues of Friedberg jump inversion theorem. The biggest difference between the skip and the Turing jump is that it is not always the case that $A \leq_e A^\lozenge$ (because not all enumeration degrees are cototal). We also study the skip operator for its own sake, noting that it has many of the nice properties of the Turing jump, even though the skip of A is not always above A. In fact, there is a set that is its own double skip. We investigate the properties of the skip operator for the class of enumeration degrees of 1-generic sets and skips of nontrivial K-pairs.

We have some open questions arising from this investigation. The main problem is: which cototality notions are first-order definable in the enumeration degrees? Is the skip first-order definable in the enumeration degrees? Kalimullin [Kal03] showed that the enumeration jump is first-order definable. Note that a positive answer to the second question would imply, that the cototal degrees are definable. Another open question: Is there a continuous enumeration degree that is not graph-cototal?

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Chapter 2

Preliminaries

2.1 Turing reducibility

The concept of Turing reducibility goes back to Turing [Tur37, Tur39]. Turing wanted to formally capture the notion of an algorithmically computable function. He developed his Turing machines, a mathematical abstract system that describes a class of functions, corresponding to the intuitive notion of algorithmically computable. In today's language, we would say that a function $f : \mathbb{N} \to \mathbb{N}$ is *partial computable* if there is a computer program that on input n halts and outputs $f(n)$, or does not halt, if $f(n)$ is not definable. A partial computable function is *computable* if it stops on every input.

An important aspect of the Turing machines is that they can be enumerated: we denote by φ_e or simple by $\{e\}$ the function computed by the e-th Turing machine. It is now easy to design a function that is not in this list using diagonalization. Let $f(e) = 0$ if $\varphi_e(e)$ does not halt and $f(e) = \varphi_e(e) + 1$ if $\varphi_e(e)$ halts. The function f, is not computable by any Turing machine. We call the set $K = \{e \mid \varphi_e(e) \text{ halts}\}$ the halting set (K from Kleene).

We write $\varphi_e(n)\downarrow$ to mean that this computation converges, that is, that it halts after a finite number of steps; and we write $\varphi_e(n)\uparrow$ to mean that it diverges, i.e., it never returns an answer. Computers, and Turing machines, run on a step-by-step basis. We use $\varphi_{e,s}(n)$ to denote the output of $\varphi_e(n)$ after s steps of computation. Notice that, given e, s, n, we can decide whether $\varphi_{e,s}(n)$ converges or not, computably: All we have to do is run $\varphi(n)$ for s steps. If f and g are partial functions, we write $f(n) = g(n)$ to mean that either both $f(n)$ and $g(n)$ are undefined, or both are defined and have the same value. Sometimes, when it is important that the functions could be partial, we use the notation $f(n) \simeq g(n)$. We say that $f \subseteq g$ if $(\forall n)(f(n)\downarrow \Rightarrow g(n)\downarrow \ \& \ f(n) = g(n))$. We write $f = g$ if $f \subseteq g \ \& \ g \subseteq f$, i.e. $f(n) \simeq g(n)$, for all n.

We identify subsets of $\mathbb{N}$ with their characteristic functions, i.e. for a set $A \subseteq \mathbb{N}$, $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \notin A$, and we will move from one viewpoint to the other without even mentioning it. For instance, a set A is said to be *computable* if its characteristic function is. An enumeration of a set A is an onto function $g : \mathbb{N} \to A$. A set A is *computably enumerable (c.e.)* if it has an enumeration that is computable or it is empty. Equivalently, a set is computably enumerable if it is the domain of a partial computable function. We denote $W_e = \{n \mid \varphi_e(n)\downarrow\}$ and $W_{e,s} = \{n \mid \varphi_{e,s}(n)\downarrow\}$.

In 1939 Turing extended his model of computability by Turing machine to allow for questions to an oracle, i.e. the Turing machine is allowed to use the function f as a primitive function during its computation; that is, the program can ask questions about the value of $f(n)$ for different n's and can use the answers to make decisions while the program is running. The function f is called the oracle of this computation. For a partial function $f : \mathbb{N} \to \mathbb{N}$ we define φ_e^f to be the function computed by the e-th Turing machine using as oracle the function f. We shall assume that if during a computation, the oracle f is called with an argument outside its domain, then the computation is unsuccessfully. For $B \subseteq \mathbb{N}$ we define φ_e^B to denote the function computed by the e-th Turing machine using as oracle the set $B \subseteq \mathbb{N}$, and actually we mean $\varphi_e^{\chi_B}$, where χ_B is the characteristic function of B.

Definition 2.1.1. A partial function f is *Turing reducible to* a partial function g (denoted $f \leq_T g$) if $f = \varphi_e^g$ for some e. We say that a set of natural numbers A is *computable from or Turing reducible to* a set of natural numbers B (denoted $A \leq_T B$) if and only if the characteristic function of the set A is φ_e^B for some natural number e.

If $x \in A = dom(\varphi_e^B) = W_e^B$ then $\varphi_e^B(x)$ halts for finitely many steps. Then there is a finite part (subfunction) $\tau \subseteq \chi_B$ such that $x \in W_e^\tau$, i.e. the questions to the oracle are finitely many. So, $x \in W_e^B \iff \exists$ finite $\tau \subseteq \chi_B(x \in W_e^\tau)$.

The relation $\leq_T$ is a preorder on the subsets of the natural numbers and induces an equivalence relation: $A \equiv_T B$ if and only if $A \leq_T B$ and $B \leq_T A$. The equivalence class of a set A under this relation is the Turing degree of A, denoted by $d_T(A)$. The Turing degrees are ordered by $d_T(A) \leq d_T(B)$ if and only if $A \leq_T B$. The least upper bound of two degrees $d_T(A) \vee d_T(B)$ is

$d_T(A \oplus B)$, where $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$ is the disjoint union of A and B, also known as join of A and B. The set $\mathbf{0}$ of all computable sets is the smallest degree. Finally relativizing the halting problem to any set A, we have $K^A = \{e \mid \varphi_e^A(e)\downarrow\}$, denoted by A'. The set K^A we call *the jump* of A and induces over degree structure *the jump operation* which maps a degree $\mathbf{a}$ to a degree $\mathbf{a}'$, such that $\mathbf{a} < \mathbf{a}'$ (see below). Thus the structure of the Turing degrees $\mathcal{D}_T$ is an upper semi-lattice with jump operation and minimal element.

Post and Kleene [KP54] established basic algebraic facts about the structure of the Turing degrees: it is an uncountable upper semi-lattice with least element and jump operation. They showed that every countable partial ordering can be embedded in the Turing degrees. Their successors, including Shoenfield, Spector, Sacks, Jockusch, Posner and many others, developed more sophisticated methods and showed further structural properties, for example the existence of minimal elements in the structure. The structure of the Turing degrees was revealed as mathematically non-trivial, rich in ideas and results. The next generation of researchers had sufficient tools to tackle problems related to first order definability in the structure. The general question is which interesting relations on $\mathcal{D}_T$ are actually definable in terms of relative computability alone. The most notable result in this direction is by Slaman and Shore [SS99]: they showed that the jump operation is first order definable in $\mathcal{D}_T$. Their solution relies on a methodology introduced by Slaman and Woodin [SW86] to analyze the automorphism group of $\mathcal{D}_T$.

All of the following properties could be found in [Rog67b, Soa87, Odi99, Coo04].

A stronger reducibility is the many-one reducibility (m-reducibility), which gives a very natural way of comparing the computability of different B— possibly incomputable B— sets of natural numbers A and B.

The set A is *many-one reducible* (m-reducible) to B ($A \leq_m B$) if there is a computable function h with the property $(\forall n)(n \in A \iff h(n) \in B)$. Let $A \leq_1 B \iff A \leq_m B$ by an one to one computable function. It is clear that if B is computable (c.e) and $A \leq_m B$ then A is computable (c.e.). Using S_n^m theorem we easily see that A is c.e. iff $A \leq_m K$. We call such sets as K complete sets for the c.e. sets.

The set A is *computably enumerable (c.e.) in* B iff for some e $A = dom(\varphi_e^B) = W_e^B$.

One can easily proves from the definitions the following properties:

1. $A \leq_T B \Rightarrow A$ is c.e. in B.

2. A is c.e. in B and $B \leq_T C \Rightarrow A$ is c.e. in C.

Theorem 2.1.2 (Post)**.** $A \leq_T B \iff A$ is c.e. in B and $\overline{A}$ is c.e. in B.

Notice that the relation "c.e. in" is not transitive. Since $\overline{K}$ is c.e. in K and K is c.e. in $\varnothing$, (K is c.e.), if we assume the transitivity of "c.e. in" then $\overline{K}$ is c.e. in $\varnothing \leq_T \mathbb{N}$. Thus $\overline{K}$ will be c.e., a contradiction.

The Turing jump $A' = K^A$ of a set A has the following properties.

Proposition 2.1.3. 1. K^A is c.e. in A.

2. Using S_n^m theorem we can prove that if B is c.e. in A then $B \leq_m K^A$.

3. $A <_T A'$, since $\overline{K^A}$ is not c.e. in A.

Here, $A <_T A'$ means that $A \leq_T A'$ & $A \not\equiv_T A'$.

Proposition 2.1.4. $A \leq_T B \iff A' \leq_m B'$.

Proof. ($\Rightarrow$) Let $A \leq_T B$. We have A' is c.e. in A and then A' is c.e. in B. Thus $A' \leq_m B'$ (by Proposition 2.1.3).

($\Leftarrow$) Let $A' \leq_m B'$. We have A is c.e. in $A \Rightarrow A \leq_m A' \leq_m B'$ and $\overline{A}$ is c.e. in $A \Rightarrow \overline{A} \leq_m A' \leq_m B'$. Then $A \leq_m B', \overline{A} \leq_m B'$. But by Proposition 2.1.3 B' is c.e. in B and so A is c.e. in B, and $\overline{A}$ is c.e. in B. By Post Theorem 2.1.2, $A \leq_T B$. ◻

Actually more stronger is true: $A \leq_T B \iff A' \leq_1 B'$.

Corollary 2.1.5 (Monotonicity of the jump)**.** $A \leq_T B \Rightarrow A' \leq_T B'$.

Definition 2.1.6. $(d_T(A))' = d_T(A')$.

Since $A <_T K^A$, then $d_T(A) < d_T(A')$.

The computably enumerable sets, and correspondingly degrees, appear in many other branches of mathematics. The solution to Hilbert's tenth problem by Davis, Putnam, Robinson and Matiyasevich [Mat93] essentially relies on the existence of a computably enumerable set that is not computable. Friedberg and Muchnik developed a powerful method used to construct c.e. degrees with specific properties, the priority method. We will use the priority method in Chapter 7.

A major theme in degree theory is the study of the local structure $\mathcal{R}$ - the computably enumerable Turing degrees, degrees that contain a c.e. set, and the local structure $\mathcal{D}_T(\leq \mathbf{0}') = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}'\}$. A recent result of Sleman and Soskova [SS18] shows a relationship between the local structure $\mathcal{D}_T(\leq \mathbf{0}') = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}'\}$ and first order arithmetic, similar to the one proved by Slaman and Woodin [SW05] for the global structure $\mathcal{D}_T$ and second order arithmetic.

The jump hierarchy, also known as the high/low hierarchy, was introduced independently by Cooper (see [Coo04]) and Soare [Soa74]. The jump classes are: $H_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \;\&\; \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}$ of $high_n$ degrees and $L_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \;\&\; \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}$ of low_n degrees. Nies, Shore and Slaman [NSS96] obtained the first order definition of the jump classes H_n (L_{n+1}) in $\mathcal{R}$, for every natural number $n \geq 1$. Later on, Shore [Sho14] showed for the local structure $\mathcal{D}_T(\leq \mathbf{0}')$ that the classes H_n and L_{n+1} for every natural number $n \geq 1$ are definable in there as well. One class of degrees which has managed to elude every attempt at definability in both local structures is that of the low_1 degrees, L_1, the degrees whose jump is the least possible degree $\mathbf{0}_T{}'$. The definability in the degree structures is in a close relationship with the automorphisms on the structures, since the definable sets are preserved under automorphism.

2.2 Genericity and forcing

Forcing and generic sets are useful tools all over computability theory. The first forcing-style argument in computability theory can be traced back to the Kleene-Post construction of two incomparable degrees [KP54], published a decade before the invention of forcing. In this section, we give an introduction to the forcing method in computable structure theory. We consider 1-generics, which have relatively low computational complexity. The notion of forcing was introduced by Cohen to prove that the continuum hypothesis does not follow from the ZFC axioms of set theory. Soon after, forcing became one of the main tools in set theory in order to prove independence results of all kinds. This implies that if a generic object satisfies a particular property, it must belong to a class where most objects have that property, and hence there is a clear reason why it has that the property. Our forcing arguments will essentially have that form: If a generic presentation has a certain computational property, then there must be a structural reason for it.

Definition 2.2.1. Every finite mapping $\tau : [0; n-1] \longrightarrow \mathbb{N}$ we call *a finite part*. We denote by $|\tau| = n$ the length of the interval, where τ is defined. For any $a \in \mathbb{N}$ and $\tau : [0; n-1] \longrightarrow \mathbb{N}$, let $\lambda x.(\tau * a)(x)$ be the finite part:

$$(\tau * a)(x) \simeq (\tau * n \to a)(x) \simeq \begin{cases} \tau(x) & \text{if } 0 \leq x < n, \\ a & \text{if } x = n. \end{cases}$$

If A is a set of natural numbers, we write $\tau \subseteq A$ instead of $\tau \subseteq \chi_A$, i.e. τ is a subfunction of the characteristic function χ_A of A.

We denote the finite parts with the Greek letters: $\sigma, \delta, \tau, \rho \ldots$. Let remain, that $\sigma \subseteq \rho$ if $(\forall x)\ (\sigma(x)\downarrow \Rightarrow \rho(x)\downarrow\ \&\ \sigma(x) = \rho(x))$.

Definition 2.2.2. The set G is *1-generic*, if for every c.e. set S of finite parts:

$$(\exists \sigma \subseteq G) \underbrace{(\sigma \in S \vee (\forall \rho \supseteq \sigma)(\rho \notin S))}_{\sigma \text{ decides } S}.$$

We will call such sets *generic* sets for short. For n-generic sets the difference is that the set of finite parts S is Σ^0_n, not only c.e.(Σ^0_1).

A set of finite parts S is called *dense in* G, if $(\forall \sigma \subseteq G)(\exists \rho \in S)(\sigma \subseteq \rho)$. Equivalently, G is generic, if whenever S is dense in G, then G meets S, i.e. $(\exists \sigma \subseteq G)(\sigma \in S)$.

Let $\mathcal{S}$ be the set of all finite parts and $S_e = W_e \cap \mathcal{S}$, $e \in \mathbb{N}$. There is a total computable function h, such that $S_e = W_{h(e)}$ for every e.

We will show for illustration how to construct a generic set.

Proposition 2.2.3. There is a generic set below $\varnothing'$.

Proof. We construct finite parts σ_s, by stages, which will approximate χ_G, $\sigma_s \subseteq \sigma_{s+1} \subseteq G$.

We start with $\sigma_0 = \varnothing$.

On stage $s+1 = e$ we define σ_{s+1} so that it decides the eth c.e. set W_e, i.e. we ask if there is an extension of σ_s in S_e. If there is, set σ_{s+1} to be the least one. If there is not then we let $\sigma_{s+1} = \sigma_s$.

Let $\chi_G = \bigcup_s \sigma_s$. The construction assures that G is generic and one can see that χ_G is a total function, since for every n the set of finite parts $\{\tau \mid |\tau| \geq n\}$ is c.e., i.e. W_e for some e. The only step in the construction that is not computable is checking whether there exists an extension of σ_s in S_e or not. This is a question that $\varnothing'$ can answer, and hence the whole construction is computable in $\varnothing'$. □

It is easy to see that if G is generic then G is not a finite set. Otherwise if $x \in G \Rightarrow x \leq n$, for some n, then the set $S = \{\sigma \mid (\exists m > n)(\sigma(m) \simeq 1)\}$ is c.e. Since G is generic then $(\exists \sigma \subseteq G)(\sigma \in S \vee (\forall \rho \supseteq \sigma)(\rho \notin S))$. Since G is finite, $\sigma \notin S$. Then $(\forall \rho \supseteq \sigma)(\forall m > n)(\rho(m) \not\simeq 1)$, which is impossible. Hence G is infinite. On the other side one can see in a similar way that if G is generic then every c.e. $V \subseteq G$ is finite, and hence every generic is not c.e.

Every c.e. $V \leq_T G$ is computable. Supose that $V \leq_T G$, V is c.e. and G is generic. We know $\overline{V} \leq_T V \leq_T G$, and hence there is an e, such that $\overline{V} = dom(\{\varphi_e^G\})$. Let $S = \{\sigma \mid (\exists x \in V)(\varphi_e^\sigma(x)\downarrow)\}$. Since S is c.e. and G is generic there is $\sigma \subseteq G$, such that $\sigma \in S \vee (\forall \rho \supseteq \sigma)(\rho \notin S)$. If $\sigma \in S$, then $(\exists x \in V)(\varphi_e^G(x)\downarrow)$. Then $x \in \overline{V}$, a contradiction. So, $(\forall \rho \supseteq \sigma)(\forall x \in V)(\varphi_e^\rho(x)\uparrow)$. If $x \in \overline{V}$ then $\varphi_e^G(x)\downarrow$. By the compactness of the computation there is $\rho \subseteq G$, $\varphi_e^\rho(x)\downarrow$. We can suppose that $\rho \supseteq \sigma$. Hence

$$x \in \overline{V} \iff (\exists \rho \supseteq \sigma)(\varphi_e^\rho(x))\downarrow,$$

i.e. $\overline{V}$ is c.e. But V is c.e., therefore V is computable. So every c.e. $V \leq_T G$ is computable.

We can relativize the notion of genericity. Let $X \subseteq \mathbb{N}$. Say that G is 1-generic relative to X (X-1-generic) if every c.e. in X set of finite parts is decided by an initial segment of G. The next lemma from [Mon] implies that the only sets that are c.e. in all generic sets are the ones that are already c.e.

Lemma 2.2.4. Suppose that G is X-1-generic. Then X is not c.e. in G, unless X is c.e. already.

In particular, we get that if G is X-1-generic, then G computes X if and only if X is computable. Thus, if G is X-1-generic, G and X form a *minimal pair*, i.e., there is no non-computable set computable from both. This is because if $Y \leq_T X$ then G is Y−1-generic too, so if also $Y \leq_T G$, then Y must be computable.

We will explain the basic relations $\models$ and $\Vdash$. We will use these relations in Chapter 3. And here we will see some basic properties including the Theorem 2.2.6.

The set G *models* the formula $F_e(x)$:

$$G \models F_e(x) \iff \{e\}^G(x)\downarrow \iff x \in W_e^G.$$

The finite part σ *forces* formula $F_e(x)$:

$$\sigma \Vdash F_e(x) \iff \{e\}^\sigma(x)\downarrow.$$

Here are some properties of these relations, following from the definitions.

1. $\sigma \subseteq G \,\&\, \sigma \Vdash F_e(x) \Rightarrow G \vDash F_e(x)$.
2. $\sigma \subseteq \rho \,\&\, \sigma \Vdash (\neg)F_e(x) \Rightarrow \rho \Vdash (\neg)F_e(x)$.
3. $G \vDash F_e(x) \Leftrightarrow (\exists \sigma \subseteq G)(\sigma \Vdash F_e(x))$.

Lemma 2.2.5. The set $\{(\sigma, e, x) \mid \sigma \Vdash F_e(x)\}$ is c.e.

$$G \vDash \neg F_e(x) \iff G \nvDash F_e(x) \iff \neg\{e\}^G(x)\downarrow.$$
$$\sigma \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \sigma)(\rho \nVdash F_e(x)).$$

Theorem 2.2.6. Let G be a generic set. Then

$$G \vDash \neg F_e(x) \iff (\exists \sigma \subseteq G)(\sigma \Vdash \neg F_e(x)).$$

Proof. ($\Leftarrow$) Let $\sigma \subseteq G \,\&\, \sigma \Vdash \neg F_e(x)$. Suppose that $G \vDash F_e(x)$. Then $(\exists \rho \subseteq G)(\rho \Vdash F_e(x))$. Let $\tau = \sigma \cup \rho$. Then $\tau \supseteq \rho$ and hence $\tau \Vdash F_e(x)$. But $\tau \supseteq \sigma$, so $\sigma \Vdash \neg F_e(x)$ and then $\tau \nVdash F_e(x)$ - a contradiction.

($\Rightarrow$) Let $G \vDash \neg F_e(x)$. We search for $\sigma \subseteq G$, $\sigma \Vdash \neg F_e(x)$, i.e. no extension of σ could forces $F_e(x)$. But G is generic. Suppose that $(\forall \sigma \subseteq G)(\sigma \nVdash \neg F_e(x))$. Hence $(\forall \sigma \subseteq G)(\exists \rho \supseteq \sigma)(\rho \Vdash F_e(x))$. Set $S_{e,x} = \{\rho \mid \rho \Vdash F_e(x)\}$. $S_{e,x}$ is c.e. and dense in G, then there is $\sigma \subseteq G, \sigma \in S_{e,x}$, i.e. $\sigma \Vdash F_e(x)$. Then $G \vDash F_e(x)$, a contradiction. So $(\exists \sigma \subseteq G)(\sigma \Vdash \neg F_e(x))$. □

Corollary 2.2.7 (Truth lemma)**.** If G is generic, then

$$G \vDash (\neg)F_e(x) \iff (\exists \sigma \subseteq G)(\sigma \Vdash (\neg)F_e(x)).$$

Notice that $\{(\sigma, e, x) \mid \sigma \Vdash \neg F_e(x)\} \le_T \varnothing'$.

Corollary 2.2.8. For every generic G we have $G' \equiv_T G \oplus \varnothing'$.

Proof. ($\Leftarrow$) G' is a upper bound of $\varnothing'$ and G. Hence $\varnothing' \oplus G \le_T G'$.

($\Rightarrow$) $G' = K^G = \{x \mid x \in W_x^G\}$ is c.e. in G. Then there is e, such that $x \in K^G \iff \{e\}^G(x)\downarrow \iff G \vDash F_e(x) \iff (\exists \sigma \subseteq G)(\sigma \Vdash F_e(x))$. Thus, K^G is c.e. in $G \oplus \varnothing'$. G is generic, then $x \in \overline{K^G} \iff \{e\}^G(x)\uparrow \iff G \nvDash F_e(x) \iff (\exists \sigma \subseteq G)(\sigma \Vdash \neg F_e(x))$. So, $\overline{K^G}$ is c.e. in $G \oplus \varnothing'$. Thus, $K^G = G' \le_T G \oplus \varnothing'$ by Post theorem. □

In Chapter 3. we will prove the Friedberg's jump inversion theorem for the structures. Here is the original theorem.

Theorem 2.2.9 (Friedberg's jump Inversion theorem)**.** [Fri57] Let $\varnothing' \leq_T B$. There exists a generic G, such that $G' \equiv_T B$, and hence $B \equiv_T G' \equiv_T G \oplus \varnothing'$.

Proof. We will construct a generic set $G \leq_T B$ by stages. Then since G will be generic, by Corollary 2.2.8, $G' \equiv_T G \oplus \varnothing'$, and hence $G' \leq_T B$. To get the other direction $B \leq_T G'$ we will code B in G. On each stage s we will define a finite part σ_s of χ_G, so that $\sigma_s \subseteq \sigma_{s+1}$. And at the end $\chi_G = \cup_s \sigma_s$. Denote by $S_{\langle e,x\rangle} = \{\rho \mid \rho \Vdash F_e(x)\} = \{\rho \mid x \in W_e^\rho\}$.

We start with $\sigma_0 = \varnothing$. Let σ_s has been constructed. On stage s we ask: "Is it true that: $(\exists \rho \supseteq \sigma_s)(\rho \in S_s)$?". Since the set $V = \{(\tau, t) \mid (\exists \rho \supseteq \tau)(\rho \in S_t)\}$ is c.e., we have $V \leq_T K = \varnothing'$. If the answer is **yes**, set σ_s' will be the minimal (with a minimal code) such ρ, if the answer is **no**, then $\sigma_s' = \sigma_s$. Thus assures that G is generic. Set $\sigma_{s+1} = \sigma_s' * \chi_B(s)$.

It is clear that $G \leq_T B$. Since $|\sigma_{s+1}| \geq s$, $s \in G \iff \sigma_{s+1}(s) = 1$. And $\sigma_{s+1} \leq_T B \oplus \varnothing' \leq_T B$.

G is generic, since σ_s' assures genericity with respect to S_s.

$B \leq_T G \oplus \varnothing'$, since we have $k \in B \iff \sigma_{k+1}(|\sigma_k'|) = 1$. We can construct B repeating the construction, changing $\chi_B(s)$ with $\chi_G(|\sigma_s'|)$. So, using oracle G and $\varnothing'$ we have $B \leq_T G \oplus \varnothing'$.

Thus G is generic and $G' \equiv_T B$. □

Corollary 2.2.10. There exists a generic $G \not\equiv_T \varnothing$ such that $G' \equiv_T \varnothing'$.

2.3 Enumeration reducibility

Enumeration reducibility was defined by Friedberg and Rogers [FR59] in the late 1950's to capture a notion of reducibility between sets in which only positive information about membership in the set is either used or computed. Actually, Uspenskiĭ, [Usp55] introduced in 1955 essentially the same concept. This notion turns out to be as natural as Turing reducibility in a number of settings, e.g., in group theory and computable model theory.

A set A is *enumeration reducible to* a set B if there is an effective uniform way, given by an *enumeration operator*, to obtain an enumeration of A given any enumeration of B. The enumeration operators are interesting in themselves, as they give the semantics of the type free λ-calculus in graph models, suggested by Plotkin [Plo72] in 1972. The interest in enumeration reducibility is also supported by the fact that the structure of the enumeration degrees contains the structure of the Turing degrees without being elementary equivalent to it. Contemporary defninability results [CGL+16, GS15, GS12, SS12] in the theory of the enumeration degrees show that the structure is useful for the study of the structure of Turing degrees.

Definition 2.3.1. Let A and B be sets of natural numbers. The set A is *enumeration reducible to* the set B, written $A \leq_e B$, if there is a c.e. set W_e, such that:

$$A = W(B) = \{x \mid (\exists D)[\langle x, D\rangle \in W_e \ \&\ D \subseteq B]\},$$

where D is a finite set coded in the standard way.

The definition above associates an effective operator on sets to every c.e. set W_e, the aforementioned enumeration operator. Let $\{\Gamma_e\}_{e\in\mathbb{N}}$ be an effective list of all enumeration operators.

Here are some examples which shows some basic properties of the enumeration reducibility.

1. If A is c.e. then $A \leq_e B$ via the c.e. set $W = \{\,\langle x, \varnothing\rangle \mid x \in A\,\}$.

2. If f is computable function for $A \leq_m B$, i.e. $A = f^{-1}(B)$, then $A \leq_e B$ via the c.e. set $W = \{\,\langle x, \{f(x)\}\rangle \mid x \in \mathbb{N}\,\}$.

Let φ and ψ are partial functions. Let $\varphi \leq_e \psi \iff G_\varphi \leq_e G_\psi$.

Proposition 2.3.2. $\varphi \leq_T \psi \ \Rightarrow \varphi \leq_e \psi$.

Just like Turing reducibility, enumeration reducibility is a pre-order on the natural numbers, it induces an equivalence relation $\equiv_e$ and a degree structure $\mathcal{D}_e$. The structure of the enumeration degrees is also an upper semi-lattice. The set $A \oplus B$ is a least upper bound of A and B with respect to $\leq_e$. Two sets A and B are *enumeration equivalent* ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$. The equivalence class of a set A under this relation is its *enumeration degree* $d_e(A)$. The set $\mathcal{D}_e$ consisting of all enumeration degrees, together with the naturally induced partial order and least upper bound operation is the *upper semi-lattice of the enumeration degrees*. It has a least element $\mathbf{0}_e$ consisting of all computably enumerable sets. For an introduction to the enumeration degrees the reader might consult Cooper [Coo90].

There is a strong relationship between the relations that we defined: $A \leq_T B$ if and only if $A \oplus \overline{A}$ is c.e. in B if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$. The set $A \oplus \overline{A}$ codes in a positive way the positive and negative information about a set A. This suggests a relationship between Turing reducibility, enumeration reducibility and the relation "c.e. in" formally expressed as follows.

Proposition 2.3.3. Let A and B be sets of natural numbers.

1. $A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.
2. A is c.e. in B if and only if $A \leq_e B \oplus \overline{B}$.

This gives the natural embedding ι of the Turing degrees into the enumeration degrees ([Med55, Myh61]):

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}).$$

A set A is called *total* if and only if $A \equiv_e A \oplus \overline{A}$. Examples of total sets are the graphs of total functions. An enumeration degree is *total* if it contains a total set. The enumeration degrees in the range of ι coincide with the total enumeration degrees.

The following theorem by Selman shows that the total enumeration degrees play an important role in the structure: an enumeration degree can be characterized by the set of total degrees above it.

Theorem 2.3.4. [Sel71] For any $A, B \subseteq \mathbb{N}$ the following are equivalent:

1. $A \leq_e B$;
2. $\{X \mid B \text{ is c.e. in } X\} \subseteq \{X \mid A \text{ is c.e. in } X\}$;
3. $\{\mathbf{x} \in \mathcal{D}_e \mid \mathbf{x} \text{ is total } \& \ d_e(B) \leq \mathbf{x}\} \subseteq \{\mathbf{x} \in \mathcal{D}_e \mid \mathbf{x} \text{ is total } \& \ d_e(A) \leq \mathbf{x}\}$.

Finally, we give the definition of a jump operator for the enumeration degrees, originally due to Cooper and studied by McEvoy [Coo84, McE85].

Definition 2.3.5. Let $K_A = \{\langle e, x\rangle \mid x \in \Gamma_e(A)\}$. The set

$$A'_e = K_A \oplus \overline{K_A}$$

is called the enumeration jump of A and $d_e(A)' = d_e(A'_e)$.

Note that $K_A = \oplus_{e \in \mathbb{N}} \Gamma_e(A) = \{\langle e, x\rangle \mid x \in \Gamma_e(A)\}$. It is clear that $K_A \equiv_e A$. Denote by $A^+ = A \oplus \overline{A}$. The enumeration jump is monotone and agrees with the Turing jump in the following sense: $(A')^+ \equiv_e (A^+)'_e$, and $A' \equiv_T (A^+)'_e$ [Coo84, McE85].

We will use Soskov's jump inversion theorem for the enumeration jump:

Theorem 2.3.6. [Sos00] For every enumeration degree $\mathbf{a}$ there exists a total enumeration degree $\mathbf{b}$, such that $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a}' = \mathbf{b}'$.

The pioneering work on the enumeration degrees dates back to Case [Cas71] and Medvedev [Med55]. In particular, Case shows that $\mathcal{D}_e$ is not a lattice as a consequence of the exact pair theorem and Medvedev proves the existence of quasi-minimal degrees: a degree is *quasi-minimal* if it bounds no nonzero total enumeration degree. Cooper laid the foundations of the study of the enumeration degrees in his survey paper [Coo84] from 1990. He established many important algebraic properties of the global and local structure, such as the lack of minimal elements, which shows that the theory of the enumeration degrees is different from the theory of the Turing degrees. McEvoy [McE85], a student of Cooper, defined the enumeration jump operation, which maps an enumeration degree $\mathbf{a}$ to a total enumeration degree $\mathbf{a}'$, such that $\mathbf{a} <_e \mathbf{a}'$. McEvoy then showed that the embedding ι preserves the jump operation. Kalimullin obtained a definable class of pairs of enumeration degrees which came to be known as Kalimullin pairs, or $\mathcal{K}$-pairs. Kalimullin [Kal03] showed that the enumeration jump is definable in $\mathcal{D}_e$. Ganchev and M. Soskova [GS15] give an alternative proof of the definability of the enumeration jump. Their proof is an instance of a more general phenomenon: they introduce the notion of a maximal $\mathcal{K}$-pair and conjecture that a nonzero enumeration degree is total if and only if it is the least upper bound of the elements of a maximal $\mathcal{K}$-pair. They show that if this conjecture is true than this would imply the first order definability the image (under the embedding of $\mathcal{D}_T$ in $\mathcal{D}_e$) of the relation on Turing degrees "c.e. in". In [GS12] they show that the first order theory of true arithmetic can be interpreted in $\mathcal{D}_e(\leq \mathbf{0'}_e)$, using coding methods based on $\mathcal{K}$-pairs, settling an open problem from Cooper's 1990 survey paper. In [GS15] they show further that the class of low enumeration degrees is first order definable. More importantly, they show that their conjecture for the first order definability of the total Σ^0_2 degrees in $\mathcal{D}_e(\leq \mathbf{0'}_e)$ using maximal $\mathcal{K}$-pairs is true for the local structure $\mathcal{D}_e(\leq \mathbf{0'}_e)$, thus settling the local version of Rogers' 1967 question. The full answer to Rogers' 1967 question is finally obtained through the collaboration of Cai, Ganchev, Lempp, Miller and M. Soskova, confirming Ganchev and Soskova's conjecture.

Theorem 2.3.7 (Cai, Ganchev, Lempp, Miller, M. Soskova)**.** [CGL+16] The total enumeration degrees are first order definable in $\mathcal{D}_e$. A nonzero enumeration degree is total if and only if it is the least upper bound of the members of a maximal Kalimullin pair.

Recent work [GS18] of Ganchev and M. Soskova shows that all classes of high enumeration degrees $H_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}'_e \;\&\; \mathbf{a}^{(n)} = \mathbf{0}_e^{(n+1)}\}$ and low enumeration degrees $L_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}'_e \;\&\; \mathbf{a}^{(n)} = \mathbf{0}_e^{(n)}\}$ are definable in $\mathcal{D}_e$, for each $n \geq 1$.

The relationship between enumeration degrees and abstract models of computability inspires a new direction in the field of computable structure theory. You could see more in our expository paper with M. Soskova [SS17].

In the last chapter we will show our latest results on a subclass of the enumeration degrees — the cototal degrees. Call a set $A \subseteq \mathbb{N}$ *cototal* if $A \leq_e \overline{A}$ and call an enumeration degree *cototal* if it contains a cototal set. We will introduce an analog of jump operation - the skip operator and we will investigate its properties.

We will investigate the skip for the class of enumeration degrees of 1-generic sets, studied by Copestake [Cop88]. We define a relativized form of 1-genericity, suitable for the context of the enumeration degrees. We use the notation "relative to $\langle X \rangle$" to denote "relative to the enumeration degree of X" (not of $X \oplus \overline{X}$ as in Turing degrees).

Definition 2.3.8. Let G and X be sets of natural numbers. G is 1-*generic relative to* $\langle X \rangle$ if and only if for every set of finite parts S such that $S \leq_{\mathrm{e}} X$:

$$(\exists \sigma \subseteq G)(\sigma \in S \vee (\forall \tau \geq \sigma)[\tau \notin S]).$$

If $X = \varnothing$, then we call G simply 1-*generic* and if $X = \overline{K}$, then G is 2-*generic*.

Note that G is 1-generic relative to X in the usual sense if and only if G is 1-generic relative to $\langle X \oplus \overline{X} \rangle$ in the sense of the definition above.

Definition 2.3.9. An enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total enumeration degree bounded by $\mathbf{a}$ is $\mathbf{0}_{\mathrm{e}}$.

McEvoy [McE85] proved that the enumeration jump restricted to the quasiminimal degrees has the same range as the unrestricted jump operator. Relativizing the notion of quasiminimality, we get the following two notions:

Definition 2.3.10. An enumeration degree $\mathbf{a}$ is a *quasiminimal cover* of an enumeration degree $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and there is no total enumeration degree $\mathbf{x}$ such that $\mathbf{b} < \mathbf{x} \leq \mathbf{a}$. The degree $\mathbf{a}$ is a *strong quasiminimal cover* of $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and every total enumeration degree $\mathbf{x}$ bounded by $\mathbf{a}$ is below $\mathbf{b}$.

The next proposition exhibits two important properties of generic enumeration degrees.

Proposition 2.3.11. Let G be 1-generic relative to $\langle X\rangle$.

1. $d_{\mathrm{e}}(G \oplus X)$ is a strong quasiminimal cover of $d_{\mathrm{e}}(X)$.
2. $\overline{G}$ is 1-generic relative to $\langle X\rangle$.

Proof. $X \leq_e G \oplus X$. To see that $G \nleq_e X$, note that G must be infinite and for every enumeration operator Γ the set $S = \{\sigma \mid (\exists n)[\sigma(n) = 0 \wedge n \in \Gamma(X)]\}$ is enumeration reducible to X. If $G \leq_e X$, then $G = \Gamma(X)$ for some enumeration operator Γ. Then $(\exists \sigma \subseteq G)(\sigma \in S \vee (\forall \tau \supseteq \sigma)[\tau \notin S])$. But $\sigma \notin S$ and hence $(\forall \tau \supseteq \sigma)[\tau \notin S]$, which is impossible.

Let Y be a set of natural numbers and assume that $Y \oplus \overline{Y} \leq_{\mathrm{e}} G \oplus X$ via the enumeration operator Γ. We will show that $Y \oplus \overline{Y} \leq_{\mathrm{e}} X$. Consider the set

$$Q = \{\sigma \mid (\exists x)(\{2x, 2x+1\} \subseteq \Gamma(\sigma \oplus X))\},$$

where we write $\sigma \oplus X$ to mean $\{n \mid \sigma(n) = 1\} \oplus X$. Note that Q is enumeration reducible to X and so, by our assumptions, G must avoid it, i.e., no $\sigma \in Q$ is an initial segment of G. Let $\sigma \subseteq G$ be a finite part with no extension in Q. Then $z \in Y \oplus \overline{Y}$ if and only if there is an extension $\tau \supseteq \sigma$ such that $z \in \Gamma(\tau \oplus X)$.

For the second part of this proposition, we introduce the following notation. If $\sigma \in 2^{<\omega}$ is a finite part, then let $\overline{\sigma}$ be the the finite part, obtained by inverting every bit of σ. For $W \subseteq 2^{<\omega}$, let $W^{-} = \{\overline{\sigma} \mid \sigma \in W\}$. Note that $\overline{\sigma} \subseteq G$ if and only if $\sigma \subseteq \overline{G}$. So if G meets W^{-} then $\overline{G}$ meets W, and if G avoids W^{-} then $\overline{G}$ avoids W. Finally, note that $W \leq_{\mathrm{e}} X$ implies that $W^{-} \leq_{\mathrm{e}} X$. □

2.4 Degree spectra

The Turing degree spectrum of a countable structure $\mathcal{A}$ provides a natural measure of the complexity of the isomorphism type of that structure. The spectrum of $\mathcal{A}$ is introduced by Richter [Ric81], as the set of those Turing degrees **a** such that for some copy $\mathcal{B}$ of $\mathcal{A}$ (that is, for some $\mathcal{B} \simeq \mathcal{A}$ with domain $\mathbb{N}$), the atomic diagram of $\mathcal{B}$ has Turing degree **a**.

Let $\mathcal{A} = (A, R_1, \ldots, R_k)$ be a countable relational structure. If in the language of the structure there are some functions symbols we represent them by their graphs. An enumeration of $\mathcal{A}$ is a total surjective mapping of $\mathbb{N}$ onto $|\mathcal{A}|$. Given an enumeration f of $\mathcal{A}$ and a subset of B of $|\mathcal{A}|^a$, let

$$f^{-1}(B) = \{\langle x_1,\dots,x_a\rangle \mid (f(x_1),\dots,f(x_a)) \in B\}.$$

Denote by $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=)$. By $D(\mathcal{A})$ we denote the atomic diagram of $\mathcal{A}$.

Definition 2.4.1. The *degree spectrum of* $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{d_T(f^{-1}(\mathcal{A})) \mid f \text{ is an enumeration of } \mathcal{A}\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathcal{A})$, then $\mathbf{a}$ is called *the degree of* $\mathcal{A}$.

We shall use the following two simple properties of the degree spectra. They are proved by Soskov in [Sos04] for enumeration degree spectra. Suppose that $\mathcal{A}$ is infinite and the domain of $\mathcal{A}$ is the set of the natural numbers.

Proposition 2.4.2. Let f be an arbitrary enumeration of $\mathcal{A}$. Then there exists an injective enumeration g of $\mathcal{A}$ such that $g^{-1}(\mathcal{A}) \leq_{\mathrm{T}} f^{-1}(\mathcal{A})$.

Proof. Let $E_f = \{\langle x,y\rangle : f(x) = f(y)\}$. Clearly $E_f \leq_T f^{-1}(\mathcal{A})$. Define the function h by means of primitive recursion as follows:

$$\begin{aligned} h(0) &\simeq 0 \\ h(n+1) &\simeq \mu z[(\forall k \leq n)(\langle h(k), z\rangle \notin E_f)]. \end{aligned}$$

Set $g(n) = f(h(n))$. Now one can easily check that g is bijective and $g^{-1}(\mathcal{A}) \oplus E_f \equiv_T f^{-1}(\mathcal{A})$. □

One noticeable difference with the standard definition of Turing degree spectra is that in the definition of the degree spectra, we use the surjective enumerations, instead of bijective enumerations. Consider the structure $\mathcal{A} = (\mathbb{N}; =)$ if we define the degree spectrum of $\mathcal{A}$ by taking into account only the bijective enumerations, then it will be equal to $\{\mathbf{0}\}$, while if we take all surjective enumerations, then $DS(\mathcal{A})$ will consist of all Turing degrees. Fortunately, this difference does not affect the notion of degree of a structure since by Proposition 2.4.2 for every enumeration f of $\mathcal{A}$ there exists a bijective enumeration g of $\mathcal{A}$ such that $g^{-1}(\mathcal{A}) \leq_T f^{-1}(\mathcal{A})$. On the other hand it allows us to show that the degree spectrum is always *closed upwards*, i.e. if $\mathbf{a} \in DS(\mathcal{A})$, and $\mathbf{a} \leq \mathbf{b}$ then $\mathbf{b} \in DS(\mathcal{A})$. This can be seen as follows: if g is an enumeration of $\mathcal{A}$ and F is a set such that $g^{-1}(\mathcal{A}) \leq_T F$ then we can define a new enumeration f of $\mathcal{A}$, which mimics g on the even numbers: $f(n/2) = g(n)$ and codes F on the odd numbers, by mapping all of them to one of two distinct members of $\mathcal{A}$ depending on membership in F.

Proposition 2.4.3. For every structure $\mathcal{A}$ the degree spectrum $\mathrm{DS}(\mathcal{A})$ is upwards closed.

Knight proved in [Kni86], that the degree spectrum using injective enumerations is closed upwards only in nontrivial structures (in a trivial structure there is a finite tuple such that every permutation of the domain fixing that tuple is an automorphism of $\mathcal{A}$).

For every computable ordinal α, following Knight [Kni86] we define *the α-th jump spectrum* $DS_\alpha(\mathcal{A})$ of a structure $\mathcal{A}$ to be the set of all αth jumps of the elements of the degree spectrum of $\mathcal{A}$. If $\mathbf{a}$ is the least element of $DS_\alpha(\mathcal{A})$, then $\mathbf{a}$ is called the *α-th jump degree* of $\mathcal{A}$. We will show in Chapter 3. that the first jump spectrum is always upwards closed.

It is very important if a structure has a degree (αth jump degree for some computable ordinal α) or not. If the degree is $\mathbf{0}$, then the structure has a simple computable presentation, i.e. with domain $\mathbb{N}$ and computable relations. In the different classes of structures the situation is different. Richter's [Ric81] proved that the Turing degree spectrum $DS(\mathcal{A})$ of a linear ordering has a degree then it is computable, i.e. this degree should be $\mathbf{0}$-the set of all computable sets. Knight [Kni86] extended Richter's result to show that the only possible first jump Turing degree of a linear ordering is $\mathbf{0}'$, so not every linear ordering has a first jump degree. Downey and Knight [DK92] proved next that for every computable ordinal α there exists a linear order $\mathcal{A}$ such that $\mathcal{A}$ has αth jump degree equal to $\mathbf{0}^{(\alpha)}$ but for all $\beta < \alpha$ there is no βth jump degree of $\mathcal{A}$. Slaman [Sla98] and independently Wehner [Weh98] gave an example of a structure $\mathcal{A}$ whose Turing degree spectrum consists of all nonzero Turing degrees, $DS(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{0} < \mathbf{a}\}$. We will give some very simple proofs of the last two results in Chapter 3. as an application of the jump Inversion theorem for structures.

The enumeration degree spectrum $DS_e(\mathcal{A})$ of a countable structure $\mathcal{A}$ is introduced by Soskov [Sos04] as the set of all enumeration degrees generated by the presentations (homomorphic copies in $\mathbb{N}$) of $\mathcal{A}$. It is also closed upwards with respect to total degrees, i.e. if $\mathbf{a} \in DS_e(\mathcal{A})$, $\mathbf{b}$ is a total e-degree and $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{b} \in DS_e(\mathcal{A})$.

Just like Turing reducibility can be expressed in terms of enumeration reducibility, the Turing degree spectrum of a structure $\mathcal{A}$ corresponds to the enumeration degree spectrum of a structure, denoted by $\mathcal{A}^+$, which codes in a positive way both the positive and negative facts about the predicates in $\mathcal{A}$. If $\mathcal{A} = (A, R_1, \dots, R_k)$ then let $\mathcal{A}^+ = (A, R_1, \dots, R_k, \neg R_1, \dots, \neg R_k)$. The image of the Turing degree spectrum of $\mathcal{A}$ under the natural embedding is exactly $DS_e(\mathcal{A}^+)$.

Note, that $DS_e(\mathcal{A}^+)$ consists only of total enumeration degrees. A structure $\mathcal{A}$ is called *total* if for every enumeration f of $\mathcal{A}$ the set $f^{-1}(\mathcal{A})$ is total. In general, if $\mathcal{A}$ is a total structure then $DS_e(\mathcal{A}) = \iota(DS(\mathcal{A}))$, so if $\mathcal{A}$ is a total structure then $\mathcal{A}$ and $\mathcal{A}^+$ have the same enumeration degree spectrum. Note that, however, not all structures whose degree spectrum consist only of total enumeration degrees are total. Consider for example, the structure $\mathcal{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function and K is the halting set. Then $DS_e(\mathcal{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathcal{A})$ is a c.e. set. Hence $\overline{K} \nleq_e f^{-1}(\mathcal{A})$. Clearly $\overline{K} \leq_e (f^{-1}(\mathcal{A}))^+$, so $f^{-1}(\mathcal{A})$ is not a total set.

A natural question arises here: if $DS_e(\mathcal{A})$ consists of total degrees does there exist a total structure $\mathcal{B}$ such that $DS_e(\mathcal{A}) = DS_e(\mathcal{B})$? In his last paper [Sos13a] Soskov, generalizing the Marker extension method of a sequence of structures, proves the following general result, giving a much stronger relationship between Turing degree spectra and enumeration degree spectra:

Theorem 2.4.4. [Sos13a] For every structure $\mathcal{A}$ there exists a total structure $\mathcal{M}$ such that $DS_e(\mathcal{M}) = \{\mathbf{a} \mid \mathbf{a} \text{ is total } \wedge (\exists \mathbf{x} \in DS_e(\mathcal{A}))(\mathbf{x} \leq \mathbf{a})\}$.

The degree spectrum of a structure measures how difficult is to present the structure. If instead we want to measure how much information is encoded in a structure, one approach is to use co-spectra. Here a set $X \subseteq \mathbb{N}$ is encoded in $\mathcal{A}$ if $X \leq_e f^{-1}(\mathcal{A})$, for every enumeration f of $\mathcal{A}$.

Co-spectrum $CS(\mathcal{A})$ of a structure $\mathcal{A}$ is the set of all lower bounds of the enumeration degree spectrum of the structure $\mathcal{A}$. If $CS(\mathcal{A})$ has a greatest element, then it is the *co-degree* of $\mathcal{A}$. For every computable ordinal α we denote by $CS_\alpha(\mathcal{A})$ the co-spectrum of $DS_\alpha(\mathcal{A})$.

An application of Selman's theorem shows that the co-spectrum of $\mathcal{A}$ depends only on the total elements of the spectrum of $\mathcal{A}$. Soskov's example of this phenomenon [Sos04] is a generalization, of a result of Rozinas [Roz78]: for every computable ordinal α and $\mathbf{b} \in DS_\alpha(\mathcal{A})$ there exist total e-degrees $\mathbf{f_0}$ and $\mathbf{f_1}$ such that : $\mathbf{f_0}^{(\alpha)} \leq \mathbf{b}$ and $\mathbf{f_1}^{(\alpha)} \leq \mathbf{b}$, and $\mathbf{f_0}^{(\beta)}, \mathbf{f_1}^{(\beta)} \notin CS_\beta(\mathcal{A})$ for

$\beta < \alpha$, and $\{\mathbf{x} \mid \mathbf{x} \in \mathcal{D}_e \ \&\ \mathbf{x} \le \mathbf{f_0}^{(\beta)} \ \&\ \mathbf{x} \le \mathbf{f_1}^{(\beta)}\} = CS_\beta(\mathcal{A})$ for every $\beta + 1 < \alpha$. He shows that there exist quasi minimal enumeration degrees for the degree spectrum, i.e. an e-degree $\mathbf{q} \notin CS(\mathcal{A})$, and every total $\mathbf{x} \le \mathbf{q} \to \mathbf{x} \in DS(\mathcal{A})$, and every total $\mathbf{x} \ge \mathbf{q} \to \mathbf{x} \in CS(\mathcal{A})$. This is an analogue of a quasi minimal degree.

Kalimullin [Kal09b], building on Wehner's result, transfers these ideas to enumeration degree spectra: There is a structure $\mathcal{A}$ such that $DS_e(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_e \ \&\ \mathbf{a} > \mathbf{0}_e\}$.

If a structure $\mathcal{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also its co-degree. The reverse is not always true. We have already seen one such example: Kalimullin's structure $\mathcal{A}$ with degree spectrum $DS_e(\mathcal{A})$ consisting of all nonzero enumeration degrees clearly has no enumeration degree, but has co-degree $\mathbf{0}_e$. As a second example, consider Richter's [Ric81] result on a linear ordering $\mathcal{A}$: the Turing degree spectrum $DS(\mathcal{A})$ always contains a minimal pair. Thus the co-degree of $DS_e(\mathcal{A}^+)$ is always $\mathbf{0}$, and non-computable linear orderings have co-degree but no degree. An analysis of Knight's proof [Kni86] generalizing the Richter's result, shows that the first jump co-spectrum of a linear ordering consists of all Σ^0_2 enumeration degrees, and so the first jump co-degree is always $\mathbf{0}'_e$, even though not every linear ordering has a first jump degree.

There are also structures with no co-degree. For example, consider $\mathcal{A} = (\mathbb{N}; G_\Psi,\ P)$, where Ψ is a function such that $\Psi(\langle n, x\rangle) = \langle n, x+1\rangle$ and the relation $P(x)$ is defined and true if $(\exists t)(x = \langle 0, t\rangle)$ or $(\exists n)(\exists t)(x = \langle n+1, t\rangle \ \&\ t \in \varnothing^{(n+1)})$. For every $X \subseteq \mathbb{N}$ we have that $d_e(X) \in CS(\mathcal{A})$ iff $(\exists n)(X \le_e \varnothing^{(n)})$. It follows that $CS(\mathcal{A})$ consists of all arithmetical degrees and hence has no greatest element, i.e. $\mathcal{A}$ has no co-degree.

The co-degree and e-degree of a structure are closely related to what Knight [Kni98] and Montalbán [Mon] call the "enumeration degree of a structure". A set $X \subseteq \mathbb{N}$ is the "enumeration degree" of a structure $\mathcal{A}$ if every enumeration of X computes a copy of $\mathcal{A}$, and every copy of $\mathcal{A}$ computes an enumeration of X. Thus by Selman's theorem the enumeration degree of X is the co-degree of the structure $\mathcal{A}^+$. This co-degree, however has an additional property: $DS(\mathcal{A}^+)$ is exactly the set of total enumeration degrees above $d_e(X)$. Thus, examples of structures with "enumeration degree" translate to examples of structures with co-degree and there are many of those: [Mon] Given $X \subseteq \mathbb{N}$, consider the group $G_X = \bigoplus_{i \in X} \mathbb{Z}_{p_i}$, where p_i is the i-th prime number. Then G_X has "enumeration degree" X, as we can easily build G_X given any enumeration of X, and for the reverse direction, we have that $n \in X$ if and only if there is an elements $g \in G_X$ of order p_n.

Example 2.4.5. A further example is the torsion free abelian group $\mathcal{G}$ of rank 1, i.e. a subgroup of $(\mathbb{Q}, +, =)$. Downey and Jockusch [Dow97] analyze the computability theoretic properties of such groups. Using results that go back to Baer, they discover a way to associate a set $S(\mathcal{G})$, called the characteristic of $\mathcal{G}$, to every torsion free abelian group $\mathcal{G}$ of rank 1, so that the Turing degree spectrum of $\mathcal{G}$ is precisely $\{d_T(Y) \mid S(\mathcal{G})$ is c.e. in $Y\}$. On the other hand, they show that for every set of natural numbers S there is a torsion free abelian group $\mathcal{G}$ of rank 1, such that $S(\mathcal{G}) \equiv_1 S$. They knew from Richter [Ric81] that this meant that not all such groups have a degree. Coles, Downey and Slaman [CDS00] use a forcing construction to show that, however, every such group has first jump degree.

Soskov [Sos04] considers the problem from the point of view of enumeration reducibility. Any subgroup of the rationals can be seen as a total structure, as the only relation involved is the graph of addition, which is a total function. Let $\mathcal{G}$ be such a group and let $\mathbf{s}_b = d_e(S(\mathcal{G}))$. It follows that

$$DS(\mathcal{G}) = \{\mathbf{b} \mid \mathbf{b} \text{ is total and } \mathbf{s}_b \leq_e \mathbf{b}\}.$$

It is an easy consequence of Selman's theorem that $\mathbf{s}_b$ is the co-degree of $\mathcal{G}$. Furthermore, $\mathcal{G}$ has degree if and only if $\mathbf{s}_b$ is total. The result of Coles, Downey and Slaman now follows from Theorem 2.3.6. There is a total enumeration degree $\mathbf{f} \geq \mathbf{s}_b$ with $\mathbf{f}' = \mathbf{s}_b'$ and so the first jump spectrum of $\mathcal{G}$ consists of all total enumeration degrees greater than or equal to $\mathbf{s}_b'$, in particular $\mathbf{s}_b'$ is the first jump degree of $\mathcal{G}$.

Another consequence of this example is that every principal ideal of enumeration degrees is a co-spectrum of a structure, namely the co-spectrum of some torsion free abelian group of rank one. Further Soskov proved [Sos04] that every countable ideal of enumeration degrees is the co-spectrum of a structure.

Understanding which subsets of the Turing degrees can be realized as degree spectra is an important open problem in the area. A natural question here: is every set of degrees that is upwards closed with respect to total elements the enumeration spectrum of a structure? The answer is, of course, 'No'. One way to see this is via the notion of a *base* and its relationship to the existence of a degree.

A subset $\mathcal{B} \subseteq \mathcal{C}$ of a set of enumeration degrees $\mathcal{C}$ is *a base of* $\mathcal{C}$ if $(\forall \mathbf{a} \in \mathcal{C})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a})$. Using generic enumerations and an argument much like that used in Selman's theorem we can show the following.

Theorem 2.4.6. [Sos04] A structure $\mathcal{A}$ has an e-degree if and only if $DS_e(\mathcal{A})$ has a countable base.

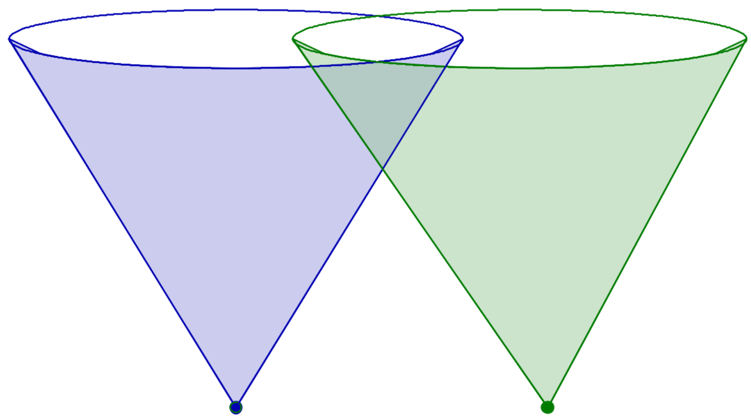

Figure 2.1: An upwards closed set with respect to total degrees which is not a degree spectra of a structure

In particular the union of two cones above incomparable degrees (and even countable cones) cannot be the enumeration degree spectrum of a structure (just like it cannot be the Turing degree spectrum of a structure). Nevertheless, degree spectra play well with co-spectra and behave structurally with respect to their elements just like the cone of total degrees above a fixed enumeration degree.

2.5 Definability in a structure

Another way to characterize the complexity of a structure $\mathcal{A}$ is to analyze the definable sets in $\mathcal{A}$. This gives a finer measure as it may happen that two structures have the same degree spectra but greatly differ in their definability power and model theoretic properties.

2.5.1 Relatively intrinsically Σ^0_α relations

We will explain first some basic facts about the hierarchies.

Arithmetical hierarchy. Let $X \subseteq \mathbb{N}$. We define classes of $\Sigma^0_n(X)$, $\Pi^0_n(X)$ and $\Delta^0_n(X)$ for $n \geq 1$ inductively. A relation $P(\overline{x})$ on the natural numbers is $\Sigma^0_1(X)$ if it can be expressed in the form $\exists y R(\overline{x}, y)$, where $R(\overline{x}, y)$ is computable in X. $P(\overline{x})$ is $\Pi^0_1(X)$ if it is a negation of a $\Sigma^0_1(X)$ relation. And it is $\Delta^0_1(X)$ if it is both $\Sigma^0_1(X)$ and $\Pi^0_1(X)$. Proceeding by induction, for $n > 1$, a relation $P(\overline{x})$ is $\Sigma^0_n(X)$ if it can be expressed in the form $\exists y R(\overline{x}, y)$, where $R(\overline{x}, y)$ is $\Pi^0_{n-1}(X)$. $P(\overline{x})$ is $\Pi^0_n(X)$ if it is a negation of a $\Sigma^0_n(X)$ relation, and is $\Delta^0_n(X)$ if it is both $\Sigma^0_n(X)$ and $\Pi^0_n(X)$.

Theorem 2.5.1. [AK00] For $n \geq 1$,

- the $\Sigma^0_n(X)$ relations are those which are c.e. relative to $X^{(n-1)}$,
- the $\Pi^0_n(X)$ relations are those which are co-c.e. (the complement is c.e.), relative to $X^{(n-1)}$,
- the $\Delta^0_n(X)$ relations are those which are computable relative to $X^{(n-1)}$.

We can approximate a Δ^0_2 function, total or partial, by a total computable function. If $g(\overline{x}, s)$ is a total function, we write $lim_{s\to\infty} g(\overline{x}, s) = y$ for the partial function $f(\overline{x})$ that is defined, with value y, just in case $g(\overline{x}, s)$ has value y for all sufficiently large s.

Theorem 2.5.2 (Limit Lemma)**.** [AK00] A function $f(\overline{x})$ is partial Δ^0_2 if and only if there exists a total computable function $g(\overline{x}, s)$ such that $f(\overline{x}) = lim_{s\to\infty} g(\overline{x}, s)$.

Actually, a relation $P(\overline{x})$ is Σ^0_2 if and only if there is a total computable function $g(\overline{x}, s)$ taking values 0 and 1, such that $P(\overline{x})$ if and only if for all sufficiently large s, $g(\overline{x}, s) = 1$.

Hyperarithmetical hierarchy. The hyperarithmetical sets can be defined using infinitely iterated Turing jumps and can be classified into a hierarchy extending the arithmetical hierarchy. Each level of the hyperarithmetical hierarchy corresponds to a countable ordinal. An ordinal notation is an effective description of a countable ordinal by a natural number. A system of ordinal notations is required in order to define the hyperarithmetic hierarchy. The fundamental property an ordinal notation must have is that it describes the ordinal in terms of smaller ordinals in an effective way. In the system of notations for ordinals, given by Kleene [Kle38], we define, simultaneously, a set of notations $\mathcal{O}$, a function $\lambda a.|a|_{\mathcal{O}}$ taking each $a \in \mathcal{O}$ to an ordinal $\alpha = |a|_{\mathcal{O}}$ and a strict partial ordering $<_{\mathcal{O}}$ on $\mathcal{O}$. Let 1 is a notation for the ordinal 0, i.e. $0 = |1|_{\mathcal{O}}$. If a is a notation for α then 2^a is a notation for $\alpha + 1$. In the partial ordering, we let $b <_{\mathcal{O}} 2^a$ if either $b <_{\mathcal{O}} a$ or $b = a$. For a limit ordinal α the notation is of the form 3.5^e, where e is the index of a total computable function φ_e with values in $\mathcal{O}$ such that $\varphi_e(0) < \varphi_e(1) < \cdots < \varphi_e(n)\ldots$, and α is the least upper bound of the sequence of ordinals $\alpha(n) = |\varphi_e(n)|_{\mathcal{O}}$. We write $\alpha = \lim \alpha(n)$. In the partial ordering, we let $b <_{\mathcal{O}} 3.5^e$ if there exists n such that $b <_{\mathcal{O}} \varphi_e(n)$. For the properties of the set of notations $\mathcal{O}$ and the relation "$<_{\mathcal{O}}$" the reader may consult [Rog67b] or [Sac90].

We define the hyperarithmetical sets $H(a)$, following [AK00], for $a \in \mathcal{O}$ by transfinite recursion on the ordinals $|a|_{\mathcal{O}}$, (we will write only $|a|$) as follows:
$H(1) = \varnothing$,
$H(2^a) = H(a)'$,
$H(3.5^e) = \{\langle u, v\rangle \mid u <_{\mathcal{O}} 3.5^e \ \&\ v \in H(u)\} =$
$\{\langle u, v\rangle \mid \exists n(u <_{\mathcal{O}} \varphi_e(n) \ \&\ v \in H(u))\}$.

Lemma 2.5.3 (Spector)**.** There is a partial computable function f such that for each $a \in \mathcal{O}$, $f(a)$ is an index for $\{b \in \mathcal{O} \mid |b| < |a|\}$ as a set computable in $H(a)'$.

Theorem 2.5.4 (Spector)**.** There is a partial computable function f such that for $a, b \in \mathcal{O}$ with $|a| < |b|$, $f(a, b)$ is an index for $H(a)$ as a set computable in $H(b)$.

Corollary 2.5.5. If $x, y \in \mathcal{O}$ and $|x| = |y|$, then $H(x) \equiv_T H(y)$.

Now, following Kleene, we define classes Σ^0_α, Π^0_α and Δ^0_α for all computable ordinals $\alpha \geq \omega$. For infinite α, a relation is said to be Σ^0_α, Π^0_α and Δ^0_α if it is, respectively, c.e., co-c.e., or computable relative to $H(a)$ for some $a \in \mathcal{O}$ with $|a| = \alpha$. By Corollary 2.5.5, such a relation will be c.e., co-c.e., or computable relative to $H(a)$ for every $a \in \mathcal{O}$ with $|a| = \alpha$. Finally, a relation is said to be *hyperarithmetical* if it is Δ^0_α for some computable ordinal α. Note that there is a lack of uniformity in the definition above when we pass from finite to infinite computable ordinals. For finite α, say $\alpha = n > 0$, by Theorem 2.5.1 Σ^0_n relations are the ones that are c.e. relative to $\varnothing^{(n-1)}$ and $\varnothing^{(n-1)} \equiv H(a)$, where $|a|$ is $n-1$, not n. We could relativize the above definition relative to an arbitrary set $X \subseteq \mathbb{N}$, starting with $H_X(1) = X$, and receiving the sets $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$ and $\Delta^0_\alpha(X)$. So, the αth-Turing jump of X for $\alpha \geq \omega$ is $\Delta^0_\alpha(X)$.

Relatively intrinsically Σ^0_α relations. Let $\mathcal{A} = (A, R_1, R_2, \dots, R_k)$ be a countable structure. For simplicity we suppose that $A = \mathbb{N}$.

Definition 2.5.6. A relation R on $\mathcal{A}$ is relatively intrinsically Σ^0_α in a structure $\mathcal{A}$ if for each $(\mathcal{B}, P) \simeq (\mathcal{A}, R)$ the relation P is Σ^0_α in the atomic diagram $D(\mathcal{B})$, which in our terms means that for every enumeration f of $\mathcal{A}$, $f^{-1}(R) \in \Sigma^0_\alpha(f^{-1}(\mathcal{A}))$.

For example, consider a linear ordering $\mathcal{A} = (A, <)$, and S-successor relation. S is relatively intrinsically Π^0_1 in $\mathcal{A}$, since $\neg S(x,y) \iff x \nless y \vee \exists z (x < z \,\&\, z < y)$ is relatively intrinsically Σ^0_1 in $\mathcal{A}$. The "block" relation $B(x,y) \iff$ there are finitely many elements $z_1, \dots, z_n$ such that $S(x, z_1)$,$S(z_1, z_2), \dots, S(z_n, y)$ is relatively intrinsically Σ^0_2 in $\mathcal{A}$, and there is no Σ^0_2 formula, which defines B. But B can be defined by a computable infinite disjunction of such formulas as we shall see in the next subsection.

Ash, Knight, Manasse and Slaman [AKMS89] and independently Chisholm [Chi90] prove that a relation is relatively intrinsically Σ^0_α in $\mathcal{A}$ iff it is definable by an computable infinitary Σ^c_α formula with finitely many parameters in $\mathcal{A}$.

2.5.2 Computable infinitary formulas

Let L be a fixed computable language. Some mathematical properties, such as the Archimedean property (true of subfields of the ordered field of reals), are expressed in a natural way by an infinitely long formula. We consider formulas of $L_{\omega_1,\omega}$ (see Keisler [Kei71]). Here ω_1 indicates that the disjunctions and conjunctions are over only countable sets, and ω indicates that there is only finite nesting of quantifiers. For example, in the language of ordered fields, there is a sentence, which adding it to the axioms of ordered fields, the models are exactly the Archimedean ordered fields $(\forall x) \bigvee_n (x < \tau_n)$, where $\tau_n = \underbrace{1 + 1 + \cdots + 1}_{n}$.

The computable infinitary Σ^c_α and Π_α formulas, denoted by Σ^c_α and Π^c_α, with free variables among $x_1, \dots, x_l$, are defined by transfinite induction on α as follows.

The Σ^c_0 and Π^c_0 formulas are quantifier free formulas on $x_1, \dots, x_l$.

For $\alpha > 0$, a Σ^c_α formula is the disjunction of a c.e. set of formulas of the form $\exists y_1 \dots \exists y_m \Psi(x_1, \dots, x_l, y_1, \dots, y_m)$, where Ψ is a Π^c_β formula, for some $\beta < \alpha$, with free variables among $x_1, \dots, x_l$, $y_1, \dots, y_m$.

A Π^c_α formula is the conjunction of a c.e. set of formulas of the form $\forall y_1 \dots \forall y_m \Psi(x_1, \dots, x_l, y_1, \dots, y_m)$, where Ψ is a Σ^c_β formula, for some $\beta < \alpha$, with free variables among $x_1, \dots, x_l, y_1, \dots, y_m$.

Formally, the c.e. disjunction above is a c.e. set of codes of such formulas. If a is a notation of α, codes are quadruples in which the first component is the symbol Σ or Π (we could use 0 and 1), the second is a notation a for α, the third is (the code for) a tuple of variables, and the fourth is a natural number e for the c.e. set W_e. (see [AK00] 7.2.)

Definition 2.5.7. A relation $R \subseteq |\mathcal{A}|^l$ is *definable* in a structure $\mathcal{A}$ by a Σ^c_α formula $\Phi(x_1, \dots, x_l, w_1, \dots, w_r)$, if there are parameters $t_1, \dots, t_r \in |\mathcal{A}|$ such that for every $a_1, \dots, a_l \in |\mathcal{A}|$ the following equivalence holds:

$$(a_1, \dots, a_l) \in R \iff \mathcal{A} \models \Phi(x_1/a_1, \dots, x_l/a_l, w_1/t_1, \dots, w_r/t_r).$$

Ash, Knight, Manasse and Slaman [AKMS89] and independently Chisholm [Chi90] prove that the relatively intrinsically Σ^0_α relations in the structure $\mathcal{A}$ are the definable ones by a computable Σ^c_α formula with finitely many parameters in $\mathcal{A}$.

Theorem 2.5.8. Let R be a relation on the structure $\mathcal{A}$. The following are equivalent:

1. R is relatively intrinsically Σ^0_α in a structure $\mathcal{A}$.

2. R is definable by a computable Σ^c_α formula with finitely many parameters in $\mathcal{A}$.

Antonio Montalbán extends in his book [Mon] this result for $\alpha = 1$ not only for relations with a fixed argument but also of those $R \subseteq |A|^{<\omega}$. For example over a Q-vector space V, the relation $LD \subseteq V^{<\omega}$ of inear dependence is always c.e. in V. To enumerate LD in a $D(V)$-computable way, go through all the possible non-trivial Q-linear combinations $q_0 v_0 + \dots + q_k v_k$ of all possible tuples of vectors $\langle v_0, \dots, v_k \rangle \in V^{<\omega}$, and if you find one that is equal to $\vec{0}$, enumerate $\langle v_0, \dots, v_k \rangle$ into LD. It is clear that we could write a Σ^c_1 formula that define this relation but the free variable will not be fixed.

Definition 2.5.9 (Generalized computable Σ^c_1-definition)**.** Let $R \subseteq \mathcal{B}^{<\omega}$, and let $\varphi_n(\overline{x}_n)_{n \in \omega}$ be a computable sequence of computable Σ^c_1 formulas, where $\varphi_n(\overline{x}_n)$ has arity n. If for each n, $\varphi_n(\overline{x}_n)$ defines $R \cap \mathcal{B}^n$, then we say that $\bigvee_n \varphi_n(\overline{x}_n)$ is a *generalized computable Σ^c_1 definition* of R.

Thus a generalized computable Σ_1^c formula allows consideration of tuples of all finite arities. Such a formula is technically not in $L_{\omega_1\omega}$, as it uses infinitely many free variables; however, it is a computable disjunction, over all $n \in \omega$, of $L_{\omega_1\omega}$ formulas φ_n with free variables $x_1, \ldots, x_n$.

Montalbán [Mon] proved that the result of Theorem 2.5.8 holds for such relations $R \subseteq |\mathcal{A}|^{<\omega}$, i.e. R is relatively intrinsically Σ_1^0 in a structure $\mathcal{A}$ if and only if R is Σ_1^0 definable in $\mathcal{A}$ with parameters. He call these relations relatively intrinsically c.e. (r.i.c.e.). He uses this result for the characterization of the effective interpretability [HTMMM17]. We will use this theorem in Chapter 5.

Chapter 3

Jump of a structure

The notion of jump of a structure contains information about the sets definable by computable infinitary Σ^c_1 formulas. This notion has been independently defined various times in the last few years. It is an analogue of the jump operation in the degree structures. The first appearance of definition of jump of structure in print is due to Vessela Baleva in [Bal02, Bal06], as part of her Ph.D. thesis under the supervision of Ivan Soskov. That definition uses, as for the jump of $\mathcal{A}$, the Moschovakis extension [Mos69] of $\mathcal{A}$, which is essentially the closure of $\mathcal{A}$ under a pairing operation, and then a universal semi-search-computable predicate is added. The notion of jump of structure is in terms of the s-reducibilities between structures, based on the notion search computability of Moschovakis. In 2007 we, with Soskov, [SS07, SS09a] use a modified definition based on Moschovakis extensions for proving two jump inversions theorems for structures in terms of degree spectra. We add an analogue of Kleene predicate to the Moschovakis extension of the structure which is universal for all relations, definable by computable infinitary Σ^c_1 formulas.

Independently Montalbán [Mon09] in 2009 suggests another approach to the definition of the jump of a structure - he does not change the domain of the structure but he adds a complete set of relations, definable by computable infinitary Π^c_1 (later he called this a structural jump). In [Mon12, Mon] he changed the added complete set of relations by those, definable by computable infinitary Σ^c_1 formulas and received an equivalent notion as ours.

First Morozov [Mor04] in 2004 and then Puzarenko [Puz09] in 2009 also define the jump of an admissible structure. Later, Alexey Stukachev [Stu09, Stu10] extended this definition to all structures $\mathcal{A}$ by considering the admissible structure $\mathbb{HF}(\mathcal{A})$ in the terms of Σ-definability in hereditarily finite extension of the structure.

The fact that all these notions and results have been rediscovered over and over is not surprising given that the authors in different countries use completely different ways to represent structures and to talk about computability on algebraic structures.

In [Sos07a, SS07, SS09a], with Soskov, we prove two jump inversion theorems, one is as the Friedberg's jump inversion Theorem 2.2.9 and the other one that shows that every jump spectrum is a spectrum of a structure. The first one says: for every structure $\mathcal{A}$, for which the degree spectrum is a subset of the jump degree spectrum of another structure $\mathcal{B}$, with $DS(\mathcal{A}) \subseteq DS_1(\mathcal{B})$, there exists a structure $\mathcal{C}$ with the property $DS_1(\mathcal{C}) = DS(\mathcal{A})$ and $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$. The structure $\mathcal{C}$ is obtained from the Marker's extension of $\mathcal{A}$. Stukachev [Stu09, Stu10] shows a similar result for the Σ definability. The second theorem shows that every jump degree spectrum $DS_1(\mathcal{A})$ is a degree spectrum of the structure - the jump of $\mathcal{A}$. Independently, later Montalbán [Mon09] proved similar result.

I want to mention that Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon [GHK+05] give an idea how the jump inversion - Friedberg's style could be generalized for a computable successor ordinal. They only do it for graphs, but we know [HKSS02] any degree spectrum can be realized as the degree spectrum of a graph. They proved the result above only as a tool to get get other results about and relative intrinsically relations. Vatev [Vat14] uses this idea and proves the jump-inversion theorem for any computable successor ordinal. Soskov proves in [Sos13b] that such theorem is not true for computable limit ordinals.

We will define first in Section 3.1 the jump of a structure and then in Section 3.2 we show that every jump spectrum is a spectrum of a structure and in Section 3.3 we will prove the jump inversion theorem. The content of these three sections are from [SS09a]. In the last Section 3.4 we will show some applications of the jump inversion theorems. The content of the last section is from [SS09b].

3.1 Jump of a structure

Let $\mathcal{A} = (A; R_1, \ldots, R_s)$ be a countable structure and let equality be among the predicates $R_1, \ldots, R_s$. We suppose that the domain A of $\mathcal{A}$ is infinite.

Following Moschovakis [Mos69] the least acceptable extension of the structure $\mathcal{A}$ is defined as follows.

Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let A^* be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under Π. Denote by N the set of all natural numbers.

We associate an element n^* of A^* with each natural number $n \in N$ by induction:

$$\begin{aligned} 0^* &= 0; \\ (n+1)^* &= \Pi(0, n^*). \end{aligned}$$

The set of all elements n^* defined above will be denoted by N^*.

Let L and R be the functions on A^* satisfying the following conditions:

$$\begin{aligned} &L(0) = R(0) = 0; \\ &(\forall t \in A)(L(t) = R(t) = 1^*); \\ &(\forall s, t \in A^*)(L(\Pi(s,t)) = s \ \& \ R(\Pi(s,t)) = t). \end{aligned}$$

The pairing function allows us to code finite sequences of elements: let $\Pi_1(t_1) = t_1$, $\Pi_{n+1}(t_1, t_2, \dots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \dots, t_{n+1}))$ for every $t_1, t_2, \dots, t_{n+1} \in A^*$.

For each predicate R_i of the structure $\mathcal{A}$ define the respective predicate R_i^* on A^* by

$$R_i^*(t) \iff (\exists a_1 \in A) \dots (\exists a_{r_i} \in A)(t = \Pi_{r_i}(a_1, \dots, a_{r_i}) \ \& \ R_i(a_1, \dots, a_{r_i})).$$

Definition 3.1.1. *Moschovakis' extension of* $\mathcal{A}$ is the structure

$$\mathcal{A}^* = (A^*; A_0, R_1^*, \dots, R_s^*, G_\Pi, G_L, G_R, =),$$

where G_Π, $G_L$and G_R are the graphs of Π, L and R respectively.

Lemma 3.1.2. Let f be an enumeration of $\mathcal{A}$. There exists an enumeration f^* of $\mathcal{A}^*$ such that $(f^*)^{-1}(\mathcal{A}^*) \equiv_T f^{-1}(\mathcal{A})$.

Proof. Let $J(x,y) = 2^{x+1}.(2y+1)$ be an effective coding of the ordered pairs of natural numbers. Denote by induction $J_1(x_1) = x_1$ and $J_{n+1}(x_1, x_2, \dots, x_{n+1}) = J(x_1, J_n(x_2, \dots, x_{n+1}))$ for any $x_1, x_2, \dots, x_{n+1} \in N$. And let l and r be computable functions satisfying the equalities:

$$l(0) = r(0) = 0,$$
$$l(2x+1) = r(2x+1) = 2 = J(0,0),$$
$$l(J(x,y)) = x \ \& \ r(J(x,y)) = y.$$

Define f^* by means of the following inductive definition:

$$\begin{array}{ll} f^*(0) & = 0^*, \\ f^*(2x+1) & = f(x), \\ f^*(J(x,y)) & = \Pi(f^*(x), f^*(y)). \end{array}$$

Clearly f^* is an enumeration of $\mathcal{A}^*$. It is easy to see that $(f^*)^{-1}(A_0) = \{2x+1 \mid x \in N\} \cup \{0\}$, $(f^*)^{-1}(G_\Pi) = \{\langle x,y\rangle : (x,y) \in G_J\}$, $(f^*)^{-1}(G_L) = \{\langle x,y\rangle : (x,y) \in G_l\}$ and $(f^*)^{-1}(G_R) = \{\langle x,y\rangle : (x,y) \in G_r\}$.

Fix a natural number i, $1 \le i \le s$. Then

$$\begin{array}{l} \langle x_1,\ldots,x_{r_i}\rangle \in f^{-1}(R_i) \iff (f(x_1),\ldots,f(x_{r_i})) \in R_i \iff \\ (f^*(2x_1+1),\ldots,f^*(2x_{r_i}+1)) \in R_i \iff \\ \Pi_{r_i}(f^*(2x_1+1),\ldots,f^*(2x_{r_i}+1))) \in R_i^* \iff \\ J_{r_i}(2x_1+1,\ldots,2x_{r_i}+1) \in (f^*)^{-1}(R_i^*). \end{array}$$

Finally, let R_1 be the equality on A. Then

$$\begin{array}{lll} \langle x,y\rangle \in (f^*)^{-1}(=) \iff [& (x,y \in (f^*)^{-1}(A) \ \& \ \langle x/2, y/2\rangle \in f^{-1}(R_1))\vee \\ & (x = y = 0)\vee \\ & (x = J(x_1,x_2) \ \& \ y = J(y_1,y_2) \ \& \\ & \langle x_1,y_1\rangle \in (f^*)^{-1}(=) \ \& \ \langle x_2,y_2\rangle \in (f^*)^{-1}(=))]. \end{array}$$

Clearly
$$\langle x,y\rangle \in f^{-1}(R_1) \iff \langle 2x+1, 2y+1\rangle \in (f^*)^{-1}(=).$$

So $(f^*)^{-1}(=) \equiv_T f^{-1}(R_1)$.

Combining all above, we get that $(f^*)^{-1}(\mathcal{A}^*) \equiv_T f^{-1}(\mathcal{A})$. □

From now on given an enumeration f of the structure $\mathcal{A}$, by f^* we shall denote the enumeration of $\mathcal{A}^*$ defined in the lemma above.

Proposition 3.1.3. $DS(\mathcal{A}) = DS(\mathcal{A}^*)$.

Proof. Let $\mathbf{a} \in DS(\mathcal{A})$ and let f be an enumeration of $\mathcal{A}$ witnessing this, i.e. $f^{-1}(\mathcal{A}) \in \mathbf{a}$. Then $(f^*)^{-1}(\mathcal{A}^*) \equiv_T f^{-1}(\mathcal{A})$ and hence $\mathbf{a} \in DS(\mathcal{A}^*)$.

Now let $\mathbf{a} \in DS(\mathcal{A}^*)$ and let h be an enumeration of $\mathcal{A}^*$ with $h^{-1}(\mathcal{A}^*) \in \mathbf{a}$. By Proposition 2.4.2, there exists an injective enumeration g of $\mathcal{A}^*$ such that $g^{-1}(\mathcal{A}^*) \leq_T h^{-1}(\mathcal{A}^*)$. Our goal is to construct an enumeration f of $\mathcal{A}$ such that $f^{-1}(\mathcal{A}) \leq_{\mathrm{T}} g^{-1}(\mathcal{A}^*)$. Then by Proposition 2.4.3 we would get that $\mathbf{a} \in DS(\mathcal{A})$.

Let $0^{\#} = g^{-1}(0^*)$. Then the set $g^{-1}(A) = g^{-1}(A_0) \setminus \{0^{\#}\}$ is computable in $g^{-1}(\mathcal{A}^*)$. Fix an element $z_0 \in g^{-1}(A)$ and let

$$m(0) = z_0;$$
$$m(i+1) = \mu z \in g^{-1}(A)[(\forall k \leq i)(m(k) \neq z)].$$

Note that $m \leq_{\mathrm{T}} g^{-1}(\mathcal{A}^*)$ is a bijective enumeration of $g^{-1}(A)$. Let

$$J(x,y) = g^{-1}(\Pi(g(x), g(y))).$$

Clearly J is computable in $g^{-1}(\mathcal{A}^*)$. As usual set $J_1(x) = x$ and

$$J_{n+1}(x_1, \dots, x_{n+1}) = J(x_1, J_n(x_2, \dots, x_{n+1})).$$

Set $f = \lambda x.g(m(x))$. Clearly f is an injective enumeration of the structure $\mathcal{A}$. Consider a predicate R_i of $\mathcal{A}$. Then

$$f^{-1}(R_i) = \{\langle x_1, \dots, x_{r_i}\rangle : J_{r_i}(m(x_1), \dots, m(x_{r_i})) \in g^{-1}(R_i^*)\}$$

and hence $f^{-1}(R_i)$ is computable in $g^{-1}(\mathcal{A}^*)$.

