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The geometry of quaternionic-contact manifolds and the Yamabe problem

D I S E R T A T I O N T H E S I S

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Introduction

In the thesis we consider a variety of problems related to the theory of quaternionic-contact (QC) manifolds. The QC geometry was first introduced by O. Biquard [Biq] to describe a type of geometric structure that appears naturally at the boundary at infinity of the quaternionic hyperbolic space. In general, a QC structure on a real (4n+3)-dimensional manifold M is a codimension three distribution H (to be called contact or horizontal distribution) which is locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$, with values in $\mathbb{R}^3$, such that the three 2-forms $d\eta_i|_H$ (the exterior derivatives of η_i, restricted to the contact distribution) are the fundamental 2-forms of some quaternionic structure on H (see also Definition 2.1 below). It is a fundamental theorem of Biquard [Biq] that a QC structure on a real analytic $(4n+3)$-dimensional manifold M is always the conformal infinity of a unique quaternionic-Kähler metric defined in a "small" $(4n+4)$-dimensional neighborhood of M. This theorem generalizes an earlier result of LeBrun [LeB82] stating that a real analytic conformal 3-manifold is always the conformal infinity of a self-dual Einstein metric. From this point of view, the QC geometry is a natural generalization of the classical concept of a conformal 3-dimensional Riemannian geometry to the higher dimensions of the type $4n + 3$.

Furthermore, the QC geometry provides a natural setting for certain Yamabe-type problem concerning the extremals and the best constant of a special L^2 Sobolev-type embedding on the quaternionic Heisenberg group known as the Folland-Stein embedding theorem [FS74]. Obtaining a solution to this problem on the Heisenberg group is one of our main goals in the theses. To explain this in some more details, let us consider a 1-form η, with values in $\mathbb{R}^3$, that defines a QC structure H. This form is not uniquely determined by the contact distribution H but it is rather determined only up to a conformal factor and the action of the group $SO(3)$ on $\mathbb{R}^3$. Therefore H is equipped with a conformal class $[g]$ of metrics and a 3-dimensional quaternionic bundle $Q \subset End(H)$ over M. The associated 2-sphere bundle $S^2(Q) \to M$ is called the twistor space of the QC-structure. The transformations preserving the QC structure, i.e., the transformations of the type $\bar{\eta} = \mu\Psi \cdot \eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries, are called *quaternionic-contact conformal (QC conformal) transformations*. If the function μ is constant, we have *quaternionic-contact homothetic (QC homothetic) transformations*. To each metric in the fixed conformal class $[g]$ on H, one can associate a linear connection preserving the QC structure [Biq] which we call Biquard connection. This connection is invariant under QC homothetic transformations, but changes in a non-trivial way under QC conformal transformations. The scalar curvature, $Scal$, with respect to the Biquard connection is one of the most important differential invariants in the QC geometry with a fixed metric tensor $g \in [g]$. In this setting, the quaternionic-contact Yamabe problem is the problem of finding all metrics $g \in [g]$ for which the associated scalar curvature $Scal$ is constant.

Already by the very appearance of the new concept of QC geometry, in the year 2000, it was clear that there exist infinitely many examples of such manifolds. The argument for this came from a paper of LeBrun [LeB91] who managed to proof, by using the deformation theory of complex manifolds, the existence of an infinite dimensional family of special complete quaternionic-Kähler metrics on the unite Ball B^{4n+4}. LeBrun observed that, if multiplied by a function that vanishes along the boundary sphere S^{4n+3} to order two, each of his special metrics on the ball extends smoothly across the boundary sphere S^{4n+3} where its rank drops to four. It was discovered later by Biquard [Biq] that the arising structure on S^{4n+3} is essentially a QC structure and therefore, on the sphere, we have infinitely many (globally defined) such structures. Clearly, the whole construction is very non-explicit and the argument of LeBrun does not help much for the construction of any explicit examples of QC structures. In fact, the number of the explicitly known examples of QC manifolds remains so far very restricted. There is essentially only one generic method for obtaining such structures explicitly. It is based on the existence of a certain very special type of Riemannian manifolds, the so called 3-Sasaki-like spaces. These are Riemannian manifolds that admit a special triple R_1, R_2, R_3 of Killing vector fields, subject to some additional requirements (see Chapter 3 and the references therein for more details), which carry a natural QC structure defined by the orthogonal complement of the triple R_1, R_2, R_3. So far, there are no explicit examples of QC structures (not even locally) for which it is proven that they can not be generated by the above construction. Investigating the relationship between the 3-Sasakian spaces and the QC geometry will be one of our main tasks here.

The first chapter of the thesis (with title "Preliminaries") is intended to be an introduction to the subject, where we explain our motivation for studying quaternionic-contact geometry and recall the main results known in this area. The rest of the thesis is built on original material most of which was published already in separate papers.

The core of the thesis is Chapter 2; it is based on results published in [IMV14]. Here we develop the basic concepts in the QC geometry and proof a number of important results upon which the rest of the thesis is build. In theorems A and B of this chapter, we obtain a partial solution to the QC Yamabe problem on the quaternionic Heisenberg group. Theorem C presents our first result relating the Riemannian geometry of 3-Sasakian manifolds to the geometry of QC Einstein spaces.

In Chapter 3, we proceed with the investigations, started in Chapter 2, concerning the geometry of QC Einstein spaces. In Theorem D we extend a result from Chapter 2 (Theorem 5.9) to the most difficult 7-dimensional case and show that the QC scalar curvature of any QC Einstein space is constant. Furthermore, in this chapter, we show that, depending on the value of the QC scalar curvature, the QC Einstein spaces are "essentially" bundles over quaternionic-Kähler or hyper-Kähler manifolds. The results presented here are published in [IMV16].

In Chapter 4, we use the techniques developed in Chapter 2 to obtain a complete solution to the QC Yamabe problem on the seven dimensional quaternionic Heisenberg group (theorems E and F). The results here are published in [IMV10].

In Chapter 5, we determine the best (optimal) constant in the L^2 Folland-Stein inequality (Theorem G) on the quaternionic Heisenberg group (in all dimensions) and the non-negative extremal functions, i.e., the functions for which equality holds. The argument presented here is purely analytical. In this respect, even though the QC Yamabe functional is involved, the QC scalar curvature is used in the proof without much geometric meaning. Rather, it is the conformal sub-laplacian that plays a central role and the QC scalar curvature appears as a constant determined by the Cayley transform and the left-invariant sub-laplacian on the quaternionic Heisenberg group. The method employed here does not give all solutions of the QC Yamabe equation on the quaternionic-contact sphere but only these that realize the infimum of the QC Yamabe functional. Therefore, if considering the seven dimensional case, the result presented here is weaker than Theorem E of Chapter 3. All results here are published in [IMV12].

Before proceeding further, I would like to say that I am very grateful and fortunate to have Prof. Stefan Ivanov as my mentor and former PhD advisor. I would also like to thank Prof. Dimiter Vassilev, Prof. Jan Slovák, Prof. Johan Davidov and Assoc. Prof. Simeon Zamkovoy for our productive collaborations in the years.

CHAPTER 1

Preliminaries

1. Quaternionic geometries and motivation

1.1. Quaternions. The algebra $\mathbb{H}$ of quaternions is by definition the vector space $\mathbb{R}^4$ endowed with a multiplication operation $(z, w) \mapsto zw$ which is associative, satisfies the left and right distributivity axioms, and for which the element

$$\mathbf{1} \overset{def}{=} (1, 0, 0, 0)$$

is the neutral element. Using the notation

$$\mathbf{i} = (0, 1, 0, 0), \qquad \mathbf{j} = (0, 0, 1, 0), \qquad \mathbf{k} = (0, 0, 0, 1),$$

the multiplication of the basis elements is defined by the following list of identities, called also the quaternionic identities:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}. \tag{1.1}$$

The algebra of quaternions is a division ring, that is, every nonzero element in $\mathbb{H}$ has an inverse. To see this, consider the conjugation $z \mapsto \bar{z}$ in $\mathbb{H}$: If

$$z = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \qquad a, b, c, d \in \mathbb{R},$$

then, by definition,

$$\bar{z} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}. \tag{1.2}$$

A simple computation shows that

$$\bar{z}z = a^2 + b^2 + c^2 + d^2 = |z|^2$$

and that $\overline{zw} = \overline{w}\,\overline{z}$. Therefore, the inverse element z^{-1} of a non-zero $z \in H$ is explicitly given by

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

1.2. Quaternionic structure on a vector space. Consider a real vector space V and a 3-dimensional subspace Q of the algebra of all real endomorphisms of V, $Q \subset End(V)$. We say that Q is a *quaternionic structure* on V, if there exists a basis I_1, I_2, I_3 of Q that satisfies the quaternionic identities

$$I_1^2 = I_2^2 = I_3^2 = -id, \qquad I_1 I_2 = -I_2 I_1 = I_3. \tag{1.3}$$

Note that if Q is a quaternionic structure on V then on Q we have canonically induced scalar product $\langle .,. \rangle$ and orientation. Indeed, the elements $I \in Q$ of unit length are precisely those that satisfy the equation $I^2 = -id$, and two elements $I, J \in Q$ are orthogonal with respect to $\langle .,. \rangle$ if and only if $IJ = -JI$.

Furthermore, it follows that a triple J_1, J_2, J_3 of elements of Q satisfies the quaternionic identities

$$J_1^2 = J_2^2 = J_3^2 = -id, \qquad J_1 J_2 = -J_2 J_1 = J_3,$$

if and only if the 3×3 matrix, obtained by expressing J_1, J_2, J_3 with respect to the initial basis I_1, I_2, I_3, is an element of the group $SO(3)$.

1.3. The group Sp(1). Observe that, since $\mathbb{H}$ is a non-commutative algebra, we must distinguish between left and right modules (vector spaces) over $\mathbb{H}$. Each finite dimensional left or right $\mathbb{H}$-vector space is, of course, isomorphic to one of the coordinate spaces $\mathbb{H}^n$ with its natural left or right multiplication operation.

We shell fix (once and for all) a real vector space isomorphism $\mathbb{H}^n \cong \mathbb{R}^{4n}$. Then, the right $\mathbb{H}$-multiplication on $\mathbb{H}^n$ defines a natural quaternionic structure $Q = span\{I_1, I_2, I_3\}$ on $\mathbb{R}^{4n}$ with $I_1(z) = z\bar{i},\ I_2(z) = z\bar{j},\ I_3(z) = z\bar{k}$ for all $z \in \mathbb{H}^n$. Obviously, Q is then a 3-dimensional Lie subalgebra of the Lie algebra $End(\mathbb{R}^{4n})$, isomorphic to the classical Lie algebra $so(3) = sp(1)$.

The unique connected Lie subgroup $G \subset GL(4n, \mathbb{R})$ with Lie algebra Q is explicitly given by

$$G = \{a_0 id + a_1 I_1 + a_2 I_2 + a_3 I_3 \mid a_s \in \mathbb{R},\ a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1\}$$

Since, obviously, G is a simply-connected Lie group, it is necessarily isomorphic to the classical group $Sp(1)$. In the sequel, we shell often identify these two groups without any further comment.

Typically one introduces a quaternionic valued, positive definite Hermitian scalar product on $\mathbb{H}^n$ by the formula $\langle \bar{z}, w\rangle_{\mathbb{H}} = \sum_s \bar{z}_s w_s,\ z, w \in \mathbb{H}^n$. The real part of the latter is just the standard real scalar product in $\mathbb{R}^{4n}$, i.e. $\langle z, w\rangle_{\mathbb{R}} = Re(\langle \bar{z}, w\rangle_{\mathbb{H}})$. Since $Sp(1)$ clearly preserves the Hermitian form $\langle \bar{z}, w\rangle_{\mathbb{H}}$, we have $Sp(1) \subset SO(4n)$.

1.4. The groups GL(n,$\mathbb{H}$) and Sp(n). By definition, $GL(n, \mathbb{H})$ is the group of the non-degenerate quaternionic $n \times n$ matrices. Clearly, $GL(n, \mathbb{H})$ acts from the left on $\mathbb{H}^n$ and, by this action, the elements of $GL(n, \mathbb{H})$ represent the endomorphisms of $\mathbb{H}^n$ that commute with the right multiplication with $\mathbb{H}$. By the real isomorphism $\mathbb{H}^n \cong \mathbb{R}^{4n}$ which we assume fixed (cf. section 1.3), we can identify the group $GL(n, \mathbb{H})$ with a certain subgroup of $GL(4n, \mathbb{R})$ by

$$GL(n, \mathbb{H}) = \{A \in GL(4n, \mathbb{R}) \mid AJ = JA, J \in Q\}.$$

Take P to be the stabilizer in $GL(n, \mathbb{H})$ of the Hermitian form $\langle \bar{z}, w\rangle_{\mathbb{H}}$, i.e.

$$P = \{A \in GL(n, \mathbb{H}) \mid \bar{A}^t A = E\} = \{B \in SO(4n) \mid BJ = JB,\ J \in Q\}.$$

Then, clearly, the Lie group P is isomorphic to the classical group $Sp(n)$ and in the sequel we shell identify these two. Let us remark explicitly here that the notation which we have just introduced brings a certain ambiguity in the case $n = 1$. In this case, the just defined group P and the previously defined group G (in section 1.3) are both identified with the same classical group $Sp(1)$ but, in fact, P and G are two different subgroups of $SO(4)$. Despite the ambiguity, this is a convenient notation that has been adopted by many authors in the area and therefore, we shell use it as well. Usually, it is quite clear from the context which of the two different copies of $Sp(1)$ in $SO(4)$ we are working with at any particular moment.

Following the above notation, the product groups $Sp(n)Sp(1) \cong Sp(n) \times Sp(1)/\mathbb{Z}_2$ and $GL(n, \mathbb{H})Sp(1)$ can be described as a subgroups of $GL(4n, \mathbb{R})$ as follows:

$$Sp(n)Sp(1) = \{A \in SO(4n) \mid A^{-1}JA \in Q,\ J \in Q\}$$

$$GL(n, \mathbb{H})Sp(1) = \{A \in GL(4n, \mathbb{R}) \mid A^{-1}JA \in Q,\ J \in Q\}.$$

1.5. Riemannian holonomy. Let (M, g) be a connected Riemannian m-manifold, and let $p \in M$ be a chosen base point. The holonomy group of (M, g) at p is the subgroup of $End(T_pM)$ consisting of those transformations that are induced by parallel transport around piecewise-smooth loops based at p. The restricted holonomy group is similarly defined, using only loops representing $1 \in \pi_1(M, p)$. The latter is automatically a connected Lie group and may be identified with a Lie subgroup of $SO(m)$ by choosing an orthogonal frame for T_pM. Changing the base point only changes the subgroup by conjugation.

Excluding Riemannian products and symmetric spaces, very few subgroups of $SO(m)$ can be restricted holonomy groups, as was first pointed by Berger [**Ber**]. In fact, the full list is as follows: $SO(m)$, $U(m/2)$, $SU(m/2)$, $Sp(m/4)Sp(1)$ (in dimension $m \geq 8$); G_2 (in dimension 7); and $Spin(7)$ (in dimension 8).

1.6. Quaternionic-Kähler manifolds. A Riemannian manifold (M, g) of dimension $m = 4n \geq 8$ is called Quaternionic-Kähler (QK) if its group of holonomy is contained in $Sp(n)Sp(1)$. Equivalently, (M, g) is a QK manifold if there exists a pointwise quaternionic structure Q (cf. section 1.2) on the tangent bundle of M such that $\nabla Q \subset Q$ holds everywhere on M, with ∇ being the Levi-Civita connection of g. If for a given QK manifold, $F \to M$ denotes the principle $Sp(n)Sp(1)$-bundle generated by parallel transport of an arbitrary orthonormal frame, then Q may be described as the vector bundle associated to F, corresponding to the adjoint representation of the group $Sp(n)Sp(1)$ on $sp(1)$.

The above definition of a QK manifold explicitly excludes the 4-dimensional case. Indeed, since $SO(4) = Sp(1)Sp(1)$, nothing interesting can generally be said about 4-manifolds with this holonomy group. The proper definition here is: A Riemannian 4-manifold is said to be Quaternionic-Kähler if it is Einstein and half-conformally flat. Recall that a Riemannian 4-manifold is called half-conformally flat if there exists an orientation with respect to which the conformal curvature satisfies $*W = W$ (or alternatively $*W = -W$), where $*$ is the Hodge star. Historically, the four dimensional case was considered first (cf. [**Pen, AHS**]) and the results achieved there gave the motivation for introducing the concept of the higher dimensional QK manifolds.

1.7. Twistor construction. Let (M, g) be a QK manifold of dimension $4n \geq 4$ and take $F \to M$ to be the corresponding principle bundle over M with structure group $Sp(n)Sp(1)$. Consiser the stabilizer $Sp(n)U(1) \subset Sp(n)Sp(1)$ of the fixed endomorphism I_1 of $\mathbb{R}^{4n}$ (as defined in 1.3) and let Z be the quotient bundle $F/(Sp(n)U(1))$. Then, $\pi : Z \to M$ is a 2-sphere bundle over M and each element I of Z corresponds to an orthogonal complex structure (to be denoted identically)

$$I : T_pM \to T_pM, \quad I^2 = -1, \quad g(IX, IY) = g(X, Y),$$

where $p = \pi(I)$ and $X, Y \in T_pM$. Each fiber $Z_p = \pi^{-1}(p)$ is topologically a 2-sphere that can be described explicitly as

$$Z_p = \{I \in Q_p \mid I^2 = -id\}.$$

Using the Levi-Civita connection ∇ of the Riemannian metric g we can split the tangent bundle of Z into horizontal and vertical parts, $TZ = D \oplus V$; the vertical part V is the kernel of the differential π_* of the projection map $\pi : Z \to M$. Since $\pi_* : D_I \to T_pM$ is an isomorphism of real vector spaces, we can lift each element $I \in Z_p$ to be an endomorphism $J'_I : D_I \to D_I$, $(J'_I)^2 = -1$, so that $D \subset TZ$ becomes a complex vector bundle, where the multiplication with $\sqrt{-1}$ is given by J'. On the other hand the fibers of π are oriented metric 2-spheres and so may be considered as Riemann surfaces. Therefore, the vertical tangent space $V = \ker \pi_*$ carries also an endomorphism $J'' : V \to V$ with $(J'')^2 = -1$. This allows us to define an almost complex structure J on the whole of $TZ = D \oplus V$ by setting $J = J' \oplus J''$. It was discovered independently by Salamon [**Sal1**] and Bérard-Bergery [**Brd**] that the almost complex structure J is always integrable, i.e, that its Nijenhuis tensor vanishes and thus, by the Newlander-Nirenberg theorem, Z is in fact a complex manifold.

Furthermore, $D \subset TZ$ is a holomorphic subbundle and the projection $TZ \to TZ/D$ gives a holomorphic line-bundle-valued 1-form $\Theta \in \Gamma(Z, \Omega^1(\mathcal{L}))$, $\mathcal{L} := TZ/D$, which satisfies $\Theta \wedge (d\Theta)^n \neq 0$, i.e., Θ is a holomorphic contact structure on Z. The map $\sigma : Z \to Z$, given by $I \mapsto -I$ and corresponding to the antipodal map on each 2-sphere $\pi^{-1}(p)$, is an antiholomorphic involution ($\sigma^2 = 1$) without fixed points. By definition, the twistor space of the QK manifold (M, g) is given by the triple (Z, Θ, σ).

1.8. Inverse twistor construction. It is essential that the above twistor construction is actually invertible [**LeB89, PP, BE**]. Indeed, let (Z, Θ, σ) be a triple, where Z is a $(2n+1)$-dimensional complex manifold, Θ is a holomorphic contact 1-form on Z that takes values in some holomorphic line bundle $\mathcal{L}$ over Z, and $\sigma : Z \to Z$ is a fixed-point-free antiholomorphic involution compatible with the contact structure Θ. Following [**LeB91**], we define M^c to be the set of all genus zero compact complex curves $C \subset Z$ which have normal bundle isomorphic to the bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$. Here $\mathcal{O}(k) \to \mathbb{CP}_1,\ k \in \mathbb{Z}$, is the line bundle of Chern class k. Since the group $H^1(\mathbb{CP}_1, \mathcal{O}(1) \otimes \mathbb{C}^{2n})$ vanishes, it follows, by a theorem of Kodaira [**Kod**], that the set M^c (if not empty) must be a $4n$-dimensional complex manifold with tangent space at any point $C \in M^c$ given by $H^0(\mathbb{CP}_1, \mathcal{O}(1) \otimes \mathbb{C}^{2n}) \cong \mathbb{C}^{4n}$. The subset $M \subset M^c$ of all $C \in M^c$ that are σ-invariant is a real analytic manifold which sits in M^c as a real slice. By Proposition 1 in [**LeB91**], the subset $S^c \subset M^c$ consisting of those $C \in M^c$ that are tangent to the contact distribution $D = \ker(\Theta)$ is (if not empty) a non-singular, complex hypersurface in M^c. Moreover, the set S of all

σ-invariant $C \in S^c$ is a real slice of S^c, and it is a smooth close hypersurface in M. The real manifold $M - S$ carries a natural pseudo Riemannian metric of holonomy $Sp(n-l)Sp(1)$, $0 \leq l \leq n$, with non-vanishing scalar curvature (cf. Theorem 1.3 in [**LeB89**]) and twistor space given by the triple (Z, Θ, σ).

This inverse construction is also unique in the following sense: If (Z, Θ, σ) is the twistor space of some QK-manifold N, then N is naturally isometric to an open subset of $M - S$. Furthermore, the germ of the geometry at any point $p \in M - S$ determines the germ of Z along the corresponding curve C_p up to a biholomorphism.

1.9. The quaternionic-contact structure on the boundary surface S. It was observed by Biquard [**Biq**] that the hypersurface $S \subset M$ (we are using the notation form the previous section 1.8), which is a $(4n-1)$-dimensional real analytic submanifold, carries a natural geometrical structure which Biquard introduced as quaternionic-contact (QC) geometry. This structure can be described in the following way: Take any point $p \in M^c$ and denote by C_p the corresponding compact complex curve in Z. By assumption $C_p \cong \mathbb{CP}_1$ (biholomorphic equivalence) and the normal bundle $N_p := TZ/TC_p$ over C_p is isomorphic to $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$. The $\mathcal{L}$-valued contact form Θ determines an isomorphism

$$\Theta \wedge d\Theta^n : \Lambda^{2n+1}(TZ) \to \mathcal{L}^{n+1}. \tag{1.4}$$

If restricting only to the curve C_p, we have

$$TZ|_{C_p} \cong TC_p \oplus N_p$$

and therefore, via the isomorphism (1.4),

$$\Lambda^{2n+1}(TZ)|_{C_p} \cong TC_p \otimes \Lambda^{2n} N_p.$$

Since $TC_p \cong \mathcal{O}(2)$ and $\Lambda^{2n} N_p \cong \Lambda^{2n}\mathcal{O}(1) \cong \mathcal{O}(2n)$, we obtain

$$(\mathcal{L}|_{C_p})^{n+1} \cong \mathcal{O}(2n+2),$$

and thus $\mathcal{L}|_{C_p} \cong \mathcal{O}(2)$. Consequently, the bundle

$$(T^*C_p) \otimes (\mathcal{L}|_{C_p}) \cong \mathcal{O}(-2) \otimes \mathcal{O}(2)$$

is trivial over each separate curve C_p. But can one find a uniform trivialization that works for all the curves C_p, $p \in M^c$ simultaneously? To answer this, denote by $\mathbb{L}_p$ the

1-dimensional space of holomorphic sections of the bundle $(T^*C_p) \otimes (\mathcal{L}|_{C_p}) \to C_p$. If $\mathbb{L} = \cup_{p \in M^c} \mathbb{L}_p$, then $\mathbb{L}$ is a holomorphic line bundle over M^c and the restriction of the contact form Θ to the tangent spaces TC_p determines a holomorphic section θ of $\mathbb{L}$, i.e., there is indeed a uniform trivialization for the bundles $(T^*C_p) \otimes (\mathcal{L}|_{C_p}) \to C_p$ only if we consider domains in M^c where θ is non-vanishing. Clearly, the hypersurface $S^c \subset M^c$ is precisely the zero locus of θ which is known to be non-degenerate by Proposition 1 in [LeB91].

In what follows, we shell often need to assume that there exists a square root $\mathcal{L}^{\frac{1}{2}}$ of the bundle $\mathcal{L}$. Since $\mathcal{L}|_{C_p} \cong \mathcal{O}(2) \cong \mathcal{O}(1)^2$, this assumption is certainly true if restricting to sufficiently small open subsets of M^c, although, even then, there will exist different possible choices for the square root $\mathcal{L}^{\frac{1}{2}}$. The final conclusions from the considerations below, however, will not depend on the local choices for $\mathcal{L}^{\frac{1}{2}}$ that we could make, and thus will remain true also without the assumption about the global existence of $\mathcal{L}^{\frac{1}{2}}$.

Consider the 2-dimensional holomorphic vector bundle $\mathcal{H}$ over M^c with fibers $\mathcal{H}_p = H^0(C_p, \mathcal{L}^{\frac{1}{2}})$ (note that $H^0(C_p, \mathcal{L}^{\frac{1}{2}}) \cong H^0(\mathbb{CP}_1, \mathcal{O}(1)) \cong \mathbb{C}^2$ for any fixed $p \in M^c$). The Wronskian

$$W : \Lambda^2(\mathcal{H}_p) \to H^0(C_p, T^*C_p \otimes \mathcal{L}) \cong \mathbb{L}_p$$

$$u \wedge v \mapsto u \otimes dv - v \otimes du$$

defines a non-degenerate $\mathbb{L}$-valued 2-form on the bundle $\mathcal{H} \to M^c$, i.e., it defines an $Sp(1,\mathbb{C})$-structure on $\mathcal{H}$, and thus also an $SO(3,\mathbb{C})$-structure on the bundle $Sym^2(\mathcal{H}) \to M^c$ of symmetric 2-tensors on $\mathcal{H}$. Since

$$Sym^2(\mathcal{H}_p) = Sym^2\Big(H^0(C_p, \mathcal{L}^{\frac{1}{2}})\Big) = H^0(C_p, \mathcal{L}),$$

the bundle $Sym^2(\mathcal{H})$ does not depend on the choice of the square root $\mathcal{L}^{\frac{1}{2}}$ and therefore, it is well defined globally over M^c.

Now let $p \in S^c = \theta^{-1}(0)$. The tangent space T_pS^c is given by $\ker(d\theta) \subset T_pM^c$. Since $\Theta(TC_p) = 0$ the contact form Θ induces a linear map $\eta : H^0(C_p, N_p) \to H^0(C_p, \mathcal{L})$, i.e., a linear map $\eta : T_pM^c \to Sym^2(\mathcal{H}_p)$. If restricting to $\ker(d\theta)$ we obtain a rank three linear map $\eta : T_pS^c \to Sym^2(\mathcal{H}_p)$. Let $H^c_p \subset T_pS^c$ be the kernel of this map. Then H^c is a holomorphic codimension three distribution on the hypersurface S^c. If we take a local $SO(3,\mathbb{C})$ equivariant trivialization $Sym^2(\mathcal{H}) \cong \mathbb{C}^3$ then we obtain a triple η_1, η_2, η_3 of local one-forms on S^c by the composition

$$T_pS^c \to Sym^2(\mathcal{H}_p) \to \mathbb{C}^3.$$

It follows from the considerations in [Biq], III.2.C that there exist a non-degenerate holomorphic symmetric 2-tensor g on H^c and a triple of holomorphic endomorphism I_1, I_2, I_3 of H^c, satisfying the quaternionic identities such that $d\eta_s(X,Y) = 2g(I_sX,Y),\ X,Y \in H^c,\ s=1,2,3$. Thus, if we go to the real slice $S \subset S^c$ (through the antiholomorphic involution σ), we obtain that (S,H) is an analytic quaternionic-contact manifold in the sense of definition 2.1.

1.10. Explicit examples of the twistor correspondence. Now, we are going to describe explicitly the twistor spaces of two typical (and well known) examples of quaternionic-Kähler manifolds, namely the quaternionic projective space $\mathbb{HP}^n = Sp(n+1)/Sp(n)Sp(1)$ and the quaternionic hyperbolic space $\mathbb{HH}^n = Sp(n,1)/Sp(n)Sp(1)$.

Let us denote by F_+ and F_- the two standard quadratic forms of signature $(4n+4,0)$ and $(4n,4)$ on the vector space $\mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$ defined by

$$F_+(q,q) = \sum_{s=1}^{n+1} |q_s|^2,$$

$$F_-(q,q) = \sum_{s=1}^{n} |q_s|^2 - |q_{n+1}|^2, \qquad q = (q_1,\dots,q_{n+1}) \in \mathbb{H}^{n+1}.$$

We define the quaternionic projective space by $\mathbb{HP}^n = (\mathbb{H}^{n+1} - \{0\})/\sim$, where

$$(q_1,\dots,q_{n+1}) \sim (q_1\lambda,\dots,q_{n+1}\lambda), \quad \text{if } \lambda \in \mathbb{H} - \{0\}.$$

The group $Sp(n+1) := GL(n,\mathbb{H}) \cap O(F_+)$ acts transitively on $\mathbb{HP}^n$ with isotropy subgroup $Sp(n)Sp(1)$. Therefore we may identify $\mathbb{HP}^n$ with the symmetric space $Sp(n+1)/Sp(n)Sp(1)$. On the other hand the group $Sp(n,1) := GL(n,\mathbb{H}) \cap O(F_-)$ acts transitively on the subset

$$\mathbb{HP}^n_- = \{[q] \in \mathbb{HP}^n \mid F_-(q,q) < 0\}$$

and the isotropy subgroup is again $Sp(n)Sp(1)$. Hence the symmetric space $\mathbb{HH}^n = Sp(n,1)/Sp(n)Sp(1)$ can be naturally identified with the open ball $\mathbb{HP}^n_- \subset \mathbb{HP}^n$. The quadratic forms F_+ and F_- induce positive definite Riemannian metrics on $\mathbb{HP}^n$ and $\mathbb{HH}^n$ respectively (which we shell denote again by F_+ and F_-). These metrics are uniquely determined (up to a constant multiple) by the requirement that they are invariant with respect to the groups $Sp(n+1)$ or $Sp(n,1)$ respectively. Both metrics are quaternionic-Kähler and therefore they are Einstein. It is well known that the scalar curvature of $\mathbb{HP}^n$ is positive and the scalar curvature of $\mathbb{HH}^n$ is negative.

The naturality of the twistor correspondence allows us to lift the action of each of the isometry groups $Sp(n+1)$ and $Sp(n,1)$ to a holomorhic action on the corresponding twistor spaces. Therefore, if we denote by Z_+ and Z_- the twistor spaces of $\mathbb{HP}^n$ and $\mathbb{HH}^n$ respectively, it follows that

$$Z_+ \cong Sp(n+1)/Sp(n)U(1), \qquad Z_- \cong Sp(n,1)/Sp(n)U(1),$$

where both Z_+ and Z_- are equipped with invariant complex structures, antiholomorphic involutions and holomorphic contact distributions. Both twistor spaces have a simple geometric realization which we shall describe next. If identifying $\mathbb{H}^{n+1} \cong \mathbb{C}^{2n+2}$, we obtain an embedding of $Sp(n+1)$ and $Sp(n,1)$ into the complex group $GL(2n+2,\mathbb{C})$. The group $Sp(n+1)$ acts transitively on $\mathbb{CP}^{2n+1}$ by biholomorphic maps and the isotropy group of this action is $Sp(n)U(1)$. Hence, as a real manifold, the twistor space Z_+ can be identified with $\mathbb{CP}^{2n+1}$ where the twistor projection $\mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ is just the Hopf map, i.e., the map that sends each complex line in $\mathbb{C}^{2n+2}$ to its own quaternionic span. Moreover, if we fix $o = (0,0,\dots,1,0) \in \mathbb{C}^{2n+2}$ then the tangent space of $\mathbb{CP}^{2n+1}$ at $[o]$ can be naturally identified with the quotient space $\mathbb{C}^{2n+2}/(\mathbb{C}o)$, which is just $\mathbb{H}^n \oplus \mathbb{C}$. It is easily seen that the action of the group $Sp(n)U(1)$—which is the stabilizer of the complex line $\mathbb{C}o$ in $Sp(n+1)$—on $\mathbb{H} \oplus \mathbb{C}$ commutes with exactly two different complex structures there, namely the standard one on $T_{[o]}\mathbb{CP}^{2n+1}$ and its complex conjugate. Since the complex structures of Z_+ and $\mathbb{CP}^{2n+1}$ are both $Sp(n+1)$ invariant, we conclude that Z_+ and $\mathbb{CP}^{2n+1}$ are isomorphic as complex manifolds as well. Further, by using the $Sp(n+1)$ invariance in a similar way, we obtain that the antiholomorphic involution σ_+ of the twistor space is given by the right multiplication $[v] \mapsto [vj]$. In order to describe the holomorphic contact distribution D_+ of the twistor space, let us interpret the tangent space of $\mathbb{CP}^{2n+1}$ at a given point $[v]$ as F_+-orthogonal complement of the complex span of v in $\mathbb{C}^{2n+2}$. Then the contact distribution D_+ ist just the F_+-orthogonal complement of the quaternionic span of v in $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$.

A similar description is available also for the twistor space (Z_-, D_-, σ_-) of the quaternionic hyperbolic space $\mathbb{HH}^n$. We have that the group $Sp(n,1)$ acts transitively on the open subset

$$\mathbb{CP}^{2n+1}_- = \Big\{[v] \in \mathbb{CP}^{2n+1} \mid F_-(v,v) < 0\Big\}$$

with isotropy group $Sp(n)Sp(1)$. Hence we can realize Z_- as an open set $\mathbb{CP}^{2n+1}_-$ in Z_+. The isotropy representation of the group $Sp(n)Sp(1)$ on the tangent space $T_{[o]}\mathbb{CP}^{2n+1}_-$ at a fixed point $[o]$ coincides with the representation discussed before which imply that the complex structures, the antiholomorphic involutions and the contact distributions of both twistor spaces (Z_+, D_+, σ_+) and (Z_-, D_-, σ_-) coincide at the origin $[o]$. The complex structure of $\mathbb{CP}^{2n+1}$ and the antiholomorphic involution σ_+ defined above are $GL(n+1,\mathbb{H})$-invariant and we have $Sp(n+1), Sp(n,1) \subset GL(n+1,\mathbb{H})$. Therefore, the complex structure of $Z_- = \mathbb{CP}^{2n+1}_-$ and the antiholomorphic involution σ_- are just the restrictions of those from (Z_+, σ_+). It is the contact distribution that makes the twistor spaces of $\mathbb{HP}^n$ and $\mathbb{HH}^n$ different. If identifying the tangent space of $\mathbb{CP}^{2n+1}_-$ at any given point $[v]$ with the F_--orthogonal complement of the complex span of v in $\mathbb{C}^{2n+2}$ (this is possible because by assumption $F_-(v,v) \neq 0$) then the contact distribution D_- is the orthogonal space of the quaternionic span of v with respect to the quadratic form F_-.

Note that the distribution D_- extends to a holomorphic distribution on the whole complex manifold $\mathbb{CP}^{2n+1}$ in an obvious way. Hence we may apply the inverse twistor construction to the triple $(\mathbb{CP}^{2n+1}, D_-, \sigma = \sigma_+)$. Each 2-dimensional complex subspace of $\mathbb{C}^{2n+2}$ gives a projective line which is a genus zero compact complex curve C in $\mathbb{CP}^{2n+1}$ with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$. The set M^c of all projective lines is a $4n$-dimensional complex manifold. Clearly, a projective line C is σ-invariant if and only if the corresponding 2-dimensional subspace of $\mathbb{C}^{2n+2} = \mathbb{H}^{n+1}$ is invariant under the right $\mathbb{H}$-multiplication. The set of all σ-invariant projective lines gives the real slice M of the complex manifold M^c which is diffeomorphic to $\mathbb{HP}^n$. Let $S \subset M$ be the hypersurface consisting of all real projective lines with tangent space contained in D_-. We have

$$S = \{v\mathbb{H} \mid v \in \mathbb{H}^{n+1} - \{0\},\ F_-(v,v) = 0\}.$$

Clearly, the complement $M - S$ has two connected components. The QK-metric on $M - S$ given by the inverse twistor construction is just the metric induced by the form F_-. Hence, one of the components of $M - S$ is exactly the quaternionic hyperbolic space $\mathbb{HH}^n$. On the other connected component we obtain metric with signature $(4n-4, 4)$ and holonomy contained in $Sp(4n-4,4)Sp(1)$.

The boundary surface S is diffeomorphic to the sphere S^{4n-1} and, according to section 1.9, there is a natural distribution H^{can} on S, such that (S, H^{can}) is a quaternionic-contact manifold as defined in definition 2.1. The distribution H^{can} could be described in the following way: For a given point $[v] \in S$ with fixed $v \in \mathbb{H}^{n+1} - \{0\},\ F_-(v,v) = 0$, we may identify the tangent space of S at $[v]$ with the quotient space $P_v/v\mathbb{H}$, where $P_v = \{w \in \mathbb{R}^{4n+4} \mid F_-(v,w) = 0\}$ is a real vector space of dimension $4n+3$. Then the distribution $H^{can}_{[v]} \subset P_v/v\mathbb{H}$ is given by

$$H^{can}_{[v]} = \{w \in \mathbb{H}^{n+1} \mid 0 = F_-(w,v) = F_-(w,vi) = F_-(w,vj) = F_-(w,vk)\}/v\mathbb{H}.$$

2. QC structures

2.1. Definition and basic properties. Let M be a $4n+3$-dimensional $(n \geq 2)$ manifold and consider any codimension three distribution H on M. Denote by L the three dimensional quotient bundle $L = TM/H$, and by L^* its dual. For a fixed point $p \in M$, each element $\eta \in L_p^*$ is, by definition, a linear map $L_p \to \mathbb{R}$. If we take the composition of η with the projection $T_pM \to L_p$, we obtain an element of T_p^*M. Therefore, we have an identification

$$L_p^* \cong \{\lambda \in T_p^*M \ : \ \lambda|_H = 0\} \tag{2.1}$$

and the sections of L^* are simply the 1-forms on M that are vanishing along the distribution H.

Definition 2.1. *A quaternionic-contact structure (QC structure) on a $(4n+3)$-dimensional $(n \geq 2)$ manifold M is a codimension three distribution H with the property that locally, around each point $p \in M$, there exist:*

i) sections η_1, η_2, η_3 of L^*;

ii) sections I_1, I_2, I_3 of the bundle $End(H)$ satisfying the quaternionic identities

$$I_1^2 = I_2^2 = I_3^2 = -id_H \quad I_1 I_2 = -I_2 I_1 = I_3;$$

iii) a symmetric and positive definite section g of the bundle $H^* \otimes H^*$, so that all these satisfy the identities

$$d\eta_s(X,Y) = 2g(I_sX,Y), \ \ s = 1,2,3. \tag{2.2}$$

Any ordered list

$$(\eta_1, \eta_2, \eta_3, I_1, I_2, I_3, g) \tag{2.3}$$

of local sections with the same type and properties as in the above Definition 2.1 will be called *admissible set* for the QC structure H on M. The 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ will be called simply – *contact form.* Neither the contact form η nor the admissible set $(\eta_1, \eta_2, \eta_3, I_1, I_2, I_3, g)$ (which we shall often abbreviate to (η_s, I_s, g), presuming that s is an index running from 1 to 3) are uniquely determined by the QC structure H. In fact, we have the following

Lemma 2.2. *Let (η_s, I_s, g) and (η'_s, I'_s, g') be two admissible sets for the same QC structure H on M, defined on some open set $U \subset M$. Then, there exists a positive* *function* $f : U \to R$ *and a matrix-valued function* $\mathcal{A} = (a_{ij}) : U \to SO(3)$ *so that*

$$(I_1', I_2', I_3') = (I_1, I_2, I_3)\mathcal{A}, \quad (\eta_1', \eta_2', \eta_3') = f(\eta_1, \eta_2, \eta_3)\mathcal{A}, \quad g' = f\, g.$$

Proof. By assumption, $H = \cap_{i=1}^3 \eta_i = \cap_{i=1}^3 \eta_i'$. Thus there exists a matrix-valued function $\mathcal{B} = (b_{ij}) : U \to GL(3)$ with $\eta_s' = \sum_{t=1}^3 b_{st}\eta_t,\ s = 1, 2, 3$. Taking the exterior derivative of the above equations we obtain

$$(d\eta_s')|_H = \sum_t b_{st}(d\eta_t)|_H. \tag{2.4}$$

Let us fix a symmetric and positive definite section h of the bundle $H^* \otimes H^*$ which we will use as a "background" metric on H. With respect to this metric, consider the restrictions of the 2-forms $(d\eta_s')|_H$ to H as endomorphisms of H, i.e., sections of the bundle $End(H) = H^* \otimes H$. This identification depends on the choice of h. However, it is easy to see that the composition of two endomorphisms of the form $((d\eta_s')|_H)^{-1} \circ (d\eta_t')|_H$, is an endomorphism independent of the choice of h. For (i, j, k) a cyclic permutation of $(1, 2, 3)$ and $h = g'$ we have

$$((d\eta_j')|_H)^{-1} \circ (d\eta_i')|_H = I_k'. \tag{2.5}$$

The above equation holds for any choice of the "background" metric h on H, in particular, also for $h = g$. Using 2.4, we conclude that

$$I_k' \;=\; ((d\eta_j')|_H)^{-1} \circ (d\eta_i')|_H \;\in\; \mathrm{span}_{\mathbb{R}}\ \{id_H, I_1, I_2, I_3\}.$$

Note that $\mathrm{span}_{\mathbb{R}}\ \{id_H, I_1, I_2, I_3\} \subset End(H)$ is an algebra with respect to the usual composition of endomorphisms, which is isomorphic to the algebra of the quaternions $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\ \{1, i, j, k\}$. If an element of $\mathbb{H}$ has square -1 then this element belongs to $Im(\mathbb{H})$. Therefore, $I_s' \in \mathrm{span}\{I_1, I_2, I_3\}$ and hence

$$\mathrm{span}_{\mathbb{R}}\ \{I_1, I_2, I_3\} \;=\; \mathrm{span}_{\mathbb{R}}\ \{I_1', I_2', I_3'\}.$$

Now, still identifying $H^* \otimes H$ with $End(H)$, using $h = g$, and recalling that the metric g is I_s- and I_s'-compatible, we observe that each of the endomorphisms $(d\eta_k')_H$ anti-commutes with both I_i' and I_j'. This implies that, as an endomorphism, $(d\eta_k')_H$ is proportional to I_k', which gives $g' = f\, g$ for some $f > 0$. The fact that the matrix valued function

$$\mathcal{A} \stackrel{def}{=} \frac{1}{f}\mathcal{B}$$

takes values in $SO(3)$ follows from the requirement that both (I_1, I_2, I_3) and (I'_1, I'_2, I'_3) satisfy the quaternionic identities. $\square$

Let us remark that the above lemma (which is a well known fact as a sort of mathematical folklore) reveals a property that is very particular for the QC geometry in contrast with the situation in the CR case. The lemma implies that with each QC manifold (M, H) we have the following list of naturally associated objects:

i) There is a 3-dimensional bundle

$$Q = \mathrm{span}\{I_1, I_2, I_3\} \subset End(H)$$

over M which we shall call *quaternionic structure* of H. Note that Q is canonically endowed with a scalar product $\langle ., . \rangle$ and orientation in such a way that for any admissible set (2.3), the basis I_1, I_2, I_3 of Q is orthonormal and oriented.

ii) The conformal class $[g]$ of symmetric sections of $H^* \otimes H^*$ from (2.3) is well defined globally on M. Using the standard partition of unity argument, it is easily seen that in $[g]$ we can always pick a globally defined positive definite representative, which we call *metric* on the contact distribution H .

iii) The bundle L^* is canonically endowed with an $CSO(3)$ structure (i.e., a conformal structure). This is done by declaring that for each admissible set (2.3) the basis η_1, η_2, η_3 is orthogonal and oriented.

2.2. Quaternionic-contact structures in dimension 7. It turns out that for $n = 1$ the definition 2.1 is too weak and one needs some further assumptions in order to make it reasonable.

To clarify the problem here, consider an arbitrary orientable 4-dimensional distribution H on a 7-dimensional manifold M, and choose some volume form ϵ on H—here, by volume form, we mean a globally defined non-vanishing section ϵ of the bundle $\Lambda^4(H^*)$ over M. Since for any two $\phi, \psi \in \Lambda^2(H^*)$, the wedge product $\phi \wedge \psi$ is proportional to ϵ, we can define a bilinear symmetric 2-form B on $\Lambda^2(H^*)$ by the equation $\phi \wedge \psi = B(\phi, \psi)\epsilon$. Then, B is non-degenerate, of signature $(3, 3)$, and it defines an inner-product on the bundle $\Lambda^2(H^*)$ over M. For any local frame η_1, η_2, η_3 of L^* (with $L = TM/H$), consider, pointwise, the subspace

$$\Gamma = span\{(d\eta_1)_H, (d\eta_2)_H, (d\eta_3)_H\} \ \subset \ \Lambda^2(H^*).$$

It is an easy observation that Γ depends only on the distribution H, but not on the particular choice of the frame η_1, η_2, η_3. Now, a simple calculation shows that the distribution H satisfies the conditions of Definition 2.1 if and only if Γ is 3-dimensional at each point of M (i.e., if Γ is non-degenerate) and the restriction of B to Γ is either positive or negative definite. Clearly, each of the above two conditions is generic (i.e., it defines an open subset in the set of all distributions on M with respect to some natural topology) which makes the set of distribution that satisfy it uncomfortably "large".

As shown in [D1], the proper definition in dimension 7 requires one to add an extra assumption to the conditions of definition 2.1; namely, the assumption about the existence of Reeb vector fields which we shall explain below.

2.3. Existence of Reeb vector fields. Assume H is a $4n$-dimensional distribution on a $(4n+3)$-dimensional manifold M that satisfies the requirements of definition 2.1 without the assumption $n \geq 2$, i.e., here, we allow also $dim(M) = 7$.

We fix some admissible set (η_s, I_s, g) for H. If V is any complementary to H distribution, i.e., such that

$$TM = H \oplus V,$$

then clearly there exists a unique frame (ξ_1, ξ_2, ξ_3) of V dual to $(\eta_1|_V, \eta_2|_V, \eta_3|_V)$.

In what follows it will be important to find a special complementary distribution $\bar{V}$ in such a way that the associated dual frame $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ would satisfy in addition the relations

$$d\eta_s(\bar{\xi}_t, X) = -d\eta_t(\bar{\xi}_s, X) \quad s, t = 1, 2, 3, \;\; X \in H. \tag{2.6}$$

If such a complement $\bar{V}$ exist, then the vector fields of the associated dual frame $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ will be called *Reeb vector fields* of the contact form $\eta = (\eta_1, \eta_2, \eta_3)$.

Following [Biq], consider the 3×3 matrix with entries the elements of H^*, given by

$$a_{st} = d\eta_s(\xi_t, .) + d\eta_t(\xi_s, .), \quad s, t = 1, 2, 3,$$

with respect to some starting complement V. One could think of the matrix a_{st} as the local representation of a certain section of the bundle $Q \otimes Q \otimes H^*$ given by the formula

$$\sum_{s,t=1}^{3} I_s \otimes I_t \otimes a_{st}. \tag{2.7}$$

Clearly, (2.7) remains unchanged if rotating the admissible set (η_s, I_s) by an $SO(3)$ matrix and therefore it depends only on the choice of the complement V but not on the choice of the admissible set. Now the problem of finding a dual basis with the properties (2.6) is equivalent to the problem of finding a complement $\bar{V}$ with vanishing associated section

$$\sum I_s \otimes I_t \otimes \bar{a}_{st} \tag{2.8}$$

of the bundle $Q \otimes Q \otimes H^*$.

Since the matrix a_{st} is symmetric, by definition, the (2.8) is actually an element of the bundle $Q \odot Q \otimes H^*$, where $Q \odot Q$ denotes the symmetric component of $Q \otimes Q$. The bundle $Q \odot Q \otimes H^*$ decomposes into exactly three irreducible components with respect to the natural $Sp(n)Sp(1)$ action. With the standard notation for the irreducible $Sp(n)Sp(1)$-representations, we have

$$Q \odot Q \otimes H^* = [\lambda^1\sigma^5] \oplus [\lambda^1\sigma^3] \oplus [\lambda^1\sigma^1].$$

Of particular interest for us is the component $[\lambda^1\sigma^5]$ that can be described explicitly by

$$[\lambda^1\sigma^5] = \{\sum_{st} I_s \otimes I_t \otimes x_{st} \in Q \otimes Q \otimes H^* \ : \ x_{st} = x_{ts}, \ \sum_t I_t x_{st} = 0\}.$$

The other two components can be described similarly:

$$[\lambda^1\sigma^3] = \{\sum_{st} I_s \otimes I_t \otimes (I_s y_t + I_t y_s) \in Q \otimes Q \otimes H^* \ : \ y_s \in H^*, \ \sum_s I_s y_s = 0\},$$

$$[\lambda^1\sigma^1] = \{\sum_{st} I_s \otimes I_t \otimes (\delta_{st} y) \in Q \otimes Q \otimes H^* \ : \ y \in H^*\}.$$

Explicitly the $[\lambda^1\sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ of a section $\sum I_s \otimes I_t \otimes a_{st}$, that has been associated to some complementary distribution V, is given by

$$b_{st} = a_{st} + \frac{1}{5}\sum_{r=1}^{3}(I_s I_r a_{tr} + I_t I_r a_{sr} - \delta_{st} a_{rr}).$$

In fact, by using some simple representation theoretic arguments or by a direct calculation, it is easy to show that the above component $\sum I_s \otimes I_t \otimes b_{st}$ is actually independent of the choice of the starting V, and therefore it is an object that characterizes the given QC-structure as a whole. With a bit more effort this section could be viewed as a component of the intrinsic torsion of the QC-structure but this will not be considered here further. For our purposes, significant is only the following:

Proposition 2.3. *There exists a complementary distribution $\bar{V}$ to H for which the associated dual frame $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ satisfies equations (2.6) if and only if the $[\lambda^1\sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ vanishes for some (and hence for each) complementary distribution V.*

Proof. Indeed, assume that $b_{st} = 0$ $s, t = 1, 2, 3$ for some fixed V with associated dual frame (ξ_1, ξ_2, ξ_3). If we define the three vector fields $r_1, r_2, r_3 \in H$ by the formula

$$g(r_s, X) = -\frac{1}{5}\sum_{p}(I_p a_{sp} + \frac{1}{2} I_s a_{pp})(X), \;\; s = 1, 2, 3, \;\; X \in H,$$

then the dual frame $\bar{\xi}_s = \xi_s - \frac{1}{2} r_s$ will have the desired property (2.6). $\square$

If $\dim(M) > 7$, the $[\lambda^1\sigma^5]$-component $\sum I_\alpha \otimes I_\beta \otimes b_{\alpha\beta}$ vanishes automatically, by a result in [**Biq**], and therefore the existence of Reeb vector fields is provided without any further assumptions about the QC structure H.

In dimension 7, however, the $[\lambda^1\sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ may not vanish in general and there are examples of distributions in dimension 7 that satisfy all the requirements of definition 2.1 (but the assumption $n \geq 2$) for which there exist no Reeb vector fields at all (cf. [**D1**]).

2.4. Special agreement in dimension 7. If the dimension of the QC-manifold M is 7 then we will assume in addition to definition 2.1 that the $[\lambda^1\sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ (cf. section 2.3) for some (and hence for each) complement V to H vanishes and thus the existence of the Reeb vector fields is provided.

2.5. Conformal infinity. There is a certain very close relationship between the geometry of quaternionic-Kähler manifolds and the QC structures. Following the discussion from section 1.8, if we are given a triple (Z, Θ, σ), where Z is a complex $(2n+1)$-manifold, Θ is a holomorphic contact form, and σ is an antiholomorphic involution, then, the family of σ-invariant complex-analytic spheres in Z with a certain fixed type of normal bundle, form a real $4n$-manifold M with boundary S. Furthermore, it was shown that on the interior of M we have a natural QK metric $\tilde{g}$, whereas on S we obtain a natural QC distribution H. The QC manifold (S, H) is called, in this context, *conformal infinity* of the QK metric $\tilde{g}$. Essentially, as shown in [**Biq**], the QC structure H shows the asymptotic behavior of the metric $\tilde{g}$ near the surface at infinity S. More precisely, there is a positive (distance) function ρ on M, vanishing to first order on S, such that on a neighborhood $(0, a] \times S \subset M$ the asymptotic behaviour of $\tilde{g}$ near S is given by

$$\tilde{g} \sim \frac{2}{\rho} g + \frac{1}{\rho^2}(d\rho^2 + \eta_1^2 + \eta_2^2 + \eta_3^2),$$

where (η_s, I_2, g) is an admissible set for the QC-structure (S, H).

It is a fundamental fact due to Biquard [**Biq**] (if $\dim(S) > 7$) and Duchemin [**D1**] (if $\dim(S) = 7$) that each real analytic QC-manifold (S, H) is the conformal infinity of a quaternionic-Kähler metric $\tilde{g}$, defined on a neighborhood of S, and admitting a real analytic extension on the boundary with pole of order 2. Moreover, the germ of the metric $\tilde{g}$ is uniquely determined by (S, H).

It is often instructive to consider the standard 3-dimensional conformal geometry as a QC-geometry in the special dimension three (i.e., with $n = 0$ and $H = \{0\}$). Then, the above result of Biquard and Duchemin could be regarded as a generalization of a previous result of LeBrun [**LeB82**] that states: Each real analytic 3-manifold S with a Riemannian conformal metric $[g]$ is naturally the conformal infinity of a germ-unique real-analytic Riemannian metric $\tilde{g}$ satisfying the self-dual Einstein equations with cosmological constant -1.

2.6. The quaternionic hyperbolic space and its conformal infinity. A standard example of a complete quaternionic-Kähler manifold of negative scalar curvature is the quaternionic hyperbolic space

$$\mathbb{H}\mathbb{H}^{n+1} = Sp(n+1, 1)/Sp(n+1) \times Sp(1),$$

where by definition the group $Sp(n+1, 1)$ consists of all endomorphisms of the vector space $\mathbb{R}^{4(n+2)} \cong \mathbb{H}^{n+1}$ contained in the intersection $O(4n+4, 4) \cap GL(n+1, \mathbb{H})$. The metric of the quaternionic hyperbolic space is uniquely determined (up to a constant multiple) by the requirement that it is $Sp(n, 1)$-invariant.

The quaternionic hyperbolic space $\mathbb{HH}^{n+1}$ can be realized also as the unit open Ball B^{4n+4} in $\mathbb{H}^{n+1}$, with the metric

$$g_{\mathbb{H}} = \frac{4Euc}{\rho} + \frac{1}{\rho^2}(d\rho^2 + (J_1 d\rho)^2 + (J_2 d\rho)^2 + (J_3 d\rho)^2), \tag{2.9}$$

where Euc is the euclidian metric on $\mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)}$, $\rho = 1 - |x|^2$ for $x \in B^{4n+4}$, and J_1, J_2, J_3 are the endomorphisms obtained by right multiplication with the conjugated imaginary quaternions $\bar{\mathbf{i}}, \bar{\mathbf{j}}$ and $\bar{\mathbf{k}}$. The sphere S^{4n+3} is the boundary of the ball B^{4n+4}. If we extend the symmetric 2-tensor $\rho^2 g_{\mathbb{H}}$ up to the boundary S^{4n+3}, we will obtain some degenerate (rank 4) symmetric tensor there. Its kernel is a codimension 3-distribution H^{can} on S^{4n+3} given by

$$H^{can} = ker(J_1 d\rho) \cap ker(J_2 d\rho) \cap ker(J_3 d\rho).$$

If we put $\eta_s = -\frac{1}{2}(J_s d\rho)|_{S^{4n+3}}$, $I_s = J_s|_{S^{4n+3}}$, $s = 1, 2, 3$ and $g = Euc|_{S^{4n+3}}$ we can easily see that $d\eta_s(X, Y) = 2g(I_s X, Y)$ for all $X, Y \in H^{can}$. Hence according to Definition 2.1 the distribution H^{can} is a QC-structure on the sphere S^{4n+3} with admissible set $(\eta_1, \eta_2, \eta_3, I_1, I_2, I_3, g)$. Moreover, the vector fields $\xi_s = J_s(x)$ on S^{4n+3} are exactly the Reeb vector fields of the contact form $\eta = (\eta_1, \eta_2, \eta_3)$. Clearly, formula 2.9 implies that (S^{4n+3}, H^{can}) is the conformal infinity of $\mathbb{HH}^{n+1}$.

2.7. Deforming the twistor space of $\mathbb{HH}^{n+1}$. As explained in section 1.10, the twistor space Z of $\mathbb{HH}^{n+1}$ can be realized as an open set in $\mathbb{CP}^{2n+1}$, where the holomorphic contact form Θ and the antiholomorphic involution σ of Z are the restrictions to Z of some globally defined $\tilde{\Theta}$ and $\tilde{\sigma}$ on $\mathbb{CP}^{2n+1}$. In [LeB89], Lebrun gives a method for constructing deformations of an open subset $\tilde{Z} \subset \mathbb{CP}^{2n+1}$ containing the closure of Z. The deformations of the complex manifold $\tilde{Z}$ are done in such a way that they preserve the holomorphic contact form $\tilde{\Theta}$ and the antiholomorphic involution $\tilde{\sigma}$. Each complex manifold in the deformation is naturally the twistor space of a complete quaternionic-Kähler manifold diffeomorphic to the open Ball B^{4n} and having as conformal infinity certain quaternionic-contact structure defined on the boundary S^{4n-1} of B^{4n}. The final result obtained in [LeB89] is that the moduli space of complete quaternionic-Kähler metrics on B^{4n} is infinite dimensional. This in particular implies that there exist infinitely many different quaternionic-contact structures on the sphere S^{4n-1}.

2.8. Biquard connection. Let (M^{4n+3}, H) be a quaternionic-contact manifold and fix some admissible set (η_s, I_s, g) for H. As explained in section 2.1, the contact form $\eta = (\eta_1, \eta_2, \eta_3)$ in the admissible set is determined only up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$. The distribution H is equipped with a conformal class $[g]$ of metrics and a 3-dimensional quaternionic bundle Q. According to Lemma 2.2 the most general transformation of the contact form η has the type $\bar\eta = \mu\Psi\cdot\eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries. We call such transformations quaternionic-contact conformal (QC conformal). If the function μ is constant we say that the transformation is quaternionic-contact homothetic. Clearly, the contact forms $\bar\eta$ which one obtains by applying homothetic quaternionic-contact transformations are precisely those for which the corresponding metric $\bar g$ (in the corresponding to $\bar\eta$ admisible set) is a constant multiple of g. If the tensor g in the natural conformal class $[g]$ is fixed we will say that (M,H,g) is a quaternionic-contact metric manifold. To every metric in the conformal class $[g]$ on H one can associate a linear connection preserving the QC structure (M,H,g) (cf. [**Biq**]) which we shall call the Biquard connection. This connection is invariant under QC homothetic transformations but changes in a non-trivial way under QC conformal transformations. If following the analogy with the 3-dimensional conformal geometry mentioned in 2.5, one should think of the Biquard connection as an analog of the Levi-Civita connection.

