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Local properties of dynamical systems

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Contents

Chapter 1

Introduction

There are known many papers studying the problem of attainability of a closed set for small time with respect to the trajectories of a given control system. This is still an open problem (but there are known already some sufficient conditions, cf. [12] and [6]). When the set is a point the problem of attainability for small time is reduced to the problem of small-time local controllability. This is also an open problem. There are known sufficient and necessary conditions only for some particular cases (cf. [10], [38] and [26]).

The small-time local controllability (STLC) property is one of the most important properties of the reachable set of nonlinear control system (cf. [2], [4], [5], [8], [9], [17]). It is crucial for solving different dynamical problems (cf. [10], [13], [15], [29], [32], [35]). For example, it is well known that the Bellman function for minimal time is in the general case only lower semicontinuous (cf. [11]). But if the general sufficient controllability condition of Sussmann holds true, then this function is Holder continuous (even Lipcshitz continuous under additional assumptions).

There exist different approaches for styding STLC property requiring different assumptions (for example, cf. [10], [18], [14], [16], [34], [36], [37]). Here we follow a general geometrical approach proposed by Sussmann (cf., [34] and [36]). This approach allows to obtain a general sufficient STLC condition which extends the most of the existing sufficient controllability conditions. It is based on a classical formula of Campbell-Baker-Hausdorff (C-B-H) formula, which is one of the basic result of Lie group theory. We have to point out that usually the Lie algebra generated by the vector fields of a smooth control system is infinite dimensional. So, to apply the C-B-H formula a suitable nilpotent approximation is needed (cf., [15]).

At the end of the XX century Arthur Krener, Henry Hermes, Hiroshi Kunita, Ronald Hirshorn and etc. proposed a set $E^+(x_0)$ of tangent vector fields to the reachable set of a smooth control system. The basic idea is to construct suitable "control variations". So, if the origin belongs to the interior of the convex hull of $E^+(x_0)$ then the corresponding control system is STLC at the point x_0. The characterization of the set $E^+(x_0)$ is still an open problem. The set $E^+(x_0)$ is defined in the second chapter following the papers [38], [25] and [26]. Moreover, different basic properties of this set are presented. Using the set $E^+(x_0)$, characterization of the STLC property is obtained in some particular cases (cf., [38] and [26]).

In the third chapter we present the main ideas of the general approach proposed by Sussman (cf. [34] and [36]). A class of bad Lie brackets is defined. The main result is the following: if the reachable set has a non empty interior and the bad Lie brackets can be "neutralized" by suitable Lie brackets, then the control system is STLC at the initial point. Next, we consider a class of polynomial control systems with drift term which is homogeneous of second degree. The general sufficient STLC condition of Sussmann cannot be applied. We study carefully the Lie algebra of the vector fields generated by this control system and obtain that some "bad Lie brackets" belong to the set $E^+(0)$ and hence they are not obstructions for STLC. The main result of this chapter is a new STLC condition.

We consider the same class of polynomial control systems in the fourth chapter. Here we study carefully the Lie algebra generated by the drift term and suitable constant vector fields of the considered control system. We define an increasing sequence of linear spaces and cones which generalize the corresponding structure known for the linear case (cf., [38] and [26]) and a suitable "weight" is defined on them. This "weight" is a natural extension of the weight used by Sussmann, Bianchini and Stefani. It is proved that the elements of this structure belong to the set E^+ and as a corollary a new sufficient STLC condition is obtained. It's remarkable that the STLC property in this case is obtained using the values of bad Lie brackets evaluated at the origin.

Suitable examples are presented in the third and the fourth chapters. They show the applicability of the obtained sufficient controllability conditions and motivate the study of the STLC property for more general control systems.

We present a new general necessary condition for STLC property of a class of non smooth control systems in the fifth chapter. When we study a smooth control system we obtain a linear subspace L determined by the values at x_0 of known elements of $E^+(x_0)$. The obtained necessary STLC condition shows when the linear space L and the values of the right-hand side on L imply that the considered control system is not STLC at x_0. Hence, this necessary condition is a natural continuation of the approach presented in the previous chapters. Moreover, it is shown the relation of this result to the known necessary conditions of Sussman (cf. [34]) and Stefani (cf. [31]).

Chapter 2

A geometric approach to study the reachable set

2.1 Small-time local controllability

We consider the following control system Σ in $\mathbb{R}^n$

$$\dot{x}(t) = f(x(t), u(t)), \tag{2.1}$$

where the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous with respect to the variables x and u, and the set U is a convex compact subset of $\mathbb{R}^m$.

Let us fix a real $T > 0$. We denote by $\mathcal{U}_T$ the set of all measurable functions u defined on $[0, T]$ such that $u(t) \in U$ for almost all $t \in [0, T]$. The elements of $\mathcal{U}_T$ are called admissible controls. An admissible trajectory of the system Σ defined on $[0, T]$ is any absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ satisfying (2.1) for almost each t of $[0, T]$ with some admissible control $u \in \mathcal{U}_T$. The reachable set $\mathbf{R}(x_0, T)$ of Σ is the set of all points reachable in time not greater than T by means of admissible trajectories of Σ starting from the point x_0.

Definition 2.1.1 *The control system Σ is called small-time locally controllable (STLC) at the point x_0 iff x_0 belongs to the interior of the set $\mathbf{R}(x_0, T)$ for each $T > 0$.*

There are many possible approaches to study the small-time local controllability, leading to different results and requiring different assumptions. Here we follow a geometrical point of view. The underlying philosophy of our approach is that the local properties of the reachable set of a control system are determined by the values of the right-hand side and its derivatives at the initial point x_0. Unfortunately, the derivatives of the right-hand side are not coordinate invariant. Therefore it's natural to consider the elements of Lie algebra generated by the vector fields.

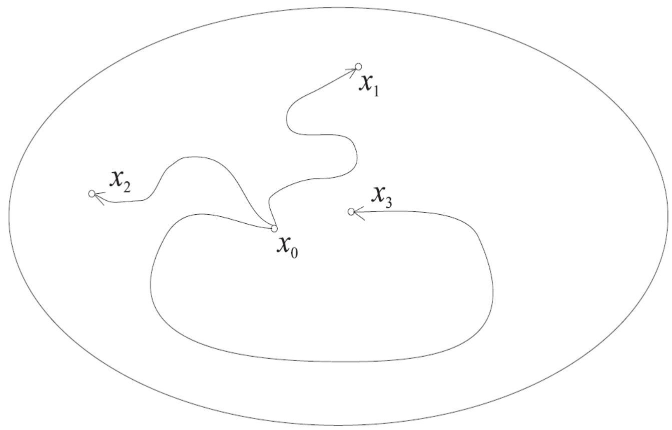

Figure 2.1: STLC property

2.2 Lie brackets and Campbell-Baker-Hausdorff formula

Let $X : \mathbb{R}^n \to \mathbb{R}^n$ and $Y : \mathbb{R}^n \to \mathbb{R}^n$ be two arbitrary vector fields. By $[X, Y]$ we denote its Lie bracket $[X, Y] : \mathbb{R}^n \to \mathbb{R}^n$ which is defined as follows

$$[X, Y](x) := Y'(x)X(x) - X'(x)Y(x), \text{ for each } x \in \mathbb{R}^n,$$

where by $X'(x)$ and $Y'(x)$ are denoted the corresponding derivatives of the maps X and Y at the point x. We set $ad^1(X, Y)(x) := [X, Y](x)$ and inductively $ad^{k+1}(X, Y)(x) := [X, ad^k(X, Y)](x)$ for each positive integer k.

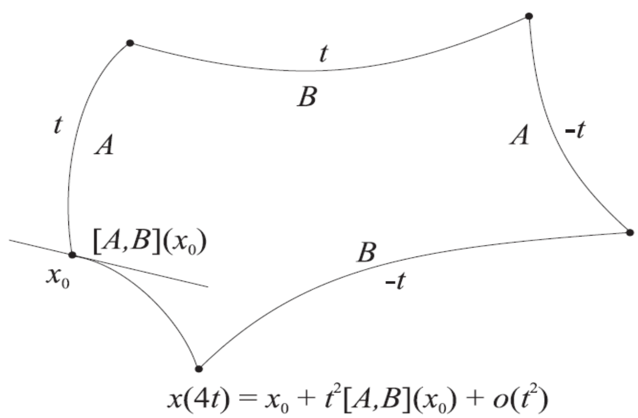

Figure 2.2: Lie brackets

It could be seen at Figure 2.2 that the Lie bracket $[A, B]$ generates new directions from the reachable set if the vector fields $A, B, -A, -B$ are admissible vector fields for the considered control system. To work with the Lie brackets we will use essentially Campbell-Baker-Hausdorff (C-B-H) formula. By means of the C-B-H formula we will find "new" tangent vectors to the reachable set. Before to define the set of tangent vector fields to the reachable set we introduce some notations:

We denote by $\mathrm{Exp}(tZ)(x_0)$ the value of the solution of the equation

$$\dot{x}(\tau) = tZ(x(\tau)),\ x(0) = x_0,$$

at time $\tau = 1$. Below we shall use the notion Exp $(Z_t)(x_0)$ for an arbitrary family of analytic vector fields $\{Z_t :\ t \in \mathbb{R}\}$, depending continuously on t, as it is defined in Sussmann (cf. [34]). Then the formula of Campbell-Baker-Hausdorff can be formulated as follows : if X and Y are analytic vector fields on $\mathbb{R}^n$, then

$$\mathrm{Exp}\ (t_1 X) \circ \mathrm{Exp}\ (t_2 Y)(x)\ =$$

$$= \mathrm{Exp}\ \left(t_1 X + t_2 Y + \frac{t_1 t_2}{2}[X, Y] + \frac{t_1 t_2^2}{12}[Y, [Y, X]] + \frac{t_1^2 t_2}{12}[X, [X, Y]] + \ldots \right)(x), \tag{2.2}$$

where $\circ$ means superposition, and the infinite sum in the right-hand side is convergent for sufficiently small $|t_1|$ and $|t_2|$.

Let us denote by $\mathcal{L} = \mathcal{L}(X, Y)$ the Lie algebra generated by the vector fields X and Y, i.e. $\mathcal{L}$ is the minimal linear subspace of vector fields which is closed under the operation Lie bracket. Let us fix $N \in \mathbb{N}$ and denote by $\mathcal{L}_N = \mathcal{L}_N(X, Y)$ the finite Lie algebra generated by the vector fields X and Y, i.e. the algebra generated by Lie brackets in order not greater than N.
If $\mathcal{S}$ is a Lie series in $\mathcal{L}$ then $\mathcal{S}_{\mathcal{N}}$ is the corresponding finite Lie series in $\mathcal{L}_{\mathcal{N}}$. Also, Exp $(\mathcal{S})_{\mathcal{N}}$ is the finite Lie series in $\mathcal{L}_{\mathcal{N}}$, corresponding to Exp $(\mathcal{S})$.
Then we say that Exp $(X) \circ$ Exp $(Y)\ \cong$ Exp $(\mathcal{S})$ iff $||\mathrm{Exp}\ (X) \circ \mathrm{Exp}\ (Y)(x) -$

$\mathrm{Exp}\ (\mathcal{S})(x)_N|| \leq C(N)t^{N+1}$ for each $x \in \mathbb{R}^n$ and each $N \in \mathbb{N}$, where $C(N) > 0$.

2.3 Tangent vector fields to the reachable set

Our approach is based on a suitable definition of tangent vector fields to the reachable set of a control system, namely we define the set E_α^+, $\alpha > 0$, of analytic vector fields. The definition of this set is related to the works of Krener (cf. [27]), Hermes (cf. [14]), Sussman (cf. [33]), Kunita (cf. [28]), Veliov and Krastanov (cf. [38]), Krastanov and Quincampoix (cf. [25]), Krastanov and Veliov (cf. [26]) and others.

A function of the type $\sum_{i=1}^{\varrho} c_i t^{d_i}$, where $c_i > 0$ and d_i are positive real numbers, $i = 1, 2, \ldots, \varrho$, is called a positive polynomial. Further, we use the notation $O(t)$ to indicate any family of analytic vector fields $O(t;x)$ parameterized by $t > 0$, continuous in (t, x) and such that the ratio $O(t;x)/t$ is bounded when t tends to zero, uniformly with respect to $x \in \mathbf{B}$. Let A^0 be the set of all families of analytic vector fields $a(t) = a(t, x)$ on $\mathbb{R}^n$, parameterized on t, continuous in (t, x) and such that there exist positive reals θ and c such that $\|a(t, x)\| \leq ct^{\theta}\|x - x_0\|$ for all $x \in \mathbf{B}$.

Definition 2.3.1 *It is said that the analytic vector field Z belongs to the set $E_\alpha^+(x_0)$ of the control system Σ iff there exist families of analytic vector fields $O(t^w)$ with $w > \alpha$, and $a(t) \in A^0(x_0)$ parameterized by $t > 0$ and a positive polynomial $p(t)$ such that*

$$Exp(t^\alpha Z + a(t) + O(t^w))(x) \in \mathcal{R}(x, p(t))$$

for each point x from a neighborhood of the point x_0.

Definition 2.3.2 *It is said that the analytic vector field Z belongs to the set $\mathcal{S}$ of the control system Σ iff there exist positive real numbers K and T, such that for every point x and each $t \in [0, T]$*

$$Exp(tZ)(x) \in \mathcal{R}(x, Kt).$$

Remark 2.3.3 *By setting $t := t^{\beta/\alpha}$ one can prove that the relation $A \in E_\alpha^+(x_0)$ implies that $A \in E_\beta^+(x_0)$ for every $\beta > \alpha$.*

Remark 2.3.4 *We denote by $E^+(x_0)$ the set $E_1^+(x_0)$.*

The importance of the set $E_\alpha^+(x_0)$ for studying the local properties of the reachable sets of a nonlinear control system can be seen from the following

Proposition 2.3.5 ***(cf. [14], [25] and [26])*** *Let $A_1, A_2, \ldots, A_k$ belong to $E_\alpha^+(x_0)$ for some $\alpha > 0$ and $0 \in int\ co\{A_1(x_0) + A_2(x_0) + \cdots + A_k(x_0)\}$. Then the control system Σ is STLC at the point x_0.*

Proof. According to Definition 2.3.1 there exists vector fields $a_i(t) \in A^0(x_0)$, positive polynomials $p_i(t)$, families of analytic vector fields $O_i(t^{w_i}), w_i > \alpha, i = 1, \ldots, k$ and $T > 0$ such that for each $t \in [0, T]$ and $x \in \mathbf{B}$

$$Exp(t^\alpha A_i + a_i(t) + O_i(t^w))(x) \in \mathcal{R}(x, p_i(t))$$

for some $\alpha > 0$. Also, it's fulfilled that $\|a_i(t,x)\| \leq c_i t^{\theta_i}\|x - x_0\|$ and $\|O_i(t,x)\| \leq c_i^{'} t^{w_i}, i = 1, \ldots, k$. Then

$$L(t_1, \ldots, t_k, x) := Exp(t_1^\alpha A_1 + a_1(t_1) + O_1(t_1^w)) \circ \cdots \circ Exp(t_k^\alpha A_k + a_k(t_k) + O_k(t_k^w))(x)$$

$$\in \mathcal{R}\left(x, \sum_{i=1}^{k} p_i(t_i)\right), \quad t_i \in [0,T], i = 1, \ldots k.$$

Let $y = (y_1, ..., y_n)$ as an arbitrary point in $\mathbf{B}$. Then exist $t(y) = (t_1(y), \ldots, t_k(y))$ such that

$$y = t_1(y)A_1(x_0) + t_2(y)A_2(x_0) + \cdots + t_k(y)A_k(x_0) \tag{2.3}$$

. Applying C-B-H formula, we obtain that

$$L(t_1, \ldots, t_k, x_0) = Exp\left(\sum_{i=1}^{k} t_i^\alpha A_i + a(t_1, \ldots, t_k) + O(t_1^{w_1}, \ldots, t_k^{w_k})\right)(x_0),$$

where

$$\max_{(t_1,\ldots,t_k,x)\in[0,T]^k, x\in\mathbf{B}} \|a(t_1, \ldots, t_k)\| \leq ct^\theta \|x\|$$

and

$$\max_{(t_1,\ldots,t_k,x)\in[0,T]^k, x\in\mathbf{B}} \|o(t_1, \ldots, t_k)\| \leq ct^w.$$

Let us define

$$\pi^*(\delta, y) := L(\delta t_1^{\frac{1}{\alpha}}(y), \ldots, \delta t_k^{\frac{1}{\alpha}}(y), x_0) =$$

$$Exp\left(\sum_{i=1}^{k} \delta^\alpha t_i(y)A_i + a(\delta t_1^{\frac{1}{\alpha}}(y), \ldots, \delta t_k^{\frac{1}{\alpha}}(y)) + O((\delta t_1^{\frac{1}{\alpha}}(y))^{w_1}, \ldots, (\delta t_k^{\frac{1}{\alpha}}(y))^{w_k})\right)(x_0).$$

$$\in \mathcal{R}\left(x, \sum_{i=1}^{k} p_i(\delta t_i^{\frac{1}{\alpha}}(y))\right), \quad \delta \in (0,1).$$