Thus $f^{-1}(\mathcal{A}) \leq_{\mathrm{T}} g^{-1}(\mathcal{A}^*)$. □

Let f be an enumeration of $\mathcal{A}$. Given natural numbers e and x let

$$f \models F_e(x) \iff x \in W_e^{f^{-1}(\mathcal{A})}$$

and let

$$f \models \neg F_e(x) \iff f \not\models F_e(x).$$

We shall connect with the modeling relation "$\models$" a forcing with conditions all finite mappings of N into A ordered in the usual way. We call these finite mappings *finite parts*. The finite parts will be denoted by the letters δ, τ.

Given a finite part δ and $R \subseteq A^n$, let $\delta^{-1}(R)$ be the finite function on the natural numbers taking values in $\{0,1\}$ such that

$$\begin{aligned} &\delta^{-1}(R)(u) \simeq 1 \iff (\exists x_1, \dots x_n \in dom(\delta))(u = \langle x_1, \dots, x_n\rangle \; \& \\ &(\delta(x_1), \dots, \delta(x_n)) \in R) \text{ and} \\ &\delta^{-1}(R)(u) \simeq 0 \iff (\exists x_1, \dots x_n \in dom(\delta))(u = \langle x_1, \dots, x_n\rangle \; \& \\ &(\delta(x_1), \dots, \delta(x_n)) \notin R). \end{aligned} \tag{3.1.1}$$

By $\delta^{-1}(\mathcal{A})$ we shall denote the finite function $\delta^{-1}(R_1) \oplus \cdots \oplus \delta^{-1}(R_s)$.

If α is a partial function and $e \in N$, then by W_e^α we shall denote the set of all x such that the computation $\{e\}^\alpha(x)$ halts successfully. We shall assume that if during a computation the oracle α is called with an argument outside it's domain, then the computation halts unsuccessfully.

Definition 3.1.4. For any $e, x \in N$ and for every finite part δ, define *the forcing relations* $\delta \Vdash F_e(x)$ and $\delta \Vdash \neg F_e(x)$ as follows:

$$\delta \Vdash F_e(x) \iff x \in W_e^{\delta^{-1}(\mathcal{A})}$$

$$\delta \Vdash \neg F_e(x) \iff (\forall \tau \supseteq \delta)(\tau \nVdash F_e(x)).$$

The following two properties of the forcing relation are obvious:

(F1) $\delta \Vdash (\neg)F_e(x)\ \&\ \delta \subseteq \tau \Rightarrow \tau \Vdash (\neg)F_e(x)$.

(F2) For every enumeration f of $\mathcal{A}$,

$$f \models F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash F_e(x)).$$

Definition 3.1.5. An enumeration f of $\mathcal{A}$ is *generic* if for every $e, x \in N$:

$$(\exists \tau \subseteq f)(\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)).$$

Note, that this is equivalent to Definition 2.2.2 for a 1-generic set, only take $G = f^{-1}(\mathcal{A})$ and S to be the set of finite parts $\{\tau \mid \tau \Vdash F_e(x)\}$. It is clear that S is c.e.

We know from Theorem 2.2.6 that for every generic enumeration f of $\mathcal{A}$ for all $e, x \in N$,

$$f \models \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash \neg F_e(x)).$$

With each finite part $\tau \neq \emptyset$ such that $\mathrm{dom}(\tau) = \{x_1, \ldots, x_n\}$ and $\tau(x_1) = s_1, \ldots, \tau(x_n) = s_n$, we associate the element $\tau^* = \Pi_n(\Pi(x_1^*, s_1), \ldots, \Pi(x_n^*, s_n))$ of A^*. Let $\tau^* = 0$ if $\tau = \emptyset$.

Define $K_{\mathcal{A}} = \{\Pi_3(\delta^*, e^*, x^*) \mid (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\ \&\ e^*, x^* \in N^*\}$.

Definition 3.1.6. *The jump of the structure* $\mathcal{A}$ is the following structure:

$$\mathcal{A}' = (A^*; A_0, R_1^*, \ldots, R_s^*, G_\Pi, G_L, G_R, =, K_{\mathcal{A}}).$$

The following proposition follows directly from Lemma 3.1.2.

Proposition 3.1.7. Let f be an enumeration of $\mathcal{A}$. Then

$$(f^*)^{-1}(\mathcal{A}') \equiv_T f^{-1}(\mathcal{A}) \oplus (f^*)^{-1}(K_{\mathcal{A}}).$$

3.2 Every Jump Spectrum is Spectrum

Theorem 3.2.1. For every structure $\mathcal{A}$ there exists a structure $\mathcal{B}$ such that $DS_1(\mathcal{A}) = DS(\mathcal{B})$.

Proof. Let $\mathcal{B} = \mathcal{A}'$ defined above. We shall prove that $DS_1(\mathcal{A}) = DS(\mathcal{B})$. We divide the proof into two parts.

Proposition 3.2.2. $DS_1(\mathcal{A}) \subseteq DS(\mathcal{B})$.

Proof. Let $\mathbf{a} \in DS_1(\mathcal{A})$ and let g be an enumeration of $\mathcal{A}$ such that $g^{-1}(\mathcal{A})' \in \mathbf{a}$. By Proposition 2.4.2, there exists an injective enumeration f of $\mathcal{A}$ such that $f^{-1}(\mathcal{A}) \leq_T g^{-1}(\mathcal{A})$. Since $f^{-1}(\mathcal{A})' \leq_T g^{-1}(\mathcal{A})'$ and $DS(\mathcal{B})$ is closed upwards, it is sufficient to show that $d_T(f^{-1}(\mathcal{A}))' \in DS(\mathcal{B})$. For we shall show that $(f^*)^{-1}(\mathcal{B}) \leq_T f^{-1}(\mathcal{A})'$ and use once more the fact that $DS(\mathcal{B})$ is closed upwards.

From the construction of the enumeration f^* in the proof of Lemma 3.1.2 it follows that f^* is also injective.

Recall the definition of the subset $N^* = \{x^* : x \in N\}$ of A^*. For every natural number x let $x^\# = (f^*)^{-1}(x^*)$ and let $N^\# = \{x^\# : x \in N\} = (f^*)^{-1}(N^*)$. Notice that $0^\# = 0$ and $(x+1)^\# = J(0, x^\#)$ and hence $N^\#$ is a computable set. Clearly there exist computable functions n_1 and n_2 such that for all natural numbers x, $n_1(x^\#) = x$ and $n_2(x) = x^\#$.

Denote by Δ the set of all finite parts in $\mathcal{A}$. Clearly for every finite part τ, there exists a unique element τ^* of A^* defined as in the previous section and a unique natural number $\tau^\# = (f^*)^{-1}(\tau^*)$.

Let $\Delta^* = \{\tau^* : \tau \in \Delta\}$ and $\Delta^\# = \{\tau^\# : \tau \in \Delta\} = (f^*)^{-1}(\Delta^*)$.

It is easy to see that a number $\tau^\#$ belongs to $\Delta^\#$ if and only if $\tau^\# = 0$ or for some $n \geq 1$ there exist n distinct elements ${x_1}^\#, \dots, {x_n}^\#$ of $N^\#$ and n odd numbers $y_1, \dots, y_n$ such that

$$\tau^\# = J_n(J({x_1}^\#, y_1), \dots, J({x_n}^\#, y_n)).$$

Therefore the set $\Delta^\#$ is also computable.

Given a $\tau^\# = J_n(J({x_1}^\#, y_1), \dots, J({x_n}^\#, y_n)) \in \Delta^\#$, let

$$dom(\tau^\#) = \{{x_1}^\#, \dots, {x_n}^\#\}$$

and for every ${x_i}^\# \in dom(\tau^\#)$, set $\tau^\#({x_i}^\#) \simeq y_i$.

We shall assume that $dom(\tau^\#) = \varnothing$ if $\tau^\# = 0$.

Notice that $dom(\tau^{\#}) = \{x^{\#} : x \in dom(\tau)\}$ and for every $x \in dom(\tau)$, $f^*(\tau^{\#}(x^{\#})) \simeq f(\tau^{\#}(x^{\#})/2) \simeq \tau(x)$.

Let $R \subseteq A^n$ and $\tau \in \Delta$. Recall the definition of the finite function $\tau^{-1}(R)$ given in the previous subsection. Clearly

$$\tau^{-1}(R)(u) \simeq 1 \iff (\exists x_1{}^{\#}, \dots, x_n{}^{\#} \in dom(\tau^{\#}))(u = \langle x_1, \dots, x_n\rangle \ \& \ \langle \tau^{\#}(x_1{}^{\#})/2, \dots, \tau^{\#}(x_n{}^{\#})/2\rangle \in f^{-1}(R)) \tag{3.2.1}$$

and

$$\tau^{-1}(R)(u) \simeq 0 \iff (\exists x_1{}^{\#}, \dots, x_n{}^{\#} \in dom(\tau^{\#}))(u = \langle x_1, \dots, x_n\rangle \ \& \ \langle \tau^{\#}(x_1{}^{\#})/2, \dots, \tau^{\#}(x_n{}^{\#})/2\rangle \notin f^{-1}(R)). \tag{3.2.2}$$

By (3.2.1) and (3.2.2), there exists a computable function ρ such that for every $\tau \in \Delta$, $\tau^{-1}(\mathcal{A}) = \{\rho(\tau^{\#})\}^{f^{-1}(\mathcal{A})}$.

It is easy to see that there exists a computable predicate P such that for all $\tau, \delta \in \Delta$, $P(\tau^{\#}, \delta^{\#}) \simeq 1 \iff \tau \subseteq \delta$.

Thus we obtain that

$$(f^*)^{-1}(K_{\mathcal{A}}) = \{J_3(\delta^{\#}, e^{\#}, x^{\#}) : (\exists \tau \in \Delta)(\delta \subseteq \tau \ \& \ \tau \Vdash F_e(x))\} =$$
$$\{J_3(\delta^{\#}, e^{\#}, x^{\#}) : (\exists \tau^{\#} \in \Delta^{\#})(P(\delta^{\#}, \tau^{\#}) \simeq 1 \ \& \ x \in W_e^{\{\rho(\tau^{\#})\}^{f^{-1}(\mathcal{A})}})\}.$$

Hence $(f^*)^{-1}(K_{\mathcal{A}})$ is c.e. in $f^{-1}(\mathcal{A})$. From here it follows that $(f^*)^{-1}(K_{\mathcal{A}}) \leq_T f^{-1}(\mathcal{A})'$. Therefore, by Proposition 3.1.7, $(f^*)^{-1}(\mathcal{B}) \leq_T f^{-1}(\mathcal{A})'$.

□

Now we turn to the proof of the reverse inclusion. We shall need the following property of the jump spectrum:

Lemma 3.2.3. Every jump spectrum is closed upwards.

Proof. Consider a structure $\mathcal{A}$. Let $\mathbf{b}$ be a degree, $\mathbf{b} \geq \mathbf{a}$ and $\mathbf{a} \in DS_1(\mathcal{A})$. Then for some $\mathbf{c} \in DS(\mathcal{A})$, $\mathbf{c}' = \mathbf{a}$. By the relativized jump inversion theorem of Friedberg, there is a degree $\mathbf{d} \geq \mathbf{c}$ such that $\mathbf{d}' = \mathbf{b}$. By Proposition 2.4.3, $\mathbf{d} \in DS(\mathcal{A})$. Thus $\mathbf{b} = \mathbf{d}' \in DS_1(\mathcal{A})$. □

Proposition 3.2.4. $DS(\mathcal{B}) \subseteq DS_1(\mathcal{A})$.

Proof. Let $\mathbf{a} \in DS(\mathcal{B})$ and m be an enumeration of $\mathcal{B}$ such that $m^{-1}(\mathcal{B}) \in \mathbf{a}$. By Proposition 2.4.2, there exists an injective enumeration f of $\mathcal{B}$ such that $f^{-1}(\mathcal{B}) \leq_T m^{-1}(\mathcal{B})$. We shall construct an enumeration g of the structure $\mathcal{A}$ such that $g^{-1}(\mathcal{A})' \leq_{\mathrm{T}} f^{-1}(\mathcal{B})$. Then, by Lemma 3.2.3, $\mathbf{a} \in DS_1(\mathcal{A})$.

Recall that $\mathcal{B} = \mathcal{A}'$. Let $f^{-1}(A) = A^\#$ and $f^{-1}(K_\mathcal{A}) = K^\#$. Clearly the sets $A^\#$ and $K^\#$ are computable in $f^{-1}(\mathcal{B})$. defnine the computable in $f^{-1}(\mathcal{B})$ function J by $J(x,y) = f^{-1}(\Pi(f(x), f(y)))$. Clearly there exist computable in $f^{-1}(\mathcal{B})$ functions l and r such that for all $x, y \in N$,

$$l(J(x,y)) = x \text{ and } r(J(x,y)) = y.$$

Set $J_1(x_1) = x_1$ and $J_{n+1}(x_1, \dots, x_{n+1}) = J(x_1, J_n(x_2, \dots, x_{n+1}))$.

For every natural number x consider the element x^* of A^* and let $x^\# = f^{-1}(x^*)$. Let $N^\# = \{x^\# : x \in N\}$. Now, we have that $N^\#$ is computable in $f^{-1}(\mathcal{B})$ and that there exist computable in $f^{-1}(\mathcal{B})$ functions n_1 and n_2 such that for all $x \in N$, $n_1(x) = x^\#$ and $n_2(x^\#) = x$.

Given a partial mapping h of N in A, by $h^\#$ we shall denote the unique mapping of $N^\#$ in $A^\#$ satisfying for all natural numbers x the equality:

$$h^\#(x^\#) \simeq f^{-1}(h(x)).$$

Clearly for all partial mappings h_1 and h_2 of N in A,

$$h_1 \subseteq h_2 \iff {h_1}^\# \subseteq {h_2}^\#.$$

For finite parts τ we shall identify $\tau^\#$ and its code $f^{-1}(\tau^*)$. Denote by Δ the set of all finite parts and let $\Delta^\# = \{\tau^\# : \tau \in \Delta\}$. Notice that the set $\Delta^\#$ is computable in $f^{-1}(\mathcal{B})$.

As in the proof of the previous proposition one can easily see that there exists a computable in $f^{-1}(\mathcal{B})$ function ρ such for every finite part τ, $\tau^{-1}(\mathcal{A}) = \{\rho(\tau^\#)\}^{f^{-1}(\mathcal{B})}$.

Now we turn to the construction of the enumeration g. We shall construct g as a generic enumeration such that $g^\#$ is computable in $f^{-1}(\mathcal{B})$.

The enumeration g will be constructed by stages. At each stage s we shall define a finite part τ_s so that $\tau_s \subseteq \tau_{s+1}$ and let $g = \bigcup_s \tau_s$.

From the construction it will follow that the function $\lambda s.{\tau_s}^\#$ is computable in $f^{-1}(\mathcal{B})$ and hence the mapping $g^\#$ is also computable in $f^{-1}(\mathcal{B})$.

We shall consider two kinds of stages. On stages $s = 2r$ we shall ensure that the mapping g is total and surjective. On stages $s = 2r + 1$ we shall ensure that g is generic.

Let $\tau_0 = \emptyset$. Suppose that we have already defined τ_s.

(a) Case $s = 2r$. Let x be the least natural number such that $x^\#$ does not belong to $dom({\tau_s}^\#)$ and let y be the least natural number in $A^\#$ which does not belong to the range of $\tau_s{}^{\#}$. Set $\tau_{s+1}(x) = f(y)$ and $\tau_{s+1}(z) \simeq \tau_s(z)$ for $z \neq x$.

(b) Case $s = 2\langle e, x\rangle + 1$. Consider the set $X_{\langle e,x\rangle} = \{\delta \mid \delta \Vdash F_e(x)\}$. Check whether there exists a finite part $\delta \in X_{\langle e,x\rangle}$ which extends τ_s. Clearly this is equivalent to $J_3(\tau_s{}^{\#}, e^{\#}, x^{\#}) \in K^{\#}$.

If the answer is negative then $\tau_s \Vdash \neg F_e(x)$. Set $\tau_{s+1} = \tau_s$.

In the case of a positive answer find a $\delta^{\#}$ such that $\tau_s{}^{\#} \subseteq \delta^{\#}$ and

$$x \in W_e^{\{\rho(\delta^{\#})\}^{f^{-1}(\mathcal{B})}}.$$

We can do that effectively in $f^{-1}(\mathcal{B})$ by enumerating all triples $(\delta^{\#}, t_1, t_2)$, where $\tau_s{}^{\#} \subseteq \delta^{\#}$, $t_1, t_2 \in N$ and checking for every such triple whether

$$x \in W_{e,t_1}^{\{\rho(\delta^{\#})\}_{t_2}^{f^{-1}(\mathcal{B})}}.$$

Set $\tau_{s+1} = \delta$.

End of the construction

By the genericity of g,

$$x \in g^{-1}(\mathcal{A})' \iff g \models F_x(x) \iff (\exists \tau \subseteq g)(\tau \Vdash F_x(x)) \iff (\exists \tau^{\#} \subseteq g^{\#})(x \in W_x^{\{\rho(\tau^{\#})\}^{f^{-1}(\mathcal{B})}}).$$

and

$$x \in N \setminus g^{-1}(\mathcal{A})' \iff g \models \neg F_x(x) \iff (\exists \tau \subseteq g)(\tau \Vdash \neg F_x(x)) \iff (\exists \tau^{\#} \subseteq g^{\#})(J_3(\tau^{\#}, x^{\#}, x^{\#}) \notin K^{\#}).$$

Since $g^{\#}$ is computable in $f^{-1}(\mathcal{B})$, we get from here that $g^{-1}(\mathcal{A})'$ and $N \setminus g^{-1}(\mathcal{A})'$ are c.e. in $f^{-1}(\mathcal{B})$ and hence $g^{-1}(\mathcal{A})' \leq_T f^{-1}(\mathcal{B})$. □

The proof of the theorem is concluded. □

3.3 Jump inversion theorem

Naturally, once we have a jump of a structure, the question of jump inversion arises: Given a structure $\mathcal{A}$ with $DS(\mathcal{A})$ consisting of total degree above $\mathbf{0}'$, is there a structure $\mathcal{C}$ such that $DS_1(\mathcal{C}) = DS(\mathcal{A})$. We will prove an even more general Friedberg's style jump inversion theorem. Let $\mathcal{A}$ and $\mathcal{B}$ be structures such that $DS(\mathcal{A}) \subseteq DS_1(\mathcal{B})$ (so, all elements of $DS(\mathcal{A})$ are above $\mathbf{0}'$). Then there exists a structure $\mathcal{C}$ such that $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_1(\mathcal{C}) = DS(\mathcal{A})$.

The proof of this theorem uses the method of Marker extensions, which will be discussed in detail in the next subsection.

3.3.1 Marker's Extensions

Marker [Mar89] presented a method of constructing for any $n \geq 1$ an $\aleph_0$-categorical almost strongly minimal theory which is not Σ_n-axiomatizable. Further Goncharov and Khoussainov [GK02] adapted the construction to the general case in order to find for any $n \geq 1$ examples of $\aleph_1$-categorical computable models as well as $\aleph_0$-categorical computable models whose theories are Turing equivalent to $\varnothing^{(n)}$. We shall give the definition of Marker's $\exists$ and $\forall$ extensions following [GK02].

Let $\mathcal{A} = (A; R_1, \ldots, R_s, =)$ be a countable structure such that each predicate R_i has arity r_i.

Marker's $\exists$-extension of R_i, denoted by $R_i^{\exists}$, is defined as follows. Consider a set X_i with new elements such that $X_i = \{x^i_{\langle a_1,\ldots,a_{r_i}\rangle} \mid R_i(a_1, \ldots, a_{r_i})\}$. We shall call the set X_i an $\exists$-fellow for R_i. We suppose that all sets A, X_1,..., X_s are pairwise disjoint.

The predicate $R_i^{\exists}$ is a predicate of arity $r_i + 1$ such that

$$R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \iff a_1, \ldots, a_{r_i} \in A \ \&\ x \in X_i \ \&\ x = x^i_{\langle a_1,\ldots,a_{r_i}\rangle}.$$

The property of $R_i^{\exists}$ is that for every $a_1, \ldots, a_{r_i} \in A$

$$(\exists x \in X_i) R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \iff R_i(a_1, \ldots, a_{r_i}). \tag{3.3.1}$$

Definition 3.3.1. The structure $\mathcal{A}^{\exists}$ is defined as follows:

$$(A \cup \bigcup_{i=1}^{s} X_i; R_1^{\exists}, \ldots, R_s^{\exists}, X_1, \ldots, X_s, =),$$

where each $R_i^{\exists}$ is the Marker's $\exists$-extension of R_i with the $\exists$-fellow X_i.

Further, Marker's $\forall$-extension of $R_i^{\exists}$, denoted by $R_i^{\exists\forall}$, is defined as follows. Consider an infinite set Y_i of new elements such that

$$Y_i = \{y^i_{\langle a_1,\ldots,a_{r_i},x\rangle} : \neg R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \ \&\ a_1, \ldots, a_{r_i} \in A,\ \&\ x \in X_i\}.$$

We shall call the set Y_i a $\forall$-fellow for $R_i^{\exists}$. We suppose that all sets A, $X_1,\dots,$ X_s and $Y_1,\dots,$ Y_s are pairwise disjoint.

The predicate $R_i^{\exists\forall}$ is a predicate of arity r_i+2 such that

1. If $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$ then $a_1,\dots,a_{r_i}\in A$, $x\in X_i$ and $y\in Y_i$;
2. If $a_1,\dots,a_{r_i}\in A,$ & $x\in$ X_i & $y\in Y_i$ then

$$\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y) \iff y = y^i_{\langle a_1,\dots,a_{r_i},x\rangle}\ .$$

From the definition of $R_i^{\exists\forall}$ it follows that if $a_1,\dots,a_{r_i}\in A$ and $x\in X_i$ then

$$(\forall y\in Y_i)R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y) \iff R_i^{\exists}(a_1,\dots,a_{r_i},x). \tag{3.3.2}$$

Definition 3.3.2. The structure $\mathcal{A}^{\exists\forall}$ is defined as follows

$$(A\cup\bigcup_{i=1}^{s}X_i\cup\bigcup_{i=1}^{s}Y_i;R_1^{\exists\forall},\dots,R_s^{\exists\forall},X_1,\dots,X_s,Y_1,\dots,Y_s,=),$$

where X_i is the $\exists$-fellow for R_i and Y_i is the $\forall$-fellow for $R_i^{\exists}$.

The structure $\mathcal{A}^{\exists\forall}$ has the following properties:

Proposition 3.3.3. 1. Let $a_1,\dots,a_{r_i}\in A$. Then:

(a) $R_i(a_1,\dots,a_{r_i}) \iff (\exists x\in X_i)(\forall y\in Y_i)R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

(b) If $R_i(a_1,\dots,a_{r_i})$ then there exists a unique $x\in X_i$ such that $(\forall y\in Y_i)R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

2. For each sequence $a_1,\dots,a_{r_i}\in A$ and $x\in X_i$ there exists at most one $y\in Y_i$ such that $\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

3. For each $y\in Y_i$ there exists a unique sequence $a_1,\dots,a_{r_i}\in A$ and $x\in X_i$ such that $\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

4. For each $x\in X_i$ there exists a unique sequence $a_1,\dots,a_{r_i}\in A$ such that for all $y\in Y_i$ the predicate $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$ is true.

Proof. 1. (a)($\Rightarrow$) Let $R_i(a_1,\dots,a_{r_i})$. Then by (3.3.1) there exists $x\in X_i$ such that $R_i^{\exists}(a_1,\dots,a_{r_i},x)$ (in fact $x = x^i_{\langle a_1,\dots,a_{r_i}\rangle}$). By (3.3.2) it follows that for every $y\in Y_i$ $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$.

($\Leftarrow$) Let $x\in X_i$ and $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$ for all $y\in Y_i$. Then by (3.3.2) $R_i^{\exists}(a_1,\dots,a_{r_i},x)$ and hence by (3.3.1) $R_i(a_1,\dots,a_{r_i})$.

1. (b) Follows from the definition of X_i and (3.3.2).
2. Follows from (3.3.2) and the definition of Y_i.
3. Follows from the definition of Y_i.
4. Let $x \in X_i$. Then $x = x^i_{\langle a_1,\dots,a_{r_i}\rangle}$ for some $a_1,\dots,a_{r_i}$ from A such that $R_i(a_1,\dots,a_{r_i})$. Hence $R_i^{\exists}(a_1,\dots,a_{r_i},x)$. Then, by (3.3.2), there is no $y \in Y_i$ such that $\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$. Clearly for every sequence $b_1,\dots,b_{r_i} \in A$ not equal to $a_1,\dots,a_{r_i}$, $R_i^{\exists}(b_1,\dots,b_{r_i},x)$ is false and hence for $y = y^i_{\langle b_1,\dots,b_{r_i},x\rangle}$ the predicate $R_i^{\exists\forall}(b_1,\dots,b_{r_i},x,y)$ is false.

□

3.3.2 Join of Two Structures

Let $\mathcal{A} = (A; R_1,\dots,R_s,=)$ and $\mathcal{B} = (B; P_1,\dots,P_t,=)$ be countable structures in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Suppose that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{=\}$ and $A \cap B = \varnothing$. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{A, B\}$, where A and B are unary predicates.

Definition 3.3.4. *The join of the structures $\mathcal{A}$ and $\mathcal{B}$* is the structure $\mathcal{A}\oplus\mathcal{B} = (A \cup B; R_1,\dots,R_s,P_1,\dots,P_t,A,B,=)$ in the language $\mathcal{L}$, where

(a) the predicate A is true only over the elements of A and similarly B is true only over the elements of B;

(b) each predicate R_i is defined on the elements of A as in the structure $\mathcal{A}$ and false if some of the arguments of R_i are not in A and similarly each predicate P_j is defined as in the structure $\mathcal{B}$ over the elements of B and false if some of the arguments of P_j are not in B.

Lemma 3.3.5. Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures and $\mathcal{C} = \mathcal{A}\oplus\mathcal{B}$. Then $DS(\mathcal{C}) \subseteq DS(\mathcal{A})$ and $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$.

Proof. We shall prove that $DS(\mathcal{C}) \subseteq DS(\mathcal{A})$. The proof of $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ is similar.

Let f be an enumeration of $\mathcal{C}$. Fix $z_0 \in f^{-1}(A)$. defnine

$$\begin{aligned} &m(0) = z_0;\\ &m(i+1) = \mu z \in f^{-1}(A)[(\forall k \le i)(\langle m(k), z\rangle \notin f^{-1}(=))]. \end{aligned}$$

Set $h = \lambda x.f(m(x))$. Note that $m \le_{\mathrm{T}} f^{-1}(\mathcal{C})$ and the enumeration h of $\mathcal{A}$ is injective and hence $h^{-1}(=)$ is computable. Moreover

$$\begin{aligned} \langle x_1,\dots,x_{r_i}\rangle \in h^{-1}(R_i) &\iff R_i(f(m(x_1)),\dots,f(m(x_{r_i})))\\ &\iff \langle m(x_1),\dots,m(x_{r_i})\rangle \in f^{-1}(R_i). \end{aligned}$$

Thus $h^{-1}(R_i) \leq_{\mathrm{T}} f^{-1}(\mathcal{C})$.

Then $h^{-1}(\mathcal{A}) \leq_{\mathrm{T}} f^{-1}(\mathcal{C})$. Since $DS(\mathcal{A})$ is closed upwards, $\mathrm{d_T}(f^{-1}(\mathcal{C})) \in DS(\mathcal{A})$. □

3.3.3 Representation of $\Sigma^0_2(D)$ Sets

Let $D \subseteq N$. A set $M \subseteq N$ is in $\Sigma^0_2(D)$ if there exists a computable in D predicate Q such that

$$n \in M \iff \exists a \forall b Q(n,a,b) \ .$$

Definition 3.3.6. [GK02] If $M \in \Sigma^0_2(D)$ then M is *one-to-one representable* if there exists a computable in D predicate Q with the following properties:

1. $n \in M \iff \exists a \forall b Q(n,a,b)$;
2. $n \in M \iff$ there exists a unique a such that $\forall b Q(n,a,b)$;
3. for every pair $\langle n,a\rangle$ there is at most one b such that $\neg Q(n,a,b)$;
4. for every b there is a unique pair $\langle n,a\rangle$ such that $\neg Q(n,a,b)$;
5. for every a there exists a unique n such that $\forall b Q(n,a,b)$.

The predicate Q from the above definition is called *an one-to-one representation of M*. Goncharov and Khoussainov [GK02] proved the following lemma:

Lemma 3.3.7. If M is a co-infinite $\Sigma^0_2(D)$ subset of N and there is an infinite computable in D subset S of M such that $M \setminus S$ is infinite, then M has an one-to-one representation.

Remark. We will use the lemma in the next subsection in the proof of Theorem 3.3.9. In order to satisfy the conditions of the lemma we need the following technical explanations.

Let $\mathcal{A} = (A; R_1, \ldots, R_s, =)$ be a countable structure. Recall that the set A is infinite. We can easily find a structure $\mathcal{A}^\#$ with the same degree spectrum as $\mathcal{A}$ and such that for every injective enumeration $f^\#$ of $\mathcal{A}^\#$ and for each predicate R of $\mathcal{A}^\#$ the set ${f^\#}^{-1}(R)$ is co-infinite and there is a computable infinite subset S of ${f^\#}^{-1}(R)$ such that ${f^\#}^{-1}(R) \setminus S$ is infinite.

One way to do this is the following. We add to the domain A of the structure $\mathcal{A}$ two new elements say "T" and "F". For each r-ary predicate R of $\mathcal{A}$ define a $(r+1)$-ary predicate $R^{\#}$ as follows:

$$R^{\#}(a_1,\dots,a_r,b) = \begin{cases} true & \text{if T} \in \{a_1,\dots,a_r,b\}; \\ false & \text{if F} \in \{a_1,\dots,a_r,b\} \ \&\ \text{T} \notin \{a_1,\dots,a_r,b\}; \\ R(a_1,\dots,a_r) & \text{if F ,T} \notin \{a_1,\dots,a_r,b\}. \end{cases}$$

Let $\mathcal{A}^{\#} = (A \cup \{\text{T},\text{F}\}; {R_1}^{\#},\dots,{R_s}^{\#},=)$.

Lemma 3.3.8. $DS(\mathcal{A}) = DS(\mathcal{A}^{\#})$ and for every injective enumeration $f^{\#}$ of $\mathcal{A}^{\#}$ and each nontrivial predicate ${R_i}^{\#}$ the set ${f^{\#}}^{-1}({R_i}^{\#})$ is co-infinite and there is a computable infinite set $S \subseteq {f^{\#}}^{-1}({R_i}^{\#})$ such that ${f^{\#}}^{-1}({R_i}^{\#}) \setminus S$ is infinite.

Proof. For each injective enumeration f of $\mathcal{A}$ we construct an enumeration $f^{\#}$ of $\mathcal{A}^{\#}$ as follows: $f^{\#}(0) =$T, $f^{\#}(1) =$F and $f^{\#}(x+2) = f(x)$. Then

$$\begin{aligned} &\langle x_1,\dots,x_{r_i},z\rangle \in {f^{\#}}^{-1}({R_i}^{\#}) \iff (0 \in \{x_1,\dots,x_{r_i},z\}) \vee \\ &(0,1 \notin \{x_1,\dots,x_{r_i},z\} \ \&\ \langle x_1-2,\dots,x_{r_i}-2\rangle \in f^{-1}(R_i)). \end{aligned}$$

It is obvious that ${f^{\#}}^{-1}({R_i}^{\#}) \leq_T^{-1} (R_i)$. Moreover let $c \neq 0,1$.

$$\langle x_1,\dots,x_{r_i}\rangle \in f^{-1}(R_i) \iff \langle x_1+2,\dots,x_{r_i}+2,c\rangle \in {f^{\#}}^{-1}({R_i}^{\#}).$$

So ${f^{\#}}^{-1}({R_i}^{\#}) \equiv_T f^{-1}(R_i)$.

For each injective enumeration $f^{\#}$ of $\mathcal{A}^{\#}$ we construct an injective enumeration f of $\mathcal{A}$ as follows. Let $tt = {f^{\#}}^{-1}(\text{T}), ff = {f^{\#}}^{-1}(\text{F})$ and $a \in {f^{\#}}^{-1}(A)$.

$$\begin{aligned} &m(0) = a; \\ &m(i+1) = \mu z[(\forall k \leq i)(z \neq m(k) \ \&\ z \neq tt \ \&\ z \neq ff]. \end{aligned}$$

Set $f = \lambda x.f^{\#}(m(x))$. Then

$$\langle x_1,\dots,x_{r_i}\rangle \in f^{-1}(R_i) \iff \langle m(x_1),\dots,m(x_{r_i}),a\rangle \in {f^{\#}}^{-1}({R_i}^{\#}).$$

$$\begin{aligned} &\langle x_1,\dots,x_{r_i},z\rangle \in {f^{\#}}^{-1}({R_i}^{\#}) \iff (tt \in \{x_1,\dots,x_{r_i},z\}) \vee \\ &(tt,ff \notin \{x_1,\dots,x_{r_i},z\} \ \&\ \langle m^{-1}(x_1),\dots,m^{-1}(x_{r_i})\rangle \in f^{-1}(R_i)). \end{aligned}$$

So $f^{-1}(R_i) \equiv_T {f^{\#}}^{-1}({R_i}^{\#})$.

In order to see that $DS(\mathcal{A}) \subseteq DS(\mathcal{A}^{\#})$ let $\mathbf{a} \in DS(\mathcal{A})$ and let h be an enumeration of $\mathcal{A}$, $h^{-1}(\mathcal{A}) \in \mathbf{a}$. By Proposition 2.4.2, there exists an injective enumeration f of $\mathcal{A}$ such that $f^{-1}(\mathcal{A}) \leq_T h^{-1}(\mathcal{A})$. Then let $f^{\#}$ be the enumeration of $\mathcal{A}^{\#}$ constructed above and so $f^{-1}(\mathcal{A}) \equiv_{\mathrm{T}} {f^{\#}}^{-1}(\mathcal{A}^{\#})$. Then by Proposition 2.4.3 we have that $\mathbf{a} \in DS(\mathcal{A}^{\#})$. The proof of $DS(\mathcal{A}^{\#}) \subseteq DS(\mathcal{A})$ is similar.

For each injective enumeration $f^{\#}$ of $\mathcal{A}^{\#}$ the set ${f^{\#}}^{-1}({R_i}^{\#})$ is co-infinite since the set $\{\langle x_1, \dots, x_{r_i}, z\rangle \mid ff \in \{x_1, \dots, x_{r_i}, z\} \;\&\; tt \notin \{x_1, \dots, x_{r_i}, z\}\}$ is infinite, here $tt = {f^{\#}}^{-1}(\mathrm{T}), ff = {f^{\#}}^{-1}(\mathrm{F})$. There is an infinite computable subset $S = \{\langle x_1, \dots, x_{r_i}, z\rangle \mid tt \in \{x_1, \dots, x_{r_i}, z\}\}$ of ${f^{\#}}^{-1}({R_i}^{\#})$. Moreover ${f^{\#}}^{-1}({R_i}^{\#}) \setminus S$ is infinite. Let $a_1, \dots, a_{r_i} \in A$ such that $R_i(a_1, \dots, a_{r_i})$. The set $\{\langle {f^{\#}}^{-1}(a_1), \dots, {f^{\#}}^{-1}(a_{r_i}), z\rangle \mid z \in N \;\&\; z \notin \{tt, ff\}\} \subseteq {f^{\#}}^{-1}({R_i}^{\#}) \setminus S$ is infinite.

Note that actually the set ${f^{\#}}^{-1}(\neg {R_i}^{\#})$ is also co-infinite and there is an infinite computable subset P of ${f^{\#}}^{-1}(\neg {R_i}^{\#})$, so that ${f^{\#}}^{-1}(\neg {R_i}^{\#}) \setminus P$ is infinite. □

3.3.4 The Jump Inversion theorem

Theorem 3.3.9. Let $\mathcal{A}$ and $\mathcal{B}$ be structures such that $DS(\mathcal{A}) \subseteq DS_1(\mathcal{B})$. then there exists a structure $\mathcal{C}$ such that $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_1(\mathcal{C}) = DS(\mathcal{A})$.

Proof. Let $\mathcal{A} = (A; R_1, \dots, R_s, =)$. For every predicate R_i consider a new predicate R_i^c which is equal to the negation of R_i, i.e.

$$R_i^c(a_1, \dots, a_{r_i}) \iff \neg R_i(a_1, \dots, a_{r_i}),$$

for every $a_1, \dots, a_{r_i} \in A$.

By Lemma 3.3.8 we may suppose that for every injective enumeration f of $\mathcal{A}$ and each nontrivial predicate R_i the sets $f^{-1}(R_i)$ and $f^{-1}(R_i^c)$ are co-infinite and there are computable infinite sets $S \subseteq f^{-1}(R_i)$ and $P \subseteq f^{-1}(R_i^c)$ such that $f^{-1}(R_i) \setminus S$ and $f^{-1}(R_i^c) \setminus P$ are infinite.

We extend the structure $\mathcal{A}$ including the negations of the predicates as follows:

$$\overline{\mathcal{A}} = (A; R_1, R_1^c, \dots, R_s, R_s^c, =).$$

We will denote the new structure by $\overline{\mathcal{A}} = (A; \overline{R}_1, \overline{R}_2, \dots, \overline{R}_{2s-1}, \overline{R}_{2s}, =)$, where $\overline{R}_{2i-1} = R_i$ and $\overline{R}_{2i} = R_i^c$ for $i = 1, \dots, s$.

It is clear that $DS(\mathcal{A}) = DS(\overline{\mathcal{A}})$ since for each enumeration f of $\mathcal{A}$ we have that $f^{-1}(\mathcal{A}) \equiv_{\mathrm{T}} f^{-1}(\overline{\mathcal{A}})$.

Consider now the structure $\overline{\mathcal{A}}^{\exists\forall}$. Let X_j be the $\exists$-fellow of $\overline{R}_j$ and Y_j be the $\forall$-fellow of $\overline{R}_j^{\exists}$, $j = 1, \dots, 2s$.

Without loss of generality we may assume that the structures $\mathcal{B} = (B; P_1, \dots, P_t, =)$ and $\overline{\mathcal{A}}^{\exists\forall}$ are disjoint.

Let $\mathcal{C} = \mathcal{B} \oplus \overline{\mathcal{A}}^{\exists\forall}$. By Lemma 3.3.5, $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$. We shall prove that $DS_1(\mathcal{C}) = DS(\overline{\mathcal{A}})$.

We start with the proof of the inclusion $DS_1(\mathcal{C}) \subseteq DS(\overline{\mathcal{A}})$.

Let $\mathbf{c} \in DS_1(\mathcal{C})$ and let g be an enumeration of $\mathcal{C}$ such that $\mathbf{c} = \mathrm{d_T}(g^{-1}(\mathcal{C}))'$. By Proposition 2.4.2 there is an injective enumeration h of $\mathcal{C}$ such that $h^{-1}(\mathcal{C}) \le_{\mathrm{T}} g^{-1}(\mathcal{C})$. We shall construct an enumeration f of $\overline{\mathcal{A}}$ such that $f^{-1}(\overline{\mathcal{A}}) \le_{\mathrm{T}} h^{-1}(\mathcal{C})'$ and hence $f^{-1}(\overline{\mathcal{A}}) \le_{\mathrm{T}} g^{-1}(\mathcal{C})'$. Then by Proposition 2.4.3, $\mathbf{c} \in DS(\overline{\mathcal{A}})$.

We have

$$z \in h^{-1}(A) \iff (\forall j \le 2s)(z \notin h^{-1}(X_j) \ \& \ z \notin h^{-1}(Y_j)) \ \& \ z \notin h^{-1}(B).$$

Thus $h^{-1}(A) \le_{\mathrm{T}} h^{-1}(\mathcal{C})$.

Fix $x_0 \in h^{-1}(A)$. Let
$m(0) = x_0$; $m(i+1) = \mu z \in h^{-1}(A)[(\forall k \le i)(m(k) \ne z)]$.

Clearly $m \le_{\mathrm{T}} h^{-1}(\mathcal{C})$.

Set $f = \lambda a.h(m(a))$. Note that the enumeration f is injective.

Let R be an r-ary predicate of $\overline{\mathcal{A}}$, X be the $\exists$-fellow of R and Y be the $\forall$-fellow of $R^{\exists}$.

By Proposition 3.3.3, we have

$$\begin{aligned}
&\langle a_1, \dots, a_r\rangle \in f^{-1}(R) \iff R(f(a_1), \dots, f(a_r)) \iff \\
&(\exists a \in X)(\forall b \in Y) R^{\exists\forall}(f(a_1), \dots, f(a_r), a, b) \iff \\
&(\exists x \in h^{-1}(X))(\forall y \in h^{-1}(Y)) R^{\exists\forall}(h(m(a_1)), \dots, h(m(a_r)), h(x), h(y)) \iff \\
&(\exists x \in h^{-1}(X))(\forall y \in h^{-1}(Y))(\langle m(a_1), \dots, m(a_r), x, y\rangle \in h^{-1}(R^{\exists\forall})) \iff \\
&(\exists x)(\forall y)(\langle m(a_1), \dots, m(a_r), x, y\rangle \in h^{-1}(R^{\exists\forall}) \ \& \ x \in h^{-1}(X) \ \& \ y \in h^{-1}(Y)).
\end{aligned}$$

Hence $f^{-1}(R) \in \Sigma^0_2(h^{-1}(\mathcal{C}))$.

Consider now the complement predicate R^c and let X^c be the $\exists$-fellow for R^c and Y^c be the $\forall$-fellow for $(R^c)^{\exists}$. We have again:

$$\begin{aligned}
&\langle a_1, \dots, a_r\rangle \in f^{-1}(R^c) \iff R^c(f(a_1), \dots, f(a_r)) \iff \\
&(\exists a \in X^c)(\forall b \in Y^c)(R^c)^{\exists\forall}(f(a_1), \dots, f(a_r), a, b) \iff \\
&(\exists x \in h^{-1}(X^c))(\forall y \in h^{-1}(Y^c))(\langle m(a_1), \dots, m(a_r), x, y\rangle \in h^{-1}(R^c)^{\exists\forall}).
\end{aligned}$$

Thus $f^{-1}(R^c) \in \Sigma^0_2(h^{-1}(\mathcal{C}))$. Therefore $f^{-1}(R) \in \Delta^0_2(h^{-1}(\mathcal{C}))$ and hence

$$f^{-1}(R) \leq_{\mathrm{T}} h^{-1}(\mathcal{C})'.$$

So, $f^{-1}(\overline{\mathcal{A}}) \leq_{\mathrm{T}} h^{-1}(\mathcal{C})'$.

Now we turn to the proof of the reverse inclusion $DS(\overline{\mathcal{A}}) \subseteq DS_1(\mathcal{C})$.

Let $\mathbf{a} \in DS(\overline{\mathcal{A}})$ and let n be an enumeration of $\overline{\mathcal{A}}$ such that $\mathbf{a} = \mathrm{d_T}(n^{-1}(\overline{\mathcal{A}}))$. By Proposition 2.4.2, there is an injective enumeration f of $\overline{\mathcal{A}}$ such that $f^{-1}(\overline{\mathcal{A}}) \leq_{\mathrm{T}} n^{-1}(\overline{\mathcal{A}})$. We are going to construct an enumeration h of $\mathcal{C}$ such that $h^{-1}(\mathcal{C})' \leq_{\mathrm{T}} f^{-1}(\overline{\mathcal{A}})$. Since, by Lemma 3.2.3, $DS_1(\mathcal{C})$ is closed upwards we shall obtain that $\mathbf{a} \in \mathrm{DS}_1(\mathcal{C})$.

Recall that $DS(\overline{\mathcal{A}}) = DS(\mathcal{A}) \subseteq DS_1(\mathcal{B})$ and $\mathrm{d_T}(f^{-1}(\overline{\mathcal{A}})) \in DS(\overline{\mathcal{A}})$. Then there is an enumeration g of $\mathcal{B}$ such that $f^{-1}(\overline{\mathcal{A}}) \equiv_{\mathrm{T}} (g^{-1}(\mathcal{B}))'$. Set $D = g^{-1}(\mathcal{B})$. Consider the predicate $\overline{R}_j$. Let $\overline{R}_j$ be r-ary. Since $f^{-1}(\overline{\mathcal{A}}) \leq_{\mathrm{T}} D'$, we have that $f^{-1}(\overline{R}_j) \leq_{\mathrm{T}} D'$. Then $f^{-1}(\overline{R}_j) \in \Sigma^0_2(D)$. Set $M_j = f^{-1}(\overline{R}_j)$. the enumeration f is injective and hence the set M_j is co-infinite and there is a computable infinite set $S \subseteq M_j$ such that $M_j \setminus S$ is infinite. So M_j satisfies all conditions from Lemma 3.3.7. Then by Lemma 3.3.7 there exists a computable in D predicate Q_j which is a one-to-one representation of M_j. Then

1. $\langle n_1, \dots, n_r\rangle \in M_j \iff$ there exists a unique a such that $(\forall b) Q_j(\langle n_1, \dots, n_r\rangle, a, b)$;

2. for every b let $r(b) = \langle\langle n_1, \dots, n_r\rangle, a\rangle$ be the unique pair such that
$$\neg Q_j(\langle n_1, \dots, n_r\rangle, a, b);$$

3. for every a let $l(a) = \langle n_1, \dots, n_r\rangle$ be the unique $\langle n_1, \dots, n_r\rangle$ such that $\forall b Q_j(\langle n_1, \dots, n_r\rangle, a, b)$.

Let $\mathbb{N}_1 = \{\langle 1, n\rangle \mid n \in \mathbb{N}\}$, $\mathbb{N}_2 = \{\langle 2, j, a\rangle \mid j \leq 2s \ \&\ a \in N\}$ and $\mathbb{N}_3 = \{\langle 3, j, b\rangle \mid j \leq 2s \ \&\ b \in N\}$. Set $\mathbb{N}_0 = \mathbb{N} \setminus (\bigcup_{i=1}^3 \mathbb{N}_i)$. Consider a computable bijection m from N onto $\mathbb{N}_0$.

The definition of the enumeration h of $\mathcal{C}$ is the following:
$h(m(n)) = g(n)$;
$h(\langle 1, n\rangle) = f(n)$;
$h(\langle 2, j, a\rangle) = x^j_{\langle f(n_1), \dots, f(n_r)\rangle}$, if $l(a) = \langle n_1, \dots, n_r\rangle$;
$h(\langle 3, j, b\rangle) = y^j_{\langle f(n_1), \dots, f(n_r), h(\langle 2, j, a\rangle)\rangle}$, if $r(b) = \langle\langle n_1, \dots, n_r\rangle, a\rangle$.

Recall that $X_j = \{x^j_{\langle a_1,\dots,a_r\rangle} \mid \overline{R}_j(a_1,\dots,a_r)\}$ is the $\exists$-fellow for $\overline{R}_j$ and $Y_j = \{y^j_{\langle a_1,\dots,a_r,x\rangle} \mid \neg\overline{R}^{\exists}_j(a_1,\dots,a_r,x)\}$ is the $\forall$-fellow for $\overline{R}^{\exists}_j$.

From the choice of Y_j it follows that

$$\begin{aligned}\neg Q_j(\langle n_1,\dots,n_r\rangle,a,b) &\iff r(b) = \langle\langle n_1,\dots,n_r\rangle,a\rangle\\ &\iff h(\langle 3,j,b\rangle) = y^j_{\langle f(n_1),\dots,f(n_r),h(\langle 2,j,a\rangle)\rangle}\\ &\iff \neg\overline{R}^{\exists\forall}_j(f(n_1),\dots,f(n_r),h(\langle 2,j,a\rangle),h(\langle 3,j,b\rangle)).\end{aligned}$$

And then

$$Q_j(\langle n_1,\dots,n_r\rangle,a,b) \iff \overline{R}^{\exists\forall}_j(\langle f(n_1),\dots,f(n_r),h(\langle 2,j,a\rangle),h(\langle 3,j,b\rangle)).$$

defnine

$$\overline{R}^{\exists\forall,h}_j(\langle\langle 1,n_1\rangle,\dots,\langle 1,n_r\rangle,\langle 2,j,a\rangle,\langle 3,j,b\rangle\rangle) \iff Q_j(\langle n_1,\dots,n_r\rangle,a,b)\ .$$

It follows that $\overline{R}^{\exists\forall,h}_j \le_{\mathrm{T}} D$. Moreover

$$\begin{aligned}\overline{R}^{\exists\forall,h}_j(\langle\langle 1,n_1\rangle,\dots,\langle 1,n_r\quad\rangle,\langle 2,j,a\rangle,\langle 3,j,b\rangle\rangle) \iff&\\ \overline{R}^{\exists\forall}_j(h(\langle 1,n_1\rangle),\dots,h(\langle 1,n_r\rangle)),h(\langle 2,j,a\rangle),h(\langle 3,j,b\rangle))&\end{aligned}$$

So $\overline{R}^{\exists\forall,h}_j = h^{-1}(\overline{R}^{\exists\forall}_j)$ and hence $h^{-1}(\overline{R}^{\exists\forall}_j) \le_{\mathrm{T}} D$.

The sets $h^{-1}(A) = \mathbb{N}_1$, $h^{-1}(X_j) = \{\langle 2,j,a\rangle \mid a \in N\}$, $h^{-1}(Y_j) = \{\langle 3,j,b\rangle \mid b \in N\}$ are computable. Then $h^{-1}(\overline{\mathcal{A}}^{\exists\forall}) \le_{\mathrm{T}} D$.

Note that

$$\begin{aligned}\overline{R}_j(f(n_1),\dots,f(n_r)) &\iff \langle n_1,\dots,n_r\rangle \in f^{-1}(\overline{R}_j)\\ &\iff (\exists a)(\forall b)Q_j(\langle n_1,\dots,n_r\rangle,a,b)\\ &\iff (\exists a)(\forall b)\overline{R}^{\exists\forall,h}_j(\langle\langle 1,n_1\rangle\dots\langle 1,n_r\rangle,\langle 2,j,a\rangle,\langle 3,j,b\rangle\rangle)\\ &\iff (\exists x\in X_j)(\forall y\in Y_j)\overline{R}^{\exists\forall}_j(f(n_1),\dots,f(n_r),x,y).\end{aligned}$$

For every predicate P_j of $\mathcal{B}$ it holds that $h^{-1}(P_j) = \{\langle m(n_1),\dots,m(n_{p_j})\rangle \mid \langle n_1,\dots,n_{p_j}\rangle \in g^{-1}(P_j)\}$ and $h^{-1}(B) = \mathbb{N}_0$. It is obvious that $h^{-1}(\mathcal{B}) \le_{\mathrm{T}} D = g^{-1}(\mathcal{B})$.

The pullback of the equality is defined over the elements which are pullbacks of elements of B as $g^{-1}(=)$. Over the other elements the equality is defined in the usual way. So, $h^{-1}(=)$ is the set:

$$\{\langle x,y\rangle \mid (\langle m^{-1}(x),m^{-1}(y)\rangle \in g^{-1}(=)\ \&\ x,y\in\mathbb{N}_0) \vee (x = y\ \&\ x,y\notin\mathbb{N}_0)\}.$$

Then $h^{-1}(=) \leq_T D$. Thus $h^{-1}(\mathcal{B} \oplus \overline{\mathcal{A}}^{\exists\forall}) = h^{-1}(\mathcal{C}) \leq_T D = g^{-1}(\mathcal{B})$. Using that $g^{-1}(\mathcal{B})' \equiv_T f^{-1}(\overline{\mathcal{A}})$, we get from here that $h^{-1}(\mathcal{C})' \leq_T f^{-1}(\overline{\mathcal{A}})$. □

The next natural questions is if one can extend the jump inversion theorem to every constructive ordinal α. As we explained at the beginning of the chapter, Goncharov, Harizanov, Knight, McCoy, Miller and Solomon [GHK⁺05] show that this is true if α is a computable successor ordinal, even though they do not state their result in terms of the jump of a structure. This result was useful later on, for instance Greenberg, Montalbán and Slaman [GMS13] use it to build a structure whose spectrum consists of the non-hyperarithetic degrees. Vatev [Vat14, Vat13, Vat15] proves the α-jump inversion theorem for a computable successor ordinal α based on the construction in [GHK⁺05]. He shows also that for any structure $\mathcal{A}$ such that the elements of $DS(\mathcal{A})$ are above $\mathbf{0}^{(\alpha)}$ for a computable successor ordinal $\alpha \geq \omega$, there is a structure $\mathcal{C}$ such that $\mathcal{C}^{(\alpha)} \equiv_w \mathcal{A}$ and moreover for every $X \subseteq A$, $(X \in \Sigma^c_\alpha(\mathcal{C}) \iff X \in \Sigma^c_1(\mathcal{A}))$.

The problem of jump inversion for $\alpha = \omega$, or, in general, any computable limit ordinal remains open for longer. In one of his last papers Soskov [Sos13b] finally proves that there is a good reason for that.

Theorem 3.3.10. [Sos13b] There is a total structure $\mathcal{A}$ with $DS(\mathcal{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}_e^{(\omega)} \leq \mathbf{b}\}$ for which there is no structure $\mathcal{M}$ with $DS_\omega(\mathcal{M}) = DS(\mathcal{A})$.

The proof relies on an analysis of the ω-jump co-spectrum of a structure.

3.4 Some Applications

We will present some applications of the jump inversion theorems [SS09a, SS09b]. The jump inversion theorem proved in the previous section can be easily generalized in the following way. Let remain the definition of the jump spectra.

Definition 3.4.1. Given a structure $\mathcal{A}$ and $n \geq 0$, let the *nth jump spectrum* $DS_n(\mathcal{A})$ be the set $\{\mathbf{a}^{(n)} : \mathbf{a} \in DS(\mathcal{A})\}$.

We denote by $\mathcal{A}^{(n)}$ the n-th jump of structure $\mathcal{A}$ defined inductively:

$$\mathcal{A}^{(0)} = \mathcal{A}; \qquad \mathcal{A}^{(n+1)} = (\mathcal{A}^n)'.$$

Clearly $DS_0(\mathcal{A}) = DS(\mathcal{A})$ and $DS_{n+1}(\mathcal{A}) = \{\mathbf{a}' : \mathbf{a} \in DS_n(\mathcal{A})\}$. Using this and Theorem 3.2.1, one can easily see by induction on n that for every n there exists a structure $\mathcal{A}^{(n)}$ such that $DS_n(\mathcal{A}) = DS(\mathcal{A}^{(n)})$.

Theorem 3.4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be structures such that $DS(\mathcal{A}) \subseteq DS_n(\mathcal{B})$. Then there exists a structure $\mathcal{C}$ such that $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_n(\mathcal{C}) = DS(\mathcal{A})$.

Proof. Induction on n. The assertion is obvious for $n = 0$. Suppose that it is true for some n. Let $DS(\mathcal{A}) \subseteq DS_{n+1}(\mathcal{B})$. Consider a structure $\mathcal{B}^{(n)}$ such that $DS(\mathcal{B}^{(n)}) = DS_n(\mathcal{B})$. Clearly $DS(\mathcal{A}) \subseteq DS_1(\mathcal{B}^{(n)})$ and hence by Theorem 3.3.9 there exists a structure $\mathcal{C}^*$ such that $DS(\mathcal{C}^*) \subseteq DS(\mathcal{B}^{(n)})$ and $DS_1(\mathcal{C}^*) = DS(\mathcal{A})$. By the induction hypothesis, there exists a structure $\mathcal{C}$ such that $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_n(\mathcal{C}) = DS(\mathcal{C}^*)$. Then $DS_{n+1}(\mathcal{C}) = DS_1(\mathcal{C}^*) = DS(\mathcal{A})$. □

Recall that degree $\mathbf{a}$ is said to be the *nth jump degree* of a structure $\mathcal{A}$ if $\mathbf{a}$ is the least element of $DS_n(\mathcal{A})$. Notice that if $\mathbf{a}$ is the nth jump degree of $\mathcal{A}$ then for all k, $\mathbf{a}^{(k)}$ is the $(n+k)th$ jump degree of $\mathcal{A}$. Hence if a structure $\mathcal{A}$ possesses a nth jump degree then it possesses $(n+k)$th jump degrees for all k.

With respect to the jump degrees of $\mathcal{A}$ it does not matter whether we consider arbitrary enumerations of $\mathcal{A}$ or only injective enumerations of $\mathcal{A}$. Indeed, by Proposition 2.4.2, if $\mathbf{a}$ is the least element of the spectrum of $\mathcal{A}$ then $\mathbf{a} = d_{\mathrm{T}}(f^{-1}(\mathcal{A}))$ for some injective enumeration f.

The definitions above of the jump spectrum can be naturally generalized for all computable ordinals α. In [DK92] Downey and Knight proved with a fairly complicated construction that for every computable ordinal α there exists a linear order $\mathcal{A}$ such that $\mathcal{A}$ has αth jump degree equal to $\mathbf{0}^{(\alpha)}$ but for all $\beta < \alpha$, there is no βth jump degree of $\mathcal{A}$. Now we can obtain this theorem for the finite ordinals as an application of Theorem 3.3.9. Consider a structure $\mathcal{B}$ such that $DS(\mathcal{B})$ consists of all degrees above $\mathbf{0}^{(n)}$ and has no least element, and such that $\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathcal{B})$. Let $\mathcal{A}$ be any total computable structure, e.g. $\mathcal{A} = (N; =)$ Clearly $DS(\mathcal{B}) \subseteq DS_n(\mathcal{A})$. By Theorem 3.4.2 there exists a structure $\mathcal{C}$ such that $DS_n(\mathcal{C}) = DS(\mathcal{B})$. Therefore $\mathcal{C}$ does not have a n-th jump degree and so no k-th jump degree for $k \leq n$. On the other hand $DS_{n+1}(\mathcal{C}) = DS_1(\mathcal{B})$ and hence the $(n+1)$-th jump degree of $\mathcal{C}$ is $\mathbf{0}^{(n+1)}$.

Why does such a structure $\mathcal{B}$ exist? We want:

(C1) $DS(\mathcal{B}) \subseteq \{\mathbf{a} \mid \mathbf{0^{(n)}} \leq \mathbf{a}\}$.

(C2) $DS(\mathcal{B})$ has no least element.

(C3) $\mathcal{B}$ has a first jump degree equal to $\mathbf{0}^{(n+1)}$.

To explain this we use a following result. There is an e-degree $\mathbf{q}$ that is quasi-minimal relative to $\mathbf{0}^{(n)}$ and with $\mathbf{q}' = \mathbf{0}^{(n+1)}$. This follows from a relativization of the Jump inversion theorem of McEvoy [McE85] :

There is a set S, such that

1. $\varnothing^{(n)} \oplus (N \setminus \varnothing^{(n)}) <_e S$;
2. $(\forall X)(X \oplus (N \setminus X) \leq_e S \Rightarrow X \leq_T \varnothing^{(n)})$;
3. $\varnothing^{(n+1)} \equiv_T S'_e$.

Let $\mathbf{q} = d_e(S)$. Let $\mathcal{B} = \mathcal{G}$ be the torsion free abelian group of rank 1 from Example 2.4.5 with characteristic $S(\mathcal{G}) = S$. So, $\mathbf{s}_b = \mathbf{q} = d_e(S)$. Recall that $DS(\mathcal{G}) = \{\mathbf{a} \mid \mathbf{s}_{\mathcal{G}} \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and the first jump degree of $\mathcal{G}$ is $\mathbf{s}'_b$. So, we have:

1. $DS(\mathcal{G}) = \{d_{\mathrm{T}}(X) \mid S$ is c.e. in $X\}$.
2. $d_{\mathrm{T}}(S'_e)$ is the first jump degree of $\mathcal{G}$.

Let $d_{\mathrm{T}}(X) \in DS(\mathcal{G})$. Then S is c.e. in X and hence $\varnothing^{(n)} \oplus N \setminus \varnothing^{(n)}$ is c.e. in X. Then $\varnothing^{(n)} \leq_{\mathrm{T}} X$. So, $\mathcal{G}$ satisfies (C1). Clearly $\mathcal{G}$ satisfies (C3).

Assume that $d_{\mathrm{T}}(X)$ is the least element of $DS(\mathcal{G})$. Then, by Selman's theorem [Sel71], $X \oplus (N \setminus X) \leq_e S$, and hence $X \leq_T \varnothing^{(n)}$. Thus S is c.e. in $\varnothing^{(n)}$. From here it follows that $S \leq_e \varnothing^{(n)} \oplus (N \setminus \varnothing^{(n)})$. A contradiction. So, $\mathcal{G}$ satisfies (C2).

An easy application of Theorem 3.2.1 is the main property of the jump of a structure. Consider a relation $R \subseteq A^n$. The relation R is relatively intrinsically Σ^0_2 on $\mathcal{A}$ if and only if R is relatively intrinsically Σ^0_1 on $\mathcal{A}'$. Indeed, let R be relatively intrinsically Σ^0_1 on $\mathcal{A}'$. If f is an enumeration of $\mathcal{A}$, we have to show that $f^{-1}(R)$ is Σ^0_2 in $f^{-1}(\mathcal{A})$, or equivalently $f^{-1}(R)$ is c.e. in $f^{-1}(\mathcal{A})'$. Consider the enumeration f^* of $\mathcal{A}'$, that extends f, defined in Lemma 3.1.2. We know that $f^{-1}(R) \equiv_T (f^*)^{-1}(R)$ and $(f^*)^{-1}(\mathcal{A}') \leq_T f^{-1}(\mathcal{A})'$. Since R is relatively intrinsically Σ^0_1 on $\mathcal{A}'$, we have that $(f^*)^{-1}(R)$ is c.e. in $(f^*)^{-1}(\mathcal{A}')$. So, $f^{-1}(R)$ is c.e. in $f^{-1}(\mathcal{A})'$.

For the other direction let R be relatively intrinsically Σ^0_2 on $\mathcal{A}$. Let f is an enumeration of $\mathcal{A}'$. We have to prove that $f^{-1}(R)$ is c.e. in $f^{-1}(\mathcal{A}')$.

We construct a generic enumeration g of $\mathcal{A}$ as in Theorem 3.2.1 and we have that $g^{-1}(\mathcal{A})' \leq_T f^{-1}(\mathcal{A}')$. Since R be relatively intrinsically Σ^0_2 on $\mathcal{A}$, $g^{-1}(R)$ is c.e in $g^{-1}(\mathcal{A})'$. So, since $f^{-1}(R) \equiv_T g^{\#^{-1}}(R^{\#})$, and the mapping $g^{\#}$ is computable in $f^{-1}(\mathcal{A}')$, we have $f^{-1}(R)$ is c.e. in $f^{-1}(\mathcal{A}')$.

Our next application is a generalization of results of Slaman [Sla98] and Wehner [Weh98]. They give an example of a structure with degree spectrum consisting of all noncomputable Turing degrees.

Theorem 3.4.3. [Weh98] There is a family of finite sets, which has no c.e. enumeration, i.e. c.e. universal set, and for each noncomputable set X there is a enumeration computable in X.

First we relativize this theorem.

Theorem 3.4.4. Let $B \subseteq N$. There is a family $\mathcal{F}$ of sets, which has no c.e. in B enumeration, and for each set $X >_T B$ there is a enumeration of the family $\mathcal{F}$, computable in X.

Following an idea of Kalimullin [Kal09b] we consider the following family of sets

$$\mathcal{F} = \{\{0\} \oplus B\} \cup \{\{1\} \oplus \overline{B}\} \cup \{\{n+2\} \oplus F \mid F \text{ finite set, } F \neq W^B_n\}.$$

Proposition 3.4.5. Let $X \subseteq N$. If a universal for $\mathcal{F}$ set U is c.e. in X then $X >_T B$.

It is clear that $B \leq_T X$.

If we assume that $B \equiv_T X$, then we can construct a computable in B function g, such that $(\forall n)(W^B_{g(n)} \neq W^B_n)$. This is a contradiction with the recursion theorem.

Proposition 3.4.6. Let $B <_T X$. There exists a universal set U for the family $\mathcal{F}$, such that $U \leq_T X$.

Since $X \nleq B$ then at least one of the sets X or $\overline{X}$ is not c.e. in B. Without loss of generality assume that X is not c.e. in B. Fix an enumeration of $X = \{x_1, \dots, x_s, \dots\}$ and denote by $\nu_s = \langle x_1, \dots, x_s \rangle$.

The set U we construct in stages. At each stage s we find an approximation U^s of U and a witness $x^s_{n,F,i}$ for every finite set F and $i, n \in N$.

Construction

$$U^0 = \{(0,0)\} \cup \{(0,2x+1) \mid x \in B\} \cup \{(1,2)\} \cup \{(1,2x+1) \mid x \notin B\} \cup \{(\langle n,F,i\rangle + 2, 2n+4)\} \cup \{(\langle n,F,i\rangle + 2, 2x+1) \mid x \in F\} \tag{3.4.1}$$

for each finite set F and $i,n \in N$ and let $x^0_{n,F,i} = -1$.

At stage s, denote by $F^s_{\langle n,F,i\rangle} = \{x \mid (\langle n,F,i\rangle + 2, 2x+1) \in U^s\}$.

- If $F^s_{\langle n,F,i\rangle} \neq W^B_{n,s}$ and $x^s_{n,F,i} \neq -1$, we set $x^{s+1}_{n,F,i} = x^s_{n,F,i}$.
- If $F^s_{\langle n,F,i\rangle} = W^B_{n,s}$ and $x^s_{n,F,i} \neq -1$, we set $x^{s+1}_{n,F,i} = -1$ and add $(\langle n,F,i\rangle + 2, 2\nu_s + 1)$ to U^{s+1}.
- If $x^s_{n,F,i} = -1$, we check if there is a z such that $z \in F^s_{\langle n,F,i\rangle} \not\Leftrightarrow z \in W^B_{n,s}$. If there is such a number z, we set $x^{s+1}_{n,F,i}$ to be the least one. If not, we add $(\langle n,F,i\rangle + 2, 2\nu_s + 1)$ to U^{s+1}.

End of construction

Let $U = \bigcup_s U^s$ and $F = \bigcup F^s$.

Consider the sequence $\{x^s_{n,F,i}\}$.

1. If this sequence has a limit a natural number, i.e. it is stable for all $s \geq s_0$ for some s_0, then the index $\langle n,F,i\rangle$ is an index of a finite set from the family $\mathcal{F}$.
2. If the sequence has a limit -1 or it does not have a limit at all, then there exists a monotone sequence of stages $s_1 < s_2 < \dots < s_k < \dots$, such that $W^B_{n,s} = \{\nu_{s_k} \mid k \in N\} \cup F$. It follows that the set $\{\nu_{s_k} \mid k \in N\}$ is c.e. in B, and hence X is c.e. in B. A contradiction.

It follows that every set with index greater than 1 in U is finite and belongs to the family $\mathcal{F}$. It is clear that every member of the family $\mathcal{F}$ has an index.

Moreover $(\langle n,F,i\rangle + 2, 2x+1) \in U$ if and only if one of the following holds:

1. $x \in F$;
2. $x = \langle \nu_0, \dots, \nu_s\rangle$, for some s.

Hence $U \leq_T X$.

So the constructed set U is universal for the family $\mathcal{F}$ and $U \leq_T X$.

Theorem 3.4.7 (Wehner, Slaman)**.** [Weh98][Sla98] There is a structure $\mathcal{C}$, for which $DS(\mathcal{C}) = \{\mathbf{x} \mid \mathbf{x} >_T \mathbf{0}\}$.

The relativized result is the following:

Theorem 3.4.8. For each $n \in N$ and every Turing degree $\mathbf{b} \geq \mathbf{0}^{(n)}$ there exists $\mathcal{C}$, for which $DS_n(\mathcal{C}) = \{\mathbf{x} \mid \mathbf{x} >_T \mathbf{b}\}$.

We construct the structure $\mathcal{A}$, such that $DS(\mathcal{A}) = \{\mathbf{x} \mid \mathbf{x} >_T \mathbf{b}\}$, using the family $\mathcal{F}$ in the same way as is done in [Weh98]. Let $\mathcal{B} = (N; =)$. It is clear that $\mathbf{b} \in DS_n(\mathcal{B})$ for each $\mathbf{b} \geq \mathbf{0}^{(n)}$. Thus $DS(\mathcal{A}) \subseteq DS_n(\mathcal{B})$. By the Jump inversion Theorem 3.4.2 there exists a structure $\mathcal{C}$, such that $DS_n(\mathcal{C}) = DS(\mathcal{A})$.

We would like to note that there is a relativized variant of Wehner's result for $\mathbf{b} = \mathbf{0}^{(n)}$ and for $\mathbf{b} = \mathbf{0}''$ as follows: Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon [GHK+05] show that for every n there is a structure $\mathcal{C}$, such that $DS(\mathcal{C}) = \{\mathbf{x} \mid \mathbf{x}^{(n)} >_T \mathbf{0}^{(n)}\}$, i.e. the degree spectrum contains exactly all non-low_n Turing degrees.

Harizanov and R. Miller [HM07] proved that there is a structure $\mathcal{C}$, such that $DS(\mathcal{C}) = \{\mathbf{x} \mid \mathbf{x}' \geq_T \mathbf{0}''\}$ and the last author made a suggestion that they can use an arbitrary Turing degree $\mathbf{b}$ in place of $\mathbf{0}''$ and thereby building structures with spectrum $\{\mathbf{x} \mid \mathbf{x}' \geq_T \mathbf{b}\}$.

In conclusion would like to point out that the Jump inversion theorem gives a method to lift some interesting results for degree spectra to the nth jump spectra.

Chapter 4

Strong jump inversion

In this chapter we will present a general result with sufficient conditions for a countable structure to admit strong ump inversion. We will show several classes of structures where these conditions apply, such as some classes of linear orderings, Boolean algebras, trees, models of theory with few types and differentially closed fields. These investigations are join with Wesley Calvert, Andrey Frolov, Valentina Harizanov, Julia Knight, Charles McCoy, Stefan Vatev, and started when most of them visited Sofia in 2013 and are published in the paper [CFH+18].

We say that a structure $\mathcal{A}$ admits *strong jump inversion* provided that for every oracle X, if X' computes $D(\mathcal{C})'$ for some $\mathcal{C} \cong \mathcal{A}$, then X computes $D(\mathcal{B})$ for some $\mathcal{B} \cong \mathcal{A}$. Jockusch and Soare [JS91] showed that there are low linear orderings without computable copies, but Downey and Jockusch [DJ94] showed that every Boolean algebra admits strong jump inversion. D. Marker and R. Miller [MM17] have shown recently that all countable models of DCF_0 (the theory of differentially closed fields of characteristic 0) admit strong jump inversion. We establish a general result with sufficient conditions for a structure $\mathcal{A}$ to admit strong jump inversion. Our conditions involve an enumeration of B_1-types, where these are made up of formulas that are Boolean combinations of existential formulas. We apply them on several classes of structures.

We start with some definitions and examples. In Section 4.2, we give a general result with sufficient conditions for strong jump inversion. In Section 4.3, we give several applications of our general result. The last of these gives the result of Marker and Miller [MM17] saying that all models of DCF_0 admit strong jump inversion. We add a result saying that the countable saturated model of DCF_0 has a decidable copy.

4.1 Canonical jump and strong jump inversion

We often identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$. We are interested in the following notion of jump inversion.

Definition 4.1.1. A structure $\mathcal{A}$ *admits strong jump inversion* provided that for all sets X, if X' computes $D(\mathcal{C})'$ for some $\mathcal{C} \cong \mathcal{A}$, then X computes $D(\mathcal{B})$ for some $\mathcal{B} \cong \mathcal{A}$.

Remark 4.1.2. The structure $\mathcal{A}$ admits strong jump inversion iff for all X, if $\mathcal{A}$ has a copy that is low over X, then it has a copy that is computable in X. Here when we say that $\mathcal{C}$ is low over X, we mean that $D(\mathcal{C})' \leq_T X'$.

The definition of strong jump inversion was motivated by the following result of Downey and Jockusch [DJ94].

Theorem 4.1.3 (Downey-Jockusch)**.** All Boolean algebras admit strong jump inversion.

Proof. [Sketch of proof]
Let $\mathcal{A}$ be a Boolean algebra that is low over X. Then X' computes the set of atoms in $\mathcal{A}$. Downey and Jockusch showed that if X' computes $(\mathcal{A}, atom(x))$, then X computes a copy of $\mathcal{A}$. The proof involves some non-uniformity. A Boolean algebra with only finitely many atoms obviously has a computable copy. Suppose $\mathcal{A}$ has infinitely many atoms. If $\mathcal{A}$ is low over X, then there is an X-computable Boolean algebra $\mathcal{B}$ with a function f, Δ^0_2 relative to X, which would be an isomorphism from $\mathcal{B}$ to $\mathcal{A}$ except that it may map a finite join of atoms in $\mathcal{B}$ to a single atom in $\mathcal{A}$. We convert f into an isomorphism by re-apportioning the atoms (see Vaught [Vau55]). □

Here are some further examples of structures that admit strong jump inversion.

Example 4.1.4 (*Equivalence structures*)**.** Each equivalence structure is characterized up to isomorphism by the number of equivalence classes of various sizes. We consider equivalence structures with infinitely many infinite classes. It is well-known, and easy to prove, that such an equivalence structure has an X-computable copy iff the set of pairs (n, k) such that there are at least k classes of size n is Σ^0_2 relative to X. (See [AK00] for a complete characterization of the equivalence structures with computable copies.)

Proposition 4.1.5. Let $\mathcal{A}$ be an equivalence structure with infinitely many infinite classes. Then $\mathcal{A}$ admits strong jump inversion.

Proof.

If $\mathcal{A}$ is low over X, then the set Q consisting of pairs (n, k) such that there are at least k classes of size n is Σ_2^0 relative to $\mathcal{A}$, so it is Σ_2^0 relative to X. Then $\mathcal{A}$ has an X-computable copy. □

Example 4.1.6 (*Abelian p-groups of length* ω)**.** By Ulm's Theorem, a countable Abelian p-group is characterized up to isomorphism by the Ulm sequence and the dimension of the divisible part. For an account of this, see [Kap69]. An Abelian p-group of length ω can be expressed as a direct sum of copies of $Z_{p^{n+1}}$, for finite n, and the Prüfer group Z_{p^∞}. Then the Ulm sequence is $(u_n(G))_{n\in\omega}$, where $u_n(G)$ is the number of direct summands of form $Z_{p^{n+1}}$. The dimension of the divisible part is the number of direct summands of form Z_{p^∞}. It is well-known, and easy to prove, that if G is an Abelian p-group of length ω with a divisible part of infinite dimension, then G has an X-computable copy iff the set $\{(n, k) : u_n(G) \geq k\}$ is Σ_2^0 relative to X.

Proposition 4.1.7. Let G be an Abelian p-group of length ω such that the divisible part has infinite dimension. Then G admits strong jump inversion.

Proof.

Suppose G itself is low over X. The set $\{(n, k) : u_n(G) \geq k\}$ is Σ_2^0 relative to G, so it is Σ_2^0 relative to X. Then G has an X-computable copy. □

Not all countable structures admit strong jump inversion.

Example 4.1.8. Jockusch and Soare [JS91] showed that there are low linear orderings with no computable copy.

Example 4.1.9. Let T be a low completion of PA. There is a model $\mathcal{A}$ such that the atomic diagram $D(\mathcal{A})$, and even the complete diagram $D^c(\mathcal{A})$, are computable in T. Then $D(\mathcal{A})'$ is Δ_2^0. By a well-known result of Tennenbaum, since $\mathcal{A}$ is necessarily non-standard, there is no computable copy.

There is a computable set of indices for computable Σ_1^c formulas, so we can enumerate, uniformly in $D(\mathcal{A})$, all relations that are relatively intrinsically Σ_1^0 on $\mathcal{A}$ (*r.i.c.e.*). Moreover, we can uniformly compute all of these relations from the Turing jump of the diagram, $D(\mathcal{A})'$. The *jump* of $\mathcal{A}$ is often defined to be a structure $\mathcal{A}'$ obtained by adding to $\mathcal{A}$ a specific named family of r.i.c.e.

relations, from which all others are effectively obtained. As we know from the previous chapter the r.i.c.e. relations on $\mathcal{A}'$ are just those which are relatively intrinsically Σ_2^0 on $\mathcal{A}$ itself.

We used in previous chapter our definition of the jump of structure when we proved the jump inversions theorems. Here we will use the Montalbán's definition from [Mon09] but modified in [Mon12], in order to show that it is equivalent to our notion.

Definition 4.1.10 (Canonical jump)**.** For a structure $\mathcal{A}$, the *canonical jump* is a structure $\mathcal{A}' = (\mathcal{A}, (R_i)_{i\in\omega})$, where $(R_i)_{i\in\omega}$ are relations from which we can uniformly compute all r.i.c.e. relations on $\mathcal{A}$, and from the index i of the relation R_i, we can compute the arity of R_i and a computable Σ_1^c formula (without parameters) that defines it in $\mathcal{A}$.

Remark 4.1.11. The set $\varnothing'$ is included in the canonical jump. We may give it by a family of relations $R_{f(e)}$, for a computable function f, where $R_{f(e)}$ is always true if $e \in \varnothing'$ and always false otherwise. We may define $R_{f(e)}$ by the computable Σ_1^c formula $\bigvee_s \tau_{e,s}$, where $\tau_{e,s}$ is $\top$ if e has entered $\varnothing'$ by step s and $\bot$ otherwise.

For some structures, there is a smaller subset of the relations that is sufficiently powerful to replace the full set.

Definition 4.1.12 (Structural jump)**.** A *structural jump* of $\mathcal{A}$ is an expansion $\mathcal{A}' = (\mathcal{A}, (R_i)_{i\in\omega})$ such that each R_i has a Σ_1^c defining formula that we can compute from i, and every relation that is relatively intrinsically Σ_2^0 on $\mathcal{A}$ is r.i.c.e. on $\mathcal{A}' \oplus \varnothing'$.

Here the structure $\mathcal{A}' \oplus \varnothing'$ is the expansion of $\mathcal{A}'$ by a family of relations that encode the set $\varnothing'$, as explained in Remark 4.1.11.

For certain classes of structures, there is a structural jump formed by adding a finite set of such relations. In particular, the relation $atom(x)$ is sufficient for Boolean algebras, and the successor relation $succ(x, y)$ is sufficient for linear orders. See [Mon09, Mon12] for further examples.

There are different statements of "jump inversion". As we know from Chapter 2. Theorem 2.2.9 the Friedberg jump inversion theorem says that if $\varnothing' \leq_T Y$, then there is a set X such that $X' \equiv_T Y \equiv \varnothing' \oplus X$. We can easily produce a structure $\mathcal{B}$ such that $X \equiv_T \mathcal{B}$, and then $Y \equiv_T \mathcal{B}'$. This is one kind of jump inversion. A more interesting kind of jump inversion theorem that we proved with Soskov [SS07, SS09a], and later (independently) Montalbán [Mon09]. This is Theorem 3.2.1 from previous chapter, formulated in other terms.

Theorem 4.1.13 (Soskov, A. Soskova, Montalbán)**.** For any countable structure $\mathcal{A}$, if Y computes a copy of the canonical jump $\mathcal{A}'$ of $\mathcal{A}$, there exists a set X such that $X' \equiv_T Y$ and X computes a copy of $\mathcal{A}$.

The proposition below shows that we can express strong jump inversion in terms of copies of the canonical jump structure $\mathcal{A}'$, as opposed to the Turing jump of the atomic diagram for various copies $\mathcal{B}$.

Proposition 4.1.14. For any structure $\mathcal{A}$, the following are equivalent:

(1) $\mathcal{A}$ admits strong jump inversion.

(2) For all sets X, if X' computes a copy of the canonical jump $\mathcal{A}'$ of $\mathcal{A}$, then X computes a copy of $\mathcal{A}$.

(3) For all sets X and Y, if $X' \equiv_T Y'$ and Y computes a copy of $\mathcal{A}$ then so does X.

Proof.

For (2) $\Rightarrow$ (1), assume $\mathcal{A}$ has a copy $\mathcal{B}$ with $(D(\mathcal{B}))' \leq_T X'$. Since $D(\mathcal{B}') \leq_T (D(\mathcal{B}))' \leq_T X'$, (2) implies that X computes a copy of $\mathcal{A}$.

For (1) $\Rightarrow$ (3), let $X' \equiv_T Y'$, where Y computes a copy $\mathcal{B}$ of $\mathcal{A}$. Then X' computes $D(\mathcal{B})'$. By (1), X computes a copy of $\mathcal{A}$.

For (3) $\Rightarrow$ (2), suppose X' computes a copy of $\mathcal{A}'$. By Theorem 4.1.13, there exists Y such that Y computes a copy of $\mathcal{A}$ and $Y' \equiv_T X'$. By (3), X computes a copy of $\mathcal{A}$. □

4.2 General result

In this section, we give a result with conditions sufficient to guarantee that a structure admits strong jump inversion. The content of this section is from [CFH+18]. The result is not difficult to prove. However, there are a number of examples where it applies. To state the result, we need some definitions.

Definition 4.2.1. Let S be a countable family of sets. An *enumeration* of S is a set R of pairs (i,k) such that S is the family of sets $R_i = \{k : (i,k) \in R\}$. If $A = R_i$, we say that i is an *R-index* for A.

Note: When we say that R is a *computable enumeration* of a family of sets, we mean that R is a computable set of pairs. This means that the sets R_i are *computable*, uniformly in i. Some researchers have used the term differently, saying that R is a *computable enumeration* if the sets R_i are uniformly *computably enumerable*.

Below, we define B_n-types precisely. We shall focus on B_1-types.

Definition 4.2.2.

1. A B_n*-formula* is a finite Boolean combination of ordinary finite elementary Σ_n-formulas.

2. A B_n*-type* is the set of B_n-formulas in the complete type of some tuple in some structure for the language.

Definition 4.2.3. Fix a structure $\mathcal{A}$. Let S be a set of B_1-types including all those realized in $\mathcal{A}$. Let R be an enumeration of S. An *R-labeling of $\mathcal{A}$* is a function taking each tuple $\overline{a}$ in $\mathcal{A}$ to an R-index for the B_1-type of $\overline{a}$.

We are interested in structures $\mathcal{A}$ with the following property.

Definition 4.2.4 (Effective type completion)**.** The structure $\mathcal{A}$ satisfies *effective type completion* if there is a uniform effective procedure that, given a B_1-type $p(\overline{u})$ realized in $\mathcal{A}$ and an existential formula $\varphi(\overline{u}, x)$ such that $(\exists x)\varphi(\overline{u}, x) \in p(\overline{u})$, yields a B_1-type $q(\overline{u}, x)$ with $\varphi(\overline{u}, x) \in q(\overline{u}, x)$, such that if $\overline{a}$ in $\mathcal{A}$ realizes $p(\overline{u})$, then some b in $\mathcal{A}$ realizes $q(\overline{a}, x)$.