Next we explain briefly the construction of the Biquard connection (for more details see [**Biq**]). Let us first consider the general situation of a manifold M with an arbitrary distribution $H\subset TM$ and a general vector bundle E over M. A partial connection on E along H is, by definition, a bilinear map $\nabla_X\sigma$ defined for vector fields X with values in H and sections σ of E such that $\nabla_{fX}\sigma = f\nabla_X\sigma$ and $\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma$ for every smooth function f on M.

If g is a metric on H then (as shown in [**Biq**, Lemma II.1.1]) for any supplementary distribution V of H in TM, there is a unique partial connection ∇ on H along H such that

(*i*) $\nabla_X g = 0, \quad X\in H$;
(*ii*) for any two sections X,Y of H, the torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ satisfies the identity $T(X,Y) = -[X,Y]_V$, where the subscript V means "the component in V".

Now let (M,H,g) be a quaternionic-contact metric manifold with admissible set (η_s, I_s) and Reeb vector fields ξ_1,ξ_2,ξ_3. Chose V to be the span of ξ_1,ξ_2,ξ_3. Each endomorphism f of H extends naturally to an endomorphism of TM by setting $f(\xi)=0,\ \xi\in V$. With this agreement we may consider the endomorphism in the basis $\{I_1,I_2,I_3\}$ of Q as endomorphisms of the tangent bundle TM. We define the fundamental 2-forms $\omega_1,\omega_2,\omega_3$ of Q by

$$\omega_s(A,B) = g(I_sA,(B)_H),\ \ A,B\in TM,\ \ s=1,2,3, \tag{2.10}$$

where the subscript H means "the component in H". Definition 2.1 imply that

$$\omega_s(A,B) = \begin{cases} \frac12 d\eta_s(A,B), & A,B\in H\\ 0, & A\in V,\ B\in TM\end{cases} \qquad \text{for } s=1,2,3. \tag{2.11}$$

.

Take ∇ to be the partial connection on H along H that corresponds to the fixed supplementary distribution V. Then according to [**Biq**, Proposition II.1.7], the partial connection ∇ preserves the bundle Q, i.e., we have

(iii) $\nabla_X Q \subset Q,\ \ X \in H.$

More precisely, for each cyclic permutation (i,j,k) of the numbers $(1,2,3)$ and any $X \in H$ there are the relations

$$\begin{aligned} \nabla_X \omega_i &= -\alpha_j(X)\omega_k + \alpha_k(X)\omega_j; \\ \nabla_X I_i &= -\alpha_j(X) I_k + \alpha_k(X) I_j, \end{aligned} \tag{2.12}$$

where $\alpha_i(X) := d\eta_k(\xi_j, X)$.

Let us introduce an inner-product $\langle .,. \rangle$ on the bundle V by setting the Reeb vector fields ξ_1, ξ_2, ξ_3 to be an orthonormal basis of V. Given a section ξ of V and a section X of H, set $\nabla_X \xi = [X, \xi]_V$. By [**Biq**, Proposition II.1.9], the latter formula defines a partial connection on V along H such that $\nabla \langle .,. \rangle = 0$.

The natural assignment

$$\xi_s \to I_s, \quad s = 1,2,3, \tag{2.13}$$

determines a bundle isomorphism $\varphi : V \to Q$. The isomorphism φ has the property that

(iv) $\nabla_X \varphi = 0$ for $X \in H$.

Indeed, by (2.12), we have

$$\nabla_X(\varphi(\xi_t)) = \nabla_X I_t = -\sum_{s=1}^3 d\eta_t(\xi_s, X) I_s = \sum_{s=1}^3 d\eta_s(\xi_t, X) I_s = \sum_{s=1}^3 \eta_s([X, \xi_t]_V) I_s = \sum_{s=1}^3 \eta_s(\nabla_X \xi_t)\varphi(\xi_s) = \varphi(\nabla_X \xi_t)$$

Set

$$P = \{A \in End(H) \mid A \text{ is skew-symmetric and } AI = IA \text{ for every } I \in Q\}.$$

This is a subbundle of $End(H)$ of rank $2n^2 + n$, orthogonal to Q and such that the commutator $[A_1, A_2]$ of two endomorphisms $A_1, A_2 \in P$ is also in P. Clearly, every fibre of P (resp. Q) is isomorphic to the Lie algebra $sp(n)$ (resp. $sp(1)$).

It is shown in [**Biq**, Lemma II.2.1] that there is a unique partial connection ∇ on H along V such that (*v*) $\nabla_\xi g = 0, \quad \xi \in H$;

(*vi*) $\nabla_\xi Q \subset Q, \quad \xi \in H$;

(*vii*) setting $T(\xi, X) = \nabla_\xi X - \nabla_X \xi - [\xi, X]$ for $\xi \in V$ and $X \in H$, every endomorphism

$$T_\xi : H \in X \to T(\xi, X) = \nabla_\xi X - [\xi, X]_H \in H$$

is an element of $(P \oplus Q)^\perp \subset End(H)$.

Note, that we have a bundle isomorphism $\{(P \oplus Q)^\perp \subset End(H)\} \cong \{(sp(n) \oplus sp(1))^\perp \subset gl(4n)\}$.

Since $\nabla_\xi Q \subset Q$ for every $\xi \in V$, we can transfer ∇_ξ from Q to V via the isomorphism $\varphi : V \to Q$. In this way get a partial connection on V along V with the property

(*viii*) $\nabla_\xi \varphi = 0, \quad \xi \in Q$.

Combining the partial connections we have defined, we obtain a connection ∇ on TM having the properties (*i*)-(*viii*). We shall call ∇ the Biquard connection of the metric QC-structure (M, H, g). This connection is uniquely determined by its properties (*i*)-(*viii*).

By using the isomorphism $\varphi : V \to Q$ we can transfer to V the metric and the orientation of Q. Then any frame ξ_1, ξ_2, ξ_3 associated to an admissible set of the QC-structure is orthonormal and positively oriented. Putting together the metric of V and the metric g of H we obtain a metric on $TM = H \oplus V$ for which H and V are orthogonal. This metric on TM will be also denoted by g and it is also parallel with respect to the Biquard connection, $\nabla g = 0$.

Summing up the facts from the above discussion, we have obtained the following result which is originally due to O. Biquard:

THEOREM 2.4. [Biq] *Let (M, H, g) be a quaternionic-contact metric manifold of dimension $4n + 3 \geq 7$. Then, there exists a unique connection ∇ with torsion T on M and a unique supplementary distribution V to H in TM, such that:*

i) ∇ preserves the decomposition $H \oplus V$ and the metric g;

ii) for $X, Y \in H$, one has $T(X, Y) = -[X, Y]_{|V}$;

iii) ∇ preserves the $Sp(n)Sp(1)$-structure on H, i.e., $\nabla g = 0$ and $\nabla Q \subset Q$;

iv) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;

v) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of endomorphisms of H, i.e., $\nabla \varphi = 0$.

2.9. Further properties of the Biquard connection. Any endomorphism Ψ of H can be naturally decomposed, with respect to some admissible set (η_s, I_s), into four parts (this we call $Sp(n)$-invariant decomposition of Ψ)

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+},$$

where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. Explicitly,

$$4\Psi^{+++} = \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3, \quad 4\Psi^{+--} = \Psi - I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3,$$
$$4\Psi^{-+-} = \Psi + I_1\Psi I_1 - I_2\Psi I_2 + I_3\Psi I_3, \quad 4\Psi^{--+} = \Psi + I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3.$$

The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}. \tag{2.14}$$

Denoting the corresponding (0,2) tensor via g by the same letter one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator

$$\dagger \;=\; I_1 \otimes I_1 \;+\; I_2 \otimes I_2 \;+\; I_3 \otimes I_3 \tag{2.15}$$

corresponding, respectively, to the eigenvalues 3 and -1, see [CSal]. If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_i, i = 1, 2, 3$ is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_3 = \frac{tr(\Psi)}{4} Id_{|H}$.

The supplementary "vertical" (sub-)space V is the linear span of the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields or fundamental vector fields. We shall extend g to a metric on M by requiring

$$span\{\xi_1, \xi_2, \xi_3\} = V \perp H \text{ and } g(\xi_s, \xi_k) = \delta_{sk}. \tag{2.16}$$

The extended metric does not depend on the action of $SO(3)$ on V, but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on TM, $\nabla g = 0$. We shall also extend the quternionic structure by setting $I_{s|V} = 0$.

Suppose the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ have been fixed. The restriction of the torsion of the Biquard connection to the vertical space V satisfies [Biq]

$$T(\xi_i, \xi_j) = \lambda \xi_k - [\xi_i, \xi_j]_{|H}, \tag{2.17}$$

where λ is a smooth function on M, which will be determined in Corollary 4.15. Here (i, j, k) is any cyclic permutation of $1, 2, 3$. Further properties of the Biquard connection are encoded in the properties of the torsion endomorphism

$$T_\xi = T(\xi, .) : H \to H, \quad \xi \in V.$$

Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part b_ξ,

$$T_\xi = T^0_\xi + b_\xi,$$

we summarize the description of the torsion due to O. Biquard in the following Proposition.

PROPOSITION 2.5. [**Biq**] *The torsion T_ξ is completely trace-free,*

$$trT_\xi = \sum_{a=1}^{4n} g(T_\xi(e_a), e_a) = 0, \quad trT_\xi \circ I = \sum_{a=1}^{4n} g(T_\xi(e_a), Ie_a) = 0, \quad I \in Q, \tag{2.18}$$

where $e_1 \dots e_{4n}$ is an orthonormal basis of H. The symmetric part of the torsion has the properties:

$$T^0_{\xi_i} I_i = -I_i T^0_{\xi_i}, \qquad i = 1, 2, 3. \tag{2.19}$$

In addition, we have

$$\begin{aligned} I_2(T^0_{\xi_2})^{+--} &= I_1(T^0_{\xi_1})^{-+-}, \qquad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \\ I_1(T^0_{\xi_1})^{--+} &= I_3(T^0_{\xi_3})^{+--}. \end{aligned} \tag{2.20}$$

The skew-symmetric part can be represented in the following way

$$b_{\xi_i} = I_i u, \quad i = 1, 2, 3, \tag{2.21}$$

where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1, I_2, I_3.

If $n = 1$ then the tensor u vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

Let $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ be the curvature tensor of the Biquard connection. The QC Ricci curvature Ric, the QC Ricci forms ρ_s and the QC scalar curvature $Scal$ are defined respectively by

$$Ric(A,B) = \sum_{a,b=1}^{4n} g(R(e_b,A)B,e_b), \ \ A,B \in TM,$$

$$\rho_s(A,B) = \frac{1}{4n}\sum_{a=1}^{4n} g(R(A,B)e_a, I_s e_a), \quad Scal = \sum_{a,b=1}^{4n} g(R(e_b,e_a)e_a,e_b),$$

where $e_1,...,e_{4n}$ is an orthonormal basis of H. The restriction of the Ricci curvature Ric to H is a symetric 2-tensor ([**Biq**]) which could be $Sp(n)Sp(1)$-invariantly decomposed in exactly three components.

2.10. Twistor construction for QC manifolds. The twistor space $\mathcal{Z}$ of a QC-manifold (M,H) is defined to be the total space of the unit sphere bundle $S^2(Q) \to M$ ([**Biq**]). Let

$$\pi : \mathcal{Z} \to M$$

be the obvious projection and take an admissible set (η_s, I_s, g) with corresponding Reeb vector fields ξ_1,ξ_2,ξ_3. Each element $I \in Z$ is an orthogonal endomorphism of the distribution H with the property $I^2 = -id_H$. We have $I = a_1 I_1 + a_2 I_2 + a_3 I_3$, with $a_1^2 + a_2^2 + a_3^3 = 1$. Consider the 1-form $\eta^{\mathcal{Z}}$ on $\mathcal{Z}$ defined at the point $I \in \mathcal{Z}$ by

$$\eta^{\mathcal{Z}} = a_1\pi^*(\eta_1) + a_2\pi^*(\eta_2) + a_3\pi^*(\eta_3).$$

If $(\bar{\eta}_s, \bar{I}_s, \bar{g})$ is another admissible set for H then $\bar{g} = fg$ for some smooth function f on M (see lemma 2.2). A small computation shows that for the corresponding 1-form $\bar{\eta}^{\mathcal{Z}}$ on the twistor space holds $\bar{\eta}^{\mathcal{Z}} = f\eta^{\mathcal{Z}}$, in particular, it follows that the kernel $\mathcal{D} = \ker \eta^{\mathcal{Z}}$ is independent of the choice of the admissible set. There is also a natural endomorphism $\mathcal{J} : \mathcal{D} \to \mathcal{D}$ with $\mathcal{J}^2 = -id_{\mathcal{D}}$, which could be described in the following way: Take any point $I \in Z$, $I = a_1 I_1 + a_2 I_2 + a_3 I_3$, $p = \pi(I)$ and denote by $\mathcal{Z}_p$ the fiber over p. The Biquard connection ∇ enables us to decompose $T_I\mathcal{Z}$ into horizontal and vertical subspaces:

$$T_I\mathcal{Z} = Hor_I \oplus T_I(\mathcal{Z}_p).$$

The differential of the projection π at the point $I \in \mathcal{Z}$ gives us the identification $Hor_I \cong T_pM = H_p \oplus V_p$ und hence also the decomposition

$$T_I\mathcal{Z} = H_p \oplus V_p \oplus T_I(\mathcal{Z}_p).$$

With the above identification, consider the vector field $R_I = a_1\xi_1 + a_2\xi_2 + a_3\xi_3$ on $\mathcal{Z}$. This vector field is invariant under QC-homothetic transformations but changes in a non-trivial way under QC-conformal transformations. In fact, R_I is precisely the Reeb vector field of the contact form $\eta^{\mathcal{Z}}$. Let W_I be the orthogonal complement of R_I in $V_p \subset Hor_I$. Then the kernel $\mathcal{D}$ of $\eta^{\mathcal{Z}}$ is given by

$$\mathcal{D}_I = H_p \oplus W_I \oplus T_I(\mathcal{Z}_p).$$

Using the canonical identification $T_I(\mathcal{Z}_p) \cong \{A \in Q \mid IA = -AI\}$, we can describe the natural complex structure of the sphere $\mathcal{Z}_p$ as the endomorphism $A \mapsto \frac{1}{2}[I, A]$.

Now we define the endomorphism $\mathcal{J} : \mathcal{D}_I \to \mathcal{D}_I$ by

$$\mathcal{J}(A) = \begin{cases} I(A), & A \in H_p \\ R_I \times A, & A \in W_I \\ \frac{1}{2}[I, A], & A \in T_I(\mathcal{Z}_p) \cong \{X \in Q \mid IX = -XI\}, \end{cases} \tag{2.22}$$

where " $\times$ " denotes the natural cross product on the 3-dimensional oriented euclidian vector space V_p. It is a theorem of Biquard ([Biq]) that the pair $(D, \mathcal{J})$ is a an integrable CR-structure on the twistor space $\mathcal{Z}$, which is also QC-conformally invariant. In particular $(D, \mathcal{J})$ does not depend on the choice of the admissible set for H.

3. Quaternionic Heisenberg group

A very basic example of a QC manifold is provided by the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. These group can be modeled on the product space $\mathbb{H}^n \times \mathrm{Im}\mathbb{H}$ $(n \geq 1)$ with a group law given by

$$(q', \omega') \;=\; (q_o, \omega_o) \circ (q, \omega) \;=\; (q_o \;+\; q, \omega \;+\; \omega_o \;+\; 2\;\mathrm{Im}\; q_o\, \bar{q}),$$

where $q,\ q_o \in \mathbb{H}^n$ and $\omega, \omega_o \in \mathrm{Im}\,\mathbb{H}$. In coordinates, with the obvious notation, a basis of left invariant horizontal vector fields $T_\alpha, X_\alpha = I_1 T_\alpha, Y_\alpha = I_2 T_\alpha, Z_\alpha = I_3 T_\alpha, \alpha = 1 \ldots, n$ is given by

$$\begin{aligned} T_\alpha &= \partial_{t_\alpha} + 2x^\alpha \partial_x + 2y^\alpha \partial_y + 2z^\alpha \partial_z \qquad & X_\alpha &= \partial_{x_\alpha} - 2t^\alpha \partial_x - 2z^\alpha \partial_y + 2y^\alpha \partial_z \\ Y_\alpha &= \partial_{y_\alpha} + 2z^\alpha \partial_x - 2t^\alpha \partial_y - 2x^\alpha \partial_z \qquad & Z_\alpha &= \partial_{z_\alpha} - 2y^\alpha \partial_x + 2x^\alpha \partial_y - 2t^\alpha \partial_z \,. \end{aligned}$$

The central (vertical) vector fields ξ_1, ξ_2, ξ_3 are

$$\xi_1 = 2\partial_x \quad \xi_2 = 2\partial_y \quad \xi_3 = 2\partial_z \,.$$

We have the following commutator relations

$$[I_j\, T_\alpha, T_\alpha] \;=\; 2\xi_j \qquad [I_j\, T_\alpha, I_i\, T_\alpha] \;=\; 2\xi_k \tag{3.1}$$

where (i, j, k) is any cyclic permutation of the indices $(1, 2, 3)$.

With respect to the local coordinates $(q', \omega) \subset \boldsymbol{G}\,(\mathbb{H})$, the standart 3-contact form $\tilde{\Theta} = (\tilde{\Theta}_1,\ \tilde{\Theta}_2,\ \tilde{\Theta}_3)$ is given by

$$2\tilde{\Theta} \;=\; d\omega \;-\; q' \cdot d\bar{q}' \;+\; dq' \cdot \bar{q}'. \tag{3.2}$$

The kernel H of the contact form $\tilde{\Theta}$ is a codimension three distribution on $\boldsymbol{G}\,(\mathbb{H})$ which is easily seen to satisfy all the conditions of Definition 2.1 and thus defines a QC structure on $\boldsymbol{G}\,(\mathbb{H})$. Since the distribution H and the contact form $\tilde{\Theta}$ are left-invariant, they are preserved by the natural left-invariant connection on $\boldsymbol{G}\,(\mathbb{H})$. Let g be the left invariant metric on H which is determined by $\tilde{\Theta}$. Then, the central vector fields ξ_1, ξ_2, ξ_3 coincide with the correponding Reeb vector fields (cf. 2.3). If V is the linear span of these Reeb vector fields. Then V is a left-invariant distribution on $\boldsymbol{G}\,(\mathbb{H})$ and $T\boldsymbol{G}\,(\mathbb{H}) = H \oplus V$. Moreover, the left-invariant connection on $\boldsymbol{G}\,(\mathbb{H})$ is easily seen to coincide with the the Biquard connection of the quaternionic-contact metric structure $(\boldsymbol{G}\,(\mathbb{H}), H, g)$.

We shall often identify $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma \ = \ \{(q', p') \in \mathbb{H}^n \times \mathbb{H} \ : \ \text{Re}\, p' \ = \ |q'|^2\},$$

by the mapping $(q', \omega) \mapsto (q', |q'|^2 - \omega)$. Since $dp' \ = \ q' \cdot d\bar{q}' \ + \ dq' \cdot \bar{q}' \ - \ d\omega$, under the identification of $\boldsymbol{G}\,(\mathbb{H})$ with Σ we have also $2\tilde{\Theta} \ = \ -dp' \ + \ 2dq' \cdot \bar{q}'$. Taking into account that $\tilde{\Theta}$ is purely imaginary, the last equation can be written in the following form

$$4\,\tilde{\Theta} \ = \ (d\bar{p}' \ - \ dp') \ + \ 2dq' \cdot \bar{q}' \ - \ 2q' \cdot d\bar{q}'.$$

3.1. Folland-Stein inequality. On the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, there is a natural left-invariant measure known as the Haar measure dH of the group. Using this and the left-invariant metric g on $\boldsymbol{G}\,(\mathbb{H})$, we have the following classical result due to Folland and Stein [FS74]:.

THEOREM 3.1 (Folland-Stein). *There exists a constant $S > 0$ such that for each $u \in C_o^\infty$ (i.e., for each smooth function u with compact support), we have the inequality*

$$\left(\int_{\boldsymbol{G}\,(\mathbb{H})} |u|^{2^*} \, dH\right)^{1/2^*} \ \leq \ S \ \left(\int_{\boldsymbol{G}\,(\mathbb{H})} |(\nabla u)_H|^2 \, dH\right)^{1/2}, \tag{3.3}$$

where $(\nabla u)_H$ denotes the H-component of the gradient of u with respect to the splitting $T\boldsymbol{G}\,(\mathbb{H}) \ = \ H \oplus V$, and 2^* stands for the constant $\frac{2n_h}{n_h - 2} \ = \ 1 + \frac{n+2}{n+1}$, where $n_h = 4n + 6$ is the so called homogeneous dimension of the group.

This theorem raises the following very natural question—known as the QC Yamabe problem—about the sharpness of the above inequality:

- $*$ What is the best possible choice for the constant S in the above inequality and for which functions u does this inequality become an equality?

Following the analogy with the classical Sobolev inequality one shows (see [GV1, Va2, IMV10]) that the above question reduces to the solvability of the following second order differential equation on the quaternionic Heisenberg group:

$$\triangle u = -Cu^{2^*-1}, \tag{3.4}$$

where C is a positive constant and $\triangle$ is the horizontal sub-Laplacian defined in terms of the Biquard connection ∇ (which in this case is just the flat, left-invariant, connection of $\boldsymbol{G}\,(\mathbb{H})$) by the formula

$$\triangle u = \sum_{s=1}^{4n} (\nabla du)(e_s, e_s), \qquad e_1, \dots, e_{4n} \text{ is any } g\text{-orthonormal basis of } H. \tag{3.5}$$

In general, the equation

$$\triangle u = -Cu^q \tag{3.6}$$

is a sort of a non-linear eigenvalue problem for the operator $\triangle$ on $\boldsymbol{G}\,(\mathbb{H})$, whose analytic properties depend on the value of the exponent q. For $q = 1$, we have the linear eigenvalue problem; if the value of q is close to 1, the analytic behavior of (3.6) is very similar to the linear case and the problem is easily solved. For large q, all the methods based on linear theory are useless. The value $2^* - 1$ of the Yamabe equation (3.4) appears to be critical for the exponent q in the senses that if q is less, then (3.6) is easy to solve and if q is more, it might be impossible to solve at all. This accounts for the complexity of the QC Yamabe equation.

3.2. QC sphere. The unit sphere $S^{4n+3} \subset \mathbb{H}^{n+1}$ $(n \geq 1)$ is given in the usual way:

$$S^{4n+3} \;=\; \Big\{(q,p) \in \mathbb{H}^n \times \mathbb{H} \;\Big|\; |q|^2 + |p|^2 = 1\Big\}.$$

The canonical QC structure on S^{4n+3} can be described as follows: By differentiating the sphere equation $p \cdot \bar{p} + q \cdot \bar{q} = 1$, we obtain that at any fixed point $x = (q,p) \in S^{4n+3}$, the tangent space of the sphere is

$$T_x S^{4n+3} \;=\Big\{(dq, dp) \in \mathbb{H}^n \times \mathbb{H} \;\Big|\; \mathrm{Re}(dq \cdot \bar{q} + dp \cdot \bar{p}) = 0\;\Big\}$$

Then, the canonical contact 1-form $\tilde{\eta}$ with values in $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$ on S^{4n+3} is defined by

$$\tilde{\eta} = \mathrm{Im}(dq \cdot \bar{q} + dp \cdot \bar{p}).$$

The kernel of $\tilde{\eta}$ gives a QC structure in S^{4n+3} since it coincides with the canonical distribution H^{can} on S^{4n+3} that have been introduced in Section 2.6 as the conformal infinity of the quternionic hyperbolic space.

3.3. Cayley transform. The Cayley transform, $\mathcal{C}$, is a natural identification between the sphere S^{4n+3} with one point deleted and the quternionic Heisenberg group. It plays a roll in the QC geometry that is very close to this of the classical stereographic projection in the conformal Riemannian geometry.

If idenitfyign $\boldsymbol{G}\,(\mathbb{H})$ with Σ as above, we have that, by definition, $\mathcal{C}$ identifies each point $(q,p) \in S^{4n+3}, p \neq 1$, with a point $(q',p') \in \Sigma$,

$$(q',p') \;=\; \mathcal{C}\,\Big((q,p)\Big), \qquad q' \;=\; (1+p)^{-1}\,q, \qquad p' \;=\; (1+p)^{-1}\,(1-p).$$

The inverse map $(q,p) \;=\; \mathcal{C}^{-1}\Big((q',p')\Big)$ is given by

$$q \;=\; 2(1+p')^{-1}\,q', \qquad p \;=\; (1+p')^{-1}\,(1-p').$$

The above equations are consistent with our definitions since

$$\text{Re}\;p' \;=\; \text{Re}\;\frac{(1+\bar{p})(1-p)}{|1+p\,|^2} \;=\; \text{Re}\;\frac{1-|p|}{|1+p\,|^2} \;=\; \frac{|q|^2}{|1+p\,|^2} \;=\; |q'|^2.$$

Writing the Cayley transform in the form: $(1+p)q' \;=\; q, \quad (1+p)p' \;=\; 1-p,$ gives

$$dp\cdot q' \;+\; (1+p)\cdot dq' \;=\; dq, \qquad dp\cdot p' \;+\; (1+p)\cdot dp' \;=\; -dp,$$

from where we find

$$\begin{aligned} dp' \;&=\; -2(1+p)^{-1}\cdot dp\cdot(1+p)^{-1} \\ dq' \;&=\; (1+p)^{-1}\cdot[dq \;-\; dp\cdot(1+p)^{-1}\cdot q]. \end{aligned} \tag{3.7}$$

The Cayley transform is a quaternionic-contact diffeomorphism between the quaternionic Heisenberg group with its standard QC structure $\tilde{\Theta}$ and the sphere minus a point with its standard QC structure H^{can}; a fact which can be seen as follows.

Equations (3.7) imply the following identities

$$
\begin{aligned}
(3.8)\quad 2\,\mathcal{C}^*\,\tilde\Theta \;=\; & -(1+\bar p)^{-1}\cdot d\bar p\cdot(1+\bar p)^{-1} \;+\; (1+p)^{-1}\cdot dp\cdot(1+p)^{-1}\\
& +\;(1+p)^{-1}\,[dq\;-\;dp\cdot(1+p)^{-1}\cdot q]\cdot\bar q\cdot(1+\bar p)^{-1}\\
& -\;(1+p)^{-1}\,q\cdot[d\bar q\;-\;\bar q\cdot(1+\bar p)^{-1}\cdot d\bar p]\cdot(1+\bar p)^{-1}\\
=\;& (1+p)^{-1}\Big[dp\cdot(1+p)^{-1}\cdot(1+\bar p)\;-\;|q|^2\,dp\cdot(1+p)^{-1}\Big](1+\bar p)^{-1}\\
+\;& (1+p)^{-1}\Big[-(1+p)\cdot(1+\bar p)^{-1}\cdot d\bar p\;+\;|q|^2\,(1+p)^{-1}d\bar p\Big](1+\bar p)^{-1}\\
& +(1+p)^{-1}\Big[dq\cdot\bar q\;-\;q\cdot d\bar q\Big](1+\bar p)^{-1}\;=\;\frac{1}{|1+p\,|^2}\,\lambda\,\tilde\eta\,\bar\lambda,
\end{aligned}
$$

where $\lambda\;=|1+p\,|\,(1+p)^{-1}$ is a unit quaternion and $\tilde\eta$ is the standard contact form on the sphere.

Since $|1+p|\;=\;\frac{2}{|1+p'|}$, we have $\lambda\;=\;\frac{1+p'}{|1+p'\,|}$ and equation (3.8) can be put in the form

$$\lambda\;\cdot(\mathcal{C}^{-1})^*\,\tilde\eta\;\cdot\bar\lambda\;=\;\frac{8}{|1+p'\,|^2}\,\tilde\Theta.$$

We see that $(\mathcal{C}^{-1})^*\tilde\eta$ and $\tilde\Theta$ correspond to the same QC structure on Σ.

3.4. Yamabe problem. Following the classical scheme, we can translate question $(*)$ from Section 3.1 to an equivalent problem on the QC sphere S^{4n+3} via Cayley transform

$$\mathcal{C}:S^{4n+3}-\{Point\}\longrightarrow \boldsymbol{G}\,(\mathbb{H})$$

(cf. [K] and [CDKR1]). It turns out that $[*]$, with a standard argumentation (cf. [IMV10]), reduces to the following question, known as the Yamabe problem on the Sphere:

$**$ What are the representatives g of the natural conformal class $[g]$ of the QC structure H^{can} on S^{4n+3} for which the QC scalar curvature $Scal$ of the associated Biquard connection is a non-zero constant?

More generally, given a QC manifold (M, H), the QC Yamabe problem is the problem of finding a global representative g of constant QC scalar curvature in the natural conformal class of metrics $[g]$ on H. This problem reduces to the solvability of the equation ([Biq]),

$$\mathcal{L}u \;:=\; 4\frac{n+2}{n+1}\,\triangle u - \; u\,Scal \;=\; -Cu^{2^*-1}\;, \tag{3.9}$$

known as the QC Yamabe equation. Here $\triangle$ is the horizontal sub-Laplacian, defined by (3.5), with respect to the Biquard connection ∇ of some fixed (arbitrary) metric g on H; $Scal$ is the QC-scalar curvature of g and C is a positive constant.

In the case of the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, the QC Yamabe equation takes the form

$$\mathcal{L}u \;\equiv\; \sum_{\alpha=1}^{n}\big(T_\alpha^2 u \;+\; X_\alpha^2 u \;+\; Y_\alpha^2 u \;+\; Z_\alpha^2 u\big) \;=\; -\frac{C(n+1)}{4(n+2)}\,u^{2^*-1}, \tag{3.10}$$

which is, up to scaling, the Euler-Lagrange equation describing the extremals in the L^2 Folland-Stein embedding theorem 3.1.

More generally, on a compact QC manifold M, the QC Yamabe equation characterizes the extremals of the Yamabe functional Υ,

$$\Upsilon(u) \;\overset{def}{=}\; \int_M 4\frac{n+2}{n+1}\,|\nabla u|^2 \;+\; Scal\cdot u^2\,dv_g, \qquad \int_M u^{2^*}\,dv_g \;=\; 1,\; u>0, \tag{3.11}$$

where dv_g is the volume form on M associated to g. Note that according to [GV2] the extremals of the above variational problem are C^∞ functions, so we will not consider regularity questions here.

The Yamabe constant $\lambda(M,H)$ of a compact QC manifold is given, by definition, as the infimum of the Yamabe functional,

$$\lambda(M,H) \;\overset{def}{=}\; \inf\{\Upsilon(u)\,:\, \int_M u^{2^*}\,dv_g \;=\; 1,\; u>0\}.$$

If $\lambda(M,H)$ is less than the Yamabe constant $\lambda(S^{4n+3}, H^{can})$ of the standard QC sphere (cf. Section 3.2), the existence of solutions of the Yamabe equation is shown in [W]. The proof of this is a straightforward generalization of the argument known from [JL2] concerning the CR case. Thefore, it is only relevant to studdy the problem on manifolds with the same Yambe constant as the sphere.

It is worth pointing out that studying the Yamabe extremals in the sub-Riemannian geometry has applications to sharp inequalities in the Euclidean setting. For example, the extremals of some *Euclidean* Hardy-Sobolev inequalities involving the distance to a $n-k$ dimensional coordinate subspace of $\mathbb{R}^n$ have been determined in [**Va3**] by relating extremals on the Heisenberg groups to extremals in the Euclidean setting. In the particular case when $k=n$ one obtains the Caffarelli-Kohn-Nirenberg inequality (cf. [**CKN**]) for which the optimal constant was found in [**GY**].

CHAPTER 2

Geometry of quaternionic contact manifolds

In this chapter, we develop the differential geometry of quaternionic-contact manifolds with an emphasis on the QC Yamabe problem (cf. Section 3.4). The origin and the form of this problem are very similar to those arising in the classical theory concerning the Riemannian [**LP**] and CR [**JL1, JL2, JL3, JL4**] cases. Both the Riemannian and the CR Yamabe problems have been a very fruitful subject in geometry and analysis and have been completely sloved. An important step in the achieved solution was the understanding of the conformally flat case, given by the correponding Heisenberg group; in the Riemannian case, this is just $\mathbb{R}^n$ (0-dimensional center); in the CR case, it is the complex Heisenberg group (1-dimensional center), whereas here we are dealing with the quaternionic Heisenberg group (three dimensional center).

In general, the quaternionic-contact Yamabe problem is about the possibility of finding, in the natural conformal class $[g]$, associated to given QC manifold (M, H), a representative of constant QC scalar curvature. The question reduces to the solvability of a certain nonlinear differential equation known as the QC Yamabe equation. In fact, if taking the conformal factor in the form $\bar{\eta} = u^{1/(n+1)}\eta$, the QC Yamabe equation reduces to

$$4\frac{n+2}{n+1}\,\triangle u - \ u\,Scal \ = \ -u^{2^*-1}\,\overline{Scal},$$

where $\triangle$ is the horizontal sub-Laplacian, defined by 3.5, whereas $Scal$ and $\overline{Scal}$ are the QC scalar curvatures corresponding to the two contact forms η and $\bar{\eta}$ respectively; the number 2^* is given by $\frac{2n_h}{n_h-2}$, where $n_h = 4n+6$ is the so called homogeneous dimension of the problem. In the case of the quaternionic Heisenberg group, this is, up to scaling, the Euler-Lagrange equation describing the extremals in the L^2 Folland-Stein embedding theorem, cf. Section 3.1.

If the Yamabe constant

$$\lambda(M) = \lambda(M,H) \overset{def}{=} \inf\{\Upsilon(u)\ :\ \int_M u^{2^*}\,dv_g \ = \ 1,\ u>0\}$$

is strictly less than that of the standard QC sphere S^{4n+3} (cf. Section 3.4), the existence of solutions is shown in [W], see also [JL2]. Therefore, it is only relevant to study the problem on manifolds with the same Yambe constant as the sphere.

In this chapter, we provide a partial solution to the Yamabe problem on the standard QC sphere or, equivalently, on the quaternionic Heisenberg group. Let us observe that [GV2] solves the same problem in a more general setting, but under the assumption that the solution is invariant under a certain group of rotation. If one is on the flat models, i.e., the groups of Iwasawa type [CDKR1] the assumption in [GV2] is equivalent to the a-priori assumption that, up to a translation, the solution is radial with respect to the variables in the first layer. The proof goes on by using the moving plane method and showing that the solution is radial also in the variables from the center, after which a very non-trivial identity is used to determine all cylindrical solutions. In this chapter, the a-priori assumption is of a different nature (see further below) and the partial solution that we obtain here serves as an intermediate step for the results presented in the subsequent chapters. The strategy, following the steps of [LP] and [JL3], is to solve the Yamabe problem on the quaternionic sphere by replacing the non-linear Yamabe equation by an appropriate geometrical system of equations which can be solved.

Our first observation is that if $n > 1$ and the QC Ricci tensor is trace-free (QC Einstein condition) then the QC scalar curvature is constant (Theorem 5.9). Studying conformal deformation of QC structures preserving the QC Einstein condition, we describe explicitly all global functions on the quaternionic Heisenberg group that deform conformally the standard flat QC structure to another QC Einstein structure. Our main result here is the following Theorem.

Theorem A. *Let*

$$\Theta = \frac{1}{2h}\tilde{\Theta}$$

be a conformal deformation of the standard QC structure $\tilde{\Theta}$ on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. If Θ is also QC Einstein, then up to a left translation the function h is given by

$$h \;=\; c\,\Big[\big(1\;+\;\nu\,|q|^2\big)^2\;+\;\nu^2\,(x^2\;+\;y^2\;+\;z^2)\Big],$$

where c and ν are positive constants. All functions h of this form have this property.

The crucial fact which allows the reduction of the Yamabe equation to a system preserving the QC Einstein condition is Proposition 9.2 which asserts that, under some "extra" conditions, QC structures with constant QC scalar curvature obtained by conformal deformations of a QC Einstein metric on a compact manifold must be again QC Einstein. The prove of this relies on detailed analysis of the Bianchi identities for the Biquard connection. Using the quaternionic Cayley transform combined with Theorem A we obtain a partial solution for the QC Yamabe problem on the sphere:

THEOREM B. *Let $\eta = f\,\tilde\eta$ be a conformal deformation of the standard QC structure $\tilde\eta$ on the sphere S^{4n+3}. Suppose η has constant QC scalar curvature.* **If the vertical space of η is integrable** *then up to a multiplicative constant η is obtained from $\tilde\eta$ by a conformal quaternionic-contact automorphism. In the case $n>1$, the same conclusion holds when the function f is a real part of anti-CRF function.*

The definition of conformal quaternionic-contact automorpism can be found in Definition 8.6. The solutions (conformal factors) we find agree with those conjectured in [GV1]. The above theorem is only a partial solution to the problem because of the "extra" assumption (printed in bold in the theorem) about the integrability of the vertical space of η. As we shall see in Chapter 3 below this condition could actually be dropped, if the dimension is 7, but the argument for this is more involved.

Studying the geometry of the Biquard connection, our main geometrical tool towards understanding the geometry of the Yamabe equation, we show that the QC Einstein condition is equivalent to the vanishing of the torsion of the Biquard connection. In our third main result here, we give a local characterization of such spaces as 3-Sasakian manifolds:

THEOREM C. *Let (M^{4n+3},H,g) be a QC manifold with positive QC scalar curvature $Scal>0$, assumed to be constant if $n=1$. The next conditions are equivalent:*

a) (M^{4n+3},H,g) is a QC Einstein manifold.

b) M^{4n+3} is locally 3-Sasakian, i.e., locally there exists an $SO(3)$-matrix Ψ with smooth entries, such that, the local contact form $\frac{16n(n+2)}{Scal}\Psi\cdot\eta$ is 3-Sasakian.

c) The torsion of the Biquard connection is identically zero.

In particular, a QC Einstein manifold of positive QC scalar curvature, assumed in addition to be constant if $n=1$, is an Einstein manifold of positive Riemannian scalar curvature.

In addition, in Theorem 8.10, we show that the above conditions are equivalent to the property that every Reeb vector field (defined by (4.1)) is an infinitesimal generator of a conformal quaternionic-contact automorphism, cf. Definition 8.7.

Finally, we also develop useful tools necessary for the geometry and analysis on QC manifolds. We define and study some special functions, which will be relevant in the geometric analysis on quaternionic-contact and hypercomplex manifolds as well as properties of infinitesimal automorphisms of QC structures. In particular, the considered anti-regular functions will be relevant in the study of QC pseudo-Einstein structures, cf. Definition 7.1.

Organization of the chapter: In the first section (Section 4) of the chapter, we develop some important properties and formulae concerning the Biquard connection that will be important for the future investigations in the thesis.

In Section 5, we write explicitly the Bianchi identities and derive a system of equations satisfied by the divergences of some important tensors. As a result we are able to show that QC Einstein manifolds, i.e., manifolds for which the restriction to the horizontal space of the QC Ricci tensor is proportional to the metric, have constant scalar curvature, see Theorem 5.9. The proof uses Theorem 5.8 in which we derive a relation between the horizontal divergences of certain $Sp(n)Sp(1)$-invariant tensors. By introducing an integrability condition on the horizontal bundle we define hyperhermitian contact structures, see Definition 5.14, and with the help of Theorem 5.8 we prove Theorem C.

Section 6 describes the conformal transformations preserving the QC Einstein condition. Note that a conformal quaternionic contact transformation between two quaternionic-contact manifold is a diffeomorphism Φ which satisfies $\Phi^*\eta = \mu\ \Psi \cdot \eta$, for some positive smooth function μ and some matrix $\Psi \in SO(3)$ with smooth functions as entries; $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an element of $\mathbb{R}^3$. One defines in an obvious manner a point-wise conformal transformation. Let us note that the Biquard connection does not change under rotations as above, i.e., the Biquard connection of $\Psi \cdot \eta$ coincides with this of η. In particular, when studying conformal transformations we can consider only transformations with $\Phi^*\eta = \mu\, \eta$. We find all conformal transformations preserving the QC Einstein condition on the quaternionic Heisenberg group or, equivalently, on the QC sphere, and prove Theorem A.

Section 7 concerns a special class of functions, which we call anti-regular, defined respectively on the quaternionic space, real hyper-surface in it, or on a quaternionic-contact manifold, cf. Definitions 7.6 and 7.15, as functions preserving the quaternionic structure. The anti-regular functions play a role somewhat similar to those played by the CR functions, but the analogy is not complete. The real parts of such functions will be also of interest in connection with conformal transformation preserving the QC Einstein tensor and should be thought of as generalization of pluriharmonic functions. Let us stress explicitly that regular quaternionic functions have been studied extensively, see [S] and many subsequent papers, but they are not as relevant for the considered geometrical structures. Anti-regular functions on hyperkähler and quaternionic Kähler manifolds are studied in [**CL1**, **CL2**, **LZ**] in a different context, namely in connection with minimal surfaces and quaternionic maps between quaternionic Kähler manifolds. The notion of hypercomplex contact structures will appear in this section again since on such manifolds the real part of anti-CRF functions, see (7.18) for the definition, have some interesting properties, cf. Theorem 7.20

In Section 8, we study infinitesimal conformal automorphisms of QC structures (QC vector fields) and show that they depend on three functions satisfying some differential conditions thus establishing a '3-hamiltonian' form of the QC vector fields

(Proposition 8.8). The formula becomes very simple expression on a 3-Sasakian manifolds (Corollary 8.9). We characterize the vanishing of the torsion of Biquard connection in terms of the existence of three vertical vector fields whose flow preserves the metric and the quaternionic structure. Among them, 3-Sasakian manifolds are exactly those admitting three transversal QC vector fields (Theorem 8.10, Corollary 8.13).

In the last section (9) of the chapter, we complete the proof of Theorem B.

4. Torsion and curvature of the Biquard connection

CONVENTION 4.1. *In this chapter, we use the following conventions:*

- *The exterior derivative of a one form θ is given by*

$$d\theta(X,Y) \;=\; X\theta(Y) \;-\; Y\theta(X) \;-\; \theta([X,Y]).$$

- *We shall denote with ∇h the horizontal gradient of the function h, see (6.1), while dh means as usual the differential of the function h.*
- *The triple $\{i,j,k\}$ will denote a cyclic permutation of $\{1,2,3\}$, unless it is explicitly stated otherwise.*

Let (M,H,g) be a QC structure on a $4n+3$-dimensional smooth manifold. Working in a local chart, we fix the vertical space

$$V = \mathrm{span}\{\xi_1,\xi_2,\xi_3\}$$

by requiring the conditions

$$d\eta_s(\xi_t,X) = -d\eta_t(\xi_s,X) \quad s,t=1,2,3,\ X\in H, \tag{4.1}$$

i.e., ξ_1,ξ_2,ξ_3 are the Reeb vector fields correponding to an admisible set (η_s,I_s) of the QC structure.

The fundamental 2-forms $\omega_i, i=1,2,3$ of the quaternionic structure

$$Q = \mathrm{span}\{I_1,I_2,I_3\}$$

are defined by

$$2\omega_{i|H} \;=\; d\eta_{i|H}, \qquad \xi\lrcorner\omega_i = 0, \quad \xi\in V. \tag{4.2}$$

Define three 2-forms $\theta_i, i = 1,2,3$ by the formulas

$$\begin{aligned} \theta_i = & \frac{1}{2}\{d((\xi_j \lrcorner d\eta_k)_{|H}) + (\xi_i \lrcorner d\eta_j) \wedge (\xi_i \lrcorner d\eta_k)\}_{|H} \\ = & \frac{1}{2}\{d(\xi_j \lrcorner d\eta_k) + (\xi_i \lrcorner d\eta_j) \wedge (\xi_i \lrcorner d\eta_k)\}_{|H} - d\eta_k(\xi_j,\xi_k)\omega_k + d\eta_k(\xi_i,\xi_j)\omega_i. \end{aligned} \tag{4.3}$$

Define, in addition, the corresponding $(1,1)$ tensors A_i by

$$g(A_i(X),Y) = \theta_i(X,Y), X,Y \in H.$$

We recall (2.16), which we shall use to define the orthogonal projections to the horizontal and vertical spaces H and V, respectively.

4.1. The torsion tensor. Due to (4.2), the torsion restricted to H has the form

$$T(X,Y) = -[X,Y]_{|V} = 2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s, \qquad X,Y \in H. \tag{4.4}$$

The next two Lemmas provide some useful technical facts.

LEMMA 4.2. *Let D be any differentiation of the tensor algebra of H. Then we have the identities*

$$\begin{aligned} D(I_i)\, I_i \; &= \; -I_i\, D(I_i),\; i=1,2,3, \\ I_1\, D(I_1)^{-+-} \; = \; I_2\, D(I_2)^{+--}, \quad I_1\, D(I_1)^{-+-} \; &= \; I_2\, D(I_2)^{+--}, \\ I_1\, D(I_1)^{-+-} \; &= \; I_2\, D(I_2)^{+--}. \end{aligned}$$

PROOF. The proof is a straightforward consequence of the next identities

$$0 = I_2(D(I_1) - I_2D(I_1)I_2) + I_1(D(I_2) - I_1D(I_2)I_1) = I_2D(I_1)^{-+-} + I_1D(I_2)^{+--},$$
$$0 = D(-\mathrm{Id}_V) = D(I_iI_i) = D(I_i)I_i + I_iD(I_i).$$

∎

With $\mathcal{L}$ denoting the Lie derivative, we shall denote by $\mathcal{L}'$ its projection on the horizontal space, i.e.,

$$\mathcal{L}'_A(X) = [\mathcal{L}_A(X)]_H, \ A \in TM, \ X \in H.$$

LEMMA 4.3. *The following identities hold true.*

$$\mathcal{L}'_{\xi_1} I_1 = -2T^0_{\xi_1} I_1 + d\eta_1(\xi_1, \xi_2) I_2 + d\eta_1(\xi_1, \xi_3) I_3, \tag{4.5}$$

$$\begin{aligned} \mathcal{L}'_{\xi_1} I_2 &= -2T^{0\ --+}_{\xi_1} I_2 - 2I_3\tilde{u} + d\eta_1(\xi_2, \xi_1) I_1 \\ &+ \frac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) I_3, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \mathcal{L}'_{\xi_2} I_1 &= -2T^{0\ --+}_{\xi_2} I_1 + 2I_3\tilde{u} + d\eta_2(\xi_1, \xi_2) I_2 \\ &- \frac{1}{2}(-d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) I_3, \end{aligned} \tag{4.7}$$

where the symmetric endomorphism $\tilde{u}$ on H, commuting with I_1, I_2, I_3, is defined by

$$2\tilde{u} = I_3((\mathcal{L}'_{\xi_1} I_2)^{--+}) + \frac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) Id_H. \tag{4.8}$$

In addition, we have six more identities, which can be obtained with a cyclic permutation of (1,2,3).

PROOF. For all $k, l = 1, 2, 3$ we have

$$\mathcal{L}_{\xi_k} \omega_l(X, Y) = \mathcal{L}_{\xi_k} g(I_l X, Y) + g((\mathcal{L}_{\xi_k} I_l) X, Y). \tag{4.9}$$

Cartan's formula yields

$$\mathcal{L}_{\xi_k} \omega_l = \xi_k \lrcorner (d\omega_l) + d(\xi_k \lrcorner \omega_l). \tag{4.10}$$

A direct calculation using (4.2) gives

$$2\omega_l = (d\eta_l)_{|H} = d\eta_l - \sum_{s=1}^{3} \eta_s \wedge (\xi_s \lrcorner d\eta_l) + \sum_{1 \le s < t \le 3} d\eta_l(\xi_s, \xi_t) \eta_s \wedge \eta_t. \tag{4.11}$$

Combining (4.11) and (4.10) we obtain, after a short calculation, the following identities

$$(\mathcal{L}_{\xi_1}\omega_1)_{|H} = (d\eta_1(\xi_1,\xi_2)\omega_2 + d\eta_1(\xi_1,\xi_3)\omega_3)_{|H} \tag{4.12}$$

$$2(\mathcal{L}_{\xi_i}\omega_j)_{|H} = (d(\xi_i \lrcorner d\eta_j) - (\xi_i \lrcorner d\eta_k) \wedge (\xi_k \lrcorner d\eta_j))_{|H}, \tag{4.13}$$

where $i \neq j \neq k \neq i$ and $i,j,k \in \{1,2,3\}$. Clearly, (4.12) and (4.9) imply (4.5).

Furthermore, using (4.1) and (4.13) twice for $i=1, j=2$ and $i=2, j=1$, we find

$$\begin{aligned}(\mathcal{L}_{\xi_1}\omega_2 \;+\; \mathcal{L}_{\xi_2}\omega_1)_{|H} \;&=\; \frac{1}{2}(d(\xi_1 \lrcorner d\eta_2) \;+\; d(\xi_2 \lrcorner d\eta_1))_{|H}\\ &= d\eta_1(\xi_2,\xi_1)\omega_1 \;+\; d\eta_2(\xi_1,\xi_2)\omega_2 \;+\; (d\eta_1(\xi_2,\xi_3) \;+\; d\eta_2(\xi_1,\xi_3))\omega_3.\end{aligned} \tag{4.14}$$

On the other hand, (4.9) implies

$$\begin{aligned}&2T^0_{\xi_1}I_2 \;+\; \mathcal{L}'_{\xi_1}I_2 \;+\; 2T^0_{\xi_2}I_1 \;+\; \mathcal{L}'_{\xi_2}I_1\\ &\qquad = d\eta_1(\xi_2,\xi_1)I_1 \;+\; d\eta_2(\xi_1,\xi_2)I_2 \;+\; (d\eta_1(\xi_2,\xi_3) \;+\; d\eta_2(\xi_1,\xi_3))I_3.\end{aligned} \tag{4.15}$$

Let us decompose (4.15) into $Sp(n)$-invariant components:

$$\begin{aligned}(\mathcal{L}'_{\xi_1}I_2)^{+--} &= -2{T^0_{\xi_1}}^{--+}I_2 + d\eta_1(\xi_2,\xi_1)I_1,\\ (\mathcal{L}'_{\xi_2}I_1)^{-+-} &= -2{T^0_{\xi_2}}^{--+}I_1 + d\eta_2(\xi_1,\xi_2)I_2,\end{aligned} \tag{4.16}$$

$$\left(\mathcal{L}'_{\xi_1}I_2 + \mathcal{L}'_{\xi_2}I_1\right)^{--+} \;=\; (d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_1,\xi_3))I_3. \tag{4.17}$$

Using (4.17) and (4.8), we obtain

$$2\tilde{u} = -I_3((\mathcal{L}'_{\xi_2}I_1)^{--+}) + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))Id_H.$$

The latter, together with (4.8), tells us that $\tilde{u}$ commutes with all $I \in Q$. Now, Lemma 4.2 with $D = \mathcal{L}'$ implies (4.6) and (4.7). The vanishing of the symmetric part of the left hand side in (4.9) for $k = 1,\ l = 2$, combined with (4.19) and (4.6) yields

$$0 = -2g(I_3\tilde{u}X, Y) - 2g(I_3\tilde{u}Y, X).$$

As $\tilde{u}$ commutes with all $I \in Q$ we conclude that $\tilde{u}$ is symmetric.

The rest of the identities can be obtained through a cyclic permutation of (1,2,3). $\square$

We describe the properties of the quaternionic-contact torsion more precisely in the next Proposition.

PROPOSITION 4.4. *The torsion of the Biquard connection satisfies the identities:*

$$T_{\xi_i} = T^0_{\xi_i} + I_i u, \quad i = 1, 2, 3, \tag{4.18}$$

$$T^0_{\xi_i} = \frac{1}{2} L_{\xi_i} g, \quad i = 1, 2.3, \tag{4.19}$$

$$u = \tilde{u} - \frac{tr(\tilde{u})}{4n} Id_H, \tag{4.20}$$

where the symmetric endomorphism $\tilde{u}$ on H commuting with I_1, I_2, I_3 satisfies

$$\begin{aligned}
\tilde{u} \ &= \ \frac{1}{2} I_1 A_1^{+--} \ + \ \frac{1}{4}\big(- d\eta_1(\xi_2, \xi_3) \ + \ d\eta_2(\xi_3, \xi_1) \ + \ d\eta_3(\xi_1, \xi_2)\big)\, Id_H \\
&= \ \frac{1}{2} I_2 A_2^{-+-} \ + \ \frac{1}{4}\big(d\eta_1(\xi_2, \xi_3) \ - \ d\eta_2(\xi_3, \xi_1) \ + \ d\eta_3(\xi_1, \xi_2)\big)\, Id_H \\
&= \ \frac{1}{2} I_3 A_3^{--+} \ + \ \frac{1}{4}\big(d\eta_1(\xi_2, \xi_3) \ + \ d\eta_2(\xi_3, \xi_1) \ - \ d\eta_3(\xi_1, \xi_2)\big)\, Id_H.
\end{aligned} \tag{4.21}$$

For $n = 1$ the tensor $u = 0$ and $\tilde{u} = \frac{tr(\tilde{u})}{4} Id_H$.

PROOF. Expressing the Lie derivative in terms of the Biquard connection, using that ∇ preserves the splitting $H \oplus V$, we see that for $X, Y \in H$ we have

$$\mathcal{L}_{\xi_i} g(X, Y) = g(\nabla_X \xi_i, Y) + g(\nabla_Y \xi_i, X) + g(T_{\xi_i} X, Y) + g(T_{\xi_i} Y, X) = 2g(T^0_{\xi_i} X, Y).$$

To show that $\tilde{u}$ satisfies (4.21), insert (4.13) into (4.3) to get

$$\theta_3 = (\mathcal{L}_{\xi_1}\omega_2)_{|H} - d\eta_2(\xi_1,\xi_2)\omega_3 + d\eta_2(\xi_3,\xi_1)\omega_3. \tag{4.22}$$

A substitution of (4.9) and (4.6) in (4.22) gives

$$\begin{aligned} A_3 = 2T^{0\ -+-}_{\xi_1} I_2 - 2I_3\tilde{u} \\ + \ d\eta_1(\xi_2,\xi_1)I_1 - d\eta_2(\xi_1,\xi_2)I_2 \\ + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))I_3. \end{aligned} \tag{4.23}$$

Now, by comparing the $(+++)$ component on both sides of (4.23) we see the last equality of (4.21). The rest of the identities can be obtained with a cyclic permutation of (1,2,3).

Turning to the rest of the identities, let Σ^2 and Λ^2 denote, respectively, the subspaces of symmetric and skew-symmetric endomorphisms of H. Let

$$skew : End(H) \to \Lambda^2$$

be the natural projection with kernel Σ^2. We have

$$\begin{aligned} 4[T_{\xi_i}]_{(\Sigma^2\oplus sp(n))^\perp} &= 3skew(T_{\xi_i}) + I_1 skew(T_{\xi_i})I_1 + I_2 skew(T_{\xi_i})I_2 + I_3 skew(T_{\xi_i})I_3 \\ &= \sum_{s=1}^{3}(skew(T_{\xi_i}) + I_s skew(T_{\xi_i})I_s). \end{aligned}$$

According to Theorem 2.4, $T_\xi X \in H$ for $X \in H, \xi \in V$. Hence,

$$T(\xi,X) = \nabla_\xi X - [\xi,X]_H = \nabla_\xi X - \mathcal{L}'_\xi(X). \tag{4.24}$$

An application of (4.24) gives

$$g(4[T_{\xi_i}]_{(\Sigma^2\oplus sp(n))^\perp}X,Y) = -\sum_{s=1}^{3} g\big((\nabla_{\xi_i}I_s)X, I_sY\big) + \frac{1}{2}\sum_{s=1}^{3}\{g\big((\mathcal{L}_{\xi_i}I_s)X, I_sY\big) - g((\mathcal{L}_{\xi_i}I_s)Y, I_sX)\}. \tag{4.25}$$

Let $B(H)$ be the orthogonal complement of

$$\Sigma^2 \oplus sp(n) \oplus sp(1) \subset \mathrm{End}(H),$$

i.e.,

$$\mathrm{End}(H) = \Sigma^2 \oplus sp(n) \oplus sp(1) \oplus B(H). \tag{4.26}$$

If Ψ is an arbitrary section of the bundle Λ^2 of M, the orthogonal projection of Ψ into $B(H)$ is given by

$$[\Psi]_{B(H)} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+} - [\Psi]_{sp(1)},$$

where $[\Psi]_{sp(1)}$ is the orthogonal projection of Ψ onto $sp(1)$. We also have

$$[\Psi]_{sp(1)} = \frac{1}{4n}\sum_{s=1}^{3}\sum_{a=1}^{4n} g(\Psi e_a, I_s e_a) I_s.$$

Theorem 2.4 - (iv) and the decomposition (4.26) yield

$$T_{\xi_i} = [T_{\xi_i}]_{(sp(n)\oplus sp(1))^\perp} = [T_{\xi_i}]_{\Sigma^2} + [T_{\xi_i}]_{B(H)} = T^0_{\xi_i} + [T_{\xi_i}]_{(\Sigma^2\oplus sp(n))^\perp} - [T_{\xi_i}]_{sp(1)}. \tag{4.27}$$

Using (4.25), Lemma 4.3 and the fact that $I_s(\nabla_{\xi_i} I_s) \in sp(1)$, we compute

$$
\begin{aligned}
(4.28)\quad & 4[T_{\xi_i}]_{(\Sigma^2 \oplus sp(n))^\perp} \; - \; [T_{\xi_i}]_{sp(1)} \\
&= -\sum_{s=1}^{3} \Big\{ skew(I_s(\mathcal{L}'_{\xi_i} I_s)) \; - \; [I_s(\mathcal{L}'_{\xi_i} I_s)]_{sp(1)} \Big\} \; = \; \sum_{s=1}^{3} skew(2I_s T^0_{\xi_i} I_s) + 4u \; = \; 4u.
\end{aligned}
$$

A substitution of (4.28) in (4.27) completes the proof. □

The $Sp(n)$-invariant splitting of (4.23) leads to the following Corollary.

COROLLARY 4.5. *The (1,1)-tensors A_i satisfy the equalities*

$$
\begin{aligned}
A_3^{+++} &= 2{T^0_{\xi_1}}^{-+-} I_2, \quad A_3^{+--} = d\eta_1(\xi_2,\xi_1) I_1, \quad A_3^{-+-} = -d\eta_2(\xi_1,\xi_2) I_2, \\
A_3^{--+} &= -2I_3\tilde{u} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)) I_3.
\end{aligned}
$$

Analogous formulas for A_1 and A_2 can be obtained by a cyclic permutation of $(1,2,3)$.