Using that $a(\delta^{\frac{1}{\alpha}} t_1^{\frac{1}{\alpha}}(y), \ldots, \delta^{\frac{1}{\alpha}} t_k^{\frac{1}{\alpha}}(y))(x_0) = 0$ we obtain that $Exp\left(a(\delta^{\frac{1}{\alpha}} t_1^{\frac{1}{\alpha}}(y), \ldots, \delta^{\frac{1}{\alpha}} t_k^{\frac{1}{\alpha}}(y))\right) = x_0$. Then $\pi^*(\delta^{\frac{1}{\alpha}}, y) =$

$$= Exp\left(\sum_{i=1}^{k} \delta t_i(y)A_i + a(\delta^{\frac{1}{\alpha}} t_1^{\frac{1}{\alpha}}(y), \ldots, \delta^{\frac{1}{\alpha}} t_k^{\frac{1}{\alpha}}(y)) + O((\delta t_1(y))^{\frac{w_1}{\alpha}}, \ldots, (\delta t_k(y))^{\frac{w_k}{\alpha}})\right)(x_0) =$$

$$Exp\left(\sum_{i=1}^{k} \delta t_i(y)A_i + a(\delta^{\frac{1}{\alpha}} t_1^{\frac{1}{\alpha}}(y), \ldots, \delta^{\frac{1}{\alpha}} t_k^{\frac{1}{\alpha}}(y)) + O((\delta t_1(y))^{\frac{w_1}{\alpha}}, \ldots, (\delta t_k(y))^{\frac{w_k}{\alpha}})\right) \circ$$

$$Exp\left(-a(\delta^{\frac{1}{\alpha}} t_1^{\frac{1}{\alpha}}(y), \ldots, \delta^{\frac{1}{\alpha}} t_k^{\frac{1}{\alpha}}(y))\right) \circ Exp\left(a(\delta^{\frac{1}{\alpha}} t_1^{\frac{1}{\alpha}}(y), \ldots, \delta^{\frac{1}{\alpha}} t_k^{\frac{1}{\alpha}}(y))\right)(x_0) =$$

$$Exp\left(\sum_{i=1}^{k} \delta t_i(y)A_i + O((\delta t_1(y))^{\frac{w_1}{\alpha}}, \ldots, (\delta t_k(y))^{\frac{w_k}{\alpha}})\right) \circ$$

$$Exp\left(a(\delta^{\frac{1}{\alpha}}t_1^{\frac{1}{\alpha}}(y),\ldots,\delta^{\frac{1}{\alpha}}t_k^{\frac{1}{\alpha}}(y))\right)(x_0)=$$

$$=Exp\left(\sum_{i=1}^{k}\delta t_i(y)A_i+O'((\delta t_1(y))^{\frac{w_1}{\alpha}},\ldots,(\delta t_k(y))^{\frac{w_k}{\alpha}})\right)(x_0)=$$

$$=x_0+\sum_{i=1}^{k}\delta t_i(y)A_i(x_0)+O''(\delta^{\frac{w_1}{\alpha}}t_1(y)^{\frac{w_1}{\alpha}},\ldots,\delta^{\frac{w_k}{\alpha}}t_k(y)^{\frac{w_1}{\alpha}})$$

Using (2.3) and $w=\min\{w_1,\ldots,w_k\}>\alpha$, we obtain that

$$\pi^*(\delta^{\frac{1}{\alpha}},y)-x_0=\delta\sum_{i=1}^{k}t_i(y)A_i(x_0)+O''(\delta^{\frac{w_1}{\alpha}}t_1(y)^{\frac{w_1}{\alpha}},\ldots,\delta^{\frac{w_k}{\alpha}}t_k(y)^{\frac{w_1}{\alpha}})=$$

$$=\delta y+\bar{O}(\delta^{\frac{w}{\alpha}}t_1(y)^{\frac{w_1}{\alpha}},\ldots,\delta^{\frac{w}{\alpha}}t_k(y)^{\frac{w_k}{\alpha}})$$

Let us denote

$$\beta_\delta(y):=\frac{1}{\delta}\left(\pi^*(\delta^{\frac{1}{\alpha}},y)-x_0\right)=y+\frac{1}{\delta}\bar{O}(\delta^{\frac{w}{\alpha}}t_1(y)^{\frac{w_1}{\alpha}},\ldots,\delta^{\frac{w}{\alpha}}t_k(y)^{\frac{w_k}{\alpha}})$$

Then β_δ converges uniformly to the identity map of B as $\delta\to 0$. Therefore $\beta_\delta(B)$ contains a neighborhood of 0, if $\delta>0$ is small enough. Hence the set $\pi^*(\delta^{\frac{1}{\alpha}},B)=x_0+\delta\beta_\delta(B)$ contains a neighborhood of the origin for each sufficiently small δ. The sum $T(\delta,y)=\sum_{i=1}^{k}p_i(\delta^{\frac{1}{\alpha}}t_i^{\frac{1}{\alpha}}(y))$ converges to 0 when $\delta\to 0$ for each $y\in B$ and $\pi^*(\delta^{\frac{1}{\alpha}},y)$ is a solution of Σ with initial point x_0 and time $T(\delta,y)$ for each sufficiently small $\delta>0$. Hence $\mathbf{R}(x_0,T(\delta,B))$ contains a neighborhood of the x_0 and so, the system Σ is STLC at x_0.

Proposition 2.3.6 ***(cf. [14], [25] and [26])*** *The set $E_\alpha^+(x_0)$ is a convex cone.*

Proof. Let A_1 and A_2 belong to E_α^+. According to Definition 2.3.1 there exist vector fields $a_i(t)\in A^0(x_0)$, positive polynomials $p_i(t)$, families of analytic vector fields $O_i(t^{w_i}), w_i>\alpha, i=1,2$ and $T>0$ such that for each $t\in[0,T]$ and $x\in\mathbf{B}$

$$Exp(t^\alpha A_i+a_i(t)+O_i(t^w))(x)\in\mathcal{R}(x,p_i(t))$$

. Let $c>0$ be an arbitrary real number. By setting $\tau:=t.c^{-\frac{1}{\alpha}}$, we obtain that

$$Exp(\tau^\alpha cA_1+a_1(\tau.c^{\frac{1}{\alpha}})+O_1(\tau.c^{\frac{w}{\alpha}}))(x)\in\mathcal{R}(x,p_1(\tau.c^{\frac{1}{\alpha}}))\tag{2.4}$$

for all $x\in\mathbf{B}$ and $\tau\in[0,T.c^{-\frac{1}{\alpha}}]$. Last inclusion implies that $cA_1\in E_\alpha^+$.
Let $T>0$ be sufficiently small such that

$$Exp(t^\alpha A_1+a_1(t)+O_1(t^w))\circ Exp(t^\alpha A_2+a_2(t)+O_2(t^w))(x)\in\mathcal{R}(x,p_1(t)+p_2(t)).$$

Applying C-B-H formula, we obtain that

$$Exp(t^{\alpha}(A_i + A_2) + a(t) + O(t^{w})) \in \mathcal{R}(x, p_1(t) + p_2(t)) \tag{2.5}$$

for suitable $a(t) \in A^0$. and $O(t^w)$.
According to (2.4) and (2.5) we can conclude that E_{α}^{+} is a convex cone.

Proposition 2.3.7 ***(cf. [14], [25] and [26])*** *Let A_1 and A_2 belong to $E_{\alpha}^{+}, \alpha > 0$, $A_1 + A_2 \in A^0$, B belong to $\mathcal{S}^{+} \cap A^0$. Then exists $\beta > \alpha$ such that the Lie brackets $[B, A_1]$ and $[B, A_2]$ belong to E_{β}^{+}.*

Proof. According to definition 2.3.1 there exist vector fields $a_i(t) \in A^0(x_0)$, positive polynomials $p_i(t)$, families of analytic vector fields $O_i(t^{w_i}), w_i > \alpha, i = 1, 2$ and $T > 0$ such that for each $t \in [0, T]$ and $x \in \mathbf{B}$

$$Exp(t^{\alpha} A_i + a_i(t) + O_i(t^{w}))(x) \in \mathcal{R}(x, p_i(t)),$$

where $a_i(t, x) \leq c_i . t^{\theta_i} \|x - x_0\|$ and $O_i(t, x) \leq C_i^{'} t^{w_i}, w_i > \alpha, i = 1, 2$. Let us choose a real number $b > 0$, such that $b > \max\left\{1, \frac{1}{\theta_i}, \frac{1}{w_i - \alpha}\right\}$. The substitution $\tau = t^{\frac{1}{b}}$ and assumption that $T > 0$ is sufficiently small lead to

$$Exp(\tau^{b\alpha} A_1 + a_1(\tau^b) + O_1(\tau^{b\alpha+1})) \circ Exp(\tau B) \circ Exp(\tau^{b\alpha} A_2 + a_2(\tau^b) + O_2(\tau^{b\alpha+1}))(x)$$
$$\in \mathcal{R}(x, p_1(\tau^b) + \tau + p_2(\tau^b))$$

for every $\tau \in [0, T^{\frac{1}{b}}]$ and $x \in \mathbf{B}$.
Applying C-B-H formula, we obtain successively existence of vector fields $\bar{a_1}(t) \in A^0$ and $\bar{O_1}(t)$ such that

$$Exp\left(\tau B + \tau^{b\alpha} A_1 + \frac{\tau^{b\alpha+1}}{2}[A_1, B] + \bar{a_1}(\tau^b) + \bar{O}_1(\tau^{b\alpha+1})\right)$$
$$\circ Exp(\tau^{b\alpha} A_2 + a_2(\tau^b) + O_2(\tau^{b\alpha+1}))(x) \in \mathcal{R}(x, p_1(\tau^b) + \tau + p_2(\tau^b))$$

. Applying one more time C-B-H formula, we obtain that exist vector fields $a(t) \in A^0$ and $O(t)$ such that

$$Exp\left(\tau B + \tau^{b\alpha}(A_1 + A_2) + \frac{\tau^{b\alpha+1}}{2}([A_1, B] + [B, A_2]) + \bar{a}(\tau^b) + \bar{O}(\tau^{b\alpha+1})\right)(x)$$
$$\in \mathcal{R}(x, p_1(\tau^b) + \tau + p_2(\tau^b)) \tag{2.6}$$

. Taking in consideration that $[A_1, B] + [B, A_2] = [A_1 + A_2, B] + 2[B, A_2]$ and $A_1 + A_2 \in A^0, b \in A^0$, we can write (2.6) in the suitable form

$$Exp\left(\tau^{b\alpha+1} A_2 + +a(\tau) + O(\tau^{b\alpha+1})\right)(x) \in \mathcal{R}(x, p(\tau)),$$

where $a(t) \in A^0$ and $O(t)$ are vector fields and $p(\tau) = p_1(\tau^b) + \tau + p_2(\tau^b)$. It means that $[B, A_2] \in E^+_{b\alpha+1}$. It can be proved analogously that $[B, A_1] \in E^+_{b\alpha+1}$.

Following propositions are also related to the elements of E^+_α. We don't present their proofs.

Proposition 2.3.8 (Sussmann, 1978). *Let $A_1, A_2, \dots, A_k$ belong to $E^+_\alpha(x_0)$ for some $\alpha > 0$ and $A_1(x_0) + A_2(x_0) + \dots + A_k(x_0) = 0$. Then $[A_i, A_j]$, $i, j = 1, \dots, k$ belong to $E^+_2(x_0)$.*

Proposition 2.3.9 (Hermes, 1978). *Let A_1 and A_2 belong to $\mathcal{S}$ and $A_1(x_0) + A_2(x_0) = 0$. Then $[A_1, [A_1, A_2]] + [A_2, [A_2, A_1]]$ belongs to $E^+_3(x_0)$.*

Some known results (cf. for example, [36]) give us elements of $E^+_\alpha(x_0)$. Applying the above written assertions we obtain new elements of the set $E^+_\alpha(x_0)$ provide constructions of elements of the $E^+_\beta, \beta > \alpha$.

2.4 Homogeneous polynomial vector fields

The main results in chapter 3 and 4 are related to control systems with right-hand side which is a map whose components are homogeneous polynomials. Because of that some useful properties of homogeneous polynomial vector fields are presented.

It is said that the vector field $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree α iff $\Gamma(\lambda x) = \lambda^\alpha \Gamma(x)$ for each $x \in \mathbb{R}^n$ and for each $\lambda > 0$. Because the vector fields f and g_{u_i}, $i = 1, \dots, \mu$, are homogeneous of degree two and zero, respectively, we present below some properties of the Lie brackets of homogeneous vector fields:

Lemma 2.4.1 *Let $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ and $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ be homogeneous polynomial vector fields of degrees α and β respectively. Then the Lie bracket $[\Gamma, \Lambda]$ is a homogeneous polynomial vector field of degree $\alpha + \beta - 1$ or Λ is identical 0.*

Proof. Indeed, $[\Gamma, \Lambda](\lambda x) = \Lambda'(\lambda x)\Gamma(\lambda x) - \Gamma'(\lambda x)\Lambda(\lambda x) = \lambda^{\beta-1+\alpha}\Lambda'(x)\Gamma(x) - \lambda^{\alpha-1+\beta}\Gamma'(x)\Lambda(x) = \lambda^{\alpha+\beta-1}[\Gamma, \Lambda](x)$. $\Diamond$

Corollary 2.4.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous polynomial vector field of degree two and $g : \mathbb{R}^n \to \mathbb{R}^n$ be a constant vector field. Let Λ be a Lie bracket in f and g which involves k times the vector field f and m times - the vector field g. Then Λ is a homogeneous vector field of degree $k - m + 1$ or Λ it is identical* 0.

Proof. The assertion holds true for $k = 1$ and $m = 0$ as well as for $k = 0$ and $m = 1$. Let us assume that this assertion holds true for some nonnegative values of k and m.

Let $\Gamma = [\Lambda, g]$, where Let Λ is a Lie bracket in f and g which involves k times the vector field f and m times - the vector field g. Then Λ is a homogeneous vector field which (according to our inductive assumption) is or identically zero or a homogeneous vector field of degree $k - m + 1$. Then, according to Lemma 2.4.1, Γ is or identically zero or a homogeneous vector field of degree $k - m + 1 + 0 - 1 = k - (m + 1) - 1$.

Let now $\Gamma = [\Lambda, f]$, where Let Λ is a Lie bracket in f and g which involves k times the vector field f and m times - the vector field g. Then Λ is a homogeneous vector field which (according to our inductive assumption) is or identically zero or a homogeneous vector field of degree $k - m + 1$. Then, according to Lemma 2.4.1, Γ is or identically zero or a homogeneous vector field of degree $k - m + 1 + 2 - 1 = (k + 1) - m + 1$.

$\Diamond$

Corollary 2.4.3 *Let Λ be a Lie bracket in f and g which involves k times the vector field f and m times the vector field g, and Λ is homogeneous of degree 1. Then $k = m$. Also, all Lie brackets in f and g which are homogeneous of degree zero are of odd length.*

Proof. Let Λ be a Lie bracket in f and g which involves k times the vector field f and m times - the vector field g, and let Λ be homogeneous of first degree. According to Corollary 2.4.2, we have that $k - m + 1 = 1$, i.e. $k = m$, and hence Λ is a Lie bracket in f and g of length $2k$, i.e. its length is an even number, or it is identically zero.

Let now Λ be a Lie bracket in f and g which involves k times the vector field f and m times - the vector field g, and let Λ be homogeneous of degree zero. According to Corollary 2.4.2, we have that $k - m + 1 = 0$, i.e. $m = k + 1$, and hence Λ is a Lie bracket in f and g of length $2k + 1$, i.e. its length is an odd number, or it is identically zero.

◊

Corollary 2.4.3 has an essential role in considered problems in chapter 3 and 4. It define whoch of the Lie brackets are constant but not zero. Exactly these brackets have to be neutralized with purpose to find new elements of E_{α}^{+}

Chapter 3

The classical approach of Sussman and a sufficient condition for small-time local controllability

3.1 A general geometrical approach

In this section we present briefly the classical Sussman's approach to study local controllability of a nonlinear control system.

Let us consider the following affine control system Σ_a in $\mathbb{R}^n$

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{3.1}$$

$$x(0) = 0,\ u(t) = (u_1, \dots, u_m) \in U \cap \bar{\mathbf{B}},$$

where vector fields $f_i, i = 0, \dots, m$ are C^∞, $f_0(0) = 0$ and $U = [-1, 1]^m$ and $\bar{\mathbf{B}}$ is the closed unit ball of $\mathbb{R}^m$ centered at the origin. As mentioned earlier the properties of the Lie algebra generated by the vector fields $f_i, i = 0, \dots, m$ are of great importance to controllability of the considered system.