Here is our general result.

Theorem 4.2.5. A structure $\mathcal{A}$ admits strong jump inversion if it satisfies the following conditions:

(1) There is a computable enumeration R of a set of B_1-types including all those realized by tuples in $\mathcal{A}$.

(2) $\mathcal{A}$ satisfies effective type completion.

(3) For all sets X, if X' computes the jump of some copy of $\mathcal{A}$, then X' computes a copy of $\mathcal{A}$ with an R-labeling.

Moreover, if $\mathcal{C}$ is a copy of $\mathcal{A}$ with an X'-computable R-labeling, then we get an X-computable copy $\mathcal{B}$ of $\mathcal{A}$ with an X'-computable isomorphism from $\mathcal{B}$ to $\mathcal{C}$.

Remark 4.2.6. For some structures $\mathcal{A}$, Condition (3) is satisfied in a strong way. For any $\mathcal{C} \cong \mathcal{A}$, $D(\mathcal{C})'$ computes an R-labeling of $\mathcal{C}$. Hence, if $\mathcal{A}$ is low, there is a Δ^0_2 isomorphism from $\mathcal{A}$ to a computable copy.

Proof. [Proof of Theorem 4.2.5]

Suppose that $\mathcal{A}$ satisfies the three conditions. Let X be a set such that X' computes the jump of some copy of $\mathcal{A}$. By Condition (3), X' computes a copy with an R-labeling. We must show that there is an X-computable copy. For simplicity, we suppose that $\mathcal{A}$ has a Δ^0_2 R-labeling, and we produce a computable copy $\mathcal{B}$, basing our construction on guesses at various portions of the R-labeling of $\mathcal{A}$. Note that once we have guessed the label for a tuple $\overline{a}$ correctly, we computably know the entire B_1-type of that tuple. We build a computable copy $\mathcal{B}$ and a Δ^0_2 isomorphism f from $\mathcal{B}$ to $\mathcal{A}$. We have the following requirements.

R_{2a}: $a \in ran(f)$

R_{2b+1}: $b \in dom(f)$

We start with an R-index for the type of $\varnothing$, where this type is the B_1-theory of $\mathcal{A}$. At each stage s, we have a tentative partial isomorphism f_s mapping a tuple $\overline{d}$ from $\mathcal{B}$ to a tuple $\overline{c}$ in $\mathcal{A}$, where the R-indices of the types of $\overline{c}$ and all of its initial segments still look correct. (At a later stage t, we may see that some of the guesses at these indices are incorrect, and we retain only the portion of f_s satisfying an initial segment of requirements based on guesses at R-indices that all look correct.) Moreover, we have enumerated a finite part $\delta(\overline{d},\overline{b})$ of the atomic diagram of $\mathcal{B}$; this can never change, since $\mathcal{B}$ must be computable. We will have checked the consistency of $\delta(\overline{d},\overline{b})$ with our guesses at the R-indices of the B_1-types of the tuple $\overline{c}$ and its initial segments. Supposing that the function taking $\overline{d}$ to $\overline{c}$ satisfies the earlier requirements, we can satisfy the requirement R_{2a} once we guess the R-index for the B_1-type $p(\overline{u},x)$ of $\overline{c},a$. We map some b, either old or new, to a so that $\delta(\overline{u},\overline{v})$ is consistent with $p(\overline{u},x)$. (Recall that the B_1-types are computable.)

Suppose that the function taking $\overline{d}$ to $\overline{c}$ satisfies the requirement R_i for all $i < 2b+1$, and R_{2b+1} is least that is unsatisfied at this stage s. Again, we assume that we have correct guesses on the R-indices for the B_1 types of $\overline{c}$ and all of its initial segments; let $p(\overline{u})$ be the B_1-type of $\overline{c}$. Finally, we have put $\delta(\overline{d}, b, \overline{b})$ in the atomic diagram of $\mathcal{B}$. Now we use the assumption of effective type completion. We determine, effectively in $p(\overline{u})$ and the existential formula $(\exists\overline{v})\delta(\overline{u}, x, \overline{v})$, a type $q(\overline{u}, x)$ appropriate for $\overline{c}$ and a putative $f_s(b)$. If $\overline{c}$ realizes $p(\overline{u})$, then some a will realize $q(\overline{c}, x)$. At step s, we can give a computable index for $q(\overline{u}, x)$, but not an R-index.

By effective type completion, if $p(\overline{u})$ really is the B_1-type of $\overline{c}$, then $q(\overline{c}, x)$ will be realized in $\mathcal{A}$. We define $f_s(b)$ as follows. We find the first a such that, based on our guess at the R-index of the B_1 type of $\overline{c}, a$, this type and $q(\overline{u}, x)$ agree on the first s formulas; then $f_s(b) = a$. Of course, this guess at the element a is likely wrong. Therefore, in order to guarantee that this requirement is satisfied, at each subsequent stage t, we need to check that, based on our guess at the R-index of the B_1 type of $\overline{c}, a$, this type and $q(\overline{u}, x)$ agree on the first t formulas. If this is not the case, then we need to re-define $f_t(b)$, but always maintaining $q(\overline{u}, x)$ as the guaranteed type of $q(\overline{c}, f(b))$, so long as our work on earlier requirements seems correct. (In particular, note that as we check consistency of the atomic diagram with the B_1 types associated with requirement R_{2b+1}, we use the computable index for $q(\overline{u}, x)$.) There is a first a realizing $q(\overline{c}, x)$, and eventually, we will have the R-index for the B_1 type of $\overline{c}, a$. Then we will have $f_s(b) = f(b) = a$. □

In several examples, $\mathcal{A}$ has effective type completion because it satisfies a property that we call *weak* 1-*saturation*. To describe this property, we need a preliminary definition.

Definition 4.2.7. Suppose $p(\overline{u})$ and $q(\overline{u}, x)$ are B_1-types. We say that $q(\overline{u}, x)$ is *generated by the formulas of* $p(\overline{u})$ *and existential formulas* provided that $q(\overline{u}, x) \supseteq p(\overline{u})$, and for any universal formula $\psi(\overline{u}, x)$ (in the indicated variables), writing $neg(\psi)$ for the natural existential formula logically equivalent to $\neg\psi$, we have $\psi(\overline{u}, x) \in q(\overline{u}, x)$ iff there is a finite conjunction $\chi(\overline{u}, x)$ of existential formulas in $q(\overline{u}, x)$ such that $(\exists x)[\chi(\overline{u}, x) \;\&\; neg(\psi(\overline{u}, x))]$ is not in $p(\overline{u})$.

Definition 4.2.8. The structure $\mathcal{A}$ is *weakly* 1-*saturated* provided that if $p(\overline{u})$ is the B_1-type of a tuple $\overline{a}$, and $q(\overline{u}, x)$ is a B_1-type generated by formulas of $p(\overline{u})$ and existential formulas, then $q(\overline{a}, x)$ is realized in $\mathcal{A}$.

The following is clear.

Lemma 4.2.9. Let $p(\overline{u})$ be a B_1-type. Suppose $q(\overline{u}, x)$ is a B_1-type that is generated by formulas of $p(\overline{u})$ and existential formulas. Then $q(\overline{u}, x)$ is consistent with all extensions of $p(\overline{u})$ to a complete type in variables $\overline{u}$.

Proposition 4.2.10. If $\mathcal{A}$ is weakly 1-saturated, then it satisfies effective type completion.

Proof.

Let $p(\overline{u})$ be a B_1-type, and suppose $\varphi(\overline{u}, x)$ is an existential formula such that $(\exists x)\varphi(\overline{u}, x) \in p(\overline{u})$. We effectively produce a type $q(\overline{u}, x)$ extending $p(\overline{u})$ and containing the formula $\varphi(\overline{u}, x)$, such that if $\overline{a}$ realizes $p(\overline{u})$, then some b realizes $q(\overline{a}, x)$. The type $q(\overline{u}, x)$ is generated by formulas of $p(\overline{u})$ and existential formulas, including the formula $\varphi(\overline{u}, x)$. We determine this B_1-type computably as follows. We start with $p(\overline{u})$ and $\varphi(\overline{u}, x)$. We have a computable list $(\varphi_n(\overline{u}, x))_{n\in\omega}$ of all existential formulas in variables $\overline{u}, x$, in order of Gödel number. We consider these formulas, in order, and we put $\varphi_n(\overline{u}, x)$ into $q(\overline{u}, x)$ iff it is consistent with what we have already put into $q(\overline{u}, x)$. (This consistency check is computable relative to $p(\overline{u})$, because it entails only asking whether the relevant B_1 formulas are in $p(\overline{u})$.) If we fail to put $\varphi_n(\overline{u}, x)$ in, then all tuples satisfying what we did put in must satisfy $neg(\varphi_n(\overline{u}, x))$, so that is in $q(\overline{u}, x)$. Knowing exactly which existential formulas are in $q(\overline{u}, x)$, we can determine which B_1 formulas are in (using truth tables). We have described an effective procedure for determining $q(\overline{u}, x)$. By weak 1-saturation, there is some b in $\mathcal{A}$ realizing $q(\overline{a}, x)$. □

4.3 Examples

The content of this section is from [CFH+18].

In this section, we consider some examples of structures that admit strong jump inversion. The examples are chosen to illustrate the use of Theorem 4.2.5. In Subsection 4.3.1, we discuss two special kinds of linear orderings. For both, we can apply Theorem 4.2.5. For the first, Condition (3) holds in a strong way, as in Remark 4.2.6. In Subsection 4.3.2, we consider Boolean algebras with no 1-atoms. The result of Downey and Jockusch says that every low Boolean algebra has a computable copy. In the case where there are no 1-atoms, our result gives a Δ^0_3 isomorphism from a low copy to a computable one. In Subsection 4.3.3, we apply Theorem 4.2.5 to some special classes of trees.

In Subsection 4.3.4, we consider models of an $\aleph_0$-categorical elementary first order theory T such that $T \cap \Sigma_2$ is computably enumerable. The fact that the B_1-types are all isolated makes it easy to produce a computable enumeration. By contrast, in Subsection 4.3.5, we consider models of the theory of differentially closed fields of characteristic 0. Here, although the theory is decidable, with all types computable, producing a computable enumeration of them is not trivial. We get a result of Marker and R. Miller [MM17] saying that all models of DCF_0 admit strong jump inversion. Moreover, a result of Morley in [Mor76] implies that, since the types of the theory have a computable enumeration, the saturated model of DCF_0 has a decidable copy.

4.3.1 Linear orderings

Frlolov proved strong jump inversion for two special classes of linear orderings, with further results on complexity of isomorphisms. The results are given in [Fro06], [Fro10], [Fro12]. Here we prove these results using Theorem 4.2.5.

First, we describe the possible B_1 types in linear orderings. Every B_1-type $p(\overline{u})$ is determined uniquely by the sizes of the intervals to the left of the first element, between successive elements, and to the right of the last element. Thus, we can define a computable enumeration R of all B_1-types realized in linear orderings so that from the index i of the B_1-type R_i, we can effectively obtain the sizes of the intervals.

Let $p(u_1, u_2)$ be a B_1-type in which the interval (u_1, u_2) is infinite. We consider B_1-types $q(u_1, u_2, x)$, with $u_1 < x < u_2$. To understand which of these are generated by formulas from $p(u_1, u_2)$ and existential formulas, it is helpful to consider the following cases.

Case 1: Let $q(u_1, u_2, x)$ be a B_1-type such that the interval (u_1, x) is finite, of size k, and the interval (x, u_2) is infinite. Let $t(u_1, u_2)$ be a complete type saying that u_1 and u_2 are infinitely far apart and u_1 belongs to a maximal discrete interval of size less than k. Clearly, $p(u_1, u_2)$ is consistent with $t(u_1, u_2)$, whereas $q(u_1, u_2, x)$ is not. By Lemma 4.2.9, $q(u_1, u_2, x)$ is not generated by $p(u_1, u_2)$ and existential formulas.

Case 2: Let $q(u_1, u_2, x)$ extend $p(u_1, u_2)$ such that $u_1 < x < u_2$, and the intervals (u_1, x) and (x, u_2) are both infinite. Then $q(u_1, u_2, x)$ is generated by $p(u_1, u_2)$ and the infinite set of existential formulas saying that for each n, there are at least n elements in the intervals (u_1, x) and (x, u_2).

Proposition 4.3.1. Let $\mathcal{A}$ be a linear ordering such that every infinite interval can be split into two infinite parts. Then $\mathcal{A}$ is weakly 1-saturated.

Proof. For a tuple $\overline{a}$, we consider the possible B_1-types $q(\overline{a}, x)$. First, suppose $q(\overline{a}, x)$ locates x in a finite interval $(-\infty, a_0)$, (a_i, a_{i+1}), or (a_n, ∞) so that the sizes of the two subintervals to the left and right of x add up properly. Then $q(\overline{u}, x)$ is generated by formulas of $p(\overline{u})$ and existential formulas saying that the subintervals have at least the desired size, and $q(\overline{a}, x)$ must be realized. Next, suppose $q(\overline{a}, x)$ locates x in an infinite interval $(-\infty, a_0)$, (a_i, a_{i+1}), or (a_n, ∞). If $q(\overline{a}, x)$ is generated by formulas of $p(\overline{u})$ and existential formulas, then x must split the interval into two infinite parts. The ordering $\mathcal{A}$ has exactly this feature. □

Here is the simpler of the two results on linear orderings.

Theorem 4.3.2. Let $\mathcal{A}$ be a linear ordering such that each element lies on a maximal discrete set that is finite. Suppose there is a finite bound on the sizes of these sets. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy with an isomorphism that is Δ^0_2 relative to X.

Proof.

Let N be the finite bound on the sizes of the maximal discrete sets. It is Δ^0_2 relative to $\mathcal{A}$ to say that the interval (a, b) has size n for some fixed n. It is Σ^0_1 relative to $\mathcal{A}$ to say that the interval is infinite—we just ask whether the interval has size greater than N.

Suppose that $\mathcal{A}$ is low over X. We can apply a procedure that is Δ^0_2 relative to X to assign an R-index to the type of any tuple $\overline{a} = (a_1, \dots, a_n)$. Any of the intervals $(-\infty, a_1)$, (a_n, ∞) and (a_i, a_{i+1}) is infinite if it has size greater than N. Using a procedure that is Δ^0_2 relative to X, we can determine whether the size is k, for $k \leq N$. We have an R-labeling of $\mathcal{A}$ that is Δ^0_2 relative to X. Then Theorem 4.2.5 gives an X-computable copy with an isomorphism that is Δ^0_2 relative to X. □

The next result, Theorem 4.3.3, is more complicated. Before we state the result, we review some well-known, basic concepts about linear orderings. Recall the *block equivalence relation* $\sim$ on a linear ordering $\mathcal{A}$, where $a \sim b$ iff $[a, b]$ is finite. For any linear ordering $\mathcal{A}$, each equivalence class under this relation is an interval that is either finite or of order type ω, ω^*, or $\zeta = \omega^* + \omega$. Furthermore, the quotient structure $\mathcal{A}/_\sim$ is itself a linear ordering, where each distinct point represents an equivalence class under $\sim$.

In Theorem 4.3.3, for a given $\mathcal{A}$ that is low over X, it is not clear that $\mathcal{A}$ itself has an R-labeling that is Δ^0_2 relative to X. However, we can build a copy $\mathcal{B}$ with such an R-labeling. We write η for the order type of the rationals.

Theorem 4.3.3. Let $\mathcal{A}$ be a linear ordering for which the quotient $\mathcal{A}/_\sim$ has order type η. Suppose also that in $\mathcal{A}$, every infinite interval has arbitrarily large finite successor chains. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_3 over X from $\mathcal{A}$ to $\mathcal{B}$.

Proof.

As in the previous result, let R be a computable enumeration of all B_1-types realized in linear orderings, such that from the index i of the type R_i, we can compute the sizes, including ∞, of the intervals. Also, as in the previous result, every infinite interval in $\mathcal{A}$ has an element that splits the interval into two infinite parts. This implies that $\mathcal{A}$ is weakly 1-saturated. Suppose $\mathcal{A}$ is low over X. We will prove the following.

Lemma 4.3.4. There is a copy $\mathcal{B}$ of $\mathcal{A}$ with an R-labeling that is Δ^0_2 over X. Moreover, there is an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ such that f is Δ^0_3 relative to X.

Assuming the lemma, we complete the proof of Theorem 4.3.3 as follows. Given $\mathcal{A}$, low over X, the lemma gives a copy $\mathcal{B}$ with an R-labeling that is Δ^0_2 relative to X, and an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ that is Δ^0_3 relative to X. By Theorem 4.2.5, there is an X-computable copy $\mathcal{C}$ with an isomorphism g from $\mathcal{C}$ to $\mathcal{B}$ that is Δ^0_2 relative to X. Then $f \circ g$ is an isomorphism from $\mathcal{C}$ to $\mathcal{A}$ that is Δ^0_3 relative to X.

Proof. [Proof of Lemma]

For simplicity, we suppose that $\mathcal{A}$ is low. We build a Δ^0_2 copy $\mathcal{B}$, along with some labels for sizes of intervals and a Δ^0_3 isomorphism f. We suppose that the universe of $\mathcal{A}$ is ω. The copy $\mathcal{B}$, also with universe ω, will have the intervals labeled by size. Throughout, we use the oracle Δ^0_2. Suppose $\mathcal{A}_n$ is the true ordering on the first n elements of $\mathcal{A}$, with the intervals correctly labeled by size. At stage s, we construct (using the Δ^0_2 oracle) an approximation $\mathcal{A}_{n,s}$ in which the intervals are either correctly labeled with a finite number at most s, or else carry the label ∞. We have a finite sub-ordering $\mathcal{B}_s$ of $\mathcal{B}$ in which the intervals are labeled by size, once and for all.

We want an isomorphism f from $\mathcal{B}$ onto $\mathcal{A}$. We must satisfy the following requirements.

R_{2a}: Put a into $ran(f)$.

R_{2b+1}: Put b into $dom(f)$.

By the end of each stage s, we have a finite function f_s that seems to satisfy the first few requirements, so that our current labels on the intervals with endpoints in $ran(f_s)$ match the labels on the corresponding intervals in $dom(f_s)$. Moreover, we ensure that if $f_s(b) = a$, then for any successor chain around b in $\mathcal{B}_s$, we also have seen, by stage s, a corresponding successor chain around a in $\mathcal{A}$.

An interval that seemed infinite at stage s may be seen to be finite at stage $s+1$. So in defining f_{s+1}, we first determine the largest initial segment of f_s (in terms of priority requirements) that can be preserved. Consider the highest priority requirement that now must be satisfied.

Suppose the next requirement to be satisfied is to put a into $ran(f)$. We have no problem finding an appropriate pre-image b and assigning the appropriate sizes to the intervals having b as an endpoint.

Suppose the next requirement to be satisfied is to put b into $dom(f)$. In the interesting sub-case, b lies in an interval (d, d'), where (d, b) and $(b, d'$ are both labeled infinite, $f_s(d) = c$ and $f_s(d') = c'$, where (c, c') appears to be infinite. We need to define $a = f_{s+1}(b)$ such that (c, a) and (a, c') both appear infinite, and whatever successor chain surrounding b is matched by one surrounding a. The naive strategy is to just look for a. This strategy may not work. Believing that we have found a, and seeing that a lies in a finite interval inside (c, c'), we may create a bigger successor chain around b, inside (d, d'). Eventually, we may discover that the interval (c, a) or (a, c') is finite. Now, we cannot map b to a. Moreover, we have made the search for $f(b)$ more difficult in that it must lie in a larger finite interval matching the one we have created around b. This can keep happening. Our current guess at the appropriate $a = f(b)$ may keep attaching itself to a successor chain around c or c'.

We need a better strategy. Instead of trying to define $a = f_{s+1}(b)$ immediately, we identify the first (relative to the standard ordering on pairs of the universe ω of $\mathcal{A}$) "buffer pair" (z, z') such that (c, z), (z, z') and (z', c') all appear infinite in $\mathcal{A}$. Once we find such a (z, z'), then we search within (z, z') for an element a and a successor chain around it sufficient to match whatever one we may have created around b; we define $f_{s+1}(b) = a$. Assuming the interval (c, c') is correctly labeled as infinite, then, at some stage, we will settle on the first correct buffer pair (z, z'), i.e., one such that (c, z), (z, z') and (z', c') all are really infinite in $\mathcal{A}$. Then, applying the hypothesis about $\mathcal{A}$, we are guaranteed to find in (z, z') an element a with a finite interval around it large enough to correspond to whatever one we may have built around b by this stage. (Recall that, in general, when we map b to a for some requirement, we vow not to locate b in a finite interval larger than the one we have seen around a.) Following this procedure, we can eventually satisfy all requirements. □ □

4.3.2 Boolean algebras

As we mentioned in the introduction, Downey and Jockusch [DJ94] showed that every low Boolean algebra has a computable copy. In [KS00], it is shown that for a low Boolean algebra $\mathcal{A}$, there is a computable copy $\mathcal{B}$ with a Δ^0_4 isomorphism. In unpublished work, Knight and Stob proved that this is best possible, in the sense that there is a low Boolean algebra with no Δ^0_3 isomorphism taking $\mathcal{A}$ to a computable copy $\mathcal{B}$.

For every element a in the Boolean algebra $\mathcal{B}$, we say that *a has size n* if it is the join of n atoms of $\mathcal{B}$. If a is not the join of finitely many atoms of $\mathcal{B}$, then we say that a has *infinite* size. Here we consider Boolean algebras with no 1-atoms, which means that every infinite element splits into two infinite elements. To describe the B_1-type of a tuple $\overline{a}$ in $\mathcal{B}$, we consider the finite sub-algebra of $\mathcal{B}$ generated by $\overline{a}$. Note that an atom in this finite sub-algebra is not necessarily an atom of $\mathcal{B}$. It is easy to see that for a tuple $\overline{a}$ in $\mathcal{B}$, the B_1-type of $\overline{a}$ is uniquely determined by the sizes in $\mathcal{B}$ of the atoms in the finite sub-algebra generated by $\overline{a}$. Thus, we can define a computable enumeration R of all B_1-types realized in Boolean algebras so that from the index i of the B_1-type R_i we can effectively obtain the sizes of the atoms in the sub-algebra generated by a tuple that satisfies this B_1-type.

Let $p(u)$ be a B_1-type saying that u is infinite. We need to know which B_1-types $q(u, x)$ are generated by $p(u)$ and existential formulas. We have two interesting cases.

Case 1: Let $q(u, x)$ be the B_1-type extending $p(u)$ in which x splits u into one finite element, say of size k, and one infinite element. Let $t(u)$ be a complete type saying that u has infinite size, but there are fewer than k atoms below it. Clearly, $p(u)$ is consistent with $t(u)$, whereas $q(u, x)$ is not. By Lemma 4.2.9, it follows that $q(u, x)$ is not generated by $p(u)$ and existential formulas.

Case 2: Let $q(u,x)$ be the B_1-type extending $p(u)$ in which x splits u into two elements of infinite sizes. Then $q(u,x)$ is generated by $p(u)$ and the infinite set of existential formulas saying that there are at least n distinct elements below x and $u \setminus x$, for every n.

The proof of the following is then straightforward.

Lemma 4.3.5. If $\mathcal{A}$ is a Boolean algebra with no 1-atoms, then $\mathcal{A}$ is weakly 1-saturated.

Proposition 4.3.6. Suppose $\mathcal{A}$ is an infinite Boolean algebra with no 1-atoms. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_3 relative to X.

Proof.

We are assuming that $\mathcal{A}$ is low over X. To show that there is an X-computable copy, it is enough to show the following.

Lemma 4.3.7. Let $\mathcal{A}$ be Boolean algebra with no 1-atom. If $\mathcal{A}$ is low over X, then X' computes a copy $\mathcal{B}$ with an R-labeling. Moreover, there is an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ that is Δ^0_3 relative to X.

Proof.

For simplicity, we suppose that $\mathcal{A}$ is low, and our entire construction uses a Δ^0_2 oracle. For notational convenience, when we write $\overline{a} \in \mathcal{A}$ or $\overline{b} \in \mathcal{B}$, we identify the tuple with the finite sub-algebra (of $\mathcal{A}$ or $\mathcal{B}$) determined by the tuple. Since $\mathcal{A}$ is low, the atom relation on $\mathcal{A}$ is Δ^0_2. Since we will guess (using the Δ^0_2 oracle) that an element of $\mathcal{A}$ is finite iff we recognize it as the join of finitely many atoms of $\mathcal{A}$, any such guess is correct. Now at a particular stage s, our guess may incorrectly assign an R-label of infinite to a finite element a of $\mathcal{A}$; however, there will be a stage t where we correctly guess the R-label of a from that stage onward. For any truly infinite element a, we guess the R-label correctly at all stages.

We must computably (relative to Δ^0_2) construct $\mathcal{B}$ with an R-labeling and an isomorphism f between $\mathcal{B}$ and $\mathcal{A}$ that is correct in the limit, so that f is Δ^0_3.

As usual, we have the following requirements.

R_{2a}: $a \in ran(f)$

R_{2b+1}: $b \in dom(f)$

At stage $s = 0$, we define $f(0_{\mathcal{B}}) = 0_{\mathcal{A}}$ and $f(1_{\mathcal{B}}) = 1_{\mathcal{A}}$; this will never change. We guess that $1_{\mathcal{A}}$ is labeled with ∞ (this will never be wrong), and we label $1_{\mathcal{B}}$ with ∞.

Assume that by the end of stage s we have defined $\overline{b} \in \mathcal{B}$ with R-labels and $f_s : \overline{d} \to \overline{c}$, where $\overline{d}$ is a subsequence of $\overline{b}$, so that the following hold:

(1) the finite algebras $\overline{d}$ and $\overline{c}$ agree;

(2) if $f_s(d) = c$, then the R-label on d matches the stage s approximation of the R-label on c;

(3) if $f_s(d) = c$, and the finite R-labels among those we have assigned to $\mathcal{B}_s$ imply that there are at least k atoms (of $\mathcal{B}$) below d, then by stage s, we have seen at least k atoms below c.

Stage $(s+1)$ approximations of R-labelings of $\mathcal{A}$ may reveal that an element in $\mathcal{A}$ with stage s approximate R-label ∞ actually is finite. So in defining f_{s+1}, we first determine the largest initial segment of f_s (in terms of priority requirements) that can be preserved. Consider the highest priority requirement that now must be satisfied.

Suppose the next requirement to be satisfied is to put a into $ran(f)$. The element a splits each atom α of the subalgebra $\overline{c}$ into α_1 and α_2, each of which has a stage $s+1$ approximation of its R-label. If $f_s(\beta) = \alpha$, then β can be split—using the other elements of $\overline{b}$ or introducing new elements into $\mathcal{B}$ if necessary—into β_1 and β_2 so that if we extend f_s by defining $f_{s+1}(\beta_1) = \alpha_1$ and $f_{s+1}(\beta_2) = \alpha_2$, then properties (1) - (3) above are maintained and R_{2a} is satisfied.

Suppose the next requirement to be satisfied is to put b into $dom(f)$. If b has not yet appeared among $\overline{b}$, then simply extend f_s to include b in any way consistent with what we've defined so far about $\overline{b}$ and consistent with conditions (1)-(3) above, and define the R-labels on the elements of $\overline{b}, b$ accordingly. Otherwise, b splits each atom β of the subalgebra $\overline{d}$ into β_1 and β_2, each of which has an R-label that must be preserved. The only interesting case is when β_1, β_2 both have R-label ∞. By conditions (1) and (2) above, α, the atom in $\overline{c}$ corresponding to β, has a current approximate R-label ∞.

Because $\mathcal{A}$ contains no 1-atom, we "look ahead" if necessary, either to discover that this R-label on α is incorrect, or to find the least (in terms of the universe ω) element α_1 below α so the approximate R-labels of both α_1 and $\alpha - \alpha_1$ are ∞. If we discover the former, then we must use a smaller initial segment of f_s and start over to satisfy a higher priority requirement. Otherwise, we are almost ready to meet the requirement R_{2b+1}. If the R-labels in $\overline{b}$ imply that there are at least k_1 atoms (of $\mathcal{B}$) below β_1 and at least k_2 atoms (of $\mathcal{B}$) below β_2, then by property (3) above, we have seen at least $k_1 + k_2$ atoms (of $\mathcal{A}$) below α. Consider the element $\alpha_1 \pm$ finitely many atoms below α so that this new element α_1' has at least k_1 atoms below it, and $\alpha - \alpha_1'$ has at least k_2 atoms below it. Extend f_s by defining $f_{s+1}(\beta_1) = \alpha_1'$ and $f_{s+1}(\beta_2) = \alpha - \alpha_1'$. Then properties (1) - (3) above are maintained and R_{2b+1} is satisfied. □

□

4.3.3 Trees

We consider some special classes of subtrees of $\omega^{<\omega}$. Our trees grow downward. The top node is $\emptyset$. For the language of trees, we use the predecessor function, where $\emptyset$—the root—is its own predecessor. We consider two special classes of trees. The first is very simple.

Proposition 4.3.8. Suppose $\mathcal{A}$ is a tree such that the top node is infinite (i.e., it has infinitely many successors), and each infinite node has only finitely many successors that are terminal, with the rest all infinite. Then $\mathcal{A}$ admits strong jump inversion.

Proof.

The B_1-type of a tuple $\overline{a}$ is determined by the subtree generated by $\overline{a}$ and labels "infinite" or "terminal" on the nodes, in particular, on the a_i. We have a computable enumeration of all possible labeled finite subtrees of trees of this kind. From this, we get a computable enumeration R of the B_1-types. Suppose that $\mathcal{A}$ is low. Then there is a Δ^0_2 R-labeling of $\mathcal{A}$.

Weak 1-saturation. Take $\overline{a}$ in $\mathcal{A}$. Consider a possible B_1-type $p(\overline{a}, x)$, generated by formulas true of $\overline{a}$ and existential formulas. The type may locate x in the subtree generated by $\overline{a}$. Then the type is realized. The type may locate x properly below some infinite a_i, or at some level not below any a_i. Again the type is realized by a new infinite element.

By Theorem 4.2.5, we get a computable copy of $\mathcal{A}$. □

The second class of trees is a bit more complicated. We use some definitions and notation. If T is a sub-tree of $\omega^{<\omega}$, and $a \in T$, we write T_a for the tree consisting of a and all nodes below.

Definition 4.3.9. For nodes a in a fixed tree T,

(1) we say that a is *finite* if T_a is finite,

(2) we say that a is *infinite* if T_a is infinite. (For the trees we consider below, if a is infinite, we will require not only that T_a is infinite, but also that a has infinitely many successors, so we will have agreement with the definition we used in Proposition 4.3.8.)

Notation. Let a be finite, with T_a the subtree below a. Let T_a^1 be a possible re-labeling of the nodes in T_a in which the nodes in a subtree are labeled ∞. We write $(T_a^1)^*$ for the infinite tree that results from extending the labeled tree T_a^1 so that all new nodes in $(T_a^1)^*$ are labeled ∞, and each node labeled ∞ has infinitely many successors labeled ∞. (No finite node in T_a^1 acquires successors in $(T_a^1)^*$.)

Here is the result for the second class of trees.

Proposition 4.3.10. Suppose T is a subtree of $\omega^{<\omega}$ such that the top node is infinite, and for any infinite node a, there are only finitely many finite successors. Suppose also that for any infinite node a, for any finite successor b, if T_b^1 is a possible re-labeling of T_b making all nodes in a certain subtree infinite, then there are infinitely many successors b_n of a such that $T_{b_n} \cong (T_b^1)^*$. Then T admits strong jump inversion.

Proof.

For simplicity, we suppose that T is low, and we apply Theorem 4.2.5 to produce a computable copy. For a tuple $\overline{a}$ in T, the B_1-type of $\overline{a}$ is determined by the subtree generated by $\overline{a}$ and formulas saying, for an element a of this subtree that it is infinite, or that it is finite with a specific finite tree T_a. We can show that T is weakly 1-saturated. Consider a B_1-type for $\overline{a}, x$, generated by B_1-formulas true of $\overline{a}$ and existential formulas. The type may put x in the subtree generated by $\overline{a}$, or in one of the trees T_{a_i}, where a_i (in the subtree) is finite. In either of these cases, the type is realized. Or, the type may put

x below some infinite a_i (in the subtree). Again, the type is realized, since there is a copy of $\omega^{<\omega}$ below a_i. This shows that $\mathcal{A}$ is weakly 1-saturated.

We have a computable enumeration of the possible finite labeled subtrees, and, hence, of the B_1-types realized in trees of this kind. Let R be this computable enumeration of B_1-types. To apply Theorem 4.2.5, we need the following.

Lemma 4.3.11. There is a copy $\mathcal{B}$ of T with a Δ^0_2 R-labeling.

Proof.

We build a Δ^0_2 copy $\mathcal{B}$ of T with nodes labeled as infinite, or with a specific finite tree below. We suppose that the ω-list of elements of T has the feature that the top element comes first, and any other element comes after its predecessor. This condition will also hold for the copy $\mathcal{B}$. For $\mathcal{B}$, we label the top node ∞. Having built a finite labeled subtree of $\mathcal{B}$, and determined a tentative partial isomorphism f from this to a subtree of T, we may find that some first node b labeled ∞ in $\mathcal{B}$ is mapped to a node a in T such that T_a is actually finite. The predecessor of b, say b', is labeled ∞, and we may still believe that the predecessor a' of a in T has an infinite tree below. In our $\mathcal{B}$, we vow to add no more terminal nodes to $\mathcal{B}_b$ and we look for a successor a'' of a with the appropriate $T_{a''}$. At a given stage, we take the first a'' that seems to work. Our first guess may not be correct—we may eventually see an unwanted finite node in $T_{a''}$. However, because of the structural properties we are assuming about T, we will eventually find a good a'', with $T_{a''}$ matching our $\mathcal{B}_b$. □

Applying Theorem 4.2.5, we get a computable copy of T. □

4.3.4 Models of a theory with few B_1-types

Lerman and Schmerl [LS79] gave conditions under which an $\aleph_0$-categorical theory T has a computable model. They assumed that the theory is arithmetical and $T \cap \Sigma_{n+1}$ is Σ^0_n for each n. In [Kni94], the assumption that T is arithmetical is dropped, and, instead, it is assumed that $T \cap \Sigma_{n+1}$ is Σ^0_n uniformly in n. The proof in [LS79] gives the following.

Theorem 4.3.12 (Lerman-Schmerl)**.** Let T be an $\aleph_0$-categorical theory that is Δ^0_N and suppose that for all $1 \leq n < N$, $T \cap \Sigma_{n+1}$ is Σ^0_n. Then T has a computable model.

To prove this, Lerman and Schmerl showed the following.

Lemma 4.3.13. For any $n < N$, if $\mathcal{A}$ is a model whose B_{n+1}-diagram is computable in X', and $T \cap \Sigma_{n+2}$ is Σ^0_1 in X, then there is a model $\mathcal{B}$ whose B_n-diagram is computable in X.

Let T be as in the Lerman-Schmerl Theorem. Let $\mathcal{A}$ be a model of T that is low over X. Then the Σ_1 diagram of $\mathcal{A}$ is computable in X'. Of course, $T \cap \Sigma_2$ is Σ^0_1, so it is Σ^0_1 relative to X. The lemma implies that $\mathcal{A}$ has an X-computable copy. In fact, we get the following.

Theorem 4.3.14. Let T be an elementary first order theory, in a computable language, such that $T \cap \Sigma_2$ is Σ^0_1. Suppose that for each tuple of variables $\overline{x}$, there are only finitely many B_1-types in variables $\overline{x}$ consistent with T. Then every model $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_2 relative to X.

Proof.
First, we show that there is a computable enumeration of all the B_1-types. Uniformly in each tuple of variables $\overline{x}$, we build a c.e. tree $P_{\overline{x}}$ whose paths represent the B_1-types in $\overline{x}$. We have a computable enumeration of B_1-formulas $(\varphi_n(\overline{x}))_{n\in\omega}$. At level n, the nodes σ in $P_{\overline{x}}$ represent the different finite sequences of formulas $\pm\varphi_k$ (in the appropriate tuple of variables), for $k < n$, that we see to be consistent with T, using the fact that $T \cap \Sigma_2$ is c.e. Note that each node $\sigma \in P_{\overline{x}}$ extends to a path. Also, $P_{\overline{x}}$ has only finitely many paths. We may suppose, running the enumeration of $T \cap \Sigma_2$ ahead, if necessary, that at step s, for the first s tuples of variables $\overline{x}$, the terminal nodes in our approximation to $P_{\overline{x}}$ all have length s.

We use all of these trees together to define the enumeration R. At stage s, we have assigned indices to the currently terminal nodes σ in $P_{\overline{x}}$ for the first s tuples of variables $\overline{x}$. For a given node σ, assigned index i, we will have put into R_i the formulas $\pm\varphi$ corresponding to this node σ. At stage 0, we assign the index 0 to the top node of $P_{\varnothing}$. At stage $s+1$, for each of the first s tuples of variables $\overline{x}$, each node σ of length s in $P_{\overline{x}}$ has at least one extension of length $s+1$. We give the index of σ to one such τ. There may be further extensions of σ or other old nodes, and we give these new indices. In addition, for the $(s+1)^{st}$ tuple of variables $\overline{x}$, we assign indices to the terminal nodes of the stage $s+1$ approximation. For the indices i assigned by stage $s+1$ to nodes σ of tree $P_{\overline{x}}$, we put into R_i all of the formulas corresponding to

σ. This process yields the desired computable enumeration of the B_1-types consistent with T.

Next, we show that $\mathcal{A}$ is weakly 1-saturated. Suppose $q(\overline{u}, x)$ is a B_1-type (consistent, of course) generated by formulas true of $\overline{a}$ and existential formulas $\varphi(\overline{u}, x)$. Since $q(\overline{u}, x)$ is isolated, it is principal, with a generating formula $\gamma(\overline{u}, x)$, of the form $\rho(\overline{u})$ & $\chi(\overline{u}, x)$, where $\rho(\overline{u})$ is in the B_1-type of $\overline{a}$, and $\chi(\overline{u}, x)$ is a finite conjunction of existential formulas. B_1 type of $\overline{a}$ includes the formula $(\exists x)\chi(\overline{u}, x)$. We have $(\exists x)\chi(\overline{u}, x)$ true of $\overline{a}$ in $\mathcal{A}$, so the type is realized.

Lemma 4.3.15. If $\mathcal{A}$ is low over X, then there is an R-labeling of $\mathcal{A}$ that is Δ^0_2 relative to X.

Proof.

For simplicity, we suppose $\mathcal{A}$ is low. For a tuple of variables $\overline{x}$, Δ^0_2 can find generating formulas for all of the B_1-types. Then Δ^0_2 can check which generating formula is true of a given tuple of elements $\overline{a}$. Then we have a Δ^0_2 R-labeling. □

Finally, we apply Theorem 4.2.5 to get an X-computable copy $\mathcal{B}$ of $\mathcal{A}$ with an isomorphism from $\mathcal{B}$ to $\mathcal{A}$ that is Δ^0_2 relative to X. □

Note: There are non-$\aleph_0$-categorical theories satisfying the conditions of Theorem 4.3.14.

Proof.

We write Θ for the ordering of type $\eta + 2 + \eta$. In [DK92], it was shown that for any linear ordering $\mathcal{A}$, $\Theta \cdot \mathcal{A}$ has a computable copy iff $\mathcal{A}$ has a Δ^0_2 copy. Let T_1 be a complete theory of linear orderings that is not $\aleph_0$-categorical. Let T be the complete theory whose models are exactly the orderings of the form $\Theta \cdot \mathcal{A}$, where $\mathcal{A}$ is a model of T_1. The theory T has a sentence saying that every element lies on an interval of type Θ. In addition, there are axioms guaranteeing that the restriction of our ordering to the set of elements that are the first in a successor pair satisfies all sentences φ in T_1.

We note that the B_1-types realized in models of T come from partitions into intervals of size 0 or ∞, with no two adjacent intervals of size 0. These are principal, so they are realized in all models of T. We note that if we replace T_1 by some other theory S_1 of infinite linear orderings, and form S in the same way, then the B_1-types realized in any and all models of S would be the same. Therefore, the Σ_2 theories are the same. If S_1 is decidable, then so is S. Thus, whether or not T_1 is decidable, $T \cap \Sigma_2$ is decidable. We chose T_1 not $\aleph_0$-categorical, so T is also not $\aleph_0$-categorical. □

4.3.5 Differentially closed fields

DF_0

A *differential field* is a field with one or more derivations satisfying the following familiar rules:

1. $\delta(u + v) = \delta(u) + \delta(v)$, and
2. $\delta(u \cdot v) = u \cdot \delta(v) + \delta(u) \cdot v$.

We consider differential fields of characteristic 0, and with a single derivation δ.

Trivially, $\mathbb{Q}$ is a differential field, under the derivation that takes all elements to 0. If a is an element of a differential field K, then a *generates* a differential field $F \subseteq K$, where the elements of F are gotten from a by closing under addition, multiplication, subtraction, division, and derivation.

DCF_0

Roughly speaking, a *differentially closed field* is a differential field in which differential polynomials have roots, where a differential polynomial is a polynomial $p(x)$ in x and its various derivatives. We write DCF_0 for the theory of differentially closed fields (of characteristic 0, with a single derivation). A. Robinson showed that the theory DCF_0 admits elimination of quantifiers. L. Blum, in her thesis, gave a nice computable set of axioms, showing that the theory is decidable. Thus, the elimination of quantifiers is effective. Blum also showed that DCF_0 is ω-stable. Then general model-theoretic results imply the existence and uniqueness of prime models over an arbitrary set. The existence and uniqueness of differential closures were not proved by algebraic methods—they really used the model theoretic results. For a discussion of differentially closed fields, emphasizing Blum's results, see Sacks [Sac10].

Differential polynomials

We consider *differential polynomials* $p(x)$ in a single variable x. A differential polynomial $p(x)$, over a differential field K, may be thought of as an algebraic polynomial in $K[x, \delta(x), \delta^{(2)}(x), \ldots, \delta^{(n)}(x)]$, for some n. We write $K\langle x\rangle$ for the set of differential polynomials over K. Initially, we let K be $\mathbb{Q}$, where $\delta(q) = 0$ for all $q \in \mathbb{Q}$. Later, K will be a finitely generated extension of $\mathbb{Q}$. Differential fields satisfy the quotient rule—this is easy to prove from the product rule. From this, it follows that if a is an element of a differential field extending K, and F is the differential subfield generated over K by a, then each element of F can be expressed in the form $\frac{p(a)}{q(a)}$, where $p(x), q(x) \in K\langle x\rangle$.

Definition 4.3.16 (Order)**.** For $p(x) \in K\langle x\rangle$, the *order* is the greatest n such that $\delta^{(n)}(x)$ appears non-trivially in $p(x)$. There are some special cases. An algebraic polynomial in x (with no derivatives) has order 0. The 0 polynomial has order ∞.

Definition 4.3.17 (degree, rank, order of ranks)**.** For $p(x) \in K\langle x\rangle$ of finite order n, the *degree* of $p(x)$ is the highest power k of $\delta^{(n)}(x)$ that appears. The *rank* of $p(x)$ is the ordered pair (n, k), where n is the order and k is the degree. We order the possible ranks of differential polynomials lexicographically.

Definition 4.3.18. A differential polynomial $p(x) \in K\langle x\rangle$ of order n is said to be *irreducible* if it is irreducible when considered as an algebraic polynomial in $K[x, \delta(x), \ldots, \delta^{(n)}(x)]$ (think of x and its derivatives as indeterminates). We count the 0 polynomial as irreducible.

Blum's axioms for DCF_0

Blum's axioms say that a differentially closed field (of characteristic 0 and with a single derivation), is a differential field K such that

(1) for any pair of differential polynomials $p(x), q(x) \in K\langle x\rangle$ such that the order of $q(x)$ is less than that of $p(x)$, there is some x satisfying $p(x) = 0$ and $q(x) \neq 0$,

(2) if $p(x)$ has order 0, then $p(x)$ has a root.

The axioms of form (2) say that K is algebraically closed.

Types

We want to understand the types, in any number of variables, realized in models of DCF_0. For a single variable x, each type over $\emptyset$ is determined by an irreducible differential polynomial $p(x) \in \mathbb{Q}\langle x\rangle$. If $p(x) \in \mathbb{Q}\langle x\rangle$ is irreducible of order n, then the corresponding type consists of formulas provable from the axioms of DCF_0, the formula $p(x) = 0$ and further formulas $q(x) \neq 0$, for $q(x) \in \mathbb{Q}\langle x\rangle$ of order less than n. The formulas $q(x) \neq 0$, for $q(x) \in \mathbb{Q}\langle x\rangle$

of order less than n, say that $x, \delta(x), \delta^{(2)}(x), \ldots, \delta^{(n-1)}(x)$ are algebraically independent over $\mathbb{Q}$. We allow the case where $p(x)$ is the 0 polynomial, which has order ∞. In this case, the corresponding type λ_p consists of the formulas provable from the axioms of DCF_0 and the formulas $q(x) \neq 0$ for $q(x)$ of all finite orders.

Similarly, for a differential field K, each type over K (to be realized in some extension of K to a model of DCF_0) is determined by an irreducible differential polynomial $p(x) \in K\langle x\rangle$. If $p(x)$ is irreducible of order n, the corresponding type $\lambda_{K,p}$ consists of formulas provable from the axioms of DCF_0, the atomic diagram of K, the formula $p(x) = 0$, and further formulas $q(x) \neq 0$, for $q(x)$ of order less than n. The formulas $q(x) \neq 0$, taken together, say that $x, \delta(x), \ldots, \delta^{(n-1)}(x)$ are algebraically independent over K.

A proof of the following result can be found in Sacks [Sac10], pp. 297-298.

Proposition 4.3.19.

1. If $p(x) \in \mathbb{Q}\langle x\rangle$ is irreducible, the corresponding type λ_p is complete over $\varnothing$. Moreover, all types over $\varnothing$ (in the variable x) have this form.

2. For a differential field K, if $p(x) \in K\langle x\rangle$ is irreducible, then $\lambda_{K,p}$ is a complete type over K, and all types over K (in the variable x) have this form.

Among the types in one variable (over $\varnothing$, or over K), there is a unique type, obtained from the 0 polynomial, that is *differential transcendental*. The other types, obtained from differential polynomials of finite rank, are *differential algebraic*.

Types in several variables

In general, we can determine a type in variables $(x_1, \ldots, x_n)$ by giving the type of x_1 (over $\varnothing$), the type of x_2 over x_1, the type of x_3 over (x_1, x_2), and so on. To describe a type in variables $(x_1, \ldots, x_n)$, we imagine a large differentially closed field M and we consider various elements and differential subfields. The type of x_1 is λ_{p_1} for some irreducible $p_1 \in \mathbb{Q}\langle x_1\rangle$. Let K_1 be the differential subfield of M generated by x_1 over $\mathbb{Q}$, where x_1 satisfies λ_{p_1} in M. The type of x_2 over K_1 is λ_{K_1,p_2} for some irreducible $p_2 \in K_1\langle x_2\rangle$. Let K_2 be the differential field generated by x_2 over K_1. In general, given K_i generated by $x_1, \ldots, x_i$, the type of x_{i+1} over K_i is $\lambda_{K_i,p_{i+1}}$ for some irreducible

$p_{i+1} \in K_i\langle x_{i+1}\rangle$, and then K_{i+1} is the differential subfield of M generated by x_{i+1} over K_i.

Toward strong jump inversion

Marker and R. Miller [MM17] showed that all models of DCF_0 admit strong jump inversion. Our goal in this subsection is to obtain this result using our Theorem 4.2.5. In the earlier applications of Theorem 4.2.5, the structures satisfied the condition of effective type completion because they were weakly 1-saturated. Among the countable models of DCF_0, only the saturated one is weakly 1-saturated. There are $2^{\aleph_0}$ non-isomorphic countable models. (In fact, Marker and Miller gave a method for coding an arbitrary countable graph in a model of DCF_0.) We will need to show effective type-completion in some other way. There is a lemma in [MM17] that does exactly this. Since we have effective quantifier elimination, we can work with quantifier-free types. Most of our effort goes into producing a computable enumeration R of the quantifier-free types realized in models of DCF_0. Once we have this, we can show easily that for any model $\mathcal{A}$, $D(\mathcal{A})'$ computes an R-labeling of $\mathcal{A}$. This puts us in position to apply Theorem 4.2.5.

Computable enumeration of types

It may at first seem that it should be easy to produce a computable enumeration of types. After all, the theory DCF_0 is decidable and all types are computable. However, T. Millar [Mil78] gave an example of a decidable theory T, with all types computable, such that there is no computable enumeration of all types. So, we have some work to do.

By quantifier elimination, we can pass effectively from a quantifier-free type $\lambda(\overline{x})$ to the complete type generated by $DCF_0 \cup \lambda(\overline{x})$. In what follows, we will enumerate quantifier-free types. We will consider realizations of the quantifier-free types in differential fields K that are not differentially closed, bearing in mind that a tuple realizing $\lambda(\overline{x})$ in K will realize the corresponding complete type generated by $DCF_0 \cup \lambda(\overline{x})$ in any extension of K to a model of DCF_0.

We eventually give a uniform procedure that, for a given tuple of variables $\overline{x}$, yields an enumeration of the types in $\overline{x}$. But first, we give a procedure for a single variable x in order to elucidate the relevant issues before proceeding to the full procedure. We determine a type $\lambda(x)$ corresponding to each differential polynomial $p(x) \in \mathbb{Q}\langle x\rangle$, irreducible or not. Let $(\varphi_s)_{s\in\omega}$ be a computable list of the atomic formulas in variable x, in order of Gödel number. At each stage, we will have put into $\lambda(x)$ finitely many formulas, always checking consistency with DCF_0.

At stage 0, we put into the type $\lambda(x)$ just the formula $p(x) = 0$, assuming that this is consistent. We also determine the order of $p(x)$—we can do this just by inspection. At stage s, we will decide φ_s, putting it or its negation into $\lambda(x)$. If $p(x)$ is irreducible, there will be a proof of exactly one of φ_s, $\neg\varphi_s$ from DCF_0, $p(x) = 0$, and the formulas $q(x) \neq 0$, for $q(x) \in \mathbb{Q}\langle x\rangle$ of order less than that of $p(x)$. So, we search for a proof. Being reducible is c.e., and if $p(x)$ is reducible, we will eventually see this.

At stage s, we search until we either find a proof of $\pm\varphi_s$ or discover that $p(x)$ is reducible. If we find a proof of φ_s (or $\neg\varphi_s$), then we add this formula to our type, provided that it is consistent to do so. If we find that $p(x)$ is reducible, then we just decide φ_s so as to maintain consistency with DCF_0. The procedure we have just described gives a type λ corresponding to each $p \in \mathbb{Q}\langle x\rangle$. If p is irreducible, then $\lambda = \lambda_p$. Thus, by considering all $p \in \mathbb{Q}\langle x\rangle$, we get all types in the variable x.

A type in one variable corresponded to a differential polynomial $p(x)$ over $\mathbb{Q}$. Intuitively, we'd like to enumerate types in n variables using all n-tuple of polynomials, according to the pattern described above in types in several variables. Unfortunately, since the fields themselves depend on the polynomials in the tuple, it is not even clear if a potential polynomial would make sense; one of its coefficients might actually be undefined. Therefore, our enumeration construction takes these obstacles into account with a more formal approach. A type in n variables will correspond to an n-tuple of formal differential polynomials $p_1(x_1), \ldots, p_n(x_n)$. Here $p_1(x_1)$ is an actual differential polynomial with coefficients in $\mathbb{Q}$. For $i \geq 1$, $p_{i+1}(x_{i+1})$ looks like a differential polynomial, but the coefficients come from a set K_i^F of formal names for possible elements of a differential field generated by elements $x_1, \ldots, x_i$. We say more about these formal names below. We define the sets K_i^F and $K_i^F\langle x_{i+1}\rangle$ by induction on i.

The many lemmas below allow us to prove Proposition 4.3.32, the computable enumeration of types, from the basic definitions and results in [Sac10].

Definition 4.3.20.

1. $K_0^F = \mathbb{Q}$, and $K_0^F\langle x_1\rangle = \mathbb{Q}\langle x_1\rangle$,

2. $K_i^F\langle x_{i+1}\rangle$ is the set of formal expressions that look like differential polynomials in the variable x_{i+1} but have coefficients in K_i^F as opposed to a well-defined differential field,

3. K_{i+1}^F consists of the expressions $\frac{r(x_{i+1})}{s(x_{i+1})}$, where $r, s \in K_i^F\langle x_{i+1}\rangle$.

Lemma 4.3.21. Uniformly in n, we can enumerate the n-tuples $p_1(x_1), \dots, p_n(x_n)$, where $p_{i+1}(x_{i+1}) \in K_i^F\langle x_{i+1}\rangle$.

Proof.
The set K_0^F is a fixed computable set with computable index, and there is a uniform, effective procedure to construct $K_i^F\langle x_{i+1}\rangle$ from K_i^F and K_{i+1}^F from $K_i^F\langle x_{i+1}\rangle$. Therefore, there is a single, computable function that gives computable indices for all of these sets. Then there is computable function that, given n, finds a computable index of $K_0^F\langle x_1\rangle \times K_1^F\langle x_2\rangle \times \dots \times K_{n-1}^F\langle x_n\rangle$.
□

Given an n-tuple of formal differential polynomials $p_1, \dots, p_n$ as above, we will obtain a type $\lambda(x_1, \dots, x_n)$ by producing a sequence of differential fields $K_0, \dots, K_n$, where $K_0 = \mathbb{Q}$, and K_{i+1} is generated over K_i by an element x_{i+1} satisfying a chosen type λ_{i+1} that depends on p_{i+1}. In the end, K_n will be generated by $x_1, \dots, x_n$, and $\lambda(x_1, \dots, x_n)$ will be the type realized by $x_1, \dots, x_n$ that generates K_n. We give several lemmas.

Lemma 4.3.22. There is a uniform effective procedure that, given a differential field K and a type $\lambda(x)$ over K, yields a differential field $K' \supseteq K$ that is generated over K by an element x realizing λ.

Proof. Uniformly in K, we construct a computable, formal set N_K that consists of names of the form $\frac{r(x)}{s(x)}$, where $r(x), s(x) \in K\langle x\rangle$. Next, we define the universe of K' from N_K and $\lambda(x)$ by induction:

1. at step 1, consider the first element of N_K, which is a formal expression of the form $\frac{r(x)}{s(x)}$. We use $\lambda(x)$ to determine if $s(x) = 0$. If so, then we do not include $\frac{r(x)}{s(x)}$ in the universe of K'; otherwise we do.

2. at step $n+1$, consider the $(n+1)^{st}$ element of N_K, which is a formal expression of the form $\frac{r(x)}{s(x)}$, where $r(x), s(x) \in K\langle x\rangle$. We use $\lambda(x)$ to determine if $s(x) = 0$. If so, then we do not include $\frac{r(x)}{s(x)}$ in the universe of K'. If not, then we use $\lambda(x)$ and simple "cross multiplication" to

determine if $\frac{r(x)}{s(x)}$ is equal to any $\frac{r_1(x)}{s_1(x)}$ that we included in K' an earlier step. If so, then we do not include $\frac{r(x)}{s(x)}$ in the universe of K'; otherwise we do.

Uniformly in K and $\lambda(x)$, the above procedure computably enumerates the elements of the universe of K' in order; therefore, the universe of K' is uniformly computable in K and $\lambda(x)$.

Finally, to define the constants and operations on the universe of K', we first use $\lambda(x)$ to identify element in the universe K' that is equal to $\frac{0_K}{1_K}$ and the element equal to $\frac{1_K}{1_K}$. Next, to calculate a sum or product of two elements $\frac{r(x)}{s(x)}$ and $\frac{r_1(x)}{s_1(x)}$ in K', or to calculate $\delta\left(\frac{r(x)}{s(x)}\right)$, we add, multiply, or differentiate formally, and then we use $\lambda(x)$ to determine what element in the universe of K' the formal expression is equal to. The definitions of these operations are uniformly computable from K' and λ, and thus ultimately from K and λ. □

Given an actual differential field K_i, generated by elements $x_1, \ldots, x_i$, some names from K_i^F have a definite value in K_i, while others do not. Recall that the names are quotients. We do not get a value if the denominator is 0.

Lemma 4.3.23. There is a uniform effective procedure that, given a differential field K_i generated by elements $x_1, \ldots, x_i$, and an element $f \in K_i^F$, determines whether f makes sense, and if so, assigns to f a definite value in K_i.

Proof.

We first form a finite set S of names such that $f \in S$, and if $g \in S \cap K_j^F$, for $0 < j \leq i$, and h is a coefficient from the numerator or denominator of g, then $h \in S$. We form K_j for $0 \leq j \leq i$. We then proceed by induction on j to determine for all $g \in S \cap K_j^F$, whether g has value in K_j, and if so, to assign the value. Then f has a value iff all elements of S have a value. □

Lemma 4.3.24. There is a uniform effective procedure that, given $p \in K_i^F\langle x_{i+1}\rangle$ and a differential field K_i generated by elements $x_1, \ldots, x_i$, determines whether p makes sense (i.e., whether the coefficients all have value in K_i), and if so, identifies p with an element of $K_i\langle x_{i+1}\rangle$.

Proof. Given the $p \in K_i^F\langle x_{i+1}\rangle$, we simply identify its coefficients as elements of K_i^F. Then we apply the previous lemma to each of the coefficients individually. If all of the co-efficients make sense, then we assign each of them a definite value in K_i and then construct the corresponding element of $K_i\langle x_{i+1}\rangle$. Otherwise, if at least one of the coefficients does not make sense, p does not make sense. □

Lemma 4.3.25. There is a uniform effective procedure that, given a differential field K and a differential polynomial $p(x)$ over K, enumerates the differential polynomials $q(x)$ of order lower than that of $p(x)$.

Proof.

First, there is a computable procedure, uniform in K, that computes orders of $p(x) \in K\langle x\rangle$. Namely, assuming $p(x)$ is written where formal "like terms" already are combined, then the procedure looks for the term with the highest derivative $\delta^n(x)$ appearing as a factor, where the coefficient in K for at least one such term is non-zero. Then, uniformly in K and $p(x)$, there is an effective procedure that lists all algebraic polynomials in $K[x, \delta(x), \ldots, \delta^{n-1}(x)]$. □

Lemma 4.3.26. There is a uniform effective procedure that, given a differential field K and a differential polynomial $p(x)$ over K, enumerates the proofs of formulas $\varphi(x)$ (with parameters in K) from DCF_0, $D(K)$, $p(x) = 0$, and $q(x) \neq 0$, for q of lower order.

Proof.

By Lemma 4.3.25, we can enumerate the polynomials $q(x)$ of lower order, so we can enumerate the axioms to use in our proofs. Then we can enumerate proofs from these axioms of formulas of the kind we are interested in. □

In Lemma 4.3.26, we did not assume that $p(x)$ is irreducible. So, the set of axioms may not generate a consistent, complete type over K.

Lemma 4.3.27. There is a uniform effective procedure that, given a differential field K, enumerates the reducible differential polynomials $p(x)$ over K.

Proof.

For a given $p(x)$ we enumerate $D(K)$, searching for a formula of form $r(x) \cdot s(x) = p(x)$, where $r(x)$ and $s(x)$ are differential polynomials over K, both non-constant. The search halts iff $p(x)$ is reducible. □

Lemma 4.3.28. Let K be a differential field. For any tuple $\overline{k}$ in K, DCF_0 together with the quantifier-free type of $\overline{k}$ generates a complete type that would be realized by $\overline{k}$ in any extension of K to a model of DCF_0.

Proof.

Let $K \subseteq M$, where M is a differentially closed field. By quantifier elimination, any formula true of $\overline{k}$ in M is proved from DCF_0 and the quantifier-free formulas true of $\overline{k}$. □

Lemma 4.3.29. There is a uniform effective procedure for determining, for a differential field K and a formula $\varphi(\overline{k}, x)$ (with parameters $\overline{k}$ in K), whether $\varphi(\overline{k}, x)$ is consistent with $DCF_0 \cup D(K)$.

Proof.

Let $\gamma(\overline{k})$ be the quantifier-free type realized by $\overline{k}$ in K. By Lemma 4.3.28, $DCF_0 \cup \gamma(\overline{k})$ generates a complete type that would be realized by $\overline{k}$ in any extension of K to a model of DCF_0. Then $\varphi(\overline{k}, x)$ is consistent with $DCF_0 \cup D(K)$ iff $(\exists x)\varphi(\overline{k}, x)$ is in this type. □

Lemma 4.3.30. There is a uniform effective procedure that, given a differential field K and $p(x) \in K\langle x\rangle$, enumerates a type $\lambda(x)$ for x over K. Moreover, if $p(x)$ is irreducible, then $\lambda(x) = \lambda_{K,p}$.

Proof.

We can determine the order of $p(x)$, just by inspection. At each step, we will have put finitely many formulas into the type $\lambda(x)$, having checked consistency with $DCF_0 \cup D(K)$ as in Lemma 4.3.29 (the parameters from K that appear in the formulas form the relevant $\overline{k}$). At step 0, we put into $\lambda(x)$ the formula $p(x) = 0$, assuming that this is consistent. We have a computable enumeration of the atomic formulas $\varphi_s(x)$ with parameters in K. At step $s+1$, we decide $\varphi_s(x)$, adding $\varphi_s(x)$ or $\neg\varphi_s(x)$ to the type $\lambda(x)$. If we have already seen that $p(x)$ is reducible, then we add $\varphi_s(x)$ to the type if it is consistent to do so, and otherwise, we add $\neg\varphi_s(x)$. Suppose that $p(x)$ appears to be irreducible. Then we simultaneously search for the following:

(1) a proof of $\pm\varphi_s$ from $DCF_0 \cup D(K)$, $p(x) = 0$, and formulas $q(x) \neq 0$ for q of order less than that of p,

(2) evidence that $p(x)$ is reducible over K.

By Lemmas 4.3.26 and 4.3.27, these are computable searches. One of the searches will halt, since if $p(x)$ is irreducible, then the formulas in (1) above generate a complete type over K. If we find that $p(x)$ is reducible, then we proceed as above, adding $\pm\varphi_s(x)$ just to maintain consistency. (We check consistency as in Lemma 4.3.29.) If we find a proof of φ_s, or $\neg\varphi_s$, then we add this formula to the type, provided that it is consistent to do so. We take inconsistency as evidence that $p(x)$ is reducible, and we proceed as above. □

Proposition 4.3.31. Uniformly in n, we can enumerate the types in n variables.

Proof.

By Lemma 4.3.21, uniformly in n, we can enumerate the n-tuples of formal differential polynomials $p_1, \dots, p_n$, where $p_1 \in \mathbb{Q}\langle x_1\rangle$, $p_{i+1} \in K_i^F\langle x_{i+1}\rangle$. The j^{th} n-tuple of differential polynomials $p_1, \dots, p_n$ will yield the j^{th} type $\lambda(x_1, \dots, x_n)$ in variables $x_1, \dots, x_n$. We describe $\lambda(x_1, \dots, x_n)$ in terms of some differential fields $K_1, \dots, K_n$ and types $\lambda_i(x_i)$ over K_{i-1}. We note that p_1 is an actual differential polynomial over $K_0 = \mathbb{Q}$. We apply Lemma 4.3.30 to p_1 and $\mathbb{Q}$, to get a type $\lambda_1(x_1)$. We apply Lemma 4.3.22 to $\mathbb{Q}$ and λ_1 to get the differential field K_1 generated by x_1 realizing λ_1.

Now, $p_2(x_2)$ is only an element of $K_1^F\langle x_2\rangle$, where K_1^F is not an actual differential field. We apply Lemma 4.3.24 to K_1 to determine whether $p_2(x_2)$ makes sense as a differential polynomial over K_1. If not, then we generate a type λ_2 for x_2 over K_1 using $DCF_0 \cup D(K_1)$ as follows. We run through the atomic formulas $\varphi_s(x_2)$ (over K_1) in order, adding φ_s if it is consistent to do so, and otherwise adding $\neg\varphi_s$. We check consistency at each step as in Lemma 4.3.29. If p_2 makes sense as a differential polynomial over K_1, then we apply Lemma 4.3.30 to get λ_2. We then apply Lemma 4.3.22 to get the differential field K_2 generated by x_2 realizing λ_2 over K_1.

In general, given K_i, we apply Lemma 4.3.24 to determine whether p_{i+1} makes sense as differential polynomial over K_i. If not, then we generate a type λ_{i+1}, using $DCF_0 \cup D(K_i)$. If p_{i+1} makes sense as a differential polynomial over K_i, then we apply Lemma 4.3.30 to get a type λ_{i+1} for x_{i+1} over K_i. From K_i and λ_{i+1}, we get K_{i+1} as in Lemma 4.3.22. After finitely many steps, calculating computable indices for the differential fields K_i and the types λ_i, we arrive at the differential field K_n. This is generated over $\mathbb{Q}$ by the elements $x_1, \dots, x_n$. The quantifier-free type we want is that realized by $x_1, \dots, x_n$ in K_n. Of course, since DCF_0 has effective quantifier elimination, we then effectively compute the complete type realized by $x_1, \dots, x_n$. □

As planned, we combine the enumerations of types in variables $x_1, \dots, x_n$, for various n.

Proposition 4.3.32. There is a computable enumeration R of all complete types realized in models of DCF_0.

Now, we can prove the result of Marker and Miller, using our Theorem 4.2.5.

Proposition 4.3.33. Every countable model of DCF_0 admits strong jump inversion.

Proof.
By Proposition 4.3.32, there is a computable enumeration R of the complete types realized in models of DCF_0, and thus, of the B_1 types. Thus, Condition (1) of Theorem 4.2.5 holds. The following lemma shows that Condition (3) holds in the strong way.

Lemma 4.3.34. Let X be a subset of ω, and let $\mathcal{A}$ be a model of DCF_0 that is low over X. Then X' computes an R-labeling of $\mathcal{A}$.

Proof. [Proof of Lemma]
Note that for each tuple $\overline{a}$, we have an $\mathcal{A}$-computable procedure for finding, at step s, the first index i such that R_i agrees with the type of $\overline{a}$ on the first s quantifier-free formulas. After some step s, this i is the first index for the B_1-type of $\overline{a}$. Thus, we have an R-labeling that is computable in $D(\mathcal{A})'$, and hence, in X', since $\mathcal{A}$ is low over X. □

We need to establish Condition (2), effective type completion. There is a uniform effective procedure for computing, from a type $p(\overline{u})$ and a formula $\varphi(\overline{u}, x)$, consistent with $p(\overline{u})$, a type $q(\overline{u}, x)$ such that if $\overline{c}$ satisfies $p(\overline{u})$, then some a satisfies $q(\overline{c}, x)$. Marker and Miller [MM17] needed this for the same reason we do. It is Lemma 4.3 in their paper. (The type $q(\overline{c}, x)$ will be realized in the differential closure of $\overline{c}$.) The conditions for Theorem 4.2.5 are all satisfied. Therefore, $\mathcal{A}$ admits strong jump inversion. □

Decidable saturated model of DCF_0

In general, a structure $\mathcal{A}$ is computable if its atomic diagram is computable, and $\mathcal{A}$ is decidable if the complete diagram is computable. By elimination of quantifiers, a model of DCF_0 is decidable iff it is computable. Using Proposition 4.3.32, we can show that the countable saturated model of DCF_0 has a decidable copy. We need the following result from Morley [Mor76].

Theorem 4.3.35. Let T be a countable complete elementary first order theory for a computable language. Then the following are equivalent:

1. T has a decidable saturated model,
2. there is a computable enumeration of all types realized in models of T.

Using Theorem 4.3.35 and Proposition 4.3.32, we get the following.

Corollary 4.3.36. The saturated model of DCF_0 has a decidable copy.

Chapter 5

Effective embeddings and interpretations

There are different notions that describe the coding (and decoding) of a structure $\mathcal{A}$ in another structure $\mathcal{B}$. The main idea is to see which classes of structures have more expressive power. We are interested in cases where there is a uniform effective procedure for coding and decoding, and in cases where there is no such procedure. We give one negative and one positive result.

Friedman and Stanley [FS89] considered a Borel embedding L of directed graphs in linear orderings. In [CCKM04], the authors relaxed the convention that the structures have universe $\mathbb{N}$, to allow finite structures. They introduced an effective version of Borel embedding. A Turing computable embedding Φ of class of structures K into another class of structures K', Φ gives a uniform effective procedure for coding each structure from K in a structure from K', which preserves the back-and-forth structure [KMVB07] and isomorphisms.

The decoding may or may not be effective. Some of the known examples of Turing computable embeddings involve uniformly defined effective interpretations. In particular, this is true of the standard codings (due to Marker, Lavrov, and Nies) of directed graphs, or structures from an arbitrary computable language, in undirected graphs. One step of decoding gives us the Medvedev reducibility. Recall that a structure $\mathcal{A}$ is Medvedev reducible to a structure $\mathcal{B}$ if there is a Turing operator Φ, that takes a copy of $\mathcal{B}$ to a copy of $\mathcal{A}$. There is a Turing computable embedding Θ of directed graphs $\mathcal{A}$ in undirected graphs (see [Mar02]). Moreover, there is a fixed tuple of existential formulas that give a *uniform* effective interpretation; i.e., for all directed graphs $\mathcal{A}$, these formulas interpret $\mathcal{A}$ in $\Theta(\mathcal{A})$. So, these existential formulas gives us the decoding. It follows that $\mathcal{A}$ is Medvedev reducible to $\Theta(\mathcal{A})$ uniformly; i.e., $\mathcal{A} \leq_s \Theta(\mathcal{A})$ with a fixed Turing operator Φ that serves for all $\mathcal{A}$.

Hirschfeldt, Khoussainov, Shore, and Slinko [HKSS02] gave conditions on Turing operators Φ from a class K to a class K' guaranteeing that $\mathcal{A} \in K$ and $\Phi(\mathcal{A})$ share many properties—having the same spectrum, the same computable dimension, etc. They found conditions that give an interpretation of $\mathcal{A}$ in $\Phi(\mathcal{A})$, using computable Σ_1^c formulas that may have parameters. There are conditions making $\Phi(\mathcal{A})$ rigid over $\mathcal{A}$. In [HKSS02], the class of undirected graphs, the class of rings, and the class of 2-step nilpotent groups lie on top.

A more general notion is considered by Montalbàn [Mon14] - the notion of effective bi-interpretability. Two structures are effectively-bi-interpretable if there are effective-interpretations of each structure in the other and the composition of the isomorphisms interpreting one structure inside the other and then interpreting the other back into the first one to be effective. He shows that the effective bi-interpretability preserves the most computability theoretic properties. A more recent result of R. Miller, Poonen, Schoutens, and Shlapentokh [MPSS18] shows that undirected graphs can be effectively interpreted in fields and fields are on top for effective-bi-interpretability.

In the next section we present our results with Julia Knight and Stefan Vatev [KSV19] for coding and decoding graphs in linear orderings. In the second section of this chapter we present an effective interpretation of fields in 2-step nilpotent groups — Heisenberg groups [ACG+20]. The last section is devoted to an interpretation of an algebraic closed field $\mathcal{C}$ with characteristic 0 in a special linear group $SL_2(\mathcal{C})$.

5.1 Coding and decoding in linear ordering

Friedman and Stanley [FS89] introduced Borel embeddings as a way of comparing classification problems for different classes of structures. A Borel embedding of a class $\mathcal{K}$ in a class $\mathcal{K}'$ represents a uniform procedure for coding structures from $\mathcal{K}$ in structures from $\mathcal{K}'$. Many Borel embeddings are actually Turing computable [CCKM04]. A Turing computable embedding of a class $\mathcal{K}$ in a class $\mathcal{K}'$ represents an effective coding procedure.

When $\mathcal{A}$ is coded in $\mathcal{B}$, effective decoding is represented by a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$. Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] considered a notion of effective interpretation of $\mathcal{A}$ in $\mathcal{B}$. They also defined a notion of computable functor, where this is a pair of Turing operators, one taking copies of $\mathcal{B}$ to copies of $\mathcal{A}$, and the other taking isomorphisms between copies of $\mathcal{B}$ to isomorphisms between the corresponding copies of $\mathcal{A}$. They showed that $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$ iff there is a computable functor from $\mathcal{B}$ to $\mathcal{A}$. The first operator is a Medvedev reduction. This uniform Medvedev reduction represents uniform effective decoding. Harrison-Trainor, R. Miller, and Montalbán [HTMM18] also considered interpretations by $L_{\omega_1\omega}$ formulas, guaranteeing Borel decoding.

The class of undirected graphs and and the class of linear orderings both lie on top under Turing computable embeddings. The standard Turing computable embeddings of directed graphs (or structures for an arbitrary computable relational language) in undirected graphs come with uniform effective interpretations. We give examples of graphs that are not Medvedev reducible to any linear ordering, or to the jump of any linear ordering. Any graph can be interpreted in a linear ordering using computable Σ^c_3 formulas. For the known Turing computable embedding of graphs in linear orderings, due to Friedman and Stanley, we show that there is no uniform interpretation defined by $L_{\omega_1\omega}$ formulas; that is, no fixed tuple of $L_{\omega_1\omega}$ formulas can interpret every graph in its Friedman-Stanley ordering.

We assume that the language of each structure is computable, where this means that the set of non-logical symbols is computable and we can effectively determine the type and arity of each symbol. We may assume that the languages are relational. We restrict our attention to structures with universe equal to $\mathbb{N}$. Let $Mod(L)$ be the class of L-structures with this universe. We identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$. We may identify this, via Gödel numbering, with a set of natural numbers, or with an element of 2^ω. Thus, we think of $Mod(L)$ as a subclass of 2^ω. For a class of structures $\mathcal{K} \subseteq Mod(L)$, we suppose that $\mathcal{K}$ is axiomatized by an $L_{\omega_1\omega}$ sentence. By a result of López-Escobar [LE65], this is the same as assuming that $\mathcal{K}$ is a Borel subclass of $Mod(L)$ closed under isomorphism.

5.1.1 Borel embeddings

The following definition is from [FS89].

Definition 5.1.1. We say that a class $\mathcal{K}$ is *Borel embeddable* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_B \mathcal{K}'$, if there is a Borel function $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

A Borel embedding of $\mathcal{K}$ into $\mathcal{K}'$ represents a uniform procedure for coding structures from $\mathcal{K}$ in structures from $\mathcal{K}'$.

Theorem 5.1.2. [FS89]
The following classes lie on top under $\leq_B$.

1. undirected graphs
2. fields of any fixed characteristic
3. 2-step nilpotent groups
4. linear orderings

Friedman and Stanley defined an embedding of graphs in fields of any fixed characteristic. They also defined an embedding of graphs in linear orderings. For the other classes listed above, Friedman and Stanley credit earlier sources. Lavrov [Lav63] defined an embedding of $Mod(L)$ (structures with a domain $\mathbb{N}$ in the language L) in undirected graphs, for any language L. There are similar constructions due to Nies [Nie96] and Marker [Mar02]. Mekler [Mek81] defined an embedding of graphs in 2-step nilpotent groups. Alternatively, we get an embedding of graphs in 2-step nilpotent groups by composing the embedding of graphs in fields with an earlier embedding by Mal'tsev [Mal60] of fields in 2-step nilpotent groups.

Example 5.1.3. Friedman and Stanley [FS89] interpreted an undirected graph in a field, say of characteristic 0. Let F^* be an algebraically closed field with transcendence basis $b_0, b_1, b_2, \ldots$. For a graph G, let $F(G)$ be the subfield generated by the following:

1. b_i, for $i \in G$,
2. elements of $acl(b_i)$,
3. $\sqrt{d+d'}$, where for some i, j joined by an edge in G, d is inter-algebraic with b_i and d' is inter-algebraic with b_j.

The formulas that define the interpretation are computable Π^0_2 or simpler. Hence, for any $F \cong F(G)$, we get a copy of G computable in F''.

Note: We have a Borel procedure for coding structures from structures of class K in structures from K'. As we shall see, there may or may not be a Borel decoding procedure.

5.1.2 Turing computable embeddings

Kechris suggested to Knight that she and her students should consider effective embeddings. This is done in [CCKM04], [KMVB07].

Definition 5.1.4. We say that a class $\mathcal{K}$ is *Turing computably embedded* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_{tc} \mathcal{K}'$, if there is a Turing operator $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

A Turing computable embedding represents an effective coding procedure. The next result is in [CCKM04].

Theorem 5.1.5. The following classes lie on top under $\leq_{tc}$.

1. undirected graphs
2. fields of any fixed characteristic
3. 2-step nilpotent groups
4. linear orderings

The reason for this is that the Borel embeddings of Friedman-Stanley, Lavrov, Nies, Marker, Mekler, and Mal'tsev are all, in fact, Turing computable.

5.1.3 Medvedev reductions

A *problem* is a subset of 2^ω or $\mathbb{N}^\omega$. Problem P is Medvedev reducible to problem Q if there is a Turing operator Φ that takes elements of Q to elements of P. The problems that interest us ask for copies of particular structures, where each copy is identified with an element of 2^ω.

Definition 5.1.6. We say that $\mathcal{A}$ is *Medvedev reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leq_s \mathcal{B}$ if there is a Turing operator that takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$.

Supposing that $\mathcal{A}$ is coded in $\mathcal{B}$, a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$ represents an effective decoding procedure.

In a number of familiar examples where $\mathcal{A} \leq_s \mathcal{B}$, the structure $\mathcal{A}$ is defined or interpreted in $\mathcal{B}$ using formulas that let us recover a copy of $\mathcal{A}$ from each copy of $\mathcal{B}$.

The notion of Medvedev reducibility captures the idea of effective recovery (decoding) of a copy of $\mathcal{A}$ from a copy of $\mathcal{B}$. It is uniform (strong) reducibility.

The other (weak) not uniform reducibility is introduced by Muchnik [Muc16]. A structure $\mathcal{A}$ is *Muchnik reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leq_w \mathcal{B}$, if every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$. It is equivalent to say $DS(\mathcal{B}) \subseteq DS(\mathcal{A})$.

Example 5.1.7. Here are some examples of Montalbán's book [Mon].

- Given a ring R, $R[x] \leq_s R$.
- For every structure $\mathcal{A}$ there is a graph $G_\mathcal{A}$ such that $\mathcal{A} \equiv_s G_\mathcal{A}$.
- In every linear ordering $\mathcal{A}$ every segment $[a,b]_\mathcal{A}$ is Muchnik reducible to $\mathcal{A}$ but not necessary Medvedev computable to $\mathcal{A}$.

5.1.4 Sample embedding

Below, we describe Marker's Turing computable embedding of directed graphs in undirected graphs.

1. For each point a in the directed graph $\mathcal{A}$, the undirected graph $\mathcal{B}$ has a point b_a connected to a triangle.
2. For each ordered pair of points (a,a') from $\mathcal{A}$, $\mathcal{B}$ has a point $p_{(a,a')}$ that is connected directly to b_a and with one stop to $b_{a'}$. The point $p_{(a,a')}$ is connected to a square if there is an arrow from a to a', and to a pentagon otherwise.

For structures $\mathcal{A}$ with more relations, the same idea works—we use more special points and more n-gons.

Fact: For Marker's embedding Φ of directed graphs in undirected graphs, there are finitary existential formulas that, for all inputs $\mathcal{A}$, define the following.

1. the set D of b_a connected to a triangle,
2. the set of ordered pairs $(b_a, b_{a'})$ such that the special point $p_{(a,a')}$ is connected to a square,
3. the set of ordered pairs $(b_a, b_{a'})$ such that the special point $p_{(a,a')}$ is connected to a pentagon.

This guarantees that any copy of $\Phi(\mathcal{A})$ computes a copy of $\mathcal{A}$.

5.1.5 Effective interpretations and computable functors

Informally, a structure $\mathcal{A}$ is effectively interpretable in a structure $\mathcal{B}$ if there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ (as in Model theory [Mar02]), but the domain of the interpretation is allowed to be a subset of $B^{<\omega}$, while in the classical definition it is required to be a subset of B^n for some n), and where all sets in the interpretation are required to be computable within the structure (while in the classical definition they should be first-order definable). The formulas defining the interpretation are *computable infinitary* Σ_1^c. A version with parameters of the effective intrepretability is introduced by Ershov [Ers85] — the Σ-definability over $\mathbb{HF}(\mathcal{B})$, the structure of hereditarily finite sets over $\mathcal{B}$. It uses the first-order logic over $\mathbb{HF}(\mathcal{B})$, and is studied in Russia over the last twenty years [EPS11, Puz09, MK08, Stu13, Kal09a]. Antonio Montalbán in [Mon, Mon12] shows that Σ-definability over $\mathbb{HF}(\mathcal{B})$ corresponds to effective interpretability in $\mathcal{B}$ with parameters.

In a number of familiar examples where $\mathcal{A} \leq_s \mathcal{B}$, the structure $\mathcal{A}$ is defined or interpreted in $\mathcal{B}$ using formulas of special kinds.

Example 5.1.8. The usual definition of the ring of integers $\mathbb{Z}$ involves an interpretation in the semi-ring of natural numbers $\mathbb{N}$. Let D be the set of ordered pairs (m, n) of natural numbers. We think of the pair (m, n) as representing the integer $m - n$. With this in mind, we can easily give finitary existential formulas that define ternary relations of addition and multiplication on D, and the complements of these relations, and a congruence relation $\sim$ on D, and the complement of this relation, such that $(D, +, \cdot)/_{\sim} \cong \mathbb{Z}$.

Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] considered a notion of effective interpretation of $\mathcal{A}$ in $\mathcal{B}$, a very general kind of interpretation, guaranteeing that $\mathcal{A} \leq_s \mathcal{B}$. The tuples in $\mathcal{B}$ that represent elements of $\mathcal{A}$ have no fixed arity. Normally, we consider formulas with a fixed tuple of variables. However, following [HTMMM17], we will consider relations $R \subseteq \mathcal{B}^{<\omega}$ in our interpretations, and we will say that such a relation R is defined in $\mathcal{B}$ by a generalized Σ_1^c formula (see Definition 2.5.9) that when there is a computable sequence of Σ_1^c formulas $\varphi_n(\overline{x}_n)$ defining $R \cap \mathcal{B}^n$. Our Σ_1^c definition of R is $\bigvee_n \varphi_n(\overline{x}_n)$. A relation R defined in this way is c.e. relative to $\mathcal{B}$.