PROPOSITION 4.6. *The covariant derivative of the quaternionic-contact structure with respect to the Biquard connection is given by*

$$
\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \tag{4.29}
$$

where the $sp(1)$-connection 1-forms α_s are determined by

$$
\alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V, \tag{4.30}
$$

$$
\begin{aligned}
(4.31)\quad \alpha_i(\xi_s) &= d\eta_s(\xi_j,\xi_k) \\
&\quad - \delta_{is}\left(\frac{tr(\tilde{u})}{2n} + \frac{1}{2}\left(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)\right)\right),
\end{aligned}
$$

for $s = 1,2,3$ and (i,j,k) any cyclic permutation of $(1,2,3)$.

PROOF. The equality (4.30) is proved by Biquard in [Biq]. Using (4.24), we obtain

$$
\nabla_{\xi_s} I_i = [T_{\xi_s}, I_i] + \mathcal{L}'_{\xi_s} I_i = [T^0_{\xi_s}, I_i] + u[I_s, I_i] + \mathcal{L}'_{\xi_s} I_i.
$$

An application of Lemma 4.3 completes the proof. $\square$

COROLLARY 4.7. *The covariant derivative of the distribution V is given by*

$$\nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j.$$

We finish this section by expressing the Biquard connection in terms of the Levi-Civita connection D^g of the metric g, namely, we have

$$\nabla_B Y = D^g_B Y + \sum_{s=1}^{3} \{((D^g_B \eta_s)Y)\xi_s + \eta_s(B)(I_s u - I_s)Y\}, \quad B \in TM, \quad Y \in H. \tag{4.32}$$

Indeed, for $B = X \in H$ formula (4.32) follows from the equation

$$\nabla_X Y = [D^g_X Y]_H.$$

If $B \in V$ we may assume $B = \xi_1$ and for $Z \in H$ we compute

$$\begin{aligned} 2g(D^g_{\xi_1} Y, Z) &= \xi_1 g(Y, Z) + g([\xi_1, Y], Z) - g([\xi_1, Z], Y) - g([Y, Z], \xi_1) = \\ &= (\mathcal{L}_{\xi_1} g)(Y, Z) + 2g([\xi_1, Y], Z) + d\eta_1(Y, Z) = 2g(T_{\xi_1} Y + [\xi_1, Y], Z) \\ &- 2g(I_1 u Y, Z) + 2g(I_1 Y, Z) = 2g(\nabla_{\xi_1} Y, Z) - 2g((I_1 u - I_1)Y, Z). \end{aligned}$$

In the above calculation we used (4.24) and Proposition 4.4.

Note that the covariant derivatives $\nabla_B \xi_s$ are also determined by (4.32) in view of the relation

$$g(\nabla_B \xi_s, \xi_k) = \frac{1}{4n} g(\nabla_B I_s, I_k), \quad s, k = 1, 2, 3$$

4.2. The Curvature Tensor. Let

$$R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$$

be the curvature tensor of ∇. For any $B, C \in \Gamma(TM)$ the curvature operator R_{BC} preserves the QC structure on M since ∇ preserves it. In particular R_{BC} preserves the distributions H and V, the quaternionic structure Q on H and the $(2,1)$ tensor

φ. Moreover, the action of R_{BC} on V is completely determined by its action on H,

$$R_{BC}\xi_i = \varphi^{-1}([R_{BC}, I_i]), \quad i = 1, 2, 3.$$

Thus, we may regard R_{BC} as an endomorphism of H and we have

$$R_{BC} \in sp(n) \oplus sp(1).$$

DEFINITION 4.8. *The Ricci 2-forms ρ_i are defined by*

$$\rho_i(B, C) = \frac{1}{4n} \sum_{a=1}^{4n} g(R(B, C)e_a, I_i e_a), \;\; B, C \in \Gamma(TM).$$

Hereafter $\{e_1, \dots, e_{4n}\}$ will denote an orthonormal basis of H. We decompose the curvature into $sp(n) \oplus sp(1)$-parts. Let $R^0_{BC} \in sp(n)$ denote the $sp(n)$-component.

LEMMA 4.9. *The curvature of the Biquard connection decomposes as follows*

$$R_{BC} = R^0_{BC} + \rho_1(B, C)I_1 + \rho_2(B, C)I_2 + \rho_3(B, C)I_3.$$

$$[R_{BC}, I_i] = 2(-\rho_j(B, C)I_k + \rho_k(B, C)I_j), \;\; B, C \in \Gamma(TM), \tag{4.33}$$

$$\rho_i = \frac{1}{2}(d\alpha_i + \alpha_j \wedge \alpha_k), \tag{4.34}$$

where the connection 1-forms α_s are determined in (4.30), (4.31).

PROOF. The first two identities follow directly from the definitions. Using (4.29), we calculate

$$\begin{aligned}[R_{BC}, I_1] &= \nabla_B(\alpha_3(C)I_2 - \alpha_2(C)I_3) - \nabla_C(\alpha_3(B)I_2 - \alpha_2(B)I_3) - (\alpha_3([B, C])I_2 \\ &\quad - \alpha_2([B, C])I_3) = -(d\alpha_2 + \alpha_3 \wedge \alpha_1)(B, C)I_3 + (d\alpha_3 + \alpha_1 \wedge \alpha_2)(B, C)I_2.\end{aligned}$$

Now (4.33) completes the proof. $\square$

Definition 4.10. *The quaternionic-contact Ricci tensor (*QC Ricci tensor *for short) and the QC scalar curvature Scal of the Biquard connection are defined by*

$$Ric(B,C)=\sum_{a=1}^{4n}g(R(e_a,B)C,e_a),\qquad Scal=\sum_{a=1}^{4n}Ric(e_a,e_a). \tag{4.35}$$

It is known, cf. [Biq], that the QC Ricci tensor restricted to H is a symmetric tensor. In addition, we define six Ricci-type tensors ζ_i, τ_i, $i=1,2,3$ as follows

$$\begin{aligned}\zeta_i(B,C)&=\frac{1}{4n}\sum_{a=1}^{4n}g(R(e_a,B)C,I_ie_a),\\ \tau_i(B,C)&=\frac{1}{4n}\sum_{a=1}^{4n}g(R(e_a,I_ie_a)B,C).\end{aligned} \tag{4.36}$$

We shall show that all Ricci-type contractions evaluated on the horizontal space H are determined by the components of the torsion. First, define the following 2-tensors on H using the tensors from Proposition 4.4

$$\begin{aligned}T^0(X,Y)&\overset{def}{=}g((T^0_{\xi_1}I_1+T^0_{\xi_2}I_2+T^0_{\xi_3}I_3)X,Y),\\ U(X,Y)&\overset{def}{=}g(uX,Y),\quad X,Y\in H.\end{aligned} \tag{4.37}$$

Lemma 4.11. *The tensors T^0 and U are $Sp(n)Sp(1)$-invariant trace-free symmetric tensors with the properties:*

$$T^0(X,Y)+T^0(I_1X,I_1Y)+T^0(I_2X,I_2Y)+T^0(I_3X,I_3Y)=0, \tag{4.38}$$

$$3U(X,Y)-U(I_1X,I_1Y)-U(I_2X,I_2Y)-U(I_3X,I_3Y)=0. \tag{4.39}$$

Proof. The lemma follows directly from (2.19), (2.21) of Proposition 2.5. □

We turn to a Lemma, which shall be used later.

LEMMA 4.12. *For any $X, Y \in H$, $\quad B \in H \oplus V$, we have*

$$Ric(B, I_iY) + 4n\zeta_i(B, Y) \; = \; 2\rho_j(B, I_kY) - 2\rho_k(B, I_jY), \tag{4.40}$$

$$\begin{aligned} \zeta_i(X, Y) \; &= \; -\frac{1}{2}\rho_i(X, Y) + \frac{1}{2n}g(I_iuX, Y) + \frac{2n-1}{2n}g(T^0_{\xi_i}X, Y) \\ &+ \; \frac{1}{2n}g(I_jT^0_{\xi_k}X, Y) - \frac{1}{2n}g(I_kT^0_{\xi_j}X, Y). \end{aligned} \tag{4.41}$$

The Ricci 2-forms evaluated on H satisfy

$$\begin{aligned} \rho_1(X, Y) \; &= \; 2g(T^{0--+}_{\xi_2}I_3X, Y) - 2g(I_1uX, Y) - \frac{tr(\tilde{u})}{n}\omega_1(X, Y), \\ \rho_2(X, Y) \; &= \; 2g(T^{0+--}_{\xi_3}I_1X, Y) - 2g(I_2uX, Y) - \frac{tr(\tilde{u})}{n}\omega_2(X, Y), \\ \rho_3(X, Y) \; &= \; 2g(T^{0-+-}_{\xi_1}I_2X, Y) - 2g(I_3uX, Y) - \frac{tr(\tilde{u})}{n}\omega_3(X, Y). \end{aligned} \tag{4.42}$$

The 2-forms τ_s evaluated on H satisfy

$$\begin{aligned} \tau_1(X, Y) \; &= \; \rho_1(X, Y) + 2g(I_1uX, Y) + \frac{4}{n}g(T^{0--+}_{\xi_2}I_3X, Y), \\ \tau_2(X, Y) \; &= \; \rho_2(X, Y) + 2g(I_2X, Y) + \frac{4}{n}g(T^{0+--}_{\xi_3}I_1X, Y), \\ \tau_3(X, Y) \; &= \; \rho_3(X, Y) + 2g(I_3X, Y) + \frac{4}{n}g(T^{0-+-}_{\xi_1}I_2X, Y). \end{aligned} \tag{4.43}$$

For $n = 1$ the above formulas hold with $U = 0$.

PROOF. From (4.33) we have

$$Ric(B, I_1Y) \ + \ 4n\zeta_1(B, Y) \ = \ \sum_{a=1}^{4n}\{R(e_a, B, I_1Y, e_a) \ + \ R(e_a, B, Y, I_1e_a)\}$$

$$= \ \sum_{a=1}^{4n}\{-2\rho_2(e_a, B)\omega_3(Y, e_a) \ + \ 2\rho_3(e_a, B)\omega_2(Y, e_a))\}$$

$$= \ 2\rho_2(B, I_3Y) \ - \ 2\rho_3(B, I_2Y),$$

Using (4.3) and (4.34) we obtain

$$\rho_1(X, Y) = A_1(X, Y) - \frac{1}{2}\alpha_1([X, Y]_V) = A_1(X, Y) + \sum_{s=1}^{3}\omega_s(X, Y)\alpha_1(\xi_s).$$

Now, Corollary 4.5 and Corollary 4.6 imply the first equality in (4.42). The other two equalities in (4.42) can be obtained in the same manner.

Letting

$$b(X, Y, Z, W) \ = \ 2\sigma_{X,Y,Z}\{\sum_{l=1}^{3}\omega_l(X, Y)g(T_{\xi_l}Z, W)\},$$

where $\sigma_{X,Y,Z}$ is the cyclic sum over X, Y, Z, we have

$$\sum_{a=1}^{4n} b(X, Y, e_a, I_1e_a) \ = \ 4g(I_1uX, Y) + 8g(I_2T^{0-+-}_{\xi_3}X, Y), \tag{4.44}$$

$$\sum_{a=1}^{4n} b(e_a, I_1e_a, X, Y) \ = \ (8n-4)g(T^0_{\xi_1}X, Y) + (8n+4)g(I_1uX, Y) \tag{4.45}$$

$$+ \ 4g(T^0_{\xi_2}I_3X, Y) - 4g(T^0_{\xi_3}I_2X, Y).$$

The first Bianchi identity gives

$$4n\left(\tau_1(X,Y)+2\zeta_1(X,Y)\right) = \sum_{a=1}^{4n}\{R(e_a,I_1e_a,X,Y)+R(X,e_a,I_1e_a,Y)+R(I_1e_a,X,e_a,Y)\} = \sum_{a=1}^{4n} b(e_a,I_1e_a,X,Y) \tag{4.46}$$

and also

$$4n(\tau_1(X,Y)-\rho_1(X,Y)) = \sum_{a=1}^{4n}\{R(e_a,I_1e_a,X,Y)-R(X,Y,e_a,I_1e_a)\} \tag{4.47}$$

$$= \frac{1}{2}\sum_{a=1}^{4n}\{b(e_a,I_1e_a,X,Y)-b(e_a,I_1e_a,Y,X)-b(e_a,X,Y,I_1e_a)+b(I_1e_a,X,Y,e_a)\}.$$

Taking into account (4.44), (4.45), (10.20) and (4.47) yield the first set of equalities in (4.43) and (4.41). The other equalities in (4.43) and (4.41) can be shown similarly. This completes the proof of Lemma 4.12. □

THEOREM 4.13. *Let* (M^{4n+3},H,g) *be a quaternionic-contact* $(4n+3)$*-dimensional manifold,* $n>1$*. For any* $X,Y\in H$ *the QC Ricci tensor and the QC scalar curvature satisfy*

$$\begin{aligned} Ric(X,Y) &= (2n+2)T^0(X,Y)+(4n+10)U(X,Y) \\ &\qquad + (2n+4)\frac{tr(\tilde u)}{n}g(X,Y), \\ Scal &= (8n+16)tr(\tilde u). \end{aligned} \tag{4.48}$$

For $n=1$*, we have*

$$Ric(X,Y) = 4T^0(X,Y)+6\frac{tr(\tilde u)}{n}g(X,Y).$$

PROOF. The proof follows from Lemma 4.12, (4.41), (4.42) and (4.40). If $n=1$, recall $U=0$ to obtain the last equality. □

COROLLARY 4.14. *The QC scalar curvature satisfies the equalities*

$$\frac{Scal}{2(n+2)} = \sum_{a=1}^{4n} \rho_i(I_i e_a, e_a) = \sum_{a=1}^{4n} \tau_i(I_i e_a, e_a) = -2\sum_{a=1}^{4n} \zeta_i(I_i e_a, e_a), \quad i=1,2,3.$$

We determine the function λ in (2.17) in the next Corollary.

COROLLARY 4.15. *The torsion of the Biquard connection restricted to V satisfies the equality*

$$T(\xi_i,\xi_j) = -\frac{Scal}{8n(n+2)}\xi_k - [\xi_i,\xi_j]_H. \tag{4.49}$$

PROOF. A small calculation using Corollary 4.7 and Proposition 4.6, gives

$$T(\xi_i,\xi_j) = \nabla_{\xi_i}\xi_j - \nabla_{\xi_j}\xi_i - [\xi_i,\xi_j] = -\frac{tr(\tilde{u})}{n}\xi_k - [\xi_i,\xi_j]_H.$$

Now, the assertion follows from the second equality in (4.48). □

COROLLARY 4.16. *The tensors $T^0, U, \tilde{u}$ do not depend on the choice of the local basis.*

5. QC Einstein quaternionic-contact structures

The goal of this section is to show that the vanishing of the torsion of the quaternionic-contact structure implies that the QC scalar curvature is constant and to prove our classification Theorem C (formulated at the begining of this chapter). The Bianchi identities will have an important role in the analysis here.

DEFINITION 5.1. *A quaternionic-contact structure is* QC Einstein *if the QC Ricci tensor is trace-free,*

$$Ric(X,Y) = \frac{Scal}{4n} g(X,Y), \quad X,Y \in H.$$

PROPOSITION 5.2. *A quaternionic-contact manifold (M,H,g) is a QC Einstein if and only if the quaternionic-contact torsion vanishes identically, $T_\xi = 0, \xi \in V$.*

Proof. If (η, Q) is QC Einstein structure then

$$T^0 = U = 0$$

because of (4.48). We will use the same symbol T^0 for the corresponding endomorphism of the 2-tensor T^0 on H. According to (4.37), we have

$$T^0 = T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3.$$

Using first (2.19) and then (2.20), we compute

$$(T^0)^{+--} = (T^0_{\xi_2})^{--+} I_2 + (T^0_{\xi_3})^{-+-} I_3 = 2(T^0_{\xi_2})^{--+} I_2. \tag{5.1}$$

Hence, $T_{\xi_2} = T^0_{\xi_2} + I_2 u$ vanishes. Similarly $T_{\xi_1} = T_{\xi_3} = 0$. The converse follows from (4.48). ☐

Proposition 5.3. *For $X \in V$ and any cyclic permutation (i, j, k) of $(1, 2, 3)$ we have*

$$\rho_i(X, \xi_i) = -\frac{X(Scal)}{32n(n+2)} + \frac{1}{2}(\omega_i([\xi_j, \xi_k], X) - \omega_j([\xi_k, \xi_i], X) - \omega_k([\xi_i, \xi_j], X)), \tag{5.2}$$

$$\rho_i(X, \xi_j) = \omega_j([\xi_j, \xi_k], X), \quad \rho_i(X, \xi_k) = \omega_k([\xi_j, \xi_k], X), \tag{5.3}$$

$$\rho_i(I_k X, \xi_j) = -\rho_i(I_j X, \xi_k) = g(T(\xi_j, \xi_k), I_i X) = \omega_i([\xi_j, \xi_k], X). \tag{5.4}$$

Proof. Since ∇ preserves the splitting $H \oplus V$, the first Bianchi identity, (4.49) and (4.33) imply

$$\begin{aligned} 2\rho_i(X, \xi_i) + 2\rho_j(X, \xi_j) &= g(R(X, \xi_i)\xi_j, \xi_k) + g(R(\xi_j, X)\xi_i, \xi_k) \\ &= \sigma_{\xi_i, \xi_j, X}\{g((\nabla_{\xi_i} T)(\xi_j, X), \xi_k) + g(T(T(\xi_i, \xi_j), X), \xi_k)\} \\ = g((\nabla_X T)(\xi_i, \xi_j), \xi_k) + g(T(T(\xi_i, \xi_j), X), \xi_k) &= -\frac{X(Scal)}{8n(n+2)} - 2\omega_k([\xi_i, \xi_j], X). \end{aligned} \tag{5.5}$$

Summing the first two equalities in (5.5) and subtracting the third one, we obtain (5.2). Similarly,

$$
\begin{aligned}
2\rho_k(\xi_j, X) = g(R(\xi_j, X)\xi_i, \xi_j) &= \sigma_{\xi_i,\xi_j,X}\{g((\nabla_{\xi_i}T)(\xi_j, X), \xi_j) + g(T(T(\xi_i, \xi_j), X), \xi_j)\} \\
&= g(T(T(\xi_i, \xi_j), X), \xi_j) = g(T(-[\xi_i, \xi_j]_H, X), \xi_j) = g([[\xi_i, \xi_j]_H, X], \xi_j) \\
&= -d\eta_j([\xi_i, \xi_j]_H, X) = -2\omega_j([\xi_i, \xi_j], X).
\end{aligned}
$$

Hence, the second equality in (5.3) follows. Analogous calculations show the validity of the first equality in (5.3). Then, (5.4) is a consequence of (5.3) and (4.49). □

The vertical derivative of the QC scalar curvature is determined in the next Proposition.

PROPOSITION 5.4. *On a QC manifold we have*

$$
\rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j) = \frac{1}{16n(n+2)}\xi_j(Scal). \tag{5.6}
$$

PROOF. Since ∇ preserves the splitting $H \oplus V$, the first Bianchi identity and (4.49) imply

$$
\begin{aligned}
&-2(\rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j)) = g(\sigma_{\xi_i,\xi_j,\xi_k}\{R(\xi_i, \xi_j)\xi_k\}, \xi_j) \\
&\qquad = g(\sigma_{\xi_i,\xi_j,\xi_k}\{(\nabla_{\xi_i}T)(\xi_j, \xi_k) + T(T(\xi_i, \xi_j), \xi_k)\}, \xi_j) = -\frac{1}{8n(n+2)}\xi_j(Scal).
\end{aligned}
$$

□

5.1. The Bianchi identities. In order to derive the essential information contained in the Bianchi identities we need the next Lemma, which is an application of a standard result in differential geometry.

LEMMA 5.5. *In a neighborhood of any point $p \in M^{4n+3}$ and a Q-orthonormal basis*

$$
\{X_1(p), X_2(p) = I_1X_1(p) \dots, X_{4n}(p) = I_3X_{4n-3}(p), \xi_1(p), \xi_2(p), \xi_3(p)\}
$$

of the tangential space at p, there exists a Q - orthonormal frame field

$$\{X_1, X_2 = I_1X_1, \dots, X_{4n} = I_3X_{4n-3}, \xi_1, \xi_2, \xi_3\},$$
$$X_{a_{|p}} = X_a(p), \xi_{s_{|p}} = \xi_s(p), a = 1, \dots, 4n, i = 1, 2, 3,$$

such that the connection 1-forms of the Biquard connection are all zero at the point p, i.e., we have

$$(\nabla_{X_a}X_b)_{|}p = (\nabla_{\xi_i}X_b)_{|}p = (\nabla_{X_a}\xi_t)_{|}p = (\nabla_{\xi_t}\xi_s)_{|}p = 0, \tag{5.7}$$

for $a, b = 1, \dots, 4n, s, t, r = 1, 2, 3$. *In particular,*

$$((\nabla_{X_a}I_s)X_b)_{|}p = ((\nabla_{X_a}I_s)\xi_t)_{|}p = ((\nabla_{\xi_t}I_s)X_b)_{|}p = ((\nabla_{\xi_t}I_s)\xi_r)_{|}p = 0.$$

Proof. Since ∇ preserves the splitting $H \oplus V$ we can apply the standard arguments for the existence of a normal frame with respect to a metric connection (see e.g. [**Wu**]). We sketch the proof for completeness.

Let

$$\{\tilde{X}_1, \dots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}$$

be a orthonormal basis around p such that $\tilde{X}_{a_{|p}} = X_a(p)$, $\quad \tilde{\xi}_{i_{|p}} = \xi_i(p)$. We want to find a modified frame $X_a = o_a^b\tilde{X}_b$, $\quad \xi_i = o_i^j\tilde{\xi}_j$, which satisfies the normality conditions of the lemma.

Let ϖ be the $sp(n) \oplus sp(1)$-valued connection 1-forms with respect to the frame $\{\tilde{X}_1, \dots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}$, i.e.,

$$\nabla\tilde{X}_b = \varpi_b^c\tilde{X}_c, \quad \nabla\tilde{\xi}_s = \varpi_s^t\tilde{\xi}_t, \quad B \in \{\tilde{X}_1, \dots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}.$$

Let $\{x^1, \dots, x^{4n+3}\}$ be a coordinate system around p such that

$$\frac{\partial}{\partial x^a}(p) = X_a(p), \quad \frac{\partial}{\partial x^{4n+t}}(p) = \xi_t(p), \quad a = 1, \dots, 4n, \quad t = 1, 2, 3.$$

One can easily check that the matrices with entries

$$o_a^b = exp\left(-\sum_{c=1}^{4n+3} \varpi_a^b(\frac{\partial}{\partial x^c})_{|p} x^c \right) \in Sp(n),$$

$$o_t^s = exp\left(-\sum_{c=1}^{4n+3} \varpi_t^s(\frac{\partial}{\partial x^c})_{|p} x^c \right) \in Sp(1)$$

are the desired matrices making the identities (5.7) true.

Now, the last identity in the lemma is a consequence of the fact that the choice of the orthonormal basis of V does not depend on the action of $SO(3)$ on V combined with Corollary 4.7 and Proposition 4.6. □

Definition 5.6. *We refer to the orthonormal frame constructed in Lemma 5.5 as a* QC normal frame*.*

Let us fix a QC normal frame

$$\{e_1, \dots, e_{4n}, \xi_1, \xi_2, \xi_3\}.$$

We shall denote with X, Y, Z horizontal vector fields $X, Y, Z \in H$ and keep the notation for the torsion of type (0,3)

$$T(B, C, D) = g(T(B, C), D), B, C, D \in H \oplus V.$$

Proposition 5.7. *On a quaternionic-contact manifold (M^{4n+3}, H, g) the following identities hold true*

$$2\sum_{a=1}^{4n}(\nabla_{e_a} Ric)(e_a, X) - X(Scal) \; = \; 4\sum_{r=1}^{3} Ric(\xi_r, I_r X) - 8n\sum_{r=1}^{3}\rho_r(\xi_r, X), \tag{5.8}$$

$$Ric(\xi_s, I_s X) \; = \; 2\rho_q(I_t X, \xi_s) + 2\rho_t(I_s X, \xi_q) + \sum_{a=1}^{4n}(\nabla_{e_a} T)(\xi_s, I_s X, e_a), \tag{5.9}$$

$$
\begin{aligned}
(5.10)\qquad & 4n(\rho_s(X,\xi_s)-\zeta_s(\xi_s,X)) \\
&\qquad = 2\rho_q(I_tX,\xi_s)+2\rho_t(I_sX,\xi_q)-\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s,X,I_se_a),
\end{aligned}
$$

$$
(5.11)\qquad \zeta_s(\xi_s,X) \;=\; -\frac{1}{4n}\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s,I_sX,e_a),
$$

where $s\in\{1,2,3\}$ *is fixed and* (s,t,q) *is an even permutation of* $(1,2,3)$.

PROOF. The second Bianchi identity implies

$$
\begin{aligned}
2\sum_{a=1}^{4n}(\nabla_{e_a}Ric)(e_a,X)-X(Scal)+2\sum_{a=1}^{4n}Ric(T(e_a,X),e_a) \\
+\sum_{a,b=1}^{4n}R(T(e_b,e_a),X,e_b,e_a)=0.
\end{aligned}
$$

An application of (4.4) to the last equality gives (5.8).

The first Bianchi identity combined with (2.18), (4.4) and the fact that ∇ preserves the orthogonal splitting $H\oplus V$ yield

$$
\begin{aligned}
Ric(\xi_s,I_sX)&=\sum_{a=1}^{4n}\left((\nabla_{e_a}T)(\xi_s,I_sX,e_a)+2\sum_{r=1}^{3}\omega_r(I_sX,e_a)T(\xi_r,\xi_s,e_a)\right)= \\
&=\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s,I_sX,e_a)+2T(\xi_s,\xi_t,I_qX)+2T(\xi_q,\xi_s,I_tX),
\end{aligned}
$$

which, together with (5.4), completes the proof of (5.9).

In a similar fashion, from the first Bianchi identity, (2.18), (4.4) and the fact that ∇ preserves the orthogonal splitting $H\oplus V$ we can obtain the proof of (5.10). Finally, take (4.40) with $B=\xi_i$ and combine the result with (5.9) to get (5.11). □

The following Theorem gives relations between $Sp(n)Sp(1)$-invariant tensors and is crucial for the solution of the Yamabe problem, which we shall undertake in the last Section. We define the horizontal divergence ∇^*P of a (0,2)-tensor field P with respect to Biquard connection to be the (0,1)-tensor defined by

$$\nabla^* P(.) = \sum_{a=1}^{4n} (\nabla_{e_a} P)(e_a, .),$$

where $e_a, a = 1, \dots, 4n$ is an orthonormal basis on H.

THEOREM 5.8. *The horizontal divergences of the curvature and torsion tensors satisfy the system* $B\, b \;=\; 0$, *where*

$$\mathbf{B} \;=\; \begin{pmatrix} -1 & 6 & 4n-1 & \frac{3}{16n(n+2)} & 0 \\ -1 & 0 & n+2 & \frac{3}{16(n+2)} & 0 \\ 1 & -3 & 4 & 0 & -1 \end{pmatrix},$$

$$\mathbf{b} \;=\; \Big(\;\nabla^* T^0, \quad \nabla^* U, \quad \mathbb{A}, \quad d\, Scal\,|_H, \quad \textstyle\sum_{j=1}^3 Ric\;(\xi_j, I_j\,.\,)\; \Big)^t,$$

with T^0 *and* U *defined in* (4.37) *and*

$$\mathbb{A}(X) = g(I_1[\xi_2, \xi_3] + I_2[\xi_3, \xi_1] + I_3[\xi_1, \xi_2], X).$$

PROOF. Throughout the proof of Theorem 5.8 (s, t, q) will denote an even permutation of $(1, 2, 3)$. Equations (5.2) and (5.4) yield

$$\sum_{r=1}^{3} \rho_r(X, \xi_r) = -\frac{3}{32n(n+2)} X(Scal) - \frac{1}{2}\mathbb{A}(X), \tag{5.12}$$

$$\sum_{s=1}^{3} \rho_q(I_t X, \xi_s) = \mathbb{A}(X). \tag{5.13}$$

Using the properties of the torsion described in Proposition 4.4 and (2.19), we obtain

$$\sum_{s=1}^{3}\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s, I_sX, e_a) = \nabla^*T^0(X) - 3\nabla^*U(X), \tag{5.14}$$

$$\sum_{s=1}^{3}\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s, X, I_se_a) = \nabla^*T^0(X) + 3\nabla^*U(X). \tag{5.15}$$

Substituting (5.13) and (5.14) in the sum of (5.9) written for $s = 1, 2, 3$, we obtain the third row of the system. The second row can be obtained by inserting (5.11) into (5.10), taking the sum over $s = 1, 2, 3$ and applying (5.12), (5.13), (5.14), (5.15).

The second Bianchi identity and applications of (4.4) give

$$\begin{aligned}\sum_{s=1}^{3}\Bigg(\sum_{a=1}^{4n}\Big[(\nabla_{e_a}Ric)(I_sX, I_se_a) + 4n(\nabla_{e_a}\zeta_s)(I_sX, e_a) \Big]\Bigg) \\ - 2\sum_{s=1}^{3}(Ric(\xi_s, I_sX) + 8n\zeta_s(\xi_s, X)) \\ + 8n\sum_{s=1}^{3}\Big[\zeta_s(\xi_t, I_qX) - \zeta_s(\xi_q, I_tX) - \rho_s(\xi_t, I_qX) + \rho_s(\xi_q, I_tX)\Big] = 0.\end{aligned} \tag{5.16}$$

Using (4.40), (4.42) as well as (2.19),(2.20) and (5.1) we obtain the next three identities

$$\begin{aligned}\sum_{s=1}^{3}\Big[Ric(I_sX, I_se_a) + 4n\zeta_s(I_sX, e_a)\Big] \\ = 2\sum_{s=1}^{3}\Big[\rho_s(I_qX, I_te_a) - \rho_s(I_tX, I_qe_a)\Big] \\ = -4T^0(X, e_a) + 24U(X, e_a) + \frac{3\,Scal}{2n(n+2)}g(X, e_a),\end{aligned} \tag{5.17}$$

$$8n\sum_{s=1}^{3}\Big[\zeta_s(\xi_t,I_qX)-\zeta_s(\xi_q,I_tX)\Big]$$
$$=\sum_{s=1}^{3}\Big[4Ric(\xi_s,I_sX)-8\rho_s(\xi_s,X)+4\rho_s(\xi_t,I_qX)-4\rho_s(\xi_q,I_tX\Big],$$
$$\sum_{s=1}^{3}\Big[-2Ric(\xi_s,I_sX)+8n\zeta_s(\xi_s,X)\Big]$$
$$=\sum_{s=1}^{3}\Big[-4Ric(\xi_s,I_sX)-4\rho_s(\xi_t,I_qX)+4\rho_s(\xi_q,I_tX)\Big],$$

A substitution of (5.17) in (5.16), and then a use (5.12) and (5.13) give the first row of the system. □

We are ready to prove one of our main observations.

THEOREM 5.9. *The QC scalar curvature of a QC Einstein quaternionic-contact manifold of dimension bigger than seven is a global constant. In addition, the vertical distribution V of a QC Einstein structure is integrable. On a seven dimensional QC Einstein manifold the constancy of the QC scalar curvature is equivalent to the integrability of the vertical distribution. In both cases the Ricci tensors are given by*

$$\rho_{t|H}=\tau_{t|H}=-2\zeta_{t|H}=-\frac{Scal}{8n(n+2)}\omega_t \quad s,t=1,2,3.,$$
$$Ric(\xi_s,X)=\rho_s(X,\xi_t)=\zeta_s(X,\xi_t)=0, \quad s,t=1,2,3.$$

PROOF. The statement for $n=1$ follows directly from Theorem 5.8 and the fact that in dimension seven $U=0$. Suppose the quaternionic-contact manifold is QC Einstein. According to Proposition 5.2, the quaternionic-contact torsion vanishes,

$$T_\xi=0, \quad \xi\in V.$$

Since $n>1$, Theorem 5.8 gives immediately that the horizontal gradient of the scalar curvature vanishes, i.e.,

$$X(Scal)=0, \quad X\in H.$$

Notice that this fact implies also

$$\xi(Scal) = 0, \xi \in V,$$

taking into account that for any $p \in M$ one has

$$[e_a, I_s e_a]_{|p} = T(e_a, I_s e_a)_{|p} = 2\xi_{s|p}.$$

Now, (5.9), (5.3), (4.41), (4.42) and (4.43) complete the proof. □

5.2. Examples of QC Einstein structures.

Example 5.10. *The flat model.*

The quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ with its standard left invariant quaternionic-contact structure (see Section 3) is the simplest example. The Biquard connection coincides with the flat left-invariant connection on $G(\mathbb{H})$. More precisely, we have the following Proposition.

Proposition 5.11. *Any quaternionic-contact manifold (M, H, g) with flat Biquard connection is locally isomorphic to $G(\mathbb{H})$.*

Proof. Since the Biquard connection ∇ is flat, there exists a local Q-orthonormal frame

$$\{T_a,\ I_1T_a,\ I_2T_a,\ I_3T_a,\ \xi_1,\ \xi_2,\ \xi_3 :\ a = 1, \dots, n\},$$

which is ∇-parallel. Theorem 5.9 shows that the quaternionic-contact torsion vanishes and the vertical distribution is integrable. In addition, (4.49) and (4.4) yield $[\xi_i, \xi_j] = 0$ with the only non-zero commutators

$$[I_iT_a, T_a] = 2\xi_i, \quad i, j = 1, 2, 3, \quad (\text{cf. } (3.1)).$$

Hence, the manifold has a local Lie group structure which is locally isomorphic to $\boldsymbol{G}\,(\mathbb{H})$ by the Lie theorems. In other words, there is a local diffeomorphism

$$\Phi :\ M \to \boldsymbol{G}\,(\mathbb{H})$$

such that $\eta = \Phi^*\Theta$, where Θ is the standard contact form on $\boldsymbol{G}\,(\mathbb{H})$, see (3.2). □

Example 5.12. *The 3-Sasakian Case.*

Suppose (M,g) is a (4n+3)-dimensional Riemannian manifold with a given 3-Sasakian structure, i.e., the cone metric on $M\times\mathbb{R}$ is a hyperkähler metric, namely, it has holonomy contained in $Sp(n+1)$ [BGN]. Equivalently, there are three Killing vector fields $\{\xi_1,\xi_2,\xi_3\}$, which satisfy:

(i) $g(\xi_i,\xi_j)=\delta_{ij},\ i,j=1,2,3$

(ii) $[\xi_i,\xi_j]=-2\xi_k$, for any cyclic permutation (i,j,k) of $(1,2,3)$

(iii) $(D_B\tilde I_i)C=g(\xi_i,C)B-g(B,C)\xi_i,\ i=1,2,3,\ B,C\in\Gamma(TM)$, where $\tilde I_i(B)=D_B\xi_i$ and D denotes the Levi-Civita connection.

A 3-Sasakian manifold of dimension $(4n+3)$ is Einstein with positive Riemannian scalar curvature $(4n+2)(4n+3)$ [Kas] and if complete it is compact with finite fundamental group due to Mayer's theorem (see [BG] for a nice overview of 3-Sasakian spaces).
Let $H=\{\xi_1,\xi_2,\xi_3\}^{\perp}$. Then

$$\tilde I_i(\xi_j)=\xi_k,\quad \tilde I_i\circ\tilde I_j(X)=\tilde I_kX,\quad \tilde I_i\circ\tilde I_i(X)=-X,\ X\in H,$$
$$d\eta_i(X,Y)=2g(\tilde I_iX,Y),\ X,Y\in H.$$

Defining

$$V=span\{\xi_1,\xi_2,\xi_3\},\quad I_{i|H}=\tilde I_{i|H},\ I_{i|V}=0,$$

we obtain a quaternionic contact structure on M [Biq]. It is easy to calculate that

$$\xi_i\lrcorner d\eta_{j|H}=0,\ d\eta_i(\xi_j,\xi_k)=2,\ d\eta_i(\xi_i,\xi_k)=d\eta_i(\xi_i,\xi_j)=0, \tag{5.18}$$
$$A_1=A_2=A_3=0 \text{ cf. } (4.3),\qquad \tilde u=\frac12 Id_H\ \text{ cf. } (4.8).$$

This quaternionic-contact structure satisfies the conditions (4.1) and therefore it admits the Biquard connection ∇. More precisely, we have

(i) $\nabla_XI_i=0, X\in H,\qquad \nabla_{\xi_i}I_i=0,\qquad \nabla_{\xi_i}I_j=-2I_k,\ \nabla_{\xi_j}I_i=2I_k,$

(ii) $T(\xi_i,\xi_j)=-2\xi_k,$

(iii) $T(\xi_i,X)=0,\ X\in H.$

From Proposition 5.2, Theorem 5.9, (4.31) and (4.34), we obtain the following Corollary.

COROLLARY 5.13. *Any 3-Sasakian manifold is a QC Einstein with positive QC scalar curvature*

$$Scal = 16n(n+2).$$

For any $s, t, r = 1, 2, 3$, the Ricci-type tensors are given by

$$\begin{aligned} \rho_{t|H} &= \tau_{t|H} = -2\zeta_{t|H} = -2\omega_t \\ Ric(\xi_s, X) &= \rho_s(X, \xi_t) = \zeta_s(X, \xi_t) = 0 \\ \rho_s(\xi_t, \xi_r) &= 0. \end{aligned} \tag{5.19}$$

The nonzero parts of the curvature R of the Biquard connection is expressed in terms of the curvature of the Levi-Civita connection R^g as follows

i) $R(X,Y,Z,W) = R^g(X,Y,Z,W)$
$+ \sum_{s=1}^{3}\{\omega_s(Y,Z)\omega_s(X,W) - \omega_s(X,Z)\omega_s(Y,W) - 2\omega_s(X,Y)\omega_s(Z,W)\}$,

ii) $R(\xi,Y,Z,W) = -R(Y,\xi,Z,W) = R^g(\xi,Y,Z,W)$,

iii) $R(\xi,\bar\xi,Z,W) = R^g(\xi,\bar\xi,Z,W)$,

iv) $R(X,Y,\xi,\bar\xi) = -4\{\eta_1\wedge\eta_2(\xi,\bar\xi)\omega_3(X,Y) + \eta_2\wedge\eta_3(\xi,\bar\xi)\omega_1(X,Y)$
newline $+ \eta_3\wedge\eta_1(\xi,\bar\xi)\omega_2(X,Y)\}$,

where $X, Y, Z, W \in H$ and $\xi, \bar\xi \in V$.

In fact, 3-Sasakian spaces are locally the only QC Einstein manifolds (cf. Theorem C). Before we turn to the proof of this fact we shall consider some special cases of QC structures suggested by the above example. These structures will be relevant in Section 6, see for example Theorem 7.20. We recall that the Nijenhuis tensor N_{I_i} corresponding to I_i on H is defined as usual by

$$N_{I_i}(X,Y) = [I_iX, I_iY] - [X,Y] - I_i[I_iX,Y] - I_i[X,I_iY], \quad X, Y \in H.$$

DEFINITION 5.14. *A quaternionic-contact structure (M, H, g) is said to be* hyperhermitian contact *(abbr.* HC structure*) if the horizontal bundle H is formally integrable with respect to I_1, I_2, I_3 simultaneously, i.e. for $i = 1, 2, 3$ and any X, $Y \in H$, we have*

$$N_{I_i}(X,Y) = 0 \quad mod \quad V. \tag{5.20}$$

In fact a QC structure is locally a HC structure exactly when two of the almost complex structures on H are formally integrable due to the next identity essentially established in [AM, (3.4.4)]

$$2N_{I_3}(X,Y) - N_{I_1}(X,Y) + I_2N_{I_1}(I_2X,Y) + I_2N_{I_1}(X,I_2Y) - N_{I_1}(I_2X,I_2Y) -$$
$$N_{I_2}(X,Y) + I_1N_{I_2}(I_1X,Y) + I_1N_{I_2}(X,I_1Y) - N_{I_2}(I_1X,I_1Y) = 0 \mod V.$$

On the other hand, the Nijenhuis tensor has the following expression in terms of a connection ∇ with torsion T satisfying (4.29)(see e.g. [Iv])

$$N_{I_i}(X,Y) = T^{0,2}_{Ii}(X,Y) + \beta_i(Y)I_jX - \beta_i(X)I_jY - I_i\beta_i(Y)I_kX + I_i\beta_i(X)I_kY, \tag{5.21}$$

where the 1-forms β_i and the (0,2)-part of the torsion $T^{0,2}_{I_i}$ with respect to the almost complex structure I_i are defined on H, correspondingly, by

$$\beta_i = \alpha_j + I_i\alpha_k, \tag{5.22}$$

$$T^{0,2}_{I_i}(X,Y) = T(X,Y) - T(I_iX,I_iY) + I_iT(I_iX,Y) + I_iT(X,I_iY). \tag{5.23}$$

Applying the above formulas to the Biquard connection and taking into account (4.4) one sees that (5.20) is equivalent to $(\beta_i)_{|H} = 0$. Hence we have the following proposition.

PROPOSITION 5.15. *A quaternionic-contact structure (M,H,g) is a hyperhermitian contact structure if and only if the connection 1-forms satisfy the relations*

$$\alpha_j(X) = \alpha_k(I_iX), \quad X \in H \tag{5.24}$$

The Nijenhuis tensors of a HC structure satisfy

$$N_{I_i}(X,Y) = T^{0,2}_{I_i}(X,Y), \quad X,Y \in H.$$

Given a QC structure (M, H, g) let us consider the three almost complex structures $(\eta_i, \tilde{I}_i)$

$$\tilde{I}_i X = I_i X, \quad X \in H, \quad \tilde{I}_i(\xi_j) = \xi_k, \quad \tilde{I}_i(\xi_i) = 0. \tag{5.25}$$

With these definitions $(\eta_i, \tilde{I}_i)$ are almost CR structures (i.e. possibly non-integrable) exactly when the QC structure is HC since the condition

$$d\eta_i(\tilde{I}_i X, \tilde{I}_i \xi_j) = d\eta_i(X, \xi_j)$$

is equivalent to

$$\alpha_k(X) = -\alpha_j(I_i X)$$

in view of (4.30). Hence, $d\eta_i$ is a (1,1)-form with respect to $\tilde{I}_i$ on

$$\xi_i^{\perp} = H \oplus \{\xi_j, \xi_k\}$$

and a HC structure supports a non integrable hyper CR-structure $(\eta_i, \tilde{I}_i)$.

A natural question is to examine when $\tilde{I}_i$ is formally integrable, i.e,

$$N_{\tilde{I}_i} = 0 \quad \text{mod} \quad \xi_i.$$

Proposition 5.16. *Let (M, H, g) be a hyperhermitian contact structure. Then the CR structures $(\eta_i, \tilde{I}_i)$ are integrable if and only if the next two equalities hold*

$$d\eta_j(\xi_k, \xi_i) = d\eta_k(\xi_i, \xi_j), \qquad d\eta_j(\xi_j, \xi_i) - d\eta_k(\xi_k, \xi_i) = 0. \tag{5.26}$$

Proof. From (4.4) it follows

$$T^{0,2}_{\tilde{I}_1}(X, Y) = 0$$

using also (5.23). Substituting the latter into (5.21) taken with respect to $\tilde{I}_i$ shows

$$N_{\tilde{I}_i}|_H = 0 \quad (\text{mod } \xi_i)$$

is equivalent to (5.24). Corollary 4.7 implies

$$\begin{aligned} N_{\tilde{I}_i}(X,\xi_j) = (\alpha_j(I_iX)+\alpha_k(X))\xi_i + (\alpha_j(\xi_k)+\alpha_k(\xi_j))I_kX \\ + (\alpha_j(\xi_j)-\alpha_k(\xi_k))I_jX + +T(\xi_k,I_iX) \\ - I_iT(\xi_k,X) - T(\xi_j,X) - I_iT(\xi_j,I_iX). \end{aligned}$$

Taking the trace part and the trace-free part in the right-hand side allows us to conclude that

$$N_{\tilde{I}_i}(X,\xi_j) = 0 \pmod{\xi_i}$$

is equivalent to the system

$$T(\xi_k,I_1X) - I_1T(\xi_k,X) - T(\xi_j,X) - I_1T(\xi_j,I_1X) = 0,$$
$$\alpha_j(\xi_k)+\alpha_k(\xi_j)=0 \quad \alpha_j(\xi_j)-\alpha_k(\xi_k)=0.$$

An application of Proposition 4.4, (2.19) and (2.20) shows the first equality is trivially satisfied, while (4.31) tells us that the other equalities are equivalent to (5.26). □

5.3. Proof of Theorem C. The equivalence of a) and c) was proved in Proposition 5.2. We are left with proving the implication a) implies b). Let $(M,H,\tilde{g})$ be a QC Einstein manifold with QC scalar curvature $\overline{Scal}$. According to Theorem 5.9, $\overline{Scal}$ is a global constant on M. We define

$$\eta = \frac{\overline{Scal}}{16n(n+2)}\,\tilde{\eta}.$$

Then (M,H,g) is a QC Einstein manifold with QC scalar curvature

$$Scal = 16n(n+2),$$

horizontal distribution $H = Ker(\eta)$ and involutive vertical distribution

$$V = span\{\xi_1, \xi_2, \xi_3\}.$$

We shall show that the Riemannian cone is a hyperkähler manifold. Consider the structures defined by (5.25). We have the relations

$$\begin{gathered}
\eta_i(\xi_j) = \delta_{ij}, \quad \eta_i \tilde{I}_j = -\eta_j \tilde{I}_i = \eta_k, \qquad \tilde{I}_i \xi_j = -\tilde{I}_j \xi_i = \xi_k \\
\tilde{I}_i \tilde{I}_j - \eta_j \otimes \xi_i = -\tilde{I}_j \tilde{I}_i + \eta_i \otimes \xi_j = \tilde{I}_k \\
\tilde{I}_i^2 = -Id + \eta_i \otimes \xi_i, \qquad \eta_i \tilde{I}_i = 0, \qquad \tilde{I}_i \xi_i = 0, \\
g(\tilde{I}_i., \tilde{I}_i.) = g(.,.) - \eta_i(.)\, \eta_i(.).
\end{gathered} \tag{5.27}$$

Let D be the Levi-Civita connection of the metric g on M determined by the structure (η, Q). The next step is to show

$$D\tilde{I}_i = Id \otimes \eta_i - g \otimes \xi_i - \sigma_j \otimes \tilde{I}_k + \sigma_k \otimes \tilde{I}_j, \tag{5.28}$$

for some appropriate 1-forms σ_s on M. We consider all possible cases.

Case 1$[X, Y, Z \in H]$ The well known formula

$$\begin{aligned}
2\, g(D_A B, C) \; &= \; Ag(B, C) \; + \; Bg(A, C) \; - \; Cg(A, B) \\
&+ g([A, B], C) - g([B, C], A) + g([C, A], B), \qquad A, B, C \in \Gamma(TM)
\end{aligned} \tag{5.29}$$

yields

$$2\, g(\, (D_X \tilde{I}_i)\, Y, Z) = d\omega_i(X, Y, Z) - d\omega_i(X, I_i Y, I_i Z) + g(N_i(Y, Z), I_i X). \tag{5.30}$$

We compute $d\omega_i$ in terms of the Biquard connection. Using (4.4), (4.29) and (5.22), we calculate

$$(5.31)\quad d\omega_i(X,Y,Z) - d\omega_i(X,I_iY,I_iZ) \;=\;= \; -2\,\alpha_j(X)\,\omega_k(Y,Z) + 2\alpha_k(X)\,\omega_j(Y,Z)$$
$$-\,\beta_i(Y)\,\omega_k(Z,X) \;-\; I_i\beta_i(Y)\,\omega_j(Z,X) \;-\; \beta_i(Z)\,\omega_k(X,Y) \;-\; I_i\beta_i(Z)\,\omega_j(X,Y).$$

A substitution of (5.21) and (5.31) in (5.30) gives

$$g((D_X\tilde{I}_i)Y,Z) = -\alpha_j(X)\omega_k(Y,Z) + \alpha_k(X)\omega_j(Y,Z). \tag{5.32}$$

Letting $\sigma_i(X) = \alpha_i(X)$, we obtain equation (5.28).

Case 2 [$\xi_s, \xi_t \in V$ and $Z \in H$] Using the integrability of the vertical distribution V and (5.29), we compute

$$2g((D_{\xi_s}\tilde{I}_i)\xi_t, Z) = 2g(D_{\xi_s}\tilde{I}_i\xi_t, Z) + 2g(D_{\xi_s}\xi_t, I_iZ) =$$
$$-\,g([\tilde{I}_i\xi_t, Z],\xi_s) - g([\xi_s, Z],\tilde{I}_i\xi_t) - g([\xi_s, I_iZ],\xi_t) - g([\xi_t, I_iZ],\xi_s).$$

An application of (4.1) allows to conclude $g((D_{\xi_s}\tilde{I}_i)\xi_t, Z) = 0$ for any $i,s,t \in \{1,2,3\}$.

Case 3 [$X,Y \in H$ and $C \in V$] First, let $C = \xi_1$. We have

$$2g((D_X\tilde{I}_1)Y,\xi_1) = 2g(D_X\tilde{I}_1Y,\xi_1)$$
$$= \; -\xi_1 g(X, I_1Y) + g([X,I_1Y],\xi_1) - g([X,\xi_1],I_1Y) - g([I_1Y,\xi_1],X)$$
$$= -(\mathcal{L}_{\xi_1}g)(X,I_1Y) + \eta_1([X,I_1Y]) = -d\eta_1(X,I_1Y) = -2g(X,Y).$$

after using (4.19), $T_{\xi_s} = 0$, $\;s = 1,2,3$, and (2.2).

For $C=\xi_2$, we calculate applying (5.27) and (5.29) that

$$
\begin{aligned}
2g((D_X\tilde I_1)Y,\xi_2) &= 2g(D_X\tilde I_1Y,\xi_2)+2g(D_XY,\xi_3)\\
&= -\xi_1 g(X,I_iY)-\xi_3 g(X,Y)+g([X,I_iY],\xi_2)-g([X,\xi_2],I_iY)\\
&\quad - g([I_iY,\xi_2],X)+g([X,Y],\xi_3)-g([X,\xi_3],Y)-g([Y,\xi_3],X)\\
&= -(\mathcal{L}_{\xi_2}g)(X,I_1Y)-(\mathcal{L}_{\xi_3}g)(X,Y)+\eta_2([X,I_1Y])+\eta_3(]X,Y])=0.
\end{aligned}
$$

The other possibilities in this case can be checked in a similar way.

Case 4 $[X\in H$ and $A,B\in V]$. We verify (5.28) for $\tilde I_1, A=\xi_1, B=\xi_2$ and $\tilde I_1, A=\xi_2, B=\xi_3$ since the other verifications are similar. Using the integrability of V, (5.29), (5.27) and (4.30), we find

$$
\begin{aligned}
2g((D_X\tilde I_1)\xi_1,\xi_2) = 2g(D_X\xi_1,\xi_3) = g([X,\xi_1],\xi_3)-g([X,\xi_3],\xi_1) = -2\alpha_2(X),\\
2g((D_X\tilde I_1)\xi_2,\xi_3) = 2g(D_X\xi_3,\xi_3)-2g(D_X\xi_2,\xi_2)=0.
\end{aligned}
$$

Case 5 $[A,B,C\in V]$ Let us extend the definition of the three 1-forms σ_s on V as follows

$$
\begin{aligned}
\sigma_i(\xi_i) &= 1+\frac{1}{2}(d\eta_i(\xi_j,\xi_k)-d\eta_j(\xi_k,\xi_i)-d\eta_k(\xi_i,\xi_j))\\
\sigma_i(\xi_j) &= d\eta_j(\xi_j,\xi_k), \qquad \sigma_i(\xi_k)=d\eta_k(\xi_j,\xi_k).
\end{aligned}
\tag{5.33}
$$

A small calculation leads to the formula

$$
g(\tilde I_iA,B)=(\eta_j\wedge\eta_k)(A,B).
\tag{5.34}
$$

On the other hand, we have

$$
\begin{aligned}
(5.35)\quad 2(D_A\eta_i)(B) &= 2g((D_A\xi_i, B) \\
&= A\eta_i(B) + \xi_i g(A,B) - B\eta_i(A) + g([A,\xi_i],B) - \eta_i([A,B]) - g([\xi_i,B],A) \\
&= \sum_{s,t=1}^{3}\Big[\eta_s(A)\,\eta_t(B)\,d\eta_i(\xi_s,\xi_t) \;-\; \eta_s(A)\,\eta_t(B)\,d\eta_t(\xi_s,\xi_i) \;-\; \eta_s(B)\,\eta_t(A)\,d\eta_t(\xi_s,\xi_i)\Big] \\
&= \;2\,\eta_j\wedge\eta_k(A,B) \;-\; 2\,\sigma_j(A)\,\eta_k(B) \;+\; 2\,\sigma_k(A)\,\eta_j(B).
\end{aligned}
$$

With the help of (5.34) and (5.35) we see

$$
\begin{aligned}
(5.36)\quad g((D_A\tilde{I}_i)B,C) &= D_A(\eta_j\wedge\eta_k)(B,C) = [D_A(\eta_j)\wedge\eta_k + \eta_j\wedge D_A(\eta_k)](B,C) \\
&= ((A\lrcorner(\eta_k\wedge\eta_i) - \sigma_k(A)\eta_i + \sigma_i(A)\eta_k)\wedge\eta_k)(B,C) \\
&\quad + (\eta_j\wedge(A\lrcorner(\eta_i\wedge\eta_j) - \sigma_i(A)\eta_j + \sigma_j(A)\eta_i))(B,C) \\
&= (\eta_k(A)\eta_i\wedge\eta_k(B,C) + \eta_j(A)\eta_i\wedge\eta_j(B,C)) - \sigma_j(A)g(\tilde{I}_kB,C) + \sigma_k(A)g(\tilde{I}_jB,C) \\
&= \eta_i(B)g(A,C) - g(A,B)\eta_i(C) - \sigma_j(A)g(\tilde{I}_kB,C) + \sigma_k(A)g(\tilde{I}_jB,C).
\end{aligned}
$$

Case 6 $[A\in V$ and $Y,Z\in H]$. Let $A=\xi_s,\ s\in\{1,2,3\}$. The right hand side of (5.28) is equal to

$$-\sigma_j(\xi_s)\omega_k(Y,Z) + \sigma_k(\xi_s)\omega_j(Y,Z).$$

On the left hand side of (5.28), we have

$$
\begin{aligned}
(5.37)\quad 2g((D_{\xi_s}\tilde{I}_i)Y,Z) &= 2g(D_{\xi_s}(I_iY),Z) + 2g(D_{\xi_s}Y, I_iZ) \\
&= \{\xi_s g(I_iY,Z) + g([\xi_s,I_iY],Z) - g([\xi_s,Z],I_iY) - g([I_iY,Z],\xi_s)\} \\
&\quad + \{\xi_s g(Y,I_iZ) + g([\xi_s,Y],I_iZ) - g([\xi_s,I_iZ],Y) - g([Y,I_iZ],\xi_s)\} \\
&= g((\mathcal{L}_{\xi_s}I_i)Y,Z) - g((\mathcal{L}_{\xi_s}I_i)Z,Y) + \omega_i(I_sY,Z) + \omega_i(Y,I_sZ)
\end{aligned}
$$

Now, recall Lemma 4.3 to compute the skew symmetric part of $g((\mathcal{L}_{\xi_s} I_i)Y,Z)$, and also use formulas (5.33), to prove the case $i=s$

$$g(\,(D_{\xi_i}\tilde{I}_i)\,Y,Z) \;=\; d\eta_i(\xi_i,\xi_j)\,\omega_j(Y,Z) \;+\; d\eta_i(\xi_i,\xi_k)\,\omega_k(Y,Z)$$
$$=\; -\sigma_j(\xi_i)\,\omega_k(Y,Z) \;+\; \sigma_k(\xi_i)\,\omega_j(Y,Z).$$

Similarly, for $s=j$ we have

$$g(\,(D_{\xi_j}\tilde{I}_i)\,Y,Z) \;=\; d\eta_j(\xi_i,\xi_j)\,\omega_j(Y,Z)$$
$$-\;\frac{1}{2}\Big(-d\eta_i(\xi_j,\xi_k) \;+\; d\eta_j(\xi_k,\xi_i) \;-\; d\eta_k(\xi_i,\xi_j)\Big)\,\omega_k(Y,Z)$$
$$-\;\omega_k(Y,Z) \;=\; -\sigma_j(\xi_j)\,\omega_k(Y,Z) \;+\; \sigma_k(\xi_j)\,\omega_j(Y,Z),$$

which completes the proof of (5.28).

At this point, consider the Riemannian cone $N=M\times\mathbb{R}^+$ with the cone metric $g_N=t^2g+dt^2$ and the almost complex structures

$$\phi_i(E,f\frac{d}{dt})=(\tilde{I}_iE+\frac{f}{t}\xi_i,-t\eta_i(E)\frac{d}{dt}),\quad i=1,2,3,\quad E\in\Gamma(TM).$$

Using the O'Neill formulas for warped product [**On**, p.206], (5.27) and the just proved (5.28) we conclude (see also [**MO**]) that the Riemannian cone $(N,g_N,\phi_i,i=1,2,3)$ is a quatrnionic Kähler manifold with connection 1-forms defined by (5.32) and (5.33). It is classical result (see e.g [**Bes**]) that a quaternionic Kähler manifolds are Einstein. This fact implies that the cone $N=M\times\mathbb{R}^+$ with the warped product metric g_N must be Ricci flat (see e.g. [**Bes**, p.267]) and therefore it is locally hyperkähler (see e.g. [**Bes**, p.397]). This means that locally there exists a $SO(3)$-matrix Ψ with smooth entries such that the triple

$$(\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\phi}_3)=\Psi\cdot(\phi_1,\phi_2,\phi_3)^t$$

is D-parallel. Consequently $(M,\Psi\cdot\eta)$ is locally 3-Sasakian. Example 5.12 and Proposition 5.2 complete the proof.

Corollary 5.17. *Let $n > 1$ and (M, H, g) be a QC structure on a (4n+3)-dimensional manifold with positive QC scalar curvature, $Scal > 0$. The following two conditions are equivalent.*

i) The structure $(M, H, \frac{16n(n+2)}{Scal} g)$ is locally 3-Sasakian.

ii) There exists a (local) 1-form η such that the connection 1-forms of the Biquard connection vanish on H, $\alpha_i(X) = -d\eta_j(\xi_k, X) = 0, X \in H$, $i, j, k = 1, 2, 3$ and $Scal$ is constant if $n = 1$.