Following [34] and [36] we consider an abstract control system: Let $X_0, X_1, X_2,$ $, \dots, X_m$ be m symbols (called "indeterminates"). We set $\vec{X} = (X_0, X_1, \dots, X_m)$ and fix a sufficiently large positive integer N. By $\mathcal{A}^N(\vec{X})$ we denote the free nilpotent associative algebra of order $N+1$: If $I = (i_1, \dots, i_k)$ is any finite sequence with $i_j \in \{0, 1\}$, then we denote by $\|I\|$ its length k and set $X_I := X_{i_1} \cdots X_{i_k}$. We let $X_\emptyset := 1$. If $I \circ J$ denotes the concatenation of I and J, then the multiplication in $\mathcal{A}^N(\vec{X})$ is given by $X_I\, X_J := X_{I \circ J}$ whenever $\|I\| + \|J\| \leq N$. If $\|I\| + \|J\| > N$, then the product $X_I X_J$ is set equal to zero. Then the basis of $\mathcal{A}^N(\vec{X})$ consists of all monomial X_I of length less than or equal to N.

We denote by $\mathcal{L}^N(\vec{X})$ the nilpotent Lie subalgebra of $\mathcal{A}^N(\vec{X})$ generated by $X_0, X_1, \ldots, X_m$ with the Lie bracket defined by

$$[P, Q] \ := \ PQ \ - \ QP.$$

The elements of $\mathcal{L}^N(\vec{X})$ will be referred to as *Lie polynomials* in $X_0, X_1, \ldots, X_m$. We apply very often the Campbell-Baker-Hausdorff formula (C-B-H formula) which says that if A and B are Lie polynomials, then there exists a Lie polynomial C such that

$$\exp\,(A)\, \exp\,(B)\ = \exp\,(C).$$

Here $\exp\,(P) \ := \ 1 + \sum_{i=1}^{N} \frac{P^i}{i!}$ for each Lie polynomial P. Let us remind that the C-B-H formula up to order three is

$$C \ = \ A \ + \ B \ + \ \frac{1}{2}\,[A, B] \ + \ \frac{1}{12}\,[A, [A, B]] \ + \ \frac{1}{12}\,[B, [B, A]] \ + \ \cdots$$

Let us define $\mathcal{G}^N(\vec{X})$ to be the set

$$\mathcal{G}^N(\vec{X}) \ = \ \left\{ \exp\,(A) : \ A \in \mathcal{L}^N(\vec{X}) \right\}.$$

Then, because of the C-B-H formula, $\mathcal{G}^N(\vec{X})$ is a group.

Following [33], we consider the following control system on $\mathcal{A}^N(\vec{X})$:

$$\dot{S}(t) = S(t)(X_0 + u(t)X_1), \text{ where } u(t) \in \mathcal{U} \ \text{ and } \ S(0) = 1. \tag{3.2}$$

Let us remind that by $\mathcal{U}$ we have denoted the set of all admissible controls, i.e. the set of all Lebesgue integrable functions u whose domain is a compact interval of the form $[0, T]$, $T > 0$, and $u(t) \in [-1, 1]$ for almost every t from $[0, T]$. The time T will be referred to as the terminal time of u and will be denoted by $T(u)$. If $u_i : [0, T(u_i)] \to [-1, 1]$, i=1,2, are admissible controls, then by $u_2 \circ u_1$ we denote an element of $\mathcal{U}$ with $T(u_2 \circ u_1) = T(u_2) + T(u_1)$ and defined as follows:

$$u_2 \circ u_1(t) := \begin{cases} u_1(t) & \text{for } t \in [0, T(u_1)), \\ u_2(t - T(u_1)) & \text{for } t \in [T(u_1), T(u_1) + T(u_2)). \end{cases} \tag{3.3}$$

It is proved in [33] that for each control $u \in \mathcal{U}$ which is defined on the interval $[0, T(u)]$, the solution $S(u)$ of (3.2) satisfying $S(u)(0) = 1$ is well defined on $[0, T(u)]$ and

$$S(u)(t) = \sum_{\|I\| \le N} s_I(u)(t) X_I, \ \forall\, t \in [0, T(u)],$$

where $s_\emptyset(u)(t) := 1$ and for each $I = (i_1, i_2, \ldots, i_k)$ with $i_j \in \{0,1\}$, $j = 1, \ldots, k$,

$$s_I(u)(t) := \int_0^t \int_0^{t_k} \int_0^{t_{k-1}} \cdots \int_0^{t_2} u^{i_k}(\tau_k) u^{i_{k-1}}(\tau_{k-1}) \cdots u^{i_2}(\tau_2) u^{i_1}(\tau_1)\, d\tau_1 \cdots d\tau_k$$

(here $u^0(t) = 1$ and $u^1(t) = u(t)$ for each $t \in [0, T(u)]$). We define $\mathrm{Ser}(u)$ to be $S(u)(T(u))$.

The reachable set $\mathcal{R}^N_{\vec{X}}(T)$ of (3.2) at time $T > 0$ is defined as the set of all points of $\mathcal{A}^N(\vec{X})$ that can be reached in time T by means of solutions of (3.2) starting from 1. Some properties of the control system (3.2) are presented in more details in [33]. Here, we shall remind only one corollary of Lemma 3.1 in [33]:

$$\text{Ser } (u_1 \circ u_2) = \text{ Ser } (u_1) \text{ Ser } (u_2) \tag{3.4}$$

for every two admissible controls u_1 and u_2. Also,

$$\text{if} \quad \exp\,(A_i) \in \mathcal{R}^N_{\vec{X}}(T_i) \quad \text{for } i = 1, \ldots, k, \quad \text{then}$$

$$\exp\,(A_1) \;\cdot\; \exp\,(A_2) \;\cdots\; \exp\,(A_k) \;\in\; \mathcal{R}^N_{\vec{X}}(T_1 + T_2 + \cdots + T_k).$$

Let us remind that $\mathcal{L}(\vec{X})$ denotes the free Lie algebra generated by the indeterminates $X_0, X_1, \ldots, X_m$, and let Λ be a Lie bracket belonging to $\mathcal{L}(\vec{X})$. We denote by $\Lambda(\vec{f})$ that Lie bracket in $f_0, f_1, \ldots, f_m$ which is obtained from Λ by substituting each $X_0, X_1, \ldots, X_m$ by f_0, f_1, f_m, respectively. Also, we set $\left(\sum_{i=1}^{k} \alpha_i \Lambda_i\right)(\vec{f}) := \sum_{i=1}^{k} \alpha_i \Lambda_i(\vec{f})$ for each Lie brackets Λ_i in $X_0, X_1, \ldots, X_m$ and each real numbers α_1, $i = 1, \ldots, k$. If S is a subset of $\mathcal{L}(\vec{X})$, then by span S we denote the minimal linear subspace of $\mathcal{L}(\vec{X})$ containing the elements of S,

$$S(\vec{f}) := \left\{\Lambda(\vec{f}) : \Lambda \in S\right\} \quad \text{and} \quad S(\vec{f})(x_0) := \left\{\Lambda(\vec{f})(x_0) : \Lambda \in S\right\}.$$

At last, by $\mathcal{B}(\vec{X})$ we denote the set of all Lie brackets in $X_0, X_1, \ldots, X_m$ of odd length in which each $X_i, i = 1, \ldots, m$ appears an even number of times. We call the elements of $\mathcal{B}(\vec{X})$ “bad Lie brackets”. The main idea of the obtained sufficient conditions in [3], [7], [33] and [36] is that the elements of $\mathcal{B}(\vec{X})$ have to be “neutralized” in order not to be obstructions for small-time local controllability. Also, we define a set of “good” elements of the set Π as follows:

$$\text{Good } := \mathcal{L}(\vec{X}) \;\setminus\; \mathcal{B}(\vec{X}),$$

i.e. good elements of the set $\mathcal{L}(\vec{X})$ are those elements of $\mathcal{L}(\vec{X})$ that are not bad Lie brackets.

Let us fix a vector $r = (r_0, r_1, \ldots, r_m)$ whose components are positive integers such that $1 \leq r_0 \leq r_i, i = 1, \ldots, m$. We set

$$\|\Lambda\|_r \ := \ r_0|\Lambda|_0 + \sum_{i=1}^{m} r_i|\Lambda|_i, \qquad \text{for every } \Lambda \in \mathcal{L}(\vec{X}),$$

where the number of times that X_i, $i = 0, 1, \ldots, m$, appears in Λ is denoted by $|\Lambda|_i$. The number $|\Lambda|_i$ is called degree of Λ with respect to X_i, $i = 0, 1, \ldots, m$. Clearly, the length $\|\Lambda\|$ of Λ is equal to $|\Lambda|_0 + |\Lambda|_1$. The positive numbers $\|\Lambda\|_r$ and $\|\Lambda\|_r^\sigma$ are called *"r-weight"* of the Lie bracket Λ.

For each positive number δ we define the sets

$$\overline{\mathcal{L}}_r^\delta = \left\{\Lambda \in \mathcal{L}(\vec{X}) : \ \|\Lambda\|_r = \delta\right\}.$$

and

$$\mathcal{L}_r^\delta = \left\{\Lambda \in \mathcal{L}(\vec{X}) : \ \|\Lambda\|_r \leq \delta\right\}.$$

Let Λ be a Lie bracket belonging to $\mathcal{L}(\vec{X})$. It is said that Λ_0 can be *r-neutralized* if

$$\Lambda_0(\vec{f}(x_0)) \ \in \ \text{span} \ \left\{\Lambda(\vec{f})(x_0) : \ \Lambda \in \mathcal{L}(\vec{X}) \text{ with } \ \|\Lambda\|_r < \|\Lambda_0\|_r\right\}.$$

Also, if $\Lambda_0(\vec{f})(x_0) = 0$, then Λ_0 is r-neutralized.

Now we shall formulate two classical results:

Theorem 3.1.1 *(Hermes controllability condition, cf. Theorem 2.1 in [34]) We consider the control system Σ with $m = 1$. We assume that*

1) *dim $\mathcal{L}(\vec{X})(\vec{f})(x_0) = n$;*
2) *$\mathcal{S}^k(\vec{X})(\vec{f})(x_0) = \mathcal{S}^{k+1}(\vec{X})(\vec{f})(x_0)$ whenever k is odd (here $\mathcal{S}^k(\vec{X})$ denotes the linear span of all Lie brackets in X_0 and X_1 which involve X_1 at most k-times).*

Then the control system Σ is STLC at x_0.

Theorem 3.1.2 *([36]) We assume that*

1) *dim $\mathcal{L}(\vec{X})(\vec{f})(x_0) = n$;*
2) *if Λ is a bad Lie bracket, then it can be r-neutralized*

Then the control system Σ is STLC at x_0.

3.2 A class of polynomial control systems

Let us consider the following control system Σ_1 in $\mathbb{R}^n$

$$\dot{x}(t) = f(x(t)) + u(t), \tag{3.5}$$

$$x(0) = 0, \ u(t) \in U \cap \bar{\mathbf{B}}$$

where U is a closed convex cone in $\mathbb{R}^n$, $\bar{\mathbf{B}}$ is the closed unit ball of $\mathbb{R}^n$ centered at the origin and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map whose components are polynomials which are homogeneous of second degree, i.e. $f(\lambda x) = \lambda^2 f(x)$ for each $\lambda > 0$ and each $x \in \mathbb{R}^n$.

Unfortunately, the general sufficient conditions from the previous section (Theorem 3.1.1 and Theorem 3.1.2) are not applicable to the considered control system (4.1). The reason of that is the fact that there exist "bad" Lie brackets that can not be neutralized in the sense of Sussmann.

The following control system Σ_A is studied in [1]:

$$\left| \begin{array}{l} \dot{x} = u \\ \dot{y} = q_1(x) + q_2(y), \end{array} \right. \tag{3.6}$$

where the state variable is $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^r$, $u \in \mathbb{R}^m$ is the control variable, and $q_1 : \mathbb{R}^m \to \mathbb{R}^r$ and $q_2 : \mathbb{R}^r \to \mathbb{R}^r$ are homogeneous quadratic polynomials, i.e. $q_1(\lambda x) = \lambda^2 q_1(x)$ and $q_2(\lambda y) = \lambda^2 q_2(y)$ for each $\lambda > 0$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^r$. We define the sets Q_1 and Q_2 as follows:

$$Q_1 := \{q_1(u) : \ u \in \mathbb{R}^m\}, \ Q_2 := \{q_2(y) : \ \pm y \in Q_1\}.$$

and denote by cone S the smallest convex closed cone containing the set $S \subset \mathbb{R}^r$. Then the result in [1] can be formulated as follows:

Theorem 3.2.1 *If*

$$\text{cone } Q_1 + \text{cone } Q_2 = \mathbb{R}^r,$$

then the system (3.6) is small-time locally controllable at the origin.

The result of Agular is considered as a corollary of the sufficient condition in the next section.

In order to show the main idea of our approach, we consider the following

Example 3.2.1 *Let us consider the following 2-dimensional control system Σ_2:*

$$\dot{x}(t) = u,\ x(0) = 0,\ u \in [-1, 1],$$
$$\dot{y}(t) = v - x^2,\ y(0) = 0,\ v \in [0, 1],$$

The below formulated Proposition 2.3.5, Proposition 2.3.6 and Proposition 2.3.7 imply that the system Σ_2 is small-time local controllable at the origin. Unfortunately, the sufficient condition in [1] is not applicable because set of admissible values of the controls should be symmetric with respect to the origin.

According to Proposition 2.3.5, it is sufficient to find a finite number of elements of the intersection of the sets $\mathcal{L}_2$ and $E^+_\alpha(0)$, for some $\alpha > 0$, that contain the origin of $\mathbb{R}^n$ in the interior of its convex convex hull.

3.3 Preliminaries

We introduce some notations. If u is an arbitrary element of $U \cap \bar{\mathbf{B}}$, then we denote by g_u the constant vector field defined by $g_u(x) := u$ for each $x \in \mathbb{R}^n$. Let us assume that the linear span $\mathcal{M}_0$ is generated by the elements $u_i \in U \cap \bar{\mathbf{B}}$, $i = 1, \ldots, \mu$, and let us denote by $\mathcal{L} = \mathcal{L}(f, g_{u_1}, \ldots, g_{u_\mu})$ the Lie algebra generated by the vector fields f and g_{u_i}, $i = 1, \ldots, \mu$. By $\mathcal{L}_{const}$ we denote the Lie subalgebra of the constant vector fields contained in $\mathcal{L}$.

Let us assume that the linear span $\mathcal{M}_0$ is generated by the elements $u_i \in U \cap \bar{\mathbf{B}}$, $i = 1, \ldots, \mu$, and let us denote by $\mathcal{L} = \mathcal{L}(f, g_{u_1}, \ldots, g_{u_\mu})$ the Lie algebra generated by the vector fields f and g_{u_i}, $i = 1, \ldots, \mu$. By $\mathcal{L}_{const}$ we denote the Lie subalgebra of the constant vector fields contained in $\mathcal{L}$. It is proved by Jurdjevic and Kupka (cf. [18]) that $\mathcal{L}_{const}$ is the smallest vector space of constant vector fields which contains the vector fields g_{u_i}, $i = 1, \ldots, \mu$, and which, in addition, contains all vector fields $[g_v, [g_w, f]]$, where g_v and g_w are arbitrary elements of $\mathcal{L}_{const}$.

We denote by $\mathbb{A} : \mathcal{L} \to \mathcal{L}$ the automorphism defined by $\mathbb{A}(f) = f$ and $\mathbb{A}(g_{u_i}) = -g_{u_i}$, $i = 1, \ldots, \mu$. We also use the "time reversal" map $\mathbb{T}$ defined by Sussmann in [36]. The maps $\mathbb{A}$ and $\mathbb{T}$ are linear, i.e. if $\mathbb{C} \in \{\mathbb{A}, \mathbb{T}\}$ and $\sum_{i \in I} \alpha_i \Lambda_i$ is a Lie series in Lie brackets of $\mathcal{L}$, then

$$\mathbb{C}\left(\textstyle\sum_{i \in I} \alpha_i \Lambda_i\right) = \textstyle\sum_{i \in I} \alpha_i \mathbb{C}(\Lambda_i).$$

and $\mathbb{C}(Exp(\mathcal{S})) = Exp(\mathbb{C}(\mathcal{S}))$ for each Lie series $\mathcal{S}$ of Lie brackets in f and g_{u_i}, $i = 1, \ldots, \mu$. Also, we have that $\mathbb{T}(\Lambda) = (-1)^{k+1}\Lambda$ for each Lie bracket Λ of $\mathcal{L}$ of length k. Further we shall use the following groups of automorphisms $\Theta^\pm :=$

$\{Id, \mathbb{A}, \mathbb{T}, \mathbb{A}\mathbb{T}\}$ and $\Theta := \{Id, \mathbb{T}\}$, where Id denotes the identity map. Clearly, the Lie brackets Λ of $\mathcal{L}$ that are invariant with respect to the action of the group Θ are of odd length while the Lie brackets Λ of $\mathcal{L}$ that are invariant with respect to the action of the group $\Theta^{\pm}$ are of odd length and each g_{u_i}, $i = 1, \ldots, \mu$, appears in Λ even number of times.