In a given structure, a relation R is *computable* Δ_1^c*-definable over* $\overline{c}$ if R and the complementary relation $\neg R$ are both defined by computable Σ_1^c formulas, with parameters in $\overline{c}$.

As we know by a result from [AKMS89], [Chi90], Theorem 2.5.8, for a relation R and a structure $\mathcal{A}$, R is relatively intrinsically c.e. (or Σ^0_α) on $\mathcal{A}$ iff it is defined in $\mathcal{A}$ by a computable Σ^c_1 (or computable Σ^c_α) formula, with a finite tuple $\overline{c}$ of parameters in $\mathcal{A}$. Actually, as we mention before, as Montalbán proved in [Mon12], a relation $R \subset A^{<\omega}$ is relatively intrinsically c.e. (r.i.c.e.) on $\mathcal{A}$ if it is defined by a generalized computable Σ^c_1 formula with no parameters but with with infinitely many free variables.

Example 5.1.9. The dependence relation on tuples in a $\mathbb{Q}$-vector space is a familiar relation with no fixed arity. It is defined by a Σ^c_1 formula $\bigvee_n \varphi_n(\overline{x}_n)$ of the kind that we use for effective interpretations. We let $\varphi_n(\overline{x}_n) = \bigvee_\lambda \lambda(\overline{x}_n) = 0$, where λ ranges over the non-trivial rational linear combinations of $\overline{x}_n = (x_1, \dots, x_n)$.

Definition 5.1.10. A structure $\mathcal{A} = (A, R_i)$ is *effectively interpreted* in a structure $\mathcal{B}$ if there is a set $D \subseteq \mathcal{B}^{<\omega}$, Σ^c_1-definable over $\varnothing$, and there are relations $\sim$ and R^*_i on D, computable Σ^c_1-definable over $\varnothing$, such that $(D, R^*_i)/_\sim \cong \mathcal{A}$.

Above, we described Marker's Turing computable embedding of directed graphs in undirected graphs, and we saw there are uniform finitary existential formulas that in the output directed graph a set D and relations $\pm R^*$ such that (D, R^*) is isomorphic to the input undirected graph. Friedman and Stanley's original embedding of graphs in fields involved a uniform interpretation by means of Σ^c_3 formulas. A more recent embedding of graphs in fields, due to R. Miller, Poonen, Schoutens, and Shlapentokh [MPSS18], gives a uniform effective interpretation.

Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] defined a second notion with gives in equivalent definition.

Definition 5.1.11. [Computable functor][HTMMM17]

A *computable functor* from $\mathcal{B}$ to $\mathcal{A}$ is a pair of Turing operators, Φ and Ψ, with the following features:

(1) For each $\mathcal{C} \cong \mathcal{B}$, we have $\Phi(\mathcal{C}) \cong \mathcal{A}$,

(2) For any $\mathcal{B}_1, \mathcal{B}_2 \cong \mathcal{B}$ and any isomorphism f from $\mathcal{B}_1$ onto $\mathcal{B}_2$, $\Psi(\mathcal{B}_1, \mathcal{B}_2, f)$ is an isomorphism from $\Phi(\mathcal{B}_1)$ onto $\Phi(\mathcal{B}_2)$. The operator Ψ is required to satisfy some natural properties.

(a) If $\mathcal{B}_1 = \mathcal{B}_2 \cong \mathcal{B}$ and f is the identity function, then $\Psi(\mathcal{B}_1, \mathcal{B}_2, f)$ is the identity on $\Phi(\mathcal{B}_1)$.

(b) For $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \cong \mathcal{B}$, and isomorphisms f from $\mathcal{B}_1$ to $\mathcal{B}_2$ and g from $\mathcal{B}_2$ to $\mathcal{B}_3$, $\Psi(\mathcal{B}_1, \mathcal{B}_3, g \circ f) = \Psi(\mathcal{B}_2, \mathcal{B}_3, g) \circ \Psi(\mathcal{B}_1, \mathcal{B}_2, f)$.

The main result from [HTMMM17] gives the equivalence of the two notions.

Theorem 5.1.12. For structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$ iff there is a computable functor Φ, Ψ from $\mathcal{B}$ to $\mathcal{A}$.

Corollary 5.1.13. If $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$, then $\mathcal{A} \leq_s \mathcal{B}$.

Proof.

We get a Medvedev reduction by taking the first half Φ of the computable functor Φ, Ψ. □

Kalimullin [Kal12] showed that the converse of the corollary fails. We can have a Turing operator Φ taking copies of $\mathcal{B}$ to copies of $\mathcal{A}$ without having a Turing operator Ψ taking triples $(\mathcal{B}_1, \mathcal{B}_2, f)$ to g, where $\mathcal{B}_1, \mathcal{B}_2$ are copies of $\mathcal{B}$ and $\mathcal{B}_1 \cong_f \mathcal{B}_2$ and $\Phi(\mathcal{B}_1) \cong_g \Phi(\mathcal{B}_2)$.

In the proof of Theorem 5.1.12, it is important that the set D in the interpretation consist of tuples from $\mathcal{B}$ of arbitrary arity. The same is true in the proof of the following.

Proposition 5.1.14. If $\mathcal{A}$ is computable, then $\mathcal{A}$ is effectively interpreted in all structures $\mathcal{B}$.

Proof.

Let $D = \mathcal{B}^{<\omega}$. Let $\bar{b} \sim \bar{c}$ if $\bar{b}, \bar{c}$ are tuples of the same length. For simplicity, suppose $\mathcal{A} = (\mathbb{N}, R)$, where R is binary. If $\mathcal{A} \models R(m, n)$, then $R^*(\bar{b}, \bar{c})$ for all $\bar{b}$ of length m and $\bar{c}$ of length n. □

Mal'tsev's embedding Φ of fields in 2-step nilpotent groups involves interpreting F in $\Phi(F)$ using formulas with parameters. Recently, we show that there is a uniform computable functor from $\Phi(F)$ to F. Hence, there is a uniform effective interpretation of F in $\Phi(F)$ in which the formulas do not have parameters. We will prove this in the next section.

5.1.6 Interpretations by more general formulas

We may consider interpretations of $\mathcal{A}$ in $\mathcal{B}$, where D, $\pm\sim$, and $\pm R_i^*$ are defined in $\mathcal{B}$ by Σ_2^c formulas formulas, and we have $(D, (R_i^*)_{i\in\mathbb{N}})/_\sim \cong \mathcal{A}$.

Recall the properties of the *jump* of $\mathcal{A}$.

1. For a structure $\mathcal{A}$, the *jump* is a structure $\mathcal{A}'$ such that the relations defined in $\mathcal{A}$ by Σ_2^c formulas are just those defined in $\mathcal{A}'$ by Σ_1^c formulas.
2. For a structure $\mathcal{A}$, the jump structure $\mathcal{A}'$ is computed by $D(\mathcal{A})'$.
3. The relations defined in $\mathcal{A}^{(2)}$ by Σ_1^c formulas are just those defined in $\mathcal{A}$ by Σ_2^c formulas.

Harrison-Trainor, R. Miller, and Montalbán [HTMM18] proved the analogue of the result from [HTMMM17] in which the interpretations are defined by formulas of $L_{\omega_1\omega}$, and the functors are Borel. Again for an interpretation of $\mathcal{A}$ in $\mathcal{B}$, the set of tuples in $\mathcal{B}$ that represent elements of $\mathcal{A}$ may have arbitrary arity. If $R \subseteq \mathcal{B}^{<\omega}$, and we have a countable sequence of $L_{\omega_1\omega}$-formulas $\varphi_n(\overline{x}_n)$ defining $R \cap \mathcal{B}^n$, then we refer to $\bigvee_n \varphi_n(\overline{x}_n)$ as an $L_{\omega_1\omega}$ definition of R.

Theorem 5.1.15. A structure $\mathcal{A}$ is interpreted in $\mathcal{B}$ using $L_{\omega_1\omega}$-formulas iff there is a Borel functor (Φ, Ψ) from $\mathcal{B}$ to $\mathcal{A}$.

5.2 Interpreting graphs in linear orderings

The content of this subsection is from [KSV19].

As we have seen, any structure can be effectively interpreted in a graph. Linear orderings do not have so much interpreting power. To show this, we use the following result of Linda Jean Richter [Ric81].

Proposition 5.2.1 (Richter)**.** For a linear ordering L, the only sets computable in all copies of L are the computable sets.

Proposition 5.2.2. There is a graph G such that for all linear orderings L, $G \nleq_s L$.

Proof.

Let S be a non-computable set. Let G be a graph such that every copy computes S. We may take G to be a "daisy" graph, consisting of a center node with a "petal" of length $2n+3$ if $n \in S$ and $2n+4$ if $n \notin S$. Now, apply Proposition 5.2.1. ☐

The following result, from [Kni86], is a lifting of Proposition 5.2.1.

Proposition 5.2.3. For a linear ordering L, the only sets computable in all copies of L' (or in the jumps of all copies of L) are the Δ^0_2 sets.

This yields a lifting of Proposition 5.2.2.

Proposition 5.2.4. There is a graph G such that for all linear orderings L, $G \nleq_s L'$.

Proof.

Let S be a non-Δ^0_2 set. Let G be a graph such that every copy computes S. Then apply Proposition 5.2.3. ☐

The pattern above does not continue. The following is well-known (see Theorem 9.12 [AK00]).

Proposition 5.2.5. For any set S, there is a linear ordering L such that for all copies of L, the second jump computes S.

Proof. [Proof sketch]

For a set A, the ordering $\sigma(A \cup \{\omega\})$ (the "shuffle sum" of orderings of type n for $n \in A$ and of type ω) consists of densely many copies of each of these orderings. The degrees of copies of $\sigma(A \cup \{\omega\})$ are the degrees of sets X such that A is c.e. relative to $X^{(2)}$. Let $A = S \oplus S^c$, where S^c is the complement of S. Consider the linear ordering $L = \sigma(A \cup \{\omega\})$. Then we have a pair of finitary Σ_3 formulas saying that $n \in S$ iff L has a maximal discrete set of size $2n$ and $n \notin S$ iff L has a maximal discrete set of size $2n+1$. It follows that any copy of $L^{(2)}$ uniformly computes the set S.

☐

Using Proposition 5.2.5, we get the following.

Proposition 5.2.6. For any graph G, there is a linear ordering L such that $G \leq_s L^{(2)}$,

Proof.

Let S be the diagram of a specific copy of G and let L be as in Proposition 5.2.5. Then $G \leq_s L^{(2)}$. ☐

5.2.1 Turing computable embedding of graphs in linear orderings

The class of linear orderings, like the class of graphs, lies on top under Turing computable embeddings. We describe the Turing computable embedding L, given in [FS89], of directed graphs in linear orderings.

Friedman-Stanley embedding. First, let $(A_n)_{n\in\omega}$ be an effective partition of $\mathbb{Q}$ into disjoint dense sets. Let $(t_n)_{1\leq n<\omega}$ be a list of the atomic types in the language of directed graphs. We let t_1 be the type of $\varnothing$, we put the types for single elements next, then the types for distinct pairs, then the types for distinct triples, etc. For a graph G, the ordering $L(G)$ is a sub-ordering of $\mathbb{Q}^{<\omega}$, with the lexicographic ordering. The elements of $L(G)$ are the finite sequences $r_0q_1r_1\ldots r_{n-1}q_nr_nk \in \mathbb{Q}^{<\omega}$ such that

1. for $i < n$, $r_i \in A_0$, and $r_n \in A_1$,

2. there is a special tuple in G, of length n, satisfying the atomic type t_m, and k is a natural number less than m,

3. if $n \geq 1$ and the special tuple is $a_1, \ldots, a_n$, then for all i with $1 \leq i \leq n$, $q_i \in A_{a_i}$.

In talks, Knight has claimed, without any proof, that this embedding does not represent an interpretation. Our goal in the rest of the section is to prove the following theorem.

Theorem 5.2.7 (Main Theorem)**.** There do not exist $L_{\omega_1\omega}$-formulas that, for all graphs G, interpret G in $L(G)$.

We begin with some definitions and simple lemmas about $L(G)$.

Definition 5.2.8. Let $b = r_0q_1r_1\ldots r_{n-1}q_nr_nk \in L(G)$. We say that b *mentions* $\overline{a}$ if $\overline{a}$ is the special tuple in G of length n, such that for $1 \leq i \leq n$, $q_i \in A_{a_i}$.

Lemma 5.2.9. Suppose $b \in L(G)$ mentions $\overline{a}$. Then b lies in a maximal discrete interval of some finite size $m \geq 1$. The number m tells us the atomic type of $\overline{a}$; in particular, it tells us the length of $\overline{a}$.

Proof.

It is clear from the definition of $L(G)$ that if b mentions $\overline{a}$, where $\overline{a}$ satisfies the atomic type t_m on our list, then b lies in a maximal discrete set of size m. Knowing just that b lies in a maximal discrete set of size m, we know the atomic type, and this tells us the length of $\overline{a}$. □

The structure of the linear ordering $L(G)$ does not directly tell us the lengths of the elements b (as elements of $\mathbb{Q}^{<\omega}$). However, if b mentions $\overline{a}$ of length n, then b has length $2n+2$.

Lemma 5.2.10. If $b \in L(G)$ has length $2n+2$, then there is an infinite interval around b that consists entirely of elements of length at least $2n+2$.

Proof.

Suppose that $b = r_0 q_1 r_1 \ldots r_{n-1} q_n r_n k$. The elements d that extend the initial segment $r_0 q_1 r_1 \ldots r_{n-1} q_n$, of length $2n$, are closer to b than those that differ on one of the first $2n$ terms. These d all have length at least $2n+2$, and they form the interval we want. □

Lemma 5.2.11. Let $b, b' \in L(G)$, where $b < b'$, and let d be an element of $[b, b']$ of minimum length. If d mentions $\overline{c}$, then all elements of $[b, b']$ mention extensions of $\overline{c}$.

Proof.

Say that d has length $2k+2$. Then b and b' are both in an interval around d consisting of elements of length at least $2k+2$. Let σ be the initial segment of d of length $2k$. Then all elements of $[b, b']$ must extend σ. Thus, all of these mention extensions of $\overline{c}$. □

Let $\overline{b}$ be a tuple in $L(G)$. For each b_i in $\overline{b}$, let $\overline{a}_i$ be the tuple in G mentioned by b_i. The formulas true of $\overline{b}$ in $L(G)$ are determined by the formulas true in G of the various $\overline{a}_i$, together with the "shape" of $\overline{b}$.

Definition 5.2.12. For a tuple $\overline{b} = (b_1, \ldots, b_n)$ in $L(G)$, with $b_1 < b_2 < \ldots < b_n$, the *shape* encodes the following information:

1. the order type of $\overline{b}$—for simplicity, we suppose that $b_1 < b_2 < \ldots < b_n$,
2. the size of each interval (b_i, b_{i+1})—we note that the interval is infinite unless b_i, b_{i+1} belong to the same finite discrete set in $L(G)$, which means that they agree on all but the last term,

3. the location of each b_i in the finite discrete interval to which it belongs,

4. the length of each b_i,

5. for $i < n$, the number k_i such that $2k_i + 2$ is the length of a shortest element d in the interval $[b_i, b_{i+1}]$—d mentions a tuple $\bar{c}$ of length k_i, and all elements of $[b_i, b_{i+1}]$ mention tuples that extend $\bar{c}$.

Proposition 5.2.13. For each n-tuple $\bar{b}$, there exist Π^c_4, and Σ^c_4 formulas in the language of linear orderings saying, in $L(G)$ for any G, that the n-tuple $\bar{x}$ has the same shape as some fixed tuple $\bar{b}$.

Proof.

We note the following.

1. For any finite n, we have a finitary d-Σ_1 formula (difference of two Σ_1 formulas) saying of an interval that it has at least n elements and it does not have at least $n+1$ elements. Thus, there are finitary Σ_2 and Π_2 formulas saying that an interval (b_i, b_{i+1}) has size n.

2. We have a finitary Σ_3 formula saying that b_i sits a specific position in a maximal discrete set of size n.

3. Assuming that our list of the atomic types $(t_n)_{1\le n<\omega}$ is as described above, we have finitary Σ_3 formulas saying that b_i has length $2n+2$—we take a finite disjunction of formulas saying that b_i lies in a maximal discrete interval of size r, where t_r is the atomic type of a tuple of length n.

4. For each k, we have a finitary Π_3 formula saying that all $z \in [b_i, b_{i+1}]$ have length at least $2k+2$.

Taking an appropriate finite conjunction of the formulas described above, we obtain a Σ^c_4 definition of the set of tuples of a specific shape, and also a Π^c_4 definition. □

Remarks on elements of length 2: Suppose d has length 2. Then $\varnothing$ is the tuple mentioned by d and the atomic type of $\varnothing$ is t_1, so d has the form $r_0 0$, where $r_0 \in A_1$. Note that d is the only element of $L(G)$ that starts with r_0. If $b < d < b'$, then b has first term r and b' has first term r', where $r < r_0 < r'$. Since all A_i are dense in $\mathbb{Q}$, essentially everything happens in the intervals (b, d) and (d, b').

Lemma 5.2.14. Suppose $c < c^* < c'$ in $L(G)$, where c^* has length 2.

(1) For any $\overline{e}$ in (c, ∞), there is an automorphism of (c, ∞) taking $\overline{e}$ to some $\overline{e}'$ in the interval (c, c^*).

(2) For any $\overline{e}$ in $(-\infty, c')$, there is an automorphism of $(-\infty, c')$ taking $\overline{e}$ to some $\overline{e}'$ in the interval (c^*, c').

Proof.

We prove (1). Note that c^* has form $r0$, where $r \in A_1$. The first term of c is some $q < r$. Let c^{**} be an element of length 2 greater than all in $\overline{e}$, with first term p. There is a permutation of $\mathbb{Q}$, say f, such that

1. f preserves the ordering and membership in the A_i's (i.e., f is an automorphism of the structure $(\mathbb{Q}, <, (A_i)_{i\in\omega})$,
2. $f(q) = q$ and $f(p) = r$.

We define an automorphism g of (c, ∞), taking each element $x\sigma$ to $f(x)\sigma$—we are changing just the first term. The fact that f preserves the ordering and membership in A_i's is needed to be sure that g has domain and range (c, ∞). □

If $a < b$ in the ordering $L(G)$, we may say that a lies *to the left of* b, or that b lies *to the right of* a.

Lemma 5.2.15. Let $\overline{b}$ be a finite tuple in $L(G)$, and let c be an element of $L(G)$.

(1) There is an automorphism of $L(G)$ taking $\overline{b}$ to a tuple $\overline{b}'$ entirely to the right of c, with elements of length 2 in between.

(2) There is also an automorphism taking $\overline{b}$ to a tuple $\overline{b}''$ entirely to the left of c, with elements of length 2 in between.

Proof.

We give the proof for (1). Suppose that c begins with r. Suppose the first element of $\overline{b}$ begins with p. Let f be a permutation of $\mathbb{Q}$ that preserves the ordering and membership in the A_i's, and such that $f(p) > r$. We have an automorphism g of $L(G)$ such that $g(x\sigma) = f(x)\sigma$. By the choice of f it follows that g has domain and range all of $L(G)$. To see that there is an element of length 2 between c and the first element of $g(\overline{b})$, we note that there is an element of A_1 between r and $f(p)$. □

5.2.2 The relations $\sim^\gamma$

Below, we recall a family of equivalence relations, defined for pairs of tuples, from the same structure, or from two different structures.

Definition 5.2.16. Let $\mathcal{A}$ and $\mathcal{B}$ be structures for a fixed finite relational language. Let $\overline{a}$ and $\overline{b}$ be tuples of the same length, where $\overline{a}$ in in $\mathcal{A}$ and $\overline{b}$ is in $\mathcal{B}$.

(1) $(\mathcal{A},\overline{a}) \sim^0 (\mathcal{B},\overline{b})$ if the tuples $\overline{a}$ and $\overline{b}$ satisfy the same atomic formulas in their respective structures.

(2) For $\gamma > 0$, $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{B},\overline{b})$ if for all $\beta < \gamma$,

   (a) for all $\overline{c} \in \mathcal{A}$, there exists $\overline{d} \in \mathcal{B}$ such that $(\mathcal{A},\overline{a},\overline{c}) \sim^\beta (\mathcal{B},\overline{b},\overline{d})$,

   (b) for all $\overline{d} \in \mathcal{B}$, there exists $\overline{c} \in \mathcal{A}$ such that $(\mathcal{A},\overline{a},\overline{c}) \sim^\beta (\mathcal{B},\overline{b},\overline{d})$.

Note: We write $\mathcal{A} \sim^\gamma \mathcal{B}$ to indicate that $(\mathcal{A},\varnothing) \sim^\gamma (\mathcal{B},\varnothing)$.

Lemma 5.2.17. Let $\mathcal{A}$ be a computable structure for a finite relational language. For any $\gamma < \omega_1^{CK}$ and for any tuple $\overline{a}$ in $\mathcal{A}$, we can effectively find a $\Pi^c_{2\gamma}$-formula $\varphi^\gamma_{\overline{a}}(\overline{x})$ such that $\mathcal{A} \models \varphi^\gamma_{\overline{a}}(\overline{b})$ iff $\overline{a} \sim^\gamma \overline{b}$.

Proof.

We proceed by induction on γ. Let $\gamma = 0$. Then

$$\varphi^0_{\overline{a}}(\overline{x}) = \bigwedge_{\varphi(\overline{x})\in B} \varphi(\overline{x}),$$

where B is the set of atomic formulas and negations of atomic formulas true of $\overline{a}$ in $\mathcal{A}$. This formula is finitary quantifier-free. Suppose $\gamma > 0$, where we have the formulas $\varphi^\beta_{\overline{a}}$ for all $\beta < \gamma$ and all $\overline{a}$. Then

$$\varphi^\gamma_{\overline{a}}(\overline{x}) = \bigwedge_{\beta<\gamma}[\bigwedge_{\overline{c}}(\exists\overline{y})\varphi^\beta_{\overline{a},\overline{c}}(\overline{x},\overline{y}) \ \& \ \bigwedge_{\overline{y}}(\forall\overline{y})\bigvee_{\overline{c}}\varphi^\beta_{\overline{a},\overline{c}}(\overline{x},\overline{y})]$$

This formula is $\Pi^c_{2\gamma}$, as required. □

Lemma 5.2.18. Let L be a fixed finite relational language. For any computable ordinal γ, and any tuples of variables $\overline{x}$, $\overline{y}$, of the same length, we can effectively find a computable $\Sigma^c_{2\gamma}$-formula $\varphi^\gamma(\overline{x},\overline{y})$ such that for any L-structure $\mathcal{A}$, and any tuples $\overline{a}$ and $\overline{b}$ from $\mathcal{A}$, $\mathcal{A} \models \varphi^\gamma(\overline{a},\overline{b})$ iff $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{A},\overline{b})$.

Proof.

Suppose that $\overline{x}$ and $\overline{y}$ have length m. Let $\gamma = 0$ and let At be the computable set of all atomic formulas on the first m variables in the language L. Then

$$\varphi^0(\overline{x},\overline{y}) = \bigwedge_{\varphi \in At} (\varphi(\overline{x}) \leftrightarrow \varphi(\overline{y})),$$

which is finitary quantifier-free. Suppose we have determined the formulas $\varphi^\beta(\overline{x},\overline{y})$ for all $\beta < \gamma$ and all appropriate pairs of tuples of variables $\overline{x},\overline{y}$. Then

$$\varphi^\gamma(\overline{x},\overline{y}) = \bigwedge_{\beta<\gamma}[\bigwedge_{\overline{u},\overline{v}}(\forall\overline{u})(\exists\overline{v})\varphi^\beta(\overline{x},\overline{u},\overline{y},\overline{v}) \ \& \ \bigwedge_{\overline{v},\overline{u}}(\forall\overline{v})(\exists\overline{u})\varphi^\beta(\overline{x},\overline{u},\overline{y},\overline{v})],$$

which is a $\Pi^c_{2\gamma}$ formula. □

The next lemma is well-known, and the proof is straightforward.

Lemma 5.2.19. Let $\mathcal{A}$ and $\mathcal{B}$ be structures for the same countable language, and let $\overline{a}$ and $\overline{b}$ be tuples of the same length, in $\mathcal{A}$ and $\mathcal{B}$,respectively. Then for any countable ordinal γ, if $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{B},\overline{b})$, then the Σ^c_γ formulas true of $\overline{a}$ in $\mathcal{A}$ are the same as the those true of $\overline{b}$ in $\mathcal{B}$.

5.2.3 $\sim^\gamma$-equivalence in linear orderings

In a linear orderings, the $\sim^\gamma$-classes of a tuple $\overline{a}$ are determined by the $\sim^\gamma$-classes of the intervals with endpoints in $\overline{a}$. Let $\mathcal{A}$ and $\mathcal{B}$ be linear orderings. Let $\overline{a} = a_1 < \ldots < a_n$ be a tuple in $\mathcal{A}$, and let $\overline{b} = b_1 < \ldots < b_n$ be a tuple in $\mathcal{B}$. Let $I_0,\ldots,I_n$ and $J_0,\ldots,J_n$ be the intervals in $\mathcal{A}$ and $\mathcal{B}$ determined by $\overline{a}$ and $\overline{b}$; i.e., I_0 is the interval $(-\infty,a_1)$ in $\mathcal{A}$, J_0 is the interval $(-\infty,b_1)$ in $\mathcal{B}$, for $i < n$, I_i is the interval (a_i,a_{i+1}) in $\mathcal{A}$, J_i is the interval (b_i,b_{i+1}) in $\mathcal{B}$, I_n is the interval (a_n,∞) in $\mathcal{A}$, and J_n is the interval (b_n,∞) in $\mathcal{B}$. The next lemma is well-known, and the proof is straightforward.

Lemma 5.2.20. $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{B},\overline{b})$ iff for $i \le n$, $I_i \sim^\gamma J_i$.

5.2.4 More on the orderings $L(G)$

We return to the orderings of form $L(G)$. In the next subsection, we will prove that there do not exist $L_{\omega_1\omega}$ formulas that, for all G, interpret G in $L(G)$. Roughly speaking, the outline is as follows. We assume that there are such formulas. The formulas are Σ_α, for some countable ordinal α. Moreover, they are X-computable Σ_α for some X such that $\alpha < \omega_1^X$. Taking G to be the ordering ω_1^X, we will produce tuples $\overline{b}, \overline{c}, \overline{b}'$ in $L(G)$ representing elements a, e, a' of G such that $\overline{b}, \overline{c} \sim^\gamma \overline{c}, \overline{b}'$, although in G, we have $a < e$ and $a' < e$. This is a contradiction. The current subsection gives several lemmas about the relations $\sim^\gamma$ on tuples in $L(G)$, and about automorphisms of $L(G)$. These lemmas are what we need to produce the tuples $\overline{b}, \overline{c}, \overline{b}'$.

To start off, we note that if $a_1, a_2 \sim^1 b_1, b_2$, then the sizes of the intervals (a_1, a_2) and (b_1, b_2) match. Moreover, if $a \sim^2 b$, then a and b belong to maximal discrete intervals of the same size.

Lemma 5.2.21. Let $I = (b, b')$, where $b < b'$, and let $J = (c, c')$, where $c < c'$. Suppose $b \sim^\gamma c$ and $b' \sim^\gamma c'$, where some $b^* \in I$ and some $c^* \in J$ each have length 2. Then $I \sim^\gamma J$.

Proof.

Suppose $\beta < \gamma$. Take $\overline{d}$ in I. We want $\overline{e}$ in J such that $(I, \overline{d}) \sim^\beta (J, \overline{e})$. We consider the cases $\beta = 0$, $\beta = 1$, and $\beta \geq 2$.

Case 1: Suppose $\beta = 0$. The fact that J contains an element of length 2 implies that it is an infinite interval. We choose $\overline{e}$ in this interval ordered in the same way as $\overline{d}$.

Case 2: Suppose $\beta = 1$. The tuple $\overline{d}$ partitions the interval $I = (b, b')$ into sub-intervals $I_0, \ldots, I_m$. We need $\overline{e}$ partitioning J into sub-intervals $J_0, \ldots, J_m$ of the same sizes. The first few intervals I_i may be finite. Since $b \sim^2 c$, we can match these intervals. Similarly, we can match the last few intervals, if these are finite. For simplicity, we suppose that the intervals I_0 and I_m are both infinite. The tuple $\overline{d}$ is automorphic to a tuple $\overline{d}'$ lying entirely to the right of c, with first element infinitely far from c. Let d' be infinitely far to the right of the last term of $\overline{d}'$. By Lemma 5.2.14, there is an automorphism of the interval (c, ∞) taking $\overline{d}', d'$ to some $\overline{e}, e'$ in the interval (c, c'). We let the J_i's be the sub-intervals of J determined by $\overline{e}$. These have the desired sizes.

Case 3: Suppose $\beta \geq 2$. We may suppose that $\overline{d} = \overline{d}_1, b^*, \overline{d}_2$. The intervals (b, ∞) and (c, ∞) are $\sim^\gamma$-equivalent. Therefore, we have $\overline{e}_1, c^{**}$ in (c, ∞) $\sim^\beta$-equivalent to $\overline{d}_1, b^*$ in (b, ∞). Since $\beta \geq 2$, we have that c^{**} has length 2. Let p be the first term of c, let r be the first term of c^*, and let q be the first term of c^{**}. Let f be a permutation of $\mathbb{Q}$, preserving the order and the A_i's, fixing p and taking q to r. We have an automorphism g of (c,∞) (or of $L(G)$) that takes $x\sigma$ to $f(x)\sigma$. Let $\overline{e}'_1$ be $g(\overline{e}_1)$. The sub-intervals of I (or of (b,∞)) determined by $\overline{d}_1, b^*$ are $\sim^\beta$ equivalent to the sub-intervals of (c,∞) determined by $\overline{e}_1, c^{**}$. These are isomorphic to the sub-intervals determined by $\overline{e}'_1, g(c^{**})$. Thus, the sub-intervals of (b,∞) determined by $\overline{d}_1, b^*$ are $\sim^\beta$-equivalent to the sub-intervals of (c,∞) determined by $\overline{e}'_1, c^*$.

In a similar way, we get $\overline{e}'_2$ such that the sub-intervals of (c^*,∞) determined by $c^*, \overline{e}'_2$ are $\sim^\beta$-equivalent to those determined by $b^*, \overline{d}_2$ in (b^*,∞). We let $\overline{e}$ be $\overline{e}'_1, \overline{e}'_2$. All together, the sub-intervals of (b,b') determined by $\overline{d}$ are $\sim^\beta$-equivalent to the corresponding sub-intervals of (c,c') determined by $\overline{e}$. □

Lemma 5.2.22. Let $\overline{b}_1, \overline{b}_2$, $\overline{c}_1, \overline{c}_2$ be increasing sequences in $L(G)$, where $\overline{b}_1 \sim^\gamma \overline{c}_1$ and $\overline{b}_2 \sim^\gamma \overline{c}_2$. Suppose further that there is an element of length 2 between the last element of $\overline{b}_1$ and the first element of $\overline{b}_2$, and there is an element of length 2 between the last element of $\overline{c}_1$ and the first element of $\overline{c}_2$. Then $\overline{b}_1, \overline{b}_2 \sim^\gamma \overline{c}_1, \overline{c}_2$.

Proof.

Say that $\overline{b}_1 = (b_1,\ldots,b_k)$, $\overline{b}_2 = (b_{k+1},\ldots,b_n)$, $\overline{c}_1 = (c_1,\ldots,c_k)$, and $\overline{c}_2 = (c_{k+1},\ldots,c_n)$. Let I_i be the intervals determined by $\overline{b}_1, \overline{b}_2$, and let J_i be the intervals determined by $\overline{c}_1, \overline{c}_2$, for $i \le n$. The fact that $\overline{b}_1 \sim^\gamma \overline{c}_1$ implies that $I_i \sim^\gamma J_i$ for $i < k$. The fact that $\overline{b}_2 \sim^\gamma \overline{c}_2$ implies that $I_i \sim^\gamma J_i$ for $k < i \le n$. It remains to show that $I_k \sim^\gamma J_k$. We have $b_k \sim^\gamma c_k$ and $b_{k+1} \sim^\gamma c_{k+1}$. We have elements of length 2 in each of the intervals I_k and J_k. Applying the previous lemma, we get the fact that $I_k \sim^\gamma J_k$. Therefore, $\overline{b}_1, \overline{b}_2 \sim^\gamma \overline{c}_1, \overline{c}_2$. □

Lemma 5.2.23. Suppose $\overline{b}$, $\overline{b}'$ are tuples in $L(G)$ of the same shape. Let $\overline{a}$, $\overline{a}'$ be the full tuples from G mentioned by the b_i's, or the b'_i's. If $\overline{a} \sim^\gamma \overline{a}'$, then $\overline{b} \sim^\gamma \overline{b}'$.

Proof.

We proceed by induction on γ. For $\gamma = 0$, the statement is trivially true. Supposing that the statement holds for $\beta < \gamma$, we show it for γ. Suppose $\overline{a} \sim^\gamma \overline{a}'$. We will have $\overline{b} \sim^\gamma \overline{b}'$ provided that for all $\beta < \gamma$,

(1) for any $\overline{d}$, there is some $\overline{d}'$ such that $\overline{b}, \overline{d} \sim^\beta \overline{b}', \overline{d}'$, and

(2) for any $\overline{d}'$, there is some $\overline{d}$ such that $\overline{b}, \overline{d} \sim^\beta \overline{b}', \overline{d}'$.

By symmetry, it is enough to prove (1). Say that $\overline{c}$ is the tuple of elements of G mentioned in the d_i's and not in $\overline{a}$. Since $\overline{a} \sim^\gamma \overline{a}'$ in G, there is a tuple $\overline{c}'$ such that $\overline{a}, \overline{c} \sim^\beta \overline{a}', \overline{c}'$. In $L(G)$, we choose $\overline{d}'$, so that the ordering and shape of $\overline{b}', \overline{d}'$ matches that of $\overline{b}, \overline{d}$, and for each d_i', the tuple $\overline{a}', \overline{c}'$ mentioned in d_i' corresponds to the one from $\overline{a}, \overline{c}$ mentioned in d_i. Using the fact that $\overline{b}'$ and $\overline{b}$ have the same shape, we can see that such $\overline{d}'$ exist. By the induction hypothesis, we have $\overline{b}, \overline{d} \sim^\beta \overline{b}'\overline{d}'$. □

Definition 5.2.24. We say that $\mathcal{A}$ is a computable infinitary substructure of $\mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$ and for all computable infinitary formulas $\varphi(\overline{x})$ and all $\overline{a}$ in $\mathcal{A}$, $\mathcal{B} \models \varphi(\overline{a})$ iff $\mathcal{A} \models \varphi(\overline{a})$. (The definition is the same as elementary substructure except that the formulas are not elementary (finitary) first order.)

Lemma 5.2.25. Let G_1 and G_2 be directed graphs such that G_1 is a computable infinitary substructure of G_2. Suppose also that G_2 is computable, so $L(G_2)$ is computable. Then $L(G_1)$ is a computable infinitary substructure of $L(G_2)$.

Proof.

Note that $L(G_1)$ is a substructure of $L(G_2)$. The Tarski-Vaught test was originally stated for elementary substructure, but it also works for computable infinitary substructure. To show that $L(G_1)$ is a computable infinitary substructure of $L(G_2)$, it is enough to show that for any computable infinitary formula $\psi(\overline{x}, u)$, if $L(G_2) \models \psi(\overline{b}, d)$, where $\overline{b}$ is in $L(G_1)$, then $L(G_2) \models \psi(\overline{b}, d')$ for some $d' \in L(G_1)$.

Say that ψ is a Π^c_α formula. Suppose $\overline{b}$ mentions $\overline{a}$ from G_1. The tuple from G_1 mentioned by d may include some elements from $\overline{a}$, plus some further elements $\overline{c}$. By Lemma 5.2.17, we have a computable infinitary formula $\varphi^\alpha_{\overline{a},\overline{c}}(\overline{x}, \overline{y})$ defining in G_2 the $\sim^\alpha$-class of $\overline{a}, \overline{c}$. By the Tarski-Vaught test, there is some $\overline{c}'$ in G_1 such that $G_2 \models \varphi^\alpha_{\overline{a},\overline{c}}(\overline{a}, \overline{c}')$. Then in G_2, $\overline{a}, \overline{c} \sim^\alpha \overline{a}, \overline{c}'$. Say that $\overline{u}$ is the tuple in G_2 mentioned by d. Each u_i is in $\overline{a}, \overline{c}$. Let $\overline{v}$ be the tuple in G_1 such that if $u_i \in \overline{a}$, then $v_i = u_i$ and if $u_i \in \overline{c}$, then v_i is the element of $\overline{c}'$ corresponding to $\overline{c}$. Thus, $\overline{u} \sim^\alpha \overline{v}$. We choose d', mentioning the tuple $\overline{v}$, such that $\overline{b}, d$ and $\overline{b}, d'$ have the same ordering and the same shape. Then by Lemma 5.2.23, $\overline{b}, d \sim^\alpha \overline{b}, d'$. By Lemma 5.2.19, we conclude that $L(G_2) \models \psi(\overline{b}, d')$, as required. □

5.2.5 Proof of Theorem 5.2.7

Theorem 5.2.7 says that there are no $L_{\omega_1\omega}$-formulas that, for all directed graphs G, define an interpretation of G in $L(G)$. We introduce the ideas of the proof in a warm-up result. Among the directed graphs are the linear orderings. The Harrison ordering H [Har68] has order type $\omega_1^{CK}(1+\eta)$. While ω_1^{CK} has no computable copy, H does have a computable copy. It is well known that H and ω_1^{CK} satisfy the same computable infinitary sentences. In fact, they satisfy the same Π_α sentences of $L_{\omega_1\omega}$ for all computable ordinals α.

Let I be the initial segment of H of order type ω_1^{CK}. Thinking of H as a directed graph, we can form the linear orderings $L(H)$ and $L(I)$. By Proposition 5.1.14, just because H has a computable copy, it is effectively interpreted in every structure $\mathcal{B}$. Our warm-up result will say that there are no computable infinitary formulas that define an interpretation of H in $L(H)$ and also define an interpretation of I in $L(I)$.

Proposition 5.2.26. $L(I)$ is a computable infinitary substructure of $L(H)$.

Proof.

Since I and H satisfy the same computable infinitary sentences and every element of I is defined by a computable infinitary formula, it follows that I is a computable infinitary substructure of H. We apply Lemma 5.2.25 to conclude that $L(I)$ is a computable infinitary substructure of H. □

Proposition 5.2.27 (Warm-up)**.** There do not exist computable infinitary formulas that define an interpretation of H in $L(H)$ and also define an interpretation of I in $L(I)$.

Proof.

Suppose there are computable infinitary formulas that define an interpretation of H in $L(H)$, and also define an interpretation of I in $L(I)$. Say D, $\sim$, and $\circledless$ are the sets of tuples defined by these formulas in $L(H)$. We note that all elements of I are represented by tuples from D that are in $L(I)$, and all tuples from D that are in $L(I)$ represent elements of I. We can translate computable infinitary formulas describing H and its elements into computable infinitary formulas about tuples in $L(H)$, referring to the formulas that define D, $\sim$, and $\circledless$.

For each computable ordinal α, we have a formula $\varphi_\alpha(x)$ saying of an element x in H that $pred(x) = \{y : y < x\}$ has order type α. Let $\psi_\alpha(\overline{x})$ be the translation formula saying of a tuple $\overline{x}$ that it is in D and the set of predecessors of the equivalence class of $\overline{x}$ has order type α. For each computable ordinal α, there is a tuple in D satisfying $\psi_\alpha(\overline{x})$ (for an appropriate $\overline{x}$). Since $L(I)$ is a computable infinitary substructure of $L(H)$, some tuple from D in $L(I)$ also satisfies $\psi_\alpha(\overline{x})$. Moreover, each tuple from D in $L(I)$ satisfies one of the formulas ψ_α. Recall that the ordering H is computable, and so is $L(H)$. We define equivalence relations $\equiv^\gamma$ on D.

Definition 5.2.28. For tuples $\overline{a}$ and $\overline{b}$ in D, let $\overline{a} \equiv^\gamma \overline{b}$ iff

1. $\overline{a}$ and $\overline{b}$ have the same shape and
2. $\overline{a} \sim^\gamma \overline{b}$.

Fact: For each computable ordinal γ and each $\overline{a}$ in D, the $\equiv^\gamma$-class of $\overline{a}$ is defined by a computable infinitary formula.

We need one more lemma.

Lemma 5.2.29. For each computable ordinal γ, there is a $\equiv^\gamma$-class C such that there are arbitrarily large computable ordinals α for which some $\overline{b}$ in C satisfies ψ_α.

Proof.

In $L(H)$, we have a tuple $\overline{b}$ in D not satisfying any of the formulas ψ_α for computable ordinals α. Let C be the $\equiv^\gamma$-class of $\overline{b}$. Since $L(I)$ is a computable infinitary substructure of $L(H)$, and C is defined by a computable infinitary formula, we must have tuples of $L(I)$ belonging to C and satisfying ψ_α for arbitrarily large computable ordinals α. □

Suppose that the formulas defining D, $\textcircled{<}$, and $\sim$ are all Σ^c_γ. Since D may have no fixed arity, we mean that there is a computable sequence of Σ^c_γ formulas defining the sets of n-tuples in D, and similarly for $\textcircled{<}$ and $\sim$. By Lemma 5.2.29, there is a set $C \subseteq D$ in which all tuples have the same shape and are in the same $\sim^\gamma$-class—in particular, the tuples in C all have the same arity. We choose tuples $\overline{b}$ and $\overline{c}$ in $L(I)$, both belonging to C, such that $\overline{b}$ satisfies ψ_α and $\overline{c}$ satisfies ψ_β, where $\alpha < \beta$.

By Lemma 5.2.15, we may suppose that all elements of the tuple $\overline{b}$ lie to the left of the $<$-first element of $\overline{c}$, and the interval between the $<$-greatest element of $\overline{b}$ and the $<$-first element of $\overline{c}$ contains an element of length 2. Also, by the same lemma, we have a tuple $\overline{b}'$, automorphic to $\overline{b}$, such that all elements of $\overline{b}'$ lie to the right of the <-greatest element of $\overline{c}$, and the interval between the <-greatest element of $\overline{c}$ and the <-first element of $\overline{b}'$ contains an element of length 2. Since $\overline{b}$ satisfies ψ_α and $\overline{c}$ satisfies ψ_β, we should have $L(I) \models \overline{b} \textcircled{<} \overline{c}$. Since $\overline{b}'$ is automorphic to $\overline{b}$, it should also satisfy ψ_α, so we should have $L(I) \models \overline{b}' \textcircled{<} \overline{c}$. Applying Lemma 5.2.22, we get the fact that $\overline{b}, \overline{c} \sim^\gamma \overline{c}, \overline{b}'$. Therefore, since $L(I) \models \overline{b} \textcircled{<} \overline{c}$, and $\textcircled{<}$ is defined by a Σ^c_γ-formula, we have $L(I) \models \overline{c} \textcircled{<} \overline{b}'$. This is the contradiction that we were expecting when we set out to prove Proposition 5.2.27. □

We have proved Proposition 5.2.27, saying that there do not exist computable infinitary formulas that define an interpretation both for the Harrison ordering H in $L(H)$ and for the well-ordered initial segment I in $L(I)$. We assumed that there were computable infinitary formulas, say Σ^c_γ, defining both interpretations, and we arrived at a contradiction. We used H and $L(H)$ to arrive at a sequence of tuples $\overline{b}_\alpha$ in $L(I)$, representing arbitrarily large elements of I, and all having the same shape and satisfying the same computable Σ^c_γ formulas. We then used automorphisms of $L(I)$ to show that our proposed interpretation failed. The next result says that, in fact, there do not exist computable infinitary formulas that define an interpretation for I in $L(I)$. Of course, I is isomorphic to ω_1^{CK}.

Proposition 5.2.30. There is no interpretation of ω_1^{CK} in $L(\omega_1^{CK})$ defined by computable infinitary formulas.

Proof. Suppose we have an interpretation of ω_1^{CK} in $L(\omega_1^{CK})$, defined by computable infinitary formulas. Say that the formulas that define the appropriate D, $\textcircled{<}$, and $\sim$ are Σ^c_γ. Our assumption gives the fact that for a Harrison ordering with well-ordered initial segment I, these formulas interpret I in $L(I)$. However, the assumption does not say that they also interpret H in $L(H)$. Thus, we are not in a position to use the important Lemma 5.2.29.

The following lemma is simple enough that we omit the proof.

Lemma 5.2.31. Let $\mathcal{A}$ be a computable structure. If $\mathcal{B}$ satisfies the computable infinitary sentences true in $\mathcal{A}$, then the formulas $\varphi^\gamma_{\overline{d}}$ that define the $\sim^\gamma$-equivalence classes of all tuples in $\mathcal{A}$ also define the $\sim^\gamma$-equivalence classes of all tuples in $\mathcal{B}$. Moreover, if $\mathcal{B} \models \varphi^\gamma_{\overline{d}}(\overline{b})$, then the Σ^c_γ-formulas true of $\overline{b}$ in $\mathcal{B}$ are the same as those true of $\overline{d}$ in $\mathcal{A}$.

The next lemma gives the conclusion of Lemma 5.2.29. The proof involves locating ω_1^{CK} inside a larger ordering similar to the Harrison ordering.

Lemma 5.2.32. In $L(\omega_1^{CK})$, there are tuples $\overline{d}_\alpha$, corresponding to arbitrarily large computable ordinals α, such that all $\overline{d}_\alpha$ are in D, all have the same length and shape, all are $\sim^\gamma$-equivalent, and $\overline{d}_\alpha$ satisfies ψ_α.

Proof. [Proof of lemma]

We use Barwise-Kreisel Compactness. Let Γ be a Π^1_1 set of computable infinitary sentences describing a structure

$$\mathcal{U} = (U_1 \cup U_2, U_1, <_1, U_2, <_2, F, c)$$

such that

1. U_1 and U_2 are disjoint sets,
2. $(U_1, <_1)$ is a linear ordering that satisfies the computable infinitary sentences true in ω_1^{CK} and H—since H is computable, this is Π^1_1,
3. $(U_2, <_2)$ satisfies the computable infinitary sentences true in $L(\omega_1^{CK})$—this is Π^1_1 since $L(H)$ is computable and $L(I)$ is a computable infinitary substructure of $L(H)$,
4. F is a function from D^{U_2} to U_1 that induces an isomorphism between $(D^{U_2}, \circledless)/_{\sim^{U^2}}$ and $(U_1, <_1)$,
5. c is a constant in U_1 such that $c >_1 \alpha$ for all computable ordinals α; i.e., there is a proper initial segment of $<_1$-$pred(c)$ of type α.

Every Δ^1_1 subset of Γ is satisfied by taking copies of ω_1^{CK}, $L(\omega_1^{CK})$, with an appropriate function F, and letting c be a sufficiently large computable ordinal. Therefore, the whole set Γ has a model. Let $\overline{b}$ be an element of D^{U_2} such that $F(\overline{b}) = c$. Let C be the set of tuples of U_2 having the shape of $\overline{b}$ and $\sim^\gamma$-equivalent to $\overline{b}$. Since $(U_2, <_2)$ satisfies the same computable infinitary sentences true in the computable structure $L(H)$, by the lemma above, the $\sim^\gamma$-equivalence class of $\overline{b}$ is defined in $(U_2, <_2)$ by a computable infinitary formula. For each computable ordinal α, we have a computable infinitary sentence χ_α saying that some tuple in C does not satisfy ψ_β for any $\beta < \alpha$. The sentence χ_α is true in our model of Γ, witnessed by $\overline{b}$ such that $F(\overline{b}) = c$. Therefore, the sentence χ_α is true also in $L(\omega_1^{CK})$, witnessed by some $\overline{b}'$. Since our formulas define an interpretation of ω_1^{CK} in $L(\omega_1^{CK})$, the witness $\overline{b}'$ for χ_α in $L(\omega_1^{CK})$ must satisfy ψ_γ for some $\gamma \geq \alpha$. □

Now, we can proceed as in the proof of Proposition 5.2.27. We are working in $L(\omega_1^{CK})$. We choose $\overline{b}, \overline{c}$, from the sequence of $\overline{d}_\alpha$'s in the lemma, such that $\overline{b} \sim^\gamma \overline{c}$, where $\overline{b}$ satisfies ψ_α and $\overline{c}$ satisfies ψ_β, for $\alpha < \beta$. By Lemma 5.2.15, we may suppose that the elements of $\overline{b}$ all lie to the left of the <-first element of $\overline{c}$, and the interval between the <-greatest element of $\overline{b}$ and the <-first element of $\overline{c}$ contains an element of length 2. Since $\alpha < \beta$, we should have $L(\omega_1^{CK}) \models \overline{b} \circledless \overline{c}$. We can take $\overline{b}'$ automorphic to $\overline{b}$ such that all elements of $\overline{b}'$ lie to the right of the <-greatest element of $\overline{c}$, and the interval between the <-greatest element of $\overline{c}$ and the <-first element of $\overline{b}'$ contains an element of length 2. Clearly, $L(\omega_1^{CK}) \models \overline{b}' \circledless \overline{c}$ since $\overline{b}'$ satisfies $\psi_\alpha(\overline{x})$. Applying Lemma 5.2.22 we get the fact that $\overline{b}, \overline{c} \sim^\gamma \overline{c}, \overline{b}'$. It follows that $L(\omega_1^{CK}) \models \overline{c} \circledless \overline{b}'$, which is a contradiction. □

We are ready to complete the proof of Theorem 5.2.7, saying that there is no tuple of $L_{\omega_1\omega}$-formulas that, for all directed graphs G, interprets G in $L(G)$.

Proof. [Proof of Theorem 5.2.7]

Suppose that we have such formulas. For some X, the formulas are X-computable infinitary. Let G be a linear ordering of type ω_1^X. Relativizing Proposition 5.2.30, we have the fact that G is not interpreted in $L(G)$ by any X-computable formulas. □

The Friedman-Stanley embedding represents a uniform effective encoding of directed graphs in linear orderings. We have seen that there is no uniform interpretation of the input graph in the output linear ordering.

Conjecture 1. Let Φ be a Turing computable embedding of directed graphs in linear orderings. There do not exist $L_{\omega_1\omega}$ formulas that, for all directed graphs G, define an interpretation of G in $\Phi(G)$.

5.3 Interpreting a field into the Heisenberg group

The Heisenberg group of a field F is the upper-triangular subgroup of $GL_3(F)$ in which all matrices have 1's along the diagonal and 0's below it. Maltsev showed that there are existential formulas with parameters, which, for every field F, define F in its Heisenberg group $H(F)$. In this section we will show that there are existential formulas without parameters, which, for every field F, interpret F in $H(F)$. Observing what is used to obtain this result, we will then formulate a general result on removing parameters from an interpretation. The results from this section are joint with Alvir, Calvert, Goodman, Harizanov, Knight, Morozov, Miller, and Weisshaar, and published in [ACG+20].

Let remind Definition 5.1.4 of "Turing computable embedding," [CCKM04], based on the earlier notion of "Borel embedding" [FS89] (Definition 5.1.1). Recall that the classes of structures have a fixed language, and are closed under isomorphism.

Definition 5.3.1. For classes K and K', where $K \leq_{tc} K'$ via Θ, we say that the structures in K are *uniformly Medvedev reducible* to their Θ-images in K', $\mathcal{A} \in K$ is *uniformly Medvedev reducible to* $\Theta(\mathcal{A})$ if there is a single Turing operator Φ such that for all $\mathcal{A} \in K$, $\mathcal{A} \leq_s \Theta(\mathcal{A})$ via Φ.

Here is a uniform definition of the effective interpretation (see Definition 5.1.10), and a uniform definition of computable functor (see Definition 5.1.11.

Definition 5.3.2. Suppose $K \leq_{tc} K'$ via Θ.

(1) We say that the structures in K are *uniformly effectively interpreted* in their Θ-images if there is a fixed collection of generalized computable Σ^c_1 formulas (without parameters) (see Definition 2.5.9) such that, for all $\mathcal{A} \in K$, define an interpretation of $\mathcal{A}$ in $\Theta(\mathcal{A})$.

(2) We say that Φ and Ψ form a *uniform computable functor* from the structures $\Theta(\mathcal{A})$ to $\mathcal{A}$ if these Turing operators serve for all $\mathcal{A} \in K$.

There is a uniform version of Theorem 5.1.12.

Theorem 5.3.3. For classes K, K' with $K \leq_{tc} K'$ via Θ, the following are equivalent:

1. there are computable Σ_1^c formulas (without parameters) which, for all $\mathcal{A} \in K$, effectively interpret $\mathcal{A}$ in $\Theta(\mathcal{A})$,

2. there are uniform Turing operators Φ, Ψ that, for all $\mathcal{A} \in K$, form a computable functor from $\Theta(\mathcal{A})$ to $\mathcal{A}$.

Maltsev defined a Turing computable embedding of fields in 2-step nilpotent groups. The embedding takes each field F to its *Heisenberg group* $H(F)$. To show that the embedding preserves isomorphism, Maltsev gave uniform existential formulas defining a copy of F in $H(F)$. The definitions involved a pair of parameters, whose orbit is defined by an existential (in fact, quantifier-free) formula. In Section 5.3.1, we recall Maltsev's definitions. In Section 5.3.2, we describe a uniform computable functor that, for all F, takes copies of $H(F)$, with their isomorphisms, to copies of F, with corresponding isomorphisms. By Theorem 5.3.3, it follows that there is a uniform effective interpretation of F in $H(F)$ with no parameters. In Section 5.3.3, we give explicit finitary existential formulas that define such an interpretation. In Section 5.3.4, we note that although F is effectively interpretable in $H(F)$ and $H(F)$ is effectively interpretable in F, we do not, in general, have effective bi-interpretability. In Section 5.3.5, we generalize what we did in passing from Maltsev's definition, with parameters, to the uniform effective interpretation, with no parameters.

5.3.1 Defining F in $H(F)$

The content of the subsections 5.3.1, 5.3.2, 5.3.3, 5.3.4, and 5.3.5 are from [ACG+20].

We recall first Maltsev's embedding of fields in 2-step nilpotent groups, and his formulas that define a copy of the field in the group.Recall that for a field F, the Heisenberg group $H(F)$ is the set of matrices of the form

$$h(a,b,c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with entries in F. Note that $h(0,0,0)$ is the identity matrix. We are interested in non-commuting pairs in $H(F)$. One such pair is $(h(1,0,0), h(0,1,0))$. For

$u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$, let

$$\Delta_{(u,v)} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}.$$

For a group G, we write $Z(G)$ for the center. For group elements x, y, the *commutator* is $[x, y] = x^{-1}y^{-1}xy$. The following technical lemma provides much of the information we need to show that F is defined, with parameters, in $H(F)$.

Lemma 5.3.4.

1. (a) For u and v, the commutator, $[u, v]$, is $h(0, 0, \Delta_{(u,v)})$, and
   (b) $[u, v] = 1$ iff $\Delta_{(u,v)} = 0$.
2. Let $u = h(u_1, u_2, u_3)$, and let $v = h(v_1, v_2, v_3)$. If $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $u \in Z(H(F))$. If $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $[u, v] = 1$ iff there exists α such that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.
3. $Z(H(F))$ consists of the elements of the form $h(0, 0, c)$.
4. If $[u, v] \neq 1$, then $x \in Z(H(F))$ iff $[x, u] = [x, v] = 1$.

Proof.

For Part 1, (a) is proved by direct computation, and (b) follows from (a). Parts 2 and 3 are easy consequences of Part 1. We prove Part 4. Suppose $[u, v] \neq 1$. If $x \in Z(H(F))$, then it commutes with both u and v. We must show that if x commutes with both u and v, then $x \in Z(H(F))$. Let $u = h(u_1, u_2, u_3)$, $v = h(v_1, v_2, v_3)$, and $x = h(x_1, x_2, x_3)$. By Part 2, since $[x, u] = 1$, there exists α such that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Similarly, since $[x, v] = 1$, there exists β such that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Since the vectors $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, are linearly independent, this implies that $\alpha = \beta = 0$. It follows that $x_1 = x_2 = 0$, so $x \in Z(H)$. □

Corollary 5.3.5. If $x \in H(F)$ is fixed by all automorphisms of $H(F)$, then $x = 1$.

Proof. Write $x = h(a, b, c)$. Lemma 5.3.4(3) shows $a = b = 0$, since all conjugations fix x. But the automorphism of $H(F)$ mapping $h(x, y, z)$ to $h(y, x, xy - z)$, which interchanges $h(1, 0, 0)$ with $h(0, 1, 0)$, maps $h(0, 0, c)$ to $h(0, 0, -c)$, hence shows that $c = 0$ as well. □

The next lemma tells us how, for any non-commuting pair u, v in the group $(H(F), *)$, we can define operations $+$ and $\cdot$, and an isomorphism f from F to $(Z(H(F)), +, \cdot)$.

Lemma 5.3.6. Let $u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$ be a non-commuting pair. Assume that $\alpha, \beta, \gamma \in F$. Let $x = h(0, 0, \alpha \cdot \Delta_{(u,v)})$, $y = h(0, 0, \beta \cdot \Delta_{(u,v)})$, and $z = h(0, 0, \gamma \cdot \Delta_{(u,v)})$. Then

1. $\alpha + \beta = \gamma$ iff $x * y = z$, where $*$ is the matrix multiplication.
2. $\alpha \cdot \beta = \gamma$ iff there exist x' and y' such that $[x', u] = [y', v] = 1$, $[u, y'] = y$, $[x', v] = x$, and $z = [x', y']$.

Proof.

For Part 1, matrix multiplication yields the fact that

$$h(0, 0, a) * h(0, 0, b) = h(0, 0, a + b) \ .$$

Then $\alpha + \beta = \gamma$ iff

$$x * y = h(0, 0, \alpha \cdot \Delta_{(u,v)}) * h(0, 0, \beta \cdot \Delta_{(u,v)}) = h(0, 0, \gamma \cdot \Delta_{(u,v)}) = z \ .$$

For Part 2, first suppose that $\alpha \cdot \beta = \gamma$. We take $x' = h(\alpha \cdot u_1, \alpha \cdot u_2, 0)$, and $y' = h(\beta \cdot v_1, \beta \cdot v_2, 0)$. Then $\Delta_{(x',u)} = 0$, so $[x', u] = h(0, 0, 0) = 1$. Similarly, $[y', v] = 1$. Also, $\Delta_{(x',v)} = \alpha \cdot \Delta_{(u,v)}$, so $[x', v] = h(0, 0, \alpha \cdot \Delta_{(u,v)}) = x$. Similarly, $\Delta_{(u,y')} = \beta \cdot \Delta_{(u,v)}$, so $[u, y'] = h(0, 0, \beta \cdot \Delta_{(u,v)}) = y$. Finally, $\Delta_{(x',y')} = \alpha \cdot \beta \cdot \Delta_{(u,v)} = \gamma \cdot \Delta_{(u,v)}$, so $[x', y'] = h(0, 0, \gamma \cdot \Delta_{(u,v)}) = z$.

Now, suppose we have x' and y' such that $[x', u] = [y', v] = 1$, $[u, y'] = y$, $[x', v] = x$, and $[x', y'] = z$. Say that $x' = h(x'_1, x'_2, x'_3)$ and $y' = h(y'_1, y'_2, y'_3)$. Since $[x', v] = x$, $\Delta_{(x',v)} = \alpha \cdot \Delta_{(u,v)}$, so $\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Since $[u, y'] = y$, $\Delta_{(u,y')} = \beta \cdot \Delta_{(u,v)}$, so $\begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \beta \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Combining these facts, we see that $\Delta_{(x',y')} = \begin{vmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{vmatrix} = \begin{vmatrix} \alpha \cdot u_1 & \beta \cdot v_1 \\ \alpha \cdot u_2 & \beta \cdot v_2 \end{vmatrix} = \alpha \cdot \beta \cdot \Delta_{(u,v)}$. Since $[x', y'] = z$, $\Delta_{(x',y')} = \gamma \cdot \Delta_{(u,v)}$. Since u and v do not commute, $\Delta_{(u,v)} \neq 0$. Therefore, $\alpha \cdot \beta = \gamma$. □

The main result of the subsection follows directly from Lemmas 5.3.4 and 5.3.6.

Theorem 5.3.7 (Maltsev, Morozov)**.** For an arbitrary non-commuting pair (u, v) in $H(F)$, we get $F_{(u,v)} = (Z(H(F)), \oplus, \otimes_{(u,v)})$ where

1. $x \in Z(H(F))$ iff $[x, u] = [x, v] = 1$,
2. $\oplus$ is the group operation from $H(F)$,
3. $\otimes_{(u,v)}$ is the set of triples (x, y, z) such that there exist x', y' with $[x', u] = [y', v] = 1$, $[x', v] = x$, $[u, y'] = y$, and $[x', y'] = z$,
4. the function $g_{(u,v)}$ taking $\alpha \in F$ to $h(0, 0, \alpha \cdot \Delta_{(u,v)}) \in H(F)$ is an isomorphism between F and $F_{(u,v)}$.

Note: From Part 4, it is clear that $h(0, 0, \Delta_{(u,v)})$ is the multiplicative identity in $F_{(u,v)}$—we may write $1_{(u,v)}$ for this element.

Proposition 5.3.8. There is a uniform Medvedev reduction Φ of F to $H(F)$.

Proof.

Given $G \cong H(F)$, we search for a non-commuting pair (u, v) in G, and then use Maltsev's definitions to get a copy of F computable from G. □

It turns out that the Medvedev reduction Φ is half of a computable functor. In the next subsection, we explain how to get the other half.

5.3.2 The computable functor

In the previous subsection, we saw that for any field F and any non-commuting pair (u, v) in $H(F)$, there is an isomorphic copy $F_{(u,v)}$ of F defined in $H(F)$ by finitary existential formulas with parameters (u, v). The defining formulas are the same for all F. Hence, there is a uniform Turing operator Φ that, for all fields F, takes copies of $H(F)$ to copies of F. In this subsection, we describe a companion operator Ψ so that Φ and Ψ together form a uniform computable functor. For any field F, and any triple (G_1, p, G_2) such that G_1 and G_2

are copies of $H(F)$ and p is an isomorphism from G_1 onto G_2, the function $\Psi(G_1, p, G_2)$ must be an isomorphism from $\Phi(G_1)$ onto $\Phi(G_2)$, and, moreover, the isomorphisms given by Ψ must preserve identity and composition. We saw in the previous subsection that for any field F, and any non-commuting pair (u,v) in $H(F)$, the function $g_{(u,v)}$ taking α to $h(0,0,\alpha\cdot\Delta_{(u,v)})$ is an isomorphism from F onto $F_{(u,v)}$. We use this $g_{(u,v)}$ below.

Lemma 5.3.9. For any F and any non-commuting pairs (u,v), (u',v') in $H(F)$, there is a natural isomorphism $f_{(u,v),(u',v')}$ from $F_{(u,v)}$ onto $F_{(u',v')}$. Moreover, the family of isomorphisms $f_{(u,v),(u',v')}$ is functorial; i.e.,

1. for any non-commuting pair (u,v), the function $f_{(u,v),(u,v)}$ is the identity,

2. for any three non-commuting pairs (u,v), (u',v'), and (u'',v''),

$$f_{(u,v),(u'',v'')} = f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')}.$$

Proof.

We let $f_{(u,v),(u',v')} = g_{(u',v')} \circ g^{-1}_{(u,v)}$. This is an isomorphism from $F_{(u,v)}$ onto $F_{(u',v')}$. It is clear that $f_{(u,v),(u,v)}$ is the identity. Consider non-commuting pairs (u,v), (u',v'), and (u'',v''). We must show that $f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')} = f_{(u,v),(u'',v'')}$. We have:

$$\begin{aligned} f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')} &= g_{(u'',v'')} \circ g^{-1}_{(u',v')} \circ g_{(u',v')} \circ g^{-1}_{(u,v)} = \\ &= g_{(u'',v'')} \circ g^{-1}_{(u,v)} = \\ &= f_{(u,v),(u'',v'')}. \end{aligned}$$

□

The next lemma says that there is a uniform existential definition of the family of isomorphisms $f_{(u,v),(u',v')}$.

Lemma 5.3.10. There is a finitary existential formula $\psi(u,v,u',v',x,y)$ that, for any two non-commuting pairs (u,v) and (u',v'), defines the isomorphism $f_{(u,v),(u',v')}$ taking $x \in F_{(u,v)}$ to $y \in F_{(u',v')}$.

Proof.

Since the operation $\otimes_{(u,v)}$ and $1_{(u',v')}$ are definable by $\exists$–formulas with parameters u,v and u',v' respectively, it suffices to prove the equivalence

$$f_{(u,v),(u',v')}(x) = y \Leftrightarrow x \otimes_{(u,v)} 1_{(u',v')} = y.$$

First assume that $f_{(u,v),(u',v')}(x) = y$, i.e., $y = g_{(u',v')} \circ g^{-1}_{(u,v)}(x)$. Let $\alpha = g^{-1}_{(u,v)}(x)$, i.e., $x = h(0,0,\alpha\cdot\Delta_{(u,v)})$. It follows that $y = h\big(0,0,\alpha\cdot\Delta_{(u',v')}\big)$. Then

$$\begin{aligned}
x \otimes_{(u,v)} 1_{(u'v')} &= h\left(0,0,\alpha\cdot\Delta_{(u,v)}\right) \otimes_{(u,v)} h\left(0,0,\Delta_{(u',v')}\right) = \\
&= h\left(0,0,\alpha\cdot\Delta_{(u,v)}\right) \otimes_{(u,v)} h\left(0,0,\frac{\Delta_{(u',v')}}{\Delta_{(u,v)}}\cdot\Delta_{(u,v)}\right) = \\
&= h\left(0,0,\alpha\cdot\frac{\Delta_{(u',v')}}{\Delta_{(u,v)}}\cdot\Delta_{(u,v)}\right) = \\
&= h\left(0,0,\alpha\cdot\Delta_{(u',v')}\right) = y.
\end{aligned}$$

Assume now that $x \otimes_{(u,v)} 1_{(u',v')} = y$ and let $x = h\left(0,0,\alpha\cdot\Delta_{(u,v)}\right)$. Then

$$\begin{aligned}
y &= x \otimes_{(u,v)} 1_{(u',v')} = h\left(0,0,\alpha\cdot\Delta_{(u,v)}\right) \otimes_{(u,v)} h\left(0,0,\Delta_{(u',v')}\right) = \\
&= h\left(0,0,\alpha\cdot\Delta_{(u',v')}\right) = g_{(u',v')} \circ g^{-1}_{(u,v)}(x) = f_{(u,v),(u',v')}(x).
\end{aligned}$$

□

We will use Lemmas 5.3.9 and 5.3.10 to prove the following.

Proposition 5.3.11. There is a uniform computable functor that, for all fields F, takes $H(F)$ to F.

Proof.

Let Φ be the uniform Medvedev reduction of F to $H(F)$. Take copies G_1, G_2 of $H(F)$ and take p such that $G_1 \cong_p G_2$. We describe $q = \Psi(G_1, p, G_2)$ as follows. Let (u,v) be the first non-commuting pair in G_1, and let (u',v') be the first non-commuting pair in G_2. Now, p takes (u,v) to a non-commuting pair $(p(u),p(v))$, and p maps $F_{(u,v)}$ isomorphically onto $F_{(p(u),p(v))}$. The function $f_{(p(u),p(v)),(u',v')}$ is an isomorphism from $F_{(p(u),p(v))}$ onto $F_{(u',v')}$. We get an isomorphism q from $F_{(u,v)}$ onto $F_{(u',v')}$ by composing p with $f_{(p(u),p(v)),(u',v')}$. For $x \in F_{(u,v)}$, we let $q(x) = f_{(p(u),p(v)),(u',v')}(p(x))$. Since $f_{(p(u),p(v)),(u',v')}$ is defined by an existential formula, with parameters $p(u), p(v), u', v'$, we can apply a uniform effective procedure to compute q from (G_1, p, G_2).

If $G_1 = G_2$ and p is the identity, then $(u,v) = (u',v')$, and by Lemma 5.3.9, $f_{(u,v),(u',v')}$ is the identity. Consider G_1, G_2, G_3, all copies of G, with functions p_1, p_2 such that $G_1 \cong_{p_1} G_2$ and $G_2 \cong_{p_2} G_3$. Then $p_3 = p_2 \circ p_1$ is an isomorphism from G_1 onto G_3. Let $q_1 = \Psi(G_1, p_1, G_2)$, $q_2 = \Psi(G_2, p_2, G_3)$, and $q_3 = \Psi(G_1, p_3, G_3)$. We must show that $q_3 = q_2 \circ q_1$. The idea is to transfer everything to G_3 and use Lemma 5.3.9. Let r_1 be the result of transferring q_1 down to G_3—$r_1 = f_{(p_3(u),p_3(v)),(p_2(u),p_2(v))}$. We have $q_1(x) = y$ iff $r_1(p_3(x)) = p_2(y)$. Let r_2 be the result of transferring q_2 down to G_3—$r_2 = f_{(p_2(u),p_2(v)),(u,v)}$. We have $q_2(y) = z$ iff $r_2(p_2(y)) = z$. We let r_3 be the result of transferring q_3 down to G_3—$r_3 = f_{(p_3(u),p_3(v)),(u,v)}$. We have $q_3(x) = z$ iff $r_3(p_3(x)) = z$. By Lemma 5.3.9, $r_3 = r_2 \circ r_1$. If $q_1(x) = y$ and $q_2(y) = z$, then $r_1(p_3(x)) = p_2(y)$, and $r_2(p_2(y)) = z$. Then $r_3(p_3(x)) = z$, so $q_3(x) = z$, as required. □

Corollary 5.3.12. There is a uniform effective interpretation of F in $H(F)$.

Proof. Apply the result from [HTMMM17]. □

The result from [HTMMM17] gives a uniform interpretation of F in $H(F)$, valid for all countable fields F, using computable Σ^c_1 formulas with no parameters. The tuples from $H(F)$ that represent elements of F may have arbitrary arity. In the next subsection, we will do better.

We note here that the uniform interpretation of F in $H(F)$ given in this subsection allows one to transfer the computable-model-theoretic properties of any graph G to a 2-step-nilpotent group, without introducing any constants. This is not a new result: in [Mek81], Mekler gave a related coding of graphs into 2-step-nilpotent groups, which, in concert with the completeness of graphs for such properties (see [HKSS02]), appears to yield the same fact, although Mekler's coding had different goals than completeness. Then, in [HKSS02], Hirschfeldt, Khoussainov, Shore, and Slinko used Maltsev's interpretation of an integral domain in its Heisenberg group with two parameters, along with the completeness of integral domains, to re-establish it. More recently, [MPSS18] demonstrated the completeness of fields, by coding graphs into fields, From that result, along with Corollary 5.3.12 and the usual definition of $H(F)$ as a matrix group given by a set of triples from F, we achieve a coding of graphs into fields, different from Mekler's coding, with no constants required.

5.3.3 Defining the interpretation directly

Our goal in this section is to give explicit existential formulas defining a uniform effective interpretation of a field in its Heisenberg group. We discovered the formulas for this interpretation by examining the infinitary formulas used in the interpretation in Corollary 5.3.12 and trimming them down to their essence, which turned out to be finitary.

Theorem 5.3.13. There are finitary existential formulas that, uniformly for every field F, define an effective interpretation of F in $H(F)$, with elements of F represented by triples of elements from $H(F)$.

We offer intuition before giving the formal proof. The domain D of the interpretation will consist of those triples (u, v, x) from $H(F)$ with $uv \neq vu$ and x in the center: for each single (u, v), we apply Maltsev's definitions, with u, v as parameters, to get $F_{(u,v)} \cong F$. We view the triples arranged as follows:

$F_{(u,v)}$	$F_{(u',v')}$	$F_{(u'',v'')}$	$\cdots$
(u, v, x_0)	(u', v', x_0)	(u'', v'', x_0)	
(u, v, x_1)	(u', v', x_1)	(u'', v'', x_1)	
(u, v, x_2)	(u', v', x_2)	(u'', v'', x_2)	
(u, v, x_3)	(u', v', x_3)	(u'', v'', x_3)	
$\vdots$	$\vdots$	$\vdots$	

Here each column can be seen as $F_{(u,v)}$ for some non-commuting pair (u, v). Now the system of isomorphisms from Lemma 5.3.9 will allow us to identify each element in one column with a single element from each other column, and modding out by this identification will yield a single copy of F.

Proof.

Let H be a group isomorphic to $H(F)$. Recalling the natural isomorphisms $f_{(u,v),(u',v')}$ defined in Lemma 5.3.9 for non-commuting pairs (u, v) and (u', v'), we define $D \subseteq H$, a binary relation $\sim$ on D, and ternary relations $\oplus$, $\odot$ (which are binary operations) on D, as follows.

1. D is the set of triples (u, v, x) such that $uv \neq vu$ and $xu = ux$ and $xv = vx$. (Notice that, no matter which non-commuting pair (u, v) is chosen, the set of corresponding elements x is precisely the center $Z(H)$, by Theorem 5.3.7.)

2. $(u, v, x) \sim (u', v', x')$ holds if and only if the isomorphism $f_{(u,v),(u',v')}$ from $F_{(u,v)}$ to $F_{(u',v')}$ maps x to x'.

3. $\oplus((u,v,x),(u',v',y'),(u'',v'',z''))$ holds if there exist $y,z \in H$ such that $(u,v,y) \sim (u',v',y')$ and $(u,v,z) \sim (u'',v'',z'')$, and $F_{(u,v)} \models x + y = z$.

4. $\odot((u,v,x),(u',v',y'),(u'',v'',z''))$ holds if there exist $y,z \in H$ such that $(u,v,y) \sim (u',v',y')$ and $(u,v,z) \sim (u'',v'',z'')$, and $F_{(u,v)} \models x \cdot y = z$.

Lemma 5.3.10 yielded a finitary existential formula defining the relation $(u,v,x) \sim (u',v',x')$. Moreover, the field addition and multiplication were defined in $F_{(u,v)}$ by finitary existential formulas using u and v, which were parameters there but here are elements of the triples in D. Finally, we must consider the negations of the relations. First, $(u,v,x) \nsim (u',v',x')$ if and only if some y' commuting with u' and v' satisfies $(u,v,x) \sim (u',v',y')$ and $y' \neq x'$ – that is, just if $f_{(u,v),(u',v')}$ maps x to some element different from x'. Likewise, since $+$ is a binary operation in $F_{(u,v)}$, the negation of $\oplus((u,v,x),(u',v',y'),(u'',v'',z''))$ is defined by saying that some $w'' \neq z''$ is the sum:

$$\exists w''([w'',u''] = 1 = [w'',v''] \;\&\; w'' \neq z'' \;\&\; \oplus((u,v,x),(u',v',y'),(u'',v'',w''))),$$

which is also existential, and similarly for the negation of $\odot$. Therefore, all of these sets have finitary existential definitions in the language of groups, with no parameters, as do the negations of $\sim$, $\oplus$, and $\odot$. (In fact, the complement of D is Σ_1^c as well.)

The functoriality of the system of isomorphisms $f_{(u,v),(u',v')}$ (across all pairs of pairs of noncommuting elements) ensures that $\sim$ will be an equivalence relation. Lemma 5.3.9 showed that $f_{(u,v),(u,v)}$ is always the identity, giving reflexivity. Transitivity follows from the functorial property in that same lemma:

$$f_{(u,v),(u'',v'')} = f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')},$$

and with $(u'',v'') = (u,v)$, this property also yields the symmetry of $\sim$.

The definitions of $\oplus$ and $\odot$ essentially say to convert all three triples into $\sim$-equivalent triples with the same initial coordinates u and v, and then to check whether the final coordinates satisfy Maltsev's definitions of $+$ and $\cdot$ in the field $F_{(u,v)}$. Understood this way, they clearly respect the equivalence $\sim$. Finally, by fixing any single noncommuting pair (u,v), we see that the set $\{(u,v,x) : x \in Z(H)\}$ contains one element from each $\sim$-class and, under $\oplus$ and $\odot$, is isomorphic to the field $F_{(u,v)}$ defined by Maltsev, which in turn is isomorphic to the original field F. □

It should be noted that, although this interpretation of F in $H(F)$ was developed using computable functors on countable fields F, it is valid even when F is uncountable (or finite). A full proof requires checking that the system of isomorphisms $f_{(u,v),(u',v')}$ remains functorial and existentially definable even in the uncountable case, but this is straightforward.

In Theorem 5.3.13, to eliminate parameters from Maltsev's definition of F in $H(F)$, we gave an interpretation of F in $H(F)$, rather than another definition. (Recall that a definition is an interpretation in which the equivalence relation on the domain is simply equality.) We now demonstrate the impossibility of strengthening the theorem to give a parameter-free definition of F in $H(F)$.

Proposition 5.3.14. There is no parameter-free definition of any field F in its Heisenberg group $H(F)$ by finitary formulas.

Proof. Suppose that there were such a definition, and let $D \subseteq (H(F))^n$ be its domain. By Corollary 5.3.5, the only $(x_1, \ldots, x_n) \in (H(F))^n$ that is fixed by all automorphisms of $H(F)$ is the tuple where every x_i is the identity element of $H(F)$. So, for every $\overline{x} \in D$ except this identity tuple, there would be an $\alpha_{\overline{x}} \in \mathrm{Aut}(H(F))$ that does not fix $\overline{x}$. With equality of n-tuples as the equivalence relation on D, $\alpha_{\overline{x}}$ yields an automorphism of the field F (viewed as D under the definable addition and multiplication) that does not fix $\overline{x}$. However, both identity elements 0 and 1 in F must be fixed by every automorphism of F. □

5.3.4 Question of bi-interpretability

If $\mathcal{B}$ is interpreted in $\mathcal{A}$, we write $\mathcal{B}^{\mathcal{A}}$ for the copy of $\mathcal{B}$ given by the interpretation of $\mathcal{B}$ in $\mathcal{A}$. The structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are uniformly relatively computable isomorphisms f from $\mathcal{A}$ onto $\mathcal{A}^{\mathcal{B}^{\mathcal{A}}}$ and g from $\mathcal{B}$ onto $\mathcal{B}^{\mathcal{A}^{\mathcal{B}}}$. In general, the isomorphism f would map each element of $\mathcal{A}$ to an equivalence class of equivalence classes of tuples in $\mathcal{A}$. We would represent f by a relation R_f that holds for $a, \overline{a}_1, \ldots, \overline{a}_r$ if f maps a to the equivalence class of the tuple of equivalence classes of the $\overline{a}_i$'s. Similarly, the isomorphism g would be represented by a relation R_g that holds for $b, \overline{b}_1, \ldots, \overline{b}_r$ if g maps b to the equivalence class of the tuple of equivalence classes of the $\overline{b}_i$'s. Saying that f and g are uniformly relatively computable is equivalent to saying that the relations R_f, R_g, have generalized computable Σ_1^c definitions without parameters.

For a field F and its Heisenberg group $H(F)$, when we define $H(F)$ in F, the elements of $H(F)$ are represented by triples from F, and we have finitary formulas, quantifier-free or existential, that define the group operation (as a relation). When we interpret F in $H(F)$, the elements of F are represented by triples from $H(F)$, and we have finitary existential formulas that define the field operations and their negations (as ternary relations). Thus, in $F^{H(F)^F}$ (the copy of F interpreted in the copy of $H(F)$ that is defined in F), the elements are equivalence classes of triples of triples. In $H(F)^{F^{H(F)}}$ (the copy of $H(F)$ defined in the copy of F that is interpreted in $H(F)$), the elements are triples of equivalence classes of triples. So, an isomorphism f from F to $F^{H(F)^F}$ is represented by a 10-ary relation R_f on F, and an isomorphism g from $H(F)$ to $H(F)^{F^{H(F)}}$—it is represented by a 10-ary relation R_g on $H(F)$.

For a Turing computable embedding Θ of K in K' we have *uniform effective bi-interpretability* if there are (generalized) computable Σ_1^c formulas with no parameters that, for all $\mathcal{A} \in K$ and $\mathcal{B} = \Theta(\mathcal{A})$, define isomorphisms from $\mathcal{A}$ to $\mathcal{A}^{\mathcal{B}^{\mathcal{A}}}$ and from $\mathcal{B}$ to $\mathcal{B}^{\mathcal{A}^{\mathcal{B}}}$. Montalbán asked the following very natural question.

Question 5.3.15. Do we have uniform effective bi-interpretability of F and $H(F)$?

The answer to this question is negative. In particular, $\mathbb{Q}$ and $H(\mathbb{Q})$ are not effectively bi-interpretable. One way to see this is to note that $\mathbb{Q}$ is rigid, while $H(\mathbb{Q})$ is not—in particular, for any non-commuting pair, $u, v \in H(\mathbb{Q})$, there is a group automorphism that takes (u, v) to (v, u). The negative answer to Question 5.3.15 then follows from [Mon, Lemma VI.26(4)], which states that if $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable, then their automorphism groups are isomorphic.

Morozov's result shows which half of effective bi-interpretability causes the difficulties.

Proposition 5.3.16 (Morozov)**.** There is a finitary existential formula that, for all F, defines in F a specific isomorphism k from F to $F^{H(F)^F}$.

Proof.

In F, we have the copy of $H(F)$, consisting of triples (a, b, c) (representing $h(a, b, c)$), for $a, b, c \in F$. The group operation, derived from matrix multiplication, is $(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + ab')$. The definitions of the universe and the operation are quantifier-free, with no parameters. We have seen how to interpret F in $H(F)$ using finitary existential formulas with no parameters. There is a natural isomorphism k from F onto $F^{H(F)^F}$ obtained as follows. In $H(F)$, let $u = h(1,0,0)$ and $v = h(0,1,0)$. Then $\Delta_{(u,v)} = 1$. We have an isomorphism mapping F to $F_{(u,v)}$ that takes α to $h(0,0,\alpha)$. We let $k(\alpha)$ be the $\sim$-class of $(u,v,h(0,0,\alpha))$. The isomorphism k is defined in F by an existential formula. The complement of k is defined by saying that $k(\alpha)$ has some other value. $\square$

The other half of what we would need for uniform effective bi-interpretability is sometimes impossible, as remarked above in the case $F = \mathbb{Q}$. We do not know of any examples where F and $H(F)$ are effectively bi-interpretable: the obstacle for $\mathbb{Q}$ might hold in all cases.

Problem 5.3.17. For which fields F, if any, are the automorphism groups of F and $H(F)$ isomorphic?