Proof. In view of Theorem C and Example 5.12 it is sufficient to prove the following Lemma. □

Lemma 5.18. *If a QC structure has zero connection one forms restricted to the horizontal space H then it is QC Einstein, or equivalently, it has zero torsion.*

Proof. If $\alpha_i(X) = 0$ for $i = 1, 2, 3$ and $X \in H$ then (4.34) together with (4.4) yield

$$2\rho_i(X,Y) = -\alpha_i([X,Y]) = \alpha_i(T(X,Y)) = 2\sum_{s=1}^{3} \alpha_i(\xi_s)\omega_s(X,Y).$$

Substitute the latter into (4.42) to conclude considering the $Sp(n)Sp(1)$-invariant parts of the obtained equalities that

$$T^0(X,Y) = U(X,Y) = \alpha_i(\xi_j) = 0, \quad \alpha_i(\xi_i) = -\frac{Scal}{8n(n+2)}.$$

□

Corollary 5.19. *Any totally umbilical hypersurface M in a quaternionic-Kähler manifold admits a canonical QC Einstein structure with a non-zero scalar curvature.*

Proof. Let M be a hypersurface in the quaternionic-Kähler manifold $(\tilde{M}, \tilde{g})$. With N standing for the unit normal vector field on M the second fundamental form is given by

$$II(A,B) = -\tilde{g}(\tilde{D}_A N, B), \quad A, B \in TM,$$

with $\tilde{D}$ being the Levi-Civita connection of $(\tilde{M}, \tilde{g})$. Since M is a totally umbilical hypersurface of an Einstein manifold $\tilde{M}$, taking a suitable trace in the Codazzi equation [**KoNo**, Proposition 4.3], we find

$$II(A,B) = -Const \cdot \tilde{g}(A,B).$$

Thus, after a homothety of $\tilde{M}$ we can assume $II(A,B) = -\tilde{g}(A,B)$.

Consider a local basis $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$ of the quaternionic structure of $\tilde{M}$ satisfying the quaternionic identities. We define the horizontal distribution H of M to be the maximal subspace of TM invariant under the action of $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$, whose restriction to M will be denoted with I_1, I_2, I_3. We claim that (H, I_1, I_2, I_3) is a QC structure on M. Defining

$$\eta_i(A) = \tilde{g}(\tilde{J}_i(N), A), \qquad \xi_i = \tilde{J}_i N,$$

a small calculation shows

$$d\eta_i(X,Y) = II(X, I_iY) - II(I_iX, Y) = 2\tilde{g}(I_iX, Y), \quad X, Y \in H,$$

Hence, the conditions in the definition of a QC structure are satisfied.

Let D be the Levi-Civita connection of the restriction g of $\tilde{g}$ to M. Then we have

$$\tilde{D}_A B = D_A B + II(A,B)N, \qquad A, B \in TM \tag{5.38}$$

Define $\tilde{I}_i(A) = \tilde{J}_i(A)_{TM}$ the orthogonal projection on TM, $A \in TM$. Since by assumption $\tilde{M}$ is a quaternionic-Kähler manifold we have

$$\tilde{D}\tilde{J}_i = -\sigma_j \otimes \tilde{J}_k + \sigma_k \otimes \tilde{J}_j.$$

This together with (5.38), after some computation gives (compare with (5.28))

$$D\tilde{I}_i = Id \otimes \eta_i - g \otimes \xi_i - \sigma_j \otimes \tilde{I}_k + \sigma_k \otimes \tilde{I}_j. \tag{5.39}$$

Using (5.39) we will show that the torsion of the QC structure is zero. The same computation as in the Case 3 in the proof of Theorem C gives

$$
\begin{aligned}
-2g(X,Y) &= 2g((D_X\tilde{I}_i)Y,\xi_i) = 2g(D_X\tilde{I}_iY,\xi_i)\\
&= -\xi_i g(X,I_iY) + g([X,I_iY],\xi_i) - g([X,\xi_i],I_iY) - g([I_iY,\xi_i],X)\\
&= -(\mathcal{L}_{\xi_i}g)(X,I_iY) - 2g(X,Y).
\end{aligned}
$$

Hence

$$
0 = (\mathcal{L}_{\xi_i}g)(X,Y) = 2T^0_{\xi_i}(X,Y).
$$

A computation analogous to the the proof of Theorem C, Case 6 gives

$$
\begin{aligned}
-2\sigma_3(\xi_1)\omega_1(Y,Z) + 2\sigma_1(\xi_1)\omega_3(Y,Z) &= 2g((D_{\xi_1}\tilde{I}_2)Y,Z)\\
&= g((\mathcal{L}_{\xi_1}I_2)Y,Z) - g((\mathcal{L}_{\xi_1}I_2)Z,Y) + \omega_2(I_1Y,Z) + \omega_2(Y,I_1Z).
\end{aligned}
\tag{5.40}
$$

From Lemma 4.3 it follows

$$
\begin{aligned}
-2\sigma_3(\xi_1)\omega_1(Y,Z) + 2\sigma_1(\xi_1)\omega_3(Y,Z) &=\\
&= -4g(I_3\tilde{u}Y,Z) + 2d\eta_1(\xi_2,\xi_1)\omega_1(Y,Z)\\
&\quad + \left(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2) - 2\right)\omega_3(Y,Z).
\end{aligned}
$$

Working similarly in the remaining cases we conclude that the traceless part u of the tensor $\tilde{u}$ vanishes. □

6. Conformal deformation of a QC structure

Let h be a positive smooth function on a QC manifold (M,H,g) and let

$$
\bar{\eta} = \frac{1}{2h}\eta
$$

be a conformal deformation of the QC structure η (to be precise we should let $\bar{g} = \frac{1}{2h}g$ on H and consider $(M, H, \bar{g})$). We denote the objects related to $\bar{\eta}$ by overlining the same object corresponding to η. Thus,

$$d\bar{\eta} = -\frac{1}{2h^2}dh \wedge \eta + \frac{1}{2h}d\eta$$

and $\bar{g} = \frac{1}{2h}g$. The new triple $\{\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\}$ is determined by (4.1). We have

$$\bar{\xi}_s = 2h\xi_s + I_s\nabla h, \quad s = 1, 2, 3, \tag{6.1}$$

where ∇h is the horizontal gradient defined by

$$g(\nabla h, X) = dh(X), X \in H.$$

The (horizontal) sub-Laplacian and the norm of the horizontal gradient are defined respectively by

$$\triangle h = tr^g_H(\nabla dh) = \sum_{\alpha=1}^{4n} \nabla dh(e_\alpha, e_\alpha), \qquad |\nabla h|^2 = \sum_{\alpha=1}^{4n} dh(e_\alpha)^2. \tag{6.2}$$

The Biquard connections ∇ and $\bar{\nabla}$ are connected by a (1,2) tensor S,

$$\bar{\nabla}_A B = \nabla_A B + S_A B, A, B \in \Gamma(TM).$$

The condition (4.4) yields

$$g(S_X Y, Z) - g(S_Y X, Z) = -h^{-1}\textstyle\sum_{s=1}^3 \omega_s(X, Y)dh(I_s Z), \quad X, Y, Z \in H.$$

From $\bar{\nabla}\bar{g} = 0$, we get

$$g(S_X Y, Z) + g(S_X Z, Y) = -h^{-1}dh(X)g(Y, Z), \quad X, Y, Z \in H.$$

The last two equations determine $g(S_XY, Z)$ for $X, Y, Z \in H$ due to the equality

$$g(S_XY,Z) = -(2h)^{-1}\{dh(X)g(Y,Z) - \sum_{s=1}^{3} dh(I_sX)\omega_s(Y,Z) + dh(Y)g(Z,X) + \sum_{s=1}^{3} dh(I_sY)\omega_s(Z,X) - dh(Z)g(X,Y)Z + \sum_{s=1}^{3} dh(I_sZ)\omega_s(X,Y)\}.$$

Using Biquard's Theorem 2.4, we obtain after some calculations that

$$g(\bar T_{\bar\xi_1}X,Y) - 2hg(T_{\xi_1}X,Y) - g(S_{\bar\xi_1}X,Y) = - \nabla dh(X,I_1Y) + h^{-1}(dh(I_3X)dh(I_2Y) - dh(I_2X)dh(I_3Y)). \tag{6.3}$$

The identity $d^2 = 0$ yields

$$\nabla dh(X,Y) - \nabla dh(Y,X) = -dh(T(X,Y)).$$

Applying (4.4), we can write

$$\nabla dh(X,Y) = [\nabla dh]_{[sym]}(X,Y) - \sum_{s=1}^{3} dh(\xi_s)\omega_s(X,Y), \tag{6.4}$$

where $[.]_{[sym]}$ denotes the symmetric part of the correspondin (0,2)-tensor.

Decomposing (6.3) into [3] and [-1] parts according to (2.14), using the properties of the torsion tensor T_{ξ_i} and (4.37) we come to the next transformation formulas:

$$\overline{T}^0(X,Y) \;=\; T^0(X,Y) \;+\; h^{-1}[\nabla dh]_{[sym][-1]}, \tag{6.5}$$

$$\bar U(X,Y) \;=\; U(X,Y) \;+\; (2h)^{-1}[\nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}, \tag{6.6}$$

$$
\begin{aligned}
g(S_{\bar\xi_1}X,Y) &= -\frac{1}{4}\Big[-\nabla dh(X,I_1Y)+\nabla dh(I_1X,Y)-\nabla dh(I_2X,I_3Y)\\
&\quad+\nabla dh(I_3X,I_2Y)\Big]-\ (2h)^{-1}\Big[dh(I_3X)dh(I_2Y)\\
&\quad-dh(I_2X)dh(I_3Y)+dh(I_1X)dh(Y)-dh(X)dh(I_1Y)\Big]\\
&+\ \frac{1}{4n}\left(-\triangle h+2h^{-1}|\nabla h|^2\right)g(I_1X,Y)-dh(\xi_3)g(I_2X,Y)+dh(\xi_2)g(I_3X,Y),
\end{aligned}
$$

where $[\,.\,]_{[sym][-1]}$ and $[\,.\,]_{[3][0]}$ denote the symmetric $[-1]$-component and the traceless $[3]$ part of the corresponding (0,2) tensors on H, respectively. Observe that for $n=1$ (6.6) is trivially satisfied.
Thus, using (4.48), we proved the following Proposition.

PROPOSITION 6.1. *Let*

$$\overline{\eta}=\frac{1}{2h}\eta$$

be a conformal deformation of a given QC structure η. Then the trace-free parts of the corresponding QC Ricci tensors are related by the equation

$$
\begin{aligned}
&Ric_0(X,Y)-\overline{Ric}_0(X,Y) \qquad (6.7)\\
&=-(2n+2)h^{-1}[\nabla dh]_{[sym][-1]}(X,Y)-(2n+5)h^{-1}\Big[\nabla dh-2h^{-1}dh\otimes dh\Big]_{[3][0]}(X,Y).
\end{aligned}
$$

For $n=1$,

$$Ric_0(X,Y)-\overline{Ric}_0(X,Y)=-4h^{-1}[\nabla dh]_{[sym][-1]}(X,Y).$$

In addition, the QC scalar curvature transforms by the formula [Biq], which is the QC Yamabe equation,

$$\overline{Scal}=2h(Scal)-8(n+2)^2h^{-1}|\nabla h|^2+8(n+2)\triangle h. \qquad (6.8)$$

6.1. Conformal deformations preserving the QC Einstein condition. In this section we investigate the question of conformal transformations, which preserve the QC Einstein condition. A straightforward consequence of (6.7) is the following

Proposition 6.2. *Let*

$$\bar{\eta} = \frac{1}{2h}\eta$$

be a conformal deformation of a given QC structure (M, H, g). Then the trace-free part of the QC Ricci tensor does not change if and only if the function h satisfies the differential equations

$$3(\nabla_X dh)Y - \sum_{s=1}^{3}(\nabla_{I_sX}dh)I_sY \;=\; -4\sum_{s=1}^{3} dh(\xi_s)\omega_s(X,Y), \tag{6.9}$$

$$(\nabla_X dh)Y - 2h^{-1}dh(X)dh(Y) + \sum_{s=1}^{3}\Big[(\nabla_{I_sX}dh)I_sY - 2h^{-1}dh(I_sX)dh(I_sY)\Big] \;=\; \lambda g(X,Y), \tag{6.10}$$

for some smooth function λ and any $X, Y \in H$.

Note that for $n = 1$ (6.10) is trivially satisfied. Let us fix a QC normal frame, cf. definition 5.6,

$$\{T_\alpha, X_\alpha = I_1T_\alpha, Y_\alpha = I_2T_\alpha, Z_\alpha = I_3T_\alpha, \xi_1, \xi_2, \xi_3\}, \quad \alpha = 1\ldots, n$$

at a point $p \in M$.

Lemma 6.3. *If h satisfies (6.9) then we have at $p \in M$ the relations*

$$c(I_j\,T_\alpha)\,T_\alpha\,h \;=\; -\;T_\alpha\,(I_j\,T_\alpha)\,h \;=\; \xi_j\,h \tag{6.11}$$

$$(I_j\,T_\alpha)\,(I_i\,T_\alpha)\,h \;=\; -\;(I_i\,T_\alpha)\,(I_j\,T_\alpha)\,h \;=\; \xi_k\,h. \tag{6.12}$$

Proof. Working with the fixed QC normal frame, equation (6.9) gives

$$4T_\alpha X_\alpha h\,(p) \;-\; [T_\alpha, X_\alpha]h\,(p) \;+\; [Y_\alpha, Z_\alpha]h\,(p) \;=\; -4\xi_1 h\,(p).$$

Lemma 5.5 and (4.4) yield $[T_\alpha, X_\alpha]h\,(p) \;-\; [Y_\alpha, Z_\alpha]h\,(p) \;=\; 0$. Hence, (6.11) follow. ☐

6.2. Proof of Theorem A. The proof of Theorem A (formulated at the begining of this chapter) will be presented as separate Propositions and Lemmas in the rest of the Section, see (6.28) for the final formula. Using the model of the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ and the notation from Section 3, we start with the follwoing Proposition in which we determine the vertical Hessian of the function h:

Proposition 6.4. *If h satisfies (6.9) on $\boldsymbol{G}\,(\mathbb{H})$ then we have the relations*

$$\xi_1^2(h) = \xi_2^2(h) = \xi_3^2(h) = 8\mu_o, \quad \xi_i\xi_j(h) = 0, \quad i \neq j = 1,2,3, \tag{6.13}$$

where $\mu_o > 0$ is a constant. In particular,

$$h(q,\omega) \;=\; g(q) \;+\; \mu_o\left[\,(x \;+\; x_o(q)\,)^2 \;+\; (y \;+\; y_o(q)\,)^2 \;+\; (z \;+\; z_o(q)\,)^2\,\right] \tag{6.14}$$

for some real valued functions g, x_o y_o and z_o on $\mathbb{H}^n$. Furthermore we have

$$T_\alpha Z_\alpha X_\alpha^2(h) = T_\alpha Z_\alpha Y_\alpha^2(h) = 0, \quad T_\alpha^2 \xi_j(h) = 0.$$

Proof. Equations (6.11) and (3.1) yield the next sequence of equalities

$$\begin{aligned} 2\xi_i\xi_j\ h \;&=\; -2T_\alpha\ (I_i\,T_\alpha)\ \xi_j\ h \;=\; -2T_\alpha\ \xi_j\ (I_i\,T_\alpha)\ h = \;\; T_\alpha\ [T_\alpha, I_j\,T_\alpha]\ (I_iT_\alpha)\ h \\ &=\; T_\alpha^2\,(I_j\,T_\alpha)\ (I_i\,T_\alpha)\ h \;-\; 2T_\alpha\ (I_j\,T_\alpha)\ T_\alpha\ (I_iT_\alpha)\ h = \;\; T_\alpha^2\ \xi_k\ h \;-\; \xi_i\ \xi_j\ h. \end{aligned}$$

Hence,

$$3\xi_i\xi_j\ h = T_\alpha^2\ \xi_k\ h.$$

Similarly, interchanging the roles of i and j together with

$$\{I_iT_\alpha, I_jT_\alpha\} = 0$$

we find

$$3\xi_j\xi_i(h) = -T_\alpha^2\ \xi_k(h).$$

Consequently,

$$\xi_i\xi_j\ h\ =\ T_\alpha^2\ \xi_k\ h\ =\ 0.$$

An analogous calculation shows that $\xi_i\xi_k\ h\ =\ 0$. Furthermore, we have

$$\begin{aligned}2\xi_1^2(h) &= 2X_\alpha T_\alpha\xi_1(h) = 2X_\alpha\xi_1 T_\alpha(h) = -X_\alpha[Y_\alpha Z_\alpha]T_\alpha(h)\\ &= X_\alpha Z_\alpha Y_\alpha T_\alpha(h) - X_\alpha Y_\alpha Z_\alpha T_\alpha(h) = \xi_2^2(h) + \xi_3^2(h).\end{aligned}$$

We derive similarly

$$2\xi_2^2(h) = \xi_1^2(h) + \xi_3^2(h), \quad 2\xi_3^2(h) = \xi_2^2(h) + \xi_1^2(h).$$

Therefore,

$$\xi_1^2(h) = \xi_2^2(h) = \xi_3^2(h), \quad \xi_i^3(h) = \xi_i\xi_j^2(h) = 0, \quad i \neq j = 1, 2, 3$$

which proves part of (6.13).

Next we prove that the common value of the second derivatives is a constant. For this we differentiate the equation

$$T_\alpha^2\,\xi_k h = 0$$

with respect to $I_k T_\alpha$ from where, (6.11) and (3.1), we get

$$\begin{aligned}0\ &=\ \xi_k\ (I_k\,T_\alpha)\ T_\alpha^2\ h\ =\ \xi_k\ T_\alpha\ (I_k\,T_\alpha)\ T_\alpha\ h\\ &\quad +\ \xi_k\ [I_k\,T_\alpha,\ T_\alpha]\ h = \ T_\alpha\ \xi_k^2\ h\ +\ 2\ T_\alpha\ \xi_k^2\ h\ =\ 3\ T_\alpha\ \xi_k^2\ h.\end{aligned}$$

In order to see the vanishing of $(I_i\,T_\alpha)\ \xi_k^2\ h$ we shall need

$$(I_i\,T_\alpha)^2\,\xi_k h = 0. \tag{6.15}$$

The latter can be seen by the following calculation.

$$
\begin{aligned}
2\xi_i\xi_j\ h \ &= \ 2\xi_i\ (I_i\,T_\alpha)\ (I_k\,T_\alpha)\ h \ = \ 2(I_i\,T_\alpha)\ \xi_i\ (I_k\,T_\alpha)\ h \\
&= \ T_\alpha\ [T_\alpha, I_j\,T_\alpha]\ (I_iT_\alpha)\ h \ = \ (I_i\,T_\alpha)^2\ T_\alpha(I_k\,T_\alpha)\ h \\
&\qquad - \ (I_i\,T_\alpha)\ T_\alpha\ (I_i\,T_\alpha)\ (I_k\,T_\alpha)\ h = \ -\ (I_i\,T_\alpha)^2\ \xi_k\ h \ - \ \xi_i\ \xi_j\ h,
\end{aligned}
$$

from where

$$0 = 3\xi_i\xi_j h = -(I_i\,T_\alpha)^2\xi_k h.$$

Differentiate

$$(I_i\,T_\alpha)^2\,\xi_k h = 0$$

with respect to I_jT_α to get

$$
\begin{aligned}
0 \ &= \ \xi_k\ (I_j\,T_\alpha)\ (I_i\,T_\alpha)^2\,\xi_k\ h = \ \xi_k\ (I_i\,T_\alpha)\ (I_k\,T_\alpha)\ T_\alpha\ h \ + \ \xi_k\ [I_j\,T_\alpha, I_i\,T_\alpha]\ (I_i\,T_\alpha)\ h \\
&= \ (I_i\,T_\alpha)\ \xi_k^2\ h \ + \ 2\ (I_i\,T_\alpha)\ \xi_k^2\ h \ = \ 3\ (I_i\,T_\alpha)\ \xi_k^2\ h.
\end{aligned}
$$

We proved the vanishing of all derivatives of the common value of $\xi_j^2 h$, i.e., this common value is a constant, which we denote by $8\mu_o$. Let us note that $\mu_o > 0$ follows easily from the fact that $h > 0$ since g is independent of x, y and z.

The rest equalities of the proposition follow easily from (6.11) and (6.13). $\square$

In view of Proposition 6.4, we define $h \ = \ g \ + \ \mu_o\,f$, where

$$f \ = \ (x + x_o(q))^2 \ + \ (y + y_o(q))^2 \ + \ (z + z_o(q))^2. \tag{6.16}$$

The following simple Lemma is one of the keys to integrating our system.

Lemma 6.5. *Let X and Y be two parallel horizontal vectors* *a) If* $\omega_s(X,Y) \;=\; 0$ *for* $s=1,2,3$ *then*

$$4XYh - 2h^{-1}\Big[dh(X)\;dh(Y) + \sum_{s=1}^{3} dh(I_sX)\;dh(I_sY)\Big] \;=\; \lambda\, g(X,Y). \tag{6.17}$$

b) If $g(X,Y) \;=\; 0$ *then*

$$2\,XYh \;-\; h^{-1}\{\,dh(X)\;dh(Y) + \sum_{s=1}^{3} dh(I_sX)\;dh(I_sY)\,\} \;=\; 2\sum_{s=1}^{3}\{\;(\xi_s h)\;\omega_s(X,Y)\;\}. \tag{6.18}$$

c) If $g(X,Y) \;=\; \omega_s(X,Y) \;=\; 0$ *for* $s=1,2,3$ *we have for any* $j\in\{1,2,3\}$

$$XY\,(\xi_j h) \;=\; 0, \tag{6.19}$$

$$8\,XYh \;=\; \mu_o\big\{(X\xi_j f)\,(Y\xi_j f) \;+\; \sum_{s=1}^{3}(I_sX\,\xi_j f)\,(I_sY\,\xi_j f)\,\big\}.$$

Proof. The equation of a) and b) are obtained by adding (6.9) and (6.10). Let us prove part c). From (6.9) and (6.10) taking any two horizontal vectors satisfying

$$g(X,Y) = \omega_s(X,Y) = 0,$$

we obtain

$$2h\nabla dh(X,Y) \;=\; dh(X)\;dh(Y) \;+\; \sum_{s=1}^{3} dh(I_sX)\;dh(I_sY).$$

If $X,\ Y$ are also parallel, differentiate along ξ_j twice to get consequently

$$
\begin{aligned}
(6.20)\quad & 2\xi_j h\, XYh \ +\ 2hXY\xi_j h \\
&= (X\xi_j h)\,(Yh) \ +\ \sum_{s=1}^{3}[(I_sX\,\xi_j h)\,(I_sY\,h) \ +\ (I_sX\,\xi_j h)\,(I_sY\,h)],\\
& 2\xi_j^2 h\, XYh \ +\ 4hXY\xi_j h \ =\ 2\big\{\,(X\xi_j h)\,(Y\xi_j h) \ +\ \sum_{s=1}^{3}(I_sX\,\xi_j h)\,(I_sY\,\xi_j h)\,\big\}.
\end{aligned}
$$

Differentiate three times along ξ_j and use $\xi_j^2 h$ =const, cf. (6.13) to get

$$2\ (\xi_j^2 h)\ XY\,(\xi_j h)\ =\ 0,$$

from where the first equality in (6.19) follows. With this information the second line in (6.20) reduces to the second equality in (6.19). □

In order to see that after a suitable translation the functions x_o y_o and z_o can be made equal to zero we prove the following proposition.

Proposition 6.6. *If h satisfies (6.9) and (6.10) on $\boldsymbol{G}\,(\mathbb{H})$ then we have*

a) For $s \in \{1,2,3\}$ and i,j,k a cyclic permutation of $1,2,3$

$$
\begin{aligned}
T_\alpha\ T_\beta\ (\xi_s h) &= (I_iT_\alpha)\ (I_iT_\beta)\ (\xi_s h) \ =\ 0 \quad \forall\ \alpha,\ \beta\\
(I_iT_\alpha)\ T_\beta\ (\xi_s h) &= (I_iT_\alpha)\ (I_iT_\beta)\ (\xi_s h) \ =\ 0,\ \ \alpha\neq\beta\\
(I_j\,T_\alpha)\,T_\alpha\ (\xi_s\, h) &= -\ T_\alpha\,(I_j\,T_\alpha)\ (\xi_s h) \ =\ 8\ \delta_{sj}\ \mu_o\\
(I_j\,T_\alpha)\,(I_i\,T_\alpha)\ (\xi_s h) &= -\ (I_i\,T_\alpha)\,(I_j\,T_\alpha)\ (\xi_s h) \ =\ 8\ \delta_{sk}\ \mu_o\ ,
\end{aligned}
\tag{6.21}
$$

i.e., the horizontal Hessian of a vertical derivative of h is determined completely.
b) There is a point $(q_o,\omega_o)\in \boldsymbol{G}\,(\mathbb{H})$, with $q_o=(q_o^1,q_o^2,\dots,q_o^n)\in\mathbb{H}^n$ and $\omega_o = ix_o+jy_o+kz_o\in Im(\mathbb{H})$ such that

$$ix_o(q)\ +\ jy_o(q)\ +\ kz_o(q)\ =\ w_o\ +\ 2\ Im\ q_o\,\bar q.$$

PROOF. a) Taking $\alpha \neq \beta$ and using $X = T_\alpha$ and $Y = T_\beta$ in (6.19), we obtain

$$T_\alpha T_\beta \xi_s h = 0, \quad \alpha \neq \beta.$$

When $\alpha = \beta$ the same equality holds by (6.15). The vanishing of the other derivatives can be obtained similarly. Finally, the rest of the second derivatives can be determined from (6.11).

b) From the identities in (6.21) all second derivatives of x_o, y_o and z_o vanish. Thus x_o, y_o and z_o are linear function. The fact that the coefficients are related as required amounts to the following system

$$\begin{aligned} T_\alpha x_o &= Z_\alpha y_o = -Y_\alpha z_o, & X_\alpha x_o &= Y_\alpha y_o = Z_\alpha z_o \\ Y_\alpha x_o &= X_\alpha y_o = -T_\alpha z_o, & Z_\alpha x_o &= -T_\alpha y_o = -X_\alpha z_o. \end{aligned}$$

By (6.14), we have

$$\begin{aligned} \xi_1 h &= 4\mu_o(x + x_o(q)), \quad \xi_2 h = 4\mu_o(y + y_o(q)), \\ \xi_3 h &= 4\mu_o(z + z_o(q)). \end{aligned}$$

Therefore, the above system is equivalent to

$$\begin{aligned} T_\alpha \xi_1\, h &= Z_\alpha \xi_2\, h = -Y_\alpha \xi_3\, h, & X_\alpha \xi_1\, h &= Y_\alpha \xi_2\, h = Z_\alpha \xi_3\, h \\ Y_\alpha \xi_1\, h &= X_\alpha \xi_2\, h = -T_\alpha \xi_3\, h & Z_\alpha \xi_1\, h &= -T_\alpha \xi_2\, h = -X_\alpha \xi_3\, h. \end{aligned}$$

Let us prove the first line. Denote

$$a = T_\alpha \xi_1\, h, \quad b = Z_\alpha \xi_2\, h, \quad c = -Y_\alpha \xi_3\, h.$$

From (6.11) and (3.1) it follows

$$\begin{aligned} a &= T_\alpha Z_\alpha Y_\alpha\, h = Z_\alpha T_\alpha Y_\alpha\, h + [T_\alpha, Z_\alpha] Y_\alpha\, h = -b + 2c \\ b &= Z_\alpha Y_\alpha T_\alpha\, h = Y_\alpha Z_\alpha T_\alpha\, h + [Z_\alpha, Y_\alpha] T_\alpha\, h = -c + 2a \\ c &= Y_\alpha T_\alpha Z_\alpha\, h = T_\alpha Y_\alpha Z_\alpha\, h + [Y_\alpha, T_\alpha] Z_\alpha\, h = -a + 2b, \end{aligned}$$

which implies $a = b = c$. The rest of the identities of the system can be obtained analogously. □

So far we have proved that if h satisfies the system (6.9) and (6.10) on $\boldsymbol{G}\,(\mathbb{H})$ then, in view of the translation invariance of the system, after a suitable translation we have

$$h(q,\omega) \;=\; g(q) \;+\; \mu_o\,(x^2 \;+\; y^2 \;+\; z^2).$$

PROPOSITION 6.7. *If h satisfies the system* (6.9) *and* (6.10) *on $\boldsymbol{G}\,(\mathbb{H})$ then after a suitable translation we have*

$$g(q) \;=\; (b \;+\; 1 \;+\; \sqrt{\mu_o}\;|q|^2)^2, \qquad b \;+\; 1 \;>\; 0.$$

PROOF. Notice that

$$\xi_1 h \;=\; 4\mu_o x, \xi_2 h \;=\; 4\mu_o y, \xi_3 h \;=\; 4\mu_o z.$$

Using this, (6.11) becoms

$$\begin{aligned} T_\alpha X_\alpha(h) = Y_\alpha Z_\alpha(h) = -X_\alpha T_\alpha(h) = -Z_\alpha Y_\alpha(h) = -4\mu_o x,\\ T_\alpha Y_\alpha(h) = Z_\alpha X_\alpha(h) = -Y_\alpha T_\alpha(h) = -X_\alpha Z_\alpha(h) = -4\mu_o y,\\ T_\alpha Z_\alpha(h) = X_\alpha Y_\alpha(h) = -Z_\alpha T_\alpha(h) = -Y_\alpha X_\alpha(h) = -4\mu_o z.\end{aligned} \tag{6.22}$$

Let us also write explicitly some of the derivatives of f, which shall be used to express the derivatives of g by the derivatives of h. For all α and β we have

$$\begin{aligned} T_\beta f \;&=\; 4\,(x^\beta x \;+\; y^\beta y \;+\; z^\beta z), & X_\beta f \;&=\; 4\,(-t^\beta x \;-\; z^\beta y \;+\; y^\beta z),\\ Y_\beta f \;&=\; 4\,(z^\beta x \;-\; t^\beta y \;-\; x^\beta z), & Z_\beta f \;&=\; 4\,(-y^\beta x \;+\; x^\beta y \;-\; t^\beta z),\\ T_\alpha T_\beta f \;&=\; 8\,(x^\alpha x^\beta \;+\; y^\alpha y^\beta \;+\; z^\alpha z^\beta), & X_\alpha X_\beta f \;&=\; 8\,(t^\alpha t^\beta \;+\; z^\alpha z^\beta \;+\; y^\alpha y^\beta),\\ Y_\alpha Y_\beta f \;&=\; 8\,(z^\alpha z^\beta \;+\; t^\alpha t^\beta \;+\; x^\alpha x^\beta), & Z_\alpha Z_\beta f \;&=\; 8\,(y^\alpha y^\beta \;+\; x^\alpha x^\beta \;+\; t^\alpha t^\beta),\end{aligned}$$

$$
\begin{aligned}
T_\alpha X_\beta f &= -4\delta_{\alpha\beta}x + 8(-x^\alpha t^\beta - y^\alpha z^\beta + z^\alpha y^\beta),\\
T_\alpha Y_\beta f &= -4\delta_{\alpha\beta}y + 8(x^\alpha z^\beta - y^\alpha t^\beta - z^\alpha x^\beta),\\
T_\alpha Z_\beta f &= -4\delta_{\alpha\beta}z + 8(-x^\alpha y^\beta + y^\alpha x^\beta - z^\alpha t^\beta),\\
X_\alpha T_\beta f &= 4\delta_{\alpha\beta}x + 8(-t^\alpha x^\beta - z^\alpha y^\beta + y^\alpha z^\beta),\\
X_\alpha Y_\beta f &= -4\delta_{\alpha\beta}z + 8(-t^\alpha z^\beta + z^\alpha t^\beta - y^\alpha x^\beta),\\
X_\alpha Z_\beta f &= 4\delta_{\alpha\beta}y + 8(t^\alpha y^\beta - z^\alpha x^\beta + y^\alpha t^\beta).
\end{aligned}
$$

From the above formulas we see that the fifth order horizontal derivatives of f vanish. In particular the fifth order derivatives of h and g coincide.

Taking $X = Y = T_\alpha$ in (6.17) we obtain

$$4T_\alpha^2 h \; - \; 2h^{-1}\{(T_\alpha h)^2 + (X_\alpha h)^2 + (Y_\alpha h)^2 + (Z_\alpha h)^2\} = \lambda. \tag{6.23}$$

Using in the same manner X_α, Y_α and Z_α we see the equality of the second derivatives

$$T_\alpha^2 h \; = \; X_\alpha^2 h \; = \; Y_\alpha^2 h \; = \; Z_\alpha^2 h. \tag{6.24}$$

Therefore, using (6.11) and (6.13), we have

$$T_\alpha^3 h = T_\alpha X_\alpha^2 h = X_\alpha T_\alpha X_\alpha h + [T_\alpha, X_\alpha] X_\alpha h = -3X_\alpha \xi_1 h = 24\mu_o t^\alpha$$

and thus $T_\alpha^4 h \; = \; 24\mu_o$. In the same fashion we conclude

$$
\begin{aligned}
T_\alpha^3 h \; &= \; 24\mu_o t^\alpha \qquad X_\alpha^3 h \; = \; 24\mu_o x^\alpha,\\
Y_\alpha^3 h \; &= \; 24\mu_o y^\alpha, \quad Z_\alpha^3 h \; = \; 24\mu_o z^\alpha.
\end{aligned} \tag{6.25}
$$

Similarly, taking $X = T_\alpha$, $Y = X_\beta$ and $j = 1$ in (6.19), we find

$$T_\alpha X_\beta h \; = \; 8\mu_o \; (-x^\alpha t^\beta \; + \; t^\alpha x^\beta \; - \; y^\alpha z^\beta \; + \; z^\alpha y^\beta), \qquad \alpha \; \neq \; \beta.$$

Plugging $X = T_\alpha$, $Y = T_\beta$ with $\alpha \neq \beta$ in (6.19) we obtain

$$T_\alpha T_\beta h \;=\; X_\alpha h X_\beta h \;=\; Y_\alpha h Y_\beta h \;=\; Z_\alpha h Z_\beta h$$
$$=\; 8\mu_o(t^\alpha t^\beta \;+\; x^\alpha x^\beta \;+\; y^\alpha y^\beta \;+\; z^\alpha z^\beta).$$

The other mixed second order derivatives when $\alpha \neq \beta$ can be obtained by taking suitable X and Y. In view of the formulas for the derivatives of f and (6.22), we conclude

$$\begin{aligned}
&T_\alpha X_\beta\, g \;=\; 8\mu_o t^\alpha x^\beta, \quad T_\alpha Y_\beta\, g \;=\; 8\mu_o t^\alpha y^\beta, \quad T_\alpha Z_\beta\, g \;=\; 8\mu_o t^\alpha z^\beta\\
&X_\alpha Y_\beta\, g \;=\; 8\mu_o x^\alpha y^\beta, \quad X_\alpha Z_\beta\, g \;=\; 8\mu_o x^\alpha z^\beta, \quad Y_\alpha Z_\beta\, g \;=\; 8\mu_o y^\alpha z^\beta,\\
&T_\alpha T_\beta g \;= 8\mu_o t^\alpha t^\beta, \quad X_\alpha X_\beta g \;= 8\mu_o x^\alpha x^\beta, \quad Y_\alpha Y_\beta g \;= 8\mu_o y^\alpha y^\beta,\\
&Z_\alpha Z_\beta g \;= 8\mu_o z^\alpha z^\beta, \quad T_\alpha X_\alpha\, g \;=\; 8\mu_o t^\alpha x^\alpha, \quad T_\alpha Y_\alpha\, g \;=\; 8\mu_o t^\alpha y^\alpha,\\
&T_\alpha Z_\alpha\, g \;=\; 8\mu_o t^\alpha z^\alpha \quad X_\alpha Y_\alpha\, g \;=\; 8\mu_o x^\alpha y^\alpha, \quad X_\alpha Z_\alpha\, g \;=\; 8\mu_o x^\alpha z^\alpha,\\
&Y_\alpha Z_\alpha\, g \;=\; 8\mu_o y^\alpha z^\alpha.
\end{aligned} \tag{6.26}$$

A consequences of the considerations so far is the fact that all second order derivative are quadratic functions of the variables from the first layer, except the pure (unmixed) second derivatives, in which case we know (6.24) and (6.25). It is easy to see then that the fifth order horizontal derivatives of h vanish. With the information so far after a small argument we can assert that g is a polynomial of degree 4 without terms of degree 3, and of the form

$$g = \mu_o \sum_{\alpha=1}^{n} (t_\alpha^4 \;+\; x_\alpha^4 \;+\; y_\alpha^4 \;+\; z_\alpha^4) \;+ p_2,$$

where p_2 is a polynomial of degree two. Furthermore, the mixed second order derivatives of g are determined, while the pure second order derivatives are equal. The latter follows from (6.23) taking $q = 0$, $\omega = 0$. Let us see that there are no terms of degree one on p_2. Taking $X = T_\alpha$, $Y = T_\beta$, $\alpha \neq \beta$ and $j = 1$ in (6.20) we find

$$
\begin{aligned}
&(4\mu_o x)\,\{\,4T_\alpha T_\beta g \;+\; 32\mu_o(x^\alpha x^\beta \;+\; y^\alpha y^\beta \;+\; z^\alpha z^\beta)\,\}\\
&= 2\Big\{(8x^\alpha)(T_\beta g + 4\mu_o(x^\beta x + y^\beta y + z^\beta z)) + (-8t^\alpha)(X_\beta g + 4\mu_o(-t^\beta x - z^\beta y + y^\beta z))\\
&+ (8z^\alpha)(Y_\beta g + 4\mu_o(z^\beta x - t^\beta y - x^\beta z)) + (-8y^\alpha)(Z_\beta g + 4\mu_o(-y^\beta x + x^\beta y - t^\beta z))\\
&+ (8x^\beta)(T_\alpha g + 4\mu_o(x^\alpha x + y^\alpha y + z^\alpha z)) + (-8t^\beta)(X_\alpha g + 4\mu_o(-t^\alpha x - z^\alpha y + y^\alpha z))\\
&+ (8z^\beta)(Y_\alpha g + 4\mu_o(z^\alpha x - t^\alpha y - x^\alpha z)) + (-8y^\beta)(Z_\alpha g + 4\mu_o(-y^\alpha x + x^\alpha y - t^\alpha z))\Big\}\\
&= 16\,\big(x^\alpha T_\beta g + x^\beta T_\alpha g - t^\alpha X_\beta g - t^\beta X_\alpha g + z^\alpha Y_\beta g + z^\beta Y_\alpha g - y^\alpha Z_\beta g - y^\beta Z_\alpha g\big)\\
&\qquad\qquad + 128\mu_o\, x\,(t^\alpha t^\beta + x^\alpha x^\beta + t^\alpha t^\beta + x^\alpha x^\beta)
\end{aligned}
$$

Taking into account (6.26), we have proved

$$
\begin{aligned}
&x^\alpha T_\beta g + x^\beta T_\alpha g - t^\alpha X_\beta g - t^\beta X_\alpha g\\
&\qquad\qquad + z^\alpha Y_\beta g + z^\beta Y_\alpha g - y^\alpha Z_\beta g - y^\beta Z_\alpha g = 0, \quad \alpha \neq \beta.
\end{aligned}
$$

Comparing coefficients in front of the linear terms implies that g has no first order terms. Thus, we can assert that g can be written in the following form

$$
g \;=\; \big(1 \;+\; \sqrt{\mu_o}\,|q|^2\big)^2 \;+\; 2a\,|q|^2 \;+\; b. \tag{6.27}
$$

Hence,

$$
h \;=\; \big(1 \;+\; \sqrt{\mu_o}\,|q|^2\big)^2 \;+\; a\,|q|^2 \;+\; b \;+\; \mu_o\,(x^2 \;+\; y^2 \;+\; z^2).
$$

Taking $X = T_\alpha$, $Y = T_\beta$ in (6.18), we obtain

$$
16\mu_0\,(1 \;+\; b) \;=\; 4\,(a \;+\; 2\sqrt{\mu_o})^2.
$$

Therefore,

$$g = \mu_o |q|^4 + (a + 2\sqrt{\mu_o}) \sqrt{\mu_o} |q|^2 + b + 1$$
$$= 2\sqrt{b+1} |q|^2 + b + 1 = \left(b + 1 + \sqrt{\mu_o} |q|^2\right)^2.$$

In turn the formula for h becomes

$$h = \left(b + 1 + \sqrt{\mu_o} |q|^2\right)^2 + \mu_o (x^2 + y^2 + z^2). \tag{6.28}$$

Setting $c = (b+1)^2$ and $\nu = \frac{\sqrt{\mu_o}}{1+b} > 0$ the solution takes the form

$$h = c \left[\left(1 + \nu |q|^2\right)^2 + \nu^2 (x^2 + y^2 + z^2)\right],$$

which completes the proof of Theorem A. □

Let us note that the final conclusion can be reached also using the fact that a QC Einstein structure has necessarily constant scalar curvature by Theorem 5.9, together with the result of [**GV1**] identifying all partially symmetric solutions of the Yamabe equation on $\boldsymbol{G}\,(\mathbb{H})$, i.e., of the equation

$$\sum_{\alpha=1}^{n} \left(T_\alpha^2 u + X_\alpha^2 u + Y_\alpha^2 u + Z_\alpha^2 u\right) = -u^{\frac{n_h+2}{n_h-2}}.$$

The fact that we are dealing with such a solution follows from (6.27). The current solution depends on one more parameter as the scalar curvature can be an arbitrary constant. This constant will appear in the argument of [**GV1**] by first using scalings to reduce to a fixed scalar curvature one for example.

7. Special functions and pseudo-Einstein QC structures

Considering only the [3]-component of the Einstein tensor of the Biquard connection due to Theorem 4.13 and by analogy with the CR-case [**L1**], it seems useful to give the following Definition.

Definition 7.1. *Let (M, H, g) be a quaternionic-contact manifold of dimension bigger than 7. We call M* QC pseudo-Einstein *if the trace-free part of the [3]-component of the QC Einstein tensor vanishes.*

Observe that for $n = 1$ any QC structure is QC pseudo-Einstein. According to Theorem 4.13 (M, H, g) is quaternionic QC pseudo-Einstein exactly when the trace-free part of the [3]-component of the torsion vanishes, $U = 0$. Proposition 6.1 yields the following claim.

PROPOSITION 7.2. *Let $\bar\eta = u\eta$ be a conformal transformation of a given QC structure. Then the trace-free part of the* [3] *component of the QC Ricci tensor (i.e. U) is preserved if and only if the function u satisfies the differential equations*

$$(\nabla_X du)Y + (\nabla_{I_1X} du)I_1Y + (\nabla_{I_2X} du)I_2Y + (\nabla_{I_3X} du)I_3Y = \frac{1}{n}\Delta u\, g(X, Y). \tag{7.1}$$

In particular, the QC pseudo-Einstein condition persists under conformal transformation $\tilde\eta = u\eta$ exactly when the function u satisfies (7.1).

PROOF. Defining $h = u^{-1}$ a small calculation shows

$$\nabla dh - 2h^{-1} dh \otimes dh = u^{-1}\nabla du. \tag{7.2}$$

Inserting (7.2) into (6.10) shows (7.1). □

Our next goal is to investigate solutions to (7.1). We shall find geometrically defined functions, which are solutions of (7.1).

7.1. Quaternionic pluriharmonic functions. We start with some analysis on the quaternion space $\mathbb{H}^n$.

7.1.1. *Pluriharmonic functions in $\mathbb{H}^n$.* Let $\mathbb{H}$ be the four-dimensional real associative algebra of the quaternions. The elements of $\mathbb{H}$ are of the form $q = t + ix + jy + kz$, where $t, x, y, z \in \mathbb{R}$ and $i,'\ j,\ k$ are the basic quaternions satisfying the multiplication rules $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. For a quaternion q we define its conjugate $\bar q = t - ix - jy - kz$, and real and imaginary parts, correspondingly, by $\Re q = t$ and $\Im q = xi + yj + zk$.The most important operator for us is the Dirac-Feuter operator $\overline{\mathcal{D}} = \partial_t + i\partial_x + j\partial_y + k\partial_z$, i.e.,

$$\overline{\mathcal{D}}\, F = \partial_t F + i\partial_x F + j\partial_y F + k\partial_z F$$

and in addition

$$\mathcal{D}F = \partial_t F - i\partial_x F - j\partial_y F - k\partial_z F.$$

Note that if F is a quaternionic valued function due to the non-commutativity of the multiplication the above expression is not the same as

$$F\overline{\mathcal{D}} \ \stackrel{def}{=}\ \partial_t F \ +\ \partial_x F i \ +\ \partial_y F j \ +\ \partial_z F k.$$

Also, when conjugating, $\overline{\overline{\mathcal{D}}\ F} \neq \mathcal{D}\overline{F}$.

Definition 7.3. *A function $F \ :\ \mathbb{H} \to \ \mathbb{H}$, which is continuously differentiable when regarded as a function of $\mathbb{R}^4$ into $\mathbb{R}^4$ is called* quaternionic anti-regular *(*quaternionic regular*), or just* anti-regular *(*regular*) for short, if $\mathcal{D}F = 0$ $(\overline{\mathcal{D}}\ F\ =\ 0)$.*

These functions were introduced by Fueter [**F**]. The reader can consult the paper of A. Sudbery [**S**] for the basics of the quaternionic analysis on $\mathbb{H}$. Let us note explicitly one of the most striking differences between complex and quaternionic analysis. As it is well known the theory of functions of a complex variable z is equivalent to the theory of power series of z. In the quaternionic case, each of the coordinates t, x, y and z can be written as a polynomial in q, see eq. (3.1) of [**S**], and hence the theory of power series of q is just the theory of real analytic functions. Our goal here is to consider functions of several quaternionic variables in $\mathbb{H}^n$ and on manifolds with quaternionic structure and present some applications in geometry.

For a point $q \in \mathbb{H}^n$ we shall write $q = (q^1, \dots, q^n)$ with

$$q^\alpha \in \mathbb{H}, \quad q^\alpha = t^\alpha + ix^\alpha + jy^\alpha + kz^\alpha, \quad \alpha = 1, \dots, n.$$

Furthermore, $q^{\overline{\alpha}} = \overline{q^\alpha}$, i.e., by definition, $q^{\overline{\alpha}} = t^\alpha - ix^\alpha - jy^\alpha - kz^\alpha$.

We recall that a function $F\ :\ \mathbb{H}^n \to\ \mathbb{H}$, which is continuously differentiable when regarded as a function of $\mathbb{R}^{4n}$ into $\mathbb{R}^4$ is called quaternionic regular, or just regular for short, if

$$\overline{\mathcal{D}}_\alpha\ F\ =\ \partial_{t_\alpha}F\ +\ i\partial_{x_\alpha}F\ +\ j\partial_{y_\alpha}F\ +\ k\partial_{z_\alpha}F\ =\ 0, \quad \alpha = 1, \dots, n.$$

In other words, a real-differentiable function of several quaternionic variables is regular if it is regular in each of the variables (see [**Per1, Per2, Joy**]. The condition that $F = f + iw + ju + kv$ is regular is equivalent to the following Cauchy-Riemann-Feuter equations

$$\begin{aligned} \partial_{t_\alpha} f - \partial_{x_\alpha} w - \partial_{y_\alpha} u - \partial_{z_\alpha} v \ &=\ 0, \quad & \partial_{t_\alpha} w + \partial_{x_\alpha} f + \partial_{y_\alpha} v - \partial_{z_\alpha} u \ &=\ 0, \\ \partial_{t_\alpha} u - \partial_{x_\alpha} v + \partial_{y_\alpha} f + \partial_{z_\alpha} w \ &=\ 0, \quad & \partial_{t_\alpha} v + \partial_{x_\alpha} u - \partial_{y_\alpha} w + \partial_{z_\alpha} f \ &=\ 0. \end{aligned} \tag{7.3}$$

DEFINITION 7.4. *A real-differentiable function* $f : \mathbb{H}^n \mapsto \mathbb{R}$ *is called* $\bar{Q}$-pluriharmonic *if it is the real part of a regular function.*

PROPOSITION 7.5. *Let* f *be a real-differentiable function* $f : \mathbb{H}^n \mapsto \mathbb{R}$. *The following conditions are equivalent:*

(i) f *is* $\bar{Q}$*-pluriharmonic, i.e., it is the real part of a regular function;*

(ii) $\overline{\mathcal{D}}_\beta\, \mathcal{D}_\alpha\, f = 0$ *for every* $\alpha, \beta \in \{1, \dots, n\}$, *where* $\mathcal{D}_\alpha = \partial_{t_\alpha} - i\partial_{x_\alpha} - j\partial_{y_\alpha} - k\partial_{z_\alpha}$*;*

(iii) f *satisfies the following system of PDEs*

$$
\begin{aligned}
f_{t_\alpha t_\beta} + f_{x_\alpha x_\beta} + f_{y_\alpha y_\beta} + f_{z_\alpha z_\beta} = 0, \quad f_{t_\alpha x_\beta} - f_{x_\alpha t_\beta} - f_{z_\alpha y_\beta} + f_{y_\alpha z_\beta} = 0 \\
-f_{y_\alpha t_\beta} + f_{z_\alpha x_\beta} + f_{t_\alpha y_\beta} - f_{x_\alpha z_\beta} = 0, \quad -f_{z_\alpha t_\beta} - f_{y_\alpha x_\beta} + f_{x_\alpha y_\beta} + f_{t_\alpha z_\beta} = 0.
\end{aligned}
\tag{7.4}
$$

PROOF. It is easy to check that $\overline{\mathcal{D}}_\beta\, \mathcal{D}_\alpha\, f = 0$ is equivalent to (7.4).

We turn to the proof of ii) implies i). Let f be real valued function on $\mathbb{H}^n$, such that,

$$\overline{\mathcal{D}}_\beta\, \mathcal{D}_\alpha\, f \;=\; 0.$$

We shall construct a real-differentiable regular function

$$F : \mathbb{H}^n \mapsto \mathbb{H}.$$

In fact, for $q \in \mathbb{H}^n$ we define

$$F(q) \;=\; f(q) \;+\; \Im \int_0^1 s^2\, (\mathcal{D}_\alpha\, f)(sq)\, q^\alpha\, ds.$$

In order to rewrite the imaginary part in a different way we compute

$$
\begin{aligned}
&\Re \int_o^1 s^2\, (\mathcal{D}_\alpha\, f)(sq)\, q_\alpha\, ds \\
&\qquad = \Re \int_0^1 s^2 \Big(\partial_{t_\alpha} f - i\partial_{x_\alpha} f - j\partial_{y_\alpha} f - k\partial_{z_\alpha} f\Big)\, (sq)\, (t_\alpha + ix_\alpha + jy_\alpha + kz_\alpha)\, ds
\end{aligned}
$$

$$= \int_0^1 s^2 \Big(\partial_{t_\alpha} f(sq) t_\alpha + \partial_{x_\alpha} f(sq) x_\alpha + \partial_{y_\alpha} f(sq) y_\alpha + \partial_{z_\alpha} f(sq) z_\alpha\Big)\, ds$$

$$= \int_0^1 s^2 \frac{d}{ds}\Big(f(sq)\Big)\, ds \;=\; s^2 f(sq)|_0^1 \,-\, 2\int_0^1 s f(sq)\, ds = \; f(q) \,-\, 2\int_0^1 s f(sq)\, ds.$$

Therefore we have

$$\Im \int_o^1 s^2 \,(\mathcal{D}_\alpha\ f)(sq)\, q^\alpha\, ds \;=\; \int_o^1 s^2 \,(\mathcal{D}_\alpha\ f)(sq)\, q^\alpha\, ds \;-\; f(q) \;+\; 2\int_0^1 s f(sq)\, ds.$$

In turn, the formula for $F(q)$ becomes

$$F(q) \;=\; \int_o^1 s^2 \,(\mathcal{D}_\alpha\ f)(sq)\, q^\alpha\, ds \;+\; 2\int_0^1 s f(sq)\, ds.$$

Hence

$$\overline{\mathcal{D}}_\beta\ F(q) \;=\; \int_o^1 s^2\, \overline{\mathcal{D}}_\beta\ \Big[(\mathcal{D}_\alpha\ f)(sq)\, q^\alpha\Big]\, ds \;+\; 2\int_0^1 s\, \overline{\mathcal{D}}_\beta\ \Big[f(sq)\Big]\, ds.$$

We compute the first term,

$$\begin{aligned}
\overline{\mathcal{D}}_\beta\ \Big[(\mathcal{D}_\alpha\ f)(sq)\, q^\alpha\Big] &= (\partial_{t_\beta} + i\partial_{x_\beta} + j\partial_{y_\beta} + k\partial_{z_\beta})\Big[(\mathcal{D}_\alpha\ f)(sq)\, q^\alpha\Big] \\
&= \overline{\mathcal{D}}_\beta\ \Big[(\mathcal{D}_\alpha\ f)(sq)\,\Big]\, q^\alpha + \ \mathcal{D}_\alpha\ f(sq)\ \partial_{t_\beta} q_\alpha + \ i\mathcal{D}_\alpha\ f(sq)\ \partial_{x_\beta} q_\alpha \\
&\qquad + \ j\mathcal{D}_\alpha\ f(sq)\ \partial_{y_\beta} q_\alpha \ + \ k\mathcal{D}_\alpha\ f(sq)\ \partial_{z_\beta} q_\alpha
\end{aligned}$$

$$= \overline{\mathcal{D}}_\beta\ \Big[(\mathcal{D}_\alpha\, f)(sq)\,\Big]\, q^\alpha + \delta_{\alpha\beta}\{\mathcal{D}_\alpha\, f(sq) + i\mathcal{D}_\alpha\, f(sq) i + j\mathcal{D}_\alpha\, f(sq) j + k\mathcal{D}_\alpha\, f(sq) k\}.$$

The last term can be simplified, using the fundamental property that the coordinates of a quaternion can be expressed by the quaternion only, as follows

$$
\begin{aligned}
&\mathcal{D}_\beta\, f(sq) \;+\; i\mathcal{D}_\beta\, f(sq)i \;+\; j\mathcal{D}_\beta\, f(sq)j \;+\; k\mathcal{D}_\beta\, f(sq)k \\
&= (\partial_{t_\beta} f \;-\; i\partial_{x_\beta} f \;-\; j\partial_{y_\beta} f \;-\; k\partial_{z_\beta} f) \;+\; i(\partial_{t_\beta} f \;-\; i\partial_{x_\beta} f \;-\; j\partial_{y_\beta} f \;-\; k\partial_{z_\beta} f)i \\
&+ j(\partial_{t_\beta} f \;-\; i\partial_{x_\beta} f \;-\; j\partial_{y_\beta} f \;-\; k\partial_{z_\beta} f)j \;+\; k(\partial_{t_\beta} f \;-\; i\partial_{x_\beta} f \;-\; j\partial_{y_\beta} f \;-\; k\partial_{z_\beta} f)k \\
&= -2\partial_{t_\beta} f \;-\; i\partial_{x_\beta} f \;-\; j\partial_{y_\beta} f \;-\; k\partial_{z_\beta} f \;+\; i\partial_{x_\beta} f \;-\; j\partial_{y_\beta} f \;-\; k\partial_{z_\beta} f \\
&\quad -\; i\partial_{x_\beta} f \;+\; j\partial_{y_\beta} f \;-\; k\partial_{z_\beta} f \;-\; i\partial_{x_\beta} f \;-\; j\partial_{y_\beta} f \;+\; k\partial_{z_\beta} f \\
&= -2\partial_{t_\beta} f \;-\; 2i\partial_{x_\beta} f \;-\; 2j\partial_{y_\beta} f \;-\; 2k\partial_{z_\beta} f \;=\; -2\overline{\mathcal{D}}_\beta\, f(sq).
\end{aligned}
$$

Going back to the computation of $\overline{\mathcal{D}}_\beta\, F(q)$, we find

$$
\begin{aligned}
\overline{\mathcal{D}}_\beta\, F(q) \;=\; &\int_0^1 \overline{\mathcal{D}}_\beta\, \big[(\mathcal{D}_\alpha\, f)(sq)\big]\, q^\alpha\, ds \;-\; 2\int_0^1 s^2 \overline{\mathcal{D}}_\beta\, f(sq)\, ds \\
&+\; 2\int_0^1 s^2 \overline{\mathcal{D}}_\beta\, f(sq)\, ds = \int_0^1 \overline{\mathcal{D}}_\beta\, \big[(\mathcal{D}_\alpha\, f)(sq)\big]\, q^\alpha\, ds.
\end{aligned}
$$

Hence, if $\overline{\mathcal{D}}_\beta\, \mathcal{D}_\alpha\, f \;=\; 0$ for every α and β we have $\overline{\mathcal{D}}_\beta\, F(q) \;=\; 0$.

Next we show that i) implies ii). Using (7.3), we have

$$
\begin{aligned}
f_{t_\alpha x_\beta} - f_{x_\alpha t_\beta} + f_{y_\alpha z_\beta} - f_{z_\alpha y_\beta} = {}& w_{x_\alpha x_\beta} + u_{y_\alpha x_\beta} + v_{z_\alpha x_\beta} + w_{t_\alpha t_\beta} + v_{y_\alpha t_\beta} - u_{z_\alpha t_\beta} \\
& - u_{t_\alpha z_\beta} + v_{x_\alpha z_\beta} - w_{z_\alpha z_\beta} + v_{t_\alpha y_\beta} + u_{x_\alpha y_\beta} - w_{y_\alpha y_\beta}.
\end{aligned}
$$

Both sides must be equal to zero by noticing that the left hand side is antisymmetric while on the right we have an expression symmetric with respect to exchanging α with β. The other identities can be obtained similarly. □

According to [Sti] there are exactly two kinds of Cauchy-Riemann equations for functions of several quaternionic variables. The second one turns out to be most suitable for the geometric purposes considered here.

DEFINITION 7.6. *A function $F : \mathbb{H}^n \to \mathbb{H}$, which is continuously differentiable when regarded as a function of $\mathbb{R}^{4n}$ into $\mathbb{R}^4$ is called* quaternionic anti-regular *(also* anti-regular*), if*

$$\mathcal{D}F \;=\; \partial_{t_\alpha}F \;-\; i\partial_{x_\alpha}F \;-\; j\partial_{y_\alpha}F \;-\; k\partial_{z_\alpha}F \;=\; 0, \quad \alpha = 1,\dots,n.$$

The condition that $F = f + iw + ju + kv$ is anti-regular function is equivalent to the following Cauchy-Riemann-Feuter equations

$$\begin{aligned} \partial_{t_\alpha}f + \partial_{x_\alpha}w + \partial_{y_\alpha}u + \partial_{z_\alpha}v &= 0, \quad & \partial_{t_\alpha}w - \partial_{x_\alpha}f - \partial_{y_\alpha}v + \partial_{z_\alpha}u &= 0, \\ \partial_{t_\alpha}u + \partial_{x_\alpha}v - \partial_{y_\alpha}f - \partial_{z_\alpha}w &= 0, \quad & \partial_{t_\alpha}v - \partial_{x_\alpha}u + \partial_{y_\alpha}w - \partial_{z_\alpha}f &= 0. \end{aligned} \tag{7.5}$$

See also (7.8) for an equivalent form of the above system.