Let us fix a compact neighborhood Ω_0 of the origin and set $\Omega := 2\Omega_0$. Then there exists $T_0 > 0$ so that each trajectory x of Σ starting from a point $x_0 \in \Omega_0$ and corresponding to some admissible control from $\mathcal{U}_T$ with $T \leq T_0$ is well defined on the interval $[0, T]$ and remains in Ω, i.e. $x(t) \in \Omega$ for all $t \in [0, T]$. Then the following finite composition of exponents

$$Exp(t_1(f + g_1)) \circ Exp(t_2(f + g_2)) \circ \cdots \circ Exp(t_k(f + g_k))(x)$$

is well defined for each $x \in \Omega_0$, each $t_i > 0$ with $T := t_1 + \cdots + t_k \leq T_0$ and each $g_i \in \{g_u : \ u \in U \cap \bar{\mathbf{B}}\}$, $i = 1, \ldots, k$. Without loss of generality we may think that $T_0 > 0$ is sufficiently small so that (according to the C-B-H formula) there exists a Lie series $\mathcal{S}$ such that

$$Exp(\mathcal{S}) \cong Exp(t_1(f + g_1)) \circ Exp(t_2(f + g_2)) \circ \cdots \circ Exp(t_k(f + g_k)). \tag{3.7}$$

We call $Exp(\mathcal{S})$ **an admissible flow of** Σ. Clearly,

$$Exp(\mathcal{S})(x) \in \mathcal{R}(x, T). \tag{3.8}$$

According to the properties of the automorphism $\mathbb{T}$ (cf. [36]), we have that

$$\mathbb{T}\left(Exp(t_1(f + g_1)) \circ \cdots \circ Exp(t_k(f + g_k))\right) = Exp(t_k(f+g_k)) \circ \cdots \circ Exp(t_1(f+g_1))$$

Remind that the set $\mathcal{M}_0$ is a linear span generated by the vectors $g_{u_1}(0), g_{u_2}(0), \ldots, g_{u_\mu}(0)$, $i = 1, \ldots, \mu$. If the vector fields g_i, $i = 1, \ldots, k$, belong to the set $\{g_{u_1}, g_{u_2}, \ldots, g_{u_\mu}\}$, then we have that

$$\mathbb{A}\left(Exp(t_1(f + g_1)) \circ \cdots \circ Exp(t_k(f + g_k))\right) = Exp(t_1(f-g_1)) \circ \cdots \circ Exp(t_k(f-g_k))$$

$$\text{and } \mathbb{A}\left(Exp(\mathcal{S})\right)(x) \in \mathcal{R}(x, T). \tag{3.9}$$

As in the previous section we define "a weight" in the Lie algebra $\mathcal{L}$. Let us fix an arbitrary vector $r := (p, q)$ whose components are positive reals satisfying the inequalities $1 \leq p \leq q$. Let Λ be an arbitrary Lie bracket in the vector fields f and g_{u_i}, $i = 1, \ldots, \mu$. We define its r-weight as follows:

$$\|\Lambda\|_r := p|\Lambda|_0 + \textstyle\sum_{i=1}^{\mu} q|\Lambda|_i,$$

where by $|\Lambda|_0$ it is denoted the number of times that the vector field f appears in Λ, and by $|\Lambda|_i$ $i = 1, \ldots, \mu$, it is denoted the number of times that the vector field g_{u_i} appears in Λ. If Λ and Γ are arbitrary Lie brackets of $\mathcal{L}$, then one can directly check that $\|[\Lambda, \Gamma]\|_r = \|\Lambda\|_r + \|\Gamma\|_r$. Each expression of the form

$$\Lambda^{w,\sigma}(\varepsilon) := \sum_{j=1}^{J} \alpha_j \varepsilon^{\|\Lambda_j\|_r} \Lambda_j \text{ with } \|\Lambda_j\|_r \in [w, \sigma],\ \alpha_j \in \mathbb{R},\ \text{and } \varepsilon \in (0,1),$$

is called **an admissible Lie polynomial in ε (with respect to the weight $\|\cdot\|_r$)**. We set

$$Bra(\Lambda^{w,\sigma}) := \{\Lambda_j :\ j = 1, \ldots, J\}.$$

Let $\tilde{\Theta}$ be a group of automorphisms defined on $\mathcal{L}$ and Λ be a Lie bracket. We say that Λ is invariant (not invariant) with respect to $\tilde{\Theta}$ if $\mathcal{C}(\Lambda) = \Lambda$ for each ($\mathcal{C}(\Lambda) \neq \Lambda$ for some) $\mathcal{C} \in \tilde{\Theta}$. We say that $\Lambda^{w,\sigma}(\varepsilon)$ is invariant (not invariant) with respect to $\tilde{\Theta}$ if all elements of $Bra(\Lambda^{w,\sigma})$ are invariant (not invariant) with respect to $\tilde{\Theta}$.

Further we use the following corollary of Proposition 5.1 from [36]:

Proposition 3.3.1 *Let $\tilde{\Theta}$ be one of the group of automorphisms Θ or $\Theta^{\pm}$ and $\varepsilon_0 > 0$. Let $Exp(\mathcal{S}(\varepsilon))$ be an arbitrary admissible flow of Σ such that*

$$Exp(\mathcal{S}(\varepsilon))(x) \in \mathcal{R}(x, T(\varepsilon)),$$

where $\mathcal{S}(\varepsilon) = \Lambda^{1,w-1}_{inv}(\varepsilon) + \Lambda^{1,\sigma-1}_{not\ inv}(\varepsilon) + \Lambda^{w,\sigma-1}_{inv}(\varepsilon) + O(\varepsilon^{\sigma})$ with $2 \leq w \leq \sigma - 1$, $\Lambda^{1,w-1}_{inv}(\varepsilon)$, $\Lambda^{1,\sigma-1}_{not\ inv}(\varepsilon)$ and $\Lambda^{w,\sigma-1}_{inv}(\varepsilon)$ are admissible Lie polynomials in ε (with respect to the weight $\|\cdot\|_r$), $T(\varepsilon)$ is a positive real polynomial with $T(\varepsilon) \in (0, T_0)$ for $\varepsilon \in (0, \varepsilon_0)$, and x belongs to a neighborhood Ω_0 of the origin. Also, we assume that $\Lambda^{1,w-1}_{inv}(\varepsilon)$ and ${\Lambda_{inv}}^{w,\sigma-1}$ are invariant, but $\Lambda^{1,w-1}_{not\ inv}(\varepsilon)$ is not invariant with respect to $\tilde{\Theta}$. Then there exist a positive integer m and an admissible flow $Exp(\mathcal{S}_{inv}(\varepsilon))$ such that

$$Exp(\mathcal{S}_{inv}(\varepsilon))(x) \in \mathcal{R}(x, mT(\varepsilon)) \text{ for each } \varepsilon \in (0, \varepsilon_0) \text{ for which } mT(\varepsilon) \in (0, T_0),$$

where

$$\mathcal{S}_{inv}(\varepsilon) = m\Lambda^{1,w-1}_{inv}(\varepsilon) + \bar{\Lambda}^{1,w-1}_{inv}(\varepsilon) + {\bar{\Lambda}_{inv}}^{w,\sigma-1}(\varepsilon) + \bar{O}(\varepsilon^{\sigma}),$$

$\bar{\Lambda}^{1,w-1}_{inv}(\varepsilon)$ and ${\bar{\Lambda}_{inv}}^{w,\sigma-1}(\varepsilon)$ are admissible Lie polynomials in ε (with respect to the weight $\|\cdot\|_r$) that are invariant with respect to $\tilde{\Theta}$. Moreover, $\bar{\Lambda}^{1,w-1}_{inv}$ is a sum of Lie brackets of the elements of $Bra(\Lambda^{1,w-1}_{inv})$ and $Bra(\Lambda^{1,\sigma-1}_{not\ inv})$.

Our aim is to study "bad" brackets in the particular case of homogeneous polynomials of second degree.

3.4 Sufficient condition

In order to formulate our main result we need some notations: Let rec C be the largest linear space contained in the convex closed cone C. Next, we define the following sets:

1. $\mathcal{K}_0 = \ U$ and $\mathcal{M}_0 = rec\ \mathcal{K}_0$;
2. $\mathcal{K}_1 = cone\ \ (\{f(u) : u \in \mathcal{M}_0\} \cup U)$ and $\mathcal{M}_1 = rec\ \mathcal{K}_1$;
3. $\mathcal{K}_2 = cone\ \{f(u) : u \in \mathcal{M}_1\}$.

Theorem 3.4.1 *If the convex cone $\mathcal{K}_1 + \mathcal{K}_2$ coincides with $\mathbb{R}^n$, then the control system Σ is small-time locally controllable at the origin.*

Remark 3.4.2 *On the assumptions of Theorem 1 (note that the controls are unbounded) we have that*

$$\mathcal{K}_0 = \mathcal{M}_0 = U = \mathbb{R}^m \times \{\mathbf{0}\},\ \mathcal{K}_1 = cone\ \ (\{f(u) : u \in \mathbb{R}^m\} \cup U) =$$

$$= cone\ \ \left(\{(0, q_1(u))^T : u \in \mathbb{R}^m\} \cup U\right) = \mathbb{R}^m \times cone\ Q_1,$$

$$\mathcal{M}_1 = rec\ \mathcal{K}_1 = \mathbb{R}^m \times rec\ cone\ Q_1$$

$$\mathcal{K}_2 = cone\ \{f(u) : u \in \mathcal{M}_1\} \supseteq cone\ \{(0, q_2(y))^T : \pm y \in \ Q_1\} = \{\mathbf{0}\} \times cone\ Q_2.$$

Clearly, the equality cone Q_1 + cone $Q_2 = \mathbb{R}^r$ implies the equality $\mathcal{K}_1 + \mathcal{K}_2 = \mathbb{R}^{m+r}$, and hence Theorem 3.2.1 is a corollary of Theorem 3.4.1. There are simple examples of control systems that are STLC (according to Theorem 3.4.1), but their small-time local controllability does not follow from Theorem 3.2.1.

3.5 Proof of the sufficient condition

Remind that each trajectory x of Σ starting from a point $x_0 \in \Omega_0$ and corresponding to some admissible control from $\mathcal{U}_T$ with $T \le T_0$ is well defined on the interval $[0, T]$ and remains in Ω.

Further we will use the notation E^+_α for $E^+_\alpha(0)$.

Let us fix an arbitrary element $u \in U \cap \bar{\mathbf{B}}$. Then, the vector field $f + g_u$ is admissible for the control system Σ. From here, using that $f(0) = \mathbf{0}$, we obtain that the vector field g_u belongs to the set E^+_1. Also, if $u \in \mathcal{M}_0$, then the vector fields $f \pm g_u$ are admissible for the control system Σ. We set $r := (p, q)$ with $p = q = 2$ and define the weight $\| \cdot \|_r$. Let

$$\Upsilon_1^u(\varepsilon) := \text{Exp }\left(\varepsilon^2(f+g_u)\right) \circ \text{Exp }\left(\varepsilon^2(f-g_u)\right).$$

Clearly,

$$\Upsilon_1^u(\varepsilon)(x) \in \mathcal{R}(x, 2\varepsilon^2) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and each } \varepsilon \in \left(0, \sqrt{\frac{T_0}{2}}\right).$$

Applying the C-B-H formula, we obtain that

$$\Upsilon_1^u(\varepsilon) \cong \text{Exp }\left(2\varepsilon^2 f + \varepsilon^4[g_u, f] + \frac{\varepsilon^6}{3}[g_u,[g_u,f]] + \Lambda_u^{7,13}(\varepsilon) + O_u^1(\varepsilon^{14})\right)$$

for some admissible Lie polynomial $\Lambda_u^{7,13} \in \mathcal{L}(f, g_u)$ in ε (with respect to $\|\cdot\|_r$). Clearly, the vector fields f and $[g_u,[g_u,f]]$ are invariant with respect to the group of automorphisms $\Theta^{\pm}$. According to Proposition 3.3.1, there exists a positive integer m so that

$$\Upsilon_m^u(\varepsilon) \cong \text{Exp }\left(2m\varepsilon^2 f + m\frac{\varepsilon^9}{3}[g_u,[g_u,f]] + \bar{\Lambda}_u^{7,13}(\varepsilon) + O_u(\varepsilon^{14})\right) \tag{3.10}$$

and

$$\Upsilon_m^u(\varepsilon)(x) \in \mathcal{R}(x, 2m\varepsilon^2) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and each } \varepsilon \in \left(0, \sqrt{\frac{T_0}{2m}}\right), \tag{3.11}$$

where $\bar{\Lambda}_u^{7,13}$ is an invariant with respect $\Theta^{\pm}$ admissible Lie polynomial in ε (with respect to $\|\cdot\|_r$). One can directly calculate that the invariant Lie brackets Λ_u^j appearing in $\bar{\Lambda}_u^{7,13}$ contain two times the vector field g_u and three times the vector field f. According to Lemma 2.4.1, all these brackets Λ_u^j are homogeneous of second degree. Hence

$$\bar{\Lambda}_u^{7,13}(\varepsilon) = \sum_{j=1}^{J_u} \alpha_j\ \varepsilon^{10}\ \Lambda_u^j =: \varepsilon^{10}\bar{\Gamma}_u^{10}, \tag{3.12}$$

where each Lie bracket Λ_u^j contains two times the vector field g_u and three times the vector field f. By setting

$$a(\varepsilon) := 2m\varepsilon^2 f + \varepsilon^{10}\bar{\Gamma}_u^{10},$$

we obtain from (3.10) and (3.11) that $[g_u,[g_u,f]] \in E_6^+$. Because u is an arbitrary element of the set $U \cap \bar{\mathbf{B}} \cap \mathcal{M}_0$, it follows that the vector field $[g_u,[g_u,f]]$ belongs to the set E_6^+ for each $u \in U \cap \bar{\mathbf{B}} \cap \mathcal{M}_0$.

Let $\bar{h} \neq 0$ belong to $\mathcal{K}_2$. According to the definition of the set $\mathcal{K}_2$, there exists $h \in \mathcal{M}_1$ such that $\bar{h} = f(h) = [g_h,[g_h,f]](0)/2$. Because $h \in \mathcal{M}_1$ there exist positive reals $\beta_j > 0$ and $u_j \in \mathcal{M}_0 \cap U \cap \bar{\mathbf{B}}$, $j = 0, 1, 2, \ldots, k$, such that $h_j := f(u_j) = \big[g_{u_j}, \big[g_{u_j}, f\big]\big]\,(0)/2$, $j = 1, 2, \ldots, k$, and

$$h = \beta_0 g_{u_0} + \sum_{j=1}^{k} \beta_j h_j. \tag{3.13}$$

According to the definition of $\mathcal{M}_0$, the vector fields $f \pm g_{u_j}$, $j = 1, 2, \ldots, k$, are admissible for Σ. Also, the definition of $\mathcal{M}_1$ implies the existence of some positive reals $\beta_j^- > 0$ and $u_j^- \in \mathcal{M}_0 \cap U \cap \bar{\mathbf{B}}$, $j = 0, 1, 2, \ldots, k^-$, such that $h_j^- := f(u_j^-) = \left[g_{u_j^-}, \left[g_{u_j^-}, f\right]\right](0)/2$, $j = 1, 2, \ldots, k^-$, and

$$-h = \beta_0^- g_{u_0^-} + \sum_{j=1}^{k^-} \beta_j^- h_j^-. \tag{3.14}$$

According to the definition of $\mathcal{M}_0$, the vector fields $f \pm g_{u_j^-}$, $j = 1, 2, \ldots, k^-$, are admissible for Σ.