Even if there are fields F such that $\mathrm{Aut}(F) \cong \mathrm{Aut}(H(F))$, we suspect that F and $H(F)$ are not effectively bi-interpretable, simply because it is difficult to see how one might give a computable Σ^c_1 formula in the language of groups that defines a specific isomorphism from $H(F)$ to $H(F)^{F^{H(F)}}$.

5.3.5 Generalizing the method

Our first general definition and proposition follow closely the example of a field and its Heisenberg group.

Definition 5.3.18. Let $\mathcal{A}$ be a structure for a computable relational language. Assume that its basic relations are R_i, where R_i is k_i-ary. We say that $\mathcal{A}$ is *effectively defined in $\mathcal{B}$ with parameters $\bar{b}$* if there exist $D(\bar{b}) \subseteq \mathcal{B}^{<\omega}$, and $\pm R_i(\bar{b}) \subseteq D(\bar{b})^{k_i}$, defined by a uniformly computable sequence of generalized computable Σ^c_1 formulas with parameters $\bar{b}$.

Proposition 5.3.19. Suppose $\mathcal{A}$ is effectively defined in $\mathcal{B}$ with parameters $\bar{b}$. For $\bar{c}$ in the orbit of $\bar{b}$, let $\mathcal{A}_{\bar{c}}$ be the copy of $\mathcal{A}$ defined by the same formulas, but with parameters $\bar{c}$ replacing $\bar{b}$. Then the following conditions together suffice to give an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters:

(1) The orbit of $\bar{b}$ is defined by a computable Σ^c_1 formula $\varphi(\bar{u})$;

(2) There is a generalized computable Σ^c_1 formula $\psi(\bar{u},\bar{v},\bar{x},\bar{y})$ such that for all $\bar{c},\bar{d}$ in the orbit of $\bar{b}$, the formula $\psi(\bar{c},\bar{d},\bar{x},\bar{y})$ defines an isomorphism $f_{\bar{c},\bar{d}}$ from $\mathcal{A}_{\bar{c}}$ onto $\mathcal{A}_{\bar{d}}$; and

(3) The family of isomorphisms $f_{\overline{c},\overline{d}}$ preserves identity and composition.

Proof.

We write $D(\overline{b})$, $\pm R_i(\overline{b})$ for the set and relations that give a copy of $\mathcal{A}$ and for the defining formulas (with parameters $\overline{b}$). We obtain a parameter-free interpretation of $\mathcal{A}$ in $\mathcal{B}$ as follows:

1. Let D consist of the tuples $(\overline{c},\overline{x})$ such that $\overline{c}$ is in the orbit of $\overline{b}$ and $\overline{x}$ is in $D(\overline{c})$. This is defined by a generalized computable Σ_1^c formula.

2. Let $\sim$ be the set of pairs $((\overline{c},\overline{x}),(\overline{d},\overline{y}))$ in D^2 such that $f_{\overline{c},\overline{d}}(\overline{x}) = \overline{y}$. This is defined by a generalized computable Σ_1^c formula. For pairs $(\overline{c},\overline{x})$, $(\overline{d},\overline{y})$ from D, it follows that $(\overline{c},\overline{x}) \nsim (\overline{d},\overline{y})$ if and only if

$$(\exists \overline{y}')((\overline{d},\overline{y}') \in D \;\&\; f_{\overline{c},\overline{d}}(\overline{x}) = \overline{y}' \;\&\; \overline{y}' \neq \overline{y}).$$

   Hence the negation of $\sim$ is also defined by a generalized computable Σ_1^c formula.

3. We let R_i^* be the set of k_i-tuples $((\overline{b}_1,\overline{x}_1),\ldots,(\overline{b}_{k_i},\overline{x}_{k_i}))$ in D^{k_i} such that for the tuple $(\overline{y}_1,\ldots,\overline{y}_{k_i})$ with $f_{\overline{b}_j,\overline{b}_1}(\overline{x}_j) = \overline{y}_j$, we have $(\overline{y}_1,\ldots,\overline{y}_{k_i}) \in R_i(\overline{b}_1)$. This is defined by a generalized computable Σ_1^c formula. The complementary relation $\neg R_i^*$ is the set of tuples $((\overline{b}_1,\overline{x}_1),\ldots,(\overline{b}_{k_i},\overline{x}_{k_i}))$ such that for $\overline{y}_1,\ldots,\overline{y}_{k_i}$ as above, $(\overline{y}_1,\ldots,\overline{y}_{k_i}) \in \neg R_i(\overline{b}_1)$. This is also defined by a generalized computable Σ_1^c formula.

The verification is identical to that of Theorem 5.3.13. □

Corollary 5.3.20. In the situation of Proposition 5.3.19, if $D(\overline{b})$ is contained in $\mathcal{B}^n$ for some single $n \in \omega$, then the ψ in item (2) and the formulas in Definition 5.3.18 will simply be computable Σ_1^c formulas (as opposed to generalized computable Σ_1^c formulas) and the interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters will also be by computable (as opposed to generalized) Σ_1^c formulas. □

The reader will have noticed that we only produced an *interpretation* of $\mathcal{A}$ in $\mathcal{B}$, even though we originally had a *definition* (with parameters) of $\mathcal{A}$ in $\mathcal{B}$. The specific example in Sections 5.3.2 and 5.3.3 suggests that this may be the best that can be done in general. On the other hand, we may extend Proposition 5.3.19 and remove parameters even in the case where $\mathcal{A}$ is interpreted (as opposed to being defined) with parameters in $\mathcal{B}$.

Definition 5.3.21 (Effective Interpretation with Parameters)**.** We say that $\mathcal{A}$, with basic relations R_i, k_i-ary, is *effectively interpreted with parameters* $\overline{b}$ in $\mathcal{B}$ if there exist $D \subseteq \mathcal{B}^{<\omega}$, $\equiv \subseteq D^2$, and $R_i^* \subseteq D^{k_i}$ such that

1. $(D, (R_i^*)_i)/_{\equiv} \cong \mathcal{A}$,
2. D, $\pm \equiv$, and $\pm R_i^*$ are defined by a computable sequence of generalized computable Σ_1^c formulas, with a fixed finite tuple of parameters $\overline{b}$.

Again, in the case where $D \subseteq \mathcal{B}^n$ for some fixed n, the formulas defining the effective interpretation are computable Σ_1^c formulas of the usual kind, with parameters $\overline{b}$.

Proposition 5.3.22. Suppose that $\mathcal{A}$ (with basic relations R_i, k_i-ary) has an effective interpretation in $\mathcal{B}$ with parameters $\overline{b}$. For $\overline{c}$ in the orbit of $\overline{b}$, let $\mathcal{A}_{\overline{c}}$ be the copy of $\mathcal{A}$ obtained by replacing the parameters $\overline{b}$ by $\overline{c}$ in the defining formulas, with domain $D_{\overline{c}}/\equiv_{\overline{c}}$ containing $\equiv_{\overline{c}}$-classes $[\overline{a}]_{\equiv_{\overline{c}}}$. Then the following conditions suffice for an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ (without parameters):

(1) The orbit of $\overline{b}$ is defined by a computable Σ_1^c formula $\varphi(\overline{x})$;

(2) There is a relation $F \subseteq \mathcal{B}^{<\omega}$, with a generalized computable Σ_1^c-definition, such that for every $\overline{c}$ and $\overline{d}$ in the orbit of $\overline{b}$, the set of pairs $(\overline{x}, \overline{y}) \in D_{\overline{c}} \times D_{\overline{d}}$ with $(\overline{c}, \overline{d}, \overline{x}, \overline{y}) \in F$ is invariant under $\equiv_{\overline{c}}$ on $\overline{x}$ and under $\equiv_{\overline{d}}$ on $\overline{y}$, and defines an isomorphism $f_{\overline{c},\overline{d}}$ from $\mathcal{A}_{\overline{c}}$ onto $\mathcal{A}_{\overline{d}}$; and

(3) The family of isomorphisms $f_{\overline{c},\overline{d}}$ preserves identity and composition.

Proof. Let the new domain D consist of those tuples $(\overline{c}, \overline{x})$ with $\overline{c}$ in the orbit of $\overline{b}$ and $\overline{x}$ in $D_{\overline{c}}$. This is defined by a generalized computable Σ_1^c formula.

Let the equivalence relation $\sim$ on D be the set of pairs $((\overline{c}, \overline{x}), (\overline{d}, \overline{y})) \in D^2$ such that $f_{\overline{c},\overline{d}}([\overline{x}]_{\equiv_{\overline{c}}}) = [\overline{y}]_{\equiv_{\overline{d}}}$. This is defined by a generalized computable Σ_1^c formula. For $(\overline{c}, \overline{x})$, $(\overline{d}, \overline{y}) \in D$, we have $(\overline{c}, \overline{x}) \not\sim (\overline{d}, \overline{y})$ if and only if

$$(\exists \overline{y}' \in D_{\overline{d}})\ (f_{\overline{c},\overline{d}}([\overline{x}]_{\equiv_{\overline{c}}}) = [\overline{y}']_{\equiv_{\overline{d}}}\ \&\ \overline{y} \not\equiv_{\overline{d}} \overline{y}').$$

Hence $\not\sim$ is also defined by a generalized computable Σ_1^c formula.

Let R_i^* be the set of k_i-tuples $((\overline{b}_1, \overline{x}_1), \ldots, (\overline{b}_{k_i}, \overline{x}_{k_i}))$ in D^{k_i} such that for the tuple $(\overline{y}_1, \ldots, \overline{y}_{k_i})$ with $f_{\overline{b}_j,\overline{b}_1}(\overline{x}_j) = \overline{y}_j$, we have $(\overline{y}_1, \ldots, \overline{y}_{k_i}) \in R_i(\overline{b}_1)$. This is defined by a generalized computable Σ_1^c-formula. The complementary relation $\neg R_i^*$ is the set of tuples $((\overline{b}_1, \overline{x}_1), \ldots, (\overline{b}_{k_i}, \overline{x}_{k_i}))$ such that for $\overline{y}_1, \ldots, \overline{y}_{k_i}$ as above, $(\overline{y}_1, \ldots, \overline{y}_{k_i}) \in \neg R_i(\overline{b}_1)$. This too is defined by a generalized computable Σ_1^c formula. Finally, as in the proofs of Theorem 5.3.13 and Proposition 5.3.19, it is clear that this yields an interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters. □

A relation $R \subseteq \mathcal{B}^{<\omega}$ may have a definition that is *generalized computable* Σ_α^c for a computable ordinal α, or *generalized* X*-computable* Σ_α for an X-computable ordinal α, or *generalized* $L_{\omega_1\omega}$, or *generalized* Σ_α for a countable ordinal α. The definition has the form $\bigvee_n \varphi_n(\overline{x}_n)$, where the sequence of disjuncts (each in $L_{\omega_1\omega}$, but of different arities n) is computable, or X-computable, or just countable. We note that each generalized $L_{\omega_1\omega}$ formula is generalized X-computable Σ_α for an appropriately chosen X and α, and each generalized Σ_α-formula is generalized X-computable Σ_α for an appropriately chosen X.

As computable structure theorists, we have focused here on effective interpretations. Nevertheless, we wish to point out that our results apply not only to effective interpretations, but to all interpretations using generalized $L_{\omega_1\omega}$ formulas. The following theorem generalizes Proposition 5.3.22 and considers every variation we can imagine.

Theorem 5.3.23. Let $\mathcal{A}$ be a relational structure with basic relations R_i that are k_i-ary. Suppose there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ by generalized $L_{\omega_1\omega}$ formulas, with parameters $\overline{b}$ from $\mathcal{B}$. For $\overline{c}$ in the orbit of $\overline{b}$, let $\mathcal{A}_{\overline{c}}$ be the copy of $\mathcal{A}$ obtained by the interpretation with parameters $\overline{c}$ replacing $\overline{b}$. Assume that there is a generalized $L_{\omega_1\omega}$-definable relation F defining, for each $\overline{c}$ and $\overline{d}$ in the orbit of $\overline{b}$, an isomorphism $f_{\overline{c},\overline{d}} : \mathcal{A}_{\overline{c}} \to \mathcal{A}_{\overline{d}}$ as in Proposition 5.3.22, and that this family is closed under composition, with the identity map as $f_{\overline{c},\overline{c}}$ for all $\overline{c}$.

Then there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ by $L_{\omega_1\omega}$ formulas without parameters. Moreover, the new interpretation satisfies all of the following.

- For each countable ordinal α, if the interpretation in $(\mathcal{B}, \overline{b})$ defines D, $\equiv$, and each R_i using Σ_α formulas from $L_{\omega_1\omega}$, and F and the orbit of $\overline{b}$ in $\mathcal{B}$ are both defined by Σ_α formulas, then the parameter-free interpretation also uses Σ_α formulas to define these sets.

- For each countable ordinal α, if the interpretation in $(\mathcal{B}, \overline{b})$ defines each of D, $\pm\equiv$, and $\pm R_i$ using Σ_α formulas, and F and the orbit of $\overline{b}$ in $\mathcal{B}$ are both defined by Σ_α formulas, then the parameter-free interpretation also uses Σ_α formulas to define its domain, its equivalence relation $\sim$, the complement $\not\sim$, and its relations $\pm R_i$. (Defining $\not\sim$ and $\neg R_i$ this way is required by the usual notion of effective Σ_α interpretation.)

- Let $X \subseteq \mathbb{N}$. If the interpretation in $(\mathcal{B}, \bar{b})$ used X-computable formulas, and F and the orbit of $\bar{b}$ in $\mathcal{B}$ are both defined by X-computable formulas, then the parameter-free interpretation also uses X-computable formulas.

  Of course, for every countable set of $L_{\omega_1\omega}$ formulas, there is an X that computes them all. If the signature of $\mathcal{A}$ is infinite, and the formulas for the interpretation of $\mathcal{A}$ in $(\mathcal{B}, \bar{b})$ are computable uniformly in X, then so are the formulas for the parameter-free interpretation of $\mathcal{A}$ in $\mathcal{B}$.

  (With $X = \varnothing$, X-computable formulas are simply computable formulas.)

- If the interpretation in $(\mathcal{B}, \bar{b})$ had domain contained in $\mathcal{B}^n$ for a single n, so that the defining formulas for this interpretation and for F in $\mathcal{B}$ are all in $L_{\omega_1\omega}$ (as opposed to generalized $L_{\omega_1\omega}$), then the parameter-free interpretation also uses (non-generalized) $L_{\omega_1\omega}$ formulas, and its domain is contained in $\mathcal{B}^{n+|\bar{b}|}$.

- If the interpretation in $(\mathcal{B}, \bar{b})$ used finitary formulas, and F and the orbit of $\bar{b}$ in $\mathcal{B}$ are both defined by finitary formulas, then the parameter-free interpretation also uses finitary formulas.

Proof. We obtain the parameter-free interpretation just as in the proof of Proposition 5.3.22. Notice that, by a result of Scott in [Sco65], the orbit of $\bar{b}$ must be definable by some $L_{\omega_1\omega}$ formula. Checking the specific claims is simply a matter of writing out the new formulas using the old ones. □

5.4 Interpreting $ACF(0)$ - C in a special linear group $SL_2(C)$

Let C be an algebraically closed field of characteristic 0 - $ACF(0)$. We write $SL_2(C)$ for the group of 2×2 matrices over C with determinant 1. Clearly, $SL_2(C)$ is defined in C without parameters. Each particular C has a computable copy, and that is effectively interpreted in $SL_2(C)$. But, there are infinitely many non-isomorphic C, differing in transcendence degree. We will give finitary existential formulas that (for all C) define C in $SL_2(C)$, with a pair of parameters. Before defining the field as a whole, we look separately at addition and multiplication. This is a work in progress together with Alvir, Knight and Miller [AKMS].

Defining $(C,+)$

Let A be the set of matrices in $SL_2(C)$ of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. Note that on A, matrix multiplication gives addition; that is,

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}.$$

We can define A using the parameter $p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Claim 1: The matrices that commute with p have the forms $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}$.

Proof. [Proof of Claim 1]

We check that if $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with p, then $c = 0$ and $a = d$. Assuming that $x \in SL_2(C)$, $a = d = 1$ or $a = d = -1$, while b can be anything. □

It is easy to check that I is the unique element of $SL_2(C)$ that is its own square. Thus, we can define I by a quantifier-free formula. Now, I has many square roots apart from $\pm I$. However, these do not commute with p—the unique square root of I that is not equal to I and commutes with p is $-I$.

Claim 2: $x \in A$ iff x commutes with p and x has a square root that commutes with p.

Proof. [Proof of Claim 2]

If $x \in A$, then x commutes with p, and x has a square root that commutes with p. Suppose x commutes with p and is not in A—$x = \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}$. We can see that nothing that commutes with p has square equal to x. □

Now, $(C,+) \cong (A,*)$, so we have a copy of $(C,+)$ defined in $SL_2(C)$ using the parameter p.

5.4.1 Defining $(C \setminus \{0\}, \cdot)$

Let M be the set of matrices of form $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$. On M, matrix multiplication gives multiplication; that is, $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} * \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & (ab)^{-1} \end{bmatrix}$. We can define M using a parameter $q = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

Claim 3: $x \in M$ iff x commutes with q.
Proof. [Proof of Claim 3]
It is clear that if $x \in M$, then $x * q = q * x$. Suppose x commutes with q, where $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $b = 0$, $c = 0$, and $ad = 1$, so $x \in M$. □

We have $(C \setminus \{0\}, \cdot) \cong (M, *)$, so $(C \setminus \{0\}, \cdot)$ is defined in $SL_2(C)$ using quantifier-free formulas with the parameter q.

5.4.2 Defining $(C, +, \cdot)$

To define the field $(C, +, \cdot)$, we represent an element $a \in C$ by a pair of matrices (x, y), where $x \in A$ and $y \in M$. The most natural choice for x is $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. If $a \neq 0$, then we let $y = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$, while if $a = 0$, then we let $y = I$. For $a = 1$, we choose (p, I), and for $a = 0$, we choose (I, I)—the same second component. Let T be the set of these pairs (x, y) chosen to represent elements of C.

Claim 4: $(x, y) \in T$ iff either $x = y = I$ (so (x, y) represents 0) or else $x \neq I$, $x \in A$, $y \in M$, and there is some z such that $z * z = y$, $z \in M$, and $z * p * z^{-1} = x$. (In the second case, (x, y) represents some $a \neq 0$, and there are just two possibilities for z, corresponding to the two possible square roots of a.)
Proof. [Proof of Claim 4]
First, suppose $(x, y) \in T$. If $x = I$, then y is also equal to I. Suppose $x \neq I$. For some $a \in C \setminus \{0\}$,
$x = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$. Clearly, $x \in A$ and $y \in M$. Let $b^2 = a$ and let $z = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}$. We have $z * z = y$, $z \in M$, and $z * p * z^{-1} = x$. □

For $(x,y) \in T$, we define addition and multiplication relations as follows:

1. $(x,y) \oplus (x',y') = (u,v)$ if $x * x' = u$ and $(u,v) \in T$,

2. $(x,y) \otimes (x',y') = (u,v)$ if either at least one of (x,y), (x',y') is (I,I) and $(u,v) = (I,I)$, or else neither of (x,y), (x',y') is (I,I), and then $y * y' = v$ and $(u,v) \in T$.

We have established the following.

Proposition 5.4.1. The field C is defined in $SL_2(C)$ using finitary existential formulas with parameters p and q. (The definition of T is existential, while the definitions of the operations are quantifier-free.)

Question 5.4.2. Are there formulas that, for all algebraically closed fields C of characteristic 0, define an effective interpretation of C in $SL_2(C)$? Are there existential formulas that serve?

Remarks. There are old model theoretic results, due to Poizat [Poi01], that give uniform definability of a copy of C in $SL_2(C)$ using elementary first order formulas without parameters. But we do not know the complexity of the defining formulas. We have a formula $\varphi(u,v)$, saying of the formulas D, $\pm\sim$, $\oplus$, and $\otimes$ that give our interpretation of C in $SL_2(C)$ that they give an field, not of characteristic 2, in which every element has a square root. For any (u,v) satisfying this formula, we get an infinite field $F_{(u,v)}$. The theory of $SL_2(C)$ is ω-stable. By an old result of Macintyre, $F_{(u,v)}$ must be algebraically closed. Poizat's results show that $F_{(u,v)}$ is isomorphic to C and that there are unique definable isomorphisms between the fields $F_{(u,v)}$ corresponding to pairs (u,v) that satisfy $\varphi(u,v)$. These isomorphisms are functorial. So, we have, not necessarily an *effective* interpretation without parameters, but one that is defined by elementary first order formulas. We do not know the complexity of the formulas.

Chapter 6

Cohesive powers

The inspiration for these investigations is Skolem's 1934 construction of a countable non-standard model of arithmetic [Sko34]. Skolem's construction can be described roughly as follows. For sets $X, Y \subseteq \mathbb{N}$, write $X \subseteq^* Y$ if $X \setminus Y$ is finite. First, fix an infinite set $C \subseteq \mathbb{N}$ that is *cohesive* for the collection of arithmetical sets: for every arithmetical $A \subseteq \mathbb{N}$, either $C \subseteq^* A$ or $C \subseteq^* \overline{A}$. Next, define an equivalence relation $=_C$ on the arithmetical functions $f\colon \mathbb{N} \to \mathbb{N}$ by $f =_C g$ if and only if $C \subseteq^* \{n : f(n) = g(n)\}$. Then define a structure on the $=_C$-equivalence classes $[f]$ by $[f] + [g] = [f + g]$, $[f] \times [g] = [f \times g]$ (where $f + g$ and $f \times g$ are computed pointwise), and $[f] < [g] \Leftrightarrow C \subseteq^* \{n : f(n) < g(n)\}$. Using the arithmetical cohesiveness of C, one then shows that this structure is elementarily equivalent to $(\mathbb{N}; +, \times, <)$. The structure is countable because there are only countably many arithmetical functions, and it has non-standard elements, such as the element represented by the identity function.

Think of Skolem's construction as a more effective analog of an ultrapower construction. Instead of building a structure from all functions $f\colon \mathbb{N} \to \mathbb{N}$, Skolem builds a structure from only the arithmetical functions f. The arithmetically cohesive set C plays the role of the ultrafilter. Feferman, Scott, and Tennenbaum [FST59] investigate the question of whether Skolem's construction can be made more effective by assuming that C is only *r-cohesive* (i.e., cohesive for the collection of computable sets) and by restricting to computable functions f. They answer the question negatively by showing that it is not even possible to obtain a model of Peano arithmetic in this way. Lerman [Ler70] investigates the situation further and shows that if one restricts to *cohesive* sets C (i.e., cohesive for the collection of c.e. sets) that are co-c.e. and to computable functions f, then the first-order theory of the structure obtained is exactly determined by the many-one degree of C. Additional results in this direction appear in [Hir75, HW75].

Dimitrov [Dim09] generalizes the effective ultrapower construction to arbitrary computable structures. These *cohesive powers* of computable structures are studied in [Dim08, DH16, DHMM14] in relation to the lattice of c.e. subspaces, modulo finite dimension, of a fixed computable infinite dimensional vector space over $\mathbb{Q}$. Here, we investigate a question dual to the question studied by Lerman. Lerman fixes a computable presentation of a computable structure (indeed, all computable presentations of the standard model of arithmetic are computably isomorphic) and studies the effect that the choice of the cohesive set has on the resulting cohesive power. Instead of fixing a computable presentation of a structure and varying the cohesive set, we fix a computably presentable structure and a cohesive set, and then we vary the structure's computable presentation. We focus on linear orders, with special emphasis on computable presentations of ω. We choose to work with linear orders because they are a good source of non-computably categorical structures and because the setting is simple enough to be able to completely describe certain cohesive powers up to isomorphism.

The results from this chapter are a joint work with Dimitrov, Harizanov, Morozov, Shafer, and Vatev, published in [DHM+19] and submitted for publication [DHM+20].

Our main results are the following, where ω, ζ, and η denote the respective order-types of the natural numbers, the integers, and the rationals.

- If C is cohesive and $\mathcal{L}$ is a computable copy of ω that is computably isomorphic to the standard presentation of ω (i.e., $\mathcal{L}$ has a computable successor function), then the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \zeta\eta$. (Corollary 6.4.6.)

- If C is co-c.e. and cohesive and $\mathcal{L}$ is a computable copy of ω, then the finite condensation of the cohesive power $\Pi_C\mathcal{L}$ has order-type $1 + \eta$. (Theorem 6.4.4. See Definition 6.3.3 for the definition of *finite condensation*.)

- If C is co-c.e. and cohesive, then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \eta$. (Corollary 6.5.2.)

- More generally, if C is co-c.e. and cohesive and $X \subseteq \mathbb{N} \setminus \{0\}$ is either a Σ_2^0 set or a Π_2^0 set, thought of as a set of finite order-types, then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \sigma(X \cup \{\omega + \zeta\eta + \omega^*\})$. Here ω^* denotes the reverse of ω, and σ denotes the shuffle operation of Definition 6.6.1. Furthermore, if X is finite and non-empty, then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \sigma(X)$. (Theorem 6.6.6.)

The above results provide many examples of pairs of isomorphic computable linear orders with non-elementarily equivalent cohesive powers. We also give examples of computable linear orders that are always isomorphic to their cohesive powers and examples of pairs of non-elementarily equivalent computable linear orders with isomorphic cohesive powers.

6.1 Basic properties

The content of this section is essentially from [DHM+20].

In this chapter $\mathbb{N}$ denotes the natural numbers, and ω denotes its order-type when thought of as a linear order. The function $\langle \cdot,\cdot \rangle\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the usual computable bijective pairing function, and π_0 and π_1 are the associated projection functions. For $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, $X{\upharpoonright}n$ denotes the set $X \cap \{0, 1, \ldots, n-1\}$. Often we consider expressions of the form $\lim_{n\in C} f(n)$, $\limsup_{n\in C} f(n)$, $\liminf_{n\in C} f(n)$, etc., where $f\colon \mathbb{N} \to \mathbb{N}$ is some function and $C \subseteq \mathbb{N}$ is an infinite set. For this, let $n_0 < n_1 < n_2 < \cdots$ be the elements of C listed in increasing order. Then $\lim_{n\in C} f(n)$ means $\lim_{i\to\infty} f(n_i)$, and $\limsup_{n\in C} f(n)$ and $\liminf_{n\in C} f(n)$ are interpreted similarly. Notice that for functions $f\colon \mathbb{N} \to \mathbb{N}$, $\lim_{n\in C} f(n) = \infty$ if and only if $\liminf_{n\in C} f(n) = \infty$.

For a partial computable function φ, $\varphi(n)\downarrow$ means that φ halts on input n and produces an output, and $\varphi(n)\uparrow$ means that φ does not halt on input n. The notation $\varphi \simeq \psi$ means that φ and ψ are equal partial functions: for every n, either $\varphi(n)\downarrow = \psi(n)\downarrow$ or both $\varphi(n)\uparrow$ and $\psi(n)\uparrow$. We also use the $\simeq$ notation to define one partial computable function in terms of another. As usual, $\varphi_{e,s}(n)$ denotes the result (if any) of running φ_e on input n for s computational steps.

Definition 6.1.1. An infinite set $C \subseteq \mathbb{N}$ is *cohesive* if for every c.e. set W, either $C \subseteq^* W$ or $C \subseteq^* \overline{W}$.

Notice that if C is cohesive and X is either c.e. or co-c.e., then $C \cap X$ being infinite implies that $C \subseteq^* X$. We use quantifiers $\forall^\infty n$ and $\exists^\infty n$ as abbreviations for 'for almost every n' and 'there are infinitely many n'. So for example, $(\forall^\infty n \in C)(n \in X)$ means $C \subseteq^* X$.

Definition 6.1.2 ([Dim09])**.** Let $\mathfrak{L}$ be a computable language. Let $\mathcal{A}$ be a computable $\mathfrak{L}$-structure with non-empty domain $A \subseteq \mathbb{N}$. Let $C \subseteq \mathbb{N}$ be cohesive. The *cohesive power of* $\mathcal{A}$ *over* C, denoted $\Pi_C \mathcal{A}$, is the $\mathfrak{L}$-structure $\mathcal{B}$ defined as follows.

- Let $D = \{\varphi : \varphi\colon \mathbb{N} \to A \text{ is partial computable and } C \subseteq^* dom(\varphi)\}$.

- For $\varphi, \psi \in D$, let $\varphi =_C \psi$ denote $C \subseteq^* \{x : \varphi(x)\downarrow = \psi(x)\downarrow\}$. The relation $=_C$ is an equivalence relation on D. Let $[\varphi]$ denote the equivalence class of $\varphi \in D$ with respect to $=_C$.

- The domain of $\mathcal{B}$ is the set $B = \{[\varphi] : \varphi \in D\}$.

- Let R be an n-ary predicate symbol of $\mathfrak{L}$. For $[\varphi_0], \dots, [\varphi_{n-1}] \in B$, define $R^{\mathcal{B}}([\varphi_0], \dots, [\varphi_{n-1}])$ by

$$R^{\mathcal{B}}([\varphi_0], \dots, [\varphi_{n-1}]) \Leftrightarrow C \subseteq^* \{x : (\forall i < n)\varphi_i(x)\downarrow \wedge R^{\mathcal{A}}(\varphi_0(x), \dots, \varphi_{n-1}(x))\}.$$

- Let f be an n-ary function symbol of $\mathfrak{L}$. For $[\varphi_0], \dots, [\varphi_{n-1}] \in B$, let ψ be the partial computable function defined by

$$\psi(x) \simeq f^{\mathcal{A}}(\varphi_0(x), \dots, \varphi_{n-1}(x)),$$

and notice that $C \subseteq^* dom(\psi)$ because $C \subseteq^* dom(\varphi_i)$ for each $i < n$. Define $f^{\mathcal{B}}$ by $f^{\mathcal{B}}([\varphi_0], \dots, [\varphi_{n-1}]) = [\psi]$.

- Let c be a constant symbol of $\mathfrak{L}$. Let ψ be the total computable function with constant value $c^{\mathcal{A}}$, and define $c^{\mathcal{B}} = [\psi]$.

We often consider cohesive powers of computable structures by co-c.e. cohesive sets. The co-c.e. cohesive sets are exactly the complements of the *maximal* sets, which are the co-atoms of the lattice of c.e. sets modulo finite difference. Such sets exist by a well-known theorem of Friedberg (see [Soa87] Theorem X.3.3). Cohesive powers are intended to be effective analogs of ultrapowers, so in light of this analogy, it makes sense to impose effectivity on the cohesive set, which plays the role of the ultrafilter, as well as on the base structure itself. Technically, it helps to be able to learn what numbers are not in the cohesive set C when building a computable structure $\mathcal{A}$ so as to influence $\Pi_C \mathcal{A}$ in a particular way. Cohesive powers by co-c.e. cohesive sets also have the helpful property that every member of the cohesive power has a total computable representative. Let $\mathcal{A}$ be a computable structure with non-empty domain A, and fix an element $a_0 \in A$. Suppose that C is co-c.e. and cohesive, and let $\varphi\colon \mathbb{N} \to A$ be a partial computable function with $C \subseteq^* dom(\varphi)$. Let N be such that $(\forall n > N)(n \in C \to \varphi(n)\downarrow)$. Define a total computable $f\colon \mathbb{N} \to A$ as follows. If $n \leq N$, then output $f(n) = a_0$. If $n > N$, then simultaneously run $\varphi(n)$ and enumerate the complement $\overline{C}$ of C. Either $\varphi(n)\downarrow$, $n \in \overline{C}$, or both. If $\varphi(n)$ halts before n is enumerated into $\overline{C}$, then output $f(n) = \varphi(n)$; and if n is enumerated into $\overline{C}$ before $\varphi(n)$ halts, then output $f(n) = a_0$. This f is total and satisfies $f =_C \varphi$.

A restricted form of Łoś's theorem holds for cohesive powers. If $\mathcal{A}$ is a computable structure, C is a cohesive set, and Φ is a Π_3 sentence, then $\Pi_C \mathcal{A} \models \Phi$ implies $\mathcal{A} \models \Phi$. In general, this version of Łoś's theorem for cohesive powers is the best possible. In Sections 6.4, 6.5, and 6.6, we see several examples of computable linear orders $\mathcal{L}$ where the Σ_3^0 sentence "there is an element with no immediate successor" is true of some cohesive power of $\mathcal{L}$ but not true of $\mathcal{L}$.

Theorem 6.1.3 ([Dim09])**.** Let $\mathcal{A}$ be a computable structure, and let C be a cohesive set.

(1) Let $t(v_0, \dots, v_{n-1})$ be a term, where all variables are displayed. Let $[\varphi_0], \dots, [\varphi_{n-1}] \in |\Pi_C \mathcal{A}|$. Let ψ be the partial computable function

$$\psi(x) \simeq t^{\mathcal{A}}(\varphi_0(x), \dots, \varphi_{n-1}(x)).$$

Then $t^{\Pi_C \mathcal{A}}([\varphi_0], \dots, [\varphi_{n-1}]) = [\psi]$.

(2) Let $\Phi(v_0, \ldots, v_{n-1})$ be a Boolean combination of Σ^0_1 and Π^0_1 formulas, with all free variables displayed. For any $[\varphi_0], \ldots, [\varphi_{n-1}] \in |\Pi_C \mathcal{A}|$,

$$\Pi_C \mathcal{A} \vDash \Phi([\varphi_0], \ldots, [\varphi_{n-1}]) \quad \Leftrightarrow \quad C \subseteq^* l\{x : (\forall i < n)\varphi_i(x)\downarrow \wedge \\ \mathcal{A} \vDash \Phi(\varphi_0(x), \ldots, \varphi_{n-1}(x))\}.$$

(3) If Φ is a Π^0_2 sentence or a Σ^0_2 sentence, then $\Pi_C \mathcal{A} \vDash \Phi$ if and only if $\mathcal{A} \vDash \Phi$.

(4) If Φ is a Π_3 sentence and $\Pi_C \mathcal{A} \vDash \Phi$, then $\mathcal{A} \vDash \Phi$.

As with structures and their ultrapowers, a computable structure $\mathcal{A}$ always naturally embeds into its cohesive powers. For $a \in A$, let ψ_a be the total computable function with constant value a. Then for any cohesive set C, the map $a \mapsto [\psi_a]$ embeds $\mathcal{A}$ into $\Pi_C \mathcal{A}$. This map is called the *canonical embedding* of $\mathcal{A}$ into $\Pi_C \mathcal{A}$. If $\mathcal{A}$ is finite and C is cohesive, then every partial computable function $\varphi\colon \mathbb{N} \to |\mathcal{A}|$ with $C \subseteq^* dom(\varphi)$ is eventually constant on C. In this case, every element of $\Pi_C \mathcal{A}$ is in the range of the canonical embedding, and therefore $\mathcal{A} \cong \Pi_C \mathcal{A}$. If $\mathcal{A}$ is an infinite computable structure, then every cohesive power $\Pi_C \mathcal{A}$ is countably infinite: infinite because $\mathcal{A}$ embeds into $\Pi_C \mathcal{A}$, and countable because the elements of $\Pi_C \mathcal{A}$ are represented by partial computable functions. See [Dim09] for further details.

Computable structures that are computably isomorphic have isomorphic cohesive powers. This fact essentially appears in [Dim09], but we include a proof here for reference.

Theorem 6.1.4. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be computable $\mathfrak{L}$-structures that are computably isomorphic, and let C be cohesive. Then $\Pi_C \mathcal{A}_0 \cong \Pi_C \mathcal{A}_1$.

Proof. For $i \in \{0, 1\}$, let $\mathcal{B}_i = \Pi_C \mathcal{A}_i$ and denote elements of $\mathcal{B}_i$ by $[\varphi]_{\mathcal{B}_i}$. Let $f\colon |\mathcal{A}_0| \to |\mathcal{A}_1|$ be a computable isomorphism. Define a function $F\colon |\mathcal{B}_0| \to |\mathcal{B}_1|$ by $F([\varphi]_{\mathcal{B}_0}) = [f \circ \varphi]_{\mathcal{B}_1}$. If $\varphi\colon \mathbb{N} \to |\mathcal{A}_0|$ is partial computable with $C \subseteq^* dom(\varphi)$, then $f \circ \varphi\colon \mathbb{N} \to |\mathcal{A}_1|$ is partial computable with $C \subseteq^* dom(f \circ \varphi)$. Furthermore, if $\varphi =_C \psi$, then $f \circ \varphi =_C f \circ \psi$. Thus F is well-defined.

To see that F is injective, suppose that $F([\varphi]_{\mathcal{B}_0}) = F([\psi]_{\mathcal{B}_0})$. Then $[f \circ \varphi]_{\mathcal{B}_1} = [f \circ \psi]_{\mathcal{B}_1}$, so $f \circ \varphi =_C f \circ \psi$. The function f is a bijection, so therefore also $\varphi =_C \psi$. Thus $[\varphi]_{\mathcal{B}_0} = [\psi]_{\mathcal{B}_0}$.

To see that F is surjective, consider $[\varphi]_{\mathcal{B}_1}$. The function $f\colon|\mathcal{A}_0| \to |\mathcal{A}_1|$ is a computable bijection between computable sets $|\mathcal{A}_0|$ and $|\mathcal{A}_1|$, so its inverse $f^{-1}\colon|\mathcal{A}_1| \to |\mathcal{A}_0|$ is also computable. The function $f^{-1} \circ \varphi\colon \mathbb{N} \to |\mathcal{A}_0|$ is thus partial computable with $C \subseteq^* dom(f^{-1} \circ \varphi)$, and $F([f^{-1} \circ \varphi]_{\mathcal{B}_0}) = [f \circ f^{-1} \circ \varphi]_{\mathcal{B}_1} = [\varphi]_{\mathcal{B}_1}$.

Let R be an n-ary relation symbol of $\mathfrak{L}$, and let $[\varphi_0], \dots, [\varphi_{n-1}] \in |\mathcal{B}_0|$. The function f is an isomorphism, so for any x, if $(\forall i < n)\varphi_i(x)\downarrow$, then

$$R^{\mathcal{A}_0}(\varphi_0(x), \dots, \varphi_{n-1}(x)) \iff R^{\mathcal{A}_1}(f(\varphi_0(x)), \dots, f(\varphi_{n-1}(x))).$$

Therefore

$$C \subseteq^* \{x : (\forall i < n)\varphi_i(x)\downarrow \wedge R^{\mathcal{A}_0}(\varphi_0(x), \dots, \varphi_{n-1}(x))\}$$

if and only if

$$C \subseteq^* \{x : (\forall i < n)\varphi_i(x)\downarrow \wedge R^{\mathcal{A}_1}(f(\varphi_0(x)), \dots, f(\varphi_{n-1}(x)))\}.$$

Thus

$$R^{\mathcal{B}_0}([\varphi_0]_{\mathcal{B}_0}, \dots, [\varphi_{n-1}]_{\mathcal{B}_0}) \iff R^{\mathcal{B}_1}(F([\varphi_0]_{\mathcal{B}_0}), \dots, F([\varphi_{n-1}]_{\mathcal{B}_0})).$$

Let g be an n-ary function symbol of $\mathfrak{L}$, and let $[\varphi_0], \dots, [\varphi_{n-1}] \in |\mathcal{B}_0|$. The function f is an isomorphism, so for any x, if $(\forall i < n)\varphi_i(x)\downarrow$, then

$$f(g^{\mathcal{A}_0}(\varphi_0(x), \dots, \varphi_{n-1}(x))) = g^{\mathcal{A}_1}(f(\varphi_0(x)), \dots, f(\varphi_{n-1}(x))).$$

Let ψ and θ be the partial computable functions given by

$$\psi(x) \simeq g^{\mathcal{A}_0}(\varphi_0(x), \dots, \varphi_{n-1}(x))$$
$$\theta(x) \simeq g^{\mathcal{A}_1}(f(\varphi_0(x)), \dots, f(\varphi_{n-1}(x))).$$

As $C \subseteq^* dom(\varphi_i)$ for each $i < n$, we therefore have that $f \circ \psi =_C \theta$. Thus

$$F(g^{\mathcal{B}_0}([\varphi_0]_{\mathcal{B}_0}, \dots, [\varphi_{n-1}]_{\mathcal{B}_0})) = F([\psi]_{\mathcal{B}_0}) = [f \circ \psi]_{\mathcal{B}_1} = [\theta]_{\mathcal{B}_1} = g^{\mathcal{B}_1}(F([\varphi_0]_{\mathcal{B}_0}), \dots, F([\varphi_{n-1}]_{\mathcal{B}_0})).$$

Finally, if c is a constant symbol of $\mathfrak{L}$ and $[\varphi]_{\mathcal{B}_0} = c^{\mathcal{B}_0}$, it is easy to check that $F([\varphi]_{\mathcal{B}_0}) = c^{\mathcal{B}_1}$. Therefore $F\colon|\mathcal{B}_0| \to |\mathcal{B}_1|$ is an isomorphism witnessing that $\mathcal{B}_0 \cong \mathcal{B}_1$. □

Recall that a computable structure $\mathcal{A}$ is called *computably categorical* if every computable structure that is isomorphic to $\mathcal{A}$ is isomorphic to $\mathcal{A}$ via a computable isomorphism. It follows from Theorem 6.1.4 that if $\mathcal{A}$ is a computably categorical computable structure and C is cohesive, then $\Pi_C\mathcal{A} \cong \Pi_C\mathcal{B}$ whenever $\mathcal{B}$ is a computable structure isomorphic to $\mathcal{A}$.

Corollary 6.1.5. Let $\mathcal{A}$ be a computably categorical computable structure, let $\mathcal{B}$ be a computable structure isomorphic to $\mathcal{A}$, and let C be cohesive. Then $\Pi_C\mathcal{A} \cong \Pi_C\mathcal{B}$.

In Theorem 6.1.4, it is essential that the two structures are isomorphic via a computable isomorphism. In the next section we present a construction with two isomorphic structures which cohesive powers are not isomorphic. In Sections 6.4, 6.5, and 6.6, we see many examples of pairs of computable linear orders that are isomorphic (but not computably isomorphic) to ω with non-elementarily equivalent cohesive powers.

6.2 Non-Isomorphic Cohesive Powers of Isomorphic Structures

The content of this section is from [DHM+19].

Theorem 6.2.1. For every co-maximal set $C \subseteq \omega$ there exist two isomorphic computable structures $\mathcal{A}$ and $\mathcal{B}$ such the cohesive powers $\prod_C \mathcal{A}$ and $\prod_C \mathcal{B}$ are not isomorphic.

Proof. Note that it suffices to prove the theorem for an arbitrary co-maximal set consisting of even numbers only. Indeed, if C is an arbitrary co-maximal set, then $C_1 = \{2s \mid s \in C\}$ is also a co-maximal set, and for any computable structure $\mathcal{M}$, we have $\prod_C \mathcal{M} \cong \prod_{C_1} \mathcal{M}$. Then, if $\mathcal{M}_0$ and $\mathcal{M}_1$ are isomorphic computable structures such that $\prod_{C_1} \mathcal{M}_0 \ncong \prod_{C_1} \mathcal{M}_1$, then $\prod_C \mathcal{M}_0 \ncong \prod_C \mathcal{M}_1$.

Let $S = \{2s \mid s \in \omega\}$. Let $A \subseteq S$ be such that $A_1 = S \setminus A$ is infinite and c.e. For every such A we will define a computable structure $\mathcal{M}_A$ with a single ternary relation.

Let $F = \{4s+1 \mid s \in \omega\}$ and $B = \{4s+3 \mid s \in \omega\}$. Fix a computable bijection f from the set $\{\langle i,j\rangle \in S \mid i < j\}$ onto F. Let also b be a computable bijection from the set $\{\langle j,i\rangle \in S \mid i < j \wedge (i \in A_1 \vee j \in A_1)\}$ onto B. For the function

f, we write f_{ij} instead of $f(i,j)$ and similarly for the function b. Define a ternary relation P as follows:

$$\begin{aligned} P &= \{(x, f_{xy}, y) \mid x, y \in S \wedge x < y\} \cup \\ &\quad \{(y, b_{yx}, x) \mid x, y \in S \wedge x < y \wedge (x \in A_1 \vee y \in A_1)\}. \end{aligned}$$

Finally, let $\mathcal{M}_A = (\omega; P)$. Informally, we can view the triples x, w, y with the property $P(x,w,y)$ as labelled arrows (e.g., $x \xrightarrow{w} y$). We start with a structure consisting of the set $S \cup F$ with arrows $i \xrightarrow{f_{ij}} j$, that connect i with j for all $i, j \in S$ such that $i < j$. These arrows can be viewed as a way of redefining the natural ordering $<$ on S. Elements of S can be thought of as "stem elements" and elements of F can be thought of as "forward witnesses." Next, we start enumerating the c.e. set $A_1 = S \smallsetminus A$. At every stage a new element k is enumerated into A_1, we add new arrows together with appropriate elements from B, the "backward witnesses," which intend to exclude k from the initial ordering on S. More precisely, we add arrows $k \xrightarrow{b_{ki}} i$ for all i, k with $i < k$, and arrows $j \xrightarrow{b_{jk}} k$, for each j, k with $j > k$. Eventually, exactly the elements of A_1 will be excluded from the ordering, and the final ordering will be an ordering on the set A.

In the resulting structure, every element $x \in A_1$ is connected with every element $y \in S$ such that $x \neq y$ with exactly two arrows: $x \xrightarrow{w} y$ and $y \xrightarrow{w_1} x$. If $x, y \in A$ are such that $x \neq y$ then they are connected with arrows of the type $x \xrightarrow{w} y$ exactly when $x < y$. In other words, the formula

$$\Phi(x,y) = \exists w P(x,w,y) \wedge \neg \exists w_1 P(y, w_1, x)$$

will be satisfied by exactly those $x, y \in A$ such that $x < y$. The formula Φ will not be satisfied by any pair (x,y) for which at least one of x or y has been excluded.

The following properties of the structure $\mathcal{M}_A$ follow immediately from the definition above.

(1) For every w there is at most one pair x, y such that $P(x,w,y)$.

(2) If $x \in S \smallsetminus A$, then for any $y \in S$, $y \neq x$ there is a unique w_1 such that $P(x, w_1, y)$ and a unique w_2 such that $P(y, w_2, x)$.

(3) If $x, y \in A$, then $x < y \Leftrightarrow \exists w P(x,w,y)$.

(4) $\mathcal{M}_A$ is computable.

To prove (4) note that the relation P is computable because

$$P(x,z,y) \Leftrightarrow x, y \in S \wedge \left[(x < y \wedge z = f_{xy}) \vee (x > y \wedge z \in B \wedge b^{-1}(z) = (x,y))\right].$$

(5) Let D, $E \subset S$ be infinite and such that $S \smallsetminus D$ and $S \smallsetminus E$ are infinite and c.e. Then $\mathcal{M}_D \cong \mathcal{M}_E$.

Since D and E are infinite, the orders $(D, <)$ and $(E, <)$, where $<$ is the natural order, are isomorphic to $\mathbb{N}$. The isomorphism between these orders, extended by any bijection between $S \smallsetminus D$ and $S \smallsetminus E$, has a unique natural extension to a map from the domain of $\mathcal{M}_D$ to the domain of $\mathcal{M}_E$. That is, the arrows in $\mathcal{M}_D$ (the elements of F and B) can be uniquely mapped to corresponding arrows in $\mathcal{M}_E$.

To continue with the proof, we let

$$\Theta(x) = (\exists t)\,[\Phi(x,t) \vee \Phi(t,x)]\,.$$

The formula $\Theta(x)$ defines the set A in $\mathcal{M}_A$.

For any structure $\mathcal{M} = (M, P)$ in the language with one ternary predicate symbol we will use the following notation:

$L_{\mathcal{M}} = \{x \in M | \mathcal{M} \vDash \Theta(x)\}$, and $<_{L_{\mathcal{M}}} = \{(x,y) \in M \times M | \mathcal{M} \vDash \Phi(x,y)\}$.

Fix $A \subseteq S$ such that $S \smallsetminus A$ is infinite and c.e.

It follows from the discussion above that the formula $\Phi(x,y)$ defines in $\mathcal{M}_A$ the restriction of the natural order $<$ to A. Clearly, $(L_{\mathcal{M}_A}, <_{L_{\mathcal{M}_A}})$ has order type ω.

Let $\mathcal{M}^\sharp_A = \prod_C \mathcal{M}_A$. For partial computable functions f and g such that $[f], [g] \in dom(\mathcal{M}^\sharp_A)$ we have:

(i) $\mathcal{M}^\sharp_A \vDash \Phi([f],[g]) \Leftrightarrow C \subseteq^* \{i | (f(i) \in A) \wedge (g(i) \in A) \wedge (f(i) < g(i))\}$

(ii) $L_{\mathcal{M}^\sharp_A} = \{[f] \in \mathcal{M}^\sharp_A | \; f(C) \subseteq^* A\}$ and $(L_{\mathcal{M}^\sharp_A}, <_{L_{\mathcal{M}^\sharp_A}})$ is a linear order.

Note that (i) follows from Theorem 6.1.3, part (2), since $\Phi(x,y)$ is a Boolean combination of Σ^0_1 and Π^0_1 formulas.

For the proof of (ii) notice that for any $[f] \in \mathcal{M}^\sharp_A$ we have either $C \subseteq^* \{i | f(i) \in A\}$ or $C \subseteq^* \{i | f(i) \in \omega \smallsetminus A\}$ because C is cohesive and $\omega \smallsetminus A$ is c.e. Since

$$[f] \in L_{\mathcal{M}^\sharp_A} \Leftrightarrow (\exists x)\,[\Phi([f],x) \vee \Phi(x,[f])]\,,$$

the equivalence in part (i) implies that $L_{\mathcal{M}^\sharp_A} = \{[f] \in \mathcal{M}^\sharp_A | \; f(C) \subseteq^* A\}$. It is easy to show that the relation $<_{L_{\mathcal{M}^\sharp_A}}$ is a linear order on $L_{\mathcal{M}^\sharp_A}$.

For any $a \in A$ let $f_a(i) = a$ for all $i \in \omega$. We will call the element $[f_a]$ in $\mathcal{M}^\sharp_A$ a constant in $\mathcal{M}^\sharp_A$.

(6) The set of constants $\{[f_a] | a \in A\}$ in the structure $\mathcal{M}^\sharp_A$ forms an initial segment of $(L_{\mathcal{M}^\sharp_A}, <_{L_{\mathcal{M}^\sharp_A}})$ of order type ω.

Clearly, if $a_0, a_1 \in A$, then $\Phi([f_{a_0}], [f_{a_1}])$ if and only if $a_0 < a_1$. Therefore, $\{[f_a] | a \in A\}$ is an ordered set of type ω. It remains to check that $\{[f_a] | a \in A\}$ is an initial segment. Suppose $[f] \in \mathcal{M}^\sharp_A$, and $a \in A$ are such that $\mathcal{M}^\sharp_A \models \Phi([f], [f_a])$. Then

$$C \subseteq^* \{i | \mathcal{M}_A \models \Phi(f(i), a)\} = \{i | f(i) \in A \wedge f(i) < a\} = \bigcup_{k \in A \wedge k < a} \{i | f(i) = k\}.$$

The last expression is a union of a finite family of mutually disjoint c.e. sets. Since C is cohesive, there exists a $k \in A$ such that $C \subseteq^* \{i | f(i) = k\}$, which means that $[f] = [f_k]$ is a constant.

We now define the following Σ^0_3 sentence

$$\Psi = (\exists x) [\Theta(x) \wedge (\forall y) [\Theta(y) \Rightarrow \Phi(y, x)]].$$

The intended interpretation of Ψ is that when $\Phi(x, t)$ defines a linear order $(L_\mathcal{M}, <_{L_\mathcal{M}})$, then the order has a greatest element. Note that $\mathcal{M}_A \models \neg\Psi$. This is because $\left(L_{\mathcal{M}_A}, <_{L_{\mathcal{M}_A}}\right)$ has order type ω and hence has no greatest element.

Before we continue with the proof we recall Proposition 2.1 from [Ler70].

Proposition 6.2.2. (Lerman [Ler70]) Let R be a co-r-maximal set, and let f be a computable function such that $f(R) \cap R$ is infinite. Then the restriction $f \upharpoonright R$ differs from the identity function only finitely.

We now fix a co-maximal (hence co-r-maximal) set $C \subseteq S$ and an infinite co-infinite computable set $D \subseteq S$. By property (5) above, we have $\mathcal{M}_C \cong \mathcal{M}_D$. Let $\mathcal{M}^\sharp_C = \prod_C \mathcal{M}_C$ and $\mathcal{M}^\sharp_D = \prod_C \mathcal{M}_D$.

It is not hard to show that, since C is co-maximal, for every partial computable function φ for which $C \subseteq^* dom(\varphi)$, there is a computable function f_φ such that $[\varphi] = [f_\varphi]$ (see [DHMM14]).

To finish the proof we will establish the following facts:

(7) $\mathcal{M}^\sharp_C \models \Psi$

(8) $\mathcal{M}^\sharp_D \models \neg\Psi$

To prove (7) recall that $L_{\mathcal{M}^\sharp_C} = \{[f] \in \mathcal{M}^\sharp_C | \ f(C) \subseteq^* C\}$. By Proposition 6.2.2 if $[f] \in \mathcal{M}^\sharp_C$ is such that $f(C) \subseteq^* C$ and $f(C)$ is infinite, then $[f] = [id]$. If $f(C)$ is finite, then f is eventually equivalent to a constant, because C is cohesive. Therefore, $L_{\mathcal{M}^\sharp_C} = \{[f_c] | c \in C\} \cup \{[id]\}$. It is easy to see that if $c \in C$, then $\Phi([f_c], [id])$. Thus, $(L_{\mathcal{M}^\sharp_C}, <_{L_{\mathcal{M}^\sharp_C}})$ has order type $\omega + 1$ with the greatest element $[id]$. Therefore, $\mathcal{M}^\sharp_C \models \Psi$.

To prove (8), let $D = \{d_0 < d_1 < \cdots\}$. The function g defined as $g(d_i) = d_{i+1}$ is computable. Suppose that $\mathcal{M}^\sharp_D \models \Psi$ and let $[f]$ be the greatest element in $(L_{\mathcal{M}^\sharp_D}, \prec_{L_{\mathcal{M}^\sharp_D}})$. Since $[f] \prec_{L_{\mathcal{M}^\sharp_D}} [g \circ f]$, it follows that $\mathcal{M}^*_D \models \neg\Psi$.

In conclusion, we defined computable isomorphic structures $\mathcal{M}_C$ and $\mathcal{M}_D$ such that $\prod_C \mathcal{M}_C$ and $\prod_C \mathcal{M}_D$ are not even elementary equivalent. The structure $\mathcal{M}_C$ also provides a sharp bound for the fundamental theorem of cohesive powers. Namely, for the Σ^0_3 sentence Ψ, $\mathcal{M}_C \models \neg\Psi$ but $\prod_C \mathcal{M}_C \models \Psi$. □

6.3 Linear orders and their cohesive powers

The content in this section, Section 6.4, Section 6.5 and Section 6.6 is from [DHM+20].

We investigate the cohesive powers of computable linear orders, with special attention to computable linear orders of type ω. A *linear order* $\mathcal{L} = (L, \prec)$ consists of a non-empty set L equipped with a binary relation $\prec$ satisfying the following axioms.

- $\forall x \forall y \, (x \prec y \;\rightarrow\; y \nprec x)$
- $\forall x \forall y \forall z \, [(x \prec y \wedge y \prec z) \;\rightarrow\; x \prec z]$
- $\forall x \forall y \, (x \prec y \vee x = y \vee y \prec x)$

Additionally, a linear order $\mathcal{L}$ is *dense* if $\forall x \forall y \exists z \, (x \prec y \rightarrow x \prec z \prec y)$ and *has no endpoints* if $\forall x \exists y \exists z \, (y \prec x \prec z)$. Rosenstein's book [Ros82] is an excellent reference for linear orders.

For a linear order $\mathcal{L} = (L, \prec)$, we use the usual interval notation $(a, b)_{\mathcal{L}} = \{x \in L : a \prec x \prec b\}$ and $[a, b]_{\mathcal{L}} = \{x \in L : a \preceq x \preceq b\}$ to denote open and closed intervals of $\mathcal{L}$. Sometimes it is convenient to allow $b \preceq a$ in this notation, in which case, for example, $(a, b)_{\mathcal{L}} = \varnothing$. The notation $|(a, b)_{\mathcal{L}}|$ denotes the cardinality of the interval $(a, b)_{\mathcal{L}}$. The notations $\min_\prec\{a, b\}$ and $\max_\prec\{a, b\}$ denote the minimum and maximum of a and b with respect to $\prec$.

As is customary, ω denotes the order-type of $(\mathbb{N}, <)$, ζ denotes the order-type of $(\mathbb{Z}, <)$, and η denotes the order-type of $(\mathbb{Q}, <)$. That is, ω, ζ, and η denote the respective order-types of the natural numbers, the integers, and the rationals, each with their usual order. We refer to $(\mathbb{N}, <)$, $(\mathbb{Z}, <)$, and $(\mathbb{Q}, <)$ as the *standard presentations* of ω, ζ, and η, respectively. Recall that every countable dense linear order without endpoints has order-type η (see [Ros82] Theorem 2.8). Furthermore, every computable countable dense linear order without endpoints is computably isomorphic to $\mathbb{Q}$ (see [Ros82] Exercise 16.4).

To help reason about order-types, we use the *sum*, *product*, and *reverse* of linear orders as well as *condensations* of linear orders.

Definition 6.3.1. Let $\mathcal{L}_0 = (L_0, \prec_{\mathcal{L}_0})$ and $\mathcal{L}_1 = (L_0, \prec_{\mathcal{L}_1})$ be linear orders.

- The *sum* $\mathcal{L}_0 + \mathcal{L}_1$ of $\mathcal{L}_0$ and $\mathcal{L}_1$ is the linear order $\mathcal{S} = (S, \prec_{\mathcal{S}})$, where $S = (\{0\} \times L_0) \cup (\{1\} \times L_1)$ and

$$(i, x) \prec_{\mathcal{S}} (j, y) \quad \text{if and only if} \quad (i < j) \lor (i = j \land x \prec_{\mathcal{L}_i} y).$$

- The *product* $\mathcal{L}_0\mathcal{L}_1$ of $\mathcal{L}_0$ and $\mathcal{L}_1$ is the linear order $\mathcal{P} = (P, \prec_{\mathcal{P}})$, where $P = L_1 \times L_0$ and

$$(x, a) \prec_{\mathcal{P}} (y, b) \quad \text{if and only if} \quad (x \prec_{\mathcal{L}_1} y) \lor (x = y \land a \prec_{\mathcal{L}_0} b).$$

  Note that, by (fairly entrenched) convention, $\mathcal{L}_0\mathcal{L}_1$ is given by the product order on $L_1 \times L_0$, not on $L_0 \times L_1$.

- The *reverse* $\mathcal{L}_0^*$ of $\mathcal{L}_0$ is the linear order $\mathcal{R} = (R, \prec_{\mathcal{R}})$, where $R = L_0$ and $x \prec_{\mathcal{R}} y$ if and only if $y \prec_{\mathcal{L}_0} x$. We warn the reader that the $*$ in the notation $\mathcal{L}_0^*$ is unrelated to the $*$ in the notation $X \subseteq^* Y$.

If $\mathcal{L}_0$ and $\mathcal{L}_1$ are computable linear orders, then one may use the pairing function $\langle \cdot, \cdot \rangle$ to compute copies of $\mathcal{L}_0 + \mathcal{L}_1$ and $\mathcal{L}_0\mathcal{L}_1$. Clearly, if $\mathcal{L}$ is a computable linear order, then so is $\mathcal{L}^*$.

Definition 6.3.2. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a linear order. A *condensation* of $\mathcal{L}$ is any linear order $\mathcal{M} = (M, \prec_{\mathcal{M}})$ obtained by partitioning L into a collection of non-empty intervals M and, for intervals $I, J \in M$, defining $I \prec_{\mathcal{M}} J$ if and only if $(\forall a \in I)(\forall b \in J)(a \prec_{\mathcal{L}} b)$.

The most important condensation is the *finite condensation*.

Definition 6.3.3. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a linear order. For $x \in L$, let $\mathbf{c}_{\mathrm{F}}(x)$ denote the set of $y \in L$ for which there are only finitely many elements between x and y:

$$\mathbf{c}_{\mathrm{F}}(x) = \{y \in L : \text{the interval } [\min_{\prec_{\mathcal{L}}}\{x, y\}, \max_{\prec_{\mathcal{L}}}\{x, y\}]_{\mathcal{L}} \text{ in } \mathcal{L} \text{ is finite}\}.$$

The set $\mathbf{c}_{\mathrm{F}}(x)$ is always a non-empty interval, as $x \in \mathbf{c}_{\mathrm{F}}(x)$. The *finite condensation* $\mathbf{c}_{\mathrm{F}}(\mathcal{L})$ of $\mathcal{L}$ is the condensation obtained from the partition $\{\mathbf{c}_{\mathrm{F}}(x) : x \in L\}$.

For example, $\mathbf{c}_{\mathrm{F}}(\omega) \cong 1$, $\mathbf{c}_{\mathrm{F}}(\zeta) \cong 1$, $\mathbf{c}_{\mathrm{F}}(\eta) \cong \eta$, and $\mathbf{c}_{\mathrm{F}}(\omega + \zeta\eta) \cong 1 + \eta$. Notice that for an element x of a linear order $\mathcal{L}$, the order-type of $\mathbf{c}_{\mathrm{F}}(x)$ is always either finite, ω, ω^*, or ζ.

We often refer to the intervals that comprise a condensation of a linear order $\mathcal{L}$ as *blocks*. For the finite condensation of $\mathcal{L}$, a block is a maximal interval I such that for any two elements of I, there are only finitely many elements of $\mathcal{L}$ between them. For elements a and b of $\mathcal{L}$, we write $a \ll_{\mathcal{L}} b$ if the interval $(a, b)_{\mathcal{L}}$ (equivalently, the interval $[a, b]_{\mathcal{L}}$) in $\mathcal{L}$ is infinite. For $a <_{\mathcal{L}} b$, we have that $a \ll_{\mathcal{L}} b$ if and only if a and b are in different blocks. See [Ros82] Chapter 4 for more on condensations.

It is straightforward to directly verify that if $\mathcal{L}$ is a computable linear order and C is cohesive, then $\Pi_C\mathcal{L}$ is again a linear order. Furthermore, one may verify that if $\mathcal{L}$ is a computable dense linear order without endpoints, then $\Pi_C\mathcal{L}$ is again a dense linear order without endpoints. These two facts also follow from Theorem 6.1.3 because linear orders are described by Π^0_1 sentences, and dense linear orders without endpoints are described by Π^0_2 sentences.

The case of $\mathbb{Q} = (\mathbb{Q}, <)$ is curious and deserves a digression. We have seen that if $\mathcal{A}$ is a finite structure, then $\mathcal{A} \cong \Pi_C\mathcal{A}$ for every cohesive set C. For $\mathbb{Q}$, $\Pi_C\mathbb{Q}$ is a countable dense linear order without endpoints, and hence isomorphic to $\mathbb{Q}$, for every cohesive set C. Thus $\mathbb{Q}$ is an example of an infinite computable structure with $\mathbb{Q} \cong \Pi_C\mathbb{Q}$ for every cohesive set C. That $\mathbb{Q}$ is isomorphic to all of its cohesive powers is no accident. By combining Theorem 6.1.3 with the theory of *Fraïssé limits* (see [Hod93] Chapter 6, for example), we see that a uniformly locally finite ultrahomogeneous computable structure for a finite language is always isomorphic to all of its cohesive powers. Recall that a structure is *locally finite* if every finitely-generated substructure is finite and is *uniformly locally finite* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every substructure generated by at most n elements has cardinality at most $f(n)$. Notice that every structure for a finite relational language is uniformly locally finite. Also recall that a structure is *ultrahomogeneous* if every isomorphism between two finitely-generated substructures extends to an automorphism of the structure.

Proposition 6.3.4. Let $\mathcal{A}$ be an infinite uniformly locally finite ultrahomogeneous computable structure for a finite language, and let C be cohesive. Then $\mathcal{A} \cong \Pi_C\mathcal{A}$.

Proof. The structure $\mathcal{A}$ is ultrahomogeneous, so it is the Fraïssé limit of its *age* (i.e., the class of all finitely-generated substructures embeddable into $\mathcal{A}$). By [Hod93] Theorem 6.4.1 and its proof, the first-order theory of $\mathcal{A}$ is $\aleph_0$-categorical and is axiomatized by a set T of Π^0_2 sentences. Thus if $\mathcal{B}$ is any countable model of T, then $\mathcal{A} \cong \mathcal{B}$. We have that $\Pi_C\mathcal{A} \models T$ by Theorem 6.1.3, so $\mathcal{A} \cong \Pi_C\mathcal{A}$. □

Proposition 6.3.4 implies that if a uniformly locally finite computable structure for a finite language is a Fraïssé limit, then it is isomorphic to all of its cohesive powers. Thus computable presentations of the Rado graph and the countable atomless Boolean algebra are additional examples of computable structures that are isomorphic to all of their cohesive powers. Examples of this phenomenon that cannot be attributed to ultrahomogeneity appear in Sections 6.4 and 6.5.

Returning to linear orders, we recall the following well-known lemma stating that a strictly order-preserving surjection from one linear order onto another is necessarily an isomorphism.

Lemma 6.3.5. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ and $\mathcal{M} = (M, \prec_{\mathcal{M}})$ be linear orders. If $f\colon L \to M$ is surjective and satisfies $(\forall x, y \in L)[x \prec_{\mathcal{L}} y \;\rightarrow\; f(x) \prec_{\mathcal{M}} f(y)]$, then f is an isomorphism.

Proof. Such an f is injective. If $x \neq y$, then either $x \prec_{\mathcal{L}} y$ or $y \prec_{\mathcal{L}} x$, so either $f(x) \prec_{\mathcal{M}} f(y)$ or $f(y) \prec_{\mathcal{M}} f(x)$. In either case, $f(x) \neq f(y)$. Similarly, if $x \nprec_{\mathcal{L}} y$, then either $x = y$, in which case $f(x) = f(y)$; or $y \prec_{\mathcal{L}} x$, in which case $f(y) \prec_{\mathcal{M}} f(x)$. In either case, $f(x) \nprec_{\mathcal{M}} f(y)$. □

Cohesive powers commute with sums, products, and reverses.

Theorem 6.3.6. Let $\mathcal{L}_0 = (L_0, \prec_{\mathcal{L}_0})$ and $\mathcal{L}_1 = (L_1, \prec_{\mathcal{L}_1})$ be computable linear orders, and let C be cohesive. Then

(1) $\Pi_C(\mathcal{L}_0 + \mathcal{L}_1) \cong \Pi_C\mathcal{L}_0 + \Pi_C\mathcal{L}_1$,

(2) $\Pi_C(\mathcal{L}_0\mathcal{L}_1) \cong (\Pi_C\mathcal{L}_0)(\Pi_C\mathcal{L}_1)$, and

(3) $\Pi_C(\mathcal{L}_0^*) \cong (\Pi_C\mathcal{L}_0)^*$.

Proof. For $i \in \{0,1\}$, let $\mathcal{B}_i = \Pi_C \mathcal{L}_i$ and denote elements of $\mathcal{B}_i$ by $[\varphi]_{\mathcal{B}_i}$.

For (1), let $\mathcal{D} = \Pi_C(\mathcal{L}_0 + \mathcal{L}_1)$ and denote elements of $\mathcal{D}$ by $[\varphi]_{\mathcal{D}}$. Define an isomorphism $F\colon |\mathcal{D}| \to |\mathcal{B}_0 + \mathcal{B}_1|$ by

$$F([\varphi]_{\mathcal{D}}) = \begin{cases} (0, [\pi_1 \circ \varphi]_{\mathcal{B}_0}) & \text{if } (\forall^\infty n \in C)(\varphi(n) \in \{0\} \times L_0) \\ (1, [\pi_1 \circ \varphi]_{\mathcal{B}_1}) & \text{if } (\forall^\infty n \in C)(\varphi(n) \in \{1\} \times L_1). \end{cases}$$

The function F is well-defined because for any $[\varphi]_{\mathcal{D}}$, exactly one of the two options occurs by cohesiveness. To check that F is an isomorphism, it suffices to check that F is surjective and order-preserving. For surjectivity, consider an $(i, [\psi]_{\mathcal{B}_i}) \in |\mathcal{B}_0 + \mathcal{B}_1|$. Let φ be the partial computable function where $\forall n[\varphi(n) \simeq \langle i, \psi(n)\rangle]$. Then $(\forall^\infty n \in C)(\varphi(n) \in \{i\} \times L_i)$, so $[\varphi]_{\mathcal{D}} \in |\mathcal{D}|$ and $F([\varphi]_{\mathcal{D}}) = (i, [\psi]_{\mathcal{B}_i})$. For order-preserving, suppose that $[\varphi]_{\mathcal{D}}$ and $[\psi]_{\mathcal{D}}$ are members of $\mathcal{D}$ with $[\varphi]_{\mathcal{D}} \prec_{\mathcal{D}} [\psi]_{\mathcal{D}}$. By cohesiveness, either $(\forall^\infty n \in C)[\pi_0(\varphi(n)) < \pi_0(\psi(n))]$ or $(\forall^\infty n \in C)[\pi_0(\varphi(n)) = \pi_0(\psi(n))]$. In the first case, it must be that

$$F([\varphi]_{\mathcal{D}}) = (0, [\pi_1 \circ \varphi]_{\mathcal{B}_0}) \prec_{\mathcal{B}_0 + \mathcal{B}_1} (1, [\pi_1 \circ \psi]_{\mathcal{B}_1}) = F([\psi]_{\mathcal{D}}),$$

as desired. In the second case, let $i \in \{0,1\}$ be such that $(\forall^\infty n \in C)[\pi_0(\varphi(n)) = \pi_0(\psi(n)) = i]$. Then it must be that

$$(\forall^\infty n \in C)[\pi_1(\varphi(n)) \in L_i \wedge \pi_1(\psi(n)) \in L_i \wedge \pi_1(\varphi(n)) \prec_{\mathcal{L}_i} \pi_1(\psi(n))],$$

so

$$F([\varphi]_{\mathcal{D}}) = (i, [\pi_1 \circ \varphi]_{\mathcal{B}_i}) \prec_{\mathcal{B}_0 + \mathcal{B}_1} (i, [\pi_1 \circ \psi]_{\mathcal{B}_i}) = F([\psi]_{\mathcal{D}}),$$

as desired.

For (2), let $\mathcal{D} = \Pi_C(\mathcal{L}_0\mathcal{L}_1)$ and again denote elements of $\mathcal{D}$ by $[\varphi]_{\mathcal{D}}$. For such a $[\varphi]_{\mathcal{D}}$, we have that $(\forall^\infty n \in C)(\varphi(n) \in L_1 \times L_0)$. Define an isomorphism $F\colon |\mathcal{D}| \to |\mathcal{B}_0\mathcal{B}_1|$ by

$$F([\varphi]_{\mathcal{D}}) = ([\pi_0 \circ \varphi]_{\mathcal{B}_1}, [\pi_1 \circ \varphi]_{\mathcal{B}_0}) \in |\mathcal{B}_1| \times |\mathcal{B}_0|.$$

To show that F is an isomorphism, again it suffices to show that F is surjective and order-preserving. For surjectivity, consider a $([\psi_1]_{\mathcal{B}_1}, [\psi_0]_{\mathcal{B}_0}) \in |\mathcal{B}_1| \times |\mathcal{B}_0|$. Let φ be the partial computable function where $\forall n[\varphi(n) \simeq \langle \psi_1(n), \psi_0(n)\rangle]$. Then $(\forall^\infty n \in C)(\varphi(n) \in L_1 \times L_0)$, so $[\varphi]_{\mathcal{D}} \in |\mathcal{D}|$ and $F([\varphi]_{\mathcal{D}}) = ([\psi_1]_{\mathcal{B}_1}, [\psi_0]_{\mathcal{B}_0})$. For order-preserving, suppose that $[\varphi]_{\mathcal{D}}$ and $[\psi]_{\mathcal{D}}$ are members of $\mathcal{D}$ with $[\varphi]_{\mathcal{D}} \prec_{\mathcal{D}} [\psi]_{\mathcal{D}}$. Then $(\forall^\infty n \in C)(\varphi(n) \prec_{\mathcal{L}_0\mathcal{L}_1} \psi(n))$. By cohesiveness, either

- $(\forall^\infty n \in C)[\pi_0(\varphi(n)) \prec_{\mathcal{L}_1} \pi_0(\psi(n))]$ or
- $(\forall^\infty n \in C)[\pi_0(\varphi(n)) = \pi_0(\psi(n)) \wedge \pi_1(\varphi(n)) \prec_{\mathcal{L}_0} \pi_1(\psi(n))]$.

In the first case, $[\pi_0 \circ \varphi]_{\mathcal{B}_1} \prec_{\mathcal{B}_1} [\pi_0 \circ \psi]_{\mathcal{B}_1}$. In the second case, $[\pi_0 \circ \varphi]_{\mathcal{B}_1} = [\pi_0 \circ \psi]_{\mathcal{B}_1}$ and $[\pi_1 \circ \varphi]_{\mathcal{B}_0} \prec_{\mathcal{B}_0} [\pi_1 \circ \psi]_{\mathcal{B}_0}$. Thus in either case,

$$F([\varphi]_{\mathcal{D}}) = ([\pi_0 \circ \varphi]_{\mathcal{B}_1}, [\pi_1 \circ \varphi]_{\mathcal{B}_0}) \prec_{\mathcal{B}_0\mathcal{B}_1} ([\pi_0 \circ \psi]_{\mathcal{B}_1}, [\pi_1 \circ \psi]_{\mathcal{B}_0}) = F([\psi]_{\mathcal{D}}),$$

as desired.

For (3) let $\mathcal{D} = \Pi_C(\mathcal{L}_0^*)$ and again denote elements of $\mathcal{D}$ by $[\varphi]_{\mathcal{D}}$. Notice that $|\mathcal{L}_0^*| = |\mathcal{L}_0| = L_0$, and therefore that $|\mathcal{D}| = |\mathcal{B}_0^*|$. Thus the function $F\colon |\mathcal{D}| \to |\mathcal{B}_0^*|$ given by $F([\varphi]_{\mathcal{D}}) = [\varphi]_{\mathcal{B}_0^*}$ is well-defined and surjective. The function F is also order-preserving. If $[\varphi]_{\mathcal{D}} \prec_{\mathcal{D}} [\psi]_{\mathcal{D}}$, then $(\forall^\infty n \in C)[\varphi(n) \prec_{\mathcal{L}_0^*} \psi(n)]$. So $(\forall^\infty n \in C)[\psi(n) \prec_{\mathcal{L}_0} \varphi(n)]$. So $[\psi]_{\mathcal{B}_0} \prec_{\mathcal{B}_0} [\varphi]_{\mathcal{B}_0}$. So $[\varphi]_{\mathcal{B}_0^*} \prec_{\mathcal{B}_0^*} [\psi]_{\mathcal{B}_0^*}$. Thus F is an isomorphism. □

Sections 6.4, 6.5, and 6.6 concern calculating the order-types of cohesive powers of computable copies of ω. To do this, we must be able to determine when one element of a cohesive power is an immediate successor or immediate predecessor of another, and we must be able to determine when two elements of a cohesive power are in different blocks of its finite condensation.

In a cohesive power $\Pi_C\mathcal{L}$ of a computable linear order $\mathcal{L}$, $[\varphi]$ is the immediate successor of $[\psi]$ if and only if $\varphi(n)$ is the immediate successor of $\psi(n)$ for almost every $n \in C$. Therefore also $[\psi]$ is the immediate predecessor of $[\varphi]$ if and only if $\psi(n)$ is the immediate predecessor of $\varphi(n)$ for almost every $n \in C$.

Lemma 6.3.7. Let $\mathcal{L}$ be a computable linear order, let C be cohesive, and let $[\psi]$ and $[\varphi]$ be elements of $\Pi_C\mathcal{L}$. Then the following are equivalent.

(1) $[\varphi]$ is the $\prec_{\Pi_C\mathcal{L}}$-immediate successor of $[\psi]$.

(2) $(\forall^\infty n \in C)(\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n))$.

(3) $(\exists^\infty n \in C)(\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n))$.

Proof. Supposing $\varphi(n)\downarrow$ and $\psi(n)\downarrow$, that $\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$ is a Π^0_1 property of $\varphi(n)$ and $\psi(n)$. Thus by cohesiveness (and the fact that $C \subseteq^* \mathit{dom}(\varphi) \cap \mathit{dom}(\psi)$), it holds that $\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$ for infinitely many $n \in C$ if and only if it holds for almost every $n \in C$. Therefore (2) and (3) are equivalent.

For (3) $\Rightarrow$ (1), suppose that $[\varphi]$ is not the $\prec_{\Pi_C \mathcal{L}}$-immediate successor of $[\psi]$. If $[\varphi] \preceq_{\Pi_C \mathcal{L}} [\psi]$, then $(\forall^\infty n \in C)(\varphi(n) \preceq_{\mathcal{L}} \psi(n))$, so for almost every $n \in C$, $\varphi(n)$ is the not the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$. If $[\psi] \prec_{\Pi_C \mathcal{L}} [\varphi]$, then there is a $[\theta]$ with $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$. Therefore $(\forall^\infty n \in C)(\psi(n) \prec_{\mathcal{L}} \theta(n) \prec_{\mathcal{L}} \varphi(n))$, so again for almost every $n \in C$, $\varphi(n)$ is the not the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$.

For (1) $\Rightarrow$ (3), suppose that for almost every $n \in C$, $\varphi(n)$ is the not the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$. If $[\varphi] \preceq_{\Pi_C \mathcal{L}} [\psi]$, then $[\varphi]$ is not the $\prec_{\Pi_C \mathcal{L}}$-immediate successor of $[\psi]$, so we may assume that $[\psi] \prec_{\Pi_C \mathcal{L}} [\varphi]$. Thus $(\forall^\infty n \in C)(\psi(n) \prec_{\mathcal{L}} \varphi(n))$. Let θ be the partial computable function which on input n, searches for an x with $\psi(n) \prec_{\mathcal{L}} x \prec_{\mathcal{L}} \varphi(n)$ and outputs the first such x found. For almost every $n \in C$, $\psi(n) \prec_{\mathcal{L}} \varphi(n)$ but $\varphi(n)$ is not the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$. For such n, $\theta(n)$ is defined and satisfies $\psi(n) \prec_{\mathcal{L}} \theta(n) \prec_{\mathcal{L}} \varphi(n)$. Therefore $(\forall^\infty n \in C)(\psi(n) \prec_{\mathcal{L}} \theta(n) \prec_{\mathcal{L}} \varphi(n))$, so $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$. Thus $[\varphi]$ is not the $\prec_{\Pi_C \mathcal{L}}$-immediate successor of $[\psi]$. $\square$

Lemma 6.3.8. Let $\mathcal{L}$ be a computable linear order, let C be cohesive, and let $[\psi]$ and $[\varphi]$ be elements of $\Pi_C \mathcal{L}$. Then the following are equivalent.

(1) $[\psi] \lessapprox_{\Pi_C \mathcal{L}} [\varphi]$.

(2) $\lim_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$.

(3) $\limsup_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$.

Proof. Let $[\varphi], [\psi] \in |\Pi_C \mathcal{L}|$. For (1) $\Rightarrow$ (2), suppose that $[\psi] \lessapprox_{\Pi_C \mathcal{L}} [\varphi]$. Given k, let $[\theta_0], \dots, [\theta_{k-1}] \in |\Pi_C \mathcal{L}|$ be such that

$$[\psi] \prec_{\Pi_C \mathcal{L}} [\theta_0] \prec_{\Pi_C \mathcal{L}} \cdots \prec_{\Pi_C \mathcal{L}} [\theta_{k-1}] \prec_{\Pi_C \mathcal{L}} [\varphi].$$

Then

$$(\forall^\infty n \in C)[\psi(n) \prec_{\mathcal{L}} \theta_0(n) \prec_{\mathcal{L}} \cdots \prec_{\mathcal{L}} \theta_{k-1}(n) \prec_{\mathcal{L}} \varphi(n)].$$

Thus for almost every $n \in C$, we have that $|(\psi(n), \varphi(n))_{\mathcal{L}}| \geq k$.
So, $\lim_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$.

The implication (2) ⇒ (3) is immediate. For (3) ⇒ (1), suppose that $\limsup_{n\in C}|(\psi(n),\varphi(n))_{\mathcal{L}}| = \infty$. Given k, let $[\theta_0],\dots,[\theta_{k-1}] \in |\Pi_C\mathcal{L}|$. We show that there is a $[\widehat{\theta}]$ with $[\psi] \prec_{\Pi_C\mathcal{L}} [\widehat{\theta}] \prec_{\Pi_C\mathcal{L}} [\varphi]$ that is different from $[\theta_i]$ for each $i < k$. It follows that the interval $([\psi],[\varphi])_{\Pi_C\mathcal{L}}$ of $\Pi_C\mathcal{L}$ is infinite, so $[\psi] \lll_{\Pi_C\mathcal{L}} [\varphi]$. To compute $\widehat{\theta}(n)$, first wait for $\psi(n)$, $\varphi(n)$, and the $\theta_i(n)$ for $i < k$ to halt. Once these computations halt, search for an $x \in L$ with $\psi(n) \prec_{\mathcal{L}} x \prec_{\mathcal{L}} \varphi(n)$ such that $x \neq \theta_i(n)$ for all $i < k$. If there is such an x, let $\widehat{\theta}(x)$ be the first such x found. If $n \in C$ is sufficiently large and $|(\psi(n),\varphi(n))_{\mathcal{L}}| > k$, then $\psi(n)\downarrow$, $\varphi(n)\downarrow$, and $(\forall i < k)\theta_i(n)\downarrow$, and such an x is found. Therefore there are infinitely many $n \in C$ with n in the domains of ψ, φ, $\widehat{\theta}$, and the θ_i for $i < k$ such that $\psi(n) \prec_{\mathcal{L}} \widehat{\theta}(n) \prec_{\mathcal{L}} \varphi(n)$ and $(\forall i < k)(\widehat{\theta}(n) \neq \theta_i(n))$. By cohesiveness, this in fact occurs for almost every $n \in C$. Thus $[\psi] \prec_{\Pi_C\mathcal{L}} [\widehat{\theta}] \prec_{\Pi_C\mathcal{L}} [\varphi]$, but $(\forall i < k)([\widehat{\theta}] \neq [\theta_i])$. □

The finite condensation of a computable linear order by a co-c.e. cohesive set is always dense.

Theorem 6.3.9. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable linear order, and let C be co-c.e. and cohesive. Then $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L})$ is dense.