Anti-regular functions on hyperkähler and quaternionic Kähler manifolds are studied in [**CL1, CL2, LZ**], under the name quaternionic maps, in connection with minimal surfaces and maps between quaternionic Kähler manifolds preserving the sphere of almost complex structures. Thus, the anti-regular functions considered here are quaternionic maps between $\mathbb{H}^n$ and $\mathbb{H}$ with a suitable choice of the coordinates.

Definition 7.7. *A real-differentiable function $f : \mathbb{H}^n \mapsto \mathbb{R}$ is called* quaternionic pluriharmonic *(* Q-pluriharmonic *for short) if it is the real part of an anti-regular function.*

The anti-regular functions and their real part play a significant role in the theory of hypercomplex manifold as well as in the theory of quaternionic-contact (hypercomplex contact) manifolds as we shall see further in the thesis. We need a real expression of the second order differential operator $\mathcal{D}_\alpha\,\overline{\mathcal{D}}_\beta\,f$ acting on a real function f.

We use the standard hypercomplex structure on $\mathbb{H}^n$ determined by the action of the imaginary quaternions

$$I_1\,dt^\alpha \;=\; dx^\alpha, \quad I_1\,dy^\alpha \;=\; dz^\alpha, \qquad I_2\,dt^\alpha \;=\; dy^\alpha, \quad I_2\,dx^\alpha \;=\; -\,dz^\alpha$$
$$I_3\,dt^\alpha \;=\; dz^\alpha, \qquad I_3\,dx^\alpha \;=\; dy^\alpha.$$

We recall a convention. For any p-form ψ we consider the p-form $I_s\psi$ and three (p+1)-forms $d_s\psi$, $s = 1,2,3$ defined by

$$I_s\psi(X_1,\dots,X_p) \overset{def}{=} (-1)^p\psi(I_sX_1,\dots,I_sX_p), \qquad d_s\psi \overset{def}{=} (-1)^p I_s d I_s\psi.$$

Consider the second order differential operators DD_{I_s} acting on the exterior algebra defined by [HP]

$$DD_{I_i} := dd_i + d_j d_k = dd_i - I_j dd_i = dd_i - I_k dd_i. \tag{7.6}$$

PROPOSITION 7.8. *Let f be a real-differentiable function $f : \mathbb{H}^n \to \mathbb{R}$. The following conditions are equivalent*

i) f is Q-pluriharmonic, i.e. it is the real part of an anti-regular function;

ii) $DD_{I_s} f \;=\; 0, \quad s = 1, 2, 3.$;

iii) $\mathcal{D}_\alpha \, \overline{\mathcal{D}}_\beta \, f \;=\; 0$ for every $\alpha, \beta \in \{1, \dots, n\}$, where

$$\mathcal{D}_\alpha \;=\; \partial_{t_\alpha} \;-\; i\partial_{x_\alpha} \;-\; j\partial_{y_\alpha} \;-\; k\partial_{z_\alpha},$$

iv) f satisfies the following system of PDEs

$$\begin{aligned} f_{t_\alpha t_\beta} + f_{x_\alpha x_\beta} + f_{y_\alpha y_\beta} + f_{z_\alpha z_\beta} \;&=\; 0, \quad -f_{t_\alpha x_\beta} + f_{x_\alpha t_\beta} - f_{z_\alpha y_\beta} + f_{y_\alpha z_\beta} \;=\; 0, \\ f_{y_\alpha t_\beta} + f_{z_\alpha x_\beta} - f_{t_\alpha y_\beta} - f_{x_\alpha z_\beta} \;&=\; 0, \quad f_{z_\alpha t_\beta} - f_{y_\alpha x_\beta} + f_{x_\alpha y_\beta} - f_{t_\alpha z_\beta} \;=\; 0. \end{aligned}$$

PROOF. A simple calculation of $\mathcal{D}_\beta \, \overline{\mathcal{D}}_\alpha \, f$ gives the equivalence between iii) and iv).

Next, we shall show that ii) is equivalent to iii). As

$$df = \partial_{t_\alpha} f \, dt^\alpha + \partial_{x_\alpha} f \, dx^\alpha + \partial_{y_\alpha} f \, dy^\alpha + \partial_{z_\alpha} f \, dz^\alpha,$$

we have

$$I_1 df = \partial_{t_\alpha} f \, dx^\alpha - \partial_{x_\alpha} f \, dt^\alpha + \partial_{y_\alpha} f \, dz^\alpha - \partial_{z_\alpha} f \, dy^\alpha.$$

A routine calculation gives the following formula

$$D\,D_{I_1} f \;=\; \sum_{\alpha,\beta} \Re(\mathcal{D}_\beta \, \overline{\mathcal{D}}_\alpha \; f) \, [\, dt^\alpha \wedge \, dx^\beta + \, dy^\alpha \wedge \, dz^\beta] \tag{7.7}$$

$$+\sum_{\alpha<\beta}\Re(i\mathcal{D}_\beta\,\overline{\mathcal{D}}_\alpha\,f)\,[-\,dt^\alpha\wedge dt^\beta - dx^\alpha\wedge dx^\beta + dy^\alpha\wedge dy^\beta + dz^\alpha\wedge dz^\beta]$$

$$+\sum_{\alpha,\beta}\Re(j\mathcal{D}_\beta\,\overline{\mathcal{D}}_\alpha\,f)\,[\,dt^\alpha\wedge dz^\beta - dx^\alpha\wedge dy^\beta] -$$

$$\sum_{\alpha,\beta}\Re(k\mathcal{D}_\beta\,\overline{\mathcal{D}}_\alpha\,f)\,[\,dt^\alpha\wedge dy^\beta + dx^\alpha\wedge dz^\beta].$$

Similar formulas hold for DD_{I_2} and DD_{I_3}. Hence, the equivalence of ii) and iii) follows.

The proof of the implication iii) implies i) is analogous to the proof of the corresponding implication in Proposition 7.5. Define

$$F(q)\ =\ f(q)\ +\ \Im\int_o^1 s^2\,(\overline{\mathcal{D}}_\beta\,f)(sq)\,q^{\overline{\beta}}\,ds,$$

and a small calculation shows that this defines an anti-regular function, i.e., $\mathcal{D}_\alpha\,F\ =\ 0$ for every α.

In order to see that iii) follows from i) we can proceed as in Proposition 7.5 and hence we skip the details. See also another proof in Proposition 7.11 □

REMARK 7.9. *We note that Proposition 7.5 and Proposition 7.8 imply that the real part of a regular function is not in the kernel of the operators DD_{I_s} which is one of the main difference between regular and anti-regular function.*

7.2. Quaternionic pluriharmonic functions on hypercomplex manifold. We recall that a hypercomplex manifold is a smooth $4n$-dimensional manifold M together with a triple (I_1, I_2, I_3) of integrable almost complex structures satisfying the quaternionic relations $I_1I_2 = -I_2I_1 = I_3$. The second order differential operators DD_{I_i} defined in [HP] by (7.21) having the origin in the papers [Sal1, Sal2, CSal] play an important róle in the theory of quaternionic plurisubharmonic functions (i.e. a real function for which $DD_{I_s}(.,I_s.)$ is positive definite) on hypercomplex manifold [A1, A2, V, AV, A3] as well as the potential theory of HKT-manifolds. We recall that Riemannian metric g on a hypercomplex manifold compatible with the three complex structures is said to be HKT-metric [HP] if the three corresponding Kähler forms $\Omega_s = g(I_s.,.)$ satisfy

$$d_1\Omega_1 = d_2\Omega_2 = d_3\Omega_3.$$

A smooth real function is a HKT-potential if locally it generates the three Kähler forms, $\Omega_s = DD_{I_s}f$ [MS, GP], in particular such a function is quaternionic plurisubharmonic. The existence of a HKT potential on any HKT metric on $\mathbb{H}^n$ is proved in [MS] and for any HKT metric in [BS].

Regular functions on hypercomplex manifold are studied from analytical point [Per1, Per2], from algebraic point [Joy, Q]. However, as we have already mentioned, regular functions are not the appropriate functions for our purposes mainly because they have no direct connection with the second order differential operator DD_{I_s}.

Here we consider anti-regular functions and their real parts on hypercomplex manifold.

Definition 7.10. *Let (M, I_1, I_2, I_3) be a hypercomplex manifold. A quaternionic valued function*

$$F : M \longrightarrow f + iw + ju + kv \in \mathbb{H}$$

is said to be anti-regular *if any one of the following relations between the differentials of the coordinates hold*

$$\begin{aligned} df &= d_1w + d_2u + d_3v \qquad & d_1f &= -dw + d_3u - d_2v \\ d_2f &= -d_3w - du + d_1v \qquad & d_3f &= d_2w - d_1u - dv. \end{aligned} \tag{7.8}$$

A real valued function $f : M \longrightarrow \mathbb{R}$ is said to be quaternionic pluriharmonic *(or* Q-pluriharmonic*) if it is the real part of an anti-regular function.*

Observe that the system (7.5) is equivalent to (7.8). We have the hypercomplex manifold analogue of Proposition 7.8

Proposition 7.11. *Let (M, I_1, I_2, I_3) be a hypercomplex manifold and let f be a real-differentiable function on M, $f : M \longrightarrow \mathbb{R}$. The following conditions are equivalent*

i) f is Q-pluriharmonic, i.e. it is the real part of an anti-regular function;

ii) $DD_{I_s}f \;=\; 0, \quad s = 1, 2, 3.$

Proof. It is easy to verify that if each I_s is integrable almost complex structure then we have the identities [HP]

$$dd_s + d_sd = 0, \qquad d_sd_r + d_rd_s = 0, \quad s, r = 1, 2, 3. \tag{7.9}$$

Using the commutation relations (7.9), we get readily that i) implies ii). For example, (7.8) yields

$$
\begin{aligned}
dd_1f + d_2d_3f + d^2w - d_2^2w - dd_3u + d_2d_1u + dd_2v + d_2dv &= 0\\
d_1df + d_3d_2f - d_1^2w + d_3^2w - d_1d_2u + d_3du - d_1d_3v - d_3d_1v &= 0.
\end{aligned}
$$

Subtracting the two equations and using the commutation relations (7.9) we get $DD_{I_1}f = 0$.

For the converse, observe that

$$DD_{I_1}f = 0 \Leftrightarrow dd_1f = I_2dd_1f.$$

The $\partial\bar{\partial}$-lemma for I_2 gives the existense of a smooth function A_1 such that

$$dd_1f = dd_2A_1.$$

Similarly, using the Poincare lemma, we obtain

$$d_1f - d_2A_1 - dB_1 = 0, \quad d_2f - d_3A_2 - dB_2 = 0, \quad d_3f - d_1A_3 - dB_3 = 0$$

for a smooth functions $A_1, A_2, A_3, B_1, B_2, B_3$. The latter implies

$$df + d_1(A_2 + B_1) + d_2(B_2 - A_3) + d_3(A_1 - B_3) = 0.$$

Set $w = -A_2 - B_1$, $u = A_3 - B_2$, $v = B_3 - A_1$ to get the equivalence between i) and ii). $\square$

7.3. The hypersurface case. In this section we shall denote with $\langle .,. \rangle$ the Euclidean scalar product in $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ and with $\tilde{I}_j$, $j = 1,2,3$, the standard almost complex structures on $\mathbb{H}^{n+1}$. Let M be a smooth hyper-surface in $\mathbb{H}^{n+1}$ with a defining function ρ,

$$M = \{\rho = 0\}, \quad d\rho \neq 0,$$

and let $i : M \hookrightarrow \mathbb{H}^{n+1}$ be the embedding. It is not hard to see that at every point $p \in M$ the subspace

$$H_p = \bigcap_{j=1}^{3} \tilde{I}_j \,(T_pM)$$

of the tangent space T_pM of M at p is the largest subspace invariant under the almost complex structures and $\dim H_p = 4n$. We shall call H_p the horizontal space at p. Thus on the horizontal space H the almost complex structures I_j, $j = 1, 2, 3$, are the restrictions of the standard almost complex structures on $\mathbb{H}^{n+1}$. In particular, for a horizontal vector X we have

$$\tilde{I}_j \, i_* X \;=\; i_* \,(I_j X). \tag{7.10}$$

Let $\tilde{\theta}^j = \tilde{I}_j \,\frac{d\rho}{|d\rho|}$. We shal drop the symbol $\sim$ from the notation of the almost complex structures when there is no ambiguity.

We define three one-forms on M by setting $\theta^j = i^*\tilde{\theta}^j = i^*(\tilde{I}_j \,\frac{d\rho}{|d\rho|})$, i.e.,

$$\theta_j \;(\;.\;) \;=\; -\frac{d\rho\,(\tilde{I}_j\;.\;)}{|d\rho|} \;=\; \langle\;.\,, \tilde{I}_j N\rangle,$$

where $N = \frac{D\rho}{|D\rho|}$ is the unit normal vector to M. We describe the hypersurfaces which inherit a natural quaternionic-contact structure from the standard structures on $\mathbb{H}^{n+1}$ (see also [**D2**]) in the next

Proposition 7.12. *If M is a smooth hypersurface of $\mathbb{H}^{n+1}$ then we have*

$$d\theta_1(I_1X, Y) = d\theta_2(I_2X, Y) = d\theta_3(I_3X, Y) \quad (X,\, Y \in H) \tag{7.11}$$

if and only if the restriction of the second fundamental form of M to the horizontal space is invariant with respect to the almost complex structures, i.e. if X and Y are two horizontal vectors we have

$$II(I_jX, I_jY) = II(X, Y).$$

Furthermore, if the restriction of the second fundamental form of M to the horizontal space is positive definite, $II(X,X) > 0$ for any non-zero horizontal vector X, then (M, θ, I_1, I_2) is a quaternionic-contact manifold.

PROOF. Let D be the Levi-Civita connection on $\mathbb{R}^{4n+4}$ and X, Y be two horizontal vectors. As the horizontal space is the intersection of the kernels of the one forms θ_j we have

$$\begin{aligned} d\theta_1(I_1X, Y) \;&=\; -\theta_1([\tilde{I}_1X, Y]) \;=\; -\langle [\tilde{I}_1X, Y], \tilde{I}_1N\rangle \\ &= -\langle D_{\tilde{I}_1X}Y - D_Y(\tilde{I}_1X), \tilde{I}_1N\rangle \;=\; -\langle D_{\tilde{I}_1X}Y, \tilde{I}_1N\rangle + \langle D_Y(\tilde{I}_1X), \tilde{I}_1N\rangle \\ &= \langle D_{\tilde{I}_1X}(\tilde{I}_1Y), N\rangle + \langle D_YX, N\rangle \;=\; II(\tilde{I}_1X, \tilde{I}_1Y) + II(X,Y). \end{aligned} \tag{7.12}$$

Therefore,

$$d\theta_1(I_1X, Y) = d\theta_2(I_2X, Y) \iff II(I_jX, I_jY) = II(X,Y).$$

The last claim of the proposition is clear from the above formula. In particular,

$$g_H(X,Y) = II(X,Y)$$

is a metric on the horizontal space when the second fundamental form is positive definite on the horizontal space and we have

$$d\theta_1(I_1X, Y) = 2g_H(X,Y).$$

Hence, (M, θ, I, J) becomes a quaternionic-contact structure. We denote the corresponding horizontal forms with ω_j, i.e.,

$$\omega_j(X,Y) = g_H(I_jX, Y).$$

□

Let us note also that in the situation as above

$$g = g_H + \Sigma_{j=1}^3 \theta_j \otimes \theta_j$$

is a Riemannian metric on M. In view of the above observations we define a QC hypersurface of $\mathbb{H}^{n+1}$ as follows.

DEFINITION 7.13. *We say that a smooth embedded hypersurface of* $\mathbb{H}^{n+1}$ *is a* QC-hypersurface *if the restriction of the second fundamental form of* M *to the horizontal space is a definite symmetric form, which is invariant with respect to the almost complex structures.*

Clearly every sphere in $\mathbb{H}^{n+1}$ is a QC hypersurface and this is true also for the ellipsoids

$$\sum_a \frac{|q^a|^2}{b_a} = 1.$$

In fact, a hypersurface of $\mathbb{H}^{n+1}$ is a QC hypersurface if and only if the (Euclidean) Hessian of the defining function ρ, considered as a quadratic form on the horizontal space, is a symmetric definite matrix from $GL(n,\mathbb{H})$, the latter being the linear group of invertible matrices which commute with the standard complex structures on $\mathbb{H}^n$. The same statement holds for hypersurfaces in quaternionic Kähler and hyperkähler manifolds.

PROPOSITION 7.14. *Let* $i : M \to \mathbb{H}^n$ *be a QC hypersurface in* $\mathbb{H}^n$, f *a real-valued function on* M. *If* $f = i^*F$ *is the restriction to* M *of a Q-pluriharmonic function* F *defined on* $\mathbb{H}^n$, *i.e.,* F *is the real part of an anti-regular function*

$$F + iW + jU + kV,$$

then

$$df = d(i^*F) = d_1(i^*W) + d_2(i^*U) + d(i^*V) \qquad mod\ \eta, \tag{7.13}$$

$$DD_{I_1} f(X, I_1Y) \;=\; -4dF(D\rho)\, g_H(X,Y) \;-\; 4(\xi_2 f)\,\omega_2(X,Y) \tag{7.14}$$

for any horizontal vector fields $X, Y \in H$.

PROOF. Let us prove first (7.13). Denote with small letters the restrictions of the functions defined on $\mathbb{H}^n$. For $X \in H$ from (7.10) we have

$$\begin{aligned}(i^*\,\tilde I_1\, dW)(X) = (\tilde I_1 dW)(i_*\,X) = -dW(\tilde I_1\, i_*\,X)\\ = -dW(i_*\,(I_1\;X)) = -dw(I_1\,X) = d_1 w\,(X).\end{aligned}$$

Applying the same argument to the functions U and V we see the validity of (7.13).

Our goal is to write the equation for f on M, using the fact that $f = i^*F$. Let us consider the function λ,

$$\lambda \; = \; \frac{dF(D\rho)}{|D\rho|^2},$$

and the one-form d_MF,

$$d_MF \; = \; dF \; - \; \lambda \, d\rho.$$

Thus the one-form df satisfies the equation

$$df \; = \; i^*(d_MF \; + \; \lambda \, d\rho) \; = \; i^*(d_MF),$$

taking into account that $(\lambda \circ i) \, d(\rho \circ i) \; = \; 0$ as ρ is constant on M. From Proposition 7.8, the assumption on F is equivalent to $DD_{\tilde{I}_j}F = 0$. Therefore, we have

$$\begin{aligned} 0 \; = \; DD_{\tilde{I}_1}F \; &= \; d\tilde{I}_1 dF \; - \; \tilde{I}_2 d\tilde{I}_1 dF \\ &= \; d\,(\tilde{I}_1 d_MF \; + \; \lambda \tilde{I}_1 d\rho) \; - \; \tilde{I}_2 d\,(\tilde{I}_1 d_MF \; + \; \lambda \tilde{I}_1 d\rho) \\ &= \; d\,\tilde{I}_1 d_MF \; + \; d\lambda \wedge \tilde{I}_1 d\rho \; + \; \lambda d\tilde{I}_1 d\rho - \; \tilde{I}_2 d\,\tilde{I}_1 d_MF \\ &\qquad - \; \tilde{I}_2(d\lambda \wedge \tilde{I}_1 d\rho) \; - \; \lambda \tilde{I}_2 \tilde{I}_1 d\rho. \end{aligned}$$

Restricting to M, and in fact, to the horizontal space H we find

$$\begin{aligned} 0 \; = \; i^*\,(DD_{\tilde{I}_1}F)|_H = \; & i^*d\,(\tilde{I}_1 d_MF)|_H \; + \; d(\lambda \circ i) \wedge i^*(\tilde{I}_1 d\rho)|_H \\ & + \; (\lambda \circ i) d i^*(\tilde{I}_1 d\rho)|_H \; - \; i^*(\tilde{I}_2 d\,\tilde{I}_1 d_MF)|_H \\ & - \; i^*(\tilde{I}_2(d\lambda \wedge \tilde{I}_1 d\rho))|_H \; - \; (\lambda \circ i)\; i^*(\tilde{I}_2 d\,\tilde{I}_1 d\rho)|_H. \end{aligned} \tag{7.15}$$

Since the horizontal space is in the kernel of the one-forms $\theta_j \; = \; \tilde{I}_j \, d\rho|_H$ it follows that $i^*\,(\tilde{I}_j \, d\rho)|_H \; = \; 0$. Hence, two of the terms in (7.15) are equal to zero, and we have

$$(7.16)\quad 0 \;=\; i^*\,(DD_{\tilde I_1}F)_{|_H} \;=\; i^*(d\,\tilde I_1 d_M F \;-\; \tilde I_2 d\,\tilde I_1 d_M F)\;_{|_H} \\ +\;(\lambda\circ i)\; i^*\big(d\,\tilde I_1 d\rho \;-\; \tilde I_2 d\,\tilde I_1 d\rho\big)_{|_H}.$$

In other words for horizontal X and Y we have

$$(7.17)\quad i^*(d\,\tilde I_1 d_M F \;-\; \tilde I_2 d\,\tilde I_1 d_M F)(X, IY) \\ =\; -(\lambda\circ i)\; i^*\big(d\,\tilde I_1 d\rho \;-\; \tilde I_2 d\,\tilde I_1 d\rho\big)(X, IY)$$

The right-hand side is proportional to the metric. Indeed, recalling

$$i^*\;(\tilde I_j\, d\,\rho)(X) \;=\; |d\rho|\;\theta_j(X) d\,\theta_j(X,Y) \;=\; 2g(I_j X, Y)$$

we obtain the identity

$$i^*\big(d\,\tilde I_1 d\rho \;-\; \tilde I_2 d\,\tilde I_1 d\rho\big)(X,Y) \;=\; 2|d\rho|\; g(I_1X,Y) \;-\; 2|d\rho|\; g(I_1I_2X, I_2Y) \\ =\; 2|d\rho|\; g(I_1X,Y) \;-\; 2g(I_3X, I_2Y) \;=\; 4|d\rho|\; g(I_1X,Y).$$

Let us consider now the term in the left-hand side of (7.17). Decomposing $d_M F$ into horizontal and vertical parts we write

$$d_M F \;=\; d_H f \;+\; F_j \tilde\theta^j.$$

By definition, for of the forms $\tilde\theta^j$, we have

$$\tilde I_1 \tilde\theta^1 = \frac{d\rho}{|d\rho|}, \quad \tilde I_1 \tilde\theta^2 = \tilde\theta^3, \quad \tilde I_1 \tilde\theta^1 \tilde\theta^3 = -\tilde\theta^2.$$

Therefore,

$$\begin{aligned} d\tilde{I}_1 d_M F \;&=\; d\tilde{I}_1 d_H F \;+\; dF_j \wedge \tilde{I}_1\theta^j \;+\; F_1 d(\frac{d\rho}{|d\rho|}) \;+\; F_2 d\tilde{\theta}^3 \;-\; F_3 d\tilde{\theta}^2 \\ &=\; d\tilde{I}_1 d_H F \;+\; dF_j \wedge \tilde{I}_1\theta^j \;-\; |d\rho|^{-2} d|d\rho| \wedge d\rho \;+\; F_2 d\tilde{\theta}^3 \;-\; F_3 d\tilde{\theta}^2 \end{aligned}$$

$$\begin{aligned} \tilde{I}_2 d\tilde{I}_1 d_M F \;=\; &\tilde{I}_2 d\tilde{I}_1 d_H F \;+\; \tilde{I}_2 dF_j \wedge \tilde{I}_2\tilde{I}_1\theta^j \\ &- \; |d\rho|^{-2}\tilde{I}_2 d|d\rho| \wedge \tilde{I}_2 d\rho \;+\; F_2 \tilde{I}_2 d\tilde{\theta}^3 \;-\; F_3 \tilde{I}_2 d\tilde{\theta}^2. \end{aligned}$$

From $I_2 d\theta^3 \;=\; -d\theta^3$, $I_2 d\theta^2 \;=\; d\theta^2$ and the above it follows

$$\begin{aligned} i^*(d\,\tilde{I}_1 d_M F - \tilde{I}_2 d\,\tilde{I}_1 d_M F)_{|H} &= DD_{I_1} f + F_2 d\theta^3 - F_3 d\theta^2 + F_2 d\theta^3 + F_3 d\theta^2 \\ &= DD_{I_1} f + 4F_2\,\omega_3. \end{aligned}$$

In conclusion, we proved $DD_{I_1} f(X,Y) \;=\; -4(\lambda \circ i)\,|\nabla\rho|\,g(I_1 X, Y) \;-\; 4F_2\,\omega_3(X,Y)$ from where the claim of the Proposition. □

7.4. Anti-CRF functions on Quaternionic contact manifold. Let (M, H, g) be a (4n+3)-dimensional quaternionic contact manifold and ∇ denote the Biquard connection on M. The equation (7.13) suggests the following

Definition 7.15. *A smooth $\mathbb{H}$-valued function*

$$F : M \longrightarrow \mathbb{H}, \quad F = f + iw + ju + kv,$$

is said to be an anti-CRF function *if the smooth real valued functions f, w, u, v satisfy*

$$df \;=\; d_1 w \;+\; d_2 u \;+\; d_3 v \;\; mod \;\; \eta, \tag{7.18}$$

where $d_i \;=\; I_i \circ d$.

Choosing a local frame

$$\{T_a, X_a = I_1 T_a, Y_a = I_2 T_a, Z_a = I_3 T_a, \xi_1, \xi_2, \xi_3\}, \quad a = 1, \dots, n$$

it is easy to check that a $\mathbb{H}$-valued function $F = f + iw + ju + kv$ is an anti-CRF function if it belongs to the kernel of the operators

$$D_{T_\alpha} \;=\; T_\alpha - iX_\alpha - jY_\alpha - kZ_\alpha, \qquad D_{T_\alpha}F \;=\; 0, \qquad \alpha = 1, \dots n. \tag{7.19}$$

Remark 7.16. *We note that anti-CRF functions have different properties than the CRF functions* [**Per1, Per2**] *which are defined to be in the kernel of the operator*

$$\overline{D}_{T_\alpha} \;=\; T_\alpha + iX_\alpha + jY_\alpha + kZ_\alpha, \qquad \overline{D}_{T_\alpha}F \;=\; 0, \qquad \alpha = 1, \dots n.$$

Equation (7.18) and a small calculation give the following Proposition.

Proposition 7.17. *A $\mathbb{H}$-valued function $F = f + iw + ju + kv$ is an anti-CRF function if and only if the smooth functions f, w, u, v satisfy the horizontal Cauchy-Riemann-Fueter equations*

$$\begin{aligned} T_\alpha f &= -X_\alpha w - Y_\alpha u - Z_\alpha v, \quad & X_\alpha f &= T_\alpha w + Z_\alpha u - Y_\alpha v, \\ Y_\alpha f &= -Z_\alpha w + T_\alpha u + X_\alpha v, \quad & Z_\alpha f &= Y_\alpha w - X_\alpha u + T_\alpha v. \end{aligned} \tag{7.20}$$

Having the quaternionic-contact form η fixed, we may extend the definitions (7.6) of DD_{I_i} to the second order differential operator DD_{I_i} acting on the real-differentiable functions $f : M \to \mathbb{R}$ by

$$DD_{I_i}f := dd_if + d_jd_kf = dd_if - I_jdd_if = d(I_idf) - I_j(d(I_idf)). \tag{7.21}$$

The following proposition provides some formulas, which shall be used later.

PROPOSITION 7.18. *On a QC manifold, for $X, Y \in H$, we have the following commutation relations*

$$DD_{I_i}f(X, I_iY) - DD_{I_k}f(X, I_kY) = -I_iN_{I_j}(X, I_iY)(f) - N_{I_k}(I_jX, I_jY)(f),$$

$$d_id_jf(X,Y) + d_jd_if(X,Y) = -N_{I_k}(I_jX, I_iY)(f), \tag{7.22}$$

$$dd_if(X,Y) + d_idf(X,Y) = N_{I_i}(I_iX, Y)f,$$

$$d_i^2f(X,Y) = -2(\xi_if)\omega_i(X,Y) + 2(\xi_jf)\omega_j(X,Y) + 2(\xi_kf)\omega_k(X,Y), \tag{7.23}$$

In particular, on a hyperhermitian contact manifold we have

$$DD_{I_i}f(X, I_iY) - DD_{I_k}(X, I_kY) = 4\xi_i(f)\omega_i(X,Y) - 4\xi_j(f)\omega_j(X,Y),$$

$$d_id_jf(X,Y) + d_jd_if(X,Y) = -4\left[(\xi_if)\omega_j(X,Y) + (\xi_jf)\omega_i(X,Y)\right], \tag{7.24}$$

$$dd_if(X,Y) + d_idf(X,Y) = 4\left[(\xi_kf)\omega_j(X,Y) - (\xi_jf)\omega_k(X,Y)\right].$$

PROOF. By the definition (7.21) we obtain the second and the third formulas in (7.22) as well as

$$DD_i(X,Y) + (dd_k - d_jd_i)f(X, I_jY) = -I_iN_{I_j}(X,Y)f.$$

The first equality in (7.22) is a consequence of the latter and the second equality in (7.22). We have

$$d_i^2f(X,Y) = -I_id(I_i^2df)(X,Y) = d(df - \sum_{s=1}^{3}(\xi_sf)\eta_s)(I_iX, I_iY)$$

which is exactly (7.23). If H is formally integrable then the formula (5.21) reduces to

$$N_i(X,Y) = T_i^{0,2}(X,Y).$$

The equation (7.24) is an easy consequences of the latter equality, (7.22) and (4.4) $\square$

Let us make the conformal change

$$\bar{\eta} = \frac{1}{2h}\eta.$$

The endomorphisms $\bar{I}_i$ will coincide with I_i on the horizontal distribution H but they will have a different kernel - the new vertical space

$$\operatorname{span}\{\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\},$$

where

$$\bar{\xi}_s = 2h\xi_s + I_s(\nabla h) \qquad \text{(cf. (6.1))}.$$

Hence, for any $P \in \Gamma(TM)$ we have

$$\begin{aligned}(7.25)\quad \bar{I}_i(P) = \bar{I}_i(P - \sum_{s=1}^{3} \tilde{\eta}_s(P)\bar{\xi}_s) = \\ I_i(P - \frac{1}{2h}\sum_{s=1}^{3} \eta_s(P)(2h\xi_s + I_s(\nabla h))) = I_i(P) \\ + \frac{1}{2h}\{\eta_i(P)\nabla h - \eta_j(P)I_k\nabla h + \eta_k(P)I_j\nabla h\}.\end{aligned}$$

Proposition 7.19. *Suppose η and $\bar{\eta}$ are two conformal to each other structures,*

$$\bar{\eta} = \frac{1}{2h}\eta.$$

a) The second order differential operator DD_{I_i} (restricted on functions) transforms as follows:

$$DD_{\bar{I}_i}f - DD_{I_i}f = -2h^{-1}df(\nabla h)\omega_i - 2h^{-1}df(I_j\nabla h)\omega_k \qquad \textit{mod } \eta.$$

b) If f is the real part of the anti-CRF function $f+iw+ju+kv$ then the two forms

$$\Omega_i \;=\; DD_{I_i} f \;-\; \lambda\,\omega_i \;+\; 4(\xi_j f)\,\omega_k \qquad mod\ \eta$$

are conformally invariant, where $\lambda \;=\; 4\big(\xi_1 w+\xi_2 u+\xi_3 v\big)$.

PROOF. a) For any $X, Y \in H$, we compute

$$d(\bar I_i df)(X,Y) = X(I_i df(Y)) - Y(I_i df(X)) - \bar I_i df[X,Y]$$
$$= d(I_i df)(X,Y) + df(\bar I_i[X,Y] - I_i[X,Y])$$

Here, we apply (7.25) to get

$$d(\bar I_i df)(X,Y) = d(I_i df)(X,Y)$$
$$+ \frac{1}{h}\{-df(\nabla h)\omega_i(X,Y) + df(I_k\nabla h)\omega_j(X,Y) - df(I_j\nabla h)\omega_k(X,Y)\}.$$

Now, we apply the defining equation (7.21) which acomplishes the proof of part a).

b) Assuming that f is the real part of an anti-CRF function, from part a) we have

$$\bar\Omega_i \;-\; \Omega_i = DD_{\bar I_i} f - DD_{I_i} f - \bar\lambda\,\bar\omega_i + \lambda\,\omega_i + 4(\bar\xi_j f)\,\bar\omega_k \;-\; 4(\xi_j f)\,\omega_k \quad \mathrm{mod}\ \eta$$
$$= -4(\xi_1 w+\xi_2 u+\xi_3 v)\omega_i - 4\big((I_1 dh)w + (I_2 dh)u + (I_3 dh)v\big)\frac{\omega_i}{2h}$$
$$-\frac{2}{h}g(df,dh)\omega_i - \frac{2}{h}g(df,d_jh)\omega_k + 4(\xi_1 w+\xi_2 u+\xi_3 v)\omega_i + 4g(df,d_jh)\frac{\omega_k}{2h}$$
$$= \;0 \quad \mathrm{mod}\ \eta,$$

taking into account (7.18). □

We restrict our considerations to hyperhermitian contact manifolds.

THEOREM 7.20. *If $f: M \to \mathbb{R}$ is the real part of an anti-CRF function*

$$f + iw + ju + kv$$

on a (4n+3)-dimensional $(n>1)$ *hyperhermitian contact manifold* (M,η,Q), *then the following equivalent conditions hold true.*

i) The next equalities hold

$$DD_{I_i}f=\lambda\omega_i-4(\xi_j f)\omega_k \qquad mod\ \eta. \tag{7.26}$$

ii) For any $X,Y\in H$ *we have the equality*

$$\begin{aligned}(\nabla_X df)(Y)+(\nabla_{I_1X}df)(I_1Y)+(\nabla_{I_2X}df)(I_2Y)+(\nabla_{I_3X}df)(I_3Y)\\ =\ \lambda g(X,Y)+df(X)\alpha_3(I_3Y)+df(I_1X)\alpha_3(I_2Y)\\ -df(I_2X)\alpha_3(I_1Y)-df(I_3X)\alpha_3(Y)+df(Y)\alpha_3(I_3X)\\ +df(I_1Y)\alpha_3(I_2X)-df(I_2Y)\alpha_3(I_1X)-df(I_3Y)\alpha_3(X).\end{aligned} \tag{7.27}$$

iii) The function f *satisfies the second order system of partial differential equations*

$$\begin{aligned}\Re(D_{T_\beta}\overline{D}_{T_\alpha}f)=\lambda g(T_\beta,T_\alpha)\\ +df(\nabla_{T_\beta}T_\alpha)+df(\nabla_{I_1T_\beta}I_1T_\alpha)+df(\nabla_{I_2T_\beta}I_2T_\alpha)+df(\nabla_{I_3T_\beta}I_3T_\alpha)\\ +df(T_\beta)\alpha_3(I_3T_\alpha)+df(I_1T_\beta)\alpha_3(I_2T_\alpha)-df(I_2T_\beta)\alpha_3(I_1T_\alpha)-df(I_3T_\beta)\alpha_3(T_\alpha)\\ +df(T_\alpha)\alpha_3(I_3T_\beta)+df(I_1T_\alpha)\alpha_3(I_2T_\beta)-df(I_2T_\alpha)\alpha_3I_1(T_\beta)-df(I_3T_\alpha)\alpha_3(T_\beta)\end{aligned} \tag{7.28}$$

$$\begin{gathered}\Re(iD_{T_\beta}\overline{D}_{T_\alpha}f)=\Re(D_{I_1T_\beta}\overline{D}_{T_\alpha}f),\quad \Re(jD_{T_\beta}\overline{D}_{T_\alpha}f)=\Re(D_{I_2T_\beta}\overline{D}_{T_\alpha}f),\\ \Re(jD_{T_\beta}\overline{D}_{T_\alpha}f)=\Re(D_{I_3T_\beta}\overline{D}_{T_\alpha}f).\end{gathered} \tag{7.29}$$

The function λ *is determined by*

$$\lambda\ =\ 4\left[(\xi_1w)+(\xi_2u)+(\xi_3v)\right]. \tag{7.30}$$

Proof. The proof includes a number of steps and occupies the rest of the section. Suppose that there exists a smooth functions w, u, v such that

$$F = f + iw + ju + kv$$

is an anti-CRF function.

i) The defining equation (7.18) yields

$$df = d_1 w + d_2 u + d_3 v + \sum_{s=1}^{3} \xi_s(f)\eta_s, \tag{7.31}$$

Since $d_s\eta_t(X,Y) = 0$, for $s,t \in \{1,2,3\}, X,Y \in H$, applying (7.23) and (2.2), we obtain from (7.31)

$$\begin{aligned}
(dd_1 f - dd_3 u + dd_2 v - 2\xi_1(w)\omega_1 - 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) &= 0,\\
(d_1 df - d_1 d_2 u - d_1 d_3 v + 2\xi_1(w)\omega_1 - 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) &= 0,\\
(d_2 d_3 f + d_2 d_1 u + d_2 dv - 2\xi_1(w)\omega_1 + 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) &= 0,\\
(d_3 d_2 f + d_3 du - d_3 d_1 v + 2\xi_1(w)\omega_1 + 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) &= 0.
\end{aligned}$$

Summing the first and the third equations, subtracting the second and the fourth and using the commutation relations (7.24) we obtain (7.26) with the condition (7.30) which proves *i*).

iii) Equations (7.19) and (7.20) yield

$$\begin{aligned}
2\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) \;&=\; 2(T_\beta T_\alpha f + X_\beta X_\alpha f + Y_\beta Y_\alpha f + Z_\beta Z_\alpha f)\\
&= (\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) + \Re(D_{T_\alpha}\overline{D}_{T_\beta} f)) + (\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) - \Re(D_{T_\alpha}\overline{D}_{T_\beta} f)) =\\
&\quad ([T_\beta, T_\alpha] + [X_\beta, X_\alpha] + [Y_\beta, Y_\alpha] + [Z_\beta, Z_\alpha])f\\
&\quad - ([T_\beta, X_\alpha] - [X_\beta, T_\alpha] + [Y_\beta, Z_\alpha] - [Z_\beta, Y_\alpha])w\\
&\quad - ([T_\beta, Y_\alpha] - [X_\beta, Z_\alpha] - [Y_\beta, T_\alpha] + [Z_\beta, X_\alpha])u\\
&\quad - ([T_\beta, Z_\alpha] + [X_\beta, Y_\alpha] - [Y_\beta, X_\alpha] - [Z_\beta, T_\alpha])v.
\end{aligned}$$

Expanding the commutators and applying (7.18), (4.4), (4.29) and (5.24) gives (7.28). Similarly, one can check the validity of (7.29)

i) $\Leftrightarrow$ *ii)* $\Leftrightarrow$ *iii)* The next lemma establishes the equivalence between i), ii) and iii).

LEMMA 7.21. *For any $X, Y \in H$ on a quaternionic-contact manifold we have the identity*

$$
\begin{aligned}
DD_{I_1}f(X, I_1Y) &= (\nabla_X df)Y + (\nabla_{I_1X} df)I_1Y + (\nabla_{I_2X} df)I_2Y \\
&+ (\nabla_{I_3X} df)I_3X - 4\xi_2(f)\omega_2(X,Y) - df(X)\alpha_3(I_3Y) + df(I_1X)\alpha_2(I_3Y) \\
&+ df(I_2X)\alpha_3(I_1Y) - df(I_3X)\alpha_2(I_1Y) - df(Y)\alpha_2(I_2X) \\
&- df(I_1Y)\alpha_3(I_2X) + df(I_2Y)\alpha_2(X) + df(I_3Y)\alpha_3(X).
\end{aligned}
$$

PROOF OF LEMMA 7.21. Using the definition and also (4.29), (4.4) and (6.4) we derive the next sequence of equalities

$$
\begin{aligned}
(dd_{I_1}f)(X,Y) &= -(\nabla_X df)(I_1Y) + (\nabla_Y df)(I_1X) - df(\nabla_X(I_1Y) \\
&- \nabla_Y(I_1X) - I_1[X,Y]) = -(\nabla_X df)(I_1Y) + (\nabla_Y df)(I_1X) \\
&+ \alpha_2(X)df(I_3Y) - \alpha_3(X)df(I_2Y) - \alpha_2(Y)df(I_3X) + \alpha_3(Y)df(I_2X) \\
&= -(\nabla_X df)I_1Y + (\nabla_{I_1X} df)Y - df(T(Y, I_1X)) \\
&+ \alpha_2(X)df(I_3Y) - \alpha_3(X)df(I_2Y) - \alpha_2(Y)df(I_3X) + \alpha_3(Y)df(I_2X).
\end{aligned}
$$

$$
\begin{aligned}
DD_{I_1}f(X, I_1Y) &= (dd_{I_1} - I_2 dd_{I_1})f\,(X, I_1Y) \\
&= (\nabla_X df)Y + (\nabla_{I_1X} df)I_1Y + (\nabla_{I_2X} df)I_2Y + (\nabla_{I_3X} df)I_3X \\
&- df\,(T\,(I_1Y,\, I_1X)) - df\,(T\,(I_3Y,\, I_3X)) - df(X)\alpha_3(I_3Y) + df(I_1X)\alpha_2(I_3Y) \\
&+ df(I_2X)\alpha_3(I_1Y) - df(I_3X)\alpha_2(I_1Y) - df(Y)\alpha_2(I_2X) \\
&- df(I_1Y)\alpha_3(I_2X) + df(I_2Y)\alpha_2(X) + df(I_3Y)\alpha_3(X).
\end{aligned}
\tag{7.32}
$$

A short calculation using (4.4) gives

$$df(T(I_1Y,\, I_1X)) \ + \ df(T(I_3Y,\, I_3X)) \ = \ 4(\xi_2 f)\,\omega_2(X,Y).$$

Inserting the last equality in (7.32) completes the proof of Lemma 7.21. □

Taking into account that the structure is hyperhermitian contact, with the help of (5.24) of Lemma 7.21 the proof of Theorem 7.20 follows. □

We conjecture that the converse of the claim of Theorem 7.20 is true. At this point we can prove Lemma 7.23, which supports the conjecture. First we prove a useful technical result.

LEMMA 7.22. *Suppose M is a quaternionic-contact manifold of dimension $(4n+3) > 7$. If ψ is a smooth closed two-form whose restriction to H vanishes, then ψ vanishes identically.*

PROOF OF LEMMA 7.22. The hypothesis on ψ show that ψ is of the form

$$\psi \ = \ \sum_{s=1}^{3} \sigma_s \wedge \eta_s,$$

where σ_s are 1-forms. Taking the exterior differential and using (4.11), we obtain for $X \in H$

$$0 = \sum_{a=1}^{4n} d\psi(e_a, I_ie_a, X) = -2\sum_{a=1}^{4n}\sum_{s=1}^{3} \sigma_s \wedge \omega_s(e_a, I_ie_a, X)$$
$$= (4n-2)\sigma_i(X) + 2\sigma_j(I_kX) - 2\sigma_k(I_jX),$$

where $e_1, \dots, e_{4n}$ is an orthonormal basis on H. For $n > 1$ the latter implies $\sigma_{s_{|H}} = 0$, $s = 1, 2, 3$. Hence, we have

$$\psi = \sum_{1 \le s < t \le} A_{st}\, \eta_s \wedge \eta_t,$$

where A_{st} are smooth functions on M. Now, the exterior derivative gives

$$0 = d\psi(e_a, I_se_a, \xi_t) \ = \ 2A_{st}.$$

□

The assumption in the next Lemma is a kind of $\partial\bar{\partial}_H$-lemma result, which we do not know how to prove at the moment, but we believe that it is true. We show how it implies the converse of Theorem 7.20.

Lemma 7.23. *Suppose, for $i = 1, 2, 3$,*

$$DD_{I_i}f \equiv dd_i f + d_j d_k f = \sum_{s=1}^{3} p_s^i \omega_s \quad (mod\ \eta)$$

implies

$$dd_i f - dd_j A_i \; = \; 2\sum_{s=1}^{3} r_s^i\, \omega_s \qquad (mod\ \eta) \tag{7.33}$$

for some function A_i on a QC manifold of dimension $(4n+3) > 7$. With this assumption, if

$$DD_{I_i}f = \sum_{s=1}^{3} p_s^i \omega_s \quad (mod\ \eta), \qquad i = 1, 2, 3,$$

then f is a real part of an anti-CRF-function.

Proof of Lemma 7.23. Consider the closed 2-forms

$$\Omega_i = d\Big(d_i f - d_j A_i - \sum_{s=1}^{3} r_s^i \eta_s\Big).$$

We have $d\Omega_i = 0$ and $\Omega_{i_{|_H}} = 0$ due to (7.33) and (4.2). Applying Lemma 7.22 we conclude $\Omega_i = 0$, after which the Poincare lemma yields

$$d_i f - d_j A_i - dB_i = 0 \quad (\text{mod } \eta)$$

for some smooth functions $A_1, A_2, A_3, B_1, B_2, B_3$. The latter implies

$$df + d_1(A_2 + B_1) + d_2(B_2 - A_3) + d_3(A_1 - B_3) = 0 \quad (\text{mod } \eta).$$

Setting $w = -A_2 - B_1, \quad u = A_3 - B_2, \quad v = B_3 - A_1$ proves the claim. $\square$

COROLLARY 7.24. *Let $f : M \to \mathbb{R}$ be a smooth real function on a (4n+3)-dimensional $(n > 1)$ 3-Sasakian manifold (M, η). If f is the real part of an anti-CRF function*

$$f + iw + ju + kv$$

then we have:

i) equation (7.26) holds true;

ii) for any $X, Y \in H$ we have the equality

$$(\nabla_X df)(Y) + (\nabla_{I_1X} df)(I_1Y) + (\nabla_{I_2X} df)(I_2Y) + (\nabla_{I_3X} df)(I_3Y) = \lambda g(X, Y).$$

The function λ is determined in (7.30).

COROLLARY 7.25. *Let $f : G(\mathbb{H}) \to \mathbb{R}$ be a smooth real function on the (4n+3)-dimensional $(n > 1)$ quaternionic Heisenberg group endowed with the standard flat quaternionic-contact structure and*

$$\{T_a, X_a, Y_a, Z_a, \quad a = 1, \dots, 4n\}$$

be ∇-parallel basis on $G(\mathbb{H})$. If f is the real part of an anti-CRF function

$$f + iw + ju + kv$$

then the following equivalent conditions hold true.

i) The equation (7.26) holds.

ii) The horizontal Hessian of f is given by

$$T_bT_af + X_bX_af + Y_bY_af + Z_aZ_b = \lambda g(T_b, T_a).$$

iii) The function f satisfies the following second order differential equation

$$D_{T_b}\overline{D}_{T_a}f = \lambda(g - i\omega_1 - j\omega_2 - k\omega_3)(T_b, T_a);$$

The function λ is given by (7.30).

Proposition 7.2, Corollary 7.24 and Example 5.12 imply the next Corollary.

Corollary 7.26. *Let (M,η) be a (4n+3)-dimensional $(n>1)$ 3-Sasakian manifold, $f: M \to \mathbb{R}$ a positive smooth real function. Then the conformally 3-Sasakian QC structure $\bar\eta = f\eta$ is QC pseudo Einstein if and only if the operators*

$$DD_{I_s}f, \quad s=1,2,3$$

satisfy (7.26). In particular, if f is real part of anti CRF function then the conformally 3-Sasakian QC structure $\bar\eta = f\eta$ is QC pseudo Einstein.

8. Infinitesimal Automorphisms

8.1. 3-contact manifolds. We start with the more general notion of 3-contact manifold (M,H), where H is an orientable codimension three distribution on M. Let $E \subset TM^*$ be the canonical bundle determined by H, i.e., the bundle of 1-forms with kernel H. Hence, M is orientable if and only if E is also orientable, i.e., E has a global non-vanishing section vol_E locally given by

$$vol_E \;=\; \eta_1\wedge\eta_2\wedge\eta_3.$$

Denote by $\eta=(\eta_1,\eta_2,\eta_3)$ the local 1 -form with values in $\mathbb{R}^3$. Clearly,

$$H = Ker\ \eta = \cap_{i=1}^3 \eta_i.$$

Definition 8.1. *A $(4n+3)$-dimensional orientable smooth manifold*

$$(M, H = Ker\ \eta)$$

is said to be a 3-contact manifold *if H is a codimnesion three distribution and the restriction of each of the 2-forms $d\eta_i$ to H is non-degenerate, i.e.,*

$$d\eta_i^{2n}\wedge\eta_1\wedge\eta_2\wedge\eta_3 \;=\; u_i\ vol_M \tag{8.1}$$

for some smooth functions $u_i>0$, $i=1,2,3$.

We shall denote by Ω_i the restriction of $d\eta_i$ to H,

$$\Omega_i = d\eta_{i_{|H}}, \quad i=1,2,3.$$

The condition (8.1) is equivalent to

$$\Omega_i^{2n} \neq 0, \quad i = 1, 2, 3$$

and the forms Ω_i^{2n} define the same orientation of H.

We remark that the notion of 3-contact structure is more general than the notion of QC structure. For example, any real hypersurface M in $\mathbb{H}^{n+1}$ with non-degenerate second fundamental form carries 3-contact structure defined in Section 7.3 (cf. Proposition 7.12 and Definition 7.13 where this structure is QC if and only if (7.11) holds, or equivalently, the second fundamental form is, in addition, invariant with respect to the hypercomplex structure on $\mathbb{H}^{n+1}$). Another examples of 3-contact structure is the so called quaternionic CR structure introduced in [**KN**] and the so called weak QC structures considered in [**D2**]. Note that in these examples the 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ are globally defined.

On any 3-contact manifold (M, η, H) there exists a unique triple (ξ_1, ξ_2, ξ_3) of vector fields transversal to H determined by the conditions

$$\eta_i(\xi_j) = \delta_{ij}, \quad (\xi_i \lrcorner d\eta_i)_{|H} = 0.$$

We refer to such a triple as fundamental vector fields or Reeb vector fields and denote

$$V = span\{\xi_1, \xi_2, \xi_3\}.$$

Hence, we have the splitting $TM = H \oplus V$.

The 3-contact structure (η, H) and the vertical space V are determined up to an action of $GL(3, \mathbb{R})$, namely for any $GL(3, \mathbb{R})$ matrix Φ with smooth entries the structure $\Phi \cdot \eta$ is again a 3-contact structure. Indeed, it is an easy algebraic fact that the condition (8.1) also holds for $\Phi \cdot \eta$. The Reeb vector field are transformed with the matrix with entries the adjunction quantities of Φ, i.e. with the inverse matrix Φ^{-1}. This leads to the next

Definition 8.2. *A diffeomorphism ϕ of a 3-contact manifold (M, η, H) is called a* 3-contact automorphism*if ϕ preserves the 3-contact structure η, i.e.,*

$$\phi^* \eta = \Phi \cdot \eta, \tag{8.2}$$

for some matrix $\Phi \in GL(3, \mathbb{R})$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an $\mathbb{R}^3$-valued one-form.

The infinitesimal versions of these notions lead to the following definition.

DEFINITION 8.3. *A vector field $\mathcal{Q}$ on a 3-contact manifold (M, η, H) is an infinitesimal generator of a 3-contact automorphism (3-contact vector field) if its flow preserves the 3-contact structure, i.e.*

$$\mathcal{L}_{\mathcal{Q}}\,\eta = \phi \cdot \eta, \qquad \phi \in gl(3, \mathbb{R}).$$

We show that any 3-contact vector field on a 3-contact manifold depend on 3-functions which satisfy certain differential relations. We begin with describing infinitesimal automorphisms of the 3-contact structure η i.e., vector fields $\mathcal{Q}$ whose flow satisfies (8.2). Our main observation is that 3-contact vector fields on a 3-contact manifold are completely determined by their vertical components in the sense of the following

PROPOSITION 8.4. *Let (M, η, H) be a 3-contact manifold. For a smooth vector field $\mathcal{Q}$ on M, consider the functions*

$$f_i = \eta_i(\mathcal{Q}), \quad i = 1, 2, 3.$$

A smooth vector field $\mathcal{Q}$ is a 3-contact vector field if and only if its horizontal part

$$\mathcal{Q}_H \stackrel{def}{=} \mathcal{Q} - f_1\xi_1 - f_2\xi_2 - f_3\xi_3,$$

is the horizontal 3-contact hamiltonian field of (f_1, f_2, f_3) defined on H by

$$\mathcal{Q}_H \lrcorner \Omega_i = [-df_i - f_j(\xi_j \lrcorner d\eta_i) - f_k(\xi_k \lrcorner d\eta_i)]_{|H}, \quad i = 1, 2, 3. \tag{8.3}$$

PROOF. A direct calculation gives (cf. (4.11))

$$d\eta_i = \Omega_i + \sum_s \eta_s \wedge (\xi_s \lrcorner d\eta_i)$$

$$-d\eta_i(\xi_j, \xi_k)\eta_j \wedge \eta_k - d\eta_i(\xi_k, \xi_i)\eta_k \wedge \eta_i - d\eta_i(\xi_i, \xi_j)\eta_i \wedge \eta_j$$

Furthermore, we compute

$$
\begin{aligned}
\mathcal{L}_{\mathcal{Q}}\,\eta_i &= \mathcal{Q}\lrcorner d\eta_i + d(\mathcal{Q}\lrcorner\eta_i) \qquad (8.4)\\
&= \mathcal{Q}_H\lrcorner\Omega_i + [d(\eta_i(\mathcal{Q})) + \eta_i(\mathcal{Q})\xi_i\lrcorner d\eta_i + \eta_j(\mathcal{Q})\xi_j\lrcorner d\eta_i + \eta_k(\mathcal{Q})\xi_k\lrcorner d\eta_i]_{|H}\\
&\quad +[\xi_i(\eta_i(\mathcal{Q})) + d\eta_i(\mathcal{Q},\xi_i)]\eta_i\\
&\quad +[\xi_j(\eta_i(\mathcal{Q})) + d\eta_i(\mathcal{Q},\xi_j)]\eta_j + [\xi_k(\eta_i(\mathcal{Q})) + d\eta_i(\mathcal{Q},\xi_k)]\eta_k.
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{Q}_H\lrcorner\Omega_i + [df_i + f_j(\xi_j\lrcorner d\eta_i) + f_k(\xi_k\lrcorner d\eta_i)]_{|H}\\
&\quad +[\xi_i(f_i) - f_j d\eta_i(\xi_i,\xi_j) - f_k d\eta_i(\xi_i,\xi_k)]\eta_i + [\xi_j(f_i) + d\eta_i(\mathcal{Q},\xi_j)]\eta_j\\
&\quad +[\xi_k(f_i) + d\eta_i(\mathcal{Q},\xi_k)]\eta_k.
\end{aligned}
$$

Suppose $\mathcal{Q}$ is a 3-contact vector field. Then (8.4) implies that f_i and $\mathcal{Q}_H$ necessarily satisfy (8.3). The converse follows from (8.4) and the conditions of the proposition. □

The last Proposition implies that the space of 3-contact vector fields is isomorphic to the space of triples consisting of smooth function f_1, f_2, f_3 satisfying the compatibility conditions (8.3).

COROLLARY 8.5. *Let (M,η) be a 3-contact manifold. Then*

a) If $\mathcal{Q}$ is a horizontal 3-contact vector field on M then $\mathcal{Q}$ vanishes identically.

b) The vector fields $\xi_i, i=1,2,3$ are 3-contact vector fields if and only if

$$\xi_i\lrcorner d\eta_j|_H = 0, \quad i,j = 1,2,3.$$

PROOF. a) With the notation of Proposition 8.4, we have

$$f_i = \eta_i(\mathcal{Q}) \equiv 0, \quad \mathcal{Q} = \mathcal{Q}_H.$$

Hence $\mathcal{Q}_H\lrcorner\Omega_i = 0$ and since Ω_i is a non-degenerate it follows $\mathcal{Q}_H = 0$.

b) The necessary and sufficient conditions are given by Proposition 8.4. For $\mathcal{Q} = \xi_s$, $s = 1,2,3$ and $\mathcal{Q}_H = 0$ equation (8.3) becomes

$$0 = \eta_j(\xi_s)d\eta_i(\xi_j, X) + \eta_k(\xi_s)d\eta_i(\xi_k, X) = d\eta_i(\xi_s, X), \quad i = 1,2,3, \; X\in H,$$

which proves the claim. □

8.2. QC vector fields. Suppose (M, H, g) is a quaternionic-contact manifold.

DEFINITION 8.6. *A diffeomorphism ϕ of a QC manifold (M, H) is called a* conformal quaternionic-contact automorphism (conformal QC automorphism) *if ϕ preserves the QC structure, i.e.*

$$\phi^*\eta = \mu\Psi \cdot \eta,$$

for some positive smooth function μ and some matrix $\Psi \in SO(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is a local 1-form considered as an element of $\mathbb{R}^3$.

In view of the uniqueness of the possible associated almost complex structures, see Lemma 2.2, a quaternionic contact automorphism will preserve also the associated (if any) almost complex structures, $\phi^*\mathbb{Q} = \mathbb{Q}$ and consequently, it will preserve the conformal class [g] on H. We note that conformal QC diffeomorphisms on S^{4n+3} are considered in [**Kam**]. The infinitesimal versions of these notions lead to the following definition.

DEFINITION 8.7. *A vector field $\mathcal{Q}$ on a QC manifold $(M, H, [g])$ is an infinitesimal generator of a conformal quaternionic-contact automorphism (*QC vector field *for short) if its flow preserves the QC structure, i.e.,*

$$\mathcal{L}_{\mathcal{Q}}\,\eta = (\nu I + O) \cdot \eta, \tag{8.5}$$

where ν is a smooth function and $O \in so(3)$ with smooth entries.