We set $\bar{u}_1 := (u_1, u_2, \ldots, u_k)$. Taking into account (3.13), we define the map

$$\Phi^{\bar{u}_1}(\varepsilon) := \Upsilon_m^{u_1}(\sqrt[6]{\beta_1}\ \varepsilon) \circ \Upsilon_m^{u_2}(\sqrt[6]{\beta_2}\ \varepsilon) \circ \cdots \circ \Upsilon_m^{u_k}(\sqrt[6]{\beta_k}\ \varepsilon).$$

According to (3.11), we have that

$$\Phi^{\bar{u}_1}(\varepsilon)(x) \in \mathcal{R}(x, c_\Phi\ \varepsilon^2) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and each } \varepsilon \in \left(0,\sqrt{\frac{T_0}{c_\Phi}}\right), \tag{3.15}$$

where
$$c_\Phi := 2m\ \left(\sqrt{\beta_1} + \sqrt{\beta_2} + \cdots + \sqrt{\beta_k}\right).$$

Using (3.10) and (3.12), we apply the C-B-H formula and obtain that

$$\Phi^{\bar{u}_1}(\varepsilon) = \prod_{j=1}^{k} \operatorname{Exp}\ \left(2m\sqrt{\beta}_j\varepsilon^2 f + \frac{m\varepsilon^6}{3}\beta_j[g_{u_j},[g_{u_j},f]] + \sqrt{\beta_j}^5\varepsilon^{10}\bar{\Gamma}^{10}_{u_j} + O_{u_j}(\varepsilon^{14})\right) \cong$$
$$\cong \operatorname{Exp}\ \left(c_\Phi\ \varepsilon^2 f + \frac{m\varepsilon^6}{3}\left(\sum_{j=1}^{k}\beta_j\ [g_{u_j},[g_{u_j},f]]\right) + \Lambda^{7,13}_{\bar{u}_1}(\varepsilon) + O^1_{\bar{u}_1}(\varepsilon^{14})\right),$$

where $\Lambda^{7,13}_{\bar{u}_1}(\varepsilon)$ is an admissible Lie polynomial in ε (with respect to $\|\cdot\|_r$).

According to Proposition 3.3.1, there exists a positive integer m_1 so that $\Phi^{\bar{u}_1}_{m_1}(\varepsilon) \cong$

$$\cong \operatorname{Exp}\ \left(m_1 c_\Phi\ \varepsilon^2 f + \frac{m_1 m\varepsilon^6}{3}\left(\sum_{j=1}^{k}\beta_j\ [g_{u_j},[g_{u_j},f]]\right) + \bar{\Lambda}^{7,13}_{\bar{u}_1}(\varepsilon) + O_{\bar{u}_1}\left(\varepsilon^{14}\right)\right) \tag{3.16}$$

and

$$\Phi^{\bar{u}_1}_{m_1}(\varepsilon)(x) \in \mathcal{R}(x, m_1 c_\Phi\ \varepsilon^2) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and each } \varepsilon \in \left(0,\sqrt{\frac{T_0}{m_1 c_\Phi}}\right) \tag{3.17}$$

where $\bar{\Lambda}^{7,13}_{\bar{u}_1} \in \mathcal{L}(f, g_{u_1}, \ldots, g_{u_k})$ is an admissible Lie polynomial in ε (with respect to $\|\cdot\|_r$) that is invariant with respect to $\Theta^{\pm}$. One can directly calculate that each of the invariant Lie brackets appearing in $\bar{\Lambda}^{7,13}_{u}$ contain two times one of the vector fields from the set $\{g_{u_j} : \ j = 1, \ldots, k\}$ and three times the vector field f. According to Lemma 2.4.1, all these brackets are homogeneous of second degree. Hence

$$\bar{\Lambda}^{7,13}_{\bar{u}_1}(\varepsilon) = \sum_{j=1}^{J_{\bar{u}_1}} \bar{\alpha}_j \ \varepsilon^{10} \ \Lambda^{j}_{\bar{u}_1} := \varepsilon^{10}\bar{\Gamma}^{10}_{\bar{u}_1},$$

where each Lie bracket $\Lambda^{j}_{\bar{u}_1}$ contains two times one of the the vector fields from the set $\{g_{u_j} : \ j = 1, \ldots, k\}$ and three times the vector field f. Then (3.16) and (3.17) imply that

$$\Phi^{\bar{u}_1}_{m_1}(\varepsilon) \cong \text{Exp} \ \left(m_1 c_\phi \ \varepsilon^2 f + \frac{m_1 m \varepsilon^6}{3} \left(\sum_{j=1}^{k} \beta_j g_{h_j} \right) + \varepsilon^{10}\bar{\Gamma}^{10}_{\bar{u}_1} + O_{\bar{u}_1}\left(\varepsilon^{14}\right) \right).$$

We set

$$\Phi^{u_0,\bar{u}_1}_{m_1}(\varepsilon) := \Phi^{\bar{u}_1}_{m_1}(\varepsilon) \circ \ \text{Exp} \ \left(\frac{m_1 m \varepsilon^6}{3} (\beta_0 f + \beta_0 g_{u_0}) \right).$$

Clearly

$$\Phi^{u_0,\bar{u}_1}_{m_1}(\varepsilon)(x) \in \mathcal{R}(x, T(\varepsilon)) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and each } \varepsilon \in \left(0, \min\left(1, \sqrt{\frac{T_0}{C}}\right)\right) \tag{3.18}$$

where $T(\varepsilon) := m_1 \left(c_\Phi \varepsilon^2 + \dfrac{m\varepsilon^6}{3}\beta_0 \right)$ and $C := m_1 \left(c_\Phi + \dfrac{m\beta_0}{3} \right)$.

On the other hand, applying the C-B-H formula, we obtain that

$$\Psi^{u_0,\bar{u}_1}(\varepsilon) := \Phi^{u_0,\bar{u}_1}_{m_1}(\varepsilon) = Exp\left(T(\varepsilon) f + \frac{m_1 m \varepsilon^6}{3} \left(\beta_0 g_{u_0} + \sum_{j=1}^{k} \beta_j g_{h_j} \right) + \right.$$

$$\left. + \varepsilon^{10}\bar{\Gamma}^{10}_{\bar{u}_1} + \Lambda^{7,13}_{u_0,\bar{u}_1} + O^{1}_{u_0,\bar{u}_1}\left(\varepsilon^{14}\right)\right),$$

where $\Lambda^{7,13}_{u_0,\bar{u}_1} \in \mathcal{L}$ is an admissible Lie polynomial in ε (with respect to $\|\cdot\|_r$).

Clearly, the elements of $Bra(\bar{\Gamma}^{10}_{\bar{u}_1})$, g_{u_0} and the Lie brackets $g_{h_j} = [g_{u_j}, [g_{u_j}, f]]$, $j = 1, \ldots, k$, are invariant with respect to Θ. According to Proposition 3.3.1, there exists a positive integer m_2 so that

$$\Psi^{u_0,\bar{u}_1}_{m_2}(\varepsilon) = Exp\left(m_2 T(\varepsilon) f + \frac{m_1 m_2 m \varepsilon^6}{3} \left(\beta_0 g_{u_0} + \sum_{j=1}^{k} \beta_j g_{h_j} \right) + \right.$$

$\left. + m_2\varepsilon^{10}\bar{\Gamma}^{10}_{\bar{u}_1} + \bar{\Lambda}^{7,13}_{u_0,\bar{u}_1} + O^{+}\left(\varepsilon^{14}\right)\right) \in \mathcal{R}(x, m_2 T(\varepsilon)) \cap \Omega$ for each $x \in \Omega_0$ and each

$$\varepsilon \in \left(0, \min\left(1, \sqrt{\frac{T_0}{m_2 C}}\right)\right),$$ where $\bar{\Lambda}^{7,13}_{u_0,\bar{u}_1}$ is an admissible Lie polynomial in ε

(with respect to $\|\cdot\|_r$) that is invariant with respect to Θ. According to the C-B-H formula, one can directly check that $Bra(\bar{\Lambda}^{7,13}_{u_0,\bar{u}_1}) = \{[f,[f,g_{u_0}]]\}$. Hence,

$$\varepsilon^{10}\bar{\Gamma}^{10}_{\bar{u}_1} + \bar{\Lambda}^{7,13}_{u_0,\bar{u}_1} = \varepsilon^{10}\sum_{j=1}^{J_{u_0,\bar{u}_1}} \Lambda^{j}_{u_0,\bar{u}_1} =: \varepsilon^{10}\Gamma^{+},$$

where each $\Lambda^{j}_{u_0,\bar{u}_1}$ is homogeneous of second degree (according to Lemma 2.4.1).

We set $c_h^{+} := mm_1m_2/3$, $T^{+}(\varepsilon) := m_2T(\varepsilon)$ and $C^{+} := m_2C$. Taking into account (3.13), we obtain that

$$\Psi^{+}(\varepsilon) := \Psi^{u_0,\bar{u}_1}_{m_2}(\varepsilon) = \text{ Exp }\left(T^{+}(\varepsilon)f + c_h^{+}\varepsilon^6 g_h + \varepsilon^{10}\Gamma^{+} + O^{+}\left(\varepsilon^{14}\right)\right) \in \tag{3.19}$$

$$\in \mathcal{R}(x, T^{+}(\varepsilon)) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and each } \varepsilon \in \left(0, \min\left(1, \sqrt{\frac{T_0}{C^{+}}}\right)\right),$$

where $Bra(\Gamma^{+})$ is a set of Lie brackets that are homogeneous of second degree.

Analogously, taking into account (3.14), one can prove the existence of an admissible flow (note that $g_{-h} = -g_h$)

$$\Psi^{-}(\varepsilon) = \text{ Exp }\left(T^{-}(\varepsilon)f - c_h^{-}\varepsilon^6 g_h + \varepsilon^{10}\Gamma^{-} + O^{-}\left(\varepsilon^{14}\right)\right) \in \tag{3.20}$$

$$\in \mathcal{R}(x, T^{-}(\varepsilon)) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and each } \varepsilon \in \left(0, \min\left(1, \sqrt{\frac{T_0}{C^{-}}}\right)\right), \text{ where } Bra(\Gamma^{-})$$

is a set of Lie brackets that are homogeneous of second degree, $T^{-}(\varepsilon) := c^{-}_{2}\varepsilon^2 + c_6^{-}\varepsilon^6$, C^{-}, c_h^{-}, c_2^{-} and c_6^{-} are positive constants. We set

$$\Psi(\varepsilon) := \Psi^{+}\left(\gamma^{+}\varepsilon\right) \circ \text{Exp }(\varepsilon f) \circ \Psi^{-}\left(\gamma^{-}\varepsilon\right),$$

where $\gamma^{+} := \sqrt[6]{\frac{1}{c_h^{+}}}$ and $\gamma^{-} := \sqrt[6]{\frac{1}{c_h^{-}}}$. Taking into account (3.19) and (3.20), we obtain that

$$\Psi(\varepsilon) \in \mathcal{R}\left(x, \bar{T}(\varepsilon)\right) \cap \Omega \text{ for each } x \in \Omega_0 \text{ and for each } \varepsilon \in \left(0, \frac{T_0}{\bar{C}}\right), \tag{3.21}$$

where $\bar{T}(\varepsilon) := \varrho_2\varepsilon^2 + \varepsilon + \varrho_6\varepsilon^6$, $\bar{C} := \varrho_2 + 1 + \varrho_6$, $\varrho_2 := \left(m_2 m_1 c_\Phi(\gamma^+)^2 + c_2^-(\gamma^-)^2\right)$ and $\varrho_6 := \left(\frac{m_2 m_1 m \beta_0(\gamma^+)^6}{3} + c_6^-(\gamma^-)^6\right)$. Applying the C-B-H formula, we obtain that

$$\Psi(\varepsilon) \cong \text{Exp}\ \Big((\varepsilon + \varrho_3\varepsilon^2 + \varrho_6\varepsilon^6)f + \tag{3.22}$$

$$+\frac{1}{12}\varepsilon^{13}(2 + \varrho_3\varepsilon + \varrho_6\varepsilon^5)[g_h,[g_h,f]] + \varepsilon^{10}((\gamma^+)^{10}\bar{\Gamma}^+ + (\gamma^-)^{10}\bar{\Gamma}^-) + \Pi(\varepsilon) + \tilde{O}(\varepsilon^{14})\Big),$$

where $\Pi(\varepsilon) := \sum_{j=1}^{\nu} \beta_j \varepsilon^{d_j} \Xi_j$, $\beta_j \in \mathbb{R}$, $d_j \geq 10$ and Ξ_j, $j = 1, \ldots, \nu$, are Lie brackets generated by f, g_h and the elements of $Bra(\bar{\Gamma}^+)$ and $Bra(\bar{\Gamma}^-)$ in which the vector field g_h appears at most one time. According to Lemma 2.4.1, all Lie brackets appearing in Π, are homogeneous of degree at last one, and hence, vanish at the origin. Also, $f(0) = 0$ and all Lie brackets from the sets $Bra(\Gamma^+)$ and $Bra(\Gamma^-)$ vanish at the origin. Then (3.21) and (3.22) imply that $[g_h,[g_h,f]] \in E_{13}^+$. Therefore, we have proved that $[g_h,[g_h,f]] \in E_{13}^+$ for each $h \in \mathcal{M}_1$. Moreover, we have already obtained that $[g_u,[g_u f]] \in E_6^+ \subseteq E_{13}^+$ for each $u \in \mathcal{M}_0$ as well as $g_u \in E_1^+ \subseteq E_{13}^+$. Because $\mathcal{K}_0 + \mathcal{K}_1 + \mathcal{K}_2 = \mathbb{R}^n$, there exists elements A_j of E_{13}^+, $j = 1, \ldots, \eta$, such that

$$\mathbf{0} \in \ int\ co\ \{A_1(0), A_2(0), \ldots, A_\eta(0)\}\,.$$

Applying Proposition 2.3.5, we obtain that the control system (4.1) is STLC at the origin. This completes the proof of the theorem. ◊

Chapter 4

Refinement of the approach based on the set E_α^+ and one more sufficient condition for small-time local controllability

The main idea in this chapter is to refine the approach based on the set of tangent vector fields of the reachable set. Here we will obtain new elements of E^+ - mixed brackets $[g_u, [g_v, f]]$. Roughtly speaking we need at least one of the Lie brackets $-[g_u, [g_u, f]]$ and $-[g_v, [g_v, f]]$ to be suitable element of $E^+(0)$.

4.1 Statement of the main result

Let us consider again the control system Σ_1 in $\mathbb{R}^n$

$$\dot{x}(t) = f(x(t)) + u(t), \tag{4.1}$$

$$x(0) = 0, \; u(t) \in U \cap \bar{\mathbf{B}}$$

where U is a closed convex cone in $\mathbb{R}^n$, $\bar{\mathbf{B}}$ is the closed unit ball of $\mathbb{R}^n$ centered at the origin and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map whose components are polynomials which are homogeneous of second degree, i.e. $f(\lambda x) = \lambda^2 f(x)$ for each $\lambda > 0$ and each $x \in \mathbb{R}^n$.

In order to present the approach and formulate our second sufficient condition, we define the following sets:

1. $\mathcal{K}_0 = U$, $\mathcal{M}_0 = rec\, \mathcal{K}_0$;

2. $\mathcal{K}_1 = cone\,(\{f(u) : u \in \mathcal{M}_0\} \cup U)$
   $\mathcal{N}_1 = \{u \in \mathcal{M}_0 : -f(u) \in \mathcal{K}_1\}$
   $\mathcal{M}_1 = span\,(\{[g_u, [g_v, f]](0) : v \in \mathcal{M}_0, u \in \mathcal{N}_1\} \cup \mathcal{M}_0)\,;$

3. For $s = 1, 2, 3, ...$ we define the sets $\mathcal{K}_{s+1}$, $\mathcal{N}_{s+1}$ and $\mathcal{M}_{s+1}$ in the following recursive way:
   $\mathcal{K}_{s+1} = cone\,(\mathcal{K}_1 \cup \mathcal{M}_s)$
   $\mathcal{N}_{s+1} = \{u \in \mathcal{M}_0 : -f(u) \in \mathcal{K}_{s+1}\}$
   $\mathcal{M}_{s+1} = span\,(\{[g_u, [g_v, f]](0) : v \in \mathcal{M}_0, u \in \mathcal{N}_{s+1}\} \cup \mathcal{M}_s)\,;$

   Finally, we set $\kappa = min\{s : \mathcal{M}_{s+1} = \mathcal{M}_s\}$. Clearly, $\kappa \leq n$.

Our main result is the following

Theorem 4.1.1 *The set $\{g_u : u \in \mathcal{K}_\kappa\}$ is a subset of E^+.*

The proof of Theorem 4.1.1 is presented in the next section.

Corollary 4.1.2 *We set*

$$\mathcal{L} = cone\ \{f(u) : u \in f\,(\mathcal{M}_0)\,, -u \in \mathcal{K}_1\}.$$

If $cone(\mathcal{L} \cup \mathcal{K}_\kappa)$ coincides with $\mathbb{R}^n$, then the control system Σ is small-time locally controllable at the origin.

Sketch of the proof of Corollary 4.1.2. Taking into account Theorem 3.4.1, we obtain that the set $\{g_u : u \in \mathcal{L}\}$ is a subset of E^+. Using Lemma 2.3.6, we obtain that $cone(\{g_u : u \in \mathcal{L} \cup \mathcal{K}_\kappa\})$ is also a subset of E^+. At last, applying Lemma 2.3.5, we complete the proof.