Proof. Let $[\varphi]$ and $[\psi]$ be elements of $\Pi_C\mathcal{L}$ with $[\psi] \lll_{\Pi_C\mathcal{L}} [\varphi]$. We partially compute a function $\theta\colon \mathbb{N} \to L$ so that $[\theta]$ is an element of $\Pi_C\mathcal{L}$ with $[\psi] \lll_{\Pi_C\mathcal{L}} [\theta] \lll_{\Pi_C\mathcal{L}} [\varphi]$.

By Lemma 6.3.8, $[\psi] \lll_{\Pi_C\mathcal{L}} [\varphi]$ means that $\limsup_{n\in C}|(\psi(n),\varphi(n))_{\mathcal{L}}| = \infty$. We define θ by enumerating $\mathrm{graph}(\theta) = \{\langle n, x\rangle : \theta(n) = x\}$. The goal is to arrange $|C \cap dom(\theta)| = \infty$ (so that $C \subseteq^* dom(\theta)$ by cohesiveness), $\limsup_{n\in C}|(\psi(n),\theta(n))_{\mathcal{L}}| = \infty$, and $\limsup_{n\in C}|(\theta(n),\varphi(n))_{\mathcal{L}}| = \infty$. It then follows that $[\psi] \lll_{\Pi_C\mathcal{L}} [\theta] \lll_{\Pi_C\mathcal{L}} [\varphi]$.

Let W denote the c.e. set $\overline{C}$, and let $(W_s)_{s\in\mathbb{N}}$ be an increasing enumeration of W. Say that n *covers* k at a stage s of our enumeration of $\mathrm{graph}(\theta)$ if

- $n \notin W_s$,
- we have already enumerated $\theta(n) = x$ for some x,
- $\varphi(n)\downarrow$ and $\psi(n)\downarrow$ within s steps each,
- $|(\psi(n), x)_{\mathcal{L}} \cap \{0,\dots,s\}| \geq k$, and
- $|(x, \varphi(n))_{\mathcal{L}} \cap \{0,\dots,s\}| \geq k$.

If there is an n that covers k at stage s, then also say that k is *covered* at stage s. Enumerate graph(θ) as follows. Start at stage $s = 0$. At stage s, let $\ell_{0,s}$ be the $<$-least number that is not covered at stage s. If $s > 0$, let $X_s = W_s \smallsetminus W_{s-1}$. Let $\ell_{1,s}$ be $<$-least (if there is such a number) such that there is an $n \in X_s$ that covered $\ell_{1,s}$ at stage $s-1$, but no $m < n$ covers $\ell_{1,s}$ at stage s. If $\ell_{1,s}$ is defined, let $k_s = \min_{<}\{\ell_{0,s}, \ell_{1,s}\}$. Otherwise, let $k_s = \ell_{0,s}$. Then check if there are $n, x \leq s$ such that:

(i) $n \notin W_s$,

(ii) $\theta(n)$ is not yet defined,

(iii) $\varphi(n)\downarrow$ and $\psi(n)\downarrow$ within s steps each,

(iv) $|(\psi(n), x)_{\mathcal{L}} \cap \{0, \ldots, s\}| \geq k_s$, and

(v) $|(x, \varphi(n))_{\mathcal{L}} \cap \{0, \ldots, s\}| \geq k_s$.

If there are such an n and x, choose the $<$-least such n and the $<$-least corresponding x, and enumerate $\theta(n) = x$. Now n covers k_s at stage s. Go to stage $s+1$. If there are no such n and x, then do nothing and go to stage $s+1$. This completes the construction of θ.

If n covers k at some stage s, there could be a later stage $t > s$ at which n does not cover k because $n \in W_t$. However, if $n \in C$, then $n \notin W_t$ for every t, so k stays covered by n forever. We show, by induction on k, that every k is eventually covered by an $n \in C$. From this, $|C \cap dom(\theta)| = \infty$, $\limsup_{n \in C} |(\psi(n), \theta(n))_{\mathcal{L}}| = \infty$, and $\limsup_{n \in C} |(\theta(n), \varphi(n))_{\mathcal{L}}| = \infty$ readily follow, as desired.

Let s_0 be a stage by which all $\ell < k$ have been covered by members of C. Let c be the $<$-maximum member of C covering an $\ell < k$ at stage s_0, and let $s_1 > s_0$ be a stage such that $W_{s_1} \upharpoonright c = W \upharpoonright c$. Then $k_s \geq k$ at all stages $s > s_1$. By assumption, $\limsup_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$. So let n_0 be the $<$-least $n_0 \in C$ with $n_0 \notin dom(\theta)$ at stage s_1 and $|(\psi(n_0), \varphi(n_0))_{\mathcal{L}}| \geq 2k+1$. If n_0 ever appears in $dom(\theta)$, it is to cover some $j \geq k$, in which case n_0 also covers k. Let $s_2 > \max_{<}\{n_0, s_1\}$ be large enough so that $W_{s_2} \upharpoonright n_0 = W \upharpoonright n_0$, $\varphi(n_0)\downarrow$ and $\psi(n_0)\downarrow$ within s_2 steps, and $|(\psi(n_0), \varphi(n_0))_{\mathcal{L}} \cap \{0, \ldots, s_2\}| \geq 2k + 1$.

Consider stage s_2. If k is not covered at stage s_2, then it must be that $\theta(n_0)$ is not defined at stage s_2. In this case, $k_{s_2} = k$, and n_0 is $<$-least for which there is an $x \leq s_2$ such that (i)–(v) hold. So $\theta(n_0)$ is defined to cover k at stage s_2.

Now suppose instead that k is covered at stage s_2. In this case, let m be $<$-least such that there is a stage $s_3 \geq s_2$ at which m covers k. If $m \in C$, then this is as desired. Otherwise, $m \in W$, in which case there is a $<$-least $s > s_3$ with $m \in W_s$. The number m covers k at stage $s-1$, but by choice of m, no $a < m$ covers k at stage s. Thus $\ell_{1,s} = k$, so $k_s = k$. If $n_0 \in dom(\theta)$ at stage s, then n_0 must already cover k, as noted above. If $n_0 \notin dom(\theta)$ at stage s, then n_0 is $<$-least for which there is an $x \leq s$ such that (i)–(v) hold. So $\theta(n_0)$ is defined to cover k at stage s. ◻

6.4 Cohesive powers of computable copies of ω

We investigate the cohesive powers of computable linear orders of type ω. Observe that an infinite linear order has type ω if and only if every element has only finitely many predecessors. We rely on this characterization throughout. Though not part of the language of linear orders, every linear order $\mathcal{L}$ of type ω has an associated successor function $S^{\mathcal{L}}: |\mathcal{L}| \to |\mathcal{L}|$ given by $S^{\mathcal{L}}(x) =$ the $<_{\mathcal{L}}$-immediate successor of x. For the standard presentation of ω, the successor function is of course given by the computable function $S(x) = x+1$. It is straightforward to check that a computable copy $\mathcal{L}$ of ω is computably isomorphic to the standard presentation if and only if $S^{\mathcal{L}}$ is computable.

We show that every cohesive power of the standard presentation of ω has order-type $\omega + \zeta\eta$ (Theorem 6.4.5). This is to be expected because $\omega + \zeta\eta$ is familiar as the order-type of every countable non-standard model of Peano arithmetic (see [Kay91] Theorem 6.4). Therefore, by Theorem 6.1.4, every cohesive power of every computable copy of ω that is computably isomorphic to the standard presentation has order-type $\omega + \zeta\eta$; or, equivalently, every cohesive power of every computable copy of ω with a computable successor function has order-type $\omega + \zeta\eta$. However, being computably isomorphic to the standard presentation (equivalently, having a computable successor function) is not a characterization of the computable copies of ω having cohesive powers of order-type $\omega + \zeta\eta$. We show that there is a computable copy of ω that is not computably isomorphic to the standard presentation, yet still has every cohesive power isomorphic to $\omega + \zeta\eta$ (Theorem 6.4.8). Thus to compute a copy of ω having a cohesive power not of type $\omega + \zeta\eta$, one must do more than simply arrange for the successor function to be non-computable. We show that for every cohesive set C, there is a computable copy $\mathcal{L}$ of ω such that the cohesive power $\Pi_C\mathcal{L}$ does not have order-type $\omega + \zeta\eta$ (Theorem 6.4.9).

However, we also show that whenever $\mathcal{L}$ is a computable copy of ω and C is a co-c.e. cohesive set, the finite condensation $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L})$ of the cohesive power $\Pi_C\mathcal{L}$ always has order-type $1+\eta$ (Theorem 6.4.4).

First, a cohesive power of a computable copy of ω always has an initial segment of order-type ω.

Lemma 6.4.1. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable copy of ω, and let C be cohesive. Then the image of the canonical embedding of $\mathcal{L}$ into $\Pi_C\mathcal{L}$ is an initial segment of $\Pi_C\mathcal{L}$ of order-type ω.

Proof. The linear order $\mathcal{L}$ has type ω, so its image in $\Pi_C\mathcal{L}$ under the canonical embedding also has type ω. We show that this image is an initial segment of $\Pi_C\mathcal{L}$. Consider $[\varphi] \in |\Pi_C\mathcal{L}|$, and suppose that $[\varphi] \prec_{\Pi_C\mathcal{L}} [\psi]$ for a $[\psi]$ in the image of the canonical embedding. We may assume that ψ is the constant function with value a for some $a \in L$. Then $(\forall^\infty n \in C)(\varphi(n) \prec_{\mathcal{L}} a)$. As $\mathcal{L} \cong \omega$, there are only finitely many elements $b_0, \dots, b_{k-1}$ of L that are $\prec_{\mathcal{L}}$-below a. Thus $(\forall^\infty n \in C)(\varphi(n) \in \{b_0, \dots, b_{k-1}\})$. By the cohesiveness of C, there is exactly one b_i for which $(\forall^\infty n \in C)(\varphi(n) = b_i)$. Therefore $[\varphi] = [$the constant function with value $b_i]$, so $[\varphi]$ is also in the image of the canonical embedding. Thus the image of the canonical embedding is an initial segment of $\Pi_C\mathcal{L}$ of order-type ω. □

Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable copy of ω, let C be cohesive, and let $\varphi\colon \mathbb{N} \to L$ be any total computable bijection. Then $[\varphi]$ is not in the image of the canonical embedding of $\mathcal{L}$ into $\Pi_C\mathcal{L}$, so it must be $\prec_{\Pi_C\mathcal{L}}$-above every element in the image of the canonical embedding. Thus $\Pi_C\mathcal{L}$ is of the form $\omega + \mathcal{M}$ for some non-empty linear order $\mathcal{M}$. By analogy with the terminology for models of Peano arithmetic, we call the elements of the ω-part of $\Pi_C\mathcal{L}$ (i.e., the image of the canonical embedding) *standard* and the elements of the $\mathcal{M}$-part of $\Pi_C\mathcal{L}$ *non-standard*.

Lemma 6.4.2. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable copy of ω, let C be cohesive, and let $[\varphi]$ be an element of $\Pi_C\mathcal{L}$. Then $[\varphi]$ is non-standard if and only if $\liminf_{n\in C}\varphi(n) = \infty$ (equivalently, $\lim_{n\in C}\varphi(n) = \infty$).

Proof. If $[\varphi]$ is standard, then φ is eventually constant on C, so $\liminf_{n\in C}\varphi(n)$ is finite. Conversely, suppose that $\liminf_{n\in C}\varphi(n) = k$ is finite. Then $(\exists^\infty n \in C)(\varphi(n) = k)$. By cohesiveness, it must therefore be that $(\forall^\infty n \in C)(\varphi(n) = k)$. That is, φ is eventually constant on C, so $[\varphi]$ is standard. □

Lemma 6.4.3. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable copy of ω, let C be cohesive, and let $[\varphi]$ be a non-standard element of $\Pi_C\mathcal{L}$. Then there are non-standard elements $[\psi^-]$ and $[\psi^+]$ of $\Pi_C\mathcal{L}$ with $[\psi^-] \lessapprox_{\Pi_C\mathcal{L}} [\varphi] \lessapprox_{\Pi_C\mathcal{L}} [\psi^+]$.

Proof. Fix any $x_0 \in L$, and define a computable sequence $x_0 \prec_{\mathcal{L}} x_1 \prec_{\mathcal{L}} x_2 \prec_{\mathcal{L}} \cdots$ by letting x_{i+1} be the $<$-least number with $x_i \prec_{\mathcal{L}} x_{i+1}$. Such an x_{i+1} always exists because $\mathcal{L}$ has no $\prec_{\mathcal{L}}$-maximum element. Furthermore, notice that $x_0 \prec_{\mathcal{L}} x_1 \prec_{\mathcal{L}} x_2 \prec_{\mathcal{L}} \cdots$ is cofinal in $\mathcal{L}$ because $\mathcal{L} \cong \omega$.

Consider a non-standard $[\varphi] \in |\Pi_C\mathcal{L}|$. Define partial computable functions $\psi^-, \psi^+ \colon \mathbb{N} \to L$ by

$$\psi^-(n) \simeq \begin{cases} x_i & \text{if } x_{2i} \preceq_{\mathcal{L}} \varphi(n) \prec_{\mathcal{L}} x_{2i+2} \\ \uparrow & \text{if } \varphi(n)\uparrow \end{cases}$$

$$\psi^+(n) \simeq \begin{cases} x_{2i} & \text{if } x_i \preceq_{\mathcal{L}} \varphi(n) \prec_{\mathcal{L}} x_{i+1} \\ \uparrow & \text{if } \varphi(n)\uparrow. \end{cases}$$

The element $[\varphi]$ is non-standard, so $(\forall i)(\forall^\infty n \in C)(x_{2i} \preceq_{\mathcal{L}} \varphi(n))$. Thus $(\forall i)(\forall^\infty n \in C)(x_i \preceq_{\mathcal{L}} \psi^-(n))$, so $[\psi^-]$ is non-standard as well. Moreover, if $x_{2i} \preceq_{\mathcal{L}} \varphi(n) \prec_{\mathcal{L}} x_{2i+2}$, then $\psi^-(n) = x_i$, and therefore $|(\psi^-(n), \varphi(n))_{\mathcal{L}}| \geq i-1$ because $x_{i+1}, \ldots, x_{2i-1} \in (\psi^-(n), \varphi(n))_{\mathcal{L}}$. Therefore $\limsup_{n \in C} |(\psi^-(n), \varphi(n))_{\mathcal{L}}| = \infty$, so $[\psi^-] \lessapprox_{\Pi_C\mathcal{L}} [\varphi]$ by Lemma 6.3.8. Similar reasoning shows that $[\varphi] \lessapprox_{\Pi_C\mathcal{L}} [\psi^+]$. Thus $[\psi^-] \lessapprox_{\Pi_C\mathcal{L}} [\varphi] \lessapprox_{\Pi_C\mathcal{L}} [\psi^+]$. □

Lemmas 6.4.1 and 6.4.3 imply that if $\mathcal{L}$ is a computable copy of ω and C is cohesive, then $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L}) \cong 1 + \mathcal{M}$ for some infinite linear order $\mathcal{M}$. We call the block corresponding to 1 the *standard block* and the blocks corresponding to $\mathcal{M}$ *non-standard blocks*. If we further assume that C is co-c.e., then we obtain that $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L}) \cong 1 + \eta$.

Theorem 6.4.4. Let $\mathcal{L}$ be a computable copy of ω, and let C be co-c.e. and cohesive. Then $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L})$ has order-type $1 + \eta$.

Proof. By Lemma 6.4.1, the standard elements of $\Pi_C\mathcal{L}$ form an initial block. By Theorem 6.3.9 and Lemma 6.4.3, the non-standard blocks of $\Pi_C\mathcal{L}$ form a countable dense linear order without endpoints. Thus $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L}) \cong 1 + \eta$. □

Thinking in terms of blocks, showing that a linear order $\mathcal{M}$ has type $\omega + \zeta\eta$ amounts to showing that $\mathcal{M}$ consists of an initial block of type ω followed by densely (without endpoints) ordered blocks of type ζ.

Theorem 6.4.5. Let $\mathbb{N}$ denote the standard presentation of ω, and let C be cohesive. Then $\Pi_C\mathbb{N}$ has order-type $\omega + \zeta\eta$.

Proof. By Lemma 6.4.1, $\Pi_C\mathbb{N}$ has an initial segment of order-type ω. To show that the non-standard blocks each have order-type ζ, it suffices to show that every element of $\Pi_C\mathbb{N}$ has an $<_{\Pi_C\mathbb{N}}$-immediate successor and that every element of $\Pi_C\mathbb{N}$ except the first element has an $<_{\Pi_C\mathbb{N}}$-immediate predecessor.

Let $[\varphi] \in |\Pi_C\mathbb{N}|$. Define partial computable functions θ and ψ by $\theta(n) \simeq \varphi(n) + 1$ and $\psi(n) \simeq \varphi(n) \ominus 1$, where $\ominus$ denotes truncated subtraction (so $0 \ominus 1 = 0$). Then $[\theta]$ is the $<_{\Pi_C\mathbb{N}}$-immediate successor of $[\varphi]$ by Lemma 6.3.7. Similarly, if $[\varphi]$ is not the least element of $\Pi_C\mathbb{N}$ (i.e., if $(\forall^\infty n \in C)(\varphi(n) \neq 0)$), then $[\psi]$ is the $<_{\Pi_C\mathbb{N}}$-immediate predecessor of $[\varphi]$.

By Lemma 6.4.3, there is neither a least nor a greatest non-standard block of $\Pi_C\mathbb{N}$. We cannot use Theorem 6.3.9 to conclude that the non-standard blocks are densely ordered because we do not assume that C is co-c.e. So suppose $[\varphi]$ and $[\psi]$ are such that $[\psi] \ll_{\Pi_C\mathbb{N}} [\varphi]$. Then $\lim_{n\in C}|(\psi(n), \varphi(n))| = \infty$ by Lemma 6.3.8. Define a partial computable function θ by $\theta(n) \simeq \lfloor(\varphi(n) + \psi(n))/2\rfloor$. Then $\lim_{n\in C}|(\psi(n), \theta(n))| = \infty$ and $\lim_{n\in C}|(\theta(n), \varphi(n))| = \infty$, so $[\psi] \ll_{\Pi_C\mathbb{N}} [\theta] \ll_{\Pi_C\mathbb{N}} [\varphi]$. Thus the non-standard blocks of $\Pi_C\mathbb{N}$ form a dense linear order without endpoints. This completes the proof that $\Pi_C\mathbb{N} \cong \omega + \zeta\eta$. □

Corollary 6.4.6. Let $\mathcal{L}$ be a computable copy of ω with a computable successor function, and let C be cohesive. Then $\Pi_C\mathcal{L}$ has order-type $\omega + \zeta\eta$.

Proof. If $\mathcal{L}$ is a computable copy of ω with a computable successor function, then $\mathcal{L}$ is computably isomorphic to the standard presentation $\mathbb{N}$ of ω. Thus $\Pi_C\mathcal{L} \cong \Pi_C\mathbb{N} \cong \omega + \zeta\eta$ by Theorems 6.1.4 and 6.4.5. □

We can calculate the order-types of the cohesive powers of many other computable presentations of linear orders by combining Theorems 6.1.4, 6.3.6, 6.4.5, and the fact that $\Pi_C\mathbb{Q} \cong \eta$.

Example 6.4.7. Let C be a cohesive set. Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the standard presentations of ω, ζ, and η.

(1) $\Pi_C\mathbb{N}^* \cong \zeta\eta + \omega^*$: This is because

$$\Pi_C\mathbb{N}^* \;\cong\; (\Pi_C\mathbb{N})^* \;\cong\; (\omega + \zeta\eta)^* \;\cong\; \zeta\eta + \omega^*.$$

(2) $\Pi_C\mathbb{Z} \cong \zeta\eta$. This is because $\mathbb{Z}$ is computably isomorphic to $\mathbb{N}^* + \mathbb{N}$, so

$$\Pi_C\mathbb{Z} \cong \Pi_C(\mathbb{N}^* + \mathbb{N}) \cong \Pi_C(\mathbb{N})^* + \Pi_C(\mathbb{N}) \cong (\zeta\eta + \omega^*) + (\omega + \zeta\eta)$$
$$\cong \zeta\eta + \zeta + \zeta\eta \cong \zeta\eta.$$

(3) $\Pi_C(\mathbb{Z}\mathbb{Q}) \cong \zeta\eta$. This is because

$$\Pi_C(\mathbb{Z}\mathbb{Q}) \cong (\Pi_C\mathbb{Z})(\Pi_C\mathbb{Q}) \cong (\zeta\eta)\eta \cong \zeta\eta.$$

(4) $\Pi_C(\mathbb{N} + \mathbb{Z}\mathbb{Q}) \cong \omega + \zeta\eta$. This is because

$$\Pi_C(\mathbb{N} + \mathbb{Z}\mathbb{Q}) \cong \Pi_C(\mathbb{N}) + \Pi_C(\mathbb{Z}\mathbb{Q}) \cong (\omega + \zeta\eta) + \zeta\eta \cong \omega + \zeta\eta.$$

Recall that, by Proposition 6.3.4, an ultrahomogeneous computable structure for a finite relational language, like the computable linear order $\mathbb{Q}$, is isomorphic to each of its cohesive powers. Notice, however, that the computable linear orders $\mathbb{Z}\mathbb{Q}$ and $\mathbb{N} + \mathbb{Z}\mathbb{Q}$ are not ultrahomogeneous, yet nevertheless are isomorphic to each of their respective cohesive powers. Thus it is also possible for a non-ultrahomogeneous computable structure to be isomorphic to each of its cohesive powers.

Notice also that $\Pi_C\mathbb{N}$ and $\Pi_C(\mathbb{N} + \mathbb{Z}\mathbb{Q})$ both have order-type $\omega + \zeta\eta$. Similarly, $\Pi_C\mathbb{Z}$ and $\Pi_C(\mathbb{Z}\mathbb{Q})$ both have order-type $\zeta\eta$. Thus it is possible for non-isomorphic linear orders to have isomorphic cohesive powers. In Section 6.5, we give an example of a pair of non-elementarily equivalent linear orders with isomorphic cohesive powers.

Now we give an example of a computable copy of ω that is not computably isomorphic to the standard presentation, yet still has all its cohesive powers isomorphic to $\omega + \zeta\eta$.

Theorem 6.4.8. There is a computable copy $\mathcal{L}$ of ω such that

- $\mathcal{L}$ is not computably isomorphic to the standard presentation of ω, yet
- for every cohesive set C, the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \zeta\eta$.

Proof. We use a classic example of a computable copy of ω with a non-computable successor function. Fix any non-computable c.e. set A, and let $f\colon \mathbb{N} \to A$ be a computable bijection. Let $\mathcal{L} = (\mathbb{N}, \prec_{\mathcal{L}})$ be the linear order obtained by ordering the even numbers according to their natural order and by setting $2a \prec_{\mathcal{L}} 2k+1 \prec_{\mathcal{L}} 2a+2$ if and only if $f(k) = a$. Specifically, define

$$\begin{aligned} 2c \prec_{\mathcal{L}} 2d &\quad\Leftrightarrow\quad 2c < 2d \\ 2c \prec_{\mathcal{L}} 2k+1 &\quad\Leftrightarrow\quad c \leq f(k) \\ 2k+1 \prec_{\mathcal{L}} 2c &\quad\Leftrightarrow\quad f(k) < c \\ 2k+1 \prec_{\mathcal{L}} 2\ell+1 &\quad\Leftrightarrow\quad f(k) < f(\ell). \end{aligned}$$

Then $\mathcal{L}$ is a computable linear order of type ω. Let $S^{\mathcal{L}}$ denote the successor function of $\mathcal{L}$. Then $A \leq_{\mathrm{T}} S^{\mathcal{L}}$ (in fact, $A \equiv_{\mathrm{T}} S^{\mathcal{L}}$) because $a \in A$ if and only if $S^{\mathcal{L}}(2a) \neq 2a+2$. Thus $S^{\mathcal{L}}$ is not computable, so $\mathcal{L}$ is not computably isomorphic to the standard presentation of ω.

Let C be cohesive. We show that $\Pi_C\mathcal{L} \cong \omega + \zeta\eta$. As in the proof of Theorem 6.4.5, it suffices to establish the following.

(a) Every element of $\Pi_C\mathcal{L}$ has a $\prec_{\Pi_C\mathcal{L}}$-immediate successor.

(b) Every element of $\Pi_C\mathcal{L}$ that is not the $\prec_{\Pi_C\mathcal{L}}$-least element has a $\prec_{\Pi_C\mathcal{L}}$-immediate predecessor.

(c) If $[\psi], [\varphi] \in |\Pi_C\mathcal{L}|$ satisfy $[\psi] \precneqq_{\Pi_C\mathcal{L}} [\varphi]$, then there is a $[\theta] \in |\Pi_C\mathcal{L}|$ with $[\psi] \precneqq_{\Pi_C\mathcal{L}} [\theta] \precneqq_{\Pi_C\mathcal{L}} [\varphi]$.

For (a), consider a $[\psi] \in |\Pi_C\mathcal{L}|$. We define a partial computable φ such that $\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$ for almost every $n \in C$. Then $[\varphi]$ is the $\prec_{\Pi_C\mathcal{L}}$-immediate successor of $[\psi]$ by Lemma 6.3.7. To define φ, observe that by the cohesiveness of C, exactly one of the following three cases occurs.

(i) $(\forall^\infty n \in C)(\psi(n) \text{ is odd})$

(ii) $(\forall^\infty n \in C)(\psi(n) = 2a, \text{ where } a \in A)$

(iii) $(\forall^\infty n \in C)(\psi(n) = 2a, \text{ where } a \notin A)$

We cannot effectively decide which case occurs, but in each case we can non-uniformly define a φ such that $[\varphi]$ is the $\prec_{\Pi_C \mathcal{L}}$-immediate successor of $[\psi]$.

If case (i) occurs, define

$$\varphi(n) \simeq \begin{cases} 2a+2 & \text{if } \psi(n) = 2k+1 \text{ and } f(k) = a \\ \uparrow & \text{otherwise.} \end{cases}$$

If case (ii) occurs, define

$$\varphi(n) \simeq \begin{cases} 2k+1 & \text{if } \psi(n) = 2a,\ a \in A, \text{ and } f(k) = a \\ \uparrow & \text{otherwise.} \end{cases}$$

If case (iii) occurs, define

$$\varphi(n) \simeq \begin{cases} 2a+2 & \text{if } \psi(n) = 2a \\ \uparrow & \text{otherwise.} \end{cases}$$

In each case, $\varphi(n)$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(n)$ for almost every $n \in C$. Thus in each case, $[\varphi]$ is the $\prec_{\Pi_C \mathcal{L}}$-immediate successor of $[\psi]$. This completes the proof of (a). The proof of (b) is analogous.

For (c), recall that for $[\psi], [\varphi] \in |\Pi_C \mathcal{L}|$,

$$[\psi] \preccurlyeq_{\Pi_C \mathcal{L}} [\varphi] \iff \lim_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty,$$

by Lemma 6.3.8. Notice that for even numbers $2a$ and $2b$, $2a \prec_{\mathcal{L}} 2b$ if and only if $2a < 2b$. So if $2a \prec_{\mathcal{L}} 2b$, then $|(2a, 2b)_{\mathcal{L}}| \geq b - a - 1$. Therefore, if $[\psi], [\varphi] \in |\Pi_C \mathcal{L}|$ are such that $\psi(n)$ and $\varphi(n)$ are even for almost every $n \in C$, then

$$\lim_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty \iff \lim_{n \in C} (\varphi(n) - \psi(n)) = \infty.$$

Furthermore, observe that if $[\psi] \in |\Pi_C \mathcal{L}|$, then by the cohesiveness of C, either $(\forall^\infty n \in C)(\psi(n) \text{ is even})$ or $(\forall^\infty n \in C)(\psi(n) \text{ is odd})$. In the case where $(\forall^\infty n \in C)(\psi(n)$ is odd$)$, the $\prec_{\Pi_C\mathcal{L}}$-immediate successor $[\varphi]$ of $[\psi]$ from case (i) above satisfies $(\forall^\infty n \in C)(\varphi(n)$ is even$)$.

Now, suppose $[\psi], [\varphi] \in |\Pi_C\mathcal{L}|$ satisfy $[\psi] \preccurlyeq_{\Pi_C\mathcal{L}} [\varphi]$. We may assume that $(\forall^\infty n \in C)(\psi(n)$ is even$)$ and that $(\forall^\infty n \in C)(\varphi(n)$ is even$)$ by replacing $[\psi]$ and/or $[\varphi]$ by their $\prec_{\Pi_C\mathcal{L}}$-immediate successors as necessary. Thus $\lim_{n\in C}(\varphi(n) - \psi(n)) = \infty$. Define a partial computable function θ by

$$\theta(n) \simeq \begin{cases} \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor & \text{if } \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor \text{ is even} \\ \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor + 1 & \text{if } \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor \text{ is odd.} \end{cases}$$

Then $(\forall^\infty n \in C)(\theta(n)$ is even$)$, $\lim_{n\in C}(\theta(n) - \psi(n)) = \infty$, and $\lim_{n\in C}(\varphi(n) - \theta(n)) = \infty$. Therefore $\lim_{n\in C}|(\psi(n), \theta(n))_{\mathcal{L}}| = \infty$ and $\lim_{n\in C}|(\theta(n), \varphi(n))_{\mathcal{L}}| = \infty$, so $[\psi] \preccurlyeq_{\Pi_C\mathcal{L}} [\theta] \preccurlyeq_{\Pi_C\mathcal{L}} [\varphi]$. This completes the proof of (c) and thus the proof that $\Pi_C\mathcal{L} \cong \omega + \zeta\eta$. □

Lastly, we show that for every cohesive set C, there is a computable copy $\mathcal{L}$ of ω such that $\Pi_C\mathcal{L}$ is not isomorphic, indeed, not elementarily equivalent, to $\omega + \zeta\eta$. The strategy is to arrange for the element [id] of $\Pi_C\mathcal{L}$ represented by the identity function id: $\mathbb{N} \to \mathbb{N}$ to have no $\prec_{\Pi_C\mathcal{L}}$-immediate successor. This exhibits an elementary difference between $\Pi_C\mathcal{L}$ and $\omega + \zeta\eta$ because every element of $\omega + \zeta\eta$ has an immediate successor. This also shows that Theorem 6.1.3 part (4) is tight: "there is an element with no immediate successor" is a Σ^0_3 sentence that is true of $\Pi_C\mathcal{L}$ but not of $\mathcal{L}$.

Theorem 6.4.9. Let C be any cohesive set. Then there is a computable copy $\mathcal{L}$ of ω for which $\Pi_C\mathcal{L}$ is not elementarily equivalent (and hence not isomorphic) to $\omega + \zeta\eta$.

Proof. Let $(\varphi_e)_{e\in\mathbb{N}}$ denote the usual effective list of all partial computable functions, and recall that $\varphi_{e,s}(n)$ denotes the result of running φ_e on input n for s computational steps. We compute a linear order $\mathcal{L} = (\mathbb{N}, \prec_{\mathcal{L}})$ of type ω such that for every φ_e:

$$(\forall^\infty n \in C)[\varphi_e(n)\downarrow \ \Rightarrow\ (\varphi_e(n) \text{ is not the } \prec_{\mathcal{L}}\text{-immediate successor of } n)]. \qquad (*)$$

By Lemma 6.3.7, achieving $(*)$ for φ_e ensures that $[\varphi_e]$ is not the $\prec_{\Pi_C\mathcal{L}}$-immediate successor of [id] in $\Pi_C\mathcal{L}$. Therefore, achieving $(*)$ for every φ_e ensures that [id] has no $\prec_{\Pi_C\mathcal{L}}$-immediate successor in $\Pi_C\mathcal{L}$. Thus $\Pi_C\mathcal{L}$ is not elementarily equivalent to $\omega + \zeta\eta$ because every element of $\omega + \zeta\eta$ has an immediate successor, which is a Π_3 property.

Fix an infinite computable set $R \subseteq \overline{C}$. Such an R may be obtained, for example, by partitioning $\mathbb{N}$ into the even numbers R_0 and the odd numbers R_1. By cohesiveness, $C \subseteq^* R_i$ for either $i = 0$ or $i = 1$, in which case $R_{1-i} \subseteq^* \overline{C}$. Thus we may take R to be an appropriate tail of R_{1-i}.

Define $\prec_{\mathcal{L}}$ in stages. By the end of stage s, $\prec_{\mathcal{L}}$ will have been defined on $X_s \times X_s$ for some finite $X_s \supseteq \{0, 1, \ldots, s\}$. At stage 0, set $X_0 = \{0\}$ and define $0 \nprec_{\mathcal{L}} 0$. At stage $s > 0$, start with $X_s = X_{s-1}$, and update X_s and $\prec_{\mathcal{L}}$ according to the following procedure.

(1) If $\prec_{\mathcal{L}}$ has not yet been defined on s (i.e., if $s \notin X_s$), then update X_s to $X_s \cup \{s\}$ and extend $\prec_{\mathcal{L}}$ to make s the $\prec_{\mathcal{L}}$-greatest element of X_s.

(2) Consider each $\langle e, n\rangle < s$ in order. For each $\langle e, n\rangle < s$, if

   (a) $\varphi_{e,s}(n)\downarrow \in X_s$,

   (b) $\varphi_e(n)$ is currently the $\prec_{\mathcal{L}}$-immediate successor of n in X_s,

   (c) $n \notin R$, and

   (d) n is not $\preceq_{\mathcal{L}}$-below any of $0, 1, \ldots, e$,

   then let m be the $<$-least element of $R \setminus X_s$, update X_s to $X_s \cup \{m\}$, and extend $\prec_{\mathcal{L}}$ so that $n \prec_{\mathcal{L}} m \prec_{\mathcal{L}} \varphi_e(n)$.

This completes the construction.

We claim that for every k, there are only finitely many elements $\prec_{\mathcal{L}}$-below k. It follows that $\mathcal{L}$ has order-type ω. Say that φ_e *acts for* n *and adds* m when $\prec_{\mathcal{L}}$ is defined on an $m \in R$ to make $n \prec_{\mathcal{L}} m \prec_{\mathcal{L}} \varphi_e(n)$ as in (2). Let s_0 be a stage with $k \in X_{s_0}$. Suppose at some stage $s > s_0$, an m is added to X_s and $m \prec_{\mathcal{L}} k$ is defined. This can only be due to a φ_e acting for an $n \notin R$ and adding m at stage s. Thus at stage s, it must be that $n \prec_{\mathcal{L}} k$ because $n \prec_{\mathcal{L}} m \prec_{\mathcal{L}} k$. Therefore it must also be that $e < k$, for otherwise k would be among $0, 1, \ldots, e$, and condition (d) would prevent the action of φ_e. Furthermore, m is chosen from R, so only elements of R are added $\prec_{\mathcal{L}}$-below k after stage s_0. All together, this means that an m can only be added $\prec_{\mathcal{L}}$-below k after stage s_0 when a φ_e with $e < k$ acts for an $n \prec_{\mathcal{L}} k$ with $n \notin R$. Each φ_e

acts at most once for each n, and no new $n \notin R$ appear $\prec_{\mathcal{L}}$-below k after stage s_0. Thus after stage s_0, only finitely many m are ever added $\prec_{\mathcal{L}}$-below k.

Finally, we claim that $(*)$ is satisfied for every φ_e. Given e, let ℓ be the $\prec_{\mathcal{L}}$-maximum element of $\{0, 1, \dots, e\}$. Observe that almost every $n \in \mathbb{N}$ satisfies $n \succ_{\mathcal{L}} \ell$ because $\mathcal{L} \cong \omega$. So suppose that $n \succ_{\mathcal{L}} \ell$ and $n \in C$. If $\varphi_e(n)\downarrow$, let s be large enough so that $\langle e, n \rangle < s$, $\varphi_{e,s}(n)\downarrow$, $n \in X_s$, and $\varphi_e(n) \in X_s$. Then either $\varphi_e(n)$ is already not the $\prec_{\mathcal{L}}$-immediate successor of n at stage $s+1$, or at stage $s+1$ the conditions of (2) are satisfied for $\langle e, n \rangle$, and an m is added such that $n \prec_{\mathcal{L}} m \prec_{\mathcal{L}} \varphi_e(n)$. This completes the proof. □

Corollary 6.4.10. Theorem 6.1.3 item (4) is tight in general: There is a cohesive set C, a computable linear order $\mathcal{L}$, and a Σ_3^0 sentence Φ such that $\Pi_C \mathcal{L} \vDash \Phi$, but $\mathcal{L} \nvDash \Phi$.

Proof. Let C be any cohesive set, and let $\mathcal{L}$ be a computable copy of ω as in Theorem 6.4.9 for C. Let Φ be a Σ_3^0 sentence in the language of linear orders expressing that there is an element with no immediate successor. Then $\Pi_C \mathcal{L} \vDash \Phi$, but $\mathcal{L} \nvDash \Phi$. □

Corollary 6.4.10 may also be deduced from Lerman's proof of Feferman, Scott, and Tennenbaum's theorem that no cohesive power of the standard model of arithmetic is a model of Peano arithmetic (see [Ler70] Theorem 2.1). Lerman gives a somewhat technical example of a Σ_3^0 sentence Φ invoking Kleene's T predicate that fails in the standard model of arithmetic but is true in every cohesive power. Our proof of Corollary 6.4.10 is more satisfying because it witnesses the optimality of Theorem 6.1.3 item (4) with a natural Σ_3^0 sentence in the simple language of linear orders.

In the next section, we enhance the construction of Theorem 6.4.9 in order to compute a copy $\mathcal{L}$ of ω with $\Pi_C \mathcal{L} \cong \omega + \eta$ for a given co-c.e. cohesive set C.

6.5 A cohesive power of order-type $\omega + \eta$

Given a co-c.e. cohesive set, we compute a copy $\mathcal{L}$ of ω for which $\Pi_C \mathcal{L}$ has order-type $\omega + \eta$. In order to help shuffle various linear orders into cohesive powers in Section 6.6, we in fact compute a linear order $\mathcal{L} = (\mathbb{N}, \prec_{\mathcal{L}})$ along with a coloring function $F: \mathbb{N} \to \mathbb{N}$ that colors the elements of $\mathcal{L}$ with countably many colors. The coloring F induces a coloring $\widehat{F}$ of $\Pi_C \mathcal{L}$ in the following way. Colors of elements of $\Pi_C \mathcal{L}$ are represented by partial computable functions $\delta: \mathbb{N} \to \mathbb{N}$ with $C \subseteq^* dom(\delta)$. As in Definition 6.1.2, write $\delta_0 =_C \delta_1$ if $(\forall^\infty n \in C)(\delta_0(n)\downarrow = \delta_1(n)\downarrow)$, and write $[\![\delta]\!]$ instead of $[\delta]$ for the $=_C$-equivalence class of δ when thinking in terms of colors. Then $\widehat{F}$ is given by $\widehat{F}([\varphi]) = [\![F \circ \varphi]\!]$. So, for example, elements $[\varphi]$ and $[\psi]$ of $\Pi_C \mathcal{L}$ have the same $\widehat{F}$-color if and only if $\varphi(n)$ and $\psi(n)$ have the same F-color for almost every $n \in C$.

Call a color $[\![\delta]\!]$ a *solid color* if there is an $x \in \mathbb{N}$ such that $(\forall^\infty n \in C)(\delta(n) = x)$. Otherwise, call $[\![\delta]\!]$ a *striped color*. Observe that if $[\![\delta]\!]$ is striped, then $\lim_{n \in C} \delta(n) = \infty$. We compute $\mathcal{L}$ and F so that $\Pi_C \mathcal{L} \cong \omega + \eta$ and every solid color occurs densely in the η-part. Between any two distinct elements of the η-part there is also an element with a striped color, but we do not ask for every striped color to occur densely. In Section 6.6, we show that replacing each point of $\mathcal{L}$ by some finite linear order depending on its color has the effect of shuffling these finite orders into the non-standard part of $\Pi_C \mathcal{L}$.

Theorem 6.5.1. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{L} = (\mathbb{N}, \prec_{\mathcal{L}})$ of ω and a computable coloring $F\colon \mathbb{N} \to \mathbb{N}$ of $\mathcal{L}$ with the following property. Let $[\varphi]$ and $[\psi]$ be any two non-standard elements of $\Pi_C \mathcal{L}$ with $[\psi] \prec_{\Pi_C \mathcal{L}} [\varphi]$. Then for every solid color $[\![\delta]\!]$, there is a $[\theta]$ in $\Pi_C \mathcal{L}$ with $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$ and $\widehat{F}([\theta]) = [\![\delta]\!]$. Also, there is a $[\theta]$ in $\Pi_C \mathcal{L}$ with $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$, where $\widehat{F}([\theta])$ is a striped color.

Proof. We are working with a co-c.e. cohesive set, so recall that in this situation every element $[\varphi]$ of $\Pi_C \mathcal{L}$ has a total representative by the discussion following Definition 6.1.2. Recall also that an element $[\varphi]$ of $\Pi_C \mathcal{L}$ is non-standard if and only if $\lim_{n \in C} \varphi(n) = \infty$ by Lemma 6.4.2.

The goal of the construction of $\mathcal{L}$ is to arrange, for every pair of total computable functions φ and ψ with $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$, that

$$(\forall^\infty n \in C)(\psi(n)\downarrow \prec_{\mathcal{L}} \varphi(n)\downarrow \Rightarrow \\ (\forall d \le \max_{<}\{\varphi(n), \psi(n)\})(\exists k)[(\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)) \wedge (F(k) = d)]). \quad (*)$$

Suppose we achieve $(*)$ for φ and ψ, where $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$ and $(\forall^\infty n \in C)(\varphi(n)\downarrow \prec_{\mathcal{L}} \psi(n)\downarrow)$. Fix any color d, and let δ be the constant function with value d. Partially compute a function $\theta(n)$ by searching for a k with $\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)$ and $F(k) = d$. If there is such a k, let $\theta(n)$ be the first such k. Property $(*)$ and the assumption $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$ ensure that there is such a k for almost every $n \in C$. Therefore $C \subseteq^* dom(\theta)$, $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$, and $\widehat{F}([\theta]) = [\![\delta]\!]$. Likewise, we could instead define $\theta(n)$ to search for a k with $\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)$ and $F(k) = \varphi(n)$ and let $\theta(n)$

be the first (if any) such k found. In this case we would have $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$ and $\widehat{F}([\theta]) = [\![\varphi]\!]$, which is a striped color because $\lim_{n \in C} \varphi(n) = \infty$. Thus achieving $(*)$ suffices to prove the theorem, provided we also arrange $\mathcal{L} \cong \omega$.

Let W denote the c.e. set $\overline{C}$, and let $(W_s)_{s \in \mathbb{N}}$ be an increasing enumeration of W. Let $(A^{i,0}, A^{i,1})_{i \in \mathbb{N}}$ be a uniformly computable sequence of pairs of sets such that

- for each i, $A^{i,0}$ and $A^{i,1}$ partition $\mathbb{N}$ into two pieces (i.e., $A^{i,1} = \overline{A^{i,0}}$) and
- $(\forall n)(\forall \sigma \in \{0,1\}^n)(\bigcap_{i<n} A^{i,\sigma(i)}$ is infinite$)$.

This can be accomplished by partitioning $\mathbb{N}$ into successive pieces of size 2^i, letting $A^{i,0}$ consist of every other piece, and letting $A^{i,1} = \overline{A^{i,0}}$.

In this proof, denote the projection functions associated to the pairing function $\langle \cdot, \cdot \rangle$ by ℓ and r, for *left* and *right*, instead of by π_0 and π_1. So $\ell(\langle x, y \rangle) = x$ and $r(\langle x, y \rangle) = y$.

The tension in the construction is between achieving $(*)$ and ensuring that for every z, there are only finitely many x with $x \prec_{\mathcal{L}} z$. Think of a $p \in \mathbb{N}$ as coding a pair $(\varphi_{\ell(p)}, \varphi_{r(p)})$ of partial computable functions for which we would like to achieve $(*)$, with $\varphi_{\ell(p)}$ playing the role of ψ and $\varphi_{r(p)}$ playing the role of φ. We assign the partition $(A^{2p,0}, A^{2p,1})$ to $\varphi_{\ell(p)}$ and the partition $(A^{2p+1,0}, A^{2p+1,1})$ to $\varphi_{r(p)}$. The sets $\{n : \varphi_{\ell(p)}(n) \in A^{2p,0}\}$ and $\{n : \varphi_{\ell(p)}(n) \in A^{2p,1}\}$ are both c.e., so if $C \subseteq^* dom(\varphi_{\ell(p)})$, then either $(\forall^\infty n \in C)(\varphi_{\ell(p)}(n) \in A^{2p,0})$ or $(\forall^\infty n \in C)(\varphi_{\ell(p)}(n) \in A^{2p,1})$; and similarly for $\varphi_{r(p)}$ and $(A^{2p+1,0}, A^{2p+1,1})$. As the construction proceeds, we eventually stabilize on a correct guess for which of $(\forall^\infty n \in C)(\varphi_{\ell(p)}(n) \in A^{2p,0})$ or $(\forall^\infty n \in C)(\varphi_{\ell(p)}(n) \in A^{2p,1})$ occurs, and similarly for $\varphi_{r(p)}$ and $(A^{2p+1,0}, A^{2p+1,1})$. Suppose we want to choose an element k to help satisfy $(*)$ for $\varphi_{\ell(q)}$ and $\varphi_{r(q)}$ for some $q > p$. If we have guessed that $(\forall^\infty n \in C)(\varphi_{\ell(p)}(n) \in A^{2p,0})$ and $(\forall^\infty n \in C)(\varphi_{r(p)}(n) \in A^{2p+1,0})$, then we choose k from $A^{2p,1} \cap A^{2p+1,1}$. That is, we choose a k that we guess does not appear in $\varphi_{\ell(p)}(C)$ or $\varphi_{r(p)}(C)$. If we are correct about the guess, then k is a safe element for $\varphi_{\ell(q)}$ and $\varphi_{r(q)}$ to use because its placement with respect to $\prec_{\mathcal{L}}$ will not incite a reaction from $\varphi_{\ell(p)}$ and $\varphi_{r(p)}$.

Define $\prec_{\mathcal{L}}$ and F in stages. By the end of stage s, $\prec_{\mathcal{L}}$ will have been defined on $X_s \times X_s$ and F will have been defined on X_s for some finite $X_s \supseteq \{0, 1, \ldots, s\}$.

At stage 0, set $X_0 = \{0\}$ with $0 \not\prec_{\mathcal{L}} 0$ and $F(0) = 0$. At stage $s > 0$, initially set $X_s = X_{s-1}$. If $s \notin X_s$, then add s to X_s, define it to be the $\prec_{\mathcal{L}}$-maximum element of X_s, and define $F(s) = 0$. Then proceed as follows.

Consider each pair $\langle p, N \rangle < s$ in order. Think of $\langle p, N \rangle$ as coding a pair $(\varphi_{\ell(p)}, \varphi_{r(p)})$ of partial computable functions as described above and a guess N of a threshold by which the cohesive behavior of $\varphi_{\ell(p)}$ and $\varphi_{r(p)}$ with respect to the partitions $(A^{2p,0}, A^{2p,1})$ and $(A^{2p+1,0}, A^{2p+1,1})$ begins. The pair $\langle p, N \rangle$ *demands action* if there is an $(a, b, n) \in \{0,1\} \times \{0,1\} \times \{N, N+1, \dots, s\}$ meeting the following conditions.

(1) For all $m \leq n$, $\varphi_{\ell(p),s}(m)\downarrow$ and $\varphi_{r(p),s}(m)\downarrow$.

(2) Both $\varphi_{\ell(p)}(n) \in A^{2p,a}$ and $\varphi_{r(p)}(n) \in A^{2p+1,b}$.

(3) For all m with $N \leq m \leq n$,

- $\varphi_{\ell(p)}(m) \in A^{2p,1-a} \to m \in W_s$, and
- $\varphi_{r(p)}(m) \in A^{2p+1,1-b} \to m \in W_s$.

(4) We have that $\varphi_{\ell(p)}(n), \varphi_{r(p)}(n) \in X_s$ and $\varphi_{\ell(p)}(n) \prec_{\mathcal{L}} \varphi_{r(p)}(n)$, but currently there is a $d \leq \max_{<}\{\varphi_{\ell(p)}(n), \varphi_{r(p)}(n)\}$ for which there is no $k \in X_s$ with $\varphi_{\ell(p)}(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi_{r(p)}(n)$ and $F(k) = d$.

(5) The element $\varphi_{\ell(p)}(n)$ is not $\preceq_{\mathcal{L}}$-below any of $0, 1, \dots, \langle p, N \rangle$.

If $\langle p, N \rangle$ demands action, let $(a_p, b_p, n) \in \{0,1\} \times \{0,1\} \times \{N, N+1, \dots, s\}$ be the lexicographically least witness to this, call (a_p, b_p, n) the *action witness* for $\langle p, N \rangle$, call the first two coordinates (a_p, b_p) of the action witness the *action sides* for $\langle p, N \rangle$, and call the last coordinate n of the action witness the *action input* for $\langle p, N \rangle$.

Let r be the $<$-greatest number for which there is an M with $\langle r, M \rangle \leq \langle p, N \rangle$. For each $q \leq r$, let (a_q, b_q) be the most recently used action sides by any pair of the form $\langle q, M \rangle$ with $\langle q, M \rangle \leq \langle p, N \rangle$. If no $\langle q, M \rangle \leq \langle p, N \rangle$ has yet demanded action, then let $(a_q, b_q) = (0, 0)$. Let $c = \max_{<}\{\varphi_{\ell(p)}(n), \varphi_{r(p)}(n)\}$, and let $k_0 < k_1 < \cdots < k_c$ be the $c + 1$ least members of

$$\bigcap_{q \leq r} \left(A^{2q,1-a_q} \cap A^{2q+1,1-b_q} \right) \smallsetminus X_s, \tag{$\star$}$$

which exist because the intersection is infinite and X_s is finite. Add $k_0, \dots, k_c$ to X_s. Let $x \in X_s$ be the current $<_{\mathcal{L}}$-greatest element of the interval $(\varphi_{\ell(p)}(n), \varphi_{r(p)}(n))_{\mathcal{L}}$ (or $x = \varphi_{\ell(p)}(n)$ if the interval is empty), and set

$$\varphi_{\ell(p)}(n) \leq_{\mathcal{L}} x <_{\mathcal{L}} k_0 <_{\mathcal{L}} \cdots <_{\mathcal{L}} k_c <_{\mathcal{L}} \varphi_{r(p)}(n).$$

Also set $F(k_i) = i$ for each $i \leq c$, and say that $\langle p, N\rangle$ *has acted and added k's*. This completes the construction.

The constructed $\mathcal{L}$ is a computable linear order. We show that $\mathcal{L} \cong \omega$ by showing that for each z, there are only finitely many elements $<_{\mathcal{L}}$-below z. So fix z. Note that z appears in X_s at stage $s = z$ at the latest.

Consider the actions of $\langle p, N\rangle$. If $\langle p, N\rangle \geq z$ and $\langle p, N\rangle$ acts at stage $s > z$ with action input n, then, by condition (5), it must be that $z <_{\mathcal{L}} \varphi_{\ell(p)}(n) <_{\mathcal{L}} \varphi_{r(p)}(n)$. In this case, the action adds elements to X_s and places them $<_{\mathcal{L}}$-between $\varphi_{\ell(p)}(n)$ and $\varphi_{r(p)}(n)$ and hence places them $<_{\mathcal{L}}$-above z. Therefore, only the actions of $\langle p, N\rangle$ with $\langle p, N\rangle < z$ can add elements $<_{\mathcal{L}}$-below z at stages $s > z$.

We show that each $\langle p, N\rangle < z$ only ever acts to add finitely many elements $k <_{\mathcal{L}} z$. It follows that there are only finitely many elements $<_{\mathcal{L}}$-below z because the $\langle p, N\rangle \geq z$ add no elements $<_{\mathcal{L}}$-below z after stage z, and each $\langle p, N\rangle < z$ adds only finitely many elements $<_{\mathcal{L}}$-below z. So let $\langle p, N\rangle < z$, and assume inductively that there is a stage $s_0 > z$ such that no pair $\langle q, M\rangle < \langle p, N\rangle$ acts to add elements $k <_{\mathcal{L}} z$ after stage s_0.

Notice that a given n can be the action input for $\langle p, N\rangle$ at most once. If $\langle p, N\rangle$ demands action with action input n at stage s, it adds elements of every color $\leq \max_{<}\{\varphi_{\ell(p)}(n), \varphi_{r(p)}(n)\}$ to X_s and places them $<_{\mathcal{L}}$-between $\varphi_{\ell(p)}(n)$ and $\varphi_{r(p)}(n)$. Thus condition (4) is never again satisfied for $\langle p, N\rangle$ with action input n at any stage $t > s$.

Suppose that either $\varphi_{\ell(p)}(m)\uparrow$ or $\varphi_{r(p)}(m)\uparrow$ for some m. Then no $n \geq m$ can be an action input for $\langle p, N\rangle$ because condition (1) always fails when $n \geq m$. Thus only finitely many numbers n can be action inputs for $\langle p, N\rangle$. Because each of these n can be an action input for $\langle p, N\rangle$ at most once, the pair $\langle p, N\rangle$ demands action only finitely many times. Thus in this case, $\langle p, N\rangle$ adds only finitely many elements $<_{\mathcal{L}}$-below z.

We now focus on the case in which both $\varphi_{\ell(p)}$ and $\varphi_{r(p)}$ are total. By cohesiveness, let $(a, b) \in \{0,1\} \times \{0,1\}$ be such that $(\forall^\infty n \in C)(\varphi_{\ell(p)}(n) \in A^{2p,a})$ and $(\forall^\infty n \in C)(\varphi_{r(p)}(n) \in A^{2p+1,b})$. Now consider all pairs $\langle p, M\rangle < z$ with this fixed p.

Claim 1. There is a stage $s_1 \geq s_0$ such that for every M with $\langle p, M\rangle < z$, whenever $\langle p, M\rangle$ demands action at a stage $s \geq s_1$, it always has action sides (a, b).

Proof. [Proof of Claim 1] There are only finitely many $\langle p, M\rangle < z$, so it suffices to show that each $\langle p, M\rangle < z$ either eventually stops demanding action or eventually always has action sides (a, b) when it does demand action.

First suppose that there is a number $m_0 \geq M$ with $m_0 \in C$, but $\varphi_{\ell(p)}(m_0) \in A^{2p,1-a}$. Let $m_1 \geq M$ be such that $\varphi_{\ell(p)}(m_1) \in A^{2p,a}$. Then no $n \geq \max_<\{m_0, m_1\}$ can be the action input for $\langle p, M\rangle$ at any stage s large enough so that $\varphi_{\ell(p),s}(m_0)\downarrow$ and $\varphi_{\ell(p),s}(m_1)\downarrow$ because condition (3) always fails at these stages. Thus there is a stage after which only numbers $n < \max_<\{m_0, m_1\}$ can be the action input for $\langle p, M\rangle$. Each $n < \max_<\{m_0, m_1\}$ can be the action input for $\langle p, M\rangle$ at most once, so in this case $\langle p, M\rangle$ demands action only finitely many times. Similarly, if instead there is a number $m_0 \geq M$ such that $m_0 \in C$, but $\varphi_{r(p)}(m_0) \in A^{2p+1,1-b}$, then $\langle p, M\rangle$ demands action only finitely many times.

Now suppose that $\varphi_{\ell(p)}(n) \in A^{2p,a}$ and $\varphi_{r(p)}(n) \in A^{2p+1,b}$ whenever $n \in C$ and $n \geq M$. Let n_0 be the $<$-least member of C with $n_0 \geq M$. Then whenever $\langle p, M\rangle$ demands action and the action witness (a_p, b_p, n) has $n \geq n_0$, it must be that $(a_p, b_p) = (a, b)$ because otherwise condition (3) would fail. Each $n < n_0$ can be the action input for $\langle p, M\rangle$ at most once, which means that there is a stage $s \geq s_0$ such that whenever $\langle p, M\rangle$ demands action at a later stage $t \geq s$, it always has action sides (a, b). □

Assume that $\langle p, N\rangle$ demands action infinitely often because otherwise we can immediately conclude that it adds only finitely many elements $\prec_{\mathcal{L}}$-below z. Let s_1 be as in Claim 1, let $t > s_1$ be a stage at which $\langle p, N\rangle$ demands action, and let $s_2 = t + 1$. Then $\langle p, N\rangle$ has action sides (a, b) at stage $t < s_2$, and whenever some $\langle p, M\rangle < z$ demands action at a stage $s \geq s_2 > s_1$, it also has action sides (a, b). Thus at every stage $s \geq s_2$, the most recently used action sides by a $\langle p, M\rangle < z$ is always (a, b).

Claim 2. Suppose that an element k is added to X_s and $k \prec_{\mathcal{L}} z$ is defined at some stage $s \geq s_2$. Then $k \in A^{2p,1-a} \cap A^{2p+1,1-b}$.

Proof. [Proof of Claim 2] We already know that if $\langle q, M\rangle \geq z$, then $\langle q, M\rangle$ does not add elements $k \prec_{\mathcal{L}} z$ after stage s_2. Thus we need only consider pairs $\langle q, M\rangle < z$. For these pairs, we have assumed inductively that if $\langle q, M\rangle < \langle p, N\rangle$, then $\langle q, M\rangle$ does not add elements $k \prec_{\mathcal{L}} z$ after stage s_2.

Thus we need only consider pairs $\langle q, M\rangle$ with $\langle p, N\rangle \leq \langle q, M\rangle < z$. Suppose such a $\langle q, M\rangle$ acts after stage s_2. When $\langle q, M\rangle$ chooses the k's to add, it uses an $r \geq p$ in the intersection ($\star$) because $\langle p, N\rangle \leq \langle q, M\rangle$. The action of pair $\langle q, M\rangle$ must use $(a_p, b_p) = (a, b)$. This is because after stage s_2, (a, b) is always the most recently used action sides by the pairs of the form $\langle p, K\rangle$ with $\langle p, K\rangle < z$. Because $\langle p, N\rangle \leq \langle q, M\rangle < z$, it is thus also the case that (a, b) is always the most recently used action sides by the pairs of the form $\langle p, K\rangle \leq \langle q, M\rangle$ at every stage after s_2. Thus when $\langle q, M\rangle$ acts at some stage $s \geq s_2$, it uses $(a_p, b_p) = (a, b)$, and therefore the k's it adds to X_s are chosen from $A^{2p,1-a} \cap A^{2p+1,1-b}$, as claimed. □

We are finally prepared to show that $\langle p, N\rangle$ adds only finitely many elements $k \prec_{\mathcal{L}} z$. Suppose that $\langle p, N\rangle$ acts at some stage $s \geq s_2$, adds an element k to X_s, and defines $k \prec_{\mathcal{L}} z$. Then at stage s, the action witness for $\langle p, N\rangle$ must be (a, b, n) for some n, where $\varphi_{\ell(p)}(n) = x$ for some $x \in A^{2p,a}$, $\varphi_{r(p)}(n) = y$ for some $y \in A^{2p+1,b}$, and $x \prec_{\mathcal{L}} y \preceq_{\mathcal{L}} z$. The action then places k's of each color $d \leq \max_<\{x, y\}$ in the interval $(x, y)_{\mathcal{L}}$. If $\langle p, N\rangle$ acts again at some later stage $t > s$ with some action input m, then again $\varphi_{\ell(p)}(m) \in A^{2p,a}$ and $\varphi_{r(p)}(m) \in A^{2p+1,b}$. However, it cannot again be that $\varphi_{\ell(p)}(m) = x$ and $\varphi_{r(p)}(m) = y$ because condition (4) would fail in this situation. Thus when adding a number $k \prec_{\mathcal{L}} z$, the action input n used by $\langle p, N\rangle$ specifies a pair $(x, y) = (\varphi_{\ell(p)}(n), \varphi_{r(p)}(n)) \in A^{2p,a} \times A^{2p+1,b}$ with $x \prec_{\mathcal{L}} y \preceq_{\mathcal{L}} z$, and each such pair can be specified by $\langle p, N\rangle$ at most once. By Claim 2, every element added $\prec_{\mathcal{L}}$-below z after stage s_2 is in $A^{2p,1-a} \cap A^{2p+1,1-b}$. Therefore there are only finitely many pairs $(x, y) \in A^{2p,a} \times A^{2p+1,b}$ with $x \prec_{\mathcal{L}} y \preceq_{\mathcal{L}} z$, and therefore $\langle p, N\rangle$ can only add finitely many elements $k \prec_{\mathcal{L}} z$. This completes the proof that $\mathcal{L} \cong \omega$.

Let φ and ψ be total computable functions with $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$. We complete the proof by showing that ($*$) is satisfied for φ and ψ. Assume that $(\forall^\infty n \in C)(\psi(n) \prec_{\mathcal{L}} \varphi(n))$, for otherwise ($*$) vacuously holds. Let p be such that $\varphi_{\ell(p)} = \psi$ and $\varphi_{r(p)} = \varphi$. By cohesiveness, let $(a, b) \in \{0, 1\} \times \{0, 1\}$ and $N \in \mathbb{N}$ be such that, for all $n \in C$ with $n > N$, $\varphi_{\ell(p)}(n) \in A^{2p,a}$ and $\varphi_{r(p)}(n) \in A^{2p+1,b}$. Let $n_0 \geq N$ be large enough so that for all $n \in C$ with $n \geq n_0$, $\varphi_{\ell(p)}(n)$ is not $\preceq_{\mathcal{L}}$-below any of $0, 1, \ldots, \langle p, N\rangle$. To choose n_0, notice that the set Z of elements that are $\preceq_{\mathcal{L}}$-below any of $0, 1, \ldots, \langle p, N\rangle$ is finite because $\mathcal{L} \cong \omega$. Then $(\forall^\infty n \in C)(\varphi_{\ell(p)}(n) \notin Z)$ because $\lim_{n \in C} \varphi_{\ell(p)}(n) = \infty$.

Suppose that $n \in C$ and $n \geq n_0$, and furthermore suppose for a contradiction that there is a $d < \max_<\{\varphi_{\ell(p)}(n), \varphi_{r(p)}(n)\}$ such that there is no k with

$\varphi_{\ell(p)}(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi_{r(p)}(n)$ and $F(k) = d$. Then conditions(1)–(5) are satisfied by (a, b, n) at all sufficiently large stages s. Condition (3) is satisfied because $\varphi_{\ell(p)}$ and $\varphi_{r(p)}$ are total. Condition (1) is satisfied because $n \geq N$ and $n \in C$. Condition (2) is satisfied by the choice of N. Condition (4) is satisfied by the assumption that there is no k with $\varphi_{\ell(p)}(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi_{r(p)}(n)$ and $F(k) = d$ and hence there is no such k at every stage s in which both $\varphi_{\ell(p)}(n)$ and $\varphi_{r(p)}(n)$ are present in X_s. Condition (5) is satisfied by the choice of n_0. Each $m < n$ can be the action input for $\langle p, N\rangle$ at most once, and, at sufficiently large stages, (a, b) is the only possible action sides for $\langle p, N\rangle$. Thus at some stage the pair $\langle p, N\rangle$ eventually demands action with action witness (a, b, n). The action of $\langle p, N\rangle$ defines $\varphi_{\ell(p)}(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi_{r(p)}(n)$ and $F(k) = d$ for some k, which contradicts that there is no such k. This shows that $(*)$ holds for $\varphi = \varphi_{r(p)}$ and $\psi = \varphi_{\ell(p)}$, which completes the proof. ◻

Let C be a co-c.e. cohesive set. The linear order $\mathcal{L}$ from Theorem 6.5.1 is an example of a computable copy of ω with $\Pi_C\mathcal{L} \cong \omega + \eta$.

Corollary 6.5.2. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \eta$.

Proof. Let C be co-c.e. and cohesive. Let $\mathcal{L}$ be the computable copy of ω from Theorem 6.5.1 for C. The cohesive power $\Pi_C\mathcal{L}$ has an initial segment of order-type ω by Lemma 6.4.1. There is neither a least nor greatest non-standard element of $\Pi_C\mathcal{L}$ by Lemma 6.4.3. Theorem 6.5.1 implies that the non-standard elements of $\Pi_C\mathcal{L}$ are dense. So $\Pi_C\mathcal{L}$ consists of a standard part of order-type ω and a non-standard part that forms a countable dense linear order without endpoints. So $\Pi_C\mathcal{L} \cong \omega + \eta$. ◻

Example 6.5.3. Let C be a co-c.e. cohesive set, and let $\mathcal{L}$ be a computable copy of ω with $\Pi_C\mathcal{L} \cong \omega + \eta$ as in Corollary 6.5.2.

(1) There is a countable collection of computable copies of ω whose cohesive powers over C are pairwise non-elementarily equivalent. Let $k \geq 1$, and let k denote the k-element linear order $0 < 1 < \cdots < k - 1$ as well as its order-type. Then $k\mathcal{L}$ has order-type ω because $\mathcal{L}$ has order-type ω, and $\Pi_C k \cong k$ by the discussion following Theorem 6.1.3. Using Theorem 6.3.6, we calculate

$$\Pi_C(k\mathcal{L}) \;\cong\; (\Pi_C k)(\Pi_C\mathcal{L}) \;\cong\; k(\omega + \eta) \;\cong\; \omega + k\eta.$$

The linear orders $\omega+k\eta$ for $k \geq 1$ are pairwise non-elementarily equivalent. The sentence "there are $x_0 \prec \cdots \prec x_{k-1}$ such that every other y satisfies either $y \prec x_0$ or $x_{k-1} \prec y$; if $y \prec x_0$, then there is a z with $y \prec z \prec x_0$; and if $x_{k-1} \prec y$, then there is a z with $y \prec z \prec x_{k-1}$" expressing that there is a maximal block of size k is true of $\omega + k\eta$, but not of $\omega + m\eta$ if $m \neq k$. Thus $1\mathcal{L}, 2\mathcal{L}, \ldots$ is a sequence of computable copies of ω whose cohesive powers $\Pi_C(k\mathcal{L})$ are pairwise non-elementarily equivalent.

(2) It is possible for non-elementarily equivalent computable linear orders to have isomorphic cohesive powers. Consider the computable linear orders $\mathcal{L}$ and $\mathcal{L} + \mathbb{Q}$. They are not elementarily equivalent because the sentence "every element has an immediate successor" is true of $\mathcal{L}$ but not of $\mathcal{L} + \mathbb{Q}$. However, using Theorem 6.3.6 and the fact that $\Pi_C\mathbb{Q} \cong \eta$, we calculate

$$\Pi_C(\mathcal{L} + \mathbb{Q}) \;\cong\; \Pi_C\mathcal{L} + \Pi_C\mathbb{Q} \;\cong\; (\omega + \eta) + \eta \;\cong\; \omega + \eta \;\cong\; \Pi_C\mathcal{L}.$$

Thus the cohesive powers $\Pi_C\mathcal{L}$ and $\Pi_C(\mathcal{L} + \mathbb{Q})$ of $\mathcal{L}$ and $\mathcal{L} + \mathbb{Q}$ are isomorphic.

6.6 Shuffling finite linear orders

The *shuffle* $\sigma(X)$ of an at-most-countable non-empty collection X of order-types is obtained by densely coloring $\mathbb{Q}$ with $|X|$-many colors, assigning each order-type in X a distinct color, and replacing each $q \in \mathbb{Q}$ by a copy of the linear order whose type corresponds to the color of q.

Definition 6.6.1. Let X be a non-empty collection of linear orders with $|X| \leq \aleph_0$, let $(\mathcal{L}_i)_{i<|X|}$ be a list of the elements of X, and write $\mathcal{L}_i = (L_i, \prec_{\mathcal{L}_i})$ for each $i < |X|$. Let $F\colon \mathbb{Q} \to |X|$ be a coloring of $\mathbb{Q}$ in which each color occurs densely. Define a linear order $\mathcal{S} = (S, \prec_{\mathcal{S}})$ by replacing each $q \in \mathbb{Q}$ by a copy of $\mathcal{L}_{F(q)}$. Formally, let $S = \{(q, \ell) : q \in \mathbb{Q} \wedge \ell \in L_{F(q)}\}$ and

$$(p, \ell) \prec_{\mathcal{S}} (q, r) \quad \text{if and only if} \quad (p < q) \vee (p = q \wedge \ell \prec_{\mathcal{L}_{F(p)}} r).$$

Because every color occurs densely, the order-type of $\mathcal{S}$ does not depend on the particular choice of F or on the order in which X is enumerated. For this reason, $\mathcal{S}$ is called the *shuffle* of X and is denoted $\sigma(X)$. We typically think of X as a collection of order-types instead of as a collection of concrete linear orders.

Let C be co-c.e. and cohesive, let $\mathcal{L}$ be the linear order from Corollary 6.5.2 for C, and consider the linear order $2\mathcal{L}$ from Example 6.5.3 item (1). We can think of $2\mathcal{L}$ as being obtained from $\mathcal{L}$ by replacing each element of $\mathcal{L}$ by a copy of 2. This operation of replacing each element by a copy of 2 is reflected in the cohesive power, and we have that $\Pi_C(2\mathcal{L}) \cong \omega + 2\eta$.

Let us now consider this same $\mathcal{L} = (L, \prec_{\mathcal{L}})$ along with its coloring $F\colon L \to \mathbb{N}$ from Theorem 6.5.1. Collapse F into a coloring $G\colon L \to \{0,1\}$, where $G(x) = 0$ if $F(x) = 0$ and $G(x) = 1$ if $F(x) \geq 1$. Then the coloring $\widehat{G}$ of $\Pi_C\mathcal{L}$ induced by G uses exactly two colors: $[\![0]\!]$ represented by the constant function with value 0, and $[\![1]\!]$ represented by the constant function with value 1. Both of these colors occur densely in the non-standard part of $\Pi_C\mathcal{L}$. Compute a linear order $\mathcal{M}$ by starting with $\mathcal{L}$, replacing each $x \in L$ with $G(x) = 0$ by a copy of 2, and replacing each $x \in L$ with $G(x) = 1$ by a copy of 3. The cohesive power $\Pi_C\mathcal{M}$ reflects this construction, and we get the linear order obtained from $\Pi_C\mathcal{L}$ by replacing each point of $\widehat{G}$-color $[\![0]\!]$ by a copy of 2 and replacing each point of $\widehat{G}$-color $[\![1]\!]$ by a copy of 3. Thus we have a computable copy $\mathcal{M}$ of ω with $\Pi_C\mathcal{M} \cong \omega + \sigma(\{2,3\})$. Using this strategy, we can shuffle any finite collection of finite linear orders into a cohesive power of a computable copy of ω.

Theorem 6.6.2. Let $k_0, \ldots, k_N$ be non-zero natural numbers. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(\{k_0, \ldots, k_N\})$.

Proof. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be the linear order from Theorem 6.5.1 for C, along with its coloring $F\colon L \to \mathbb{N}$. Collapse F into a coloring $G\colon L \to \{0,1,\ldots,N\}$ by setting $G(x) = F(x)$ if $F(x) < N$ and $G(x) = N$ if $F(x) \geq N$. Consider the induced coloring $\widehat{G}$ on the cohesive power $\Pi_C\mathcal{L}$. For any partial computable φ, $G \circ \varphi$ only takes values $a \leq N$. Thus by cohesiveness, if $C \subseteq^* dom(\varphi)$, then $G \circ \varphi$ is eventually constant on C. Therefore $[\![G \circ \varphi]\!] = [\![a]\!]$ (i.e., the color represented by the constant function with value a) for some $a \leq N$. So $\widehat{G}$ colors $\Pi_C\mathcal{L}$ with colors $[\![0]\!], [\![1]\!], \ldots, [\![N]\!]$, and each color occurs densely in the non-standard part of $\Pi_C\mathcal{L}$.

Let $\mathcal{M} = (M, \prec_{\mathcal{M}})$ be the computable linear order obtained by replacing each $x \in L$ by a copy of $k_{G(x)}$. Formally, we define

$$M = \{\langle x, i\rangle : x \in L \wedge i < k_{G(x)}\}$$

and

$$\langle x, i\rangle \prec_{\mathcal{M}} \langle y, j\rangle \quad \text{if and only if} \quad (x \prec_{\mathcal{L}} y) \vee (x = y \wedge i < j).$$

It is straightforward to check that $\mathcal{M} \cong \omega$, as $\mathcal{M}$ is infinite and every element has only finitely many $\prec_{\mathcal{M}}$-predecessors.

To calculate the order-type of $\Pi_C\mathcal{M}$, we consider what we call the *projection condensation* of $\Pi_C\mathcal{M}$. For a $[\varphi] \in |\Pi_C\mathcal{M}|$, let

$$\mathbf{c}_\pi([\varphi]) = \{[\psi] \in |\Pi_C\mathcal{M}| : \pi_0 \circ \psi =_C \pi_0 \circ \varphi\}.$$

If $\mathbf{c}_\pi([\varphi])$ and $\mathbf{c}_\pi([\psi])$ are distinct, then they are disjoint because $=_C$ is an equivalence relation. To see that $\mathbf{c}_\pi([\varphi])$ is an interval of $\Pi_C\mathcal{M}$, suppose that $[\psi_0]$ and $[\psi_1]$ are in $\mathbf{c}_\pi([\varphi])$ and that $[\psi_0] \prec_{\Pi_C\mathcal{M}} [\theta] \prec_{\Pi_C\mathcal{M}} [\psi_1]$. Then $(\forall^\infty n \in C)[\pi_0(\psi_0(n)) \preceq_{\mathcal{L}} \pi_0(\theta(n)) \preceq_{\mathcal{L}} \pi_0(\psi_1(n))]$. However, $[\psi_0], [\psi_1] \in \mathbf{c}_\pi([\varphi])$ means that $(\forall^\infty n \in C)[\pi_0(\psi_0(n)) = \pi_0(\varphi(n)) = \pi_0(\psi_1(n))]$. Thus it must also be that $(\forall^\infty n \in C)[\pi_0(\theta(n)) = \pi_0(\varphi(n))]$, so $[\theta] \in \mathbf{c}_\pi([\varphi])$. The *projection condensation* $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ of $\Pi_C\mathcal{M}$ is the condensation obtained from the partition $\{\mathbf{c}_\pi([\varphi]) : [\varphi] \in |\Pi_C\mathcal{M}|\}$.

Observe that the map $\mathbf{c}_\pi([\varphi]_{\mathcal{M}}) \mapsto [\pi_0 \circ \varphi]_{\mathcal{L}}$ is an isomorphism between $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ and $\Pi_C\mathcal{L}$, where we now write $[\cdot]_{\mathcal{M}}$ and $[\cdot]_{\mathcal{L}}$ to distinguish between members of $\Pi_C\mathcal{M}$ and of $\Pi_C\mathcal{L}$. Thus we can think of $\widehat{G}$ as coloring $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ by $\widehat{G}(\mathbf{c}_\pi([\varphi]_{\mathcal{M}})) = \widehat{G}([\pi_0 \circ \varphi]_{\mathcal{L}}) = [\![G \circ \pi_0 \circ \varphi]\!]$. Call an element $\mathbf{c}_\pi([\varphi]_{\mathcal{M}})$ of $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ *non-standard* if the corresponding $[\pi_0 \circ \varphi]_{\mathcal{L}}$ is a non-standard element of $\Pi_C\mathcal{L}$. Then the non-standard elements of $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ form a linear order of type η colored by $\widehat{G}$, and every color occurs densely. To finish the proof, it suffices to show that if $\mathbf{c}_\pi([\varphi]_{\mathcal{M}})$ has color $[\![a]\!]$, then its order-type is k_a. It follows that the non-standard elements of $\Pi_C\mathcal{M}$ have order-type $\sigma(\{k_0, \dots, k_N\})$ and therefore that $\Pi_C\mathcal{M}$ has the desired order-type $\omega + \sigma(\{k_0, \dots, k_N\})$.

Consider some $\mathbf{c}_\pi([\varphi]_{\mathcal{M}})$, and suppose that $\widehat{G}(\mathbf{c}_\pi([\varphi]_{\mathcal{M}})) = \widehat{G}([\pi_0 \circ \varphi]_{\mathcal{L}}) = [\![a]\!]$. Let n_0 be such that $G(\pi_0(\varphi(n))) = a$ for all $n \in C$ with $n \geq n_0$. Then $\langle \pi_0(\varphi(n)), i \rangle \in M$ whenever $n \geq n_0$ and $i < k_a$. Define partial computable functions ψ_i for $i < k_a$ by $\psi_i(n) = \langle \pi_0(\varphi(n)), i \rangle$ if $n \geq n_0$ and $\psi_i(n)\uparrow$ if $n < n_0$. Then $[\psi_i]_{\mathcal{M}} \in \mathbf{c}_\pi([\varphi]_{\mathcal{M}})$ for each $i < k_a$, and

$$[\psi_0]_{\mathcal{M}} \prec_{\Pi_C\mathcal{M}} [\psi_1]_{\mathcal{M}} \prec_{\Pi_C\mathcal{M}} \cdots \prec_{\Pi_C\mathcal{M}} [\psi_{k_a-1}]_{\mathcal{M}}.$$

Thus to show that $\mathbf{c}_\pi([\varphi]_{\mathcal{M}})$ has order-type k_a, we need only show that if $[\theta]_{\mathcal{M}} \in \mathbf{c}_\pi([\varphi]_{\mathcal{M}})$, then $[\theta]_{\mathcal{M}} = [\psi_i]_{\mathcal{M}}$ for some $i < k_a$.

So suppose that $[\theta]_{\mathcal{M}} \in \mathbf{c}_\pi([\varphi]_{\mathcal{M}})$, and let $n_1 > n_0$ be such that $\pi_0(\theta(n)) = \pi_0(\varphi(n))$ for all $n \in C$ with $n \geq n_1$. Then also $G(\pi_0(\theta(n))) = a$ for all $n \in C$ with $n \geq n_1$. Thus by the definition of M, it must be that $\pi_1(\theta(n)) < k_a$ for all $n \in C$ with $n \geq n_1$. By cohesiveness, there is therefore an $i < k_a$ such that $(\forall^\infty n \in C)(\pi_1(\theta(n)) = i)$. So $[\theta]_\mathcal{M} = [\psi_i]_\mathcal{M}$, which completes the proof. □

For the remainder of this section, let α denote the order-type $\omega + \zeta\eta + \omega^*$. Ultimately, we want to use the method of Theorem 6.6.2 to show that if $X \subseteq \mathbb{N} \setminus \{0\}$ is either Σ^0_2 or Π^0_2, then, thinking of X as a set of finite order-types, there is a cohesive power of ω with order-type $\omega + \sigma(X \cup \{\alpha\})$. We first consider the particular case $X = \mathbb{N} \setminus \{0\}$ to illustrate how α naturally appears when shuffling infinitely many finite order-types.

Theorem 6.6.3. Let X be the set of all finite non-zero order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$.

Proof. Let $\mathcal{L} = (L, \prec_\mathcal{L})$ be the linear order from Theorem 6.5.1 for C, along with its coloring $F\colon L \to \mathbb{N}$. Let $\mathcal{M} = (M, \prec_\mathcal{M})$ be the computable linear order obtained by replacing each $x \in L$ by a copy of $x+1$ if $F(x) = 0$ and by a copy of $F(x)$ if $F(x) > 0$. Formally, define

$$M = \{\langle x, i\rangle : x \in L \wedge [(F(x) = 0 \wedge i \leq x) \vee (F(x) > 0 \wedge i < F(x))]\}$$

and

$$\langle x, i\rangle \prec_\mathcal{M} \langle y, j\rangle \quad \text{if and only if} \quad (x \prec_\mathcal{L} y) \vee (x = y \wedge i < j).$$

Then $\mathcal{M}$ is a computable linear order of type ω.

As in the proof of Theorem 6.6.2, consider the projection condensation $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ of $\Pi_C\mathcal{M}$ as colored by $\widehat{F}$. By Theorem 6.5.1, the non-standard elements of $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ form a linear order of type η in which the solid $\widehat{F}$-colors occur densely. Furthermore, between any two distinct non-standard elements of $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ there is a non-standard element with a striped color. As in the proof of Theorem 6.6.2, if $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has solid color $[\![k]\!]$ for some $k > 0$, then its order-type is k. We show that if a non-standard $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has either solid color $[\![0]\!]$ or a striped color, then its order-type is α. It follows that the non-standard elements of $\Pi_C\mathcal{M}$ have order-type $\sigma(X \cup \{\alpha\})$, so $\Pi_C\mathcal{M}$ has the desired order-type $\omega + \sigma(X \cup \{\alpha\})$. We give the proof for the striped color case and then indicate the small modification that is needed for the color $[\![0]\!]$ case.

Suppose that $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has striped color $[\![\delta]\!]$. To show that $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has order-type $\alpha \cong \omega + \zeta\eta + \omega^*$, it suffices to show the following for the interval $\mathbf{c}_\pi([\chi]_\mathcal{M})$ of $\Pi_C\mathcal{M}$.

(1) There is a $\prec_{\Pi_C\mathcal{M}}$-least element $[\lambda]_{\mathcal{M}}$.

(2) There is a $\prec_{\Pi_C\mathcal{M}}$-greatest element $[\rho]_{\mathcal{M}}$.

(3) If $[\varphi]_{\mathcal{M}}$ is not $\prec_{\Pi_C\mathcal{M}}$-greatest, then it has an $\prec_{\Pi_C\mathcal{M}}$-immediate successor.

(4) If $[\varphi]_{\mathcal{M}}$ is not $\prec_{\Pi_C\mathcal{M}}$-least, then it has an $\prec_{\Pi_C\mathcal{M}}$-immediate predecessor.

(5) We have that $[\lambda]_{\mathcal{M}} \preccurlyeq_{\Pi_C\mathcal{M}} [\rho]_{\mathcal{M}}$.

(6) If $[\psi]_{\mathcal{M}} \preccurlyeq_{\Pi_C\mathcal{M}} [\varphi]_{\mathcal{M}}$, then there is a $[\theta]_{\mathcal{M}}$ with $[\psi]_{\mathcal{M}} \preccurlyeq_{\Pi_C\mathcal{M}} [\theta]_{\mathcal{M}} \preccurlyeq_{\Pi_C\mathcal{M}} [\varphi]_{\mathcal{M}}$.

Claim 1. $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has a $\prec_{\Pi_C\mathcal{M}}$-least element $[\lambda]_{\mathcal{M}}$ and a $\prec_{\Pi_C\mathcal{M}}$-greatest element $[\rho]_{\mathcal{M}}$.