In view of the discussion above a QC vector field on a QC manifold (M, H, g) satisfies the conditions.

$$\mathcal{L}_{\mathcal{Q}}\,g = \nu g, \tag{8.6}$$

$$\mathcal{L}_{\mathcal{Q}}\,I = O \cdot I, \qquad O \in so(3), \quad I = (I_1, I_2, I_3)^t, \tag{8.7}$$

If the flow of a vector field $\mathcal{Q}$ is a conformal diffeomorphism of the horizontal metric g, i.e. (8.6) holds, we shall call it *infinitesimal conformal isometry.* If the function $\nu = 0$ then $\mathcal{Q}$ is said to be *infinitesimal isometry.*

A QC vector field on a QC manifold is a 3-contact vector field of special type. Indeed, let $\sharp$ be the musical isomorphism between T^*M and TM with respect to the fixed Riemannian metric g on TM and recall that the forms α_j were defined in (4.30). (4.31). We have

PROPOSITION 8.8. *Let (M, H, g) be a quaternionic-contact manifold. The vector field $\mathcal{Q}$ is a QC vector field if and only if*

$$\mathcal{Q} \;=\; -\frac{1}{2}\big(f_j\, I_i\, \alpha_k^\sharp \;-\; f_k\, I_i\, \alpha_j^\sharp \;-\; I_i\,(df_i)^\sharp\big) \;+\; \sum_{s=1}^{3} f_s\, \xi_s, \tag{8.8}$$

for some functions f_1, f_2 and f_3 such that for any positive permutation (i, j, k) of $(1, 2, 3)$ we have

$$\begin{aligned} f_j\, d\eta_i(\xi_j, \xi_i) + \; f_k\, d\eta_i(\xi_k, \xi_i) \;+\; \xi_i\, f_i & \\ = \; f_k\, d\eta_j(\xi_k, \xi_j) + \; f_i\, d\eta_j(\xi_i, \xi_j) \;+\; \xi_j\, f_j, \end{aligned} \tag{8.9}$$

$$\begin{aligned} f_i\, d\eta_i(\xi_i, \xi_j) + \; f_k\, d\eta_i(\xi_k, \xi_j) \;+\; \xi_j\, f_i & \\ = \; -\, f_j\, d\eta_j(\xi_j, \xi_i) \;-\; f_k\, d\eta_j(\xi_k, \xi_i) \;-\; \xi_i\, f_j, \end{aligned} \tag{8.10}$$

$$\begin{aligned} f_j\, I_i(\alpha_k)^\sharp \;-\; f_k\, I_i(\alpha_j)^\sharp \;-\; I_i(df_i)^\sharp & \\ = \; f_i\, I_k(\alpha_j)^\sharp \;-\; f_j\, I_k(\alpha_i)^\sharp \;-\; I_k(df_k)^\sharp \end{aligned} \tag{8.11}$$

Conversely, any three smooth functions satisfying the compatibility conditions (8.9), (8.10) and (8.11) determine a QC vector field by (8.8).

PROOF. Notice that (8.8) implies $f_i = \eta_i(\mathcal{Q})$. By Cartan's formula (8.5) is equivalent to

$$\mathcal{Q} \lrcorner d\eta_i \;+\; df_i \;=\; \nu\, \eta_i \;+\; o_{is}\, \eta_s.$$

In other words, both sides must be the same when evaluated on ξ_t, $t = 1, 2, 3$ and also when restricted to the horizontal bundle. Let

$$\mathcal{Q} = \mathcal{Q}_H + \sum_{s=1}^{3} f_s \xi_s.$$

Consider first the action on the vertical vector fields. Pairing with ξ_t and taking successively $t = i, j, k$ gives

$$
\begin{aligned}
f_j\, d\eta_i(\xi_j, \xi_i) + \ f_k\, d\eta_i(\xi_k, \xi_i) \ + \ \xi_i\, f_i \ &= \ \nu \ + \ o_{ii} \\
\alpha_k(\mathcal{Q}_H) \ + \ f_i\, d\eta_i(\xi_i, \xi_j) + \ f_k\, d\eta_i(\xi_k, \xi_j) \ + \ \xi_j\, f_i \ &= \ o_{ij} \\
-\alpha_j(\mathcal{Q}_H) \ + \ f_i\, d\eta_i(\xi_i, \xi_k) + \ f_j\, d\eta_i(\xi_j, \xi_k) \ + \ \xi_k\, f_i \ &= \ o_{ik}.
\end{aligned} \tag{8.12}
$$

Equating the restrictions to the horizontal bundle, i.e.,

$$
d\eta_i(\mathcal{Q}, .)|_H \ + \ df_i|_H \ = \ 0,
$$

gives

$$
\Big(f_j\, d\eta_i(\xi_j, .) \ + \ f_k\, d\eta_i(\xi_k, .) \ + \ d\eta_i(\mathcal{Q}_H, .) \ + \ df_i\Big)|_H \ = \ 0.
$$

Since $g(A, .)|_H \ = \ 0 \Leftrightarrow A \ = \ \sum_{s=1}^{3} \eta_s(A)\xi_s$, the last equation is equivalent to

$$
-f_j\alpha_k^\sharp \ + \ f_k\alpha_j^\sharp \ + \ 2I_i\mathcal{Q}_H \ + \ (df_i)^\sharp \ = \ \sum_{s=1}^{3} \big(-f_j\alpha_k(\xi_s) \ + \ f_k\alpha_j(\xi_s) \ + \ \xi_s f_i\big)\, \xi_s. \tag{8.13}
$$

Acting with I_i determines $2\mathcal{Q}_H \ = \ f_j\, I_i(\alpha_k)^\sharp \ - \ f_k\, I_i(\alpha_j)^\sharp \ - \ I_i(df_i)^\sharp$, which implies (8.8). In addition we have

$$
\alpha_j(\mathcal{Q}_H) \ = \ -\frac{1}{2}\Big(f_j\, \alpha_j(I_i(\alpha_k)^\sharp) \ - \ f_k\, \alpha_j(I_i(\alpha_j)^\sharp) \ - \ \alpha_j(I_i(df_i)^\sharp)\Big)
$$

On the other hand, $o \in so(3)$ is equivalent to o being a skew symmetric which is equivalent to (8.9) and (8.10), by the above computations. Therefore, if we are given three functions f_1, f_2, f_3 satisfying (8.9), (8.10) and (8.11), then we define $\mathcal{Q}$ by (8.8). Using (8.12) we define ν and o with $o \in so(3)$ with smooth entries. With these definitions $\mathcal{Q}$ is a QC vector field. □

Using the formulas in Example 5.13 we obtain from Proposition 8.8 the following '3-hamiltonian' form of a QC vector field on 3-Sasakian manifold.

COROLLARY 8.9. *Let* (M,η) *be a 3-Sasakian manifold. Any QC vector field* $\mathcal{Q}$ *has the form*

$$\mathcal{Q} = \mathcal{Q}_H + f_1\xi_1 + f_2\xi_2 + f_3\xi_3,$$

where the smooth functions f_1, f_2, f_3 *satisfy the conditions*

$$d_i f_i = d_j f_j, \quad \xi_i(f_i) = \xi_j(f_j), \quad \xi_i(f_j) = -\xi_j(f_i), \quad i,j = 1,2,3,$$

and the horizontal part $\mathcal{Q}_H \in H$ *is determined by*

$$\mathcal{Q}_H \lrcorner d\eta_i = d_i f_i, \quad i \in \{1,2,3\}.$$

The matrix in (8.5) *has the form*

$$\nu I_{d_3} + O = \begin{pmatrix} \xi_1(\eta_1(\mathcal{Q})) & -\xi_1(\eta_2(\mathcal{Q})) - 2\eta_3(\mathcal{Q}) & -\xi_1(\eta_3(\mathcal{Q})) + 2\eta_2(\mathcal{Q}) \\ \xi_1(\eta_2(\mathcal{Q})) + 2\eta_3(\mathcal{Q}) & \xi_1(\eta_1(\mathcal{Q})) & -\xi_2(\eta_3(\mathcal{Q})) - 2\eta_1(\mathcal{Q}) \\ \xi_1(\eta_3(\mathcal{Q})) - 2\eta_2(\mathcal{Q}) & \xi_2(\eta_3(\mathcal{Q})) + 2\eta_1(\mathcal{Q}) & \xi_1(\eta_1(\mathcal{Q})) \end{pmatrix}.$$

In particular, the Reeb vector fields ξ_1, ξ_2, ξ_3 *are infinitesimal isometries.*

We note that on any QC structure homothetic to a 3-Sasakian structure, the Reeb vector fields are also infinitesimal isometries, i..e, (8.6) with $\nu = 0$ and (8.7) hold for $\mathcal{Q} = \xi_i, i = 1,2,3$. This follows easily from Corollary 8.9. Our next goal is to characterize QC structures for which the Reeb vector fields are QC vector fields. It turns out that the just mentioned setting is the only possible. More precisely, we have the following Theorem.

THEOREM 8.10. *Let* (M,H,g) *be a QC manifold with positive QC scalar curvature, assumed constant in dimension seven. The following conditions are equivalent.*

i) Each of the Reeb vector fields is a QC vector field.

ii) The QC structure is homothetic to a 3-Sasakian structure. In particular, the Reeb vector fields are infinitesimal isometries.

PROOF. We note that Corollary 8.5 shows that on a QC manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are 3-contact exactly when the connection 1-forms vanish on H. By Lemma 5.18, Theorem 5.9 together with the made assumptions on the QC scalar curvature we see that the QC scalar curvature is a positive constant. Now, Corollary 5.17 shows that the given QC stricture is homothetic to a 3-Sasakian structure. The converse direction was already explained before the statement of the Theorem. □

For the remaining of this section we prove other useful properties of QC vector fields. The next three Lemmas are of independent interest. Given a vector field $\mathcal{Q}$, we define the symmetric tensor $T^0_{\mathcal{Q}}$ and the skew-symmetric tensor $u_{\mathcal{Q}}$

$$T^0_{\mathcal{Q}} \;=\; \sum_{s=1}^{3}\eta_s(\mathcal{Q})\,T^0_{\xi_s}, \qquad u_{\mathcal{Q}} \;=\; \sum_{s=1}^{3}\eta_s(\mathcal{Q})\,I_s u, \tag{8.14}$$

respectively such that

$$T(\mathcal{Q},X,Y) \;=\; g(T^0_{\mathcal{Q}}X,Y) \;+\; g(u_{\mathcal{Q}}X,Y),$$

LEMMA 8.11. *The tensors $T^0_{\mathcal{Q}}$ and $u_{\mathcal{Q}}$ lie in the $[-1]$ component associated to the operator † cf.* (2.15) *and* (2.14)*.*

PROOF. By the definition of $u_{\mathcal{Q}}$, we have

$$g(u_{\mathcal{Q}}I_1X,I_1Y) \;=\; \sum_{s=1}^{3}\eta_s(\mathcal{Q})\,g(I_suX,Y)$$

and after summing we find

$$\sum_{j=1}^{3}g(u_{\mathcal{Q}}I_jX,I_jY) \;=\sum_{j=1}^{3}\eta_j(\mathcal{Q})\,g(I_juX,Y) \;=\; -g(u_{\mathcal{Q}}X,Y).$$

We turn to the second claim. Recall that $T^0_{\xi_j}$ anti-commutes with I_j, see (2.19). Hence,

$$\begin{aligned} g(T^0_{\mathcal{Q}}I_1X,I_1Y) \;=\; -\eta_1(\mathcal{Q})\,g(T^0_{\xi_1}X,Y) - \;\eta_2(\mathcal{Q})\,[g(T^0_{\xi_2}{}^{--+}X,Y) \;-g(T^0_{\xi_2}{}^{+--}X,Y)]\\ -\;\eta_3(\mathcal{Q})\,[g(T^0_{\xi_2}{}^{-+-}X,Y) \;-g(T^0_{\xi_3}{}^{+--}X,Y)], \end{aligned}$$

$$\begin{aligned} g(T^0_{\mathcal{Q}} I_2X, I_2Y) \;&=\; -\eta_2(\mathcal{Q})\, g(T^0_{\xi_2}X, Y) - \;\eta_1(\mathcal{Q})\,[g(T^0_{\xi_1}{}^{--+}X, Y) \;-g(T^0_{\xi_1}{}^{-+-}X, Y)] \\ &\quad - \;\eta_3(\mathcal{Q})\,[g(T^0_{\xi_3}{}^{+--}X, Y) \;-g(T^0_{\xi_3}{}^{-+-}X, Y)], \\ g(T^0_{\mathcal{Q}} I_3X, I_3Y) \;&=\; -\eta_3(\mathcal{Q})\, g(T^0_{\xi_3}X, Y) - \;\eta_1(\mathcal{Q})\,[g(T^0_{\xi_1}{}^{-+-}X, Y) \;-g(T^0_{\xi_2}{}^{--+}X, Y)] \\ &\quad - \;\eta_2(\mathcal{Q})\,[g(T^0_{\xi_2}{}^{+--}X, Y) \;-g(T^0_{\xi_2}{}^{--+}X, Y)]. \end{aligned}$$

Summing the above three equations we come to

$$\sum_{j=1}^{3} g(T^0_{\mathcal{Q}} I_jX, I_jY) \;=\; -\sum_{j=1}^{3} g(\mathcal{Q}, \xi_j)\, g(T^0_{\xi_j}X, Y) \;=\; -g(T^0_{\mathcal{Q}}X, Y),$$

which finishes the proof of Lemma 8.11. $\square$

LEMMA 8.12. *If $\mathcal{Q}$ is a QC vector field then the next two equalities hold*

$$g(\nabla_X \mathcal{Q}, Y) \;+\; g(\nabla_Y \mathcal{Q}, X) \;+\; 2g(T^0_{\mathcal{Q}}X, Y) = \nu\, g(X, Y), \tag{8.15}$$

$$\begin{aligned} 3g(\nabla_X \mathcal{Q}, Y) - \sum_{s=1}^{3} g(\nabla_{I_sX}\mathcal{Q}, I_sY) + 4g(T^0_{\mathcal{Q}}X, Y) + 4g(u_{\mathcal{Q}}X, Y) \\ = -2\sum_{(ijk)} L_{ij}(\mathcal{Q})\omega_k(X, Y), \end{aligned} \tag{8.16}$$

where the sum is over all even permutation of (1,2,3) and

$$\begin{aligned} L_{ij}(\mathcal{Q}) = -L_{ji}(\mathcal{Q}) = \xi_j(\eta_i(\mathcal{Q})) - \eta_j(\mathcal{Q}) d\eta_j(\xi_i, \xi_j) \\ + \frac{1}{2}\eta_k(\mathcal{Q})\left(\frac{Scal}{8n(n+2)} + d\eta_j(\xi_k, \xi_i) - d\eta_i(\xi_j, \xi_k) - d\eta_k(\xi_i, \xi_j)\right). \end{aligned} \tag{8.17}$$

PROOF. In terms of the Biquard connection (8.6) reads exactly as (8.15). Furthermore, from (8.7), (8.12) and (4.29) it follows

$$
\begin{aligned}
(8.18)\quad o_{ij}I_jX + o_{ik}I_kX = \ & (\mathcal{L}_{\mathcal{Q}}\, I_i)(X) = \\
= \ & -\nabla_{I_iX}\mathcal{Q} \ + I_i\nabla_X\mathcal{Q} - \alpha_j(\mathcal{Q})I_kX + \alpha_k(\mathcal{Q})I_jX - \ T(\mathcal{Q}, I_iX) \ + \ I_iT(\mathcal{Q}, X).
\end{aligned}
$$

A use of (8.12), (4.30) and (4.31) allows us to write the last equation in the form

$$
\begin{aligned}
g(\nabla_X\mathcal{Q}, Y) \ - \ g(\nabla_{I_iX}\mathcal{Q}, I_iY) + \ T(\mathcal{Q}, X, Y) \ - \ T(\mathcal{Q}, I_iX, I_iY) \\
= (o_{ij} - \alpha_k(\mathcal{Q}))\omega_k(X, Y) - (o_{ik} + \alpha_j(\mathcal{Q}))\omega_j(X, Y) \\
= -L_{ij}(\mathcal{Q})\omega_k(X, Y) + L_{ik}(\mathcal{Q})\omega_j(X, Y),
\end{aligned}
$$

where $L_{ij}(\mathcal{Q})$ satisfy (8.17). Summing the above identities for the three almost complex structures and applying Lemma 8.11, we obtain (8.16), which completes the proof of Lemma 8.12. $\square$

COROLLARY 8.13. *Let (M, H, g) be a QC manifold with positive QC scalar curvature, assumed to be constant in dimension seven. The following conditions are equivalent*

i) There exists a local 3-Sasakian structure compatible with $H = Ker\ \eta$;

ii) There are three linearly independent vertical QC vector fields.

PROOF. Let $\gamma_1, \gamma_2, \gamma_3$ be linearly independent vertical QC vector fields. Then (8.15) for $\mathcal{Q} = \gamma_i$ yields

$$T^0_{\gamma_i} = 0, \quad i = 1, 2, 3, \quad \nu = 0$$

since T_{γ_i} is trace-free. Thus, for any cyclic permutation (i, j, k) of $(1, 2, 3)$, (8.16) implies

$$u_{\gamma_i} = 0, \qquad L_{ij}(\gamma_s) = 0$$

by comparing the trace and the trace-free part. Hence, we get

$$T_{\xi_s} = u_{\xi_s} = 0,$$

since γ_s are linearly independent vertical vector fields. Now, Theorem C shows that the given QC stricture is homothetic to a 3-Sasakian structure. ☐

In the particular case when the vector field $\mathcal{Q}$ is the gradient of a function defined on the manifold M, we have the following formulas.

Corollary 8.14. *If h is a smooth real valued function on M and $\mathcal{Q} = \nabla h$ is a QC vector field, then for any horizontal vector fields X and Y we have*

a) $[(\nabla dh)]_{[3][0]}(X,Y) \;=\; 0$

b) $[\nabla dh\,]_{[sym][-1]}(X,Y) \;=\; -T^0_{\mathcal{Q}}(X,Y)$ (*cf.* (8.14))

c) $u_{\mathcal{Q}}(X,Y) \;=\; 0$ (*cf.* (8.14)), $\qquad L_{ij}(\nabla h) = 0.$

Proof. Use (8.15) and (6.4) to get

$$2\nabla dh(X,Y) \;+\; 2dh(\xi_j)\,\omega_j(X,Y) \;+\; 2g(T^0_{\mathcal{Q}}X,Y) \;=\; \nu g(X,Y).$$

Decomposing in the $[-1]$ and $[3]$ components completes the proof of a) and b), taking into account (8.16) and Lemma 8.11. The skew-symmetric part of (8.16) gives

$$2u_{\mathcal{Q}} \;+\; \sum_{(ijk)} L_{ij}(\nabla h)\omega_k \;=\; 0,$$

where the sum is over all even permutations of $(1,2,3)$. Hence, c) follows by comparing the trace and trace-free parts of the last equality. ☐

We finish the section with another useful observation.

Lemma 8.15. *Let (M,H) be QC manifold and $\mathcal{Q}$ be a QC vector field determined by* (8.5) *and* (8.12). *The next equality hold*

$$d\eta_i([\mathcal{Q}, I_iX]^{\perp}, Y) + d\eta_i(I_iX, [\mathcal{Q}, Y]^{\perp}) = 0$$

Proof. We have using (8.5) that

$$\begin{aligned}(8.19)\quad \mathcal{L}_{\mathcal{Q}}\, d\eta_i(I_iX,Y) &= 2(\mathcal{L}_{\mathcal{Q}}\,\omega_i)(I_iX,Y) - d\eta_i([\mathcal{Q},I_iX]^{\perp},Y)\\ &- d\eta_i(I_iX,[\mathcal{Q},Y]^{\perp}) = -2(\mathcal{L}_{\mathcal{Q}}\,g)(X,Y) + 2g((\mathcal{L}_{\mathcal{Q}}\,I_i)I_iX,Y)\\ &- d\eta_i([\mathcal{Q},I_iX]^{\perp},Y) - d\eta_i(I_iX,[\mathcal{Q},Y]^{\perp}) = (d\mathcal{L}_{\mathcal{Q}}\,\eta_i)(I_iX,Y)\\ &= (d\nu\wedge\eta_i + \nu d\eta_i + do_{ij}\wedge\eta_j + o_{ij}d\eta_j + do_{ik}\wedge\eta_k + o_{ik}d\eta_k)(I_iX,Y)\\ &= -2\nu g(X,Y) - 2o_{ij}\omega_k(X,Y) + 2o_{ik}\omega_j(X,Y),\end{aligned}$$

where o_{st} are the entries of the matrix O given by (8.12). An application of (8.6) and (8.7) to (8.19) gives the assertion. □

9. QC Yamabe problem

9.1. The Divergence Formula. Let (M,η) be a quaternionic-contact manifold with a fixed globally defined contact form η. For a fixed $j\in\{1,2,3\}$ the form

$$Vol_\eta = \eta_1\wedge\eta_2\wedge\eta_3\wedge\omega_j^{2n} \tag{9.1}$$

is a volume form. Note that Vol_η is independent of j. We define the (horizontal) divergence of a horizontal vector field/one-form $\sigma\in\Lambda^1(H)$ by

$$\nabla^*\sigma \;= tr|_H\nabla\sigma = \sum_{a=1}^{4n}(\nabla_{e_\alpha}\sigma)(e_\alpha). \tag{9.2}$$

Clearly the horizontal divergence does not depend on the basis and is an $Sp(n)Sp(1)$-invariant. For any horizontal 1-form $\sigma\in\Lambda^1(H)$ we denote with $\sigma^\#$ the corresponding horizontal vector field via the horizontal metric defined with the equality $\sigma(X)=g(\sigma^\#,X)$. It is justified to call the function $\nabla^*\sigma$ divergence of σ in view of the following Proposition.

Proposition 9.1. *Let (M,η) be a quaternionic-contact manifold of dimension $(4n+3)$ and*

$$\eta\wedge\omega_s^{2n-1} \overset{def}{=} \eta_1\wedge\eta_2\wedge\eta_3\wedge\omega_s^{2n-1}.$$

For any horizontal 1-form $\sigma \in \Lambda^1(H)$ we have

$$d(\sigma^{\#} \lrcorner (Vol_\eta)) \;=\; -(\nabla^* \sigma)\, \eta \wedge \omega^{2n} \;.$$

Therefore, if M is compact,

$$\int_M (\nabla^* \sigma)\, \eta \wedge \omega^{2n} \;=\; 0.$$

Proof. We work in a QC normal frame at a point $p \in M$ constructed in Lemma 5.5. Since σ is horizontal, we have $\sigma^{\#} = g(\sigma^{\#}, e_a)e_a$. Therefore, we calculate

$$\sigma^{\#} \lrcorner (Vol_\eta) = \sum_{a=1}^{4n} (-1)^a g(\sigma^{\#}, e_a)\eta \wedge e^{\#}_{a_1} \wedge \cdots \wedge \hat{e^{\#}_a} \wedge \cdots \wedge e^{\#}_{4n},$$

where $\hat{e^{\#}_a}$ means that the 1-form $e^{\#}_a$ is missing in the above wedge product. The exterior derivative of the above expression gives

$$d(\sigma^{\#} \lrcorner (Vol_\eta)) = -\sum_{a=1}^{4n} e_a g(\sigma^{\#}, e_a) Vol_\eta = -(\nabla^* \sigma)\, Vol_\eta.$$

Indeed, since the Biquard connection preserves the metric, the middle term is calculated as follows:

$$e_a g(X, e_a) = g(\nabla_{e_a} X, e_a) + g(X, \nabla_{a_a} e_a)$$

which evaluated at the point p gives

$$e_a g(X, e_a)_{|_p} = g(\nabla_{e_a} X, e_a)_{|_p}) = ((\nabla_{e_a} \sigma) e_a)_{|_p}.$$

In order to obtain the last equality we also used the definition of the Reeb vector fields, (4.1), and the following sequence of identities

$$de^{\#}_b(e_b, e_a)_{|_p} = e^{\#}_b([e_b, e_a])_{|_p} = e^{\#}_b(\nabla_{e_b} e_a - \nabla_{e_a} e_b - T(e_b, e_a))_{|_p} = 0$$

since $T(e_b, a_a)$ is a vertical vector field. This proves the first formula. If the manifold is compact, then Stoke's theorem completes the proof. □

We note that the integral formula of the above theorem was essentially proved in [[W], Proposition 2.1].

9.2. Partial solutions of the QC Yamabe problem. Given a QC structure $\tilde{\eta}$, the QC Yamabe problem is to find all QC structures η that are QC conformal to $\tilde{\eta}$ and have constant QC scalar curvature. The relation between the two QC scalar curvatures is given by the QC Yamabe equation (6.8) and the problem is to find all solutions of this equation. In this Section we shall present a partial solution of the QC Yamabe problem on the quaternionic sphere. Equivalently, using the Cayley transform this provides a partial solution of the QC Yamabe problem on the quaternionic Heisenberg group. The extra assumption under which we classify the solutions of the QC Yamabe equation consists of assuming that the "new" quaternionic structure has an integrable vertical space. The change of the vertical space is given by (6.1). Of course, the standard quaternionic contact structure has an integrable vertical distribution. A note about the Cayley transform is in order. We shall define below the explicit Cayley transform for the considered case, but one should keep in mind the more general setting of groups of Heisenberg type [CDKR1]. In that respect, the solutions of the QC Yamabe equation on the quaternionic Hesenberg group, which we describe, coincide with the solutions on the groups of Heisenebrg type [GV1].

As in Section 5 we are considering a conformal transformation $\tilde{\eta} = \frac{1}{2h}\eta$, where $\tilde{\eta}$ represents a fixed quaternionic-contact structure and η is the "new" structure conformal to the original one. In fact, eventually, $\tilde{\eta}$ will stand for the standard quaternionic-contact structure on the quaternionic sphere. In this case the QC Yamabe problem , up to a homothety, is to find all structures η, which are conformal to $\tilde{\eta}$ and have constant scalar curvature equal to $16n(n+2)$, see Corollary 5.13.

PROPOSITION 9.2. *Let $(M, \bar{\eta})$ be a compact QC Einstein manifold of dimension $(4n+3)$. Let $\bar{\eta} = \frac{1}{2h}\eta$ be a conformal transformation of the QC structure $\bar{\eta}$ on M. Suppose η has constant scalar curvature.*

a) If $n > 1$, then any one of the following two conditions implies that η is a QC Einstein structure:
  i) the vertical space of η is integrable;
  ii) the QC structure η is QC pseudo Einstein.

b) If $n = 1$ and the vertical space of η is integrable than η is a QC Einstein structure.

PROOF. The proof follows the steps of the solution of the Riemannian Yamabe problem on the standard unit sphere, see [LP]. Theorem C shows that $\tilde{\eta}$ is a QC

Einstein structure. Theorem 4.13 and equations (6.7), (6.5), and (6.6) imply

$$
\begin{aligned}
(9.3)\quad [Ric_0]_{[-1]}(X,Y) &= (2n+2)T^0(X,Y)\\
&= -(2n+2)h^{-1}[\nabla dh]_{[sym][-1]}(X,Y)
\end{aligned}
$$

$$
\begin{aligned}
(9.4)\quad [Ric_0]_{[3]}(X,Y) &= 2(2n+5)U(X,Y)\\
&= -(2n+5)h^{-1}[\nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}(X,Y).
\end{aligned}
$$

Furthermore, when the scalar curvature of η is a constant then Theorem 5.8 gives

$$
\nabla^* T^0 = (n+2)\mathbb{A}, \qquad \nabla^* U = \frac{(1-n)}{2}\mathbb{A}. \tag{9.5}
$$

If $n > 1$ and either the vertical space of η is an integrable distribution or η is QC pseudo Einstein $U = 0$, then (9.5) shows that $\mathbb{A} = 0$ and the divergences of T^0 and U vanish $\nabla^* T^0 = 0$ and $\nabla^* U = 0$. The same conclusion can be reached in the case $n = 1$ assuming the integrability of the vertical space (recall that always $U = 0$ when $n = 1$). We shall see that, in fact, T^0 and U vanish, i.e., η is also QC Einstein. Consider first the $[-1]$ component. Taking norms, multiplying by h and integrating, the divergence formula gives

$$
\begin{aligned}
\int_M h\,|\,[Ric_o]_{[-1]}\,|^2\,\eta\wedge\omega^{2n} &= (2n+2)\int \langle [Ric_o]_{[-1]}, \nabla dh]\rangle\,\eta\wedge\omega^{2n}\\
&= (2n+2)\int_M \langle \nabla^*\,[Ric_o]_{[-1]}, \nabla h]\rangle\,\eta\wedge\omega^{2n} = 0.
\end{aligned}
$$

Thus, the $[-1]$ component of the QC Einstein tensor vanishes $|\,[Ric_o]_{[-1]}\,| = 0$. Define $h = \frac{1}{2u}$, inserting (7.2) into (9.4) one gets

$$
[Ric_0]_{[3]} = 2(2n+5)U = -(2n+5)[\nabla du]_{[3][0]},
$$

from where, arguing as before we get $[Ric_0]_{[3]} = 0$. Theorem C completes the proof. $\square$

Corollary 9.3. *Let $\bar\eta = \frac{1}{2h}\eta$ be a conformal transformation of a compact QC Einstein manifold of dimension $(4n+3)$ and suppose $\bar\eta$ has constant QC scalar curvature.*

i) If $n>1$ and either the gradient ∇h or the gradient $\nabla(\frac{1}{h})$ is a QC vector fields then h is a constant.

ii) If $n=1$ and the gradient $\nabla(\frac{1}{h})$ is a QC vector fields then h is a constant.

Proof. Suppose ∇h is a QC vector field. Corollary 8.14, b) yields

$$[\nabla dh]_{[sym][-1]} = 0$$

since the torsion of Biquard connection vanishes due to Proposition 5.2. Then Proposition 9.2 and a) in Corollary 8.14 imply that on H we have

$$dh\otimes dh + d_1h\otimes d_1h + d_2h\otimes d_2h + d_3h\otimes d_3h = \frac{|dh|^2}{n}g.$$

If $n>1$ then $dh_{|H}=0$, which implies $dh=0$ using the bracket generating condition.

Suppose $\nabla(\frac{1}{h})$ is a QC vector field. Then Proposition 9.2, (7.2) combined with b) in Corollary 8.14 show that on H we have

$$3dh\otimes dh - d_1h\otimes d_1h - d_2h\otimes d_2h - d_3h\otimes d_3h = 0.$$

Define $X=I_1X, Y=I_1Y$ etc. to get

$$dh\otimes dh = d_1h\otimes d_1h = d_2h\otimes d_2h = d_3h\otimes d_3h.$$

Hence, $dh_{|H}=0$ since dim $Ker\, dh \;=\; 4n-1$ and $dh=0$ as above. □

9.3. Proof of Theorem B.

Proof. We start the proof with the observation that from Proposition 9.2 and Corollary 7.26 the new structure η is also QC Einstein. Next we bring into consideration the quaternionic Heisenberg group (cf. Section 3) and the correponding Cayley transform (cf. Section 3.3) which is a conformal quaternionic-contact diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic-contact structure $\tilde\Theta$ and the sphere minus a point with its standard structure $\tilde\eta$.

Since, $\tilde\Theta$ is QC Einstein by definition, while η and hence also $(\mathcal{C}^{-1})^*\eta$ are QC Einstein as we observed at the beginning of the proof. Now we can apply Theorem A according to which up to a multiplicative constant factor the forms $(\mathcal{C}^{-1})^*\tilde\eta$ and

$(\mathcal{C}^{-1})^*\eta$ are related by a translation or dilation on the Heisenebrg group. Hence, we conclude that up to a multiplicative constant, η is obtained from $\tilde{\eta}$ by a conformal quaternionic-contact automorphism, see Definition 8.6. $\square$

Let us note that the Cayley transform defined in the setting of groups of Heisenberg type is also a conformal transformation on H, see cf. [**ACD**, Lemma 2.5]. One can write the above transformation formula in this more general setting.

CHAPTER 3

Quaternionic-contact Einstein manifolds

An extensively studied class of QC structures is provided by the 3-Sasakian spaces. In Therem C of Chapter 2, we have shown that a QC manifold is locally 3-Sasakian iff it is QC Einstein with poitive and constant QC scalar curvature. Furthermore, as a consequence of the Bianchi identities, in Theorem 5.9 (Chapter 2), we have shown that the QC scalar curvature of a QC Einstein manifold of dimension at least eleven is constant while the seven dimensional case was left open. In this chapter, we shall extend this two results. We begin with:

Theorem D. *The QC scalar curvature of a 7-dimensional QC Einstein manifold is always a constant.*

The proof of Theorem D makes use of the QC conformal curvature tensor [**IV1**] that characterizes the QC conformally flat structures. We shall need also a result of Kulkarni [**Kul**] on the algebraic properties of curvature tensors in four dimensions, and an extension of Theorem A (Chapter 2) describing explicitly the different QC Einstein structures defined locally on the quaternionic Heisenberg group which are also point-wise QC conformal to the flat one. The main application of Theorem D is the removal of the a-priori assumption of constancy of the QC scalar curvature in some previous results concerning seven dimensional QC Einstein manifolds.

The remaining parts of this chapter are motivated by some known properties of the QC Einstein manifolds with non-vanishing QC scalar curvature, for which we prove here corresponding results in the case of vanishing QC scalar curvature. With this goal in mind and because of its independent interest, in Section 11, we define a certain connection on the canonical three dimensional vertical distribution of a QC manifold. We show that the QC Einstein spaces can be characterized by the flatness of this vertical connection. Using this, we write the structure equations of a QC Einstein manifold in terms of the defining 1-forms, their exterior derivatives and the QC scalar curvature (see Theorem 12.1).

Recall that the complete, and regular 3-Sasakian and nS-spaces (called negative 3-Sasakian here) have a canonical fibering with fiber $Sp(1)$ or $SO(3)$, and base a quaternionic-Kähler manifold. We show that if $S > 0$ (resp. $S < 0$), the QC Einstein manifolds are "essentially" $SO(3)$ bundles over quaternionic-Kähler manifolds with positive (resp. negative) scalar curvature. In section 13 we show that in the "regular" case, similar to the non-zero QC scalar curvature cases, a QC Einstein manifold of zero scalar curvature fibers over a hyper-Kähler manifold (cf. Proposition 13.3).

We conclude the chapter with a brief section where we show that every QC Einstein manifold of non-zero QC scalar curvature carries two Einstein metrics. Note that the corresponding result concerning the 3-Sasakian case is well known, see [BGN]. In the negative QC scalar curvature case both Einstein metrics are of signature $(4n, 3)$ of which the first is (locally) negative 3-Sasakian, while the second "squashed' metric is not 3-Sasakian, see Proposition 13.4.

10. Proof of Theorem D

Convention 10.1. *Throughout this chapter, unless explicitly stated otherwise, we will use the following conventions:*

a) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$ while s, t will denote any numbers from the set $\{1, 2, 3\}$, $s, t \in \{1, 2, 3\}$.

b) For a decomposition $TM = V \oplus H$ we let $[.]_V$ and $[.]_H$ be the corresponding projections to V and H.

c) A, B, C, etc. will denote sections of the tangent bundle of M, i.e., $A, B, C \in TM$.

d) X, Y, Z, U will denote horizontal vector fields, i.e., $X, Y, Z, U \in H$.

e) ξ, ξ', ξ'' will denote vertical vector fields, i.e., $\xi, \xi', \xi'' \in V$.

f) $\{e_1, \dots, e_{4n}\}$ denotes an orthonormal frame for the horizontal distribution H;

g) The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ is used. For example, $k = P(e_b, e_a, e_a, e_b)$ means $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$;

h) We shall denote by S the (normalized) QC scalar curvature given by

$$S = \frac{Scal}{8n(n+2)}. \tag{10.1}$$

The proof of Theorem D is achieved with the help of the following Lemma 10.2 in which we calculate the curvature $R(Z, X, Y, V)$ of the Biquard connection at points where the horizontal gradient of the (normalized) QC scalar curvature S does not vanish, $\nabla S \neq 0$. The proof of Theorem D proceeds by showing that on any open set where S is not locally constant M is locally QC conformally flat. In fact, on any open set where $\nabla S \neq 0$ the QC conformal curvature W^{qc} defined in [IV1] will be seen to vanish, hence by [IV1, Theorem 1.2] the QC manifold is locally QC conformally flat. The final step involves a generalization of Theorem A (Chapter 2), which follows by a slight modification of its proof, allowing the explicit description of all QC Einstein structures defined on open sets of the quaternionic Heisenberg group that are point-wise QC conformally-flat. It turns out that all such QC structures are of constant QC scalar curvature, which allows the completion of the proof of Theorem D.

Lemma 10.2. *On a seven dimensional QC Einstein manifold we have the following formula for the horizontal curvature of the Biquard connection on any open set where the QC scalar curvature is not constant,*

$$R(Z,X,Y,V)=2S\Big[g(Z,V)g(X,Y)-g(X,V)g(Z,Y)\Big]. \tag{10.2}$$

Proof of Lemma 10.2. Our first goal is to show the next identity,

$$R(Z,X,Y,\nabla S)=2S\Big[dS(Z)g(X,Y)-dS(X)g(Z,Y)\Big], \tag{10.3}$$

where ∇S is the horizontal gradient of S defined by $g(X,\nabla S)=dS(X)$. For this, recall the general formula proven in [**IV1**, Theorem 3.1, (3.6)],

$$\begin{aligned} R(\xi_i,\xi_j,X,Y)&=(\nabla_{\xi_i}U)(I_jX,Y)-(\nabla_{\xi_j}U)(I_iX,Y)\\ &-\frac14\Big[(\nabla_{\xi_i}T^0)(I_jX,Y)+(\nabla_{\xi_i}T^0)(X,I_jY)\Big]\\ &+\frac14\Big[(\nabla_{\xi_j}T^0)(I_iX,Y)+(\nabla_{\xi_j}T^0)(X,I_iY)\Big]\\ &-(\nabla_X\rho_k)(I_iY,\xi_i)-\frac{Scal}{8n(n+2)}T(\xi_k,X,Y)\\ &\qquad-T(\xi_j,X,e_a)T(\xi_i,e_a,Y)+T(\xi_j,e_a,Y)T(\xi_i,X,e_a), \end{aligned} \tag{10.4}$$

where the Ricci two forms are given by, cf. [**IV1**, Theorem 3.1],

$$\begin{aligned} 6(2n+1)\rho_s(\xi_s,X)&=(2n+1)X(S)+\\ &\frac12\Big[(\nabla_{e_a}T^0)(e_a,X)-3(\nabla_{e_a}T^0)(I_se_a,I_sX)\Big]-2(\nabla_{e_a}U)(e_a,X), \end{aligned} \tag{10.5}$$

$$6(2n+1)\rho_i(\xi_j, I_kX) = -6(2n+1)\rho_i(\xi_k, I_jX)$$
$$= (2n-1)(2n+1)X(S) - \frac{1}{2}\Big[(4n+1)(\nabla_{e_a}T^0)(e_a, X) + 3(\nabla_{e_a}T^0)(I_ie_a, I_iX)\Big]$$
$$- 4(n+1)(\nabla_{e_a}U)(e_a, X).$$

In our case $T^0 = U = 0$, hence (10.4) takes the form

$$R(\xi_i, \xi_j, X, Y) = -(\nabla_X \rho_k)(I_iY, \xi_i). \tag{10.6}$$

Letting $n = 1$ and $T^0 = U = 0$ in (10.5) it follows

$$\rho_i(I_kY, \xi_j) = -\frac{1}{6}dS(Y),$$

which after a cyclic permutation of ijk and a replacement of Y by I_kY yields

$$\rho_k(I_iY, \xi_i) = -\frac{1}{6}dS(I_kY). \tag{10.7}$$

Taking the covariant derivative of (10.7) with respect to the Biquard connection and applying (4.29) we calculate

$$\begin{aligned}(\nabla_X\rho_k)(I_iY,\xi_i) &- \alpha_i(X)\rho_j(I_iY,\xi_i)\\ &+ \alpha_j(X)\rho_i(I_iY,\xi_i) - \alpha_j(X)\rho_k(I_kY,\xi_i) + \alpha_k(X)\rho_k(I_jY,\xi_i)\\ &- \alpha_j(X)\rho_k(I_iY,\xi_k) + \alpha_k(X)\rho_k(I_iY,\xi_j) = -\frac{1}{6}\nabla^2 S(X, I_kY)\\ &\qquad + \frac{1}{6}\alpha_i(X)dS(I_jY) - \frac{1}{6}\alpha_j(X)dS(I_iY).\end{aligned} \tag{10.8}$$

Applying (10.5) with $n = 1$ and $T^0 = U = 0$ we see that the terms involving the connection 1-forms cancel and (10.8) turns into

$$(\nabla_X\rho_k)(I_iY, \xi_i) = -\frac{1}{6}\nabla^2 S(X, I_kY). \tag{10.9}$$

A substitution of (10.9) into (10.6) and taking into account the skew-symmetry of $R(\xi_i, \xi_j, X, Y)$ with respect to X and Y, allows us to obtain the following identity for the horizontal Hession of S:

$$\nabla^2 S(X, I_s Y) + \nabla^2 S(Y, I_s X) = 0. \tag{10.10}$$

The trace of (10.10) together with the Ricci identity yield

$$\begin{aligned} 0 = 2\nabla^2 S(e_a, I_k e_a) &= \nabla^2 S(e_a, I_k e_a) - \nabla^2 S(I_k e_a, e_a) \\ &= -2\sum_{s=1}^{3} \omega_s(e_a, I_k e_a) dS(\xi_s) = -8dS(\xi_k), \end{aligned}$$

i.e., we have

$$dS(\xi_s) = 0, \qquad \nabla^2 S(\xi_s, \xi_t) = 0. \tag{10.11}$$

The equality (10.11) shows that S is constant along the vertical directions, $dS(\xi_s) = 0$, hence, in view of (4.29), the second equation of (10.11) holds as well. In addition, we have

$$\nabla^2 S(X, \xi_s) = XdS(\xi_s) - dS(\nabla_X \xi_s) = 0$$

since ∇ preserves the vertical directions due to (4.29). Moreover, the Ricci identity

$$\nabla^2 S(\xi_s, X) - \nabla^2 S(X, \xi_s) = dS(T(\xi_s, X)) = 0$$

together with the above equality leads to

$$\nabla^2 S(\xi_s, X) = \nabla^2 S(X, \xi_s) = 0. \tag{10.12}$$

Next, we show that the horizontal Hessian of S is symmetric. Indeed, we have the identity

$$\nabla^2 S(X,Y) - \nabla^2 S(Y,X) = d^2S(X,Y) - dS(T(X,Y)) = -2\sum_{s=1}^{3}\omega_s(X,Y)dS(\xi_s) = 0 \tag{10.13}$$

where we applied (10.11) to conclude the last equality. Now, (10.10) and (10.13) imply

$$\nabla^2 S(X,Y) - \nabla^2 S(I_sX, I_sY) = 0 \tag{10.14}$$

which shows that the $[-1]$-component of the horizontal Hessian vanishes. Hence, the horizontal Hessian of S is proportional to the horizontal metric since $n=1$, i.e.,

$$\nabla^2 S(X,Y) = \frac{\nabla^2 S(e_a,e_a)}{4} g(X,Y) = -\frac{\triangle S}{4} g(X,Y), \tag{10.15}$$

where $\triangle S = -\nabla^2 S(e_a,e_a)$ is the sub-Laplacian of S. We have the following Ricci identity of order three (see e.g. [IPV])

$$\nabla^3 S(X,Y,Z) - \nabla^3 S(Y,X,Z) = - R(X,Y,Z,\nabla S) - 2\sum_{s=1}^{3}\omega_s(X,Y)\nabla^2 S(\xi_s,Z). \tag{10.16}$$

Applying (10.12), we conclude from (10.16) that

$$\nabla^3 S(X,Y,Z) - \nabla^3 S(Y,X,Z) = -R(X,Y,Z,\nabla S). \tag{10.17}$$

Combining (10.17) and (10.15), we obtain the next expression for the curvature

$$R(Z,X,Y,\nabla S) = \frac{\nabla^3 S(X,e_a,e_a)}{4} g(Z,Y) - \frac{\nabla^3 S(Z,e_a,e_a)}{4} g(X,Y). \tag{10.18}$$

The trace of (10.18) together with the first equality of (4.48) computed for $n = 1, T^0 = 0$ and $U = 0$ yield

$$Ric(Z, \nabla S) = 6SdS(Z) = -\frac{3}{4}\nabla^3 S(Z, e_a, e_a).$$

Thus, we have

$$\nabla^3 S(Z, e_a, e_a) = -8SdS(Z). \tag{10.19}$$

Now, a substitution of (10.19) in (10.18) gives (10.3).

Turning to the general formula (10.2), we note that the horizontal curvature of the Biquard connection, in the QC Einstein case, satisfies the identity

$$R(X, Y, Z, V) + R(Y, Z, X, V) + R(Z, X, Y, V) = 0. \tag{10.20}$$

This follows from the first Bianchi identity since $(\nabla T)(X, Y) = 0$ and

$$T(T(X, Y), Z) = \sum_{s=1}^{3} 2\omega_s(X, Y)T(\xi_s, Z) = 0.$$

Thus, the horizontal curvature has the algebraic properties of a Riemannian curvature tensor; namely, it is skew-symmetric with respect to the first and the last pairs and satisfies the Bianchi identity (10.20). Therefore, it also has the fourth Riemannian curvature property

$$R(X, Y, Z, V) = R(Z, V, X, Y). \tag{10.21}$$

The equalities (10.3) and (10.21) imply

$$\begin{aligned} 0 &= R(I_i\nabla S, I_j\nabla S, I_k\nabla S, \nabla S) = R(I_k\nabla S, \nabla S, I_i\nabla S, I_j\nabla S), \\ 0 &= R(I_i\nabla S, I_j\nabla S, I_j\nabla S, \nabla S) = R(I_j\nabla S, \nabla S, I_i\nabla S, I_j\nabla S). \end{aligned} \tag{10.22}$$

Moreover, using (4.33) and the second equality in (4.48) with $T^0 = U = 0$ we calculate

$$R(I_j\nabla S, I_i\nabla S, I_i\nabla S, I_k\nabla S) - R(I_j\nabla S, I_i\nabla S, \nabla S, I_j\nabla S)$$
$$= -2\rho_j(I_j\nabla S, I_i\nabla S)\omega_k(\nabla S, I_k\nabla S) + 2\rho_k(I_j\nabla S, I_i\nabla S)\omega_j(\nabla S, I_k\nabla S) = 0 \tag{10.23}$$

The second equality of (10.22) together with (10.23) yields

$$R(I_j\nabla S, I_i\nabla S, I_i\nabla S, I_k\nabla S) = 0. \tag{10.24}$$

Finally, (10.3), (10.22), (10.23), (10.24) together with (4.33) and (4.48) imply for any $s \neq t$ the identities

$$R(I_s\nabla S, I_t\nabla S, I_t\nabla S, I_s\nabla S) = R(I_s\nabla S, \nabla S, \nabla S, I_s\nabla S) = 2S|\nabla S|^4. \tag{10.25}$$

In a neighborhood of any point where $\nabla S \neq 0$ the quadruple

$$\left\{\frac{\nabla S}{|\nabla S|}, \frac{I_1\nabla S}{|\nabla S|}, \frac{I_2\nabla S}{|\nabla S|}, \frac{I_3\nabla S}{|\nabla S|}\right\}$$

is an orthonormal basis of H, hence after a small calculation taking into account (10.22), (10.24) and (10.25), we see that for any orthonormal basis $\{Z, X, Y, V\}$ of H, we have

$$R(Z, X, Y, V) = 0, \qquad R(Z, X, Z, V) - R(Y, X, Y, V) = 0, \tag{10.26}$$

where the second equation follows from the first using the orthogonal basis

$$\{Z + Y, X, Z - Y, V\}.$$

For the "sectional curvature" $K(Z,X) = R(Z,X,Z,X)$ we have then the identities

$$
\begin{aligned}
&K(Z,X) + K(Y,V) - K(Z,V) - K(Y,X) \\
&\qquad = R(Z,X,Z,X) + R(Y,V,Y,V) - R(Z,V,Z,V) - R(Y,X,Y,X) \\
&\qquad = R(Y,X,Y,X) + R(Y,X,Y,V) - R(Y,V,Y,X) + R(Y,V,Y,V) \\
&\qquad - R(Z,X,Z,X) - R(Z,X,Z,V) + R(Z,V,Z,X) - R(Z,V,Z,V) \\
&\qquad\qquad = R(Z,X+V,Z,X-V) - R(Y,X+V,Y,X-V) = 0
\end{aligned}
$$

using (10.21) in the second equality and (10.26) in the last equality. Now, [**Kul**, Theorem 3], shows that the Riemannian conformal tensor of the horizontal curvature R vanishes. In view of $Ric = 6S \cdot g$, we conclude that the curvature restricted to the horizontal space is given by (10.2) which proves the lemma. $\square$

Proof of Theorem D. Let M be a QC Einstein manifold of dimension seven with a local $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ defining the given QC structure. Suppose the QC scalar curvature is not a locally constant function. We shall reach a contradiction by showing that M is locally QC conformally flat.

To prove the first claim we prove that if the QC scalar curvature is not locally constant then the QC conformal curvature W^{qc} of [**IV1**] vanishes at the points where $\nabla S \neq 0$. For this we recall the formula for the QC conformal curvature W^{qc} given in [**IV1**, Prposition 4.2] which with the assumptions $T^0 = U = 0$ simplifies to

$$
\begin{aligned}
W^{qc}(Z,X,Y,V) &= \frac{1}{4}\Big[R(Z,X,Y,V) + \sum_{s=1}^{3} R(I_sZ,I_sX,Y,V)\Big] \\
&\quad + \frac{S}{2}\Big[g(Z,Y)g(X,V) - g(Z,V)g(X,Y) \\
&\qquad + \sum_{s=1}^{3}\Big(\omega_s(Z,Y)\omega_s(X,V) - \omega_s(Z,V)\omega_s(X,Y)\Big)\Big].
\end{aligned}
\tag{10.27}
$$

A substitution of (10.2) in (10.27) shows $W^{qc} = 0$ on $\nabla S \neq 0$.

Now, [**IV1**, Theorem 1.2] shows that the open set $\nabla S \neq 0$ is locally QC conformaly flat, i.e., every point p, $\nabla S(p) \neq 0$ has an open neighborhood O and a QC conformal transformation $F : O \to \boldsymbol{G}\,(\mathbb{H})$ to the quaternionic Heisenberg group

$\boldsymbol{G}\,(\mathbb{H})$ equipped with the standard flat QC structure $\tilde{\Theta}$. Thus,

$$\Theta \overset{def}{=} F^*\eta = \frac{1}{2\mu}\tilde{\Theta}$$

for some positive smooth function μ defined on the open set $F(O)$. By its definition Θ is a QC Einstein structure, hence the proof of Theorem A (Chapter 2) shows that, with a small change of the parameters, μ is given by

$$\mu(q,\omega) \;=\; c_0 \left[\left(\sigma \;+\; |q+q_0|^2\right)^2 \;+\; |\omega \;+\; \omega_o \;+\; 2\ \mathrm{Im}\ q_o\,\bar{q}|^2 \right], \tag{10.28}$$

for some fixed $(q_o,\omega_o) \in \boldsymbol{G}\,(\mathbb{H})$ and constants $c_0 > 0$ and $\sigma \in \mathbb{R}$. A small calculation using (10.28) and the Yamabe equation (6.8) (Chapter 2) shows

$$Scal_\Theta = 128n(n+2)c_0\sigma = const.$$

Since η is QC conformal to Θ via the map F, it follows that

$$Scal_\eta = const$$

on O, which is a contradiction. $\square$

An immediate consequence of Theorem D and Theorem 5.9 (Chapter 2) is the next

Corollary 10.3. *The vertical space V of a seven dimensional QC Einstein manifold is integrable.*

We note that the integrability of the vertical distribution of a $4n+3$ dimensional QC Einstein manifold in the case $n > 1$, and when $S = const$ and $n = 1$, was proven earlier in Theorem 5.9 (Chapter 2). Thus, in any dimension, the vertical distribution V of a QC Einstein manifold is integrable and we have

$$\rho_s(X,Y) = -S\omega_s(X,Y), \qquad Ric(\xi_s,X) = \rho_s(X,\xi_t) = 0, \qquad [\xi_s,\xi_t] \in V. \tag{10.29}$$

Another Corollary of Theorem D and the analysis of the corresponding results in the case $n > 1$[IV2] is

Corollary 10.4. *If M is a seven dimensional QC Einstein manifold then $d\Omega = 0$, where Ω is the fundamental 4-form, defined by*

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

We note that also the converse of Corollary 10.4 is true if $n > 1$, see [IV2], while in the case $n = 1$ a counterexample was found in [CFS].

11. A characterization based on vertical flat connection

In this section we show that for any QC manifold M there is a natural linear connection $\tilde{\nabla}$, defined on the vertical distribution V (considered as a vector bundle over M). This connection has the remarkable property of being flat exactly when M is QC Einstein, see Theorem 11.3. It will be a very useful technical tool when studding the Riemannian geometry of QC Einstein spaces below in this chapter.

We start by introducing a cross-product on the vertical space V. Using the triple of Reeb vector fields ξ_1, ξ_2, ξ_3 we extend the horizontal metric g to a metric h on M by requiring $\mathrm{span}\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and

$$h|_H = g, \qquad h|_V = \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3. \tag{11.1}$$

The Riemannian metric h as well as the Biquard connection do not depend on the action of $SO(3)$ on V, but both change if η is multiplied by a conformal factor. Since the distribution V has a natural orientation and an inner product (the restriction of h to V, denoted by $\langle .,. \rangle$ here), we can introduce a cross-product operation $\times : \Lambda^2(V) \to V$ in the standard way: $\xi_i \times \xi_j = \xi_k$, $\xi_i \times \xi_i = 0$. The cross product operation is parallel with respect to any connection on V preserving the inner product $\langle .,. \rangle$, in particular, with respect to the Biquard connection ∇. For any $\xi, \xi', \xi'' \in V$, we have the standard relations

$$\begin{aligned}
&(\xi \times \xi') \times \xi'' = \langle \xi, \xi'' \rangle \xi' - \langle \xi', \xi'' \rangle \xi, \\
&\xi \times (\xi' \times \xi'') = (\xi \times \xi') \times \xi'' + \xi' \times (\xi \times \xi''), \\
&\nabla_A(\xi \times \xi') = (\nabla_A \xi) \times \xi' + \xi \times (\nabla_A \xi').
\end{aligned} \tag{11.2}$$

In the next lemma we collect a few formulas that will be useful later, in the proof of Theorem 11.3.

LEMMA 11.1. *The curvature R and torsion T of the Biquard connection ∇ of a QC Einstein manifold satisfy the following identities*

$$
\begin{aligned}
&T(\xi,\xi') = -S\xi\times\xi', \quad T(\xi,X)=0,\\
&R(A,B)\xi = -2S\sum_{s=1}^{3}\omega_s(A,B)\xi_s\times\xi.
\end{aligned}
\tag{11.3}
$$

PROOF. The torsion of the Biquard connection ∇ is described by the formulas (cf. Section 4, Chapter 2)

$$
\begin{aligned}
&T(X,Y) = -[X,Y]_V = 2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s,\\
&T(\xi_s,X) = \frac{1}{4}(I_sT^0 - T^0I_s)(X) + I_sU(X),\\
&T(\xi_i,\xi_j) = -S\xi_k - [\xi_i,\xi_j]_H,
\end{aligned}
\tag{11.4}
$$

The first two identities of Lemma 11.1 follow directly from (11.4) and the integrability of the vertical distribution V, see Corollary 10.3 and the paragraph after it. The last identity follows from (10.4) and (10.29). In particular, the three Ricci 2-forms $\rho_s(A,B)$ vanish unless A and B are both horizontal, in which case we have (10.29). The proof is complete. □

DEFINITION 11.2. *We define a connection $\widetilde{\nabla}$ on the (vertical) vector bundle V over M as follows*

$$
\widetilde{\nabla}_X\xi := \nabla_X\xi, \qquad \widetilde{\nabla}_\xi\xi' := \nabla_\xi\xi' + S(\xi\times\xi'). \tag{11.5}
$$

The main result of this section is

THEOREM 11.3. *A QC manifold M is QC Einstein iff the connection $\tilde{\nabla}$ is flat.*

PROOF. We start by relating the curvature of $\widetilde{\nabla}$ to the curvature of the Biquard connection ∇. To this end, let

$$
L \;=\; (\widetilde{\nabla}-\nabla) \quad \in\; \Gamma(M, T^*M\otimes V^*\otimes V)
$$

be the difference between the two connections on V. Then (11.5) implies

$$L_A\xi = L(A,\xi) = S[A]_V \times \xi,$$

where $[A]_V$ is the orthogonal projection of A on V. The curvature tensor $R^{\widetilde{\nabla}}$ of the new connection $\widetilde{\nabla}$ is given in terms of R and L by the well known general formula

$$R^{\widetilde{\nabla}}(A,B)\xi = R(A,B)\xi + \big(\nabla_A L\big)(B,\xi) - \big(\nabla_B L\big)(A,\xi) + \big[L_A, L_B\big]\xi + L\big(T(A,B),\xi\big). \tag{11.6}$$

We proceed by considering each of the terms on the right hand side of (11.6) separately. We have

$$R(A,B)\xi = \left(\sum_{s=1}^{3} 2\rho_s(A,B)\xi_s\right) \times \xi. \tag{11.7}$$

Using (11.2) and the obvious identity $\nabla_A\big([B]_V\big) = \big[\nabla_A B\big]_V$, we obtain

$$\big(\nabla_A L\big)(B,\xi) = \nabla_A\big(L(B,\xi)\big) - L\big(\nabla_A B,\xi\big) - L\big(B,\nabla_A\xi\big) = dS(A)[B]_V \times \xi. \tag{11.8}$$

From (11.2) it follows

$$\big[L_A, L_B\big]\xi = \big(L_A \times L_B\big) \times \xi = S^2\Big([A]_V \times [B]_V\Big) \times \xi. \tag{11.9}$$

The torsion identities (11.4) imply

$$L\big(T(A,B),\xi\big) = S\big[T(A,B)\big]_V \times \xi = S\Big(-S[A]_V \times [B]_V + 2\sum_{s=1}^{3}\omega_s(A,B)\xi_s\Big) \times \xi. \tag{11.10}$$

Finally, a substitution of (11.7), (11.8), (11.9) and (11.10) in the right hand side of formula (11.6) gives the equivalent relation

$$
\begin{aligned}
(11.11)\quad & R^{\widetilde{\nabla}}(A,B)\xi \\
&= \left(\sum_{s=1}^{3} 2\rho_s(A,B)\xi_s + dS(A)[B]_V - dS(B)[A]_V + 2S\sum_{s=1}^{3}\omega_s(A,B)\xi_s\right)\times\xi.
\end{aligned}
$$

We are now ready to complete the proof of the theorem. Suppose first that M is a QC Einstein manifold. By Theorem D when $n=1$ and Theorem 5.9 (Chapter 2) when $n>1$, it follows that the QC scalar curvature is constant. Lemma 11.1 implies that

$$\sum_{s=1}^{3}\rho_s(A,B)\xi_s = -S\sum_{s=1}^{3}\omega_s(A,B)\xi_s.$$

Since $dS=0$, (11.11) gives $R^{\widetilde{\nabla}}=0$, and thus $\widetilde{\nabla}$ is a flat connection on V.