4.2 Examples

Example 4.2.1

Let us consider the following control system:

$$\begin{aligned}&\dot{x}(t) = u(t),\ \ u(t) \in [-1, 1],\\&\dot{y}(t) = v(t),\ \ v(t) \in [-1, 1],\\&\dot{z}(t) = x^2(t) - y^2(t),\\&\dot{p}(t) = z^2(t) - x(t)y(t).\end{aligned}$$

We set $f(x, y, z, p) := (0, 0, x^2 - y^2, z^2 - xy)^T$, $g_1(x, y, z, p) := (1, 0, 0, 0)^T$ and $g_2(x, y, z, p) := (0, 1, 0, 0)^T$.

We have that

$$\mathcal{M}_0 = \mathcal{K}_0 = \{(x, y, 0, 0)^T : x \in \mathbb{R}, y \in \mathbb{R}\},$$

$$\mathcal{K}_1 \supset \{(x, y, z, 0)^T : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\},$$

$$\mathcal{N}_1 \supset \{(x, 0, 0, 0)^T : x \in \mathbb{R}\} \cup \{(0, y, 0, 0)^T : y \in \mathbb{R}\}$$

$$\mathcal{M}_1 \supset \{(x, y, 0, p)^T : x \in \mathbb{R}, y \in \mathbb{R}, p \in \mathbb{R}\},$$

$$\mathcal{K}_2 = \{(x, y, z, p)^T : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}, p \in \mathbb{R}\}.$$

Applying Theorem 4.1.1, we obtain that $\{g_u(0) : u \in \mathcal{K}_2\} \subset E^+$. Because $\mathcal{K}_2 = \mathbb{R}^4$, we can conclude (taking into account Lemma 2.3.6) that this control system is small-time local controllable at the origin.

Example 4.2.2

Let us consider the following control system:

$$\begin{aligned}&\dot{x}(t) = u(t),\ \ u(t) \in [-1, 1],\\&\dot{y}(t) = v(t),\ \ v(t) \in [-1, 1],\\&\dot{z}(t) = x^2(t) - y^2(t),\\&\dot{p}(t) = z^2(t) - y^2(t).\end{aligned}$$

Using Theorem 4.1.1 and Corollary 4.1.2, we can not conclude that this control system is STLC at the origin.

In fact, the reachable set of this system is a linear subspace of a half space and this system is not STLC at the origin. Indeed, let us fix an arbitrary T from the interval $(0,1)$ and let us assume that $z(T)=0$. Since

$$z(t)=\int_0^t (x^2(s)-y^2(s))ds,$$

we have that

$$0=z(T)=\int_0^T (x^2(t)-y^2(t))dt \text{ and hence } \int_0^T x^2(t)dt=\int_0^T y^2(t)dt \tag{4.2}$$

Because

$$p(T)=\int_0^T (z^2(t)-y^2(t))dt,$$

the following equalities hold true

$$p(T)=\int_0^T \left(\left(\int_0^t x^2(s)ds-\int_0^t y^2(s)ds\right)^2-y^2(t)\right)dt=$$

$$=\int_0^T \left(\int_0^t x^2(s)ds\right)^2 dt+\int_0^T \left(\int_0^t y^2(s)ds\right)^2 dt$$

$$-2\int_0^T \left(\int_0^t x^2(s)ds\right)\left(\int_0^t y^2(s)ds\right)dt-\int_0^T y^2(t)dt.$$

Taking into account (4.2) and the inequalities

$$|x(T)|\le\int_0^T |u(t)|dt\le T,$$

we obtain that

$$p(T)\le 4T\left(\int_0^T x^2(t)dt\right)^2-\int_0^T x^2(t)dt=$$

$$=\left(-\int_0^T x^2(t)dt\right)\left(1-4T\int_0^T x^2(t)dt\right).$$

If T is sufficiently small, then $p(T)\le 0$, and hence this system is not small-time local controllable at the origin.

4.3 Proof of Theorem 4.1.1

Let us fix a compact neighborhood Ω_0 of the origin and set $\Omega := 2\Omega_0$. Then there exists $T_0 > 0$ so that each trajectory x of Σ starting from a point $x_0 \in \Omega_0$ and corresponding to some admissible control from $\mathcal{U}_T$ with $T \leq T_0$ is well defined on the interval $[0, T]$ and remains in Ω, i.e. $x(t) \in \Omega$ for all $t \in [0, T]$. Then the following finite composition of exponents

$$Exp(t_1(f + g_1)) \circ Exp(t_2(f + g_2)) \circ \cdots \circ Exp(t_k(f + g_k))(x)$$

is well defined for each $x \in \Omega_0$, each $t_i > 0$ with $T := t_1 + \cdots + t_k \leq T_0$ and each $g_i \in \{g_u : \ u \in U \cap \bar{\mathbf{B}}\}$, $i = 1, \ldots, k$. Without loss of generality we may assume here and further that $T_0 > 0$ is sufficiently small so that (according to the C-B-H formula) there exists a Lie series $\mathcal{S}$ such that

$$Exp(\mathcal{S}) = Exp(t_1(f + g_1)) \circ Exp(t_2(f + g_2)) \circ \cdots \circ Exp(t_k(f + g_k)). \qquad (4.3)$$

Let us define the sets

$$\mathcal{U}^+ := \Bigg\{ u : (0, \varepsilon_u) \to U \cap \bar{\mathbf{B}} : u(\varepsilon) := \sum_{i=1}^{m} \varepsilon^{\alpha_i} u_i,$$

$$u_i \in U \cap \bar{\mathbf{B}}, \alpha_i > 1, i = 1, \ldots, m, \varepsilon_u \in (0, 1), \ m \in I\!N \Bigg\},$$

and

$$\mathcal{U} := \Bigg\{ u : (0, \varepsilon_u) \to (\text{rec } U) \cap \bar{\mathbf{B}} : u(\varepsilon) := \sum_{i=1}^{m} \varepsilon^{\alpha_i} u_i,$$

$$u_i \in (\text{rec } U) \cap \bar{\mathbf{B}}, \ \alpha_i > 1, i = 1, \ldots, m, \varepsilon_u \in (0, 1), \ m \in I\!N \Bigg\},$$

where by $I\!N$ it is denoted the set of all positive integer numbers. Clearly, if $u \in \mathcal{U}$, then $-u$ also belongs to $\mathcal{U}$.

Let us fix $u \in \mathcal{U}$, $\varepsilon \in (0, \min(\varepsilon_u, T_0/4))$ and a real $\alpha > 1$. Applying the equality $-g_{u(\varepsilon)} = g_{-u(\varepsilon)}$ and the C-B-H formula, we obtain that

$$Exp(\varepsilon^\alpha(f \pm g_{u(\varepsilon)})) \circ Exp(\varepsilon^\alpha(f \mp g_{u(\varepsilon)})) =$$

$$= Exp\left(2\varepsilon^\alpha f \pm \varepsilon^{2\alpha}[g_{u(\varepsilon)}, f] + \frac{\varepsilon^{3\alpha}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + O^{\pm}(\varepsilon^{4\alpha})\right). \qquad (4.4)$$

Our choice of ε implies that for each $x \in \Omega_0$ the trajectory $\mathcal{P}_u(\varepsilon)(x)$ is well defined, where

$$\mathcal{P}_u(\varepsilon)(x) := Exp(\varepsilon^\alpha(f + g_{u(\varepsilon)})) \circ Exp(\varepsilon^\alpha(f - g_{u(\varepsilon)}))\circ$$
$$\circ Exp(\varepsilon^\alpha(f - g_{u(\varepsilon)})) \circ Exp(\varepsilon^\alpha(f + g_{u(\varepsilon)}))(x) \in \mathcal{R}(x, 4\varepsilon^\alpha). \tag{4.5}$$

Taking into account (4.4) and the C-B-H formula, we obtain that

$$\mathcal{P}_u(\varepsilon) = Exp\left(2\varepsilon^\alpha f + \varepsilon^{2\alpha}[g_{u(\varepsilon)}, f] + \frac{\varepsilon^{3\alpha}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + O^+(\varepsilon^{4\alpha})\right) \circ$$

$$\circ Exp\left(2\varepsilon^\alpha f - \varepsilon^{2\alpha}[g_{u(\varepsilon)}, f] + \frac{\varepsilon^{3\alpha}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + O^-(\varepsilon^{4\alpha}))\right) =$$

$$= Exp\left(4\varepsilon^\alpha f + \frac{2\varepsilon^{3\alpha}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + 2\varepsilon^{3\alpha}[f, [f, g_{u(\varepsilon)}]] + O(\varepsilon^{4\alpha}))\right). \tag{4.6}$$

We choose arbitrary elements $u_0 \in \mathcal{U}^+$ and $u_1, \ldots, u_s$ from $\mathcal{U}$, set $\hat{u} := (u_0, u_1, \ldots, u_s)$, $\varepsilon_{\hat{u}} = min\{\varepsilon_{u_i}, i = 1, \ldots, s\} > 0$ and consider the function

$$(0, \varepsilon_{\hat{u}}) \ni \varepsilon \to g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f\left(u_i(\varepsilon)\right).$$

We call this function *an admissible sum of Lie brackets.* Then, taking into account (4.5) and (4.6), we obtain for each $x \in \Omega_0$ and for each $\varepsilon \in (0, \min(\varepsilon_{\hat{u}}, T_0/(4s+1)))$ that

$$\mathcal{P}_{\hat{u}}(\varepsilon)(x) := Exp\left(\frac{4}{3}\varepsilon^{3\alpha}\left(f + g_{u_0(\varepsilon)}\right)\right) \circ$$
$$\circ \mathcal{P}_{u_1}(\varepsilon) \circ \cdots \circ \mathcal{P}_{u_s}(\varepsilon)(x) \in \mathcal{R}\left(x, \frac{4}{3}\varepsilon^{3\alpha} + 4s\varepsilon^\alpha\right). \tag{4.7}$$

Applying the C-B-H formula, we have that

$$\mathcal{P}_{\hat{u}}(\varepsilon) = Exp\left(\left(\frac{4}{3}\varepsilon^{3\alpha} + 4s\varepsilon^\alpha\right) f + \frac{4\varepsilon^{3\alpha}}{3}\left(g_{u_0(\varepsilon)} + \frac{1}{2}\sum_{i=1}^{s}[g_{u_i(\varepsilon)}, [g_{u_i(\varepsilon)}, f]]\right) + \right.$$

$$\left. + 2\varepsilon^{3\alpha}\sum_{i=1}^{s}[f, [f, g_{u_i(\varepsilon)}]] + O^s(\varepsilon^{4\alpha})\right) =$$

$$= Exp\left(\left(\frac{4}{3}\varepsilon^{3\alpha} + 4s\varepsilon^\alpha\right) f + \frac{4\varepsilon^{3\alpha}}{3}\left(g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f\left(u_i(\varepsilon)\right)\right) + \right.$$

$$\left. + 2\varepsilon^{3\alpha}\sum_{i=1}^{s}[f, [f, g_{u_i(\varepsilon)}]] + O^s(\varepsilon^{4\alpha})\right). \tag{4.8}$$

Because the vector field f is homogeneous of second degree, one can check that the vector fields $[f,[f,g_{u_i(\varepsilon)}]]$, $i=1,\dots,s$, are also homogeneous of second degree, and hence $[f,[f,[g_{u_i(\varepsilon)}]](0)=0$, $i=1,\dots,s$, for each ε from the interval $(0,\varepsilon_{\hat{u}})$.

So, we have shown that the inclusion (4.7) and the equality (4.8) hold true for the admissible sum of Lie brackets

$$(0,\varepsilon_{\hat{u}})\ni\varepsilon\to g_{u_0(\varepsilon)}+\sum_{i=1}^{s}f(u_i(\varepsilon)).$$

Let us fix the real numbers $\alpha>1$ and $\beta>0$ such that the following inequalities hold true

$$1<2^{\kappa-1}\beta<2^{\kappa+1}\beta<\alpha. \tag{4.9}$$

We set $\mu:=2^{\kappa}\beta$,

$$\mu_1:=2^{\kappa-1}\beta,\mu_2:=2^{\kappa-1}\left(1+\frac{1}{2}\right)\beta,\mu_s:=2^{\kappa-1}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{s-1}}\right)\beta,$$

for each $s=1,\dots,\kappa$. Clearly, the inequalities (4.9) imply that

$$1<\mu_1<\mu_2<\cdots<\mu_\kappa<\mu \text{ and } 2\mu<\alpha. \tag{4.10}$$

First, we show that for each elements p and q of the set $\{1,\dots,\kappa\}$ with $p<q$ the following inequality holds true

$$\mu_p+\mu\le 2\mu_q. \tag{4.11}$$

Indeed, we have that

$$2\mu_q-\mu-\mu_p=2^{\kappa}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{q-1}}\right)\beta-2^{\kappa}\beta-2^{\kappa-1}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{p-1}}\right)\beta=$$

$$=2^{\kappa}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{q-1}}\right)\beta-2^{\kappa}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{p-1}}+\frac{1}{2^{p}}\right)\beta\ge 0.$$

Next, we prove the following

Claim. For each positive integer $q\in\{1,2,\dots,\kappa\}$, and for each element $\big[g_{u_q},\big[g_{v_q},f\big]\big]\in\mathcal{M}_q$ there exist $\varepsilon_{u_qv_q}\in(0,1)$, $v_{qi}\in\mathcal{M}_0$ and $\delta_{qi}\ge 0$, $i=1,\dots,\bar{q}$, such that the function

$$(0,\varepsilon_{u_qv_q})\ni\varepsilon\to\varepsilon^{\mu_q+\mu}\big[g_{u_q},\big[g_{v_q},f\big]\big]+\varepsilon^{2\mu}\sum_{i=1}^{\bar{q}}\varepsilon^{\delta_{qi}}f(v_{qi}) \tag{4.12}$$

is an admissible sum of Lie brackets, i.e. there are $u_0 \in \mathcal{U}^+$ and $u_\alpha \in \mathcal{U}$, $\alpha = 1, \ldots, s$, such that

$$\varepsilon^{\mu_q+\mu} \left[g_{u_q}, \left[g_{v_q}, f\right]\right] + \varepsilon^{2\mu} \sum_{i=1}^{\bar{q}} \varepsilon^{\delta_{qi}} f(v_{qi}) = g_{u_0(\varepsilon)} + \sum_{\alpha=1}^{s} f(u_\alpha(\varepsilon)),$$

for each $\varepsilon \in (0, \varepsilon_{u_q v_q})$.

Proof of the Claim. The proof will be done by induction. First, we show that the claim holds true for $q = 1$. Indeed, let $[g_{u_1}, [g_{v_1}, f]] \in \mathcal{M}_1$. Then $v_1 \in \mathcal{M}_0, u_1 \in \mathcal{N}_1$. According to the definition of $\mathcal{N}_1$,

$$-f(u_1) = g_{u_{1,0}} + \sum_{j=1}^{p_1} f(u_{1j}) \tag{4.13}$$

where $u_{1,0} \in U$ and $u_{1j} \in \mathcal{M}_0$, $j = 1, \ldots, p_1$.

Clearly, there exists $\varepsilon_{u_1 v_1} \in (0, 1)$ such that for each $\varepsilon \in (0, \varepsilon_{u_1 v_1})$ the sum $\varepsilon^{\mu_1} u_1 + \varepsilon^{\mu} v_1$ belongs to (rec U) $\cap \bar{B}$ and

$$f(\varepsilon^{\mu_1} u_1 + \varepsilon^{\mu} v_1) + \varepsilon^{2\mu_1} g_{u_{1,0}} + \sum_{j=1}^{p_1} f(\varepsilon^{\mu_1} u_{1j}) =$$

$$= \varepsilon^{2\mu_1} f(u_1) + \varepsilon^{\mu_1+\mu} [g_{u_1}, [g_{v_1}, f]] + \varepsilon^{2\mu} f(v_1) + \varepsilon^{2\mu_1} g_{u_{1,0}} + \varepsilon^{2\mu_1} \sum_{j=1}^{p_1} f(u_{1j})$$

Applying (4.13) we obtain that

$$\varepsilon^{\mu_1+\mu} [g_{u_1}, [g_{v_1}, f]] + \varepsilon^{2\mu} f(v_1) =$$

$$= \varepsilon^{2\mu_1} g_{u_{1,0}} + f(\varepsilon^{\mu_1} u_1 + \varepsilon^{\mu} v_1) + \sum_{j=1}^{p_1} f(\varepsilon^{\mu_1} u_{1j}).$$

Hence, the function

$$(0, \varepsilon_{u_1 v_1}) \ni \varepsilon \to \varepsilon^{\mu_1+\mu} [g_{u_1}, [g_{v_1}, f]] + \varepsilon^{2\mu} f(v_1)$$

is an admissible sum of Lie brackets. So, we obtain that the Claim holds true for $q = 1$.