Proof. [Proof of Claim 1] Define partial computable functions λ and ρ to make $\pi_1(\lambda(n))$ as small as possible and $\pi_1(\rho(n))$ as large as possible while respecting $\pi_0(\lambda(n)) \simeq \pi_0(\rho(n)) \simeq \pi_0(\chi(n))$ for every n:

$$\lambda(n) \simeq \langle \pi_0(\chi(n)), 0\rangle$$
$$\rho(n) \simeq \begin{cases} \langle \pi_0(\chi(n)), \pi_0(\chi(n))\rangle & \text{if } F(\pi_0(\chi(n))) = 0 \\ \langle \pi_0(\chi(n)), F(\pi_0(\chi(n))) - 1\rangle & \text{if } F(\pi_0(\chi(n))) > 0. \end{cases}$$

If $[\theta]_{\mathcal{M}} \in \mathbf{c}_\pi([\chi]_{\mathcal{M}})$, then, for almost every $n \in C$, we have that:

- $\pi_0(\theta(n)) = \pi_0(\chi(n))$,
- $\pi_1(\theta(n)) \geq 0$,
- $F(\pi_0(\chi(n))) = 0 \rightarrow \pi_1(\theta(n)) \leq \pi_0(\chi(n))$, and
- $F(\pi_0(\chi(n))) > 0 \rightarrow \pi_1(\theta(n)) \leq F(\pi_0(\chi(n))) - 1$.

Therefore $[\lambda]_{\mathcal{M}} \preceq_{\Pi_C\mathcal{M}} [\theta]_{\mathcal{M}} \preceq_{\Pi_C\mathcal{M}} [\rho]_{\mathcal{M}}$, which means that $[\lambda]_{\mathcal{M}}$ is $\prec_{\Pi_C\mathcal{M}}$-least and $[\rho]_{\mathcal{M}}$ is $\prec_{\Pi_C\mathcal{M}}$-greatest. □

Claim 2. If $[\varphi]_{\mathcal{M}}$ is not $\prec_{\Pi_C\mathcal{M}}$-greatest, then it has an $\prec_{\Pi_C\mathcal{M}}$-immediate successor; and if $[\varphi]_{\mathcal{M}}$ is not $\prec_{\Pi_C\mathcal{M}}$-least, then it has an $\prec_{\Pi_C\mathcal{M}}$-immediate predecessor.

Proof. [Proof of Claim 2] Suppose that $[\varphi]_{\mathcal{M}} \in \mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is not the $\prec_{\Pi_C \mathcal{M}}$-greatest element $[\rho]_{\mathcal{M}}$ from Claim 1. Then for almost every $n \in C$, we have that $F(\pi_0(\varphi(n))) = 0 \to \pi_1(\varphi(n)) < \pi_0(\varphi(n))$ and that $F(\pi_0(\varphi(n))) > 0 \to \pi_1(\varphi(n)) < F(\pi_0(\varphi(n))) - 1$. Thus define a partial computable θ by

$$\theta(n) \simeq \begin{cases} \langle \pi_0(\varphi(n)), \pi_1(\varphi(n)) + 1 \rangle & \text{if } F(\pi_0(\varphi(n))) = 0 \wedge \pi_1(\varphi(n)) < \pi_0(\varphi(n)) \\ & \text{or } F(\pi_0(\varphi(n))) > 0 \wedge \pi_1(\varphi(n)) < F(\pi_0(\varphi(n))) - 1 \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $[\theta]_{\mathcal{M}} \in \mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is the $\prec_{\Pi_C \mathcal{M}}$-immediate successor of $[\varphi]_{\mathcal{M}}$ by Lemma 6.3.7.

Similarly, if $[\varphi]_{\mathcal{M}} \in \mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is not the $\prec_{\Pi_C \mathcal{M}}$-least element $[\lambda]_{\mathcal{M}}$, then $\pi_1(\varphi(n)) > 0$ for almost every $n \in C$. In this case, define a partial computable ψ by $\psi(n) \simeq \langle \pi_0(\varphi(n)), \pi_1(\varphi(n)) \ominus 1 \rangle$. Then $[\psi]_{\mathcal{M}} \in \mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is the $\prec_{\Pi_C \mathcal{M}}$-immediate predecessor of $[\varphi]_{\mathcal{M}}$. ☐

Claim 3. $[\lambda]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\rho]_{\mathcal{M}}$.

Proof. [Proof of Claim 3] The color $[\![\delta]\!] = \widehat{F}(\mathbf{c}_\pi([\chi]_{\mathcal{M}})) = [\![F \circ \pi_0 \circ \chi]\!]$ is striped, so $\lim_{n \in C} F(\pi_0(\chi(n))) = \infty$. Therefore $\lim_{n \in C} \pi_1(\rho(n)) = \infty$ as well, so $\lim_{n \in C} |(\lambda(n), \rho(n))_{\mathcal{M}}| = \infty$. Therefore $[\lambda]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\rho]_{\mathcal{M}}$ by Lemma 6.3.8.
☐

Claim 4. If $[\psi]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\varphi]_{\mathcal{M}}$, then there is a $[\theta]_{\mathcal{M}}$ with $[\psi]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\theta]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\varphi]_{\mathcal{M}}$.

Proof. [Proof of Claim 4] Suppose that $[\psi]_{\mathcal{M}}$ and $[\varphi]_{\mathcal{M}}$ are members of $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ with $[\psi]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\varphi]_{\mathcal{M}}$. Then $\lim_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{M}}| = \infty$ by Lemma 6.3.8. As $\pi_0 \circ \varphi =_C \pi_0 \circ \psi =_C \pi_0 \circ \chi$, it must therefore be that $\lim_{n \in C} |\pi_1(\varphi(n)) - \pi_1(\psi(n))| = \infty$. Define a partial computable θ by

$$\theta(n) = \begin{cases} \left\langle \pi_0(\chi(n)), \left\lfloor \frac{\pi_1(\psi(n)) + \pi_1(\varphi(n))}{2} \right\rfloor \right\rangle & \text{if } \pi_0(\varphi(n)) = \pi_0(\psi(n)) = \pi_0(\chi(n)) \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $\pi_0 \circ \theta =_C \pi_0 \circ \chi$ as well, and also $\lim_{n \in C} |\pi_1(\varphi(n)) - \pi_1(\theta(n))| = \infty$ and $\lim_{n \in C} |\pi_1(\theta(n)) - \pi_1(\psi(n))| = \infty$. Therefore $[\psi]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\theta]_{\mathcal{M}} \lessapprox_{\Pi_C \mathcal{M}} [\varphi]_{\mathcal{M}}$.
☐

Claims 1–4 show that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ satisfies items (1)–(6). Therefore $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has order-type α.

If instead $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is a non-standard element of solid color $[\![0]\!]$, then essentially the same argument shows that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ satisfies (1)–(6) and thus has order-type α. The only adjustment needed is to showing that $\lim_{n \in C} \pi_1(\rho(n)) = \infty$ in Claim 3. This time we have that $(\forall^\infty n \in C)[F(\pi_0(\chi(n))) = 0]$, so $(\forall^\infty n \in C)[\pi_1(\rho(n)) = \pi_0(\chi(n))]$. However, $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is non-standard, which means that $\lim_{n \in C} \pi_0(\chi(n)) = \infty$. So $\lim_{n \in C} \pi_1(\rho(n)) = \infty$, and Claim 3 holds in this case as well. This completes the proof. □

In the proof of Theorem 6.6.3, it was not necessary to use color $[\![0]\!]$ to shuffle copies of α into $\Pi_C \mathcal{M}$ because the striped colors shuffle in α automatically. However, suppose instead that $k_0, \dots, k_{N-1}$ is a (possibly empty) finite list of non-zero natural numbers, and we want to obtain a cohesive power with order-type $\omega + \sigma(\{k_0, \dots, k_{N-1}, \alpha\})$. To do this, let C be co-c.e. and cohesive, and let $\mathcal{L}$ and F be as in Theorem 6.5.1 for C. Collapse F to $N+1$ colors $\{0, 1, \dots, N\}$ as in Theorem 6.6.2. Compute $\mathcal{M}$ by replacing points x of $\mathcal{L}$ of color 0 by copies of $x+1$ as in Theorem 6.6.3, and by replacing points x of $\mathcal{L}$ of color a with $1 \le a \le N$ by copies of k_{a-1} as in Theorem 6.6.2. Then $\Pi_C \mathcal{M}$ has order-type $\omega + \sigma(\{k_0, \dots, k_{N-1}, \alpha\})$.

Finally, to shuffle Σ^0_2 or Π^0_2 sets of finite order-types into cohesive powers of ω, it is convenient to work with linear orders whose domains are c.e. To this end, let a *partial computable structure* for a computable language $\mathfrak{L}$ consist of a non-empty c.e. domain A along with uniformly partial computable interpretations of the symbols of $\mathfrak{L}$. For example, a partial computable linear order $(L, \prec)$ consists of a non-empty c.e. set L and a partial computable $\varphi\colon L \times L \to \{0,1\}$ computing the characteristic function of the $\prec$-relation on $L \times L$.

Cohesive powers of partial computable structures may be defined exactly as in Definition 6.1.2, the only difference being that the domain A of the partial computable structure $\mathcal{A}$ is now c.e. instead of computable. Suppose that $\mathcal{A}$ is a partial computable structure with infinite c.e. domain A, and let $f\colon \mathbb{N} \to A$ be a one-to-one computable enumeration of A. As usual, we can pull back f to define a computable structure $\mathcal{B}$ with domain $\mathbb{N}$ that is isomorphic to $\mathcal{A}$ via f. For each n-ary relation symbol R, define $R^{\mathcal{B}}(x_0, \dots, x_{n-1})$ to hold if and only if $R^{\mathcal{A}}(f(x_0), \dots, f(x_{n-1}))$ holds. For each n-ary function symbol g, define $g^{\mathcal{B}}(x_0, \dots, x_{n-1})$ to be $f^{-1}(g^{\mathcal{A}}(f(x_0), \dots, f(x_{n-1})))$. For each constant symbol c, define $c^{\mathcal{B}} = f^{-1}(c^{\mathcal{A}})$. Furthermore, if C is any cohesive set, then $\Pi_C \mathcal{A} \cong \Pi_C \mathcal{B}$. This is proved exactly as in Theorem 6.1.4, with $\mathcal{B}$ playing the role of $\mathcal{A}_0$, $\mathcal{A}$ playing the role of $\mathcal{A}_1$, and f being the f enumerating the domain of $\mathcal{A} = \mathcal{A}_1$ as discussed above. The only difference is that now f^{-1} is partial computable instead of computable, but this is inessential because all that matters is that f^{-1} has domain A. Therefore, if we wish to show that there is a computable copy of ω having a cohesive power of a certain order-type, it suffices to show that there is a partial computable copy of ω having a cohesive power of the desired order-type.

Theorem 6.6.4. Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Π^0_2 set, thought of as a set of finite order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$.

Proof. Assume that $X \neq \varnothing$, as otherwise we can compute a copy $\mathcal{M}$ of ω with $\Pi_C\mathcal{M} \cong \omega + \sigma(\{\alpha\})$ by combining the proofs of Theorems 6.6.2 and 6.6.3 in the way described above. Let R be a computable predicate for which $X = \{k : \forall a \exists b R(k,a,b)\}$. Let $k_0 > 0$ be the $<$-least element of X. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be the linear order from Theorem 6.5.1 for C, along with its coloring $F\colon L \to \mathbb{N}$. By the above discussion, it suffices to produce a partial computable copy $\mathcal{M}$ of ω with $\Pi_C\mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$. We define $\mathcal{M}$ from $\mathcal{L}$ as follows. If $x \in L$ has $F(x) < k_0$, then replace x by a copy of $x+1$ as is done with color 0 in the proof of Theorem 6.6.3. If $x \in L$ has $F(x) \geq k_0$, then first replace x by a copy of k_0. Then for each $a \leq x$, search for a b such that $R(F(x),a,b)$. If $(\forall a \leq x)(\exists b)R(F(x),a,b)$, then add further elements to replace x by a copy of $F(x)$ instead of by a copy of k_0. The ultimate effect of this procedure is that if $F(x) \in X$, then we shuffle $F(x)$ into $\Pi_C\mathcal{M}$; whereas if $F(x) \notin X$, then we shuffle k_0 into $\Pi_C\mathcal{M}$. Formally, define

$$\begin{aligned} M = &\{\langle x,i\rangle : x \in L \wedge F(x) < k_0 \wedge i \leq x\} \\ &\cup \{\langle x,i\rangle : x \in L \wedge i < k_0 \leq F(x)\} \\ &\cup \{\langle x,i\rangle : x \in L \wedge (\forall a \leq x)(\exists b)R(F(x),a,b) \wedge k_0 \leq i < F(x)\}. \end{aligned}$$

and

$$\langle x,i\rangle \prec_{\mathcal{M}} \langle y,j\rangle \quad \text{if and only if} \quad (x \prec_{\mathcal{L}} y) \vee (x = y \wedge i < j),$$

where $\prec_{\mathcal{M}}$ is restricted to $M \times M$. Then $\mathcal{M}$ is a partial computable copy of ω. We need to show that $\Pi_C\mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$.

As in the proofs of Theorems 6.6.2 and 6.6.3, consider the projection condensation $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ of $\Pi_C\mathcal{M}$ as colored by $\widehat{F}$. Suppose that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is non-standard.

If $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has solid color $[\![k]\!]$ for some $k < k_0$, then $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has order-type α by the same argument as in the color $[\![0]\!]$ case of the proof of Theorem 6.6.3.

Suppose that $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has solid color $[\![k]\!]$ for some $k \geq k_0$ with $k \in X$. Then $(\forall^\infty n \in C)[F(\pi_0(\chi(n))) = k]$ and $\forall a \exists b R(k, a, b)$. Thus for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \dots, \langle \pi_0(\chi(n)), k-1\rangle$. So $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has order-type k by the same argument as in the proof of Theorem 6.6.2.

Suppose that $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has solid color $[\![k]\!]$ for some $k \geq k_0$ with $k \notin X$. Then $(\forall^\infty n \in C)[F(\pi_0(\chi(n))) = k]$, but $\exists a \forall b \neg R(k, a, b)$. Thus for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \dots, \langle \pi_0(\chi(n)), k_0-1\rangle$. So $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has order-type k_0 by the same argument as in the proof of Theorem 6.6.2.

Finally, suppose that $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has striped color $[\![\delta]\!] = [\![F \circ \pi_0 \circ \chi]\!]$. Then $\lim_{n \in C} F(\pi_0(\chi(n))) = \infty$. There are two cases, depending on how the cohesiveness of C falls with respect to the c.e. set

$$S = \{n : (\forall a \leq \pi_0(\chi(n)))(\exists b) R(F(\pi_0(\chi(n))), a, b)\}.$$

If $C \subseteq^* S$, then for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \dots, \langle \pi_0(\chi(n)), F(\pi_0(\chi(n))) - 1\rangle$. So $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has order-type α by the same argument as in the striped $[\![\delta]\!]$ case in the proof of Theorem 6.6.3.

If $C \subseteq^* \overline{S}$, then for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \dots, \langle \pi_0(\chi(n)), k_0 - 1\rangle$. So $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has order-type k_0 by the same argument as in the proof of Theorem 6.6.2.

The non-standard elements of $\mathbf{c}_\pi(\Pi_C \mathcal{M})$ form a linear order of type η in which the solid $\widehat{F}$-colors occur densely. We have seen that the order-type of a non-standard $\mathbf{c}_\pi([\chi]_\mathcal{M})$ is:

- α if $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has solid color $[\![k]\!]$ with $k < k_0$, which includes 0 because $k_0 > 0$;
- k if $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has solid color $[\![k]\!]$ with $k \geq k_0$ and $k \in X$;
- k_0 if $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has solid color $[\![k]\!]$ with $k \geq k_0$ and $k \notin X$;
- either α or k_0 if $\mathbf{c}_\pi([\chi]_\mathcal{M})$ has a striped color.

Recalling that k_0 is the $<$-least element of X, we therefore have that $\Pi_C \mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$. □

Theorem 6.6.5. Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Σ^0_2 set, thought of as a set of finite order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$.

Proof. The proof is similar to that of Theorem 6.6.4. In the proof of Theorem 6.6.4, we arrange $\mathcal{M}$ to shuffle k into $\Pi_C\mathcal{M}$ when a Π^0_2 property holds of k and to shuffle a fixed k_0 into $\Pi_C\mathcal{M}$ when a Π^0_2 property fails of k. In this proof, we want to shuffle k into $\Pi_C\mathcal{M}$ when a Π^0_2 property fails of k and to shuffle α into $\Pi_C\mathcal{M}$ when a Π^0_2 property holds of k.

Let $X \subseteq \mathbb{N} \setminus \{0\}$ be Σ^0_2. Let R be a computable predicate for which $\overline{X} = \{k : \forall a \exists b R(k,a,b)\}$. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be the linear order from Theorem 6.5.1 for C, along with its coloring $F\colon L \to \mathbb{N}$. Again, it suffices to produce a partial computable copy $\mathcal{M}$ of ω with $\Pi_C\mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$. We define $\mathcal{M}$ from $\mathcal{L}$ as follows. If $x \in L$ has $F(x) = 0$, then replace x by a copy of $x+1$ as is done in the proof of Theorem 6.6.3. If $x \in L$ has $F(x) > 0$, then first replace x by a copy of $F(x)$. Then for each $a \leq x$, search for a b such that $R(F(x),a,b)$. If $x \geq F(x)$ and $(\forall a \leq x)(\exists b)R(F(x),a,b)$, then add further elements to replace x by a copy of $x+1$ instead of a copy of $F(x)$. Formally, define

$$
\begin{aligned}
M = {} & \{\langle x,i\rangle : x \in L \wedge F(x) = 0 \wedge i \leq x\} \\
& \cup \{\langle x,i\rangle : x \in L \wedge F(x) > 0 \wedge i < F(x)\} \\
& \cup \{\langle x,i\rangle : x \in L \wedge F(x) > 0 \wedge (\forall a \leq x)(\exists b)R(F(x),a,b) \wedge i \leq x\}.
\end{aligned}
$$

and

$$\langle x,i\rangle \prec_{\mathcal{M}} \langle y,j\rangle \quad \text{if and only if} \quad (x \prec_{\mathcal{L}} y) \vee (x = y \wedge i < j),$$

where $\prec_{\mathcal{M}}$ is restricted to $M \times M$. Then $\mathcal{M}$ is a partial computable copy of ω. We need to show that $\Pi_C\mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$.

As in the previous proofs, consider the projection condensation $\mathbf{c}_\pi(\Pi_C\mathcal{M})$ of $\Pi_C\mathcal{M}$ as colored by $\widehat{F}$. Suppose that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is non-standard.

If $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has solid color $[\![0]\!]$, then $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has order-type α by the same argument as in the color $[\![0]\!]$ case of the proof of Theorem 6.6.3.

Suppose that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has solid color $[\![k]\!]$ for some $k > 0$ with $k \in X$. Then $(\forall^\infty n \in C)[F(\pi_0(\chi(n))) = k]$, but $\exists a \forall b \neg R(k,a,b)$. Thus for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \ldots, \langle \pi_0(\chi(n)), k-1\rangle$. So $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has order-type k by the same argument as in the proof of Theorem 6.6.2.

Suppose that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has solid color $[\![k]\!]$ for some $k > 0$ with $k \notin X$. Then $(\forall^\infty n \in C)[F(\pi_0(\chi(n))) = k]$ and $\forall a \exists b R(k, a, b)$. Thus for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \ldots, \langle \pi_0(\chi(n)), \pi_0(\chi(n))\rangle$. So $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has order-type α by the same argument as in the color $[\![0]\!]$ case of the proof of Theorem 6.6.3.

Finally, suppose that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has striped color $[\![\delta]\!] = [\![F \circ \pi_0 \circ \chi]\!]$. Then $\lim_{n \in C} F(\pi_0(\chi(n))) = \infty$. There are two cases, depending on how the cohesiveness of C falls with respect to the c.e. set

$$S = \{n : F(\pi_0(\chi(n))) \leq \pi_0(\chi(n)) \ \wedge\ (\forall a \leq \pi_0(\chi(n)))(\exists b) R(F(\pi_0(\chi(n))), a, b)\}.$$

We show that $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has order-type α in both cases.

If $C \subseteq^* S$, then for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \ldots, \langle \pi_0(\chi(n)), \pi_0(\chi(n))\rangle$. So $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has order-type α by the same argument as in the color $[\![0]\!]$ case of the proof of Theorem 6.6.3.

If $C \subseteq^* \overline{S}$, then for almost every $n \in C$, the elements of M of the form $\langle \pi_0(\chi(n)), i\rangle$ are exactly $\langle \pi_0(\chi(n)), 0\rangle, \ldots, \langle \pi_0(\chi(n)), F(\pi_0(\chi(n))) - 1\rangle$. So $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has order-type α by the same argument as in the striped $[\![\delta]\!]$ case in the proof of Theorem 6.6.3.

The non-standard elements of $\mathbf{c}_\pi(\Pi_C \mathcal{M})$ form a linear order of type η in which the solid $\widehat{F}$-colors occur densely. We have seen that the order-type of a non-standard $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ is:

- α if $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has solid color $[\![0]\!]$;
- k if $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has solid color $[\![k]\!]$ with $k \in X$;
- α if $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has solid color $[\![k]\!]$ with $k > 0$ and $k \notin X$;
- α if $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has a striped color.

We therefore have that $\Pi_C \mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$. □

We combine the results of this section into a single statement.

Theorem 6.6.6. Let $X \subseteq \mathbb{N} \smallsetminus \{0\}$ be either a Σ_2^0 set or a Π_2^0 set, thought of as a set of finite order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C \mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$. Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C \mathcal{M}$ has order-type $\omega + \sigma(X)$.

Chapter 7

On Cototality and the Skip Operator

In this chapter we will present the notions of cototality and skip operator in the enumeration degrees. The degree structures as $\mathcal{D}_T$ and $\mathcal{D}_e$ with $\leq$, $\oplus$ and jump operator are also abstract structures. Here we will consider a subclass of the enumeration degrees $\mathcal{D}_e$ called cototal degrees. We started this project in 2015 together with Hristo Ganchev, Steffen Lempp, Jouseph Miller, and Mariya Soskova in Sofia, after the CiE 2015 in Bucharest, when Joseph Miller and Steffen Lempp from University of Madison, Wisconsin, visited Sofia. In 2016 Uri Andrews and Rutger Kuyper from the same university also joined the project and we will present in this chapter the results from the paper [AGK+19], in the journal *Transaction of the American Mathematical Society*. The content of his chapter is almost from [AGK+19], with a small exception in Subsections 7.4.2 and 7.5.1.

A set $A \subseteq \mathbb{N}$ is *cototal* if it is enumeration reducible to its complement, $\overline{A}$. The *skip* of A is the uniform upper bound of the complements of all sets enumeration reducible to A. These are closely connected: A has cototal degree if and only if it is enumeration reducible to its skip. We study cototality and related properties, using the skip operator as a tool in our investigation. We give many examples of classes of enumeration degrees that either guarantee or prohibit cototality. We also study the skip for its own sake, noting that it has many of the nice properties of the Turing jump, even though the skip of A is not always above A (i.e., not all degrees are cototal). In fact, there is a set that is its own double skip.

For an arbitrary set $A \subseteq \mathbb{N}$, the enumeration degree of A and the enumeration degree of $\overline{A}$, the complement of A, need not be comparable. By requiring that they are comparable, we can isolate two interesting subclasses of the enumeration degrees. The first was introduced at the same time as the enumeration degrees themselves. We call a set $A \subseteq \mathbb{N}$ *total* if $\overline{A} \leq_e A$, and we call an enumeration degree *total* if it contains a total set. Note that A is total if and only if $A \equiv_e A \oplus \overline{A}$, where $\oplus$ denotes the effective disjoint union of sets. Since every set of the form $A \oplus \overline{A}$ is total, the total degrees are exactly the degrees of sets $A \oplus \overline{A}$ for some $A \subseteq \mathbb{N}$. We know that the map $\iota : A \mapsto A \oplus \overline{A}$ induces an order-preserving isomorphism between the Turing degrees and the total enumeration degrees. The name "total" is due to the fact that an enumeration degree is total if and only if it contains the graph of a total function. In particular, if A is a total set, then $d_e(A)$ contains the graph of the characteristic function of A.

It is important to note that total degrees[1] always contain nontotal sets as well. For example, all c.e. sets have total degree because they are all enumeration equivalent to the empty set, but only computable c.e. sets are total.

7.1 Cototality

What happens if we reverse the relationship between A and $\overline{A}$? Call a set $A \subseteq \mathbb{N}$ *cototal* if $A \leq_e \overline{A}$, and call an enumeration degree *cototal* if it contains a cototal set. While we are the first to isolate this property under this name, both the property and the name have appeared in the literature. The name was essentially first used, as far as we are aware, in an abstract of A.V. Pankratov from 2000 [Pan00]. Pankratov used "'кототальное' (Russian for "cototal") to refer to what *we* call the graph-cototal degrees, which turns out to be a proper subclass of the cototal degrees: For any total function $f : \mathbb{N} \to \mathbb{N}$, let $G_f = \{\langle n, m\rangle : f(n) = m\}$ be the graph of f. It is easy to see that $\overline{G_f} \leq_e G_f$, so $\overline{G_f}$ is a cototal set. If an enumeration degree contains a set of the form $\overline{G_f}$, then we call it *graph-cototal*.

The graph-cototal sets and degrees were further studied by Solon, Pankratov's advisor. In [Sol06], he used "co-total" to refer to what we call "graph-cototal". However, in the Russian version [Sol05] of the same paper, Solon used "кототальное" for a different property: Call a degree **a** *weakly cototal* if it contains a set A such that $\overline{A}$ has total enumeration degree. It is clear that

[1] We will sometimes use the term *degree* to refer to an enumeration degree.

every cototal degree is weakly cototal, since if $A \leq_e \overline{A}$, then $\overline{A}$ is a total set. So we have

$$\text{graph-cototal} \implies \text{cototal} \implies \text{weakly cototal}.$$

We show that these three properties are distinct. The harder separation is given in Section 7.6, where we use an infinite-injury argument relative to $\mathbf{0}'$ to construct a cototal degree that is not graph-cototal. In Section 7.5, we give examples of weakly cototal degrees that are not cototal, as well as enumeration degrees that are not weakly cototal. Of these properties, we believe that there is a strong case that *cototal* is the most fundamental.

Our study of cototality was motivated by two examples of cototal sets that were pointed out to us by Jeandel [Jea15]. He showed that the set of non-identity words in a finitely generated simple group is cototal (see also Thomas and Williams [TW16]). Jeandel also gave an example from symbolic dynamics: The set of words that appear in a minimal subshift is cototal. This is particularly interesting because the Turing degrees of elements of a minimal subshift are exactly the degrees that enumerate the set of words that appear in the subshift, so understanding the enumeration degree of this set is closely related to understanding the Turing degree spectrum of the subshift.

In Section 7.3, we explain Jeandel's examples in more detail, and we give several other examples of cototal sets and degrees. We show that every Σ^0_2-set is cototal, in fact, graph-cototal. We show that the complement of a maximal independent subset of a computable graph is cototal, and that every cototal degree contains the complement of a maximal independent subset of $\omega^{<\omega}$. Ethan McCarthy proved that the same is true of complements of maximal antichains in $\omega^{<\omega}$. We show that joins of nontrivial K-pairs are cototal, and that the natural embedding of the continuous degrees into the enumeration degrees maps into the cototal degrees. Finally, we note that Harris [Har10] proved that sets with a good approximation have cototal degree.

The earliest reference to a cototality notion seems to be in Case's dissertation [Cas69, p. 14] from 1969; he wrote "The author does not know if there are sets A such that A lies in a total partial degree and $\overline{A}$ lies in a non-total partial degree, but he conjectures that there are no such sets." In our language, Case is conjecturing that if $\overline{A}$ has weakly cototal degree, then it has total degree. The same question also appears in the journal version [Cas71, p. 426]. Gutteridge [Gut71, Chapter II] disproved this conjecture by constructing a quasiminimal graph-cototal degree. Recall that an enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total degree below $\mathbf{a}$ is $\mathbf{0}_e = d_e(\varnothing)$;

in particular, quasiminimal degrees are nontotal. At least two other independent constructions of nontotal cototal degrees appear in the literature: Sanchis [San78], apparently unaware of Case's conjecture, gave an explicit construction of a cototal set that is not total. Aware of Case's conjecture but not Gutteridge's example, Sorbi [Sor88] constructed a quasiminimal cototal degree. Neither of these constructions explicitly produce a graph-cototal degree.

In the abstract mentioned above, Pankratov [Pan00] claimed that there is a graph-cototal Σ^0_2-enumeration degree that forms a minimal pair with every incomplete Π^0_1-enumeration degree.[2] The graph-cototal degrees were studied more extensively by Solon [Sol05, Sol06].[3] He proved that every total enumeration degree above $\overline{K}$ contains the graph G_f of a total function $f\colon\mathbb{N}\to\mathbb{N}$ such that $\overline{G_f}$ is quasiminimal. He also showed that for every total enumeration degree $\mathbf{b}$, there is a graph-cototal enumeration degree $\mathbf{a}$ that is quasiminimal over $\mathbf{b}$. Finally, Solon proved that for every total enumeration degree $\mathbf{b}$ above $\overline{K}$, there is a graph-cototal quasiminimal enumeration degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{b}$ (see below for more about the enumeration jump). This strengthens a result of McEvoy [McE85], who proved that the quasiminimal enumeration degrees have all possible enumeration jumps. Note that all three of Solon's results can also be seen as generalizations of Gutteridge's construction of a quasiminimal graph-cototal degree.

7.2 The skip

Cototality is closely related to the other main subject: the skip operator. Let $\{\Gamma_e\}_{e\in\omega}$ be an effective list of all enumeration operators and let $K_A = \bigoplus_{e\in\omega}\Gamma_e(A) = \{\langle e,x\rangle \mid x\in\Gamma_e(A)\}$. Note that $K_A \equiv_{\mathrm{e}} A$. We define the *skip* of A to be $A^\Diamond = \overline{K_A}$. It is easy to see that the skip is degree invariant, so it induces an operator on enumeration degrees. We use $\mathbf{a}^\Diamond$ to denote the skip of $\mathbf{a}$. Note that the complements of elements of $d_{\mathrm{e}}(A)$ are enumeration reducible to $A^\Diamond$; indeed, they are columns of $A^\Diamond$. In other words, $d_{\mathrm{e}}(A^\Diamond)$ is the maximum possible degree of the complement of an element of $d_{\mathrm{e}}(A)$. One

[2] This result does not appear to be published and we do not know the proof that Pankratov had in mind, but note that graph-cototality is free because every Σ^0_2-enumeration degree is graph-cototal.

[3] We note here a slight confusion in Solon's papers between cototal sets and cototal degrees, which does not, however, affect his main results.

consequence of this characterization is the connection between the skip and cototality:

Proposition 7.2.1. A set $A \subseteq \mathbb{N}$ has cototal degree if and only if $A \leq_e A^\diamond$.

Proof. If A has cototal degree, then there is $B \equiv_e A$ such that $B \leq_e \overline{B}$. So $A \equiv_e B \leq_e \overline{B} \leq_e A^\diamond$. For the other direction, assume that $A \leq_e A^\diamond$. So $K_A \equiv_e A \leq_e A^\diamond = \overline{K_A}$, hence A has cototal degree. □ This connection is quite useful; the separations we prove in Section 7.5 rely on our study of the skip operator in Section 7.4.

In some ways, the skip is analogous to the jump operator in the Turing degrees. For example, a standard diagonalization argument shows that $A^\diamond \nleq_e A$. In Proposition 7.4.1, we restate the well-known fact that $A \leq_e B$ if and only if $A^\diamond \leq_1 B^\diamond$, mirroring the jump in the Turing degrees. Finally, in Theorem 7.4.3, we prove a skip inversion theorem analogous to Friedberg jump inversion: If $S \geq_e \overline{K}$, then there is a set A such that $A^\diamond \equiv_e S$.

The biggest difference between the skip and the Turing jump is that it is not always the case that $A \leq_e A^\diamond$ (because not all enumeration degrees are cototal). In fact, as we will see in Section 7.4.2, there is a *skip 2-cycle*, i.e., a set $A \subseteq \mathbb{N}$ such that $A = A^{\diamond\diamond}$. If we modify the skip to ensure that it is increasing in the enumeration degrees, then we recover the definition of the enumeration jump as introduced by Cooper[4] [Coo84].

The *enumeration jump* of a set $A \subseteq \mathbb{N}$ is $A'_e = K_A \oplus \overline{K_A} \equiv_e A \oplus A^\diamond$. (We will also use A' to denote A'_e). So A has cototal degree if and only if $A'_e \equiv_e A^\diamond$. Of course, the enumeration jump is degree invariant and induces an operator on the enumeration degrees; we use $\mathbf{a}'$ for the jump of $\mathbf{a}$. The definition of the enumeration jump ensures that $A <_e A'_e$, as we expect from a jump. On the other hand, we lose two of the properties that the skip shares with the Turing jump. The enumeration jump is always total, so it cannot possibly map onto all enumeration degrees above $\mathbf{0}'_e$. However, by Friedberg jump inversion, it does map onto the total degrees above $\mathbf{0}'_e$. We will also see, in Proposition 7.4.20, that $A'_e \leq_1 B'_e$ does not necessarily imply that $A \leq_e B$. So neither the skip nor the enumeration jump is the perfect analogue of the Turing jump; we believe that both have a role in the study of the enumeration degrees.

[4]Cooper [Coo84] thanks his student McEvoy for helping provide the correct definition of the enumeration jump operator. Sorbi recalled (in private communication) that Cooper's original "incorrect" definition was actually our definition of the skip operator.

7.3 Examples of cototal sets and degrees

7.3.1 Total degrees

For any set $A \subseteq \mathbb{N}$, the set $A \oplus \overline{A}$ is clearly cototal. Therefore, every total degree is cototal.

7.3.2 The complement of the graph of a total function

As we have noted, if $f{:}\,\mathbb{N} \to \mathbb{N}$ is total, then $\overline{G_f}$, the complement of the graph of f, is a cototal set. This is because $\langle n, m\rangle \in \overline{G_f}$ if and only if there is $m' \neq m$ such that $\langle n, m'\rangle \in G_f$. The class of graph-cototal enumeration degrees turns out to lie strictly between the total degrees and the cototal degrees. The hard part will be showing that there is a cototal degree that is not graph-cototal. We will do that in Section 7.6. To see that every total degree is graph-cototal, recall that each total degree contains the graph of the characteristic function χ_A of some total set A; it also contains the complement of the graph of χ_A. We already saw that $\overline{G_{\chi_A}} \leq_{\mathrm{e}} G_{\chi_A}$. But now since $\langle n, m\rangle \in G_{\chi_A}$ if and only if $m \in \{0,1\}$ and $\langle n, 1-m\rangle \in \overline{G_{\chi_A}}$, we have that $\overline{G_{\chi_A}} \equiv_{\mathrm{e}} G_{\chi_A}$. The next result implies that there are nontotal graph-cototal degrees.

Proposition 7.3.1. Every enumeration degree $\mathbf{a} \leq \mathbf{0}'_{\mathrm{e}}$ is graph-cototal.

Proof. The enumeration degrees below $\mathbf{0}'_{\mathrm{e}}$ consist entirely of Σ^0_2-sets. So, fix an enumeration degree $\mathbf{a} \leq \mathbf{0}'_{\mathrm{e}}$ and a Σ^0_2-set $A \in \mathbf{a}$. We must show that there is a set $\overline{G} \equiv_{\mathrm{e}} A$ that is the complement of the graph G of a total function. We can think of $\overline{G}$ as an infinite table such that each column contains all but one element.

Fix a Σ^0_2-approximation $\{A_s\}_{s<\omega}$ to the set A. This is a uniformly computable sequence of finite sets such that $a \in A$ if and only if $a \in A_s$ for all but finitely many s. So, to every $a \in A$ we can associate the first stage s_a such that $a \in A_s$ for every $s \geq s_a$. We may assume that $A_0 = \varnothing$. Consider the set

$$U = \{\langle a, s\rangle \mid s \neq s_a\} = \{\langle a, s\rangle \mid a \in A_{s-1} \vee (\exists t \geq s)[a \notin A_t]\}.$$

Note that U is a c.e. table such that the a-th column of U contains all natural numbers if $a \notin A$ and all but one natural number if $a \in A$. We combine U and A to define the set $\overline{G}$:

$$\langle a, m\rangle \in \overline{G} \text{ if and only if } m = 0 \;\&\; a \in A \vee m > 0 \;\&\; \langle a, m-1\rangle \in U.$$

The set $\overline{G}$ is clearly in the degree $\mathbf{a}$ and is the complement of the graph of the total function $g : \mathbb{N} \to \mathbb{N}$ such that $g(a) = s_a + 1$ if $a \in A$ and $g(a) = 0$ if $a \notin A$. □

It is worth pointing out that the argument above cannot be extended to further levels of the arithmetical hierarchy. In Section 7.5, we will show that there are Π^0_2-sets that do not even have cototal enumeration degree. Another way to see this is to use a theorem of Badillo and Harris [BH12] proving the existence of a Π^0_2-enumeration degree that contains only properly Π^0_2-sets. Such a degree must have skip equal to $\mathbf{0}'_{\mathrm{e}}$ and hence cannot be cototal. On the other hand, it is easy to see that every Π^0_2-set has weakly cototal degree. This is because every set A is enumeration equivalent to $A \oplus K$, where K is the halting set. So, if A is Π^0_2 then $\overline{A \oplus K} = \overline{A} \oplus \overline{K} \equiv_{\mathrm{e}} \overline{K} \in \mathbf{0}'_{\mathrm{e}}$. As for higher levels of the arithmetical hierarchy, we will see in Section 7.5 that there are Δ^0_3-sets that are not even weakly cototal.

Let $\overline{G}$ be the complement of the graph G of a total function. Notice that the reduction Γ witnessing that $\overline{G} \leq_{\mathrm{e}} G$ described above has the following interesting feature: If $x \in \overline{G}$, then there is a unique axiom in Γ that enumerates x into $\Gamma(G)$. We say that $\overline{G}$ reduces to G via *a unique axiom reduction*. We will next see that this property characterizes the graph-cototal enumeration degrees among all cototal enumeration degrees.

Proposition 7.3.2 (Unique Axiom Characterization)**.** An enumeration degree $\mathbf{a}$ is graph-cototal if and only if it contains a cototal set A that reduces to $\overline{A}$ via a unique axiom reduction.

Proof. We have already seen that graph-cototal degrees have this property. For the reverse direction, let $\mathbf{a}$ be an enumeration degree and let $A \in \mathbf{a}$ be a cototal set that reduces to $\overline{A}$ via a unique axiom reduction Γ. We will, in this case as well, construct an infinite table $\overline{G}$, the first row of which will contain only elements in columns corresponding to members of A. For the remaining rows, we will use the c.e. set Γ. Note that if $\langle a, D\rangle \in \Gamma$ and $a \notin A$, then D must contain an element of A, and if $a \in A$, then there is a unique axiom $\langle a, D\rangle$ such that $D \cap A = \varnothing$. Intuitively, the idea is to assign the axioms of Γ to the remaining undecided elements in each column and enumerate into $\overline{G}$ an element in the a-th column unless it corresponds to the unique correct axiom for a. We formalize this idea below.

Fix a computable function s that lists Γ without repetitions. Without loss of generality, we may assume that Γ is infinite, as a finite unique axiom reduction can enumerate only a finite set and we already know that $\mathbf{0}_e$ is graph-cototal. We define the set $\overline{G}$ as follows:

$$\begin{aligned}\langle a,m\rangle \in \overline{G} \text{ if and only if } & [m=0 \,\&\, a\in A] \\ \text{or } & \Bigg[m>0 \,\&\, \Big[s(m-1) \text{ is not an axiom for } a \\ & \qquad \text{or } (s(m-1)=\langle a,D\rangle \,\&\, D\cap A\neq\varnothing)\Big]\Bigg].\end{aligned}$$

The set $\overline{G}$ is clearly in the degree $\mathbf{a}$ and is the complement of the graph of the total function $g\colon \mathbb{N}\to\mathbb{N}$ such that $g(a)=d+1$, where d codes the unique correct axiom for a if $a\in A$, and $g(a)=0$ if $a\notin A$. □

We can make this characterization even tighter by noting that the reduction Γ used to witness that $\overline{G}\leq_e G$ is furthermore a *singleton operator*: every axiom in Γ is of the form $\langle a,\{b\}\rangle$ where $a\neq b$.

We will therefore be interested in finding examples of cototal enumeration degrees that do not satisfy the Unique Axiom Characterization, as we would like to separate the cototal degrees from the graph-cototal degrees. The next example, which comes from graph theory, is motivated by this desire.

7.3.3 Complements of maximal independent sets

Recall that an (undirected) graph is a pair $G=(V,E)$, where V is a set of vertices and E is a set of unordered pairs of vertices, called the edge relation.

Definition 7.3.3. An *independent set* for a graph $G=(V,E)$ is a set of vertices $S\subseteq V$ such that no pair of distinct vertices in S is connected by an edge. An independent set is *maximal* if it has no proper independent superset.

In other words, an independent set S is maximal if and only if every vertex $v\in V$ is either in S or is connected by an edge to an element of S. The maximal independent sets for the graph of the cube are illustrated in the figure below, courtesy of David Eppstein and Wikipedia.

Consider an infinite graph $G=(\mathbb{N},E)$ with a computable edge relation. For example, we can think of the tree $\omega^{<\omega}$ as a computable graph on the

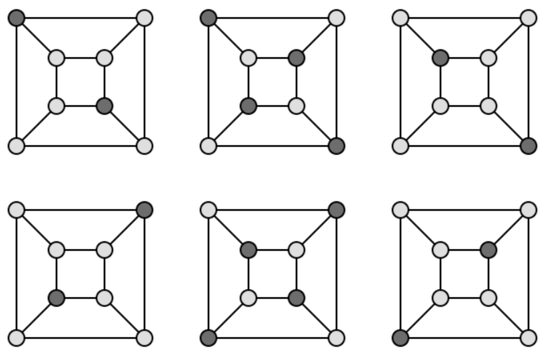

Figure 7.1: Maximal independent sets for the cube

natural numbers by fixing an effective coding of the finite sequences of natural numbers and putting an edge between any non-root node and its immediate predecessor. If S is a maximal independent set for G, then S can enumerate its complement:

$$\overline{S} = \{v \mid (\exists u \in S)[\{u, v\} \in E]\}.$$

It follows that complements of maximal independent sets in computable graphs on $\mathbb{N}$ are cototal. Our main reason for considering this example is that, in general, this reduction does not have the unique axiom property. This is well illustrated by Figure 7.1: the maximal independent set in the middle of the first row, for example, would enumerate each element of its complement with three distinct correct axioms. Hence we might hope that complements of maximal independent sets allow us to move beyond the graph-cototal degrees. They do, and in fact, they are universal for the cototal enumeration degrees.

Theorem 7.3.4. Every cototal degree contains the complement of a maximal independent set for $\omega^{<\omega}$.

Proof. Fix a cototal set A and let $A = \Gamma(\overline{A})$. We will build a set $G \subseteq \omega^{<\omega}$ which will be the complement of a maximal independent set for $\omega^{<\omega}$. In this case, we will again assume that A is not c.e. and so Γ is an infinite c.e. set, as there are easy examples of computable maximal independent sets, e.g., the set of all odd-length strings in $\omega^{<\omega}$. So let g be a computable listing of Γ without repetitions. We will further assume that no axiom in Γ is of the form $\langle a, \varnothing\rangle$. We can easily replace Γ with an operator that fits this description by replacing every such axiom by $\langle a, \{b_0\}\rangle$, where b_0 is some fixed member of $\overline{A}$. We will also fix a number $a_0 \in A$.

To every node $\sigma \in \omega^{<\omega}$ other than the root $\langle\rangle$, we will computably assign a finite set D_σ. The set G will then be defined as

$$G = \{\sigma \mid D_\sigma \cap A \neq \varnothing\} \cup \{\langle\rangle\}.$$

The assignment is defined by induction:

1. If $\sigma = n$ is a length-1 string then $D_\sigma = \{n\}$.
2. If $\sigma = \tau n$. Then we have two cases:
   (a) If $g(n)$ is not an axiom for any member of D_τ then we let $D_\sigma = \{a_0\}$.
   (b) If $g(n) = \langle a, D\rangle$ is an axiom for $a \in D_\tau$ then we let $D_\sigma = D$.

From the definition, it follows that $G \leq_e A$, and from part 1 in particular, that also $A \leq_e G$, as $n \in G$ if and only if $\{n\} \cap A \neq \varnothing$ if and only if $n \in A$. It remains to be shown that $\overline{G}$ is a maximal independent set.

Fix $\tau, \sigma \in \omega^{<\omega}$ such that $\sigma = \tau n$. We must show that either $\sigma \in G$ or $\tau \in G$ to ensure that $\overline{G}$ is independent. If $\tau \notin G$ then $\tau \neq \langle\rangle$ and $D_\tau \subseteq \overline{A}$. If $g(n)$ is not an axiom for any element in D_τ, then $D_\sigma = \{a_0\} \subseteq A$ and hence $\sigma \in G$. Otherwise $g(n) = \langle a, D_\sigma\rangle$ and $a \notin A$. As $A = \Gamma(\overline{A})$ it must be that $D_\sigma \nsubseteq \overline{A}$ and so $D_\sigma \cap A \neq \varnothing$, hence $\sigma \in G$.

Finally, we must show that every $\tau \in G$ has a neighbor σ in $\overline{G}$ to ensure that $\overline{G}$ is maximal. If $\tau = \langle\rangle$, then σ can be chosen as any of its length-1 neighbors corresponding to elements $b \in \overline{A}$. Suppose that $\tau \neq \langle\rangle$ and let $a \in D_\tau \cap A$. Then $a \in \Gamma(\overline{A})$ and hence there is an axiom $\langle a, D\rangle \in \Gamma$ such that $D \subseteq \overline{A}$. Fix n such that $\langle a, D\rangle = g(n)$. We assign the set D to the string $\sigma = \tau n$; it follows that $\sigma \notin G$. □

Note, that the proof above holds even if we restrict ourselves to *singleton degrees*, the degree structure induced by restricting reductions to singleton operators. The singleton degree of a set that is cototal with respect to singleton reduction contains the complement of a maximal independent set for $\omega^{<\omega}$.

7.3.4 Complements of maximal antichains in $\omega^{<\omega}$

A closely related example comes from simply considering maximal antichains in $\omega^{<\omega}$. In this case, the partial ordering on finite sequences of natural numbers is defined by $\sigma \leq \tau$ if and only if $\sigma \preceq \tau$. An *antichain* is a subset of $\omega^{<\omega}$ such that no two elements in it are comparable, and an antichain is *maximal* if it cannot be extended to a proper superset that is also an antichain. Examples of computable maximal antichains are easy to come up with: For any fixed n, the set of all elements of $\omega^{<\omega}$ of length n is a maximal antichain.

If S is a maximal antichain, then $\overline{S} \leq_e S$ as $\sigma \in \overline{S}$ if an only if there is some $\tau \in S$ that is comparable with σ. As in the example above, this reduction does not have the unique axiom property. Consider for example the maximal antichain of all strings of length n. Then every string of length $m < n$ has infinitely many reasons to be enumerated into the complement of this maximal antichain. Ethan McCarthy has shown that complements of maximal antichains are also universal for the cototal enumeration degrees.

Theorem 7.3.5 (McCarthy [McC18])**.** Every cototal degree contains the complement of a maximal antichain in $\omega^{<\omega}$.

7.3.5 The set of words that appear in a minimal subshift

We will next give a more detailed account of our motivating examples, introduced by Jeandel [Jea15]. The first one requires us to recall some definitions from symbolic dynamics.

Definition 7.3.6. Let $X \subseteq 2^\omega$ be closed in the usual topology on Cantor space.

1. X is a *subshift* if X is closed under the shift operation, which removes the first bit in a binary sequence, i.e., $a\alpha \in X$ implies $\alpha \in X$.

2. If X is a subshift then the *language of* X is the set

$$\mathcal{L}_X = \{\sigma \in 2^{<\omega} : (\exists \alpha \in X)[\sigma \text{ is a subword of } \alpha]\}.$$

   The set $\overline{\mathcal{L}_X}$ is called the *set of forbidden words*.

3. A subshift X is *minimal* if it has no nonempty proper subset that is also a subshift. This is equivalent to saying that every $\sigma \in \mathcal{L}_X$ is a subword of every $\alpha \in X$.

Jeandel discovered an interesting relationship between the enumeration degree of the language of a minimal subshift and the Turing degrees of the elements of the subshift: The Turing degrees of elements in X are exactly the

Turing degrees that enumerate $\mathcal{L}_X$. This fact is particularly interesting if one takes into account Selman's characterization of enumeration reducibility. For an arbitrary set A, let $\mathcal{E}_A$ denote the set of all Turing degrees whose elements compute enumerations of A. Selman [Sel71] proved that $A \leq_e B$ if and only if $\mathcal{E}_B \subseteq \mathcal{E}_A$. Thus, the enumeration degree of the set $\mathcal{L}_X$ can be characterized by $\mathcal{E}_{\mathcal{L}_X}$, which turns out to be exactly the set of Turing degrees that compute elements of the minimal subshift X. It is then natural to ask what additional properties an enumeration degree must have in order to be the enumeration degree of the language of a minimal subshift. The following theorem shows that it must be cototal.

Theorem 7.3.7 (Jeandel [Jea15])**.** $\mathcal{L}_X \leq_e \overline{\mathcal{L}_X}$.

Ethan McCarthy has very recently shown that, in fact, cototality precisely characterizes the enumeration degrees of languages of minimal subshifts.

Theorem 7.3.8 (McCarthy [McC18])**.** If A is cototal, then $A \equiv_e \mathcal{L}_X$ for some minimal subshift X.

7.3.6 The non-identity words in a finitely generated simple group

The second example from Jeandel [Jea15] is related to group theory.

Definition 7.3.9. Let G be a group.

1. G is *finitely generated* if there are finitely many elements in G, called generators, such that every element in G can be expressed as a product of these generators. (For convenience, we will assume that the set of generators is closed under inverses.)

2. G is *simple* if its only normal subgroups are G and the trivial group.

3. The set of *identity words* of G is the set $\mathcal{W}_G$ of all words (i.e., finite sequences of generators) that represent the identity element.

4. A *presentation* of G is a pair $\langle F \mid R \rangle$ such that F is a set of generators and $\mathcal{W}_G$ is the normal closure of $R \subset \mathcal{W}_G$.

The word problem for a group G is the problem of deciding the set $\mathcal{W}_G$. Kuznetsov [Kuz58] showed that if G is a finitely generated simple group with a presentation $\langle F \mid R\rangle$ such that R is computable, then it has a decidable word problem. Jeandel considered the collection of all finitely generated simple groups without restricting the complexity of their presentation. He showed that the set of non-identity words in a finitely generated simple group is cototal. This was also independently observed by Thomas and Williams [TW16].

Theorem 7.3.10 (Jeandel [Jea15]; Thomas and Williams [TW16])**.** If G is a finitely generated simple group then $\overline{\mathcal{W}_G} \leq_e \mathcal{W}_G$.

This generalizes Kuznetsov's result, as if a group $G = \langle F \mid R\rangle$ has a computable set of relations R, then $\mathcal{W}_G$ is automatically c.e. The fact that $\overline{\mathcal{W}_G} \leq_e \mathcal{W}_G$ shows that $\overline{\mathcal{W}_G}$ is also c.e. and hence $\mathcal{W}_G$ is computable.

7.3.7 Joins of nontrivial $\mathcal{K}$-pairs

Our next example relates to a class of pairs of enumeration degrees that have been recently shown to play an important role when it comes to the first-order definability of relations on $\mathcal{D}_e$.

Definition 7.3.11. A pairs of sets $\{A, B\}$ form a *$\mathcal{K}$-pair* if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. A $\mathcal{K}$-pair is *nontrivial* if neither of its components is c.e.

$\mathcal{K}$-pairs were introduced by Kalimullin [Kal03]. He showed that they are first-order definable in the structure of the enumeration degrees and used them to give a first-order definition of the enumeration jump. Cai, Ganchev, Lempp, Miller, and M. Soskova [CGL+16] used $\mathcal{K}$-pairs to define the class of total enumeration degrees. It is therefore reasonable to always keep an eye on the class of $\mathcal{K}$-pairs as it might hold the key to the first-order definability of relations that we are considering here as well: cototal enumeration degrees and the skip operator. In the next section, $\mathcal{K}$-pairs will give us a wide variety of examples of sets that do not have cototal degree. When one considers the join $A \oplus B$, however, of a nontrivial $\mathcal{K}$-pair $\{A, B\}$, one always gets a cototal set. To see this, we will need to review an important property of $\mathcal{K}$-pairs.

Proposition 7.3.12 (Kalimullin [Kal03])**.** If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair then

- $A \le_e \overline{B}$ and $B \le_e \overline{A}$;
- $\overline{B} \le_e A \oplus \overline{K}$ and $\overline{A} \le_e B \oplus \overline{K}$.

It follows from the first part that if $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then $A \oplus B \le_e \overline{B} \oplus \overline{A} \equiv_e \overline{A \oplus B}$.

We would like to point out that this example generalizes the fact that every total degree is cototal, as by Cai, Ganchev, Lempp, Miller, and M. Soskova [CGL⁺16], the total degrees are exactly the ones that contain the join of a particular kind of a $\mathcal{K}$-pair. The joins of nontrivial $\mathcal{K}$-pairs therefore form a first-order definable class of cototal enumeration degrees that contains the total enumeration degrees. Unfortunately, they do not contain all cototal degrees. Ahmad [Ahm89] showed that there are nonsplitting Σ^0_2-enumeration degrees, i.e. degrees that are not the least upper bound of any pair of strictly smaller degrees. So, even though, as we have already seen, all Σ^0_2-enumeration degrees are cototal, the nonsplitting ones cannot be joins of nontrivial $\mathcal{K}$-pairs.

7.3.8 Continuous degrees

Motivated by a question of Pour-El and Lempp from computable analysis, Miller [Mil04] introduced a degree structure that captures the complexity of elements of computable metric spaces, such as $\mathcal{C}[0,1]$ and $[0,1]^\omega$. This structure naturally embeds into the enumeration degrees, and the range of this embedding is strictly between the class of total enumeration degrees and the class of all enumeration degrees.

As an example, consider the metric space $\mathcal{C}[0,1]$ of continuous functions on the unit interval with the standard metric

$$d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

A computable presentation of a metric space $\mathcal{M}$ consists of a fixed dense sequence $Q^{\mathcal{M}} = \{q_n\}_{n<\omega}$ on which the metric is computable as a function on indices. For a computable presentation of $\mathcal{C}[0,1]$ we can fix, for example, a reasonable enumeration of the polygonal functions having segments with rational endpoints. A *name* n_f for a continuous function f is a code (say, as an element of ω^ω) that gives a way to approximate f. Specifically, a name n_f should code a function taking a rational number $\varepsilon > 0$ and producing an index $n_f(\varepsilon)$ such that $d(f, q_{n_f(\varepsilon)}) < \varepsilon$. For $f, g \in \mathcal{C}[0,1]$, we say that f is *reducible* to g if every name for g computes a name for f. In the same way, we can compare the complexity of elements from arbitrary metric spaces. This reducibility induces a degree structure, the *continuous degrees*. It turns out that every continuous degree contains an element of $\mathcal{C}[0,1]$.

In order to understand the embedding of the continuous degrees into the enumeration degrees, it is easier to focus on another computable metric space: The *Hilbert cube* is $[0,1]^\omega$ along with the metric

$$d(\alpha,\beta) = \sum_{n\in\omega} 2^{-n}|\alpha(n) - \beta(n)|.$$

A dense set witnessing that $[0,1]^\omega$ is computable is, for example, a reasonable enumeration of the rational sequences with finite support. As was the case with $\mathcal{C}[0,1]$, every continuous degree contains an element of $[0,1]^\omega$.

Miller gave a way to assign to a sequence $\alpha \in [0,1]^\omega$ a set A_α such that $\mathcal{E}_{A_\alpha}$ (defined in Section 7.3.5) is the set of all Turing degrees that compute names of α. This induces an embedding of the continuous degrees into the enumeration degrees.

Definition 7.3.13 (Miller [Mil04])**.** For $\alpha \in [0,1]^\omega$, let

$$A_\alpha = \bigoplus_{i<\omega}(\{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q >_{\mathbb{Q}} \alpha(i)\}).$$

It is not hard to see that A_α has the desired property: Computing a name for α is exactly as hard as enumerating A_α. We say that the enumeration degree of A_α is *continuous*. By showing that there is a nontotal continuous enumeration degree, Miller proved that there are continuous functions that do not have a name of least Turing degree, which answered Pour-El and Lempp's question.

Note that if α does not have any rational entries, then A_α is a total set. If, on the other hand, α does have rational entries, then every component of A_α is nonuniformly equivalent to a total set. The existence of nontotal continuous enumeration degrees shows that this nonuniformity is significant. We are nevertheless able to show that all continuous degrees are cototal.

Proposition 7.3.14. Every continuous degree is cototal.

Proof. Let $\alpha \in [0,1]^\omega$ and $A_\alpha = \bigoplus_{i<\omega}(\{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q >_{\mathbb{Q}} \alpha(i)\})$. By rearranging the odd and even elements in every column of $\overline{A}_\alpha$, we obtain the set $B_\alpha \equiv_e \overline{A}_\alpha$ defined by

$$B_\alpha = \bigoplus_{i<\omega}(\{q \in \mathbb{Q} \mid q \leq_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q \geq_{\mathbb{Q}} \alpha(i)\}).$$

It is now easy to see that q is a member of the i-th even column of A_α if and only if there is an $r >_{\mathbb{Q}} q$ such that r is in the i-th even column of B_α. Similarly, q is a member of the i-th odd column of A_α if and only if there is an $r <_{\mathbb{Q}} q$ such that r is in the i-th odd column of B_α. It follows that $A_\alpha \leq_e B_\alpha \equiv_e \overline{A_\alpha}$. □

7.3.9 Sets with good approximations have cototal degree

Lachlan and Shore [LS92] introduced the following general notion of an approximation to a set.

Definition 7.3.15. Let A be a set of natural numbers. A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ (given by canonical indices) is a *good approximation to* A if

- for every n, there is a stage s such that $A \upharpoonright n \subseteq A_s \subseteq A$; and
- for every n, there is a stage s such that for every $t > s$, if $A_t \subseteq A$ then $A \upharpoonright n \subseteq A_t$.

This definition can be seen as a generalization of Cooper's notion of a Σ^0_2-approximation with infinitely many thin stages, used to show the density of the Σ^0_2-enumeration degrees [Coo84]. Lachlan and Shore [LS92] introduced the hierarchy of the n-c.e.a. sets. A set is 1-c.e.a. if it is c.e., and $(n+1)$-c.e.a. if it is the join of an n-c.e.a. set X and a set Y c.e. in X. It is not difficult to see that the enumeration degrees of the 2-c.e.a. sets are exactly the Σ^0_2-enumeration degrees. Lachlan and Shore proved that every set that is n-c.e.a. has a good approximation and then showed that the enumeration degrees of the n-c.e.a. sets are dense. Harris [Har10] proved that sets that have good approximations always have cototal enumeration degrees. We outline his proof below for completeness.

Proposition 7.3.16 (Harris [Har10, Proposition 4.1])**.** If A has a good approximation, then $K_A \leq_e \overline{K_A}$.

Proof.

Let $\{A_s\}_{s<\omega}$ be a good approximation to A. Consider the set C defined by

$$C = \{\langle x, s\rangle \mid (\exists t > s)[A_t \subseteq A \;\&\; x \notin A_t]\}.$$

It follows from the definition that $C \leq_e A$. Using the fact that $\overline{K_A} = \bigoplus_{e<\omega} \overline{\Gamma_e(A)}$ is a uniform upper bound of the set of complements of all sets that are enumeration reducible to A, we obtain that $\overline{C} \leq_e \overline{K_A}$. Now, let us take a closer look at $\overline{C}$:

$$\overline{C} = \{\langle x, s\rangle \mid (\forall t > s)[A_t \subseteq A \rightarrow x \in A_t]\}.$$

Using the second property of good approximations, notice that $x \in A$ if and only if there is a stage s such that $\langle x, s\rangle \in \overline{C}$. It follows that $A \leq_e \overline{C}$. This now gives us that $K_A \equiv_e A \leq_e \overline{C} \leq_e \overline{K_A}$. □

In particular, we obtain that the enumeration degrees of n-c.e.a. sets are cototal.

7.4 The skip

In the previous section, we saw many examples of cototal sets and enumeration degrees. In this section, we study the skip operator, in part to provide a wide variety of examples of degrees that are not cototal. Recall that the *skip* of a set $A \subseteq \mathbb{N}$ is $A^\diamond = \overline{K_A}$. As we saw in the introduction, the skip gives us an easy way to determine whether or not a degree is cototal. For the reader's convenience, we restate that result:

Proposition 7.2.1. *A set $A \subseteq \mathbb{N}$ has cototal degree if and only if $A \leq_e A^\diamond$.*

In addition to being a tool in our study of cototality, the skip is a natural operator in its own right. As we discussed in the introduction, the enumeration jump fails to have some of the nice properties of the Turing jump. For example, it is well-known that $A \leq_T B$ if and only if $K^A \leq_1 K^B$, where K^A denotes the halting set relative to A. The analogous property does not hold, in general, for the enumeration jump. It is true that $A \leq_e B$ implies $K_A \oplus \overline{K_A} \leq_1 K_B \oplus \overline{K_B}$, but the reverse implication can fail; we will see in Proposition 7.4.20. The skip, on the other hand, gives us an embedding of the enumeration degrees into the 1-degrees.

Proposition 7.4.1. $A \leq_e B$ if and only if $A^\diamond \leq_1 B^\diamond$.

Proof. If $A \leq_e B$, then $K_A \leq_e B$ and hence K_A is a fixed column of $K_B = \bigoplus_{e<\omega} \Gamma_e(B)$, where $\{\Gamma_e\}_{e\in\omega}$ is the standard listing of all enumeration operators. It follows that $\overline{K_A}$ is a fixed column in $\overline{K_B}$ and hence $\overline{K_A} \leq_1 \overline{K_B}$.

If $\overline{K_A} \leq_1 \overline{K_B}$ then $K_A \leq_1 K_B$ and hence $A \equiv_e K_A \leq_e K_B \equiv_e B$. □

This shows that we can define the skip operator on degrees.

Definition 7.4.2. The *skip* of the enumeration degree $\mathbf{a}$ is $\mathbf{a}^\Diamond = d_e(A^\Diamond)$ for any member $A \in \mathbf{a}$.

7.4.1 Skip inversion

It follows from Proposition 7.2.1 that an enumeration degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^\Diamond$, if and only if $\mathbf{a}^\Diamond = \mathbf{a}'$. The definition of the enumeration jump operator restricts its range to the total enumeration degrees and by monotonicity to the total enumeration degrees in the cone above $\mathbf{0}'_e$. By transferring the Friedberg Jump Inversion Theorem through the standard embedding into the enumeration degrees, we see that every total enumeration degree above $\mathbf{0}'_e$ is in the range of the jump operator. The range of the skip operator is also restricted by monotonicity to enumeration degrees above $\mathbf{0}_e^\Diamond = \mathbf{0}'_e$. We show that this is the only restriction on the range of the skip operator, thereby providing a further analogy between the skip and the Turing jump. Recall that $\overline{K}$, the complement of the halting set, is a representative of the degree $\mathbf{0}'_e$.

Theorem 7.4.3. For any set $S \geq_e \overline{K}$, there is a set A such that $A^\Diamond \equiv_e S$. (In fact, we also have $S \equiv_e \overline{A} \equiv_e \overline{A} \oplus \overline{K}$ and $\overline{S} \leq_e A \oplus \overline{K}$.)

Proof. Given a set $S \geq_e \overline{K}$, we build a set A such that $S \equiv_e \overline{A} \leq_e A^\Diamond \leq_e \overline{A} \oplus \overline{K}$. For a set $X \subseteq \mathbb{N}$ and a natural number e, let $X^{[e]} = \{\langle e, x\rangle : x \in \mathbb{N}\} \cap X$. We will build A meeting two types of requirements:

$$\mathcal{R}_e \colon e \in S \iff A^{[e]} \neq \mathbb{N}^{[e]},$$
$$\mathcal{P}_e \colon \text{"force } e \text{ into } K_A \text{ subject to higher-priority restraints".}$$

The $\mathcal{R}_e$-requirements ensure that $S \leq_e \overline{A}$, as $e \in S$ if and only if there is an $x \in \mathbb{N}$ such that $\langle e, x\rangle \in \overline{A}$. The basic strategy for $\mathcal{R}_e$ is quite simple: If $e \notin S$ then enumerate all of $\mathbb{N}^{[e]}$ into A. Otherwise, withhold one number $a_e \in \mathbb{N}^{[e]}$ from A and enumerate $\mathbb{N}^{[e]} \smallsetminus \{a_e\}$ into A.

The $\mathcal{P}_e$-requirements will let us prove that $A^\Diamond$ can be enumerated from $\overline{K}$ and S. The basic strategy for $\mathcal{P}_e$ is to try to force e into K_A by adding a finite set to the current version of A so that $e \notin K_A$ can only be caused by the finitely many numbers a_i that higher priority $\mathcal{R}$-requirements use for coding the values of S. We will use the 1-equivalent form of the set K_A, namely, $\{e \mid e \in \Gamma_e(A)\}$, where $\{\Gamma_e\}_{e\in\omega}$ is our fixed listing of all enumeration operators.

We now proceed in stages as follows:

Stage 0: Set $a_e = \langle e, 0\rangle$ for all $e \in \omega$, and set $A_0 = \varnothing$.

Stage $s = e + 1$: For each subset $D \subseteq \{i{:}\, i < e\}$, check if there are a finite subset $F_D \subseteq \mathbb{N} \smallsetminus \{a_i{:}\, 0 \le i < e\}$ and a stage t such that $e \in \Gamma_{e,t}(F_D \cup \{a_i{:}\, i \in D\})$. If so, take F_D from the least such pair; otherwise, set $F_D = \varnothing$. Set

$$F = \bigcup_{D \subseteq \{i:i<e\}} F_D \text{ and}$$

$$G = \{z \mid z \text{ is the least member of } \mathbb{N}^{[i]} \smallsetminus (A_s \cup F \cup \{a_i\}) \text{ for some } i < e\}.$$

Enumerate $F \cup G$ into A_{s+1}. For each $j \ge e$ with $a_j \in F$, we reset $a_j \in \omega^{[j]}$ to be a fresh number outside F.

Denote the resulting set after ω many stages by A_ω. Finally, let

$$A = A_\omega \cup \{a_e{:}\, e \notin S\}.$$

In order to make the proof more compact, we introduce the following definition and prove a lemma about it:

Definition 7.4.4. For sets $A, B \subseteq \mathbb{N}$, we say $A \le_{e'} B$ if there is a "K-c.e. enumeration operator reducing A to B", i.e., a K-c.e. set Φ such that for all x, $x \in A$ if and only if there is a finite set $F \subseteq B$ (given by a canonical index) with $\langle x, F\rangle \in \Phi$.

Lemma 7.4.5. For any sets $A, B \subseteq \mathbb{N}$, we have $A \le_{e'} B$ if and only if $A \le_{\mathrm{e}} B \oplus \overline{K}$.

Proof. If $A \le_{e'} B$ via a K-c.e. operator $\Phi = W^K$, say, then each axiom $\langle x, F\rangle \in \Phi$ can be rewritten into axioms $\langle x, F, P, N\rangle$ where $\langle x, F\rangle \in W^K$ via computations requiring $P \subseteq K$ and $N \subseteq \overline{K}$, and these axioms $\langle x, F, P, N\rangle$ can be combined into a single c.e. enumeration operator Ψ witnessing $A \le_{\mathrm{e}} B \oplus K \oplus \overline{K} \equiv_{\mathrm{e}} B \oplus \overline{K}$.

Conversely, suppose $A \le_{\mathrm{e}} B \oplus K \oplus \overline{K} (\equiv_{\mathrm{e}} B \oplus \overline{K})$ via a c.e. enumeration operator Ψ, then we can define a K-c.e. enumeration operator Φ by enumerating $\langle x, F\rangle$ into Φ for any $\langle x, F \oplus P \oplus N\rangle \in \Psi$ with $P \subseteq K$ and $N \subseteq \overline{K}$. □

From the construction and the definition of A, it is now clear that all $\mathcal{R}_e$-requirements are satisfied, and so $\overline{K} \le_{\mathrm{e}} S \le_{\mathrm{e}} \overline{A} \le_{\mathrm{e}} A^\diamond$.

We next observe that

$$\{a_e\}_{e\in\omega} \le_T K. \tag{7.4.1}$$

Using (7.4.1) and that $e \in S$ if and only if $a_e \notin A$, it is now clear that both $\overline{S} \le_{e'} A$ and $\overline{A} \le_{e'} S$, and so by Lemma 7.4.5, we have both $\overline{S} \le_{\mathrm{e}} A \oplus \overline{K}$ and

$\overline{A} \leq_e S \oplus \overline{K} \equiv_e S$. The last inequality combined with the already established $S \leq_e \overline{A}$ gives us that $\overline{A} \equiv_e S$.

Finally, using (7.4.1) and the action of the $\mathcal{P}_e$-requirements, we also have $A^\lozenge \leq_{e'} \overline{A}$. This is because K can figure out for which D we found an F_D at stage $s = e+1$ such that $e \in \Gamma_e(F_D \cup \{a_i : i \in D\})$. Then $e \in A^\lozenge$ if and only if $\overline{A}$ intersects $\{a_i : i \in D\}$ for every such D. So again by Lemma 7.4.5, we have that $A^\lozenge \leq_e \overline{A} \oplus \overline{K} \equiv_e S$. □

Notice that the proof of Theorem 7.4.3 directly gives us the following result.

Theorem 7.4.6. Let $n \geq 2$. For any Π^0_n-set $S \geq_e \overline{K}$, there is a Σ^0_n-set A such that $A^\lozenge \equiv_e S$. Furthermore, for any Σ^0_n-set $S \geq_e \overline{K}$, there is a Π^0_n-set A such that $A^\lozenge \equiv_e S$.

Proof. This follows directly from the proof of Theorem 7.4.3, noting that A as built there is equal to $A_\omega \cup \{\langle e,k\rangle : e \notin S\}$, that A_ω is Δ^0_2, and that $\{\langle e,k\rangle : e \notin S\}$ is of the same complexity as the complement of S. □

Definition 7.4.7. An enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total enumeration degree bounded by $\mathbf{a}$ is $\mathbf{0}_e$.

McEvoy [McE85] proved that the enumeration jump restricted to the quasiminimal degrees has the same range as the unrestricted jump operator. We show that the skip has the same property. Actually, we prove with Soskov [SS13] the same property for the degree spectrum: every element of the jump spectrum is a jump of a quasi minimal degree with respect to the spectrum and co-spectrum.

Corollary 7.4.8. For any set $S \geq_e \overline{K}$, there is a set A of quasiminimal degree such that $A^\lozenge \equiv_e S$.

Proof. We modify the construction in Theorem 7.4.3 slightly. We add additional requirements $\mathcal{Q}_e$ that ensure that A is quasiminimal:

$$\mathcal{R}_e : e \in S \iff A^{[e]} \neq \mathbb{N}^{[e]},$$
$$\mathcal{P}_e : \text{“force } e \text{ into } K_A \text{ subject to higher-priority restraints”},$$
$$\mathcal{Q}_e : \Gamma_e(A) = X \oplus \overline{X} \Rightarrow X \text{ is computable.}$$

At stage $s = e+1$, after we have defined the set F and G for the sake of the requirement $\mathcal{P}_e$, we will handle the requirement $\mathcal{Q}_e$. The procedure is similar. For any subset $D \subseteq \{i : i < e\}$, check if there are a finite subset $E_D \subseteq \mathbb{N} \setminus \{a_i : 0 \le i < e\}$, a number x, and a stage t such that $\{2x, 2x+1\} \subseteq \Gamma_{e,t}(\{a_i : i \in D\} \cup E_D)$; for any such D, choose the set E_D from the least such triple; if there is no such triple, set $E_D = \varnothing$. Set

$$E = \bigcup_{D \subseteq \{i : i < e\}} E_D.$$

Enumerate $F \cup G \cup E$ into A_{s+1} and then redefine the values of a_j appropriately: If $j \ge e$ and $a_j \in F \cup E$, we reset $a_j \in \mathbb{N}^{[j]}$ to be a fresh number outside $F \cup E$.

If $\Gamma_e(A)$ turns out to be a total set $X \oplus \overline{X}$, then we can compute X: Let $A^* = (\mathbb{N}^{[<e]} \cap A) \cup \mathbb{N}^{[\ge e]}$. As every column of A is finitely different from $\mathbb{N}$, it follows that A^* is a computable set and $A \subseteq A^*$. Now, $x \in X$ if and only if $2x \in \Gamma_e(A^*)$ and $x \notin X$ if and only if $2x+1 \in \Gamma_e(A^*)$. □

7.4.2 Further properties of the skip operator and examples

We will now investigate the possible behavior of the iterated skip operator.

Definition 7.4.9. Fix $A \subseteq \mathbb{N}$. We inductively define $A^{\langle n \rangle}$, the *n-th skip of* A.

- $A^{\langle 0 \rangle} = A$,
- $A^{\langle n+1 \rangle} = (A^{\langle n \rangle})^{\Diamond}$.

The *n-th skip of* $d_e(A)$ is $d_e(A)^{\langle n \rangle} = d_e(A^{\langle n \rangle})$.

If $\mathbf{a}$ is a cototal enumeration degree, then every iteration of the skip of $\mathbf{a}$ agrees with the corresponding iteration of the jump of $\mathbf{a}$, i.e., for all $n < \omega$, we have that $\mathbf{a}^{\langle n \rangle} = \mathbf{a}^{(n)}$. Theorem 7.4.3 proves that there are non-cototal enumeration degrees, e.g., the skip invert of a nontotal enumeration degree. It is natural to ask what we can say in general about the sequence $\{\mathbf{a}^{\langle n \rangle}\}_{n \in \omega}$. One immediate observation is that even though the skip of A need not be above A, its double skip always is: For any set A, we know that $\overline{A} \le_1 A^{\Diamond}$. Applying this twice, we have $A \le_1 \overline{A^{\Diamond}} \le_1 A^{\Diamond\Diamond}$, so a fortiori $A \le_e A^{\Diamond\Diamond}$. It follows that $\mathbf{a}^{\langle n \rangle} \le \mathbf{a}^{\langle n+2 \rangle}$ for all n. In addition, by monotonicity, we have that for every n, $\mathbf{0}_e^{(n)} \le \mathbf{a}^{\langle n \rangle}$. If $\mathbf{a}^{\langle n \rangle}$ is not cototal for every natural number n, then we have a form of *zig-zag* behavior of the skip, illustrated in Figure 7.2. We will search for examples of degrees whose skips have this general behavior.

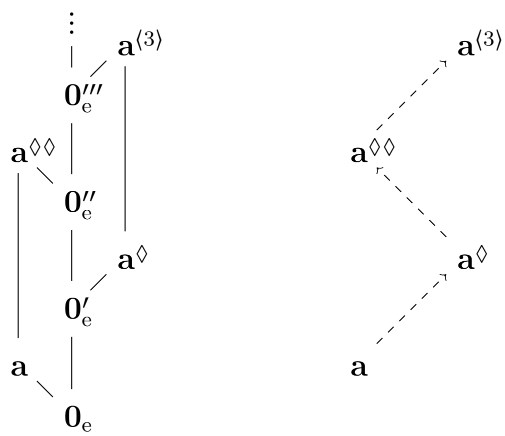

Figure 7.2: Iterated skips of a degree: the zig-zag

Skips of generic sets

We will start by investigating the skip for the class of enumeration degrees of 1-generic sets. Definition 2.3.8 defines a relativized form of 1-genericity, suitable for the context of the enumeration degrees. Let me remind that we use the notation "relative to $\langle X\rangle$" to denote "relative to the enumeration degree of X".

The set G is 1-*generic relative to* $\langle X\rangle$ if and only if for every $S \subseteq 2^{<\omega}$ such that $S \leq_{\mathrm{e}} X$:

$$(\exists \sigma \subseteq G)[\sigma \in S \vee (\forall \tau \supseteq \sigma)[\tau \notin S]].$$

If $X = \varnothing$, then we call G simply 1-*generic* and if $X = \overline{K}$, then G is 2-*generic*.

From (Definition 2.3.10) we know that the degree $\mathbf{a}$ is a *strong quasiminimal cover* of $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and every total enumeration degree $\mathbf{x}$ bounded by $\mathbf{a}$ is below $\mathbf{b}$.

We proved in Proposition 2.3.11 the following properties of 1-generic relative to $\langle X\rangle$ set G:

1. $d_{\mathrm{e}}(G \oplus X)$ is a strong quasiminimal cover of $d_{\mathrm{e}}(X)$.

2. $\overline{G}$ is 1-generic relative to $\langle X\rangle$.

We know from Corollary 2.2.8 that the Turing jump of a 1-generic set has a nice characterization: $K_G \equiv_T G \oplus K$, or, in other words, G is generalized low. This property relativizes: If G is 1-generic relative X, then $K_{G\oplus X} \equiv_T G \oplus K_X$. A similar property is true of the skip of a 1-generic set G relative to $\langle X\rangle$.

Proposition 7.4.10. If G is 1-generic relative to $\langle X \rangle$, then $(G \oplus X)^\Diamond \equiv_e \overline{G} \oplus X^\Diamond$.

Proof. Note that we always have $\overline{G} \oplus X^\Diamond \leq_e (G \oplus X)^\Diamond$, no matter what the sets G and X are, simply from the monotonicity of the skip operator. The nontrivial reduction is the reverse one. Suppose $\langle e, x \rangle \in (G \oplus X)^\Diamond$, i.e., $x \notin \Gamma_e(G \oplus X)$. Consider the set

$$D_{e,x} = \{\sigma \in 2^{<\omega} \mid x \in \Gamma_e(\sigma \oplus X)\}.$$

This set is enumeration reducible to X uniformly in e and x, and so there must be a finite part $\sigma \subseteq G$ such that no extension of σ is in $D_{e,x}$. The set

$$E_{e,x} = \{\sigma \mid (\exists \tau \supseteq \sigma)[\tau \in D_{e,x}]\}$$

is also uniformly enumeration reducible to X, and so its complement is uniformly enumeration reducible to $X^\Diamond$. We claim that:

$$\langle e, x \rangle \in (G \oplus X)^\Diamond \text{ if and only if } (\exists \sigma)[\{n \mid \sigma(n) = 0\} \subseteq \overline{G} \ \&\ \sigma \in \overline{E}_{e,x}].$$

The implication from left to right has already been established: If $\langle e, x \rangle \in (G \oplus X)^\Diamond$, then the initial segment of G with no extension in $D_{e,x}$ witnesses that the statement on the right is true. So let $\langle e, x \rangle$ be such that there is a σ with $\{n \mid \sigma(n) = 0\} \subseteq \overline{G}$ and such that $\sigma \in \overline{E}_{e,x}$. Towards a contradiction, suppose that $\langle e, x \rangle \notin (G \oplus X)^\Diamond$, i.e., $x \in \Gamma_e(G \oplus X)$. Let $\tau \prec G$ be such that $x \in \Gamma_e(\tau \oplus X)$ and define σ^* of length $\max(|\sigma|, |\tau|)$ as follows

$$\sigma^*(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma|, \\ \tau(n) & \text{if } |\sigma| \leq n < |\tau|. \end{cases}$$

Then σ^* is an extension of σ. Furthermore, if $\tau(n) = 1$ then $\sigma^*(n) = 1$. Indeed, this is obvious for $n \geq |\sigma|$, and for $n < |\sigma|$, this follows from the fact that $\{n \mid \sigma(n) = 0\} \subseteq \overline{G}$ and $\tau \subseteq G$. Thus $\sigma^* \in D_{e,x}$, contradicting our assumption that σ has no extension in $D_{e,x}$. □

Now, we can easily give an example of a set G whose iterated skips form a zig-zag. Consider G to be a set that is arithmetically generic, i.e., G is 1-generic relative to $\langle \varnothing^{(n)} \rangle$ for every natural number n. Note that $\overline{G}$ has the same property. Then by induction using the characterization above we can show that for all $n < \omega$:

- If n is odd then $G^{\langle n \rangle} \equiv_e \overline{G} \oplus \varnothing^{(n)}$ and $(\overline{G})^{\langle n \rangle} \equiv_e G \oplus \varnothing^{(n)}$.

- If n is even then $G^{\langle n\rangle} \equiv_e G \oplus \varnothing^{(n)}$ and $(\overline{G})^{\langle n\rangle} \equiv_e \overline{G} \oplus \varnothing^{(n)}$.

Furthermore, all iterates of the skip for both sets G and $\overline{G}$ are not total, as their degrees are quasiminimal covers of the corresponding iterate of the jump of $\mathbf{0}_e$. It follows that they also do not have cototal degree, as by Proposition 7.2.1 sets H of cototal degree have total skips: $K_H \equiv_e H \leq_e H^\Diamond = \overline{K_H}$. This gives an example of a double zig-zag as in Figure 7.3. It is worth noting that only the reductions implied by the diagram occur. For example, $G \nleq_e G^{\langle 3\rangle}$; otherwise $G^{\langle 3\rangle} \equiv_e G \oplus G^{\langle 3\rangle} \equiv_e G \oplus \overline{G} \oplus \varnothing^{\langle 3\rangle}$ would be total.

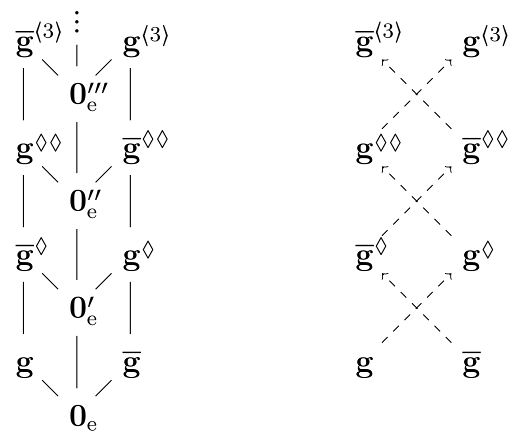

Figure 7.3: The iterated skips of the degrees of an arithmetically generic set and its complement: double zig-zag

Skips of nontrivial $\mathcal{K}$-pairs.

Kalimullin [Kal03] relativized the notion of a $\mathcal{K}$-pair in a way similar to how we relativized the notion of 1-genericity.