Conversely, if $\widetilde{\nabla}$ is flat, then by applying (11.11) with $(A,B)=(X,Y)$ we obtain $\rho_s(X,Y)=-S\omega_s(X,Y)$. Applying the second formula of (4.48) we derive $T^0=0$ and $U=0$ by comparing the $Sp(n)Sp(1)$ components of the obtained equalities. Thus, (M,η) is a QC Einstein manifold taking into account the first formula in (4.48). $\square$

12. The structure equations of a QC Einstein manifold

Let M be a QC manifold with normalized QC scalar curvature S (cf. (10.1)). From [IV2, Proposition 3.1] we have the structure equations

$$
\begin{aligned}
& d\eta_i = 2\omega_i - \eta_j\wedge\alpha_k + \eta_k\wedge\alpha_j - S\eta_j\wedge\eta_k, \\
(12.1)\qquad & d\omega_i = \omega_j\wedge(\alpha_k + S\eta_k) - \omega_k\wedge(\alpha_j + S\eta_j) \\
& \qquad\qquad - \rho_k\wedge\eta_j + \rho_j\wedge\eta_k + \frac{1}{2}dS\wedge\eta_j\wedge\eta_k,
\end{aligned}
$$

where (η_1,η_2,η_3) is a local $\mathbb{R}^3$-valued 1-form defining the given QC structure and α_s are the corresponding connection 1-forms. If, locally, there is an $\mathbb{R}^3$-valued 1-form

$\eta = (\eta_1, \eta_2, \eta_3)$ that satisfies structure equations

$$d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k$$

with $S = const$ or such that the corresponding connection 1-forms vanish on H, $\alpha_i|_H = 0$, then M is a QC Einstein manifold of normalized QC scalar curvature S, see [**IV2**, Proposition 3.1] and Lemma 5.18 (Chapter 2).

Conversely, on a QC Einstein manifold of nowhere vanishing QC scalar curvature the structure equations (12.2) hold true by [**IV2**] and taking into account Corollary 13.1. The purpose of this section is to give the corresponding results in the case $Scal = 0$. The proof of Theorem 12.1 which is based on the connection $\widetilde{\nabla}$, defined in Section 11, rather than on the cone over a 3-Sasakian manifold employed in [**IV2**], works also in the case $Scal \neq 0$, thus in the statement of the Theorem we will not make an explicit note on the condition $Scal = 0$.

THEOREM 12.1. *Let M be a QC manifold. The following conditions are equivalent:*

a) M is a QC Einstein manifold;

b) locally, the given QC structure is defined by 1-form (η_1, η_2, η_3) such that for some constant S we have

$$d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k; \tag{12.2}$$

c) locally, the given QC structure is defined by 1-form (η_1, η_2, η_3) such that the corresponding connection 1-forms vanish on H, $\alpha_s = -S\eta_s$.

PROOF. As explained above, the implication c) $\Rightarrow$ a) is known, while b) $\Rightarrow$ c) is an immediate consequence of (12.1). Thus, only the implication a) implies b) needs to be proven, see also the paragraph preceding the Theorem.

Assume a) holds. We will show that the structure equation in b) are satisfied. By Theorem D when $n = 1$ and Theorem 5.9 (Chapter 2) when $n > 1$, it follows M is of constant QC scalar curvature. Let V be the vertical distribution. Clearly, the connection $\widetilde{\nabla}$ defined in Theorem 11.3 is a flat metric connection on V with respect to the inner product $\langle ., . \rangle$. Therefore the bundle V admits a local orthonormal oriented frame K_1, K_2, K_3 which is $\widetilde{\nabla}$-parallel, i.e., we have

$$\nabla_A K_i = -S[A]_V \times K_i. \tag{12.3}$$

There exists a triple of local 1-forms (η_1, η_2, η_3) on M vanishing on H, which satisfy $\eta_s(K_t) = \delta_{st}$. We rewrite (12.3) as

$$\nabla_A K_i = S\big(\eta_j(A)K_k - \eta_k(A)K_j\big). \tag{12.4}$$

Since K_1, K_2, K_3 is an orthonormal and oriented frame of V, we can complete the dual triple (η_1, η_2, η_3) to one defining the given QC structure. By differentiating the equalities $\eta_s(K_i) = \delta_{si}$ we obtain using (12.4) that

$$\begin{aligned} 0 \;=\; \big(\nabla_A\eta_s\big)(K_i) + \eta_s\big(\nabla_A K_i\big) \;&=\; \big(\nabla_A\eta_s\big)(K_i) + \eta_s\Big(S\big(\eta_j(A)K_k - \eta_k(A)K_j\big)\Big) \\ &=\; \big(\nabla_A\eta_s\big)(K_i) + S\Big(\eta_j(A)\delta_{sk} - \eta_k(A)\delta_{sj}\Big). \end{aligned}$$

Hence, $\big(\nabla_A\eta_i\big)(B) \;=\; S\eta_j \wedge \eta_k(A,B)$, which together with Lemma 11.1 allows the computation of the exterior derivative of η_i,

$$\begin{aligned} d\eta_i(A,B) &= \big(\nabla_A\eta_i\big)(B) - \big(\nabla_B\eta_i\big)(A) + \eta_i\big(T(A,B)\big) \\ &= S\eta_j\wedge\eta_k(A,B) - S\eta_j\wedge\eta_k(B,A) + \eta_i\Big(- S[A]_V \times [B]_V + 2\sum_s \omega_s(A,B)\xi_s\Big) \\ &= \Big(2\omega_i + S\eta_j\wedge\eta_k\Big)(A,B), \end{aligned} \tag{12.5}$$

which proves (12.2). Now $\alpha_s|_H = 0$ shows that K_s satisfy (4.1) and therefore K_s are the Reeb vector fields, which completes the proof of the Theorem. □

We finish the section with another condition characterizing QC Einstein manifolds, which is useful in some calculations.

Proposition 12.2. *Let M be a QC manifold. M is QC Einstein iff for some η compatible with the given QC structure*

$$d\omega_s(X,Y,Z) = 0. \tag{12.6}$$

Proof. If (12.2) are satisfied, then we have $0 \;=\; d(d\eta_i) \;=\; d\Big(2\omega_i + S\eta_j \wedge \eta_k\Big)$, which implies (12.6).

Conversely, suppose the given QC structure is locally defined by 1-form (η_1, η_2, η_3) which satisfies (12.6). By (12.1) we have

$$\Big(\omega_j \wedge \alpha_k - \omega_k \wedge \alpha_j\Big)|_H = 0,$$

which after a contraction with the endomorphism I_i gives

$$\begin{aligned}
0 = (\omega_j \wedge \alpha_k - \omega_k \wedge \alpha_j)(X, e_a, I_i e_a) \\
= \omega_j(X, e_a)\alpha_k(I_i e_a) + \omega_j(e_a, I_i e_a)\alpha_k(X) + \omega_j(I_i e_a, X)\alpha_k(e_a) \\
- \omega_k(X, e_a)\alpha_j(I_i e_a) - \omega_k(e_a, I_i e_a)\alpha_j(X) - \omega_k(I_i e_a, X)\alpha_j(e_a) \\
= 2\omega_j(X, e_a)\alpha_k(I_i e_a) - 2\omega_k(X, e_a)\alpha_j(I_i e_a) \\
= 2\alpha_k(I_k X) \ + \ 2\alpha_j(I_j X).
\end{aligned}$$

Since the above calculation is valid for any even permutation (i, j, k), it follows that $\alpha_s(X) = 0$ which completes the proof of the Proposition. $\square$

13. Related Riemannian geometry

A (4n + 3)-dimensional (pseudo) Riemannian manifold (M, g) is 3-Sasakian if the cone metric is a (pseudo) hyper-Kähler metric [**BG, BGN**]. We note explicitly that in this chapter 3-Sasakian manifolds are to be understood in the wider sense of positive (the usual terminology) or negative 3-Sasakian structures, cf. [**IV2**, Section 2] where the "negative" 3-Sasakian term was adopted in the case when the Riemannian cone is hyper-Kähler of signature $(4n, 4)$. Every 3-Sasakian manifold is a QC Einstein manifold of constant QC scalar curvature, [**Biq**]. As well known, a positive 3-Sasakian manifold is Einstein with a positive Riemannian scalar curvature [**Kas**] and, if complete, it is compact with finite fundamental group due to Myers theorem. The negative 3-Sasakian structures are Einstein with respect to the corresponding pseudo-Riemannian metric of signature $(4n, 3)$ [**Kas, Tan**]. In this case, by a simple change of signature, we obtain a positive definite nS metric on M, [**Tan, Jel, Kon**].

By Therem C of Chapter 2 when $Scal > 0$, and [**IV2**] when $Scal < 0$ a QC Einstein manifold of dimension at least eleven is locally QC homothetic to 3-Sasaki. The corresponding result in the seven dimensional case was proven with the extra assumption that the QC scalar is constant. Thanks to Theorem D, the additional hypothesis is redundant, hence we have the following

Corollary 13.1. *A seven dimensional QC Einstein manifold of nowhere vanishing QC scalar curvature is locally QC homothetic to a 3-Sasakian structure.*

There are many known examples of positive 3-Sasakian manifold, see [**BG**] and references therein for a nice overview of 3-Sasakian spaces. On the other hand, certian SO(3)-bundles over quaternionic-Kähler manifolds with negative scalar curvature constructed in [**Kon**, **Tan**, **Jel**] are examples of negative 3-Sasakian manifolds. Other, explicit examples of negative 3-Sasakian manifolds are constructed also in [**AFIV**].

Complete and regular 3-Sasakian manifolds, resp. nS-structures, fiber over a quaternionic-Kähler manifold with positive, resp. negative, scalar curvature [**Is**, **BGN**, **Tan**, **Jel**] with fiber $SO(3)$. Conversely, a quaternionic-Kähler manifold with positive (resp. negative) scalar curvature has a canonical $SO(3)$ principal bundle, the total space of which admits a natural 3-Sasakian (resp. nS-) structure [**Is**, **Kon**, **Tan**, **BGN**, **Jel**].

In this section we describe the properties of QC Einstein structures of zero QC scalar curvature, which complement the well known results in the 3-Sasakian case. A common feature of the $Scal = 0$ and $Scal \neq 0$ cases is the existence of Killing vector fields.

LEMMA 13.2. *Let M be a QC Einstein manifold with zero QC scalar curvature. If (η_1, η_2, η_3) is an $\mathbb{R}^3$-valued local 1-form defining the QC structure as in (12.2), then the corresponding Reeb vector fields ξ_1, ξ_2, ξ_3 are Killing vector fields for the Riemannian metric h, cf. (11.1).*

PROOF. By Theorem 12.1 c) we have $\alpha_i = 0$, hence $\nabla_A \xi_i = 0$ while Lemma 11.1 yields $T(\xi_s, \xi_t) = 0$. Therefore,

$$[\xi_s, \xi_t] = \nabla_{\xi_s}\xi_t - \nabla_{\xi_t}\xi_s - T(\xi_s, \xi_t) = 0,$$

which implies that for any $i, s, t \in \{1, 2, 3\}$, we have

$$(\mathcal{L}_{\xi_i} h)(\xi_s, \xi_t) = -h([\xi_i, \xi_s], \xi_t) - h(\xi_s, [\xi_i, \xi_t]) = 0.$$

Furthermore, using $d\eta_j(\xi_i, X) = \alpha_k(X) = 0$, we compute

$$(\mathcal{L}_{\xi_s} h)(\xi_t, X) = -h(\xi_t, [\xi_s, X]) = d\eta_t(\xi_s, X) = 0.$$

Finally, (4.18) gives

$$(\mathcal{L}_{\xi_i} h)(X, Y) = (\mathcal{L}_{\xi_i} g)(X, Y) = 2T^0_{\xi_i}(X, Y) = 0,$$

which completes the proof. □

13.1. Quotient a QC Einstein manifold with $S = 0$. The total space of an $\mathbb{R}^3$-bundle over a hyper-Kähler manifold with closed and locally exact Kähler forms $2\omega_s = d\eta_s$ with connection 1-forms η_s is a QC structure determined by the three 1-forms η_s, which is QC Einstein of vanishing QC scalar curvature, see [**IV2**]. In fact, we characterize QC Einstein manifold with vanishing QC scalar curvature as $\mathbb{R}^3$-bundle over hyper-Kähler manifold.

Let M be a QC Einstein manifold. As observed in Corollary 10.3 and the paragraph after it the vertical distribution V is completely integrable hence defines a foliation on M. We recall, taking into account [**Pal**], that the quotient space $P = M/V$ is a manifold when the foliation is regular and the quotient topology is Hausdorff.

If P is a manifold and all the leaves of V are compact, then by Ehresmann's fibration theorem [**Ehr, Pal**] it follows that $\Pi : M \to P$ is a locally trivial fibration and all the leaves are isomorphic. By [**Pal**], examples of such foliations are given by regular foliations on compact manifolds. In the case of a QC Einstein manifold of non-vanishing QC scalar curvature, the leaves of the foliation generated by V are Riemannian 3-manifold of positive constant curvature. Hence, if the associated (pseudo) Riemannian metrics on M is complete, then the leaves of the foliation are compact. On the other hand, in the case of vanishing QC scalar curvature, the leaves of the foliation are flat Riemannian manifolds that may not be compact as is, for example, the case of the quaternionic Heisenberg group. We summarize the properties of the Reeb foliation on a QC Einstein manifold of vanishing QC scalar curvature case in the following

Proposition 13.3. *Let M be a QC Einstein manifold with zero QC scalar curvature.*

a) If the vertical distribution V is regular and the space of leaves $P = M/V$ with the quotient topology is Hausdorff, then P is a locally hyper-Kähler manifold.
b) If the leaves of the foliation generated by V are compact then there exists an open dense subset $M_o \subset M$ such that $P_o := M_o/V$ is a locally hyper-Kähler manifold.

Proof. We begin with the proof of a). By Theorem 12.1 we can assume, locally, the structure equations given in Theorem 12.1. This, together with Lemma 4.3 & Theorem 4.13 (Chapter 2) implies that the horizontal metric g, see also (4.18), and the closed local fundamental 2-forms ω_s, see (12.2) with $S = 0$, are projectable. The claim of part a) follows from Hitchin's lemma [**Hit**].

We turn to the proof of part b). Lemma 13.2 implies that, in particular, the Riemannian metric h on M is bundle-like, i.e., for any two horizontal vector fields X and Y in the normalizer of V under the Lie bracket, the equation $\xi h(X,Y) = 0$ holds for any vector field ξ in V. Since all the leaves of the vertical foliation are assumed to be compact, we can apply [**Mo**, Proposition 3.7], which shows that $P = M/V$ is a 4n-dimensional orbifold. In particular P is a Hausdorff space. The regular points of any orbifold are an open dens set. Thus, if we let P_o be the set of all regular points of P, then P_o is an open dens subset of P which is also a manifold.

It follows that if $M_o := \Pi^{-1}(P_o)$ then all the leaves of the restriction of the vertical foliation to M_o are regular and hence the claim of b) follows. $\square$

13.2. Riemannian curvature. Let M be a QC Einstein manifold. Note that, by applying an appropriate QC homothetic transformation, we can aways reduce a general QC Einstein structure to one whose normalized QC scalar curvature S equals $0, 2$ or -2. Consider the one-parameter family of (pseudo) Riemannian metrics $h^\lambda,\ \lambda \neq 0$ on M by letting

$$h^\lambda(A,B) \ := \ h(A,B) + (\lambda-1)h|_V.$$

Let ∇^λ be the Levi-Civita connection of h^λ. Note that h^λ is a positive-definite metric when $\lambda > 0$ and has signature $(4n,3)$ when $\lambda < 0$.

Let us recall that, if $S = 2$ and $\lambda = 1$ the Riemannian metric $h = h^\lambda$ is a 3-Sasakian metric on M. In particular, it is an Einstein metric of positive Riemannian scalar curvature (4n + 2)(4n + 3) [Kas]. There is also a second Einstein metric, the "squashed" metric, in the family h^λ when $\lambda = 1/(2n+3)$, see [BG]. The case $S = -2$ is completely analogous. Here we have two distinct pseudo-Riemannian Einstein metrics corresponding to $\lambda = -1$ and $\lambda = -1/(2n+3)$. The first one defines a negative 3-Sasaskian structure. On the other hand, the metric h^λ with $\lambda = 1$ (assuming $S = -2$) gives an nS structure on M. In [Tan], it was shown that the Riemannian Ricci tensor of the latter has precisely two constant eigenvalues, $-4n-14$ (of multiplicity $4n$) and $4n+2$ (of multiplicity 3), and that the Riemannian scalar curvature is the negative constant $-16n^2-44n+6$. In particular, in this case, (M, h^λ) is an example of an A-manifold in the terminology of [Gr].

The following proposition addresses the case $S = 0$. However, the argument is valid for all values of S and $\lambda \neq 0$. In particular, we obtain new proofs of the above mentioned results concerning the cases of positive and negative 3-Sasakian structures.

PROPOSITION 13.4. *Let M be a QC Einstein manifold with normalized QC scalar curvature S. For a vector field A, let $[A]_V$ denote the orthogonal projection of A onto the vertical space V.*

The (pseudo) Riemannian Ricci and scalar curvatures of h^λ are given by

$$Ric^\lambda(A,B) = \Big(4n\lambda + \frac{S^2}{2\lambda}\Big) h^\lambda\Big([A]_V,[B]_V\Big) + \Big(2S(n+2) - 6\lambda\Big) h^\lambda\Big([A]_H,[B]_H\Big) \tag{13.1}$$

$$Scal^\lambda = \frac{1}{\lambda}\Big(-12n\lambda^2 + 8n(n+2)S\lambda + \frac{3}{2}S^2\Big). \tag{13.2}$$

In particular, if $S = 0$, the Ricci curvature of each metric in the family h^λ has exactly two different constant eigenvalues of multiplicities $4n$ and 3 respectively.

Proof. We start by noting that the difference

$$L = \nabla^\lambda - \nabla$$

between the Levi-Cevita connection ∇^λ and the Biquard connection ∇ is given by

$$L(A,B) = \nabla^\lambda_A B - \nabla_A B = \frac{S}{2}[A]_V \times [B]_V + \sum_{s=1}^{3}\Big\{ -\omega_s(A,B)\xi_s + \lambda\eta_s(A)I_sB + \lambda\eta_s(B)I_sA\Big\}. \tag{13.3}$$

Indeed, if we let

$$D_AB = \nabla_AB + L(A,B),$$

then $h^\lambda(L(A,B),C)$ is skew symmetric in B and C, hence the connection D preserves the metric h^λ. Furthermore, the torsion tensor of D vanishes since

$$h^\lambda(L(A,B),C) - h^\lambda(L(B,A),C) = -h^\lambda(T(A,B),C).$$

The latter follows from the formula for T in Lemma 11.1. Thus D is the Levi-Civita connection of h^λ.

The well known formula for the difference $R^\lambda - R$ between the curvature tensors of two connections ∇^λ and ∇ gives

$$R^\lambda(A,B)C \;-\; R(A,B)C = (\nabla_AL)(B,C) \;-\; (\nabla_BL)(A,C) \;+\; [L_A,L_B]C + L(T(A,B),C). \tag{13.4}$$

From (13.3), it follows L is ∇-parallel. Thus, in the right hand side of the above formula only the last two terms are non-zero. Furthermore, we have that

$$[L_A,L_B]C = L(A,L(B,C)) - L(B,L(A,C)).$$

A straightforward computation gives

$$
\begin{aligned}
(13.5)\quad R^\lambda(A,B)C &= R(A,B)C \\
&+ h^\lambda\Big([B]_V,[C]_V\Big)\Big(\frac{S^2}{4\lambda}[A]_V+\lambda[A]_H\Big) - h^\lambda\Big([A]_V,[C]_V\Big)\Big(\frac{S^2}{4\lambda}[B]_V+\lambda[B]_H\Big) \\
&+ \sum_{(i,j,k)-\text{cyclic}} \Big\{ \Big(\frac{S}{2}-\lambda\Big)\eta_k(A)\omega_j(B,C) - \Big(\frac{S}{2}-\lambda\Big)\eta_k(B)\omega_j(A,C) \\
&- \Big(\frac{S}{2}-\lambda\Big)\eta_j(A)\omega_k(B,C) + \Big(\frac{S}{2}-\lambda\Big)\eta_j(B)\omega_k(A,C) \\
&+ (S+2\lambda)\eta_k(C)\omega_j(A,B) - (S+2\lambda)\eta_j(C)\omega_k(A,B) \\
&- \lambda\eta_i(B)h^\lambda\Big([A]_H,[C]_H\Big) + \lambda\eta_i(A)h^\lambda\Big([B]_H,[C]_H\Big) \Big\}\xi_i \\
&+ \sum_{(i,j,k)-\text{cyclic}} \Big\{ \Big(\frac{\lambda S}{2}-\lambda^2\Big)\eta_j\wedge\eta_k(B,C)I_iA \\
&- \Big(\frac{\lambda S}{2}-\lambda^2\Big)\eta_j\wedge\eta_k(A,C)I_iB - (\lambda S-2\lambda^2)\eta_j\wedge\eta_k(A,B)I_iC \\
&- \lambda\omega_i(B,C)I_iA + \lambda\omega_i(A,C)I_iB + 2\lambda\omega_i(A,B)I_iC \Big\}.
\end{aligned}
$$

After taking the trace with respect to A and D in equation (13.5), we obtain

$$
\begin{aligned}
Ric^\lambda(B,C) &= Ric\Big([B]_H,[C]_H\Big) \\
&+ \Big(4n\lambda+\frac{S^2}{2\lambda}\Big)h^\lambda\Big([B]_V,[C]_V\Big) - 6\lambda h^\lambda\Big([B]_H,[C]_H\Big).
\end{aligned}
$$

Since M is assumed to be QC Einstein, we have

$$
(13.6)\qquad Ric\Big([B]_H,[C]_H\Big) = \frac{Scal}{4n}g\Big([B]_H,[C]_H\Big) = 2(n+2)Sh^\lambda\Big([B]_H,[C]_H\Big),
$$

which yields (13.1). Taking one more trace in (13.1) gives the formula for the scalar curvature. $\square$

CHAPTER 4

Solving the QC Yamabe equation on S^7

The QC Yamabe problem on S^7 is about the determinantion of all contact 1-forms η of the canonical QC structure on the sphere that have constant QC scalar curvature. In Chapter 2, we conjectured that these are precisely the forms that can be obtained as pull-back $\phi^*(\tilde{\eta})$ of the standard contact form $\tilde{\eta}$, where ϕ is a conformal quaternionic-contact automorphism of the sphere. In Theorem B (Chapter 2), we have shown a weaker result, namely, that the same conclusion holds provided the vertical space of η is integrable. The purpose of this chapter is to remove this extra assumption and to prove the conjecture when the dimension is seven:

THEOREM E. *Let $\tilde{\eta} = \frac{1}{2h}\eta$ be a conformal deformation of the standard qc-structure $\tilde{\eta}$ on the unit sphere S^7. If η has constant QC scalar curvature, then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic-contact automorphism. In particular, the Yamabe constant $\lambda(S^7)$ of the sphere is $48\,(4\pi)^{1/5}$ and this minimum value is achieved only by $\tilde{\eta}$ and its images under conformal quaternionic-contact automorphisms.*

An important motivation for studying the QC Yamabe problem on the sphere comes from its connection with the determination of the norm and extremals of the related Folland-Stein embedding on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, cf. Theorem 3.1. Using Theorem E, we obtain:

THEOREM F. *Let*

$$\boldsymbol{G}\,(\mathbb{H}) \;=\; \mathbb{H} \times Im\,\mathbb{H}$$

be the seven dimensional quaternionic Heisenberg group. The best constant in the L^2 Folland-Stein embedding theorem is

$$S_2 \;=\; \frac{2\sqrt{3}}{\pi^{3/5}}$$

An extremal is given by the function

$$v = \frac{2^{11}\sqrt{3}}{\pi^{3/5}}[(1+|q|^2)^2 + |\omega|^2]^{-2}, \ (q,\omega) \in \boldsymbol{G}\,(\mathbb{H})$$

Any other non-negative extremal is obtained from v by translations (16.7) *and dilations* (16.8).

Our result confirms the Conjecture made after [**GV1**, Theorem 1.1]. In [**GV1**, Theorem 1.6], a similar result is obtained in all dimensions, but with the extra assumption of partial-symmetry. Here with a completely different method, we show that the symmetry assumption is superfluous in the case of the first quaternionic Heisenberg group.

A key step in the present result is the establishment of a suitable divergence formula, Theorem 15.4, see [**JL3**] for the CR case and [**Ob**], [**LP**] for the Riemannian case. With the help of this divergence formula we show that the 'new' structure is also QC Einstein, thus we reduce the Yamabe problem on S^7 from solving the non-linear Yamabe equation to a geometrical system of differential equations describing the QC Einstein structures conformal to the standard one. Invoking the (quaternionic) Cayley transform, which is a contact conformal diffeomorphism (cf. Section 3.3), we turn the question to the corresponding system on the quaternionic Heisenberg group. On the latter, all global solutions were explicitly described in Theorem A (Chapter 2) and this is enough to conclude the proof of the result.

REMARK 13.5. *With the left invariant basis of Theorem F the Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ is not a group of Heisenberg type. If we consider $\boldsymbol{G}\,(\mathbb{H})$ as a group of Heisenberg type then the best constant in the L^2 Folland-Stein embedding theorem is, cf.* [**GV1**, Theorem 1.6],

$$S_2 = \frac{15^{1/10}}{\pi^{2/5}\,2\,\sqrt{2}}.$$

and an extremal is given by the function

$$F(q,\omega) = \gamma\ \left[(1+|q|^2)^2 + 16|\omega|^2)\right]^{-2}, \ (q,\omega) \in \boldsymbol{G}\,(\mathbb{H})$$

where

$$\gamma = 32\,\pi^{-17/50}\,2^{1/5}\,15^{2/5}.$$

14. Conformal transformations

Convention 14.1. *We use the following conventions:*

- $\{e_1, \dots, e_{4n}\}$ *denotes an orthonormal basis of the horizontal space* H.
- *The summation convention over repeated vectors from the basis* $\{e_1, \dots, e_{4n}\}$ *will be used. For example, for a (0,4)-tensor* P, *the formula* $k = P(e_b, e_a, e_a, e_b)$ *means*

$$k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b).$$

- *The triple* (i, j, k) *denotes any cyclic permutation of* $(1, 2, 3)$.

Note that a conformal quaternionic-contact transformation of a QC manifold is a diffeomorphism Φ which satisfies

$$\Phi^*\eta = \mu\ \Psi \cdot \eta,$$

for some positive smooth function μ and some matrix $\Psi \in SO(3)$ with smooth functions as entries and η is an $\mathbb{R}^3$-valued one form, $\eta = (\eta_1, \eta_2, \eta_3)^t$ is a column vector with entries one-forms. The Biquard connection does not change under rotations, i.e., the Biquard connection of $\Psi \cdot \eta$ and η coincide. Hence, studying conformal transformations we may consider only transformations

$$\Phi^*\eta \ = \ \mu\ \eta.$$

Let h be a positive smooth function on a QC manifold (M, η). Let

$$\bar\eta = \frac{1}{2h}\eta$$

be a conformal deformation of the QC structure η. We will denote the objects related to $\bar\eta$ by over-lining the same object corresponding to η. Thus,

$$d\bar\eta = -\frac{1}{2h^2}\, dh \wedge \eta \ + \ \frac{1}{2h} d\eta$$

and $\bar g = \frac{1}{2h} g$. The new triple $\{\bar\xi_1, \bar\xi_2, \bar\xi_3\}$ is determined by the conditions defining the Reeb vector fields. We have

$$\bar\xi_s \;=\; 2h\,\xi_s \;+\; I_s\nabla h, \qquad s=1,2,3, \tag{14.1}$$

where ∇h is the horizontal gradient defined by

$$g(\nabla h, X) = dh(X), \qquad X \in H.$$

The components of the torsion tensor transform according to the following formulas from Section 6

$$\overline{T}^0(X,Y) \;=\; T^0(X,Y) \;+\; h^{-1}\,[\nabla dh]_{[sym][-1]}(X,Y), \tag{14.2}$$

$$\bar U(X,Y) \;=\; U(X,Y) \;+\; (2h)^{-1}[\,\nabla dh - 2h^{-1} dh\otimes dh]_{[3][0]}(X,Y), \tag{14.3}$$

where the symmetric part is given by, cf. (14.9),

$$[\,\nabla dh]_{[sym]}(X,Y) \;=\; \nabla dh(X,Y) \;+\; \sum_{s=1}^{3} dh(\xi_s)\,\omega_s(X,Y)$$

and $_{[3][0]}$ indicates the trace free part of the [3]-component of the corresponding tensor. In addition, the qc-scalar curvature changes according to the formula (cf. Section 6)

$$\overline{\mathrm{Scal}} \;=\; 2h\,(\mathrm{Scal}) \;-\; 8(n+2)^2\,h^{-1}|\nabla h|^2 \;+\; 8(n+2)\,\triangle h. \tag{14.4}$$

The following vectors will be important for our considerations,

$$A_i \;=\; I_i[\xi_j,\xi_k], \qquad \text{hence} \quad A \;=\; A_1 \;+\; A_2 \;+\; A_3. \tag{14.5}$$

Lemma 14.2. *Let h be a positive smooth function on a QC manifold (M,H,g) with constant QC scalar curvature $Scal = 16n(n+2)$ and $\bar\eta \;=\; \frac{1}{2h}\,\eta$ a conformal deformation of the QC structure η. If $\bar\eta$ is a 3-Sasakian structure, then we have the* *formulas*

$$\begin{aligned}(14.6)\quad A_1(X) \;=\; & -\frac{1}{2}h^{-2}dh(X) \;-\; \frac{1}{2}h^{-3}|\nabla h|^2 dh(X)\\ & -\; \frac{1}{2}h^{-1}\Big(\;\nabla dh(I_2X,\xi_2) \;+\; \nabla dh(I_3X,\xi_3)\Big)\\ & +\; \frac{1}{2}h^{-2}\Big(dh(\xi_2)\,dh(I_2X) \;+\; dh(\xi_3)\,dh(I_3X)\Big)\\ & +\; \frac{1}{4}h^{-2}\Big(\;\nabla dh(I_2X,I_2\nabla h) \;+\; \nabla dh(I_3X,I_3\nabla h)\Big).\end{aligned}$$

The expressions for A_2 *and* A_3 *can be obtained from the above formula by a cyclic permutation of* $(1,2,3)$*. Thus, we have also*

$$\begin{aligned}A(X) \;=\; & -\frac{3}{2}h^{-2}dh(X) \;-\; \frac{3}{2}h^{-3}|\nabla h|^2 dh(X)\\ & -\; h^{-1}\sum_{s=1}^{3}\;\nabla dh(I_sX,\xi_s) \;+\; h^{-2}\sum_{s=1}^{3}dh(\xi_s)\,dh(I_sX)\\ & +\; \frac{1}{2}h^{-2}\sum_{s=1}^{3}\;\nabla dh(I_sX,I_s\nabla h)\end{aligned}$$

Proof. First we calculate the $sp\,(1)$-connection 1-forms of the Biquard connection ∇. For a 3-Sasaki structure we have

$$d\bar\eta_i(\bar\xi_j,\bar\xi_k) = 2, \quad \bar\xi_i \lrcorner d\bar\eta_i = 0,$$

the non-zero $sp(1)$-connection 1-forms are

$$\bar\alpha_i(\bar\xi_i) = -2, \quad i = 1,2,3,$$

and the QC scalar curvature $\overline{Scal} = 16n(n+2)$. Then (14.1) and Proposition 4.30 yield

$$
\begin{aligned}
2d\eta_i(\xi_j,\xi_k) &= 2h^{-1} + h^{-2}\|dh\|^2,\\
\alpha_i(X) &= -h^{-1}dh(I_iX),\\
\alpha_i(\xi_j) &= -h^{-1}dh(\xi_k) = -\alpha_j(\xi_i),\\
4\alpha_i(\xi_i) &= -4 - 2h^{-1} - h^{-2}\|dh\|^2.
\end{aligned}
\tag{14.7}
$$

From the 3-Sasakian assumption the commutators are

$$[\bar{\xi}_i, \bar{\xi}_j] = -2\bar{\xi}_k.$$

Thus, for $X \in H$ taking also into account (14.1), we have

$$
\begin{aligned}
g([\bar{\xi}_1,\bar{\xi}_2],\, I_3X) &= -2\,g(\bar{\xi}_3,\, I_3X)\\
&= -2\,g(2\,h\,\xi_3 + I_3\,\nabla h,\, I_3\,X) = -2\,dh\,(X).
\end{aligned}
$$

Therefore, using again (14.1), we obtain

$$
\begin{aligned}
-2\,dh(X) &= g([\bar{\xi}_1,\,\bar{\xi}_2],\, I_3X) = g\big([2h\xi_1 + I_1\,\nabla h, 2h\xi_2 + I_2\,\nabla h],\, I_3X\big)\\
&= -4h^2\,A_3(X) + 2hg([\xi_1,\, I_2\nabla h],\, I_3\,X) + 2hg([I_1\nabla h, \xi_2],\, I_3\,X)\\
&\quad + g([I_1\,\nabla h,\, I_2\,\nabla h], I_3\,X).
\end{aligned}
\tag{14.8}
$$

The last three terms are transformed as follows. The first equals

$$
\begin{aligned}
g\,([\xi_1,\, I_2\nabla h], I_3\,X) &= g\,((\nabla_{\xi_1}\,I_2)\,\nabla h + I_2\,\nabla_{\xi_1}\,\nabla h,\, I_3X)\\
&\quad - g\,(T(\xi_1,\, I_2\nabla h),\, I_3X) = -\alpha_3(\xi_1)\,dh(I_2X) + \alpha_1(\xi_1)\,dh(X)\\
&\quad - \nabla dh\,(\xi_1, I_1X) - g\,(T(\xi_1,\, I_2\nabla h),\, I_3X),
\end{aligned}
$$

where we use (4.29) and the fact that ∇ preserves the splitting $H \oplus V$. The second term is

$$g\,(\,[I_1\nabla h, \xi_2], I_3\,X) \;=\; \alpha_2(\xi_2)\,dh(X) \;+\; \alpha_3(\xi_2)\,dh(I_1X) \;-\; \nabla dh\,(\xi_2, I_2X) \\ -\; g\,(\,T(I_1\nabla h,\,\xi_2),\,I_3X),$$

and finally

$$g\,(\,[I_1\nabla h, I_2\nabla h], I_3\,X) \;=\; -\alpha_3(I_1\nabla h)\,dh(I_2X) \;+\; \alpha_1(I_1\nabla h)\,dh(X) \\ -\; \nabla dh\,(I_1\nabla h, I_1X) \;+\; \alpha_2(I_2\nabla h)\,dh(X) \;+\; \alpha_3(I_2\nabla h)\,dh(I_1X) \\ -\; \nabla dh\,(I_2\nabla h, I_2X).$$

Next we apply (14.7) to the last three equalities, then substitute their sum into (14.8), after which we use the commutation relations

$$\begin{aligned} \nabla dh\,(X,Y) - \nabla dh\,(Y,X) \;&=\; -dh(T(X,Y)) = \;-2\sum_{s=1}^{3}\omega_s(X,Y)\,dh(\xi_s), \\ \nabla dh\,(X,\xi) - \nabla dh\,(\xi,X) \;&=\; -dh(T(X,\xi)), \qquad X,\,Y\in H, \quad \xi\in V. \end{aligned} \tag{14.9}$$

The result is the following identity

$$\begin{aligned} 4h^2A_3(X) \;=\; &(-4h \;+\; h^{-1}\|\nabla h\|^2)\,dh(X) \\ &-2h\,\big[\,\nabla dh\,(I_1X,\xi_1) \;+\; \nabla dh\,(I_2X,\xi_2)\big] \\ &-\;\big[\nabla dh\,(I_1X, I_1\nabla h)) \;+\; \nabla dh\,(I_2X, I_2\nabla h)\big] \\ &+\;2\,\big[\,dh(\xi_1)\,dh(I_1X) \;+\; dh(\xi_2)\,dh(I_2X) \\ &+\;2\,dh(\xi_3)\,dh(I_3X)\big] +\; 2h\,\big[\,T(\xi_1, I_1X, \nabla h) \;+\; T(\xi_2, I_2X, \nabla h) \\ &-\; T(\xi_1, I_2X, I_3\nabla h) \;+\; T(\xi_2, I_1X, I_3\nabla h)\big], \end{aligned} \tag{14.10}$$

where $T(\xi, X, Y) = g(T_\xi X, Y)$ for a vertical vector ξ and horizontal vectors X and Y. With the help of Proposition 2.5 we decompose the torsions into symmetric and anti-symmetric part

$$T_{\xi_i} = T^0_{\xi_i} + I_i U, \quad i = 1,\ 2,\ 3,$$

and then express the symmetric parts of the torsion terms in the form

$$T^0_{\xi_1} = (T^0_{\xi_1})^{--+} + (T^0_{\xi_1})^{-+-}, \quad T^0_{\xi_2} = (T^0_{\xi_2})^{--+} + (T^0_{\xi_2})^{+--}.$$

Hence, using

$$T^{0^{--+}} = 2(T^0_{\xi_2})^{+--} I_2 = 2(T^0_{\xi_1})^{-+-} I_1 \quad \text{etc.},$$

which follows again from Proposition 2.5, the sum of the torsion terms in (14.10) can be seen to equal

$$2T^{0^{--+}}(X,\, \nabla h) \ -\ 4\,U(X, \nabla h).$$

This allows us to rewrite (14.10) in the form

$$\begin{aligned}
(14.11) \quad 4A_3(X) \ = \ & (-4h^{-1} \ + \ h^{-3}\|\nabla h\|^2)\, dh(X) \\
& - \ 2\,h^{-1}\left[\nabla dh\,(I_1 X, \xi_1) \ + \ \nabla dh\,(I_2 X, \xi_2)\right] \\
& + \ 2\,h^{-2}\left[\,dh(\xi_1)\, dh(I_1 X) \ + \ dh(\xi_2)\, dh(I_2 X)\right. \\
& \left. + \ 2\,dh(\xi_3)\, dh(I_3 X)\right] - \ h^{-2}\left[\nabla dh\,(I_1 X, I_1 \nabla h)\right. \\
& \left. + \ \nabla dh\,(I_2 X, I_2 \nabla h)\right] \ + \ 4h^{-1}\left[\,(T^{0^{--+}}(\nabla h, X) \ - \ 2U(\nabla h, X)\right].
\end{aligned}$$

Using (14.2) the T^{0--+} component of the torsion can be expressed by h as follows, see (2.14) and (4.38),

$$
\begin{aligned}
4T^{0--+}(\nabla h, X) \;=\; & T^0(\nabla h, X) \\
& - \; T^0(I_1\nabla h, I_1X) \;-\; T^0(I_2\nabla h, I_2X) + \; T^0(I_3\nabla h, I_3X) \\
= \; & -h^{-1}\Big\{[\nabla dh]_{[-1]}(\nabla h, X) - [\nabla dh]_{[-1]}(I_1\nabla h, I_1X) \\
& - [\nabla dh]_{[-1]}(I_2\nabla h, I_2X) + [\nabla dh]_{[-1]}(I_3\nabla h, I_3X)\Big\} \\
& - h^{-1}\sum_{s=1}^{3}\Big\{dh(\xi_s)\Big[g(I_s\nabla h, X) \;-\; g(I_sI_1\nabla h, I_1X) \\
& - \; g(I_sI_2\nabla h, I_2X) \;+\; g(I_sI_3\nabla h, I_3X)\Big]\Big\} \\
= \; & -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;-\; \nabla dh\,(I_1\nabla h, I_1X) \\
& - \; \nabla dh\,(I_2\nabla h, I_2X) \;+\; \nabla dh\,(I_3\nabla h, I_3X)\Big\} + \; 4h^{-1}\,dh(\xi_3)\,dh(I_3X).
\end{aligned}
$$

Invoking equation (14.9) we can put ∇h in second place in the Hessian terms, thus, proving the formula

$$
\begin{aligned}
4T^{0--+}(\nabla h, X) \;=\; & -\;4h^{-1}\,dh(\xi_3)\,dh(I_3X) \\
& - h^{-1}\Big\{\nabla dh\,(X, \nabla h) \;-\; \nabla dh\,(I_1X, I_1\nabla h) \\
& - \; \nabla dh\,(I_2X, I_2\nabla h) \;+\; \nabla dh\,(I_3X, I_3\nabla h)\Big\}.
\end{aligned}
\tag{14.12}
$$

On the other hand, (4.39), (14.3) and the Yamabe equation (14.4) give

$$
\begin{aligned}
(14.13)\quad 8\,U(\nabla h, X) \;&=\; -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s\nabla h, I_s X)\\
&\qquad -2h^{-1}\|\nabla h\|^2\,dh(X) - \frac{\triangle h}{n}\,dh(X) + 2h^{-1}\frac{\|\nabla h\|^2}{n}\,dh(X)\Big\}\\
&= -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s\nabla h, I_s X)\Big\}\\
&\qquad -\; h^{-1}\Big\{-2h^{-1}\|\nabla h\|^2\,dh(X)\\
&\qquad - \frac{2n - 4nh + (n+2)h^{-1}\|\nabla h\|^2}{n}\,dh(X) \;+\; 2h^{-1}\frac{\|\nabla h\|^2}{n}\,dh(X)\Big\}\\
&= -h^{-1}\Big\{\nabla dh\,(X, \nabla h) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s X, I_s\nabla h)\Big\}\\
&\qquad -\; h^{-1}\big(-3h^{-1}\|\nabla h\|^2 \;-\; 2 \;+\; 4h\big)dh(X).
\end{aligned}
$$

Substituting the last two formulas in (14.11) gives A_3 in the form of (14.6) written for A_1, cf. the paragraph after (14.6). □

15. Divergence formulas

We shall need the divergences of various vector/forms through the almost complex structures, so we start with a general formula valid for any horizontal vector/form A. Let $\{e_1, \dots, e_{4n}\}$ be an orthonormal basis of H. The divergence of $I_1 A$ is

$$\nabla^*(I_1 A) \;\equiv\; (\nabla_{e_a}(I_1 A))(e_a) \;=\; -(\nabla_{e_a} A)(I_1 e_a) \;-\; A((\nabla_{e_a} I_1)e_a),$$

recalling $I_1 A(X) = -A(I_1 X)$.

It follows form Theorem 5.8 that if (M, H, g) is of constant QC scalar curvature then

$$(15.1)\qquad \nabla^* T^0 = (n+2)A, \qquad \nabla^* U = \frac{1-n}{2}A$$

We say that an orthonormal frame

$$\{e_1, e_2 = I_1 e_1, e_3 = I_2 e_1, e_4 = I_3 e_1, \dots, e_{4n} = I_3 e_{4n-3}, \xi_1, \xi_2, \xi_3\}$$

is a QC normal frame (at a point) if the connection 1-forms of the Biquard connection vanish (at that point). Lemma 5.5 asserts that a QC normal frame exists at each point of a QC manifold. With respect to a QC normal frame the above divergence reduces to

$$\nabla^*(I_1 A) \quad = \quad -(\nabla_{e_a} A)(I_1 e_a).$$

LEMMA 15.1. *Suppose (M, H, g) is a quaternionic contact manifold with constant QC scalar curvature. For any function h we have the following formulas, writen with respect to some admissible set (η_s, I_s, g) with Reeb vector filds ξ_s,*

$$\begin{aligned}
\nabla^* \Big(\sum_{s=1}^{3} dh(\xi_s) I_s A_s \Big) \quad &= \quad \sum_{s=1}^{3} \ \nabla dh\,(I_s e_a, \xi_s) A_s(e_a) \\
\nabla^* \Big(\sum_{s=1}^{3} dh(\xi_s) I_s A \Big) \quad &= \quad \sum_{s=1}^{3} \ \nabla dh\,(I_s e_a, \xi_s) A(e_a).
\end{aligned}$$

PROOF. Using the identification of the 3-dimensional vector spaces spanned by $\{\xi_1, \xi_2, \xi_3\}$ and $\{I_1, I_2, I_3\}$ with $\mathbb{R}^3$, the restriction of the action of $Sp(n)Sp(1)$ to this spaces can be identified with the action of the group $SO(3)$. With this in mind, one verifies easily that the 1-forms A,

$$\sum_{s=1}^{3} dh(\xi_s) I_s A_s = -\sum_{i=1}^{3} dh(\xi_i)[\xi_j, \xi_k]$$

and

$$\sum_{s=1}^{3} dh(\xi_s) I_s A$$

are $Sp(n)Sp(1)$ invariant on H. Thus, it is sufficient to compute their divergences in a QC normal frame. To avoid the introduction of new variables, in this proof, we shall assume that

$$\{e_1, \dots, e_{4n}, \xi_1, \xi_2, \xi_3\}$$

is a QC normal frame.

We apply (4.49). Using that the Biquard connection preserves the splitting of TM, we find

$$\begin{aligned}\nabla^*[\xi_1,\xi_2] &= -g(\nabla_{e_a}\big(T(\xi_1,\xi_2)\big), e_a)\\ &= -g(\,(\nabla_{e_a}T)\,(\xi_1,\xi_2), e_a) \;-\; g(\,T(\nabla_{e_a}\xi_1,\xi_2), e_a) \;-\; g(\,T(\xi_1,\nabla_{e_a}\xi_2), e_a).\end{aligned}$$

From Bianchi's identity we have ($\sigma_{A,B,C}$ means a cyclic sum over (A,B,C))

$$\begin{aligned}g((\nabla_{e_a}T)(\xi_1,\xi_2), e_a) &= -g((\nabla_{\xi_1}T)(\xi_2,e_a), e_a) \;-\; g((\nabla_{\xi_2}T)(e_a,\xi_1), e_a)\\ &\quad -\; g(\sigma_{e_a,\xi_1,\xi_2}\Big\{T(T(e_a,\xi_1),\xi_2)\Big\}, e_a) \;+ g(\sigma_{e_a,\xi_1,\xi_2}\Big\{R(e_a,\xi_1)\xi_2\Big\}, e_a)\\ &= -g(T(\,T(e_a,\xi_1),\xi_2), e_a) \;-\; g(T(\,T(\xi_1,\xi_2), e_a), e_a)\\ &\quad -\; g(T(T(\xi_2,e_a),\xi_1), e_a) = \; g(\,T(T(\xi_1,e_a),\xi_2), e_a)\\ &\quad -\; g(\,T(T(\xi_2,e_a),\xi_1), e_a) \;-\; g(\,T(T(\xi_1,\xi_2), e_a), e_a),\end{aligned}$$

taking into account that as mappings on H the torsion tensors $T(\xi_i, X)$ and the curvature tensor $R(\xi_1,\xi_2)$ are traceless, so

$$g((\nabla_{\xi_1}T)(\xi_2, e_a), e_a)$$

and

$$g(R(\xi_1,\xi_2)e_a, e_a) = 0,$$

while the connection preserves the splitting, to obtain the next to last line. The last term is equal to zero as

$$
\begin{aligned}
g(T(T(\xi_1,\xi_2),e_a),e_a) \;&=\; g\big(T(-\frac{\mathrm{Scal}}{8n(n+2)}\xi_3 \;-\; [\xi_1,\xi_2]_H, e_a), e_a\big)\\
&=\; -\frac{\mathrm{Scal}}{8n(n+2)}\, g(T(\xi_3,e_a),e_a) \;=0,
\end{aligned}
$$

taking into account that the torsion T_{ξ_3} is traceless and $T([\xi_1,\xi_2]_H,e_a)$ is a vertical vector. On the other hand,

$$
\begin{aligned}
&g\,(\,T(T_{\xi_1}e_a,\xi_2),e_a) \;-\; g\,(\,T(T_{\xi_2}e_a,\xi_1),e_a)\\
&\quad=\; -\;\Big[g(T(e_b,\xi_2),e_a)\,g(T(\xi_1,e_a),e_b) \;-\; g(T(e_b,\xi_1),e_a)\,g(T(\xi_2,e_a),e_b)\Big]\\
&\quad=\;\Big[g(T(\xi_2,e_b),e_a)\,g(T(\xi_1,e_a),e_b) \;-\; g(T(\xi_1,e_b),e_a)\,g(T(\xi_2,e_a),e_b)\Big] \;=\; 0.
\end{aligned}
$$

The equalities

$$
\nabla^*(I_1A_1)=\nabla^*(I_2A_2)=0
$$

with respect to a QC normal frame can be obtained similarly. Hence, the first formula in Lemma 15.1 follows.

We are left with proving the second divergence formula. Since the scalar curvature is constant, (5.2) implies

$$
A(X)=-2\sum_{s=1}^{3}\rho_s(X,\xi_s). \tag{15.2}
$$

Fix an $s\in\{1,2,3\}$. Working again in a QC normal frame we have

$$
(\nabla_{e_a}A)(I_se_a) \;=\; -2\sum_{t=1}^{3}(\nabla_{e_a}\rho_t)(I_se_a,\xi_t).
$$

A calculation involving the expressions (4.42) and the properties of the torsion shows that

$$\operatorname{tr}(\rho_t \circ I_s) = -\frac{1}{2(n+2)}\,\delta_{st}\,\mathrm{Scal}. \tag{15.3}$$

The second Bianchi identity

$$\begin{aligned} 0 = {} & g((\nabla_{e_a} R)(I_s e_a, \xi_t)e_b, I_t e_b) + g((\nabla_{I_s e_a} R)(\xi_t, e_a)e_b, I_t e_b) \\ & + g((\nabla_{\xi_t} R)(e_a, I_s e_a)e_b, I_t e_b) + g(R(T(e_a, I_s e_a), \xi_t)e_b, I_t e_b) \\ & + g(R(T(I_s e_a, \xi_t), e_a)e_b, I_t e_b) + g(R(T(\xi_t, e_a), I_s e_a)e_b, I_t e_b), \end{aligned}$$

together with the constancy of the QC scalar curvature and (15.3) show that the third term on the right is zero and thus

$$\sum_{t=1}^{3}\Big\{ 2\,(\nabla_{e_a}\rho_t)(I_s e_a, \xi_t) - 2\,\rho_t(T(\xi_t, I_s e_a), e_a) + \rho_t(T(e_a, I_s e_a), \xi_t)\Big\} = 0.$$

Substituting (4.4) in the above equality we come to the equation

$$\sum_{t=1}^{3}(\nabla_{e_a}\rho_t)(I_s e_a, \xi_t) = \sum_{t=1}^{3}\rho_t(T(\xi_t, I_s e_a), e_a) - 4n\sum_{t=1}^{3}\rho_t(\xi_s, \xi_t) = 0, \tag{15.4}$$

where the vanishing of the second term follows from (5.6), while the vanishing of the first term is seen as follows. Using the standard inner product on $End(H)$

$$g(C, B) = tr(B^* C) = \sum_{a=1}^{4n} g(C(e_a), B(e_a)),$$

where $C, B \in End(H)$, $\{e_1, ..., e_{4n}\}$ is a g-orthonormal basis of H, the definition of $T^0_{\xi_s}$, the formulas in Lemma 4.12 and Proposition 2.5 imply

$$
\begin{aligned}
&\sum_{s=1}^{3} \rho_s(T(\xi_s, I_1 e_a), e_a) \\
&\quad = g(\rho_1, T^0_{\xi_1} I_1) + g(\rho_2, T^0_{\xi_2} I_1) + g(\rho_3, T^0_{\xi_3} I_1) - g(\rho_1, u) - g(\rho_2, I_3 u) \\
&\qquad + g(\rho_3, I_2 u) = g(\rho_1, T^0_{\xi_1} I_1) + g(\rho_2, T^0_{\xi_2} I_1) + g(\rho_3, T^0_{\xi_3} I_1) \\
&\qquad = g(2(T^0_{\xi_2})^{--+} I_3 - 2I_1 u - \frac{Scal}{8n(n+2)} I_1, T^0_{\xi_1} I_1) \\
&\qquad + g(2(T^0_{\xi_3})^{+--} I_1 - 2I_2 u - \frac{Scal}{8n(n+2)} I_2, T^0_{\xi_2} I_1) \\
&\qquad + g(2(T^0_{\xi_1})^{-+-} I_2 - 2I_3 u - \frac{Scal}{8n(n+2)} I_3, T^0_{\xi_3} I_1) \\
&\quad = -2g((T^0_{\xi_2})^{--+} I_2, T^0_{\xi_1}) + 2g((T^0_{\xi_3})^{+--}, T^0_{\xi_2}) + 2g((T^0_{\xi_1})^{-+-} I_3, T^0_{\xi_3}) \\
&\qquad = 2g((T^0_{\xi_3})^{+--}, (T^0_{\xi_2})^{+--}) + 2g((T^0_{\xi_1})^{-+-}, I_3 (T^0_{\xi_3})^{+--}) \\
&\qquad = 2g(I_2 (T^0_{\xi_3})^{+--}, I_2 (T^0_{\xi_2})^{+--}) - 2g(I_1 (T^0_{\xi_1})^{-+-}, I_2 (T^0_{\xi_3})^{+--}) = 0.
\end{aligned}
$$

Renaming the almost complex structures shows that the same conclusion is true when we replace I_1 with I_2 or I_3 in the above calculation.

Finally, the second formula in Lemma 15.1 follows from (15.2) and (15.4). $\square$

We shall also need the following one-forms

$$
\begin{aligned}
D_1(X) &= -h^{-1} T^{0^{+--}}(X, \nabla h) \\
D_2(X) &= -h^{-1} T^{0^{-+-}}(X, \nabla h) \\
D_3(X) &= -h^{-1} T^{0^{--+}}(X, \nabla h)
\end{aligned}
\tag{15.5}
$$

For simplicity, using the musical isomorphism, we will denote with D_1, D_2, D_3 the corresponding (horizontal) vector fields, for example

$$g(D_1, X) = D_1(X) \qquad \forall X \in H.$$

Finally, we set

$$D = D_1 + D_2 + D_3 = -h^{-1}T^0(X, \nabla h). \tag{15.6}$$

LEMMA 15.2. *Suppose (M, η) is a quaternionic-contact manifold with constant QC scalar curvature $Scal = 16n(n+2)$. Suppose*

$$\bar{\eta} = \frac{1}{2h}\eta$$

has vanishing $[-1]$-torsion component $\overline{T}^0 = 0$. We have

$$D(X) = \frac{1}{4}h^{-2}\Big(3\,\nabla dh(X, \nabla h) - \sum_{s=1}^{3} \nabla dh(I_sX, I_s\nabla h)\Big) + h^{-2}\sum_{s=1}^{3} dh(\xi_s)\,dh(I_sX).$$

and the divergence of D satisfies

$$\nabla^* D = |T^0|^2 - h^{-1}g(dh, D) - h^{-1}(n+2)\,g(dh, A).$$

PROOF. a) The formula for D follows immediately from (14.2).

b) We work in a QC normal frame. Since the scalar curvature is assumed to be constant we use (15.1) to find

$$\begin{aligned}
\nabla^* D &= -h^{-1}\,dh(e_a)D(e_a) - h^{-1}\nabla^* T^0(\nabla h) - h^{-1}T^0(e_a, e_b)\,\nabla dh(e_a, e_b) \\
&= -h^{-1}\,dh(e_a)D(e_a) - h^{-1}(n+2)\,dh(e_a)A(e_a) - g(T^0,\, h^{-1}\,\nabla dh) \\
&= |T^0|^2 - h^{-1}dh(e_a)D(e_a) - h^{-1}(n+2)\,dh(e_a)A(e_a),
\end{aligned}$$

using (14.2) in the last equality. □

Let us also consider the following one-forms (and corresponding vectors)

$$F_s(X) \;=\; -h^{-1}\, T^0(X, I_s\nabla h), \quad X\in H \quad s=1,2,3.$$

From the definition of F_1 and (15.5) we find

$$\begin{aligned}
F_1(X) \;&=\; -h^{-1}T^0(X, I_1\nabla h)\\
&= -h^{-1}T^{0+--}(X, I_1\nabla h) \;-\; h^{-1}T^{0-+-}(X, I_1\nabla h) \;-h^{-1}T^{0--+}(X, I_1\nabla h)\\
&= h^{-1}T^{0+--}(I_1X, \nabla h) \;-\; h^{-1}T^{0-+-}(I_1X, \nabla h) \;-h^{-1}T^{0--+}(I_1X, \nabla h)\\
&= -D_1(I_1X) \;+\; D_2(I_1X) \;+\; D_3(I_1X).
\end{aligned}$$

Thus, the forms F_s can be expressed by the forms D_s as follows

$$\begin{aligned}
F_1(X) \;&=\; -D_1(I_1X) \;+\; D_2(I_1X) \;+\; D_3(I_1X)\\
F_2(X) \;&=\; D_1(I_2X) \;-\; D_2(I_2X) \;+\; D_3(I_2X)\\
F_3(X) \;&=\; D_1(I_3X) \;+\; D_2(I_3X) \;-\; D_3(I_3X).
\end{aligned} \tag{15.7}$$

LEMMA 15.3. *Suppose (M,η) is a quaternionic-contact manifold with constant QC scalar curvature $Scal = 16n(n+2)$. Suppose*

$$\bar\eta = \frac{1}{2h}\eta$$

has vanishing $[-1]$-torsion component, $\overline{T}^0 = 0$. We have

$$\begin{aligned}
\nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)F_s\Big) \;&=\; \sum_{s=1}^3\Big[\;\nabla dh\,(I_se_a,\xi_s)F_s(I_se_a)\Big]\\
&\quad+\; h^{-1}\sum_{s=1}^3\Big[dh(\xi_s)dh(I_se_a)D(e_a) \;+(n+2)\,dh(\xi_s)dh(I_se_a)\,A(e_a)\Big].
\end{aligned}$$

PROOF. We note that the vector

$$\sum_{s=1}^{3} dh(\xi_s)F_s$$

is an $Sp(n)Sp(1)$ invariant, hence, we may assume that

$$\{e_1, \dots, e_{4n}, \xi_1, \xi_2, \xi_3\}$$

is a QC normal frame. Since the scalar curvature is assumed to be constant we can apply (15.1), thus

$$\nabla^* T^0 \;=\; (n+2)A.$$

Turning to the divergence, we compute

$$\begin{aligned}
(15.8)\quad \nabla^*\Big(\sum_{s=1}^{3} dh(\xi_s)F_s\Big) \;&=\; \sum_{s=1}^{3}\Big[\,\nabla dh\,(e_a,\xi_s)F_s(e_a)\Big] \\
&- \sum_{s=1}^{3} h^{-1}\,dh(\xi_s)\,\nabla^* T^0(I_s\nabla h) + \sum_{s=1}^{3}\Big[\,h^{-2}\,dh(\xi_s)\,dh(e_a)T^0(e_a, I_s e_b)\,dh(e_b) \\
&\qquad - \; h^{-1}\,dh(\xi_s)\,T^0(e_a, I_s e_b)\;\nabla dh\,(e_a, e_b)\Big] \\
&= \sum_{s=1}^{3}\Big[\,\nabla dh\,(e_a,\xi_s)F_s(e_a)\Big] \;-\; \sum_{s=1}^{3} h^{-1}\,dh(\xi_s)\,\nabla^* T^0(I_s\nabla h) \\
&+ \sum_{s=1}^{3}\Big[\,h^{-1}\,dh(\xi_s)\,dh(I_s e_a)\,D(e_a)\Big] = \sum_{s=1}^{3}\Big[\,\nabla dh\,(e_a,\xi_s)F_s(e_a) \\
&\quad + \; h^{-1}\,dh(\xi_s)\,dh(I_s e_a)\,D(e_a) \;+\; h^{-1}(n+2)\,dh(\xi_s)\,dh(I_s e_a)\,A(e_a)\Big],
\end{aligned}$$

using the symmetry of T^0 in the next to last equality, and the fact

$$T^0(e_a, I_1 e_b)\ \nabla dh\,(e_a, e_b)$$

$$= -\ h^{-1}\nabla dh_{[sym][-1]}(e_a, I_1 e_b)\Big[\nabla dh_{[sym]}(e_a, e_b) - \sum_{s=1}^{3} dh(\xi_s)\omega_s(e_a, e_b)\Big] = 0,$$

which is a consequence of (14.2), the formula for the symmetric part of ∇dh given after (14.3) and the zero traces of the [-1]-component. Switching to the basis

$$\{I_s e_a : a = 1, \dots, 4n\}$$

in the first term of the right-hand-side of (15.8) completes the proof. □

At this point we restrict our considerations to the 7-dimensional case, i.e., for $n = 1$. Following is our main technical result. As mentioned in the introduction, we were motivated to seek a divergence formula of this type based on the Riemannian and CR cases of the considered problem. The main difficulty was to find a suitable vector field with non-negative divergence containing the norm of the torsion. The fulfilment of this task was facilitated by the results of Chapter 2, which in particular showed that similarly to the CR case, but unlike the Riemannian case, we were not able to achieve a proof based purely on the Bianchi identities.