Let us assume that the Claim holds true for each positive integer r satisfying the inequality $r \leq q$ for some positive integer $q < \kappa$. We prove that it holds true also for $p := q + 1$.

Indeed, let us fix an element $\left[g_{u_p},\left[g_{v_p},f\right]\right]\in\mathcal{M}_p\setminus\mathcal{M}_q$. Then $v_p\in\mathcal{M}_0, u_p\in\mathcal{N}_p$. According to the definition of $\mathcal{N}_p$,

$$-f(u_p)=g_{u_{\alpha_p}}+\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma_p}}\left[g_{u_{\gamma_pj_p}},\left[g_{v_{\gamma_pj_p}},f\right]\right], \tag{4.14}$$

where $u_{\alpha_p}\in U$, $u_{\beta_p}\in\mathcal{M}_0$ and each Lie bracket $\left[g_{u_{\gamma_pj_p}},\left[g_{v_{\gamma_pj_p}},f\right]\right]\in\mathcal{M}_{\gamma_p}$ with $\gamma_p<p$. Clearly, there exists $\varepsilon_{u_p^0v_p^0}\in(0,1)$ such that for each $\varepsilon\in(0,\varepsilon_{u_p^0v_p^0})$ the sum $\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p$ belongs to $(\operatorname{rec}U)\cap\bar B$ and

$$f\left(\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p\right)=\varepsilon^{2\mu_p}f\left(u_p\right)+\varepsilon^{\mu_p+\mu}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}f\left(v_p\right).$$

We add

$$\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma_p}}\left[g_{u_{\gamma_pj_p}},\left[g_{v_{\gamma_pj_p}},f\right]\right]$$

to both sides of this equality and obtain

$$f\left(\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p\right)+\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma_p}}\left[g_{u_{\gamma_pj_p}},\left[g_{v_{\gamma_pj_p}},f\right]\right]=$$

$$=\varepsilon^{2\mu_p}f\left(u_p\right)+\varepsilon^{\mu_p+\mu}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}f\left(v_p\right)+$$

$$+\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma_p}}\left[g_{u_{\gamma_pj_p}},\left[g_{v_{\gamma_pj_p}},f\right]\right].$$

Taking into account (4.14), we obtain that

$$f\left(\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p\right)+\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\beta_p}f(u_{\bar\beta_p})+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma_p}}\left[g_{u_{\gamma_pj_p}},\left[g_{v_{\gamma_pj_p}},f\right]\right]=$$

$$=\varepsilon^{\mu_p+\mu}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}f\left(v_p\right). \tag{4.15}$$

According to the inductive assumption, for each multi-index γ_pj_p there exists $\varepsilon_{\gamma_pj_p}\in(0,1)$ and a function

$$(0,\varepsilon_{\gamma_pj_p})\ni\varepsilon\to\varepsilon^{2\mu}\sum_{k_p=1}^{p_{\gamma_pj_p}}\varepsilon^{\delta_{\gamma_pj_pk_p}}f(v_{\gamma_pj_pk_p})$$

(here each $v_{\gamma_p j_p k_p} \in \mathcal{M}_0$ and each real $\delta_{\gamma_p j_p k_p} \geq 0$) such that the function

$$(0, \varepsilon_{\gamma_p j_p}) \ni \varepsilon \to \varepsilon^{\mu+\mu_{\gamma_p}} \left[g_{u_{\gamma_p j_p}}, \left[g_{v_{\gamma_p j_p}}, f\right]\right] + \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p})$$

is an admissible sum of Lie brackets, i.e. for each $\varepsilon \in (0, \varepsilon_{\gamma_p j_p})$ we have that

$$\varepsilon^{\mu+\mu_{\gamma_p}} \left[g_{u_{\gamma_p j_p}}, \left[g_{v_{\gamma_p j_p}}, f\right]\right] + \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) =$$

$$= g_{u^0_{\gamma_p j_p}(\varepsilon)} + \sum_{i_{\gamma_p j_p}=1}^{\bar{i}_{\gamma_p j_p}} f\left(u_{i_{\gamma_p j_p}}(\varepsilon)\right),$$

where $u^0_{\gamma_p j_p} \in \mathcal{U}^+$ and $u_{i_{\gamma_p j_p}} \in \mathcal{U}$ for each $i_{\gamma_p j_p} = 1, \ldots, \bar{i}_{\gamma_p j_p}$.

Taking this into account and setting

$$\varepsilon_{u^1_p v^1_p} = min\{\varepsilon_{u^0_p v^0_p}, \varepsilon_{\gamma_p j_p}, j_p = 1, \ldots, \bar{j}_{\gamma_p}, \gamma_p = 1, \ldots, \bar{\gamma}_p\} > 0,$$

we obtain from (4.15) that for each $\varepsilon \in (0, \varepsilon_{u^1_p v^1_p})$ the following equality holds true

$$f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \varepsilon^{2\mu_p} g_{u_{\alpha_p}} + \varepsilon^{2\mu_p} \sum_{\beta_p=1}^{\bar{\beta}_p} f(u_{\beta_p}) +$$

$$+ \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p - \mu_{\gamma_p} - \mu} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \varepsilon^{\mu_{\gamma_p}+\mu} \left[g_{u_{\gamma_p j_p}}, \left[g_{v_{\gamma_p j_p}}, f\right]\right] +$$

$$+ \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p - \mu_{\gamma_p} - \mu} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) =$$

$$= \varepsilon^{\mu_p+\mu} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p - \mu - \mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}).$$

Hence

$$\varepsilon^{\mu_p+\mu} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \varepsilon^{2\mu} \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p - \mu - \mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) =$$

$$= f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \varepsilon^{2\mu_p} g_{u_{\alpha_p}} + \sum_{\beta_p=1}^{\bar{\beta}_p} f(\varepsilon^{\mu_p} u_{\beta_p}) +$$

$$+ \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_p} \left(\varepsilon^{\mu+\mu_{\gamma_p}} \left[g_{u_{\gamma_p}}, \left[g_{v_{\gamma_p}}, f\right]\right] + \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) \right) =$$

$$= f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \varepsilon^{2\mu_p} g_{u_{\alpha_p}} + \sum_{\beta_p=1}^{\bar{\beta}_p} f(\varepsilon^{\mu_p} u_{\beta_p}) + \tag{4.16}$$

$$+ \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \left(g_{u^0_{\gamma_p j_p}(\varepsilon)} + \sum_{i_{\gamma_p j_p}=1}^{\bar{i}_{\gamma_p j_p}} f\left(u_{i_{\gamma_p j_p}}(\varepsilon)\right) \right).$$

Clearly there exists $\varepsilon_{u_p v_p} \in (0, \varepsilon_{u_p^1 v_p^1})$ such that the sum

$$u_p^0(\varepsilon) := \varepsilon^{2\mu_p} u_{\alpha_p} + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} u^0_{\gamma_p j_p}(\varepsilon)$$

belongs to the set $\mathcal{U}^+$ for each $\varepsilon \in (0, \varepsilon_{u_p v_p})$. Then (4.16) can be written as follows:

$$\varepsilon^{\mu_p+\mu} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \varepsilon^{2\mu} \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) =$$

$$= g_{u_p^0(\varepsilon)} + f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \sum_{\beta_p=1}^{\bar{\beta}_p} f(\varepsilon^{\mu_p} u_{\beta_p}) +$$

$$+ \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{i_{\gamma_p j_p}=1}^{\bar{i}_{\gamma_p j_p}} f\left(u_{i_{\gamma_p j_p}}(\varepsilon)\right).$$

Because $2\mu_p - \mu - \mu_{\gamma_p} \geq 0$, the last equality implies that the function $(0, \varepsilon_{u_p v_p}) \ni \varepsilon \to \Lambda(\varepsilon)$, where $\Lambda(\varepsilon) =$

$$= \varepsilon^{\mu_p+\mu} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \varepsilon^{2\mu} \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}),$$

is also an admissible sum of Lie brackets. Hence, the inductive assumption holds true for $\left[g_{u_{\gamma_p}}, \left[g_{v_{\gamma_p}}, f\right]\right]$. Because $\left[g_{u_{\gamma_p}}, \left[g_{v_{\gamma_p}}, f\right]\right]$ is an arbitrary Lie bracket from $\mathcal{M}_p$, we can conclude that the inductive assumption holds true also for $p := q+1$. Therefore, the inductive assumption holds true for each $p \in \{1, 2, \ldots, \kappa\}$. This completes the proof of the Claim.

Let us fix an integer number p from the set $\{1,2,\dots,\kappa\}$ and an arbitrary Lie bracket $\big[g_{u_p},\big[g_{v_p},f\big]\big]\in\mathcal{M}_p$. Then the Claim implies the existence of $\varepsilon_{u_pv_p}\in(0,1)$, $v_i\in\mathcal{M}_0$ and $\delta_i\geq 0$, $i=1,\dots,\bar\rho$, such that

$$(0,\varepsilon_{u_pv_p})\ni\varepsilon\to\varepsilon^{\mu+\mu_p}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}\sum_{i=1}^{\bar\rho}\varepsilon^{\delta_i}f(v_i)$$

is an admissible sum of Lie brackets, i.e.

$$(0,\varepsilon_{u_pv_p})\ni\varepsilon\to\varepsilon^{\mu+\mu_p}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}\sum_{i=1}^{\bar\rho}\varepsilon^{\delta_i}f(v_i)=g_{u_0(\varepsilon)}+\sum_{\alpha=1}^{s}f(u_\alpha(\varepsilon)),$$

where $u_0\in\mathcal{U}^+$ and each u_α belongs to $\mathcal{U}$ for $\alpha=1,\dots,s$. This equality and the relations (4.7) and (4.8) imply that

$$Exp\left(\left(\frac{4}{3}\varepsilon^{3\alpha}+4s\varepsilon^{\alpha}\right)f+\frac{4\varepsilon^{3\alpha}}{3}\left(\varepsilon^{\mu+\mu_p}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}\sum_{i=1}^{\bar\rho}\varepsilon^{\delta_i}f(v_i)\right)+\right.$$

$$\left.+2\varepsilon^{3\alpha}\sum_{i=1}^{s}[f,[f,g_{u_i(\varepsilon)}]]+O^s(\varepsilon^{4\alpha})\right)(x)=$$

$$=Exp\left(\left(\frac{4}{3}\varepsilon^{3\alpha}+4s\varepsilon^{\alpha}\right)f+\frac{4\varepsilon^{3\alpha}}{3}\left(g_{u_0(\varepsilon)}+\sum_{\alpha=1}^{s}f(u_\alpha(\varepsilon))\right)+\right.$$

$$\left.+2\varepsilon^{3\alpha}\sum_{i=1}^{s}[f,[f,g_{u_i(\varepsilon)}]]+O^s(\varepsilon^{4\alpha})\right)(x)=$$

$$=Exp\left(\frac{4}{3}\varepsilon^{3\alpha}\left(f+g_{u_0(\varepsilon)}\right)\right)\circ\mathcal{P}_{u_1}(\varepsilon))\circ\cdots\circ\mathcal{P}_{u_s}(\varepsilon)(x)\in$$

$$\in\mathcal{R}\left(x,\frac{4}{3}\varepsilon^{3\alpha}+4s\varepsilon^{\alpha}\right)$$

for each $\varepsilon\in(0,\min(\varepsilon_{u_pv_p},T_0/(4s+1)))$ and for each $x\in\Omega_0$. Here $\mathcal{P}_u(\varepsilon)$ is defined by (4.5). Our choice of μ_p and μ (cf. the inequalities (4.10) and (4.11)) implies the inequalities $\mu_p<\mu$ and $2\mu<\alpha$. From here we obtain that $3\alpha+\mu_p+\mu<3\alpha+2\mu<4\alpha$. Taking into account that f and $[f,[f,g_{u_i(\varepsilon)}]]$, $i=1,\dots,s$, are homogeneous of second degree for each $\varepsilon\in(0,\varepsilon_{u_pv_p})$, we obtain that $f(0)=0$ and $[f,[f,[g_{u_i(\varepsilon)}]](0)=0$ for each $\varepsilon\in(0,\varepsilon_{u_pv_p})$. Hence $\big[g_{u_p},\big[g_{v_p},f\big]\big]\in E^+$. This completes the proof of Theorem 4.1.1. $\triangle$

Chapter 5

A necessary condition for small-time local controllability

5.1 Motivation and statement of the main result

We consider the following control system Σ_n in $\mathbb{R}^n$

$$\dot{x}(t) = f(x(t), u(t)), \tag{5.1}$$

under the assumptions:

A1. The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous with respect to the variables x and u;

A2. The set U is a compact subset of $\mathbb{R}^m$;

A3. There exist real numbers $\rho \in (0, 1)$ and $K > 0$, such that the following inequality holds true

$$\|f(x_2, u) - f(x_1, u)\| \leq K\|x_2 - x_1\|$$

for each u of U and for each two elements x_2 and x_1 of $\rho\bar{\mathbf{B}}$ (by $\bar{\mathbf{B}}$ it is denoted the closed unit ball of $\mathbb{R}^n$ centered at the origin).

In order to show the main idea of our approach, we consider the following

Example 5.1.1 *Let us consider the following 3-dimensional control system Σ_3:*

$$\begin{aligned}
&\dot{x}(t) = u(t),\ \ x(0) = 0,\ \ u(t) \in [-1, 1],\\
&\dot{y}(t) = v(t),\ \ y(0) = 0,\ \ v(t) \in [-1, 1],\\
&\dot{z}(t) = ax^2(t) + bx(t)y(t) + cy^2(t) + dx(t)z(t), z(0) = 0,
\end{aligned}$$

where a, b, c and d are constants.

To study the small-time local controllability of this system at the origin, it is natural to consider its linearization at the origin:

$$\begin{aligned} \dot{x}(t) &= u(t), & x(0) &= 0, \\ \dot{y}(t) &= v(t), & y(0) &= 0, \\ \dot{z}(t) &= 0, & z(0) &= 0. \end{aligned} \tag{5.2}$$

According to the Kalman criterion this linear system is not small-time local controllable at the origin. On the other hand, one can easily prove that for each positive time $T > 0$ there exists a real number $\mu > 0$ such that each point of the linear space

$$L := \{(x, y, z)^T \in \mathbb{R}^3 : z = 0\}$$

(by p^T is denoted the transpose vector of p) whose norm is less than μ is reachable from the origin in time not greater than T by means of trajectories of the linearization (5.2).

This fact motivate us to study the values of the right-hand side of the system Σ_3 on the linear space L and explains the meaning of the linear span L that appears in the formulation of the below written Assumption 4. We shall continue the study of the local properties of the reachable set of the system Σ_3 after while.

In order to formulate our main result we need the following assumption:

A4. There exists a proper linear subspace L of $\mathbf{R}^n$ (i.e. $L \neq \mathbf{R}^n$), such that

$$\text{rec}(\text{ cone } (\{f(x, u) : \ x \in \rho\bar{\mathbf{B}} \cap L, u \in U\} \ \cup \ L \ ) \ ) \subseteq L.$$

Here by cone (S) it is denoted the smallest convex closed cone containing the set S, and by rec (C) - the largest linear space contained in the convex closed cone C.

Theorem 5.1.1 *Let the Assumptions A1, A2, A3 and A4 hold true. Then the control system* Σ_n *is not STLC at the origin.*

Theorem 5.1.1 implies directly the following corollary which generalizes the necessary controllability condition (cf. Theorem 1) from [33]. Indeed, let us consider the case when $L = \{\mathbf{0}\}$. Then Assumption A4 takes the form:

A4'. $\text{rec} \ (\text{ cone } (\ \{f(\mathbf{0}, u) : \ u \in U\} \ ) \ ) = \{\mathbf{0}\}$

and we obtain the following

Corollary 5.1.2 *Let the Assumptions A1, A2, A3 and A4' be fulfilled. Then the control system* Σ_n *is not STLC at the origin.*

Example 5.2 (continuation) In order to apply Theorem 3 to this example, one has to verify Assumption 4. It is easy to check that the set on the left-hand side of Assumption 4 is *rec* (*cone* $(D \cup L)$), where

$$D := \left\{ (u, v, ax^2 + bxy + cy^2)^T : \begin{array}{l} u \in [-1, 1], \\ v \in [-1, 1] \\ x^2 + y^2 \leq \rho^2 \end{array} \right\}$$

The following cases are possible:

Case I: $b^2 - 4ac \leq 0$. Then the sign of the third coordinate of all elements of the set D is one and the same, and hence, *rec* (*cone* $(D \cup L)) \subseteq L$. Applying Theorem 3, we obtain that the control system Σ_3 is not small-time locally controllable at the origin. We would like to point out that this conclusion does not follow from the known necessary controllability conditions obtained in Sussmann ([34]), Stefani ([31]), Kawski ([19]) and Krastanov ([24]), because these necessary STLC conditions concern only the scalar input case.