Definition 7.4.11. A pair of sets of natural numbers $\{A, B\}$ forms a *$\mathcal{K}$-pair relative to* $\langle X\rangle$ if there is a set $W \leq_e X$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. The pair $\{A, B\}$ is a *nontrivial $\mathcal{K}$-pair relative to* $\langle X\rangle$ if, in addition, $A \nleq_e X$ and $B \nleq_e X$.

Note that if $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to every $\langle X\rangle$ such that $A, B \nleq_e X$. We summarize some properties of relativized $\mathcal{K}$-pairs below.

Proposition 7.4.12 (Kalimullin [Kal03])**.** Let $A, B, X \subseteq \mathbb{N}$ and suppose that $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$.

1. If $C \leq_e B$ then $\{A, C\}$ forms a $\mathcal{K}$-pair relative to $\langle X\rangle$.
2. $A \leq_e \overline{B} \oplus X$.
3. $\overline{A} \leq_e B \oplus X^\Diamond$.
4. $d_e(A \oplus X)$ and $d_e(B \oplus X)$ are strong quasiminimal covers of $d_e(X)$.
5. For every $Z \subseteq \mathbb{N}$, the degrees $d_e(A \oplus X \oplus Z)$ and $d_e(B \oplus X \oplus Z)$ have a greatest lower bound, and it is $d_e(X \oplus Z)$.

Note that items (1), (2) and (3) are symmetrically true if we swap A and B.

The skip of a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$ has the following properties:

Proposition 7.4.13. If $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$, then

$$(A \oplus X)^\Diamond \leq_e B \oplus X^\Diamond \quad \text{and} \quad (B \oplus X)^\Diamond \leq_e A \oplus X^\Diamond.$$

The oracle set X is of cototal degree if and only if for every nontrivial $\mathcal{K}$-pair $\{A, B\}$ relative to $\langle X\rangle$,

$$(A \oplus X)^\Diamond \equiv_e B \oplus X^\Diamond \quad \text{and} \quad (B \oplus X)^\Diamond \equiv_e A \oplus X^\Diamond.$$

Proof. If $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$, then $\{A \oplus X, B\}$ also forms a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$: Replace the witnessing set W by

$$W^* = \{\langle 2a, b\rangle \mid \langle a, b\rangle \in W\} \cup \{\langle 2a+1, b\rangle \mid a \in X\}.$$

As $K_{A\oplus X} \equiv_e A \oplus X$, it follows that $\{K_{A\oplus X}, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$, and so

$$(A \oplus X)^\Diamond = \overline{K_{A\oplus X}} \leq_e B \oplus X^\Diamond.$$

On the other hand, if X is of cototal degree, then using the monotonicity of the skip operator we get that $B \leq_e \overline{A} \oplus X \leq_e A^\Diamond \oplus X^\Diamond \leq_e (A \oplus X)^\Diamond$, and hence $B \oplus X^\Diamond \leq_e (A \oplus X)^\Diamond$.

Finally, consider the oracle set X and let $\{A, B\}$ be a nontrivial (unrelativized) $\mathcal{K}$-pair such that $A, B \nleq_e X$. Note that both $\{A, B\}$ and $\{A \oplus X, B \oplus$

$X\}$ are nontrivial $\mathcal{K}$-pairs relative to $\langle X \rangle$. If the characterization of the skip operator holds for both pairs, then we have that

$$(A \oplus X)^{\Diamond} \equiv_{\mathrm{e}} B \oplus X^{\Diamond} \equiv_{\mathrm{e}} B \oplus X \oplus X^{\Diamond}, \text{ and}$$
$$(B \oplus X)^{\Diamond} \equiv_{\mathrm{e}} A \oplus X^{\Diamond} \equiv_{\mathrm{e}} A \oplus X \oplus X^{\Diamond}.$$

Now, using the last property from Proposition 7.4.12, we have that

$$d_{\mathrm{e}}(X^{\Diamond}) = d_{\mathrm{e}}(A \oplus X^{\Diamond}) \wedge d_{\mathrm{e}}(B \oplus X^{\Diamond}) = d_{\mathrm{e}}(A \oplus X \oplus X^{\Diamond}) \wedge d_{\mathrm{e}}(B \oplus X \oplus X^{\Diamond}) = d_{\mathrm{e}}(X \oplus X^{\Diamond}).$$

It follows that X is of cototal degree. □

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair and both A and B are not arithmetical, then $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to $\langle \varnothing^{(n)} \rangle$ for every natural number n. As every set $\varnothing^{(n)}$ is of (co)total enumeration degree, it follows by Proposition 7.4.13 that the iterated skips of A and B also form a double zigzag: For all $n < \omega$,

- if n is odd then $A^{\langle n \rangle} \equiv_{\mathrm{e}} B \oplus \varnothing^{(n)}$ and $B^{\langle n \rangle} \equiv_{\mathrm{e}} A \oplus \varnothing^{(n)}$, and
- if n is even then $A^{\langle n \rangle} \equiv_{\mathrm{e}} A \oplus \varnothing^{(n)}$ and $B^{\langle n \rangle} \equiv_{\mathrm{e}} B \oplus \varnothing^{(n)}$.

Furthermore, by Proposition 7.4.12, for every natural number n, $\{d_{\mathrm{e}}(A)^{\langle n \rangle}, d_{\mathrm{e}}(B)^{\langle n \rangle}\}$ forms a minimal pair of quasiminimal degrees above $\mathbf{0}_{\mathrm{e}}^{(n)}$.

A pair of enumeration degrees $\{\mathbf{a}, \mathbf{b}\}$ forms a $\mathcal{K}$-*pair* (relative to $\mathbf{x}$) if there are representatives $A \in \mathbf{a}$ and $B \in \mathbf{b}$ that form a $\mathcal{K}$-pair (relative to $\mathbf{x}$). We will use the characterization of the skips of $\mathcal{K}$-pairs along with the following theorem of Ganchev and Sorbi [GS16] to give an example of degrees whose iterated skips behave quite differently.

Theorem 7.4.14 (Ganchev, Sorbi [GS16])**.** For every enumeration degree $\mathbf{x} > \mathbf{0}_{\mathrm{e}}$, there is a degree $\mathbf{a} \leq \mathbf{x}$ such that $\mathbf{a}$ is half of a nontrivial $\mathcal{K}$-pair and such that $\mathbf{a}' = \mathbf{x}'$.

One of the main ingredients in the proof of the theorem above is the following observation, which follows easily from Proposition 7.4.13. If $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then A and B have equivalent enumeration jumps:

$$A'_{\mathrm{e}} \equiv_{\mathrm{e}} A \oplus A^{\Diamond} \equiv_{\mathrm{e}} A \oplus B \oplus \varnothing' \equiv_{\mathrm{e}} B \oplus B^{\Diamond} \equiv_{\mathrm{e}} A'_{\mathrm{e}}.$$

Now consider a nonzero enumeration degree $\mathbf{x} \leq_{\mathrm{e}} \mathbf{0}'_{\mathrm{e}}$, and let $\mathbf{a} \leq \mathbf{x}$ be half of a nontrivial $\mathcal{K}$-pair such that $\mathbf{a}' = \mathbf{x}'$. Let $\mathbf{b}$ be such that $\{\mathbf{a}, \mathbf{b}\}$ forms a nontrivial $\mathcal{K}$-pair. Then $\mathbf{b}^{\Diamond} = \mathbf{a} \vee \mathbf{0}'_{\mathrm{e}} = \mathbf{0}'_{\mathrm{e}}$ and $\mathbf{b}' = \mathbf{a}' = \mathbf{x}'$. In particular, if we take $\mathbf{x}$ to be high, i.e., such that $\mathbf{x}' = \mathbf{0}''_{\mathrm{e}}$, then we have an example of an enumeration degree such that all iterations of its skip are total enumeration degrees, but mismatch its iterations of the jump by one iteration:

$$\mathbf{b}^{\Diamond} < \mathbf{b}' = \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3\rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1\rangle} < \cdots.$$

If we take $\mathbf{x}$ to be an intermediate degree, i.e., a degree such that for all n, $\mathbf{0}_{\mathrm{e}}^{(n)} < \mathbf{x}^{(n)} < \mathbf{0}_{\mathrm{e}}^{(n+1)}$ then we get the following:

$$\mathbf{b}^{\Diamond} < \mathbf{b}' < \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' < \mathbf{b}^{\langle 3\rangle} < \cdots < \mathbf{b}^{(n)} < \mathbf{b}^{\langle n+1\rangle} < \cdots.$$

We end this discussion with some thoughts about the definability of the skip operator. Kalimullin [Kal03] proved that the relation "$\{\mathbf{a}, \mathbf{b}\}$ forms a $\mathcal{K}$-pair relative to $\mathbf{x}$" is first-order definable with parameter $\mathbf{x}$. Using this result, he showed that the enumeration jump operator is first-order definable. Combining these results with the characterization of the skip operator for nontrivial $\mathcal{K}$-pairs, we immediately obtain the following result.

Corollary 7.4.15. The relation

$$SK = \{(\mathbf{a}, \mathbf{a}^{\Diamond}) \mid \mathbf{a} \text{ is half of a nontrivial } \mathcal{K}\text{-pair } \}$$

is first-order definable in $\mathcal{D}_{\mathrm{e}}$.

Proof. If $\mathbf{a}$ is half of a nontrivial pair, then $\mathbf{a}^{\Diamond} = \mathbf{0}'_{\mathrm{e}} \vee \mathbf{b}$ where $\mathbf{b}$ is some nonzero degree that forms a $\mathcal{K}$-pair with $\mathbf{a}$. □

It remains an open question whether or not the skip operator is first-order definable in $\mathcal{D}_{\mathrm{e}}$.

A skip 2-cycle

As seen above, the skip can exhibit a form of *zig-zag* behavior. We now show that there is another extreme case that could occur: The double skip $\mathbf{a}^{\Diamond\Diamond}$ of an enumeration degree $\mathbf{a}$ could be equal to $\mathbf{a}$ itself. Perhaps surprisingly, this degree is not constructed in a way that is common in computability theory. Instead, we use the following theorem due to Knaster and Tarski.

Theorem 7.4.16 (Knaster–Tarski Fixed Point Theorem)**.** Let L be a complete lattice and let $f\colon L \to L$ be monotone, i.e., for all $x, y \in L$, we have that $x \le y$ implies that $f(x) \le f(y)$. Then f has a fixed point. In fact, the fixed points of f form a complete lattice.

We apply the Knaster–Tarski theorem to a function on 2^ω, which we view as the power set lattice of $\mathbb{N}$, ordered by subset inclusion.

Theorem 7.4.17. There is a set A such that $A^{\Diamond\Diamond} = A$.

Proof. Let $f\colon 2^\omega \to 2^\omega$ be the double skip operator, i.e., $f(A) = A^{\Diamond\Diamond}$. Note that if $A \subseteq B$, then $K_A \subseteq K_B$, so $A^\Diamond \supseteq B^\Diamond$. Applied twice, we obtain $A^{\Diamond\Diamond} \subseteq B^{\Diamond\Diamond}$, so f is monotone. Hence, by the Knaster–Tarski Fixed Point Theorem, there is an A such that $A^{\Diamond\Diamond} = A$. □

Note that we do not just have that A and $A^{\Diamond\Diamond}$ are enumeration equivalent, but they are equal as sets. However, we will mainly be interested in the fact that the enumeration degree $\mathbf{a}$ of A satisfies $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$. If we have such a degree $\mathbf{a}$, then we will say that $\mathbf{a}$ and $\mathbf{a}^\Diamond$ form a *skip* 2*-cycle*.

As we show next, skip 2-cycles are computationally very complicated; namely, they compute all hyperarithmetic sets.

Proposition 7.4.18. Let $\mathbf{a}$ and $\mathbf{a}^\Diamond$ form a skip 2-cycle. Then $\mathbf{a} \ge \mathbf{b}$ for every total hyperarithmetic degree $\mathbf{b}$.

Proof. Let A be a set of degree $\mathbf{a}$. We build an enumeration operator Φ such that $\Phi(A, p) = H(p) \oplus \overline{H(p)}$ for every ordinal notation p, where $H(p)$ we defined in Chapter 2 (see also [Sac90, Chapter 2]). By the Recursion Theorem, we may assume that we know an index for Φ. We let $\Phi(A, 1) = H(1) \oplus \overline{H(1)} = \varnothing \oplus \mathbb{N}$.

Now assume that $|p|$ is a successor, say, $p = 2^q$ and so $|p| = |q| + 1$. Note that if $C \ge_{\mathrm{e}} D \oplus \overline{D}$, then $C^\Diamond \ge_{\mathrm{e}} K_D \oplus \overline{K_D}$ uniformly in an index for the first reduction. Furthermore, we inductively assume that $\Phi(A, q) = H(q) \oplus \overline{H(q)}$. Combining these facts,

$$A \equiv_{\mathrm{e}} A^{\Diamond\Diamond} \ge_{\mathrm{e}} H(2^{2^q}) \oplus \overline{H(2^{2^q})} \ge_{\mathrm{e}} H(p) \oplus \overline{H(p)}$$

uniformly, so let $\Phi(A, p) = H(p) \oplus \overline{H(p)}$.

Finally, assume that $|p|$ is a limit ordinal, say, $p = 3 \cdot 5^e$ and so $|p|$ is the limit of $|q_0|, |q_1|, \ldots$, where $q_i = \varphi_e(i)$. Using the inductive assumption that $\Phi(A, q_i) = H(q_i) \oplus \overline{H(q_i)}$, we can set $\Phi(A, p) = H(p) \oplus \overline{H(p)}$, where $H(p) = \bigoplus_{i \in \omega} H(q_i)$. □

Given the fact that we have shown the existence of a skip 2-cycle, it is only natural to consider whether (proper) skip n-cycles exist for any other natural number $n \geq 1$. This turns out to be false.

Proposition 7.4.19. Let $n \in \omega$ be nonzero such that $\mathbf{a}^{\langle n \rangle} = \mathbf{a}$. Then $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$.

Proof. First, observe that $\mathbf{a}^{\langle 2n \rangle} = \mathbf{a}^{\langle n \rangle} = \mathbf{a}$, so without loss of generality we may assume that n is even. By monotonicity of the double skip, we then have that

$$\mathbf{a} \leq \mathbf{a}^{\langle 2 \rangle} \leq \cdots \leq \mathbf{a}^{\langle n-2 \rangle} \leq \mathbf{a}^{\langle n \rangle} = \mathbf{a},$$

so

$$\mathbf{a} = \mathbf{a}^{\langle 2 \rangle} = \cdots = \mathbf{a}^{\langle n-2 \rangle} = \mathbf{a}^{\langle n \rangle}.$$

□

The set A we obtained in Theorem 7.4.17 allows us to give the example of a pair of sets A and B that illustrate the flaw in the enumeration jump mentioned in the last paragraph of Section 7.2.

Proposition 7.4.20. $A'_{\mathrm{e}} \equiv_1 B'_{\mathrm{e}}$ does not necessarily imply $A \equiv_{\mathrm{e}} B$.

Proof. Let A be the set we obtained in Theorem 7.4.17, and let $B = A^{\Diamond} = \overline{K_A}$, so $A = B^{\Diamond}$. Then $K_A = \overline{B} \leq_1 \overline{K_B}$ since B is a column of K_B, and similarly, $K_B = \overline{A} \leq_1 \overline{K_A}$. It follows that

$$K_A \oplus \overline{K_A} \leq_1 \overline{K_B} \oplus B \leq_1 \overline{K_B} \oplus K_B \equiv_1 K_B \oplus \overline{K_B},$$

and similarly

$$K_B \oplus \overline{K_B} \leq_1 \overline{K_A} \oplus A \leq_1 K_A \oplus \overline{K_A}.$$

Thus $A'_{\mathrm{e}} = K_A \oplus \overline{K_A} \equiv_1 K_B \oplus \overline{K_B} = B'_{\mathrm{e}}$, but clearly A is not enumeration equivalent to B. □

7.5 Separating cototality properties

7.5.1 Degrees that are not weakly cototal

Let us begin by showing that the weakest cototality property we introduced, aptly named *weakly cototal*, is nontrivial, i.e., that there are degrees that are not weakly cototal. We will present three different examples in this section. First, we note that sufficiently generic sets are not weakly cototal.

Proposition 7.5.1. If $\mathbf{a}$ is a 2-generic enumeration degree, then $\mathbf{a}$ is not weakly cototal.

Proof. Let G be 2-generic and let $A \equiv_e G$. Towards a contradiction, let us assume that $\overline{A}$ has total enumeration degree. Then by Proposition 7.4.10 with $X = \varnothing$, we have that

$$\overline{G} \oplus \overline{K} \equiv_e G^\lozenge \geq_e \overline{A}.$$

By Proposition 2.3.11(2) with $X = \overline{K}$, $\overline{G}$ is 2-generic, so by Proposition 2.3.11(1) with $X = \overline{K}$ and the totality of $d_e(\overline{A})$, we obtain that $\overline{A} \leq_e \overline{K}$. It follows that $G \oplus \overline{K} \equiv_e A \oplus \overline{K} \geq_e \overline{A} \oplus K$, and so $d_e(G \oplus \overline{K})$ is total. This contradicts Proposition 2.3.11(1), that $d_e(G \oplus \overline{K})$ is a quasiminimal cover for $d_e(\overline{K})$ and so cannot be a total enumeration degree. □

Next, we show that we can also get such examples using $\mathcal{K}$-pairs.

Proposition 7.5.2. Let $\mathbf{a}, \mathbf{b} \nleq_e \mathbf{0}_e'$ form a nontrivial $\mathcal{K}$-pair. Then $\mathbf{a}$ is not weakly cototal.

Proof. Let $\{A, B\}$ form a nontrivial $\mathcal{K}$-pair with $A, B \nleq_e \overline{K}$. It follows that $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle\overline{K}\rangle$, and so by Proposition 7.4.12, the degree of $B \oplus \overline{K}$ is a strong quasiminimal cover of $\mathbf{0}_e'$. Towards a contradiction, suppose that A has weakly cototal degree. As $\mathcal{K}$-pairs are closed with respect to enumeration equivalence, we may assume that $\overline{A}$ is of total enumeration degree. By the same Proposition 7.4.12, we have, on the one hand, that $\overline{A} \leq_e B \oplus \overline{K}$ and so $\overline{A} \leq_e \overline{K}$, and on the other hand, that $B \leq_e \overline{A}$. It follows that $B \leq_e \overline{K}$, contradicting our choice of B. □

For our final example of a degree that is not weakly cototal, recall from Theorem 7.4.17 that there is a degree $\mathbf{a}$ such that $\mathbf{a}^{\lozenge\lozenge} = \mathbf{a}$. Such a degree is not weakly cototal.

Proposition 7.5.3. Let $\mathbf{a}$ be such that $\mathbf{a}^{\lozenge\lozenge} = \mathbf{a}$. Then $\mathbf{a}$ is not weakly cototal.

Proof. Towards a contradiction, assume that A in the degree $\mathbf{a}$ is such that $\overline{A}$ has total enumeration degree. Then $A^\lozenge \geq_e \overline{A}$ implies that

$$A^{\lozenge\lozenge} \geq_e \left(\overline{A}\right)^\lozenge \geq_e A,$$

so $d_e(A)$ is the skip of the total degree $d_e(\overline{A})$ and hence total. But then $A^{\lozenge\lozenge} >_e A$, which is a contradiction. □

7.5.2 Weakly cototal degrees that are not cototal

We will prove the next separation using the skip inversion we proved in Theorem 7.4.3 above.

Proposition 7.5.4. There is a degree $\mathbf{a}$ that is weakly cototal, but not cototal.

Proof. Let $B \geq_e \overline{K}$ be any total set, and let $S = K_B$. Then note that $S \equiv_e B$, so the *degree* of S is total, but S is not total *as a set.* Now apply Theorem 7.4.3 to obtain an A such that $A^\Diamond \equiv_e S$ and $\overline{S} \leq_e A \oplus \overline{K}$.

Then A is weakly cototal since $A \equiv_e K_A$ and $\overline{K_A} = A^\Diamond \equiv_e S$, which has total degree. Let $\mathbf{a}$ be the degree of A. We claim that $\mathbf{a}$ is not cototal. By Proposition 7.2.1, it suffices to show that $A \not\leq_e A^\Diamond$. Towards a contradiction, assume that $A \leq_e A^\Diamond$. Since $A^\Diamond \geq_e \overline{K}$ always holds, we now see that

$$S \equiv_e A^\Diamond \geq_e A \oplus \overline{K} \geq_e \overline{S}$$

so S would be a total set, which is a contradiction. □

The proof above combined with Theorem 7.4.6 yields the promised Π^0_2 degree that is not cototal. Of course, as noted earlier, such a degree can be obtained using a theorem of Badillo and Harris [BH12] proving the existence of a Π^0_2-enumeration degree that contains only properly Π^0_2-sets. As all Π^0_2 enumeration degrees are weakly cototal, this gives us a more concrete separation result.

An alternative way to separate the weakly cototal degrees from the cototal degrees is given by the following proposition.

Proposition 7.5.5. If $\mathbf{b} \nleq \mathbf{0}'_e$ but forms a nontrivial $\mathcal{K}$-pair with $\mathbf{a} \leq \mathbf{0}'_e$, then $\mathbf{b}$ forms a minimal pair with $\mathbf{b}^\Diamond$.

Proof. Towards a contradiction, assume there is a nonzero degree $\mathbf{c}$ such that $\mathbf{c} \leq \mathbf{b}$ and $\mathbf{c} \leq \mathbf{b}^\Diamond$. The fact that $\mathbf{c} \leq \mathbf{b}$ gives us that $\mathbf{a}$ and $\mathbf{c}$ form a $\mathcal{K}$-pair by Proposition 7.4.12(1). Using this, Proposition 7.4.12(2), and Proposition 7.4.13 twice, we have

$$\mathbf{b} \leq \mathbf{a}^\Diamond = \mathbf{c} \oplus \mathbf{0}'_e \leq \mathbf{b}^\Diamond = \mathbf{a} \oplus \mathbf{0}'_e = \mathbf{0}'_e.$$

So $\mathbf{b} \leq \mathbf{0}'_e$, which is a contradiction. □

Corollary 7.5.6. If $\mathbf{b}$ is as in the previous proposition, then $\mathbf{b}$ is weakly cototal, but not cototal.

Proof. By the previous proposition combined with Proposition 7.2.1, **b** is not cototal. On the other hand, from Proposition 7.4.13 we know that $B^\Diamond \equiv_e A \oplus \overline{K} \equiv_e \overline{K}$, since $A \leq_e \overline{K}$. Also as $\overline{K}$ has total degree, as in the proof of Proposition 7.5.4, this implies that B is weakly cototal. □

The only separation left to prove is the separation of the cototal degrees from the graph-cototal degrees. We will prove this result in the next section.

7.6 There is a cototal degree that is not graph-cototal

Theorem 7.6.1. There is a cototal enumeration degree that is not graph-cototal.

Proof.

We fix the undirected graph $\mathcal{G} = (\omega^{<\omega}, E)$, where the edge relation is given by $E(a, b)$ if and only if $a^- = b$ or $a = b^-$ (i.e., a is an immediate successor of b or the immediate predecessor of b). We will build the complement of a maximal independent set for the graph $\mathcal{G}$. Recall that this is a subset $A \subseteq \omega^{<\omega}$ with the property that every element $a \in \omega^{<\omega}$ is either outside A or is connected by an edge to an element outside A, but not both.

Our other condition on the set A will be that it is not enumeration equivalent to a graph-cototal set. We construct A as such using a construction in the framework of a $0'''$-priority construction over $0'$. We start by listing an infinite sequence of requirements that collectively ensure that we meet our goal. We then make use of a tree of strategies. Strategies on the tree inherit the standard ordering of nodes: We use $\alpha \leq \beta$ to denote that α is a prefix of β and $\alpha <_L \beta$ to denote that α is to the left of β in the tree. Every strategy is assigned one of the requirements. At every stage we build a finite path through this tree, activating strategies along it and injuring all strategies to the right of it. Activated strategies perform actions towards satisfaction of their requirements. Injured strategies are *initialized*—they must start over as if they were never activated before. The intention is that there will be a *true path*, a leftmost infinite path of nodes visited at infinitely many stages, such that every strategy along this path succeeds in satisfying the requirement that is assigned to it. We refer the reader to Soare [Soa87] for a more detailed introduction to priority arguments and the tree method. We warn the reader that our argument differs from standard infinite-injury arguments in a couple of ways: There will be some strategies α which intentionally injure other strategies β with $\alpha \prec \beta$, and this will cause injury along the true path. Also, we will have strategies β which cause strategies $\alpha \prec \beta$ to revert to a previous state in α's construction, though for every α each state in α's construction will only be susceptible to reversion by finitely many $\beta \succ \alpha$. Finally, we will make use of the notion of *moment* to refer to substages in the construction. We assume that actions that strategies make, such as injury and initialization, have immediate effect during moments in the construction, rather than at the end of a stage.

At every moment in the construction, we will say some strategies restrain elements in A and some restrain elements out of A. When we refer to the set A at any given moment in the dynamic construction, we mean

$$\omega^{<\omega} \smallsetminus \{a \colon \text{some strategy } \beta \text{ currently restrains } a \text{ out of } A\}.$$

Our set $A \subseteq \omega^{<\omega}$ now needs to satisfy the following requirements, for all $a \in \omega^{<\omega}$ and all enumeration operators Φ and Ψ.

Requirements:

$$\text{global:} (\forall x, y \in \omega^{<\omega} \smallsetminus A)[\neg\, xEy]$$

$$\mathcal{N}_a \colon a \notin A \text{ or } (\exists x)[xEa \wedge x \notin A]$$

$$\mathcal{R}_{\Phi,\Psi} \colon A = \Psi(\Phi(A)) \implies \Phi(A) \neq \overline{G_f} \text{ for any total function } f \colon \mathbb{N} \to \mathbb{N}$$

Clearly, our global requirement and the $\mathcal{N}_a$-requirements and $\mathcal{R}_{\Phi,\Psi}$-requirements will ensure that A is of cototal (see Section 7.3.3) but not of graph-cototal enumeration degree.

Construction:

We define a priority tree as follows: Each $\mathcal{N}_a$-strategy has only one outcome, d. Each $\mathcal{R}_{\Phi,\Psi}$-strategy has infinitely many possible outcomes: stop $< \infty < \cdots <$ wait$_1$ $<$ wait$_0$. We assign all nodes on a given level of the tree to the same requirement, and every non-global requirement is associated to some level. Finally, if the last coordinate in $a \in \omega^{<\omega}$ is k, then we ensure that the $\mathcal{N}_a$-strategy does not appear in the first k levels of the tree.

The main difficulty in this construction is in performing the strategy for an $\mathcal{R}_{\Phi,\Psi}$-requirement while allowing lower-priority requirements to succeed.

As we will see, one $\mathcal{R}$-requirement may restrain infinitely many elements into A, while lower-priority requirements may need to extract some of these elements from A.

Let us describe the $\mathcal{R}_{\Phi,\Psi}$-strategy for a node α on the priority tree. The strategy has parameters x_α, F^n_α, H^n_α, y^n_α, z^n_α, and D^n_α, whose meaning we now explain. The goal of the strategy is to ensure that some column of $\Phi(A)$ is either complete or misses two elements and thus $\Phi(A)$ cannot be the complement of the graph of a total function. The parameter x is the column that the strategy uses. The superscript n on the parameters F, H, y, z, and D refers to the values of these parameters under the assumption that the true outcome of α is the outcome wait_n. The parameter F is a set that the strategy restrains in A, i.e., the strategy makes sure that no lower-priority strategy removes any element of F from A. The set $H \cup D$ is a finite set that, if we remove it from A and all higher-priority restraints remain, will cause the element $\langle x, y\rangle$ (for our parameter y) to be removed from $\Phi(A)$. The set $H \cup D$ is partitioned into two pieces as the two pieces will relate to other strategies in different ways. The set H is comprised of elements that at some prior stage were restrained out of A by strategies below some outcome wait_m of α (except when α is activated for the first time after initialization, when H contains a single fresh element), and D is comprised of elements that at some prior stage were restrained out of A by a strategy below the outcome ∞ of α. We only extract the set $H \cup D$ if we see some other number w such that we can also ensure that $\langle x, w\rangle \notin \Phi(A)$. The number z is the least number other than y for which we currently do not know that $\langle x, z\rangle \in \Phi(A)$. We try to ensure that z increases infinitely often, thus making the entire column contained in $\Phi(A)$, ensuring that $\Phi(A)$ cannot be the complement of the graph of a total function. The general idea is that if it ever happens that we cannot increase z, i.e., we cannot put $\langle x, z\rangle$ into $\Phi(A)$, then by removing $H \cup D$ from A, we can ensure that two elements are missing from the x-th column of $\Phi(A)$. In this case, as before, $\Phi(A)$ is not the complement of the graph of a total function.

Step −1: When first activated (or after initialization) the strategy starts from Step −1. Pick a large a_0 with $a_0^-, a_0 \in A$ and $a_0{}^\frown\mathfrak{m} \in A$ for all m. (Here by *large* we mean that neither the string a_0, nor any of its components have been mentioned in the construction so far.) Check, using oracle $\mathbf{0}'$, if there are finite sets F and G (given by canonical indices) such that

$$a_0 \in \Psi(G) \text{ and } G \subseteq \Phi(F) \text{ and } F \subseteq\sim A, \qquad (7.6.1)$$

where $\sim A$ is the set of those a for which there is no strategy $\gamma \leq \alpha$, $\gamma <_L \alpha$, or $\gamma \geq \alpha^\frown\infty$ that currently restrains a out of A. If no such F and G exist, then the $\mathcal{R}_{\Phi,\Psi}$-requirement is trivially satisfied since $a_0 \in A$ but $a_0 \notin \Psi(\Phi(A))$. In this case, place a restraint keeping $a_0 \in A$, place a restraint $b \notin A$ for any b for which some strategy $\gamma \geq \alpha^\frown\infty$ is currently restraining $b \notin A$, and take outcome stop. As long as it is not initialized, the strategy will never act again and when visited from now on will take outcome stop. If we can find such F and G with $a_0 \notin F$, then we again take the outcome stop and satisfy the $\mathcal{R}_{\Phi,\Psi}$-requirement by restraining a_0 out of A while ensuring $F \subseteq A$ and thus $a_0 \in \Psi(\Phi(A))$ by restraining F in A. Otherwise, possibly enlarge the finite set F so as to maximize $|G \cap \Phi(F \setminus \{a_0\})|$ for this fixed G. Fix any pair $\langle x, y^0\rangle \in G \setminus \Phi(F \setminus \{a_0\})$ (which is a nonempty set by our assumption) such that for this fixed x, we have that y^0 is least such that $\langle x, y^0\rangle \in G \setminus \Phi(F \setminus \{a_0\})$. Fix x from now on as the parameter x. Let s be the current stage. (Note, that we may assume that $y^0 < s$ by speeding up the construction if necessary.) Let $F^0 = F \setminus \{a_0\}$ and let $H^0 = \{a_0\}$, let D^0 be the set of all elements restrained out by some strategy extending $\alpha^\frown\infty$, and let z^0 be the least number z other than y^0 that is $\leq s$ such that $\langle x, z\rangle \notin \Phi(F^0)$, if such a number exists; let z^0 be s otherwise. We define $F^i = F^0$, $H^i = H^0$, $D^i = D^0$, $z^i = z^0$ for all $i \leq s$. Go to Step s. (This is to ensure that if α is initialized infinitely often, then it visits each outcome wait$_n$ only finitely often. We will design the construction so that from now on, α cannot be reverted to Step i for $i < s$.) We take outcome ∞.

Regardless of which outcome we took, we initialize all nodes that are strictly to the right of the outcome we took.

Step n: Being in Step n means that n is largest such that F^n, H^n, D^n, y^n, and z^n are defined. We say a node β is on the *n-subtree* if the length of β is strictly less than n and for all $\gamma \leq \beta$, $\gamma^\frown wait_k \leq \beta$implies that $k < n$. Note that the n-subtree is finite for every n. If $\beta \geq \alpha^\frown\infty$ and β is not on the n-subtree and (when last visited) β did not take outcome stop, then we initialize β. (Strategies $\beta \geq \alpha^\frown\infty$ can revert α to a previous step. This action ensures that only finitely many strategies can revert α to a step smaller than n.)

Let W be the set of elements restrained out of A by some strategy γ, such that $\gamma \leq \alpha$ or $\gamma <_L \alpha$. Let B_∞ be the set of elements restrained out of A by nodes extending $\alpha^\frown\infty$ and let B_n be the set of elements restrained out of A by nodes extending $\alpha^\frown wait_n$ along with the elements that are in $H^k_\beta \cup D^k_\beta$ for some k and some $\beta \geq \alpha^\frown wait_n$.

Let us say that a set $Y \subseteq \omega^{<\omega}$ is *consistent* if it does not contain any pair of elements which are connected to each other. For example, it will follow inductively that $W \cup B_\infty$ is consistent. Furthermore, since B_n consists of elements that are introduced in the construction after W and B_∞ are defined, it will follow that if $Y \subseteq B_n$ is consistent then so is $W \cup B_\infty \cup Y$. Using oracle $\mathbf{0}'$, we check if $\langle x, z^n \rangle \in \Phi(\mathbb{N} \smallsetminus (W \cup B_\infty \cup Y))$ for all consistent subsets $Y \subseteq B_n$. If so, then we let X_0 be a finite set such that X_0 is disjoint from $W \cup B_\infty \cup B_n$ and $\langle x, z^n \rangle \in \Phi(X_0 \cup (B_n \smallsetminus Y))$ for every consistent $Y \subseteq B_n$. We then redefine F^n to be $F^n \cup X_0$, redefine z^n to be the least number z other than y^n which is $\leq s$ (where s is the current stage) such that $\langle x, z \rangle \notin \Phi(F^n)$, if such a number exists, and let z^n be s otherwise. Leave all other parameters the same, and take outcome wait$_n$.

If $\langle x, z^n \rangle \notin \Phi(\mathbb{N} \smallsetminus (W \cup B_\infty \cup Y))$ for some consistent set $Y \subseteq B_n$, then fix any such set Y. We now check whether $\langle x, z^n \rangle \in \Phi(\mathbb{N} \smallsetminus (W \cup B_\infty))$. If so, then we let X_1 be some set disjoint from $W \cup B_\infty$ such that $\langle x, z^n \rangle \in \Phi(X_1)$. Define $H^{n+1} = Y$, $D^{n+1} = B_\infty$, $y^{n+1} = z^n$, define $F^{n+1} = F^n \cup (X_1 \smallsetminus Y) \cup H^n$, and let z^{n+1} be the least number z other than y^{n+1} that is $\leq s$ (where s is the current stage) such that $\langle x, z \rangle \notin \Phi(F^{n+1})$, if such a number exists, and let z^{n+1} be s otherwise. (This means that next time α is visited, it will be in Step $n+1$, unless it is reverted back to a Step $\leq n$.) We take outcome ∞.

If $\langle x, z^n \rangle \notin \Phi(\mathbb{N} \smallsetminus (W \cup B_\infty))$, then we place a negative restraint so that $(H^n \cup D^n \cup B_\infty) \cap A = \varnothing$. For each γ such that $\gamma^\frown\infty \leq \alpha$ and such that $(H^n \cup D^n \cup B_\infty) \cap F^k_\gamma \neq \varnothing$ or such that $H^n \cup D^n \cup B_\infty$ contains an element a that is the predecessor of an element $b \in H^k_\gamma \cup D^k_\gamma$ (i.e., $b = a^\frown\mathfrak{m}$ for some m), we undefine all of γ's parameters with superscript $\geq k$. Note that γ now reverts to being in a smaller step, say Step ℓ; we say we have *reverted γ to Step ℓ.* (This action is necessary to ensure that there are no conflicting restraints on a and that the set $H^k_\gamma \cup D^k_\gamma \cup B^\gamma_\infty$ is consistent, should γ later on need to restrain it out of A. It will follow from the proof that we need not worry about the possibility of a being the successor of an element $b \in H^k_\gamma \cup D^k_\gamma$. We also point out that if $\alpha \geq \beta^\frown wait_k$, then we do not need to revert β to a previous step as all of β's parameters that can potentially be restrained out of A are defined before α was accessible.) We also initialize all strategies below $\gamma^\frown\infty$ which are not on the ℓ-subtree except for α and which (when last visited) did not take outcome stop. We undefine all of α's parameters (since α will not be reverted back to any Step k, unless it is initialized) and take the outcome stop; unless initialized, we will forever take the outcome stop from now on with no further action.

Regardless of which outcome we took, we initialize all nodes which are strictly to the right of the outcome we took.

The behavior of an $\mathcal{N}_a$-strategy β is simple: First it tries to assess whether a will end up an element of A or not. If a is not mentioned by any strategy of higher priority, then β safely assumes that $a \in A$. If $a \in (D^n_\alpha \cup H^n_\alpha) \cap A$ and $\beta \geq \alpha^\frown wait_n$, then β can assume that a will remain in A (unless β is initialized). If $a \in H^n_\alpha$ for a node α with $\alpha^\frown\infty \leq \beta$ that is in Step n (i.e., n is largest so that H^n_α is defined), then end the stage. (Here β believes that a will not remain in H^n_α and so simply waits.) Otherwise, if $a \in D^n_\alpha$ for a node α with $\alpha^\frown\infty \leq \beta$ that is in Step n, or if $a \notin A$, then do nothing and take the outcome d. (It will follow from the construction that if β is on the true path and $a \in D^n_\alpha$ for a node α with $\alpha^\frown\infty \leq \beta$ at infinitely many stages at which β is visited, then $a \notin A$.) If $a \in A$, then we pick a fresh number m and place a restraint to prevent $a^\frown\mathfrak{m}$ from being in A. In that case, if any $\gamma \geq \beta$ is an $\mathcal{R}_{\Phi,\Psi}$-strategy that (when last visited) did not take outcome stop, then initialize γ. We do this to ensure that the global requirement is met, i.e., no strategy can restrain a out of A unless it initializes β.

Verification:

We now show that our construction ensures the satisfaction of all requirements.

Lemma 7.6.2. If α is a strategy that is reverted to Step n, it is because a node $\beta \geq \alpha^\frown\infty$ that is on the $(n+1)$-subtree takes the outcome stop.

Proof. If α was in Step -1 when the parameters F^{n+1}_α, H^{n+1}_α, and D^{n+1}_α were last modified before the current stage then all strategies extending $\alpha^\frown\infty$ that had not yet stopped were in their initial state. Otherwise, at the moment when F^{n+1}_α was last modified before the current moment, α was in Step at most $n+1$. At the moment when H^{n+1}_α and D^{n+1}_α were defined, α was at most in Step n. In any case, all strategies extending $\alpha^\frown\infty$ that were not on the $(n+1)$-subtree and had not yet taken outcome stop were initialized or in initial state. We claim that if $\beta \geq \alpha^\frown\infty$ is not on the $(n+1)$-subtree, then (unless α is initialized) at no later stage will β have $(H^k_\beta \cup D^k_\beta \cup B_\infty) \cap F^{n+1}_\alpha \neq \emptyset$ or will $H^k_\beta \cup D^k_\beta \cup B_\infty$ contain a predecessor of an element in $H^{n+1}_\alpha \cup D^{n+1}_\alpha$. We prove this by induction on moments of the construction. At no later stage will an $\mathcal{N}_c$-strategy place a restraint that takes an element out of A that is in F^{n+1}_α or that is the predecessor of an element in $H^{n+1}_\alpha \cup D^{n+1}_\alpha$. This is because the $\mathcal{N}_c$-strategy extracts an element of the form $c^\frown \mathfrak{m}$ where m is new, so it could not have appeared in F_α^{n+1} or be a predecessor of an element in $H_\alpha^{n+1} \cup D_\alpha^{n+1}$. The same is true for $\mathcal{R}_{\Phi,\Psi}$-strategies that extract a new element because they take the outcome stop in Step −1, because this is also a new element. Now, any H_β^k or D_β^k is formed from elements that are in $H_\gamma^\ell \cup D_\gamma^\ell$ for $\gamma > \beta$, from elements restrained out by $\mathcal{N}_c$-strategies extending β, or from some $\{a_0\}$ introduced by an $\mathcal{R}_{\Phi,\Psi}$-strategy in Step −1 extending β (or equal to β if β is in Step −1). So, by induction, no $H_\gamma^\ell \cup D_\gamma^\ell$ can contain an element in F_α^{n+1} or an element that is the predecessor of an element in $H_\alpha^{n+1} \cup D_\alpha^{n+1}$, thus neither can $H_\beta^k \cup D_\beta^k$. □

Lemma 7.6.3. If α is along the true path and is an $\mathcal{R}_{\Phi,\Psi}$-strategy, and $n \in \omega$, then there are only finitely many stages at which α is reverted to Step n.

Proof. We prove the result by simultaneous induction on n for all strategies combined. Suppose towards a contradiction that some strategy α on the true path is reverted to Step n infinitely often. Since there are only finitely many elements in the $(n+1)$-subtree below $\alpha^\frown\infty$, there must be some β doing this infinitely often. In particular, α is on the $(n+1)$-subtree. Suppose in addition that α is the longest strategy on the true path and in the $(n+1)$-subtree that is reverted to Step n infinitely often.

Every time β reverts α to Step n, it must take the outcome stop. To be initialized infinitely often after taking outcome stop, it must be that we visit a node left of β infinitely often, as this is the only way in which we initialize stopped strategies. Since α is on the true path, there must be some shortest $\gamma \geq \alpha$ along the true path which takes an outcome left of β infinitely often.

Case 1: γ is an $\mathcal{R}$-strategy and β is below the outcome wait_k of γ for some $k < n+1$ (as β is on the $(n+1)$-subtree). It follows that $\alpha \neq \gamma$, because $\beta \geq \alpha^\frown\infty$. Then to visit β again, γ must be reverted to a step $\leq n$ infinitely often. By our choice of α and by the induction hypothesis, this is impossible.

Case 2: γ is an $\mathcal{R}$-strategy and β is below the outcome ∞ of γ. Then γ infinitely often visits the outcome stop, but is initialized. Since we initialize a strategy that has taken outcome stop only by visiting a node to the left of it, this contradicts our choice of γ as a strategy on the true path.

Case 3: γ is an $\mathcal{R}$-strategy and β is below the outcome stop of γ. Then γ cannot take an outcome left of the outcome stop.

Case 4: γ is an $\mathcal{N}$-strategy. This is impossible because $\mathcal{N}$-strategies only have one outcome. □

Lemma 7.6.4. Every strategy along the true path is initialized only finitely often.

Proof. For α to be initialized, either some node to its left is visited, or it is not in the k-subtree and some node $\beta^\frown\infty \leq \alpha$ is in Step k, or some $\mathcal{N}$-node above it places a restraint. The first case happens only finitely often, since α is along the true path. The second case happens only finitely often since, if α is on the ℓ-subtree for the least number l, and along the true path, then each β with $\beta^\frown\infty \leq \alpha$ eventually is not reverted to a step $< \ell$. The last case can only happen finitely often since, by the inductive hypothesis, every node $\beta < \alpha$ is initialized only finitely often. □

Lemma 7.6.5. For each node α, if H^k_α becomes defined at stage s, then for every $a \in H^{k+1}_\alpha$, the last number in the string a is $> k, s$.

Proof. For H^k_α to become defined at stage s, this requires α to take outcome ∞ at stage s. Thus every strategy below $\alpha^\frown wait_k$ is initialized. Since before an element can enter H^{k+1}_α, it must first be in $H^\ell_\beta \cup D^\ell_\beta$ for some $\beta \geq \alpha^\frown wait_k$ and some $\ell \in \omega$, be restrained out of A by some $\mathcal{N}$-strategy extending $\alpha^\frown wait_k$, or be in some $\{a_0\}$ introduced by an $\mathcal{R}_{\Phi,\Psi}$-strategy β in Step -1 with $\beta \geq \alpha^\frown wait_k$, we see that it must be introduced by such a strategy at a stage $\geq s$. Furthermore, the number k has been mentioned. So, when β restrains an element a out of A, its last number is new, thus greater than k and s. □

We will say that a node is *active* at the current moment m if it has been visited at a moment $n \leq m$ and has not been initialized at any moment in the interval $[n, m]$.

Lemma 7.6.6. If α is an $\mathcal{R}$-strategy that first takes the outcome stop (since its last initialization) at stage s, then there is no active node below $\alpha^\frown stop$ at the beginning of stage s.

Proof. The statement is clearly true if this is the first time when α takes outcome stop. Suppose that this is not the case and consider the stage s at which α last took the outcome stop. Let $t > s$ be the first time that α was initialized after stage s. Then this initialization must be the result of visiting a node to the left of α, since this is the only way we initialize stopped strategies. Thus any $\gamma \geq \alpha^\frown stop$ would also be initialized. □

The following lemma ensures that at no stage of the construction do we take any contradictory actions.

Lemma 7.6.7. At every moment of the construction, the following hold:

1. No two different strategies place conflicting restraints on a string (i.e., it is impossible that one restrains it in A and the other restrains it out of A, where being restrained into A means being restrained in by Step -1 or being in the current set F).

2. It is not the case that any H^n_α contains an element that is restrained out of A.

3. If $a \in D^n_\alpha$, and either a is restrained out of A by β or $a \in H^k_\beta$, then $\alpha^\frown stop <_L \beta$. (So, if α ever restrains a out of A, then β is initialized.)

4. If $a \in H^n_\alpha \cup D^n_\alpha$ and a is restrained in A by β, and $\alpha \npreceq \beta$, then either $\alpha^\frown stop <_L \beta$ or $\beta^\frown\infty \leq \alpha$. (So, if α restrains a out of A, then β is initialized or reverted to a smaller step.)

5. If $a \in D^n_\alpha$ and $a \in D^m_\beta$ for $\alpha \neq \beta$, then $\alpha \geq \beta^\frown\infty$ or $\beta \geq \alpha^\frown\infty$. (So, $D^n_\alpha \subseteq D^n_\beta$ or $D^n_\beta \subseteq D^n_\alpha$.)

6. If $(n, \alpha) \npreceq (m, \beta)$, then $H^n_\alpha \cap H^m_\beta = \varnothing$.

7. It is not the case that two different strategies restrain a out of A at the same time.

Proof. We will show that all statements hold at all times. Consider the first moment when one of the claims fails, and suppose that it is (1). Let α restrain a into A, while β restrains a out of A.

Case 1: α placed its restraint first. Then β places its negative restraint in one of two ways: Either β is an $\mathcal{N}_c$-strategy for some c, or β is an $\mathcal{R}_{\Phi,\Psi}$-strategy which takes the outcome stop. In the first case, the $\mathcal{N}_c$-strategy restrains an element of the form $c^\frown\mathfrak{m}$ where m is fresh, in particular ensuring that $c^\frown\mathfrak{m}$ is not already restrained in A. In the second case, the only new element restrained out by β in Step -1 is a fresh element a_0 which cannot happen for the same reason, or β restrains out elements in $H^n_\beta \cup D^n_\beta \cup B_\infty$ (where β is in Step n). But a cannot be in B_∞, as otherwise, at the previous moment, we would have restrained a both in and out of A. If a is in $H^n_\beta \cup D^n_\beta$, then this must have happened at a previous moment. So by the inductive hypothesis, by (4), either $\beta^\frown stop <_L \alpha$ or $\beta \geq \alpha^\frown\infty$. In the former case, we have that α is initialized as β decides to restrain a out of A, thus there is no conflict. In the second case, α is reverted to a previous Step k at which $a \notin F^k_\alpha$, again showing there is no conflict.

Case 2: β placed its restraint first. Then α is an $\mathcal{R}_{\Phi,\Psi}$-strategy that adds a to F, while β already restrains a out of A. We have three cases where this can happen. First, if α is in Step -1, then by the definition of F^0 it can only be that $\beta \geq \alpha^\frown wait_k$ for some $k \in \omega$. (Note that this is nontrivially possible, because it could be that α was initialized for not being on some subtree, while β is left active, as it had already reached its outcome stop.) However, then β is initialized, because α has outcome stop or ∞, both to the left of wait$_k$, so there is no conflict. Otherwise, either α remains in Step n and increases F by including X_0, or α moves to Step $n+1$ and defines F^{n+1} to be $F^n \cup (X_1 \setminus Y) \cup H^n$. In the former case, since X_0 is explicitly chosen to be comprised of elements that are not restrained out of A by any strategy at all, we see that a cannot be added to F^n at this moment. In the latter case, H^n is disjoint from anything restrained out of A by (2) (and our assumption of this being the first moment when any of the conditions is violated). Suppose we then have $a \in X_1 \setminus Y$. Then it is either in $X_1 \setminus B_n$ and thus not restrained out of A by any strategy, or it is in B_n. If it is in B_n, then it is either in H^k_γ or D^k_γ for some γ extending $\alpha^\frown wait_n$ or is restrained out of A by such a γ. In the first case, this contradicts (2). In the second case, we would have $\gamma^\frown stop <_L \beta$ by (3), thus $\alpha^\frown\infty <_L \beta$ and β is initialized when α places a into F^{n+1}_α. In the third case, either a is restrained out of A by two different strategies at a previous moment, contradicting (7), or $\gamma = \beta$ and β is initialized when α takes outcome ∞.

Suppose the first moment where any of the claims fails is one where (2) fails, i.e., a appears both in H^n_α and is restrained out of A by a node β.

Case 1: α placed a into H^n_α first. Then β places its negative restraint in one of two ways: Either β is an $\mathcal{N}_c$-strategy for some c, or β is an $\mathcal{R}_{\Phi,\Psi}$-strategy that takes the outcome stop. In the first case, the $\mathcal{N}_c$-strategy restrains an element of the form $c^\frown\mathfrak{m}$ where m is fresh, in particular ensuring that $c^\frown\mathfrak{m}$ is not already in H^n_α. In the second case, the only new element restrained out in Step -1 is a fresh element, which we have just mentioned does not cause a conflict, or we have $a \in H^m_\beta \cup D^m_\beta \cup B_\infty$ where β is in Step m, and β takes the outcome stop. If $a \in B_\infty$, then a was previously restrained out of A, which is a contradiction. If $a \in H^m_\beta$, then we would have $H^m_\beta \cap H^n_\alpha \neq \varnothing$ at a previous moment, contradicting (6). If $a \in D^m_\beta$, then by (3), $\beta^\frown stop <_L \alpha$, and so α is initialized when β restrains a out of A; thus there is no conflict.

Case 2: β restrains a out of A first and α places a into H^n_α at the current moment. If α is in Step -1 then H^0 contains only one element, chosen as a fresh number by α and hence not restrained by β. It follows that α is in Step $n-1 \geq 0$ and a must have either been restrained out of A by some strategy $\gamma \geq \alpha^\frown wait_{n-1}$ or must have been in $H^k_\gamma \cup D^k_\gamma$ for some $\gamma \geq \alpha^\frown wait_{n-1}$. It would violate (2) for a to be in H^k_γ at the previous moment, and if $a \in D^k_\gamma$ at the previous moment, then by (3), we would have $\gamma^\frown stop <_L \beta$, so $\alpha^\frown\infty <_L \beta$ and β is initialized when a is added into H^n_α. So suppose γ restrains a out of A. When we redefine $H^n_\alpha = Y$ (for some $Y \subseteq B_n$), we take the outcome $\alpha^\frown\infty$, thus injuring this γ. Thus if $\beta = \gamma$, then we have that β has relinquished its restraint, and if $\beta \neq \gamma$, then at the previous moment, β and γ restrained the same element out of A, violating (7) at the previous moment.

Suppose the first moment where any of the claims fails is one where (3) fails, i.e., suppose $a \in D^n_\alpha$ and a is restrained out of A by β or is contained in H^k_β while $\alpha^\frown stop \nless_L \beta$.

Case 1: α places a into D^n_α first. Suppose a is restrained out of A by β. Again, this cannot happen if β is an $\mathcal{N}$-strategy. Suppose β is an $\mathcal{R}$-strategy that takes the outcome stop in Step -1. Then the only new element restrained out is fresh, so a must have already been restrained out of A by some $\gamma \geq \beta^\frown\infty$. Then (c) applied at a previous moment implies that $\alpha^\frown stop <_L \gamma$, so either $\alpha^\frown stop <_L \beta$ or $\beta \leq \alpha$. Furthermore, α cannot be below $\beta^\frown stop$, because the last time β was initialized, so were all non-stopped $\mathcal{R}$-strategies below it, and hence D^n_α for such an α is empty. So, if $\beta \prec \alpha$, then when β takes the outcome stop, α is initialized. If $\alpha = \beta$, then D^n_α becomes undefined when β takes the outcome stop. In both cases there is no conflict.

Now, suppose we have $a \in H^k \cup D^k \cup B_\infty$ for β. If $a \in H^k$, we have that $\alpha^\frown stop <_L \beta$, by (3) at the previous moment. If $a \in D^k$, then by (5) at the previous moment, either $\alpha \geq \beta^\frown\infty$ or $\beta \geq \alpha^\frown\infty$. In the first case, when β restrains a out of A, it takes the outcome stop, injuring α, while in the second case, $\alpha^\frown stop <_L \beta$. If $a \in B_\infty$, then at the previous moment, we had a restrained out of A by some $\gamma \geq \beta^\frown\infty$. Thus $\alpha^\frown stop <_L \gamma$, by (3) at the previous moment. Either $\alpha^\frown stop <_L \beta$ or $\beta^\frown\infty \leq \alpha$ (noting that, if $\alpha = \beta$, we would undefine D^n_α when β takes the outcome stop and that, as before, $\alpha \geq \beta^\frown stop$ is not possible). In the latter case, when β restrains a out of A, it takes the outcome stop, injuring α.

Now, suppose α places a into D^n_α first and at the current moment a enters H^k_β. It cannot be that β is in Step -1 as the only element that enters H^0 is a fresh number, hence different from a. It follows that β is in step $k-1 \geq 0$ and at the previous moment, a was either in $H^\ell_\gamma \cup D^\ell_\gamma$ for some $\gamma \geq \beta^\frown wait_{k-1}$ or was restrained out of A by some $\gamma \geq \beta^\frown wait_{k-1}$. If it was in H^ℓ_γ or restrained out of A by γ, then by (3) at the previous moment, $\alpha^\frown stop <_L \gamma$. Thus either $\alpha^\frown stop <_L \beta$, $\beta^\frown wait_{k-1} \leq \alpha$, or $\alpha = \beta$. Note that $\beta^\frown \infty \leq \alpha$ is not a possible case, because all elements in H^k_β have been introduced into the construction by strategies extending outcome wait_{k-1} after α placed $a \in D^n_\alpha$. In the second case, when β adds a to H^k, it initializes α, avoiding conflict. If $\alpha = \beta$ and $n < k$, then a must have been introduced by some strategy $\delta \geq \alpha$ which is an $\mathcal{N}$-strategy or an $\mathcal{R}$-strategy which took the outcome stop at Step -1 at some stage after a entered D^n_α, which contradicts freshness. If $\alpha = \beta$ and $n = k$, then both events really happen at the same time, and by definition of D^n_α, a was restrained out at the previous moment by a strategy $\delta \geq \alpha^\frown\infty$, which cannot be true by (2). Lastly, suppose a was in D^ℓ_γ for $\gamma \geq \beta^\frown wait_{k-1}$. Then by (5) (at the previous moment), $\alpha \geq \gamma^\frown\infty$ or $\gamma \geq \alpha^\frown\infty$. If $\alpha \geq \beta^\frown wait_{k-1}$, then when β takes the outcome ∞ to place a into H^k_β, α is initialized. So, we may suppose rather that $\alpha^\frown\infty \leq \beta \leq \gamma$. Thus $\alpha^\frown stop <_L \beta$.

Case 2: β restrains a out of A or places a into H^k_β first. If $\alpha = \beta$ these events really occur at the same time, and we already discussed this situation in the previous case. So, let us assume $\alpha \neq \beta$. When a joins D^n_α, D^n_α is defined to be the set of elements restrained out of A by nodes extending $\alpha^\frown\infty$. Thus at the previous moment, β restrains a out of A, and hence β is below $\alpha^\frown\infty$ by (7) or a was in H^k_β, and it was restrained out of A by some node $\gamma \geq \alpha^\frown\infty$, contradicting (2). It follows that $\alpha^\frown stop <_L \beta$.

Consider the first moment when one of the claims fails, and suppose that it is (4). Let $a \in H^n_\alpha \cup D^n_\alpha$ be such that a is restrained in A by some node β.

Case 1: a is placed into $H^n_\alpha \cup D^n_\alpha$ first. Then β cannot be left of α, as otherwise α would be initialized. Furthermore, α cannot currently be in the outcome stop, otherwise H^n_α would be undefined. Thus, $\alpha^\frown stop <_L \beta$ or $\beta \leq \alpha$. If $\beta^\frown wait_m \leq \alpha$, then it must be that β is in step m and expands the definition by adding X_0 to F^m. Indeed, if β is in Step -1 it has outcome stop or ∞, initializing α; if β is in Step $k < m$ then β was reverted to a smaller step $l \leq k$ after α placed a in $H^n_\alpha \cup D^n_\alpha$ by a strategy $\gamma \geq \beta^\frown\infty$ and at that moment α must have been initialized; if β is in Step $k > m$ or if β is in Step m and defines F^{m+1} then it will have outcome to the left of α at this moment and so initialize α. The set X_0 added to F^m is disjoint from $H^n_\alpha \cup D^n_\alpha$ for any $\alpha \geq \beta^\frown wait_m$, so a is not in X_0. The strategy α cannot be below β's outcome stop, as β cannot restrain any elements in A after α was first accessible unless it is initialized and then α would be initialized as well. Thus, the only possibility left is $\beta^\frown\infty \leq \alpha$, as desired.

Case 2: a is restrained in A by β first. For a to enter $H^n_\alpha \cup D^n_\alpha$, it must have already been either restrained out of A by some strategy below α, or α is in Step -1 and $a = a_0$, or it must have been in $H^k_\gamma \cup D^k_\gamma$ for some γ below $\alpha^\frown wait_n$. In the first case, this contradicts (1). The second case contradicts the freshness of a_0. So suppose a was in $H^k_\gamma \cup D^k_\gamma$, and we have either $\gamma^\frown stop <_L \beta$ or $\gamma \geq \beta^\frown\infty$ by (4) at a previous moment. In the first case, we then also have $\alpha^\frown stop <_L \beta$. In the second case, either $\alpha \geq \beta^\frown\infty$ or $\alpha^\frown wait_n \leq \beta^\frown\infty \leq \gamma$. In the former case, we have what the claim allows, and in the latter case, again $\alpha^\frown stop <_L \beta$.

Consider the first moment when one of the claims fails, and suppose that it is (5). Let $a \in D^n_\alpha \cap D^m_\beta$ for $\alpha \neq \beta$. For a to be in D^n_α, some strategy γ_1 below $\alpha^\frown\infty$ that is an $\mathcal{N}$-strategy or $\mathcal{R}$-strategy in Step -1 must have at some point proposed a into the construction. Similarly, for a to be in D^m_β, some strategy γ_2 below $\beta^\frown\infty$ that is an $\mathcal{N}$-strategy or $\mathcal{R}$-strategy in Step -1 must have, at some point proposed a in the construction. Now, since in both cases the strategy proposes a fresh element, it is impossible that $\gamma_1 \neq \gamma_2$. Thus $\gamma = \gamma_1 = \gamma_2$ is below both $\alpha^\frown\infty$ and $\beta^\frown\infty$, showing that either $\alpha \geq \beta^\frown\infty$ or $\beta \geq \alpha^\frown\infty$.

Consider the first moment when one of the claims fails, and suppose that it is (6). Let $a \in H^n_\alpha \cap H^m_\beta$. Without loss of generality, H^m_β is the one defined at this moment. Again, it cannot be that β is in Step -1 by freshness of a_0, so β is in Step $m-1 \geq 0$. Then some strategy γ below $\beta^\frown wait_{m-1}$ either previously had $a \in H^k_\gamma \cup D^k_\gamma$, or was previously restraining a out of A. If a was in H^k_γ or γ was restraining a out of A, we would already contradict (6) or (2), unless $(n,\alpha) = (k,\gamma)$. However, if $\alpha = \gamma$, then we have that $\alpha \geq \beta^\frown wait_{m-1}$, and hence when β defines H^m_β and takes the outcome ∞, it initializes α. Finally, assume $a \in D^k_\gamma$. Then, by (3), $\gamma^\frown stop <_L \alpha$. But $\beta^\frown\infty <_L \gamma^\frown stop <_L \alpha$, so again, when β defines H^m_β and takes the outcome ∞, it initializes α.

Consider the first moment when one of the claims fails, and suppose that it is (7). Let α and β both restrain a out of A, and let us assume that α places its restraint first.

Case 1: β is an $\mathcal{N}_b$-strategy. Then since β restrains an element out of A of the form $b^\frown\mathfrak{m}$ where m is fresh, it cannot restrain a out of A if it is already restrained out of A.

Case 2: β is an $\mathcal{R}_{\Phi,\Psi}$-strategy. Suppose a is restrained out of A by β in Step -1. Then since a is not fresh, it must have been restrained out of A by some $\gamma \geq \beta^\frown\infty$ not below the outcome stop. By (7) at the previous moment, $\gamma = \alpha$. Thus, when β takes the outcome stop, α is initialized, so there is no conflict. Thus, at the previous moment, $a \in H^n_\beta \cup D^n_\beta \cup B_\infty$. If $a \in D^n_\beta$, then by (3) at the previous moment, we have $\beta^\frown stop <_L \alpha$. Thus, when β takes the outcome stop, it initializes α, showing that there is no conflict. It is impossible that $a \in H^n_\beta$ by (2) at the previous moment. If a is in B_∞, then some strategy γ below $\beta^\frown\infty$ restrains a out of A. By (7) at the previous moment, we have that $\alpha = \gamma \geq \beta^\frown\infty$. So again, when β takes the outcome stop, α is initialized, showing that there is no conflict. □

Lemma 7.6.8. The set A is $\Delta^0_2(\mathbf{0}')$. That is, for every a, there is a stage t such that $a \in A$ at every moment after stage t or $a \notin A$ at every moment after stage t.

Proof. Every restraint of a number a being out of A begins with the introduction of a by an $\mathcal{N}$-strategy or an $\mathcal{R}$-strategy in Step −1. If this never happens for a, then we have $a \in A$ at every moment of the construction.

Suppose some γ introduces a. If γ is never subsequently initialized and it restrains a out of A, then $a \notin A$ for every moment after this restraint is placed. If γ is an $\mathcal{R}$-strategy and $a \in H^0$, but γ is never subsequently initialized and never restrains a out of A, then by Lemma 7.6.7(2) no other strategy does either, so $a \in A$ at every moment after a is introduced.

Otherwise, let s be a stage such that γ has introduced a and is initialized before stage s. After γ introduces a, no $\mathcal{N}$-strategy or $\mathcal{R}$-strategy in Step −1 can re-introduce a by freshness. At stage $t \geq s$, a may be in $H^n_\beta \cup D^n_\beta$ for some $\beta \preceq \gamma$ and some $n \in \omega$. Let Z_t be the set of such β. Furthermore, let U_t be the set of $\beta \preceq \gamma$ that restrain a out at stage t. It follows by Lemma 7.6.7(7) that at all times, U_t contains at most one element. If at stage $t \geq s$, the set $U_t = \{\beta_t\}$, then by Lemma 7.6.7(2) and (3), Z_t contains only strategies α that would initialize β_t if they had outcome stop, in particular, only strategies above β_t (as all strategies in $Z_t \cup U_t$ are initial segments of γ, hence comparable). If at a stage $t \geq s$, a enters $H^n_\alpha \cup D^n_\alpha$ for $\alpha \notin Z_{t-1}$ then it does so in one of two ways: by joining H^n_α if there is a $\beta \in Z_{t-1} \cup U_{t-1}$ such that $\alpha^\frown wait_{n-1} \preceq \beta$, in which case this β is initialized and hence leaves $Z_t \cup U_t$, or by joining D^n_α if there is a $\beta \in U_{t-1}$ with $\beta \geq \alpha^\frown\infty$. In both cases, the minimum priority of strategies in $Z_{t-1} \cup U_{t-1}$ does not decrease, while $Z_t \cup U_t$ only contains nodes that are (possibly non-proper) initial segments of nodes in $Z_{t-1} \cup U_{t-1}$. If at stage t, the strategy β enters U_t, then it must be that β takes the outcome stop and $\beta \in Z_{t-1}$.

As we argued above, at every stage t we have that $Z_t \cup U_t$ only contains nodes that are initial segments of nodes in $Z_{t-1} \cup U_{t-1}$. So, there are only finitely many possibilities for U_t. Furthermore, it also follows from our argument above that, if the strategy that restrains a out of A changes between stages t_1 and t_2, then the strategy restraining a out at the later stage t_2 has higher priority than the strategy restraining a out at stage t_1. Therefore, let $t > s$ be a stage at which $U_r = U_t$ for all $r > t$. If U_r is empty, then $a \in A$ at every moment after t, and if it is not, then $a \notin A$ at every moment after t. □

Lemma 7.6.9. If β is an $\mathcal{N}_a$-strategy along the true path, then β ensures that $\mathcal{N}_a$ is satisfied.

Proof. Let β be an $\mathcal{N}_a$-strategy along the true path. By Lemma 7.6.4, let s be a stage at which β is visited and such that no $\gamma \preceq \beta$ is ever initialized at a stage $t \geq s$. Furthermore, by Lemma 7.6.3, we can suppose that s is large enough such that, if $\alpha^\frown\infty \preceq \beta$, then α will never be in Step k for $k \leq a(|a|-1)$. Then it follows from Lemma 7.6.5 that at any stage $t \geq s$, $a \notin H^\ell_\alpha$ for any node α such that $\alpha^\frown\infty \preceq \beta$ with α in Step ℓ. If, at some stage $t > s$ when β is visited, we have both $a \notin D^\ell_\alpha$ for all nodes α such that $\alpha^\frown\infty \preceq \beta$ with α in Step ℓ, and $a \in A$, then β will place a permanent restraint, ensuring that $a^\frown\mathfrak{m} \notin A$ for some fresh m. Otherwise, at any stage $t > s$ we have $a \notin A$ or $a \in D^\ell_\alpha$ for some node α as in the previous sentence, in which case at the last stage $r < t$ when we visited α, we defined D^ℓ_α to only contain elements that are restrained out at that moment. Thus, if β fails to place a permanent restraint, then $a \notin A$ at infinitely many moments when β is visited; thus $a \notin A$ follows by Lemma 7.6.8. □

Lemma 7.6.10. Let a be a string restrained out of A at stage t. Then all successors of a that are introduced by stage t can never again be restrained out of A.

Proof. Towards a contradiction, suppose that some successor b of a that was already introduced at stage t is restrained out of A at some later stage $s > t$. When we introduce a number, we always select it as a new number. It follows that a cannot be introduced after b and that no $\mathcal{R}$-strategy can introduce b, so b is introduced by an $\mathcal{N}_a$-strategy δ after a is introduced and at a stage $r < t$ when a is in A. Since a has been introduced and will eventually be restrained out of A and since δ initialized all lower priority strategies at stage r, the element a must be hiding in $H^n_\gamma \cup D^n_\gamma$ for some strategy γ of higher priority than δ. At stage t, the string a is restrained out of A by some $\mathcal{R}$-strategy $\beta \preceq \gamma$ that takes the outcome stop. At this stage, the strategy δ

is initialized, so the only way in which b can be restrained out of A at stage $s > t$ is if it, in turn, is hiding at stage t in $H^k_\alpha \cup D^k_\alpha$ for some $\mathcal{R}$-strategy α of higher priority than β. Note that strategies of lower priority than β are either in initial state or initialized at stage t. Furthermore, as β is not initialized at any stage in the interval $(r, t]$ (or else every possible strategy that could be keeping a as part of its parameters, such as γ, would also be initialized) and α was visited at a stage q in that interval (when it defined $H^k_\alpha \cup D^k_\alpha$), β must be an extension of α. If $\beta \geq \alpha^\frown\infty$, then at stage t the strategy β reverts α to a previous Step l such that $H^l_\alpha \cup D^l_\alpha$ does not contain any successors of a. Otherwise, $\beta \geq \alpha^\frown wait_i$, for some i. But then at the stage $q < t$ at which b entered $H^k_\alpha \cup D^k_\alpha$ the strategy α had outcome ∞, initializing β, γ and all other strategies that could be protecting a, so a could not be restrained out of A at stage t. This gives us the desired contradiction. □

Lemma 7.6.11. At no moment are there any two strings that are edge-related and both restrained out of A. Furthermore, if α is an $\mathcal{R}_{\Phi,\Psi}$-strategy that is currently in Step $n \geq 0$, then at any moment we have that, if $a, b \in B_\infty \cup H^n_\alpha \cup D^n_\alpha \cup B_n$ and a and b are edge-related, then $a, b \notin B_\infty$ and $a \notin H^n_\alpha \cup D^n_\alpha$ or $b \notin H^n_\alpha \cup D^n_\alpha$.

Proof. We prove these two facts simultaneously by induction on moments. Let us first consider the first claim. Suppose towards a contradiction that two strings a and b are edge-related and both restrained out of A. Let α restrain a and β restrain b out of A, and suppose α places its restraint first (or $\alpha = \beta$ thus α and β place their restraint simultaneously).

Case 1: β is an $\mathcal{N}_c$-strategy. Then β restrains an element out of the form $c^\frown\mathfrak{m}$ where m is fresh. Thus, it is impossible that a is a successor of $c^\frown\mathfrak{m}$. So, the only possibility is that $a = c$, but then β would not restrain any element out of A at all, since $c \notin A$ when β is visited.

Case 2: β is an $\mathcal{R}_{\Phi,\Psi}$-strategy that takes the outcome stop. If β restrains $b = a_0$ out in Step -1, then a and b are not edge-related by construction. If β restrains b out of A, because b is restrained by some strategy $\gamma \geq \beta^\frown\infty$, then the claim follows inductively as a and b would both be restrained out of A at a previous moment. So, we can assume that β is in Step $n \geq 0$ and, at the previous moment, a was restrained out of A by α and b was either in H^n_β, D^n_β or B_∞. If $\alpha = \beta$, then a and b were restrained out at the same time, since an $\mathcal{R}_{\Phi,\Psi}$-strategy only places a negative restraint when it takes the outcome stop. Thus from the second claim at the previous moment it now follows that a and b are not edge-related.

Next, let us assume that $\alpha \neq \beta$. It follows by the inductive hypothesis that $b \notin B^{\infty}$. Suppose $b \in D^n_\beta \cup H^n_\beta$. Then since α is not initialized by β taking outcome stop, we can conclude that either $\alpha \prec \beta$ or $\alpha <_L \beta$. Either way, α has not acted since β and all $\mathcal{R}$-strategies below β that have not yet stopped were initialized. (If α is an $\mathcal{N}$-strategy, then when it acted, it would have initialized β unless it had already stopped, in which case it must have been initialized since then in order to stop again. If α is an $\mathcal{R}$-strategy, then it places a negative restraint by taking the outcome stop, so β was either initialized then for being right of the outcome stop or is below the outcome stop and was first visited then.) Consider the moment t at which α acted to take a out of A. At that moment, no $\mathcal{R}$-strategy below β had any parameters, unless already in outcome stop. Thus, for b to be in $D^n_\beta \cup H^n_\beta$ now and not out of A at stage t (which is excluded by our inductive hypothesis), it was, at some stage $r > t$, restrained out of A by an $\mathcal{N}$-strategy or introduced by an $\mathcal{R}$-strategy in Step -1 below β. The second case cannot happen, because we explicitly pick a b which is not connected to a. Thus, at some stage, we had an $\mathcal{N}$-strategy restraining an element that is edge-related to an element not in A, which is impossible by Case 1.

Next, let us consider the second claim. Assume that $a, b \in Z_\alpha = B_\infty \cup H^n_\alpha \cup D^n_\alpha \cup B_n$ and that a and b are edge-related. Let us assume that a entered Z_α first, and that b just entered Z_α. We now have several cases.

First, if b enters $H^n_\alpha \cup D^n_\alpha$, then the strategy α is currently moving from Step $n-1$ to Step n, b is a member of $B_\infty \cup B_{n-1}$ at the previous moment, and B_n is empty. In particular, Z_α at the current moment contains only elements from $B_\infty \cup B_{n-1}$, as $D^n_\alpha = B_\infty$ and $H^n_\alpha \subseteq B_{n-1}$. Since $a \in Z_\alpha$ at the current moment, it follows that $a \in B_\infty \cup B_{n-1}$ at the previous moment as well. Thus, by the induction hypothesis, $a, b \in B_{n-1}$, and therefore by construction, we can only have that $a, b \in H^n_\alpha$. However, we explicitly defined H^n_α as a consistent subset of B_{n-1}, hence not containing any edge-related elements.

Next, if b enters B_∞, it means that at the previous stage $s^- < s$ when α was visited, we took the outcome ∞ and some strategy extending this outcome restrained b out of A. It follows that currently $B_n = \varnothing$ once again. It cannot be that $a \in B_\infty$, by the first claim at the current moment, which we have just proven. So $a \in H^n_\alpha \cup D^n_\alpha$ from stage s^- onwards until the current moment at stage s. It follows that at stage s^-, no $\mathcal{N}_a$-strategy $\beta \succeq \alpha^\frown\infty$ acts, and hence b cannot have been restrained out for this reason. Suppose that b is restrained out by an $\mathcal{R}$-strategy $\beta \succeq \alpha^\frown\infty$ taking outcome stop at stage s^-. If a is the successor of b, then β would revert the strategy α to a Step l such that $a \notin H^l_\alpha \cup D^l_\alpha$ and so $a \notin Z_\alpha$ at the current moment. It follows that b is the successor of a and b was introduced by an $\mathcal{N}_a$-strategy δ extending β at an earlier stage $r < s$ when a was already introduced but not restrained out of A. At stage r, the strategy α was in a previous step $k < n$. (Otherwise the $\mathcal{N}_a$-strategy δ would not impose a restraint on b at stage r.) This means that $a \in D^n_\alpha$, as the elements of H^n_α are introduced after stage r by strategies extending $\alpha^\frown wait_{n-1}$ and all these strategies are initialized at stage r (as $\delta \geq \alpha^\frown\infty$ and so α took outcome ∞ at stage r). In order for a to enter D^n_α, it must be restrained out of A when D^n_α is defined at stage t such that $r < t < s$. But then by Lemma 7.6.10 it follows that b cannot be restrained out of A at stage s.

Finally, let us assume b enters B_n. Then it has to be introduced by some node $\beta \geq \alpha^\frown wait_n$, since otherwise it was already in B_n at the previous moment. Now, by freshness, the only reason it can be connected to a is if $b^- = a$ and a is currently not restrained out. Thus, $a \notin B^\infty$. □

Lemma 7.6.12. Let α be an $\mathcal{R}_{\Phi,\Psi}$-strategy along the true path. For all $n \in \omega$ and at any moment, for all $w < z^n_\alpha$, either $w = y^n_\alpha$ or $\langle x, w\rangle \in \Phi(F^n_\alpha)$.

Proof. Note that this is true when we first enter Step 0, since z^0 is selected as the least z other than y^0 such that $\langle x, z\rangle \notin \Phi(F^0_\alpha)$. This statement is preserved when we revert back to a previous step (since it holds for the previous step). This statement is also preserved when we stay in the same step and increase F and z: We have one new value of w to consider, the old value of z. But we add X_0 to F to ensure that $\langle x, w\rangle \in \Phi(F)$. Similarly, this statement is preserved when we move to a new step; this is guaranteed by the choice of y^{n+1}, z^{n+1}, and the inclusion of H^n in F^{n+1}. □

Lemma 7.6.13. If α is an $\mathcal{R}_{\Phi,\Psi}$-strategy along the true path, then α ensures that $\mathcal{R}_{\Phi,\Psi}$ is satisfied.

Proof. Let s be large enough so that α is never initialized after stage s and let s be least with that property. After stage s, at the first time at which α is visited, if it fails to find sets F and G in Step -1, then it places a finite restraint $a_0 \in A$ and it restrains out of A elements that were already restrained out of A by nodes below α that are not below the outcome stop of α, which are never injured. Note that strategies below the outcome stop of α are in their initial state and so any node $\leq \alpha$ or $<_L \alpha$ that is currently restraining an element out of A will permanently do so, since the only way to injure this node is to visit some node to its left, which would injure α again as well. Thus the true set A is a subset of the set A that α looks at when it sees that $a_0 \notin \Psi(\Phi(A))$. Thus $a_0 \in A \smallsetminus \Psi(\Phi(A))$, and the requirement is satisfied. If it finds sets F and G with $a_0 \notin F$, then again, it places a finite restraint $a_0 \notin A$, $F \subseteq A$, and the requirement is permanently satisfied since $a_0 \in \Psi(\Phi(A)) \smallsetminus A$.

Now, suppose that the true outcome of α is wait_n for some n. By Lemma 7.6.3 we can also assume that s is large enough such that α is in Step n at stage s and after stage s, α is never reverted to any Step k with $k \leq n$. It follows that α has outcome wait_n at every stage after stage s. (Note that if α has any other outcome after stage s, it must either be initialized or be reverted to a Step $k \leq n$ in order to get to outcome wait_n again). Thus, in the algorithm, we always have $\langle x, z^n \rangle \in \Phi(\mathbb{N} \smallsetminus (W \cup B_\infty \cup Y))$ for every consistent set $Y \subseteq B_n$. (Note that we define B_∞ and B_n at the moment when z^n is first defined.) In particular, this is true for $Y = (\mathbb{N} \smallsetminus A) \cap B_n$ because Y is consistent by Lemma 7.6.11 (using Lemma 7.6.8 for the definition of A). Furthermore, the algorithm then replaces F^n by $F^n \cup X_0$, which ensures that $\langle x, z^n \rangle \in \Phi(X_0 \cup (B_n \smallsetminus Y)) \subseteq \Phi(A)$. Thus, for all $z \neq y^n$ (which never changes after stage s), we have that $\langle x, z \rangle \in \Phi(A)$. Next, by Lemma 7.6.7(2), we see that no element of H^n (which never changes after stage s) is ever restrained out of A by any strategy, thus $F^n \cup H^n \subseteq A$, so $\langle x, y^n \rangle \in \Phi(A)$, showing that the entire x-th column is contained in $\Phi(A)$, so $\mathcal{R}_{\Phi,\Psi}$ is satisfied.

Next, suppose that the true outcome of α is ∞. Fix any $k \in \omega$; we argue that $\{\langle x, z \rangle \mid z \in [0, k]\} \subseteq \Phi(A)^{[x]}$. Assume s is large enough so that α has been visited by stage s after its final initialization. Let $t > s$ be a stage at which $\alpha^\frown\infty$ is visited and which is large enough so that α is never reverted to Step n for $n \leq k + s + 1$ after stage t. Then, since every time the step is increased, y is increased as well, it follows that at every stage after t, if α is in Step n, then $z^n > y^n \geq n - s$. By Lemma 7.6.12, and since at every future stage, α restrains F^n_α into A, we see that $[0, k] \subseteq \Phi(A)^{[x]}$. Since this holds for every k, we have that every element of the x-th column is contained in $\Phi(A)$, thus $\mathcal{R}_{\Phi,\Psi}$ is satisfied.

Finally, suppose that the true outcome of α is stop and it is achieved via Step n. Let $t \geq s$ be a stage at which α takes outcome stop and no higher-priority strategy is ever initialized after stage t. Then α places a restraint keeping $H^n \cup D^n \cup B_\infty$ out of A which is never lifted. Since no higher-priority strategy is initialized, every element of W is permanently restrained out of A and W has remained the same since α's last initialization. Thus $A \subseteq \mathbb{N} \smallsetminus (W \cup B_\infty)$. This outcome means that $\langle x, z^n \rangle \notin \Phi(\mathbb{N} \smallsetminus (W \cup B_\infty))$, thus $\langle x, z^n \rangle \notin \Phi(A)$. Furthermore, since $H^n \cup D^n$ is restrained out of A, we have $\langle x, y^n \rangle \notin \Phi(A)$. Thus, $\mathcal{R}_{\Phi,\Psi}$ is satisfied. □

This completes the proof of the theorem. □

7.7 Open questions

In this section, we collect the open questions arising from this chapter, some of which have already been asked.

7.7.1 Definability

As mentioned above, Kalimullin [Kal03] showed that the enumeration jump is first-order definable. Is this also true for the skip?

Question 7.7.1. Is the skip first-order definable in the enumeration degrees?

Furthermore, we have discussed several cototality notions here. Which of these are definable?

Question 7.7.2. Which cototality notions are first-order definable in the enumeration degrees?

Note that a positive answer to the first question would imply, by Proposition 7.2.1, that the cototal degrees are definable.

7.7.2 Arithmetical zigzag

In Section 7.4.2, we have shown that the skip can exhibit a form of *zigzag behavior*: There are degrees $\mathbf{a}$ such that none of the finite skips of $\mathbf{a}$ are total. However, the examples constructed there are not arithmetical. We suspect that this is not a coincidence.

Conjecture 2. If $\mathbf{a}$ is an arithmetical enumeration degree, then $\mathbf{a}^{\langle n \rangle}$ is total for some $n \in \omega$.

7.7.3 Graph-cototal degrees

Theorem 7.6.1 constructed a cototal Δ^0_3-degree that is not graph-cototal. On the other hand, Proposition 7.3.1 proves that every Σ^0_2-degree is graph-cototal. This leaves the following open:

Question 7.7.3. Is every Π^0_2 cototal enumeration degree graph-cototal?

We do not know of a simpler proof of the existence of a cototal enumeration degree that is not graph-cototal. A more informative separation result would be derived from a positive answer to the following question:

Question 7.7.4. Is there a continuous enumeration degree that is not graph-cototal?

7.7.4 Skip cototality

Let us say that a degree $\mathbf{a}$ is *skip cototal* if $\mathbf{a}^\Diamond$ is total. Notice that every skip cototal degree $\mathbf{a}$ is weakly cototal, and that every cototal degree is skip cototal. Furthermore, note that in the proofs of Proposition 7.5.4 and Corollary 7.5.6, we in fact constructed a degree $\mathbf{a}$ that is skip cototal but not cototal. Even the alternative example of a weakly cototal degree given by Badillo and Harris [BH12]—the degree that is entirely composed of properly Π^0_2-sets—is also a skip cototal degree.

Conjecture 3. Every weakly cototal degree $\mathbf{a}$ is skip cototal.

As mentioned above, every Π^0_2-degree is weakly cototal. Therefore, a proof of our conjecture would in particular imply that the skip of every Π^0_2-degree is total, which is also open.

Chapter 8

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