THEOREM 15.4. *Suppose (M^7, η) is a quaternionic-contact structure conformal to a 3-Sasakian structure $\tilde\eta$, $\tilde\eta = \frac{1}{2h}\,\eta$. If*

$$Scal_\eta = Scal_{\tilde\eta} = 16n(n+2),$$

then with f given by

$$f\ =\ \frac{1}{2}\ +\ h\ +\ \frac{1}{4}h^{-1}|\nabla h|^2,$$

the following identity holds

$$\nabla^*\Big(fD \;+\; \sum_{s=1}^{3} dh(\xi_s)\,F_s \;+\; 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s \;-\; \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big)$$
$$= \; f|T^0|^2 \;+\; h\,\langle \mathcal{Q}V,\,V\rangle.$$

Here, $\mathcal{Q}$ *is a positive semi-definite matrix and*

$$V = (D_1, D_2, D_3, A_1, A_2, A_3)$$

with A_s*,* D_s *defined, correspondingly, in* (14.5) *and* (15.5)*.*

Proof. Using the formulas for the divergences of D,

$$\sum_{s=1}^{3} dh(\xi_s)\,F_s, \quad \sum_{s=1}^{3} dh(\xi_s) I_s A_s \quad \text{and} \quad \sum_{s=1}^{3} dh(\xi_s)\,I_s A$$

given correspondingly in Lemmas 15.2, 15.3 and 15.1 we have the identity ($n = 1$ here)

$$\nabla^*\Big(\,fD \;+\; \sum_{s=1}^{3} dh(\xi_s)\,F_s + \;4\sum_{s=1}^{3} dh(\xi_s) I_s A_s \;-\; \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big) \tag{15.9}$$
$$= \Big(dh(e_a) \;-\; \frac{1}{4}h^{-2} dh(e_a)|\nabla h|^2 \;+\; \frac{1}{2}h^{-1}\;\nabla dh(e_a,\nabla h)\Big)\,D(e_a)$$
$$+\; f\,\Big(|T^0|^2 \;-h^{-1}dh(e_a)D(e_a) \;-\; h^{-1}(n+2)\,dh(e_a)A(e_a)\Big)$$
$$+\;\sum_{s=1}^{3}\;\nabla dh(I_s e_a,\xi_s)F_s(I_s e_a) \;+\; h^{-1}\sum_{s=1}^{3}\Big[\,dh(\xi_s)\,dh(I_s e_a)\,D(e_a)$$
$$+\;(n+2)\,dh(\xi_s)\,dh(I_s e_a)\,A(e_a)\Big]$$

$$+\ 4\sum_{s=1}^{3}\nabla dh(I_se_a,\xi_s)\,A_s(e_a)\ -\frac{10}{3}\ \sum_{s=1}^{3}\ \nabla dh(I_se_a,\xi_s)\,A(e_a)$$

$$= \Big(dh(e_a)\ -\ \frac{1}{4}h^{-2}dh(e_a)|\nabla h|^2\ +\ \frac{1}{2}h^{-1}\ \nabla dh(e_a,\nabla h)\Big)\sum_{t=1}^{3}D_t(e_a)$$

$$+\ f\left(|T^0|^2\ -h^{-1}dh(e_a)\right)\big(\sum_{t=1}^{3}D_t(e_a)\big)\ -\ fh^{-1}(n+2)\,dh(e_a)\big(\sum_{t=1}^{3}A_t(e_a)\big)$$

$$+\ \nabla dh(I_1e_a,\xi_1)\,(D_1(e_a)\ -\ D_2(e_a)\ -\ D_3(e_a))$$

$$+\ \nabla dh(I_2e_a,\xi_2)\,(-D_1(e_a)\ +D_2(e_a)\ -\ D_3(e_a))$$

$$+\ \nabla dh(I_3e_a,\xi_3)\,(-D_1(e_a)\ -\ D_2(e_a)\ +D_3(e_a))$$

$$+\ h^{-1}\big(\sum_{s=1}^{3}dh(\xi_s)\,dh(I_se_a)\big)\,\big(\sum_{t=1}^{3}D_t(e_a)\big)$$

$$+\ h^{-1}(n+2)\big(\sum_{s=1}^{3}\ dh(\xi_s)\,dh(I_se_a)\big)\,\big(\sum_{t=1}^{3}A_t(e_a)\big)$$

$$+\ 4\sum_{s=1}^{3}\nabla dh(I_se_a,\xi_s)\,A_s(e_a)\ -\frac{10}{3}\ \big(\sum_{s=1}^{3}\ \nabla dh(I_se_a,\xi_s)\big)\,\big(\sum_{t=1}^{3}A_t(e_a)\big),$$

where the last equality uses (15.7) to express the vectors F_s by D_s, and the expansions of the vectors A and D according to (14.5) and (15.6). Since the dimension of M is seven it follows

$$U\ =\ \bar{U}\ =\ [\ \nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}\ =\ 0.$$

This, together with the Yamabe equation (14.4) , which when $n=1$ becomes

$$\triangle h = 2 - 4h + 3h^{-1}|\nabla h|^2,$$

yield the formula, cf. (14.13),

$$\nabla dh\,(X,\nabla h)\ +\ \sum_{s=1}^{3}\nabla dh\,(I_sX,I_s\nabla h) - \big(\,2\ -\ 4h\ +3h^{-1}|\nabla h|^2\big)\,dh(X)\ =\ 0. \tag{15.10}$$

From equations (15.5) and (14.12) we have

$$\begin{aligned}
D_1(X)\ &=\ h^{-2}\,dh(\xi_1)\,dh(I_1X)\ +\ \frac{1}{4}h^{-2}\big[\ \nabla dh\,(X,\nabla h)+\ \nabla dh\,(I_1X,I_1\nabla h)\\
&\qquad -\ \nabla dh\,(I_2X,I_2\nabla h)\ -\ \nabla dh\,(I_3X,I_3\nabla h)\big],\\
D_2(X)\ &=\ h^{-2}\,dh(\xi_2)\,dh(I_2X)\ +\ \frac{1}{4}h^{-2}\big[\nabla dh\,(X,\nabla h)\ -\ \nabla dh\,(I_1X,I_1\nabla h)\\
&\qquad +\ \nabla dh\,(I_2X,I_2\nabla h)\ -\ \nabla dh\,(I_3X,I_3\nabla h)\big],\\
D_3(X)\ &=\ h^{-2}\,dh(\xi_3)\,dh(I_3X)\ +\ \frac{1}{4}h^{-2}\big[\nabla dh(X,\nabla h)-\ \nabla dh\,(I_1X,I_1\nabla h)\\
&\qquad -\ \nabla dh\,(I_2X,I_2\nabla h)\ +\ \nabla dh\,(I_3X,I_3\nabla h)\big].
\end{aligned}$$

Expressing the first term in (15.10) by the rest and substituting with the result in the above equations we come to

$$\begin{aligned}
D_i(e_a)\ &=\frac{1}{4}h^{-2}\big(2\ -\ 4h\ +3h^{-1}|\nabla h|^2\big)\,dh(e_a)+h^{-2}\,dh(\xi_i)\,dh(I_ie_a)+\\
&\qquad \frac{1}{2}h^{-2}\left[-\ \nabla dh\,(I_je_a,I_j\nabla h)\ -\nabla dh\,(I_ke_a,I_k\nabla h)\right].
\end{aligned} \tag{15.11}$$

At this point, by a purely algebraic calculation, using Lemma 14.2 and (15.11) we find:

$$\frac{22}{3}A_1 - \frac{2}{3}A_2 - \frac{2}{3}A_3 + \frac{11}{3}D_1 - \frac{1}{3}D_2 - \frac{1}{3}D_3$$
$$= -3h^{-1}\left(1+\frac{1}{2}h^{-1}dh(e_a)+\frac{1}{4}h^{-2}|\nabla h|^2\right)dh(e_a) + 3h^{-2}\left(\sum_{s=1}^{3} dh(\xi_s)\,dh(I_s e_a)\right)$$
$$+\frac{2}{3}h^{-1}\nabla dh(I_1e_a,\xi_1) - \frac{10}{3}h^{-1}\nabla dh(I_2e_a,\xi_2) - \frac{10}{3}h^{-1}\nabla dh(I_3e_a,\xi_3).$$

Similarly,

$$3\,A_1 - A_2 - A_3 + 2D_1$$
$$= \left(-2h^{-1}+\frac{1}{2}h^{-2}+h^{-3}|\nabla h|^2\right)dh(e_a) - \frac{1}{2}h^{-2}\sum_{s=1}^{3}\nabla dh(I_se_a, I_s\nabla h)$$
$$+ h^{-1}\,\nabla dh(I_1e_a,\xi_1) - h^{-1}\,\nabla dh(I_2e_a,\xi_2) - h^{-1}\,\nabla dh(I_3e_a,\xi_3)$$
$$+ h^{-2}\sum_{s=1}^{3} dh(\xi_s)\,dh(I_se_a).$$

On the other hand, the coefficient of $A_1(e_a)$ in (15.9) is found to be, after setting $n=1$,

$$h\Big[-3\left(1+\frac{1}{2}h^{-1}+\frac{1}{4}h^{-2}|\nabla h|^2\right)h^{-1}\,dh(e_a) + 3h^{-2}\left(\sum_{s=1}^{3} dh(\xi_s)\,dh(I_se_a)\right)$$
$$+\frac{2}{3}h^{-1}\nabla dh(I_1e_a,\xi_1) - \frac{10}{3}h^{-1}\nabla dh(I_2e_a,\xi_2) - \frac{10}{3}h^{-1}\nabla dh(I_3e_a,\xi_3)\Big],$$

while the coefficient of $D_1(e_a)$ in (15.9) is

$$dh(e_a) - \frac{1}{4}h^{-2}dh(e_a)|\nabla h|^2 + \frac{1}{2}h^{-1}\nabla dh(e_a,\nabla h) - f\,h^{-1}dh(e_a) + \nabla dh(I_1e_a,\xi_1) - \nabla dh(I_2e_a,\xi_2) - \nabla dh(I_3e_a,\xi_3)D_1(e_a) + h^{-1}\big(\sum_{s=1}^{3} dh(\xi_s)\,dh(I_se_a)\big). \tag{15.12}$$

Substituting $\nabla dh(e_a,\nabla h)$ according to (15.10), i.e.,

$$\nabla dh\,(e_a,\nabla h) = -\sum_{s=1}^{3}\nabla dh\,(I_se_a,I_s\nabla h) + \big(2 - 4h + 3h^{-1}|\nabla h|^2\big)\,dh(e_a)$$

and using the definition of f transforms the above expression into

$$dh(e_a) - \frac{1}{4}h^{-2}dh(e_a)|\nabla h|^2 - \left(\frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2\right)h^{-1}dh(e_a) + \frac{1}{2}h^{-1}\left(-\sum_{s=1}^{3}\nabla dh\,(I_se_a,I_s\nabla h) + \big(2 - 4h + 3h^{-1}|\nabla h|^2\big)\,dh(e_a)\right) + \nabla dh(I_1e_a,\xi_1) - \nabla dh(I_2e_a,\xi_2) - \nabla dh(I_3e_a,\xi_3)D_1(e_a) + h^{-1}\left(\sum_{s=1}^{3} dh(\xi_s)\,dh(I_se_a)\right).$$

Simplifying the above expression shows that the coefficient of $D_1(e_a)$ in (15.9) is

$$\Big(-2 + \frac{1}{2}h^{-1} + h^{-2}|\nabla h|^2\Big)\, dh(e_a) - \frac{1}{2}h^{-1}\left(\sum_{s=1}^{3} \nabla dh\,(I_s e_a, I_s \nabla h)\right)$$
$$+ \nabla dh(I_1 e_a, \xi_1) - \nabla dh(I_2 e_a, \xi_2) - \nabla dh(I_3 e_a, \xi_3)$$
$$+ h^{-1}\left(\sum_{s=1}^{3} dh(\xi_s)\, dh(I_s e_a)\right)$$

Hence, we proved that the coefficient of $D_1(e_a)$ in (15.9) is

$$h\,(3\,A_1 - A_2 - A_3 + 2D_1)\,(e_a),$$

while those of $A_1(e_a)$ is

$$h\Big(\frac{22}{3}\,A_1 - \frac{2}{3}\,A_2 - \frac{2}{3}\,A_3 + \frac{11}{3}\,D_1 - \frac{1}{3}\,D_2 - \frac{1}{3}\,D_3\Big)(e_a).$$

A cyclic permutation gives the rest of the coefficients in (15.9). With this, the divergence (15.9) can be written in the form

$$\nabla^*\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\,F_s + 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big)$$
$$= f|T^0|^2 + h\,\sigma_{1,2,3}\Big\{\, g\,\big(D_1,\, 3A_1 - A_2 - A_3 + 2\,D_1\big)$$
$$+ g\,\big(A_1,\, \frac{22}{3}\,A_1 - \frac{2}{3}\,A_2 - \frac{2}{3}\,A_3 + \frac{11}{3}\,D_1 - \frac{1}{3}\,D_2 - \frac{1}{3}\,D_3\big)\Big\},$$

where $\sigma_{1,2,3}$ denotes the sum over all positive permutations of $(1,2,3)$. Let $\mathcal{Q}$ be equal to

$$\mathcal{Q} := \begin{bmatrix} 2 & 0 & 0 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 2 & 0 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ 0 & 0 & 2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} \\ \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}$$

so that

$$\nabla^*\Big(fD \;+\; \sum_{s=1}^{3} dh(\xi_s)\, F_s \;+\; 4dh(\xi_s)\, I_s A_s \;-\; \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\, I_s A\Big)$$
$$= \; f\,|T^0|^2 \;+\; h\,\langle \mathcal{Q}V,\, V\rangle,$$

with $V = (D_1, D_2, D_3, A_1, A_2, A_3)$. It is not hard to see that the eigenvalues of $\mathcal{Q}$ are given by

$$\{0, \quad 0, \quad 2\,(2+\sqrt{2}), \quad 2\,(2-\sqrt{2}), \quad 10, \quad 10\},$$

which shows that $\mathcal{Q}$ is a non-negative matrix. $\square$

16. Proof of the main theorems

Here we prove the two theorems E and F that have been formulated at the beginning of this chapter. The proof relies on Theorem 15.4 and Theorem A (Chapter 2).

16.1. Proof of Theorem E. Integrating the divergence formula of Theorem 15.4 we see that according to the divergence theorem established in Proposition 9.1 (Chapter 2) the integral of the left-hand side is zero. Thus, the right-hand side vanishes as well, which shows that the quaternionic-contact structure η has vanishing torsion, i.e., it is also QC Einstein according to Theorem 4.13.

Next we bring into consideration the 7-dimensional quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H}) \cong \Sigma$ and the quaternionic Cayley transform, $\mathcal{C}$, as described in Section 3. The Cayley transform is a conformal quaternionic-contact diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic-contact structure $\tilde{\Theta}$ and $S^7 \setminus \{(-1,0)\}$ with its standard structure,

$$\lambda \cdot (\mathcal{C}_* \,\tilde{\eta}) \cdot \bar{\lambda} \;=\; \frac{8}{|1+p_1\,|^2}\,\tilde{\Theta}, \tag{16.1}$$

where

$$\lambda \;=\; \frac{1+p_1}{|1+p_1\,|}$$

is a unit quaternion and $\tilde{\eta}$ is the standard quaternionic-contact form on the sphere,

$$\tilde{\eta} \;=\; dq\cdot\bar{q} \;+\; dp\cdot\bar{p} \;-\; q\cdot d\bar{q} - \; p\cdot d\bar{p}.$$

Hence, up to a constant multiplicative factor and a quaternionic-contact automorphism the forms $\mathcal{C}_*\tilde{\eta}$ and $\tilde{\Theta}$ are conformal to each other. It follows that the same is true for $\mathcal{C}_*\eta$ and $\tilde{\Theta}$. In addition, $\tilde{\Theta}$ is QC Einstein by definition, while η and hence also $\mathcal{C}_*\eta$ are QC Einstein as we observed at the beginning of the proof. According to Theorem A (Chapter 2), up to a multiplicative constant factor, the forms $\mathcal{C}_*\tilde{\eta}$ and $\mathcal{C}_*\eta$ are related by a translation or dilation on the Heisenberg group. Hence, we conclude that up to a multiplicative constant, η is obtained from $\tilde{\eta}$ by a conformal quaternionic-contact automorpism which proves the first claim of Theorem E. From the conformal properties of the Cayley transform and [**Va1**, **Va2**] it follows that the minimum of the Yamabe functional (cf. also Section 3.4) is achieved by a smooth 3-contact form, which due to the Yamabe equation is of constant QC scalar curvature. This shows the second claim of Theorem E.

16.2. Proof of Theorem F. Let $\mathcal{D}^{1,2}$ be the space of functions $u \in L^{2^*}(\boldsymbol{G}\,(\mathbb{H}))$ having distributional horizontal gradient

$$|\nabla u|^2 = |T_1 u|^2 + |X_1 u|^2 + |Y_1 u|^2 + |Z_1 u|^2 \in L^2(\boldsymbol{G}\,(\mathbb{H}))$$

with respect to the Lebesgue measure dH on $\mathbb{R}^7$, which is the Haar measure on the group. Let us define the constant ($2^* = 5/2$ here)

$$\Lambda \overset{def}{=} inf \left\{ \int_{\boldsymbol{G}\,(\mathbb{H})} |\nabla v|^2 \, dH \ : \ v \in \mathcal{D}^{1,2}, \ v \geq 0, \ \int_{\boldsymbol{G}\,(\mathbb{H})} |v|^{2^*} \, dH \ = \ 1 \right\}.$$

Let v be a function for which the infimum is achieved. Note that such function exists by [Va1] or [Va2]. Furthermore, $\Lambda = S_2^{-2}$, where S_2 is the best constant in the L^2 Folland-Stein inequality (3.3), since $v \in \mathcal{D}^{1,2}$ implies $|v| \in \mathcal{D}^{1,2}$ and the gradient is the same a.e.. From the choice of v we have

$$\Lambda \ = \int_{\boldsymbol{G}\,(\mathbb{H})} |\nabla v|^2 \, dH, \qquad \int_{\boldsymbol{G}\,(\mathbb{H})} v^{2^*} \, dH \ = \ 1.$$

Writing the Euler-Lagrange equation of the constrained problem we see that v is a non-negative entire solution of

$$\left(T_1^2 \ + \ X_1^2 \ + \ Y_1^2 \ + \ Z_1^2\right) v = -\Lambda \ v^{3/2}.$$

By [GV2, Lemma 10.2] (see [Va2] or [Va1, Theorem 10.3] for further details) v is a bounded function. Similarly to [FS74, Theorem 16.7] it follows v is a Lipschits continuous function in the sense of non-isotropic Lipschits spaces [F]. Iterating this argument and using [F, Theorem 5.25] we see that v is a C^∞ smooth function on the set where it is positive, while being of class $\Gamma^{2,\beta}_{loc}$, the non-isotropic Lipschits space, for some $\beta > 0$. In particular v is continuously differentiable function by [F, Theorem 5.25]. Applying the Hopf lemma [GV1, Theorem 2.13] on the set where v is positive shows that v cannot vanish, i.e., it is a positive entire solution to the Yamabe equation. The positivity can also be seen by the Harnack inequality, see [W] for example. Let

$$u \overset{def}{=} \Lambda^{\frac{1}{2^*-2}} \ v,$$

then u is a positive entire solution of the Yamabe equation

$$\left(T_1^2 \ + \ X_1^2 \ + \ Y_1^2 \ + \ Z_1^2\right) u \ = \ -u^{3/2} \tag{16.2}$$

From the definition of u, we have

$$\Lambda \;=\; \Big(\int_{\boldsymbol{G}\,(\mathbb{H})} |\nabla u|^2 \, dH \Big)^{\frac{1}{5}} \;=\; \Big(\int_{\boldsymbol{G}\,(\mathbb{H})} u^{5/2} \, dH \Big)^{\frac{1}{5}}.$$

We shall compute the last integral by determining u with the help of the divergence formula.

As before, let $\tilde\Theta$ be the standard contact form on $\boldsymbol{G}\,(\mathbb{H})$ identified with Σ. Using the inversion and the Kelvin transform on $\boldsymbol{G}\,(\mathbb{H})$, cf. [GV2, Sections 8 and 9], we can see that if $\Theta = \frac{1}{2h}\tilde\Theta$ has constant scalar curvature, then the Cayley transform lifts the QC structure defined by Θ to a QC structure of constant QC scalar curvature on the sphere, which is conformal to the standard. The details are as follows. Let us define two contact forms Θ_1 and Θ_2 on Σ setting

$$\Theta_1 \;=\; u^{4/(n_h-2)}\tilde\Theta, \quad \text{and} \quad \Theta_2 \;=\; (\mathcal{K}u)^{4/(n_h-2)} \, \frac{\bar p'}{|p'|} \, \tilde\Theta \, \frac{p'}{|p'|},$$

where u is as in (16.2), $\mathcal{K}u$ is its Kelvin transform, see (16.5) for the exact formula, and n_h is the homogeneous dimension of the group (for the seven dimensional Heseinber group, we have n=1 and $n_h = 10$). Notice that $\frac{\bar p'}{|p'|} \, \tilde\Theta \, \frac{p'}{|p'|}$ defines the same QC structure on the Heisenberg group as $\tilde\Theta$ and $\mathcal{K}u$ is a smooth function on the whole group according to [GV2, Theorem 9.2]. We are going to see that using the Cayley transform these two contact forms define a contact form on the sphere, which is conformal to the standard and has constant QC scalar curvature.

Let $P_1 = (-1, 0)$ and $P_2 = (1, 0)$ be correspondingly the 'south' and 'north' poles of the unit sphere

$$S \;=\; \{|q|^2 + |p|^2 = 1\}.$$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the corresponding Cayley transforms defined, respectively, on $S \setminus \{P_1\}$ and $S \setminus \{P_2\}$. Note that $\mathcal{C}_1$ was defined in Section 3.3, while $\mathcal{C}_2$ is given by

$$(16.3) \quad (q_2, p_2) \;=\; \mathcal{C}_2 \Big((q, p) \Big), \quad q_2 \;=\; -(1-p)^{-1} \, q, \quad p_2 \;=\; (1-p)^{-1} \, (1+p).$$

In order that Θ_1 and Θ_2 define a contact form η on the sphere it is enough to see that

$$\Theta_1(p,q) \;=\; \Theta_2 \circ \mathcal{C}_2 \circ \mathcal{C}_1^{-1}(p,q), \quad \text{i.e.,} \quad \Theta_1 \;=\; (\mathcal{C}_2 \circ \mathcal{C}_1^{-1})^* \Theta_2. \tag{16.4}$$

A calculation shows that $\mathcal{C}_2 \circ \mathcal{C}_1^{-1} : \Sigma \to \Sigma$ is given by

$$q_2 \;=\; -p_1^{-1}\, q_1, \qquad p_2 \;=\; p_1^{-1},$$

or, equivalently, in the model $\boldsymbol{G}\,(\mathbb{H})$

$$q_2 \;=\; -(|q_1|^2 - \omega_1)^{-1}\, q_1, \qquad \omega_2 \;=\; -\frac{\omega_1}{|q_1|^4 + |\omega_1|^2}.$$

Hence, $\sigma \;=\; \mathcal{C}_2 \circ \mathcal{C}_1^{-1}$ is an involution on the group. Furthermore, with the help of (16.1) we calculate

$$\mathcal{C}_{1*} \circ \mathcal{C}_2{}^* \,\Theta \;=\; \frac{1}{|p_1|^2}\, \bar{\mu}\, \Theta\, \mu, \qquad \mu \;=\; \frac{p_1}{|p_1|},$$

which proves the identity (16.4). Using the properties of the Kelvin transform, [**GV2**, Sections 8 and 9],

$$(\mathcal{K}u)\,(q', p') \;\overset{def}{=}\; |p'^{-(Q-2)/2}\, u\big(\sigma(q', p')\big), \tag{16.5}$$

we see that u and $\mathcal{K}u$ are solutions of the Yamabe equation (16.2). This implies that the contact form η has constant QC scalar curvature, equal to $\frac{4(n_h+2)}{n_h-2}$.

Notice that η is conformal to the standard form $\tilde{\eta}$ and the arguments in the preceding proof imply then that η is QC Einstein. A small calculation shows that this is equivalent to the fact that if we set

$$\bar{u} \;=\; 2^{10}\; [(1 + |q|^2)^2 \;+\; |\omega|^2]^{-2}, \tag{16.6}$$

then $\bar{u}$ satisfies the Yamabe equation (16.2) and all other nonnegative solutions of (16.2) in the space $\mathcal{D}^{1,2}$ are obtained from $\bar{u}$ by translations and dilations,

$$\tau_{(q_o,\omega_o)}\bar{u}\ (q,\omega) \ \stackrel{def}{=}\ \bar{u}(q_o+q,\omega+\omega_o), \tag{16.7}$$

$$\bar{u}_\lambda\,(q) \ \stackrel{def}{=}\ \lambda^4\bar{u}(\lambda q,\lambda^2\omega), \qquad\qquad \lambda>0. \tag{16.8}$$

Thus, u which was defined in the beginning of the proof is given by equation (16.6) up to translations and dilations. This allows the calculation of the best constant in the Folland-Stein inequality, see [**GV1**, (4.52)],

$$\Lambda^5 \ =\ \int\limits_{\boldsymbol{G}\,(\mathbb{H})} \frac{2^{25}}{[(1+|q|^2)^2\ +\ |\omega|^2)]^5}\,dH \ =\ 2^{25}\,\pi^{7/2}\frac{\Gamma(\frac{7}{2})}{\Gamma(7)} \ =\ \frac{\pi^{12/10}}{12},$$

where Γ is the Gamma function. Hence

$$S_2 \ =\ \Lambda^{-1/2} \ =\ \frac{2\sqrt{3}}{\pi^{3/5}}.$$

Recalling the relation between u and v we find that the extremals in the Folland-Stein embedding are given by

$$v \ =\ \frac{2^{11}\sqrt{3}}{\pi^{3/5}}[(1+|q|^2)^2\ +\ |\omega|^2]^{-2}$$

and its translations and dilations. The proof of Theorem F is complete.

CHAPTER 5

The optimal constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group

In this chapter we determine the best (optimal) constant in the L^2 Folland-Stein inequality (Theorem 3.1, Chapter 1) on the quaternionic Heisenberg group (in all dimensions) and the non-negative extremal functions, i.e., the functions for which equality holds:

THEOREM G.
a) Let $\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}^n \times Im\mathbb{H}$ be the quaternionic Heisenberg group. The best constant in the L^2 Folland-Stein embedding inequality (3.3) is

$$S_2 = \frac{\left[2^{-2n}\,\omega_{4n+3}\right]^{-1/(4n+6)}}{2\sqrt{n(n+1)}},$$

where $\omega_{4n+3} = 2\pi^{2n+2}/(2n+1)!$ is the volume of the unit sphere $S^{4n+3} \subset \mathbb{R}^{4n+4}$. The non-negative functions for which (3.3) becomes an equality are given by the functions of the form

$$F = \gamma\left[(1+|q|^2)^2 + |\omega|^2\right]^{-(n+1)}, \qquad \gamma = const, \tag{16.9}$$

and all functions obtained from F by translations (18.2) and dilations (18.3).
b) The QC Yamabe constant of the standard QC structure of the sphere is

$$\lambda(S^{4n+3}, H^{can}) = 16\,n(n+2)\,\left[((2n)!)\,\omega_{4n+3}\right]^{1/(2n+3)}. \tag{16.10}$$

The proof relies on a realization of Branson, Fontana and Morpurgo [BFM], used also by Frank and Lieb [FL], that the old idea of Szegö [Sz], see also Hersch [He], can be used to find the sharp form of (logarithmic) Hardy-Littlewood-Sobolev type inequalities on the Heisenberg group. The argument presented here is purely analytical. In this respect, even though the QC Yamabe functional is involved, the QC scalar curvature is used in the proof without much geometric meaning. Rather, it is the conformal sub-laplacian that plays a central role and the QC scalar curvature appears as a constant determined by the Cayley transform and the left-invariant sub-laplacian on the quaternionic Heisenberg group. This method does not give all solutions of the QC Yamabe equation on the quaternionic-contact sphere but only these that realize the infimum of the QC Yamabe functional. Therefore, if considering the seven dimensional case, the result presented here is clearly weaker than Theorem E of Chapter 3.

17. The model quaternionic-contact structures

CONVENTION 17.1. *We use the following conventions:*

- n_h *will denote the homogeneous dimension* $4n+6$ *of the quaternionic Heisenberg group* $\boldsymbol{G} = \boldsymbol{G}\,(\mathbb{H}) \cong \Sigma$ *whose topological dimension is* $4n+3$
- $\tilde{\Theta}$ *will denote the standard left-invariant Im*$(\mathbb{H})$*-valued contact form on* $\boldsymbol{G}$ *as introduced in Section 3 (Chapter 1).*
- $\tilde{\eta}$ *will denote the standard QC form on the unit sphere* S^{4n+3}*;*
- Vol_η *will denote the volume form determined by the QC form* η*, thus* $Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (\omega_1)^{2n}$*.*

We begin with a calculation concerning the value of the QC scalar curvature on the sphere:

LEMMA 17.2. *The QC scalar curvature* $\tilde{S}$ *of the standard QC form* $\tilde{\eta}$ *on* S^{4n+3} *is*

$$\tilde{S} \;=\; \frac{1}{2}(n_h+2)(n_h-6) = 8n(n+2). \tag{17.1}$$

REMARK 17.3. *Notice that the standard contact form we consider here is twice the 3-Sasakian form on* S^{4n+3}*, which has QC scalar curvature equal to 16n(n+2) (cf. Corollary 5.13, Chapter 2).*

PROOF. Using the coordinates $(q',p') \in \Sigma \subset \mathbb{H}^n \times \mathbb{H}$, $\quad p' = |q'|^2 + \omega'$, we introduce the functions

$$\begin{aligned} h &= \frac{1}{16}|1+p'|^2 = \frac{1}{16}\left[(1+|q'|^2)^2 + |\omega'|^2\right] \quad \text{and} \\ \Phi &= (2h)^{-(n_h-2)/4} = 8^{(n_h-2)/4}\left[(1+|q'|^2)^2 + |\omega'|^2\right]^{-(n_h-2)/4}, \end{aligned} \tag{17.2}$$

so that now we have

$$\Theta = \frac{1}{2h}\tilde{\Theta} = \Phi^{4/(n_h-2)}\tilde{\Theta}.$$

A small calculation shows that the sub-laplacian of h w.r.t. $\tilde{\Theta}$ is given by

$$\triangle h = \frac{n_h - 6}{4} + \frac{n_h + 2}{4}|q'|^2$$

and thus Φ is a solution of the QC Yamabe equation on the Heisenberg group Σ

$$\triangle \Phi = -K\,\Phi^{2^*-1}, \qquad K = (n_h - 2)(n_h - 6)/8, \tag{17.3}$$

where $\triangle$ is the sub-laplacian on the quaternionic Heisenberg group. Denoting with $\mathcal{L}$ and $\tilde{\mathcal{L}}$ the conformal sub-laplacians of Θ and $\tilde{\Theta}$, respectively, we have

$$\Phi^{-1}\mathcal{L}(\Phi^{-1}u) = \Phi^{-2^*}\tilde{\mathcal{L}}u,$$

with

$$\mathcal{L} = a\triangle_\Theta - S_\Theta, \qquad a = 4\frac{Q+2}{Q-2}.$$

Here $\triangle_\Theta$ is the sub-laplacian associated to Θ, i.e., $\triangle_\Theta u = tr(\nabla^\Theta du)$–the horizontal trace of the Hessian of u, using the Biquard connection ∇^Θ of Θ; S_Θ is the QC scalar curvature of Θ. Thus, letting $u = \Phi$ we come to $\mathcal{L}(1) = \Phi^{1-2^*}\tilde{\mathcal{L}}\Phi$, which shows $-S_\Theta = -4\frac{n_h+2}{n_h-2}K$. The latter is the same as that of $\tilde{\eta}$ since the two structures are isomorphic via the diffemorphism $\mathcal{C}$, or rather its extension, since we can consider $\mathcal{C}$ as a quaternionic-contact conformal transformation between the whole sphere S^{4n+3} and the compactification $\hat{\Sigma}\cup\infty$ of the quaternionic Heisenberg group by adding the point at infinity. $\square$

We turn to the task of determining the first eigenvalue of the sub-laplacian on S^{4n+3}. In fact, we shall need only the fact that the restriction of every coordinate function is an eigenfunction. The proof of this fact can be seen directly without any reference to the Biquard connection, but this will require setting a lot of notation, so we prefer to use a result from Chapter 2.

Lemma 17.4. *If ζ is any of the (real) coordinate functions in $\mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}$, then*

$$\tilde{\triangle}\zeta = -\lambda_1\zeta, \quad \lambda_1 = \frac{\tilde{S}}{n_h + 2} = 2n \tag{17.4}$$

where $\tilde{\triangle}$ *is the sub-laplacian of the standard QC form* $\tilde{\eta}$ *of* S^{4n+3}.

Proof. It is enough to furnish a proof for the sub-laplacian on the 3-Sasakain sphere since the two QC forms defer by a constant. We can see that every ζ of the considered type is an eigenfunction by using Corollary 7.24 (Chapter 2). It will be enough to see it for one coordinate function provided the sub-laplacian on the sphere is rotation invariant. Thus, let us take $\zeta = t_1$. Notice that ζ is quaternionic pluri-harmonic (cf. Definition 7.7, Chapter 2) since it is the real part of the anti-regular function $t_1 + ix_1 - jy_1 - kz_1$. So, its restriction to the 3-Sasakain sphere is the real part of an anti-CRF function. Therefore, we apply Corollary 7.24 which gives $tr(\nabla d\zeta) = 4\lambda n$ for the sub-laplacian of the 3-sasakain QC structure on the sphere. Next, we compute λ, which can be found in Theorem 7.20. Using that the sphere is 3-Sasakian it follows the Reeb vector fields are obtained from the outward pointing unit normal vector N as follows, $\xi_1 = iN$, $\xi_2 = jN$ and $\xi_3 = kN$, where for a point on the sphere we have $N(q) = q \in \mathbb{H}^{n+1}$. Therefore $\lambda = -t_1 = -\zeta$. To see this easier notice that only the first four coordinates of N matter. So, if we assume $n = 0$ we have $iN = -x + it + ky - jz$, $jN = -y + iz + jt - kx$ and $kN = -z - iy + jx + kt$, so we need to sum the *real dot product* of these vectors with i, j and k, respectively, which gives $-t$. Thus, for the sub-laplacian on the 3-Sasakian sphere we have

$$tr(\nabla d\zeta) = -4n\zeta,$$

where ζ is the restriction any of the coordinate functions of $\mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}$. Since the QC contact form $\tilde{\Theta}$ is twice the 3-Sasakain QC contact form on the sphere it follows $\tilde{\triangle}$ is 1/2 of the 3-Sasakain sub-laplacian. Thus

$$\tilde{\triangle} = -2n\zeta,$$

which shows $\lambda_1 = 2n = \frac{1}{2}(n_h - 6) = \tilde{S}/(n_h + 2)$. $\square$

We finish this section with a simple Lemma which will be used to relate the various explicit constants. Its claim also follows from the conformal invariance of the Yamabe equation, but we prefer to give a proof, which is independent of the notion of QC scalar curvature. For a QC form η we let $|\nabla^\eta F|^2 = \sum_{\alpha=1}^{4n} |dF(e_\alpha)|^2$ be the square of the length of the horizontal gradient of a function F taken with respect to an orthonormal basis of the horizontal space $H = Ker\,\eta$ and the metric determined by η.

LEMMA 17.5. *Let* $F \in \overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})$, *cf.* (18.1), *be a positive function with* $\int_{\boldsymbol{G}} F^{2^*} Vol_{\tilde{\Theta}} = 1$. *Then we have*

$$\int_{\boldsymbol{G}} a|\nabla^{\tilde{\Theta}} F|^2 \, Vol_{\tilde{\Theta}} \;=\; \int_{S^{4n+3}} \left(a|\nabla^{\tilde{\eta}} g|^2 + \tilde{S} g^2\right) \, Vol_{\tilde{\eta}}, \qquad a = 4(2^* - 1), \tag{17.5}$$

and

$$\int_{\boldsymbol{G}} g^{2^*} Vol_{\tilde{\eta}} = 1,$$

where

$$g = \mathcal{C}^*(F\Phi^{-1}), \tag{17.6}$$

and, as before, $\mathcal{C} : S^{4n+3} \to \Sigma$ *is the Cayley transform,* $\Theta = \Phi^{4/(Q-2)}\tilde{\Theta}$, *cf.* (17.2).

REMARK 17.6. *Notice that* $Vol_{\tilde{\Theta}} = 2^{-3}\,(2n)!\,dH$, *where* dH *is the Lebesgue measure in* $\mathbb{R}^{4n+3}$, *which is a Haar measure on the group.*

PROOF. It will be convenient for the remaining of this proof to denote by small letters the pull-back by the Cayley transform of a function denoted with the corresponding capital letter. Thus, $f = \mathcal{C}^* F = F \circ \mathcal{C}$, $\phi = \mathcal{C}^*(\Phi)$ and $g = f\phi^{-1}$. By the conformality of the QC structures on the group and the sphere we have

$$Vol_{\Theta} = \Phi^{2^*} Vol_{\tilde{\Theta}} \tag{17.7}$$

By (17.7) we have $F^{2^*} Vol_{\tilde{\Theta}} = f^{2^*} \phi^{-2^*} Vol_{\tilde{\eta}}$, which motivates the definition (17.6) of the function g which is defined on the sphere and should be regarded as corresponding to the function F. Thus, we have for example $F = G\Phi$. By definition we have

$$\int_{\boldsymbol{G}} g^{2^*} Vol_{\tilde{\eta}} = 1,$$

so our next task is to see that the Yamabe integral is preserved

$$\int_{\boldsymbol{G}} |\nabla^{\tilde{\Theta}} F|^2 \, Vol_{\tilde{\Theta}} \;=\; \int_{S^{4n+3}} \left(|\nabla^{\tilde{\eta}} g|^2 + K g^2\right) \, Vol_{\tilde{\eta}}. \tag{17.8}$$

Here is where we shall exploit that a power of the conformal factor of the Cayley transform is a solution of the Yamabe equation. Let $\left\langle \nabla^{\Theta}\Phi, \nabla^{\Theta}G \right\rangle = \sum_{a=1}^{4n} (e_a\Phi)\,(\mathbf{e}_a G)$ where $\{e_1, \dots, e_{4n}\}$ is an orthonormal basis of the horizontal space H. Using the divergence formula from Section 9.1 (Chapter 2) we find

$$\int_{\boldsymbol{G}} |\tilde{\nabla}^{\Theta} F|^2 \, Vol_{\tilde{\Theta}} = \int_{\boldsymbol{G}} |\nabla^{\tilde{\Theta}}(G\Phi)|^2 \, Vol_{\tilde{\Theta}}$$

$$= \int_{\boldsymbol{G}} \left(G^2 |\nabla^{\tilde{\Theta}}\Phi|^2 + \Phi^2 |\nabla^{\tilde{\Theta}} G|^2 + \left\langle \Phi\nabla^{\tilde{\Theta}}\Phi, \nabla^{\tilde{\Theta}} G^2 \right\rangle \right) Vol_{\tilde{\Theta}}$$

$$= \int_{\boldsymbol{G}} \left(\Phi^2 |\nabla^{\tilde{\Theta}} G|^2 - G^2 \Phi \triangle_{\tilde{\Theta}} \Phi \right) Vol_{\tilde{\Theta}}.$$

Now, the Yamabe equation (17.3) gives

$$\int_{\boldsymbol{G}} |\nabla^{\tilde{\Theta}} F|^2 \, Vol_{\tilde{\Theta}} = \int_{\boldsymbol{G}} \left(\Phi^2 |\nabla^{\tilde{\Theta}} G|^2 + K G^2 \Phi^{2^*} \right) Vol_{\tilde{\Theta}}$$

$$= \int_{S^{4n+3}} \left(\phi^{2-2^*} (|\nabla^{\tilde{\Theta}} G| \circ \mathcal{C})^2 + K g^2 \right) Vol_{\tilde{\eta}} = \int_{S^{4n+3}} \left(|\nabla^{\tilde{\eta}} g|^2 + K g^2 \right) Vol_{\tilde{\eta}},$$

taking into account that $\mathcal{C}$ is a QC conformal map. Finally, a glance at (17.3) and (17.1) shows $\tilde{S}/K = 4(2^* - 1) = (4(n_h + 2)/(n_h - 2)$ which allows to put (17.8) in the form (17.5). $\square$

18. The best constant in the Folland-Stein inequality

In this section, following [FL], we prove the main Theorem. It is important to observe that a suitable adaptation of the method of concentration of compactness due to P. L. Lions [Lio1], [Lio2] allows to prove that the Yamabe constant and optimal constant in the Folland-Stein inequality is achieved in the space $\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})$, see [Va1] and [Va2]. Here

$$\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G}) = \overline{C_o^{\infty}(\boldsymbol{G})}^{\|\cdot\|_{\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})}}.$$

The space $\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})$ is endowed with the norm

$$||u||_{\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})} \;=\; ||\,|\nabla u|\,||_{L^{2^*}(\boldsymbol{G})}. \tag{18.1}$$

where ∇u is the horizontal gradient of u and $|\nabla u|^2 = \sum_{a=1}^{4n}(e_a u)^2$ for an orthonormal basis $\{e_1, \dots, e_{4n}\}$ of horizontal left invariant vector fields.

In this regard an elementary, yet crucial observation, is that if u is an entire solution to the Yamabe equation, then such are also the two functions

$$\tau_h u \;\overset{def}{=}\; u \circ \tau_h, \qquad h \in \boldsymbol{G}, \tag{18.2}$$

where $\tau_h : \boldsymbol{G} \to \boldsymbol{G}$ is the operator of left-translation $\tau_h(g) = hg$, and

$$u_\lambda \;\overset{def}{=}\; \lambda^{(n_h-2)/2}\; u \circ \delta_\lambda, \qquad \lambda > 0. \tag{18.3}$$

The Heisenberg dilations are defined by

$$\delta_\lambda\left((q', \omega')\right) = \big((\lambda q', \lambda^2 \omega')\big)\,, \qquad (q', \omega') \in \boldsymbol{G}$$

It is also well known, [Va1] and [Va2], that there are smooth positive minimizer of the Folland-Stein inequality on the quaternionic Heisenberg group $\boldsymbol{G}$. These facts will be used without further notice on regularity and existence.

We start with the "new" key, see [BFM] and [FL], allowing the ultimate solution of the considered problem.

LEMMA 18.1. *For every $v \in L^1(S^{4n+3})$ with $\int_{S^{4n+3}} v \; Vol_{\tilde\eta} = 1$ there is a quaternionic-contact conformal transformation ψ such that*

$$\int_{S^{4n+3}} \psi\, v \; Vol_{\tilde\eta} = 0.$$

PROOF. Let $P \in S^{4n+3}$ be any point of the quaternionic sphere and N be its antipodal point. Let us consider the local coordinate system near P defined by the Cayley transform $\mathcal{C}_N$ from N. It is known that $\mathcal{C}_N$ is a quaternionic-contact conformal transformation between $S^{4n+3} \setminus N$ and the quaternionic Heisenberg group. Notice that in this coordinate system P is mapped to the identity of the group. For every r, $0 < r < 1$, let $\psi_{r,P}$ be the QC conformal transformation of the sphere, which in the fixed coordinate chart is given on the group by a dilation with center the identity by a factor δ_r. If we select a coordinate system in $\mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}$ so that $P = (1,0)$ and $N = (-1,0)$ and then apply the formulas for the Cayley transform from (cf. Section 3.3), the formula for $(q^*, p^*) = \psi_{r,P}(q,p)$ becomes

$$
\begin{aligned}
q^* &= 2r\left(1 + r^2(1+p)^{-1}(1-p)\right)^{-1}(1+p)\,q\\
p^* &= \left(1 + r^2(1+p)^{-1}(1-p)\right)^{-1}\left(1 - r^2(1+p)^{-1}(1-p)\right), i.e,
\end{aligned}
$$

We can define then the map $\Psi : B \to \bar{B}$, where B ($\bar{B}$) is the open (closed) unit ball in $\mathbb{R}^{4n+4}$, by the formula

$$\Psi(rP) = \int_{S^{4n+3}} \psi_{1-r,P}\, v\ Vol_{\tilde{\eta}}.$$

Notice that Ψ can be continuously extended to $\bar{B}$ since for any point P on the sphere, where $r = 1$, we have $\psi_{1-r,P}(Q) \to P$ when $r \to 1$. In particular, $\Psi = id$ on S^{4n+3}. Since the sphere is not a homotopy retract of the closed ball it follows that there are r and $P \in S^{4n+3}$ such that $\Psi(rP) = 0$, i.e.,

$$\int_{S^{4n+3}} \psi_{1-r,P}\, v\ Vol_{\tilde{\eta}} = 0.$$

Thus, $\psi = \psi_{1-r,P}$ has the required property. $\square$

In the next step we prove that we can assume that the minimizer of the Folland-Stein inequality satisfies the zero center of mass condition. A number of well known invariance properties of the Yamabe functional will be exploited.

LEMMA 18.2. *Let v be a smooth positive function on the sphere with*

$$\int_{S^{4n+3}} v^{2^*}\ Vol_{\tilde{\eta}} = 1.$$

There is a smooth positive function u such that

$$\int_{S^{4n+3}} \left(4\frac{n_h+2}{n_h-2}\,|\nabla u|^2\ +\ \tilde{S}\,u^2\right) Vol_{\tilde{\eta}}\ = \int_{S^{4n+3}} \left(4\frac{n_h+2}{n_h-2}\,|\nabla v|^2\ +\ \tilde{S}\,v^2\right) Vol_{\tilde{\eta}}$$

and

$$\int_{S^{4n+3}} u^{2^*}\, Vol_{\tilde\eta} = 1$$

In addition,

$$\int_{S^{4n+3}} P\, u^{2^*}(P)\, Vol_{\tilde\eta} = 0, \qquad P \in \mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}. \tag{18.4}$$

In particular, the Yamabe constant $\lambda(S^{4n+3}, [\tilde\eta])$*, given by*

$$\inf\{\int_{S^{4n+3}} \Big(4\frac{n_h+2}{n_h-2}\, |\nabla v|^2 \ + \ \tilde S\, v^2\Big) Vol_{\tilde\eta} : \int_{S^{4n+3}} v^{2^*}\ Vol_{\tilde\eta} \ = \ 1, \ v>0\}, \tag{18.5}$$

is achieved for a positive function u *with a zero center of mass, i.e., for a function* u *satisfying* (18.4).

Proof. By Section 9.1 (Chapter 2), $Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (\omega_1)^{2n}$ is a volume form on a QC manifold with contact form η. Thus if η is a QC structure on the sphere which is QC conformal to the standard QC structure $\tilde\eta$, $\eta = \phi^{4/(n_h-2)}\tilde\eta$, then $Vol_\eta = \phi^{2^*} Vol_{\tilde\eta}$. This allows to cast the Yamabe equation (3.9) in the form

$$\phi^{-1} v\, \mathcal{L}(\phi^{-1} v)\ Vol_\eta = v \tilde{\mathcal{L}}(v)\ Vol_{\tilde\eta}.$$

Therefore, if we take a positive function v on the sphere $\int_{S^{4n+3}} v^{2^*}\, Vol_{\tilde\eta} = 1$ and then consider the function

$$u = \phi^{-1}(v \circ \psi^{-1}), \tag{18.6}$$

where ψ is the QC conformal map of Lemma 18.1, $\eta \equiv (\psi^{-1})^*\tilde\eta$, and ϕ is the corresponding conformal factor of ψ, we can see that u achieves the claim of the Lemma. $\square$

We shall call a function u on the sphere a *well centered* function when (18.4) holds true. In the next step we show that a well centered minimizer has to be constant.

Lemma 18.3. *If* u *is a well centered local minimum of the problem* (18.5)*, then* $u \equiv const.$

PROOF. Let ζ be a smooth function on the sphere S^{4n+3}. After applying the divergence formula from Section 9.1 (Chapter 2), we obtain the formula

$$\Upsilon(\zeta u) = \int_{S^{4n+3}} \zeta^2 \Big(4\frac{n_h+2}{n_h-2}\, |\tilde\nabla u|^2 \;+\; \tilde S\, u^2 \Big)\, Vol_{\tilde\eta} \\ -\; 4\frac{n_h+2}{n_h-2} \int_{S^{4n+3}} u^2 \zeta \tilde\triangle \zeta \; Vol_{\tilde\eta}. \tag{18.7}$$

This suggests to take as a test function ζ an eigenfunction of the sub-laplacian $\tilde\triangle$ of the standard QC structure. In particular, we can let ζ be any of the coordinate functions in $\mathbb{H}^n \times \mathbb{H}$ in which case $\tilde\triangle\zeta = -\lambda_1\zeta$.

It will be useful to introduce the functional $N(v) = \big(\int_{S^{4n+3}} v^{2^*} \; Vol_{\tilde\eta}\big)^{2/2^*}$ so that

$$\lambda(S^{4n+3}, [\tilde\eta]) = \inf\{\mathcal{E}(v) : \, v \in D\,(S^{4n+3})\}, \qquad \mathcal{E}(v) \overset{def}{=} \Upsilon(v)/N(v). \tag{18.8}$$

Computing the second variation $\delta^2\mathcal{E}(u)v = \frac{d^2}{dt^2}\mathcal{E}(u+tv)_{|t=0}$ of $\mathcal{E}(u)$ we see that the local minimum condition $\delta^2\mathcal{E}(u)v \geq 0$ implies

$$\Upsilon(v) - (2^*-1)\Upsilon(u) \int_{S^{4n+3}} u^{2^*-2} v^2 \; Vol_{\tilde\eta} \;\; \geq 0$$

for any function v such that $\int_{S^{4n+3}} u^{2^*-1} v \; Vol_{\tilde\eta} = 0$. Therefore, for ζ being any of the coordinate functions in $\mathbb{H}^n \times \mathbb{H}$ we have

$$\Upsilon(\zeta u) - (2^*-1)\Upsilon(u) \int_{S^{4n+3}} u^{2^*} \zeta^2 \; Vol_{\tilde\eta} \;\; \geq 0,$$

which after summation over all coordinate functions taking also into account (18.7) gives

$$\Upsilon(u) - (2^*-1)\Upsilon(u) - 4\lambda_1(2^*-1) \int_{S^{4n+3}} u^2 \; Vol_{\tilde\eta} \geq 0,$$

which implies, recall $2^* - 1 = (n_h + 2)/(n_h - 2)$,

$$0 \le 4(2^* - 1)\,(2^* - 2) \int_{S^{4n+3}} |\tilde{\nabla} u|^2 \; Vol_{\tilde{\eta}}$$
$$\le \left(4\lambda_1(2^* - 1) - (2^* - 2)\,\tilde{S}\right) \int_{S^{4n+3}} u^{2^*} \; Vol_{\tilde{\eta}}.$$

Thus, our task of showing that u is constant will be achieved once we see that

$$4\lambda_1(2^* - 1) - (2^* - 2)\,\tilde{S} \le 0, \quad \text{i.e,} \quad \lambda_1 \le \tilde{S}/(n_h + 2). \tag{18.9}$$

By Lemma 17.4 we have actually equality $\lambda_1 = \tilde{S}/(n_h + 2)$, which completes the proof. It is worth observing that inequality (18.9) can be written in the form

$$\lambda_1\, a \le (2^* - 2)\,\tilde{S},$$

where a is the constant in front of the (sub-)laplacian in the conformal (sub-)laplacian, i.e., $a = 4\frac{n_h+2}{n_h-2}$ in our case. $\square$

At this point the proof of our main Theorem G follows easily as follows.

Proof of Theorem G. Let F be a minimizer (local minimum) of the Yamabe functional $\mathcal{E}$ on $\boldsymbol{G}$ and g the corresponding function on the sphere defined in Lemma 17.5. By Lemma 18.2 and (18.6) the function $g_0 = \phi^{-1}(g \circ \psi^{-1})$ will be well centered and a minimizer (local minimum) of the Yamabe functional $\mathcal{E}$ on S^{4n+3}. The latter claim uses also the fact that the map $v \mapsto u$ of equation (18.6) is one-to-one and onto on the space of smooth positive functions on the sphere. Now, from Lemma 18.3 we conclude that $g_o = const$. Looking back at the corresponding functions on the group we see that

$$F_0 = \gamma \left[(1 + |q'|^2)^2 + |\omega'|^2\right]^{-(n_h-2)/4}$$

for some $\gamma = const > 0$. Furthermore, the proof of Lemma 18.1 shows that F_0 is obtained from F by a translation (18.2) and dilation (18.3). Correspondingly, any positive minimizer (local maximum) of problem (18.11) is given up to dilation or translation by the function

$$F = \gamma \left[(1 + |q'|^2)^2 + |\omega'|^2\right]^{-(n_h-2)/4}, \qquad \gamma = const. > 0. \tag{18.10}$$

Of course, translations (18.2) and dilations (18.3) do not change the value of $\mathcal{E}$. Incidentally, this shows that any local minimum of the Yamabe functional $\mathcal{E}$ on the sphere or the group has to be a global one.

We turn to the determination of the best constant. Let us define the constants (18.11)

$$\Lambda_{\tilde{\Theta}} \stackrel{def}{=} inf \left\{ \int_{\boldsymbol{G}\,(\mathbb{H})} |\nabla v|^2 \, Vol_{\tilde{\Theta}} \; : \; v \in \overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G}), \; v \geq 0, \; \int_{\boldsymbol{G}\,(\mathbb{H})} |v|^{2^*} \, Vol_{\tilde{\Theta}} \; = \; 1 \right\}$$

and

$$\Lambda \stackrel{def}{=} inf \left\{ \int_{\boldsymbol{G}\,(\mathbb{H})} |\nabla v|^2 \, dH \; : \; v \in \overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G}), \; v \geq 0, \; \int_{\boldsymbol{G}\,(\mathbb{H})} |v|^{2^*} \, dH \; = \; 1 \right\}.$$

Clearly, $\Lambda_{\tilde{\Theta}} = S_{\tilde{\Theta}}^{-2}$, where $S_{\tilde{\Theta}}$ is the best constant in the L^2 Folland-Stein inequality

$$\left(\int_{\boldsymbol{G}\,(\mathbb{H})} |u|^{2^*} \; Vol_{\tilde{\Theta}} \right)^{1/2^*} \leq \; S_{\tilde{\Theta}} \; \left(\int_{\boldsymbol{G}\,(\mathbb{H})} |\nabla^{\tilde{\Theta}} u|^2 \; Vol_{\tilde{\Theta}} \right)^{1/2}, \tag{18.12}$$

while $\Lambda = S_2^{-2}$ is the best constant in the L^2 Folland-Stein inequality (3.3) (taken with respect to the Lebesgue measure). By Remark 17.6, we have

$$\Lambda_{\tilde{\Theta}} \; = \; \left[2^{-3} (2n)! \right]^{1/(2n+3)} \; \Lambda.$$

Furthermore, by Lemma 18.3 and equations (17.5) and (17.6) with $g = const$, we have

$$\begin{aligned}
\Lambda_{\tilde{\Theta}} \; = \; \frac{1}{S_2^2} = \frac{\int_{\boldsymbol{G}} \; |\nabla^{\tilde{\Theta}} F|^2 \, Vol_{\tilde{\Theta}}}{\left[\int_{\boldsymbol{G}} |F|^{2^*} \, Vol_{\tilde{\Theta}} \right]^{2/2^*}} \\
= \frac{\int_{S^{4n+3}} \left(|\nabla^{\tilde{\eta}} g|^2 + \frac{\tilde{S}}{a} \, g^2 \right) \, Vol_{\tilde{\eta}}}{\left[\int_{S^{4n+3}} |g|^{2^*} \, Vol_{\tilde{\eta}} \right]^{2/2^*}} \; = \; 4n(n+1) \left[((2n)!) \, \omega_{4n+3} \right]^{1/(2n+3)}.
\end{aligned}$$

Here,

$$\omega_{4n+3} = 2\pi^{2n+2}/\Gamma(2n+2) = 2\pi^{2n+2}/(2n+1)!$$

is the volume of the unit sphere $S^{4n+3} \subset \mathbb{R}^{4n+4}$ and we also took into account Remark 17.3 which shows that $Vol_{\tilde{\eta}}$ gives $2^{2n+3}\,((2n)!)\,\omega_{4n+3}$ for the volume of S^{4n+3}. Thus,

$$S_{\tilde{\Theta}} \;=\; \Big(4n(n+1)\,[((2n)!)\,\omega_{4n+3}]^{1/(2n+3)}\Big)^{-1/2} \;=\; \frac{[((2n)!)\,\omega_{4n+3}]^{-1/(4n+6)}}{2\sqrt{n(n+1)}},$$

which completes the proof of part a).

b) The Yamabe constant of the sphere is calculated immediately by taking a constant function in (18.8)

$$\lambda(S^{4n+3},[\tilde{\eta}]) = a\,\Lambda_{\tilde{\Theta}}, \qquad a = 4\frac{n_h+2}{n_h-2} = 4\frac{n+2}{n+1}. \tag{18.13}$$

This completes the proof of Theorem G. ∎

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