Case II: $b^2 - 4ac > 0$. We show that in this case the control system Σ_3 is small-time locally controllable at the origin. Note that this conclusion does not follow from the general result in Sussmann ([36]). Observing that Σ_3 is a control-affine system with quadratic drift term and constant control-input vector fields, one can try to apply the sufficient controllability condition obtained in Aguilar ([1]). Unfortunately, the STLC property of this example can not be obtained as a corollary of this result. For this reason we apply another differential-geometrical approach to determine the possible directions of expansion of the reachable set of smooth control system. This approach is based on a suitable definition of tangent vector fields to the reachable set of a control system, namely the set E^+ defined in chapter 2 of analytic vector fields. The definition of this set is related to the works of Krener ([27]), Hermes ([14]), Sussman ([33]), Kunita ([28]), Veliov and Krastanov ([38]), Hirshorn ([16]), Krastanov and Quincampoix ([25]), Krastanov and Veliov ([26]) and others.

We continue with the considered example. We fix $\varepsilon_0 > 0$, and a compact neighborhood Ω of the origin. Without loss of generality, we may think that Ω and $\varepsilon_0 > 0$ are sufficiently small so that the compositions Υ, Υ^2 and Υ^4 appearing below are well defined.

We set $p := (x, y, z)^T$ and define the vector fields $f : R^3 \to R^3$ and $g : R^3 \to R^3$ as follows

$$f(p) = (0, 0, ax^2 + bxy + cy^2 + dxz)^T, g_{\alpha,\beta}(p) := (\alpha, \beta, 0)^T,$$

where α and β are arbitrary elements of the interval $[-1, 1]$. Because

$$\text{Exp } (\varepsilon(f + g))(p) \in \mathbf{R}(p, \varepsilon)$$

for each point $p \in \Omega$ and each $\varepsilon \in [0, \varepsilon_0]$, and $f(\mathbf{0}) = (\mathbf{0})$, we can conclude that the vector field $g_{\alpha,\beta}$ belongs to the set $E^+(\mathbf{0})$.

Let us fix an arbitrary $\varepsilon \in [0, \varepsilon_0]$, and arbitrary $\bar{\alpha}$ and $\bar{\beta}$ from the interval $[-1, 1]$. We set $g := g_{\bar{\alpha},\bar{\beta}}$ and $\Upsilon(\varepsilon) := \text{Exp } (\varepsilon(f + g)) \circ \text{Exp } (\varepsilon(f - g))$. Then $\Upsilon(\varepsilon)(p)$ belongs to $\mathbf{R}(p, 2\varepsilon)$ for each point $p \in \Omega$. Applying the C-B-H formula, we obtain that

$$\Upsilon(\varepsilon) = \text{Exp } \left(2\varepsilon f + \varepsilon^2[g, f] + \frac{\varepsilon^3}{3}[g, [g, f]] + O(\varepsilon^4)\right).$$

Following the proof of Proposition 5.1 in Sussmann (1987) (cf., also, Sussmann (1983)), we consider the automorphism λ which sends f to f and g to $-g$. If we set

$$\Upsilon^2(\varepsilon) := \Upsilon(\varepsilon) \circ \lambda\left(\Upsilon(\varepsilon)\right),$$

then we have that

$$\Upsilon^2(\varepsilon)(p) \in \mathbf{R}(p, 4\varepsilon) \text{ for each } p \in \Omega.$$

On the other hand, applying the C-B-H formula, we obtain that $\Upsilon^2(\varepsilon) =$

$$= \text{Exp } \left(4\varepsilon f + 2\varepsilon^3[f, [f, g]] + \frac{2\varepsilon^3}{3}[g, [g, f]] + O_1(\varepsilon^4)\right).$$

By setting $\Upsilon^4(\varepsilon)) := \Upsilon^2(\varepsilon) \circ \lambda\left(\Upsilon^2(\varepsilon)\right)$, it follows that

$$\Upsilon^4(\varepsilon)(p) \in \mathbf{R}(p, 8\varepsilon) \text{ for each } p \in \Omega.$$

Applying again the C-B-H formula, we obtain that

$$\Upsilon^4(\varepsilon)) = \text{Exp } \left(8\varepsilon f + \frac{4\varepsilon^3}{3}[g, [g, f]] + O_2(\varepsilon^4)\right).$$

Because $f(\mathbf{0}) = \mathbf{0}$, the vector field $[g, [g, f]]$ belongs to the set $E^+(\mathbf{0})$ for each choice of the parameters $\bar{\alpha} \in [-1, 1]$ and $\bar{\beta} \in [-1, 1]$. Clearly we have that

$$[g, [g, f]](x, y, z) = 2(0, 0, a\bar{\alpha}^2 + b\bar{\alpha}\bar{\beta} + c\bar{\beta}^2)^T.$$

We set

$$A := \left\{(\alpha, \beta, 0)^T : \ \alpha, \beta \in [-1, 1]\right\}$$

and

$$B := \left\{2(0, 0, a\bar{\alpha}^2 + b\bar{\alpha}\bar{\beta} + c\bar{\beta}^2)^T : \ \bar{\alpha}, \bar{\beta} \in [-1, 1]\right\}.$$

Because the origin $\mathbf{0}$ of $\mathbf{R^n}$ belongs to the interior of the convex hull of the set $A \cup B$, we can find finite number of elements $c_i \in A \cup B$, $i = 1, \ldots, k$, such that $\mathbf{0}$ belongs to the interior of the convex hull of the set $\{c_i : \ i = 1, \ldots, k\}$. Because the elements of $A \cup B$ are values of elements of $E^+(\mathbf{0})$ at the origin, there exist elements $Z_i \in E^+$ with $Z_i(\mathbf{0}) = c_i$ for each $i = 1, \ldots, k$. Applying Lemma 2.3.5, we obtain that the control system Σ_3 is small-time locally controllable at the origin. We would like to point out that the small-time local controllability of the system Σ_3 is determined completely by the values of the parameters a, b and c, and does not depend on the parameter d. □

5.2 Some corollaries

Let $A_i : \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, \ldots, k$ are linear mappings and C_i, $i = 1, \ldots, k$ are convex closed cones in $\mathbf{R}^n$. We consider the control system Σ_s (called "switching system"):

$$\dot{x}(t) \in A_i x(t) + C_i, i = 1, \ldots, k.$$

An admissible trajectory of the system Σ_s defined on $[0, T]$ is any absolutely continuous function $x : [0, T] \to R^n$, with finitely number of indexes $i_1, i_2, \ldots, i_p$, numbers $0 = t_0 < t_1 < \ldots t_p \leq T$ and integrable functions $u_j : [t_{j-1}, t_j] \to C_{i_j}$, $j = 1, \ldots p$, satisfying

$$\dot{x}(t) = A_{i_j} x(t) + u_j(t) \text{ for almost each } t \in [t_{j-1}, t_j].$$

Corollary 5.2.1 *Let L be a proper linear subspace of R^n such that*

$$rec \ (\ cone \ (\ \{\bigcup_{i=1}^{k} (A_i x + C_i) \,|\, x \in \rho\bar{\mathbf{B}} \cap L, u \in U\} \ \cup \ L \ ) \ ) \subseteq L.$$

Then the switching system Σ_s is not STLC at the origin.

Let L be linear subspace of R^n, such that:

B1. L is invariant with respect to A_i, $i = 1, \ldots, k$, i.e. $A_i x \in L$ for each $x \in L$;

B2. $\text{rec} \left(\text{cone} \left(\bigcup_{i=1}^{k} C_i \ \cup \ L \right) \right) \subseteq L.$

Corollary 5.2.2 *Let B1 and B2 are fulfilled. Then the control system Σ_s is not STLC at the origin.*

The corollary coincides with necessary condition of Krastanov & Veliov (2005). Moreover, if the conditions B1 and B2 are not fulfilled, then the system is STLC at the origin.

Let us consider the system Σ_S

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{k} u_i(t) g_i(t), \tag{5.3}$$

We consider the case $L = \{\mathbf{0}\}$. Then the Assumption A4 takes the form:

A4'. rec (cone ($\{f(\mathbf{0}, u) : \ u \in U\}$)) $= \{\mathbf{0}\}$

and we obtain directly the following corollary

Corollary 5.2.3 *Let the Assumptions A1, A2, A3 and A4' be fulfilled. Then the control system Σ is not STLC at the origin.*

This corollary coincides with Theorem 1 from [33]).

Theorem 5.2.4 (Sussman (1978)) *Let the origin does not belong to the convex hull of the values $f(\mathbf{0}, u)$ that corresponds to those $u \in U$ for which $f(\mathbf{0}, u) \neq 0$. Then the control system Σ_S is not STLC at the origin.*

Let us consider the following control system: Σ_{2b}:

$$\dot{x}(t) = f(x(t)) + u(t)b,$$

where f is polynomial map homogeneous of degree 2, b is a fixed vector in $\mathbf{R}^n$ and $u \in [-1, 1]$.

Then we can formulate the next corollary of the Theorem 5.1.1.

Corollary 5.2.5 *Let $\hat{L}_1 := \{\alpha b : \alpha \in \mathbb{R}\}$ be a proper linear subspace of $\mathbb{R}^n$. Let us assume that rec (cone ($\{ad^2(g_\xi, f)(0) + 2ub : \ \xi \in \rho B \cap \hat{L}_1, u \in [-1, 1]\} \cup \hat{L}_1$)) $\subseteq \hat{L}_1$. Then the control system Σ_{2b} is not STLC at the origin.*

This corollary leads to the following theorem of Sussman, preformulated for this case.

Theorem 5.2.6 (Sussman (1983)) *Let L_1 be the linear span of all Lie brackets in f and b which involves b at most one time. We assume that $[g_b, [g_b, f]](0) \notin L_1(0)$. Then the control system Σ_2 is not STLC at the origin.*

Let us consider the following control system $\Sigma_{2k'}$:

$$\dot{x}(t) = f(x(t)) + u(t)b,$$

where f is polynomial map homogeneous of degree $2k$, b is a fixed vector in $\mathbf{R}^n$ and $u \in [-1, 1]$. The theorem 5.1.1 takes the form:

Corollary 5.2.7 *Let $\hat{L}_{2k-1} := \{\alpha b : \alpha \in \mathbb{R}\}$ be a proper linear subspace of $\mathbb{R}^n$. Let us assume that rec (cone ($\{ad^{2k}(g_\xi, f)(0) + 2ub : \xi \in \rho B \cap \hat{L}_{2k-1}, u \in [-1, 1]\} \cup \hat{L}_{2k-1}$)) $\subseteq \hat{L}_{2k-1}$. Then the control system $\Sigma_{2k'}$ is not STLC at the origin.*

and the necessary optimality condition of Stefani (cf. [31]), applied to the system $\Sigma_{2k'}$ is a corollary of Theorem 5.2.7 .

Theorem 5.2.8 (Stefani (1986)) *Let L_{2k-1} be the linear span of all Lie brackets in f and b which involves b at most $2k-1$ times.We assume that $ad^{2k}(b, f)(0) \notin L_{2k-1}(0)$. Then the system $\Sigma_{2k'}$ is not STLC at the origin.*

Moreover, the necessary condition 5.1.1 is applicable also in case with more than one control system $\Sigma_{2k''}$:

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t)b_i,$$

Corollary 5.2.9 *Let $\hat{L}_{2k-1} := \left\{ \sum_{i=1}^{m} \alpha_i b_i : \alpha_i \in \mathbb{R}, i = 1, \ldots, m \right\}$ be a proper linear span of $\mathbb{R}^n$. If*
rec (cone ($\{ad^{2k}(g_b, f)(0) + \sum_{i=1}^{m} u_i b_i : x \in \rho B \cap \hat{L}_{2k-1}, u \in U\} \cup \hat{L}_{2k-1}$)) $\subseteq \hat{L}_{2k-1}$, then the control system $\Sigma_{2k''}$ is not STLC at the origin.

The last corollary is a new result and it's not a particular case of a known result.

5.3 Proof of the necessary condition

To prove the sufficiency we apply the differential-geometrical approach based on the concept of tangent vector fields to the reachable set of a control system, namely the set E^+ which is defined in chapter 2. To prove the necessity, we use some ideas that can be found in [20], [30], [33], [38] and [26].

We choose the linear independent vectors $p_1, p_2, \ldots, p_k$, so that they generate the linear span L. Next we add the linear vectors $p_{k+1}, \ldots, p_n$ so that the vectors $p_1, p_2, \ldots, p_n$ form a basis of $\mathbb{R}^n$. We denote by M the matrix which columns are the vectors $p_1, p_2, \ldots, p_n$. Let e_i, $i = 1, \ldots, n$, be the vector of $\mathbb{R}^n$ whose i-th component is one and all other components are zero. One can directly check that $p_i = Me_i$, $i = 1, \ldots, n$.

We introduce the new coordinates $(y^T,\ z^T)^T := M^{-1}x$, where the components of y (of z) are the first k (the last $n-k$) components of $M^{-1}x$. Let $y = (y_1, \ldots, y_k)^T$ be an arbitrary vector of $\mathbb{R}^k$ and $\mathbf{0}$ be the origin of $\mathbb{R}^{n-k}$. Then we have that

$$M \begin{pmatrix} y \\ \mathbf{0} \end{pmatrix} = \sum_{i=1}^{k} y_i M e_i = \sum_{i=1}^{k} y_i p_i \in L. \tag{5.4}$$

Let $c := \max \{f(x,u) : \ x \in \rho\bar{\mathbf{B}}, u \in U\}$. Because of the compactness of the set U, the Weierstrass theorem and the continuity of f, the following inequality holds true $c < +\infty$. Also, the compactness of the set U implies the existence of $\bar{T}_0 > 0$ such that each admissible trajectory of Σ starting from the origin at the moment of time zero and corresponding to some arbitrary admissible control from $\Omega_{\bar{T}_0}$ is well defined on the interval $[0, \bar{T}_0]$. Let us fix an arbitrary positive number $T_0 \in (0, \bar{T}_0)$. Without loss of generality we may think that $T_0 > 0$ is so small that each admissible trajectory of Σ defined on $[0, T_0]$ and starting from the origin remains in $\rho\bar{\mathbf{B}}$.

Let us fix an arbitrary admissible control $u \in \Omega_{T_0}$ of Σ and let $x(t)$, $t \in [0, T_0]$, be the corresponding trajectory of Σ starting from the origin and remaining in $\rho\bar{\mathbf{B}}$. We set $(y(t)^T,\ z(t)^T)^T := M^{-1}x(t)$, $t \in [0, T_0]$. Then

$$\begin{pmatrix} \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = M^{-1}\dot{x}(t) = M^{-1}f(x(t), u(t)) =$$
$$= M^{-1} f \left(M \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}, u(t) \right) =: \begin{pmatrix} g(y(t), z(t), u(t)) \\ h(y(t), z(t), u(t)) \end{pmatrix}.$$

Without loss of generality we may think that $T_0 > 0$ is so small that

$$M \begin{pmatrix} y(t) \\ \mathbf{0} \end{pmatrix}, \ M \begin{pmatrix} \mathbf{0} \\ z(t) \end{pmatrix} \text{ and } M \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \tag{5.5}$$

belong to $\rho\bar{\mathbf{B}}$ for each $t \in [0, T_0]$. From here we obtain that

$$\begin{aligned} &\|h(y(t), z(t), u(t)) - h(y(t), \mathbf{0}, u(t))\| \leq \\ &\leq \|M^{-1}\| \left\| f\left(M\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}, u(t)\right) - \right. \\ &\qquad \left. f\left(M\begin{pmatrix} y(t) \\ \mathbf{0} \end{pmatrix}, u(t)\right)\right\| \leq \\ &\leq \|M^{-1}\|\ K \left\| M\begin{pmatrix} \mathbf{0} \\ z(t) \end{pmatrix}\right\| \leq \|M^{-1}\|\ K\ \|M\|\ \|z(t)\|. \end{aligned} \tag{5.6}$$

We set $\tilde{K} := \|M^{-1}\|\ K\ \|M\|$ and let us assume that

$$\mathbf{0} \in \text{ co } \left(\left\{\lambda\ h(y, \mathbf{0}, u) :\ \lambda \geq 0, u \in U \text{ and } y \in \mathbb{R}^k,\right.\right.$$

$$\left.\left.\text{such that } M\begin{pmatrix} y \\ \mathbf{0} \end{pmatrix} \in L \cap \rho\bar{B}\right\} \cap S_{R^{n-k}}\right),$$

where $S_{R^{n-k}}$ is the unit sphere of $\mathbb{R}^{n-k}$ centered at the origin.

Then there exist reals $\gamma_i > 0$ and $\alpha_i > 0$, $i = 1, \ldots, s$, so that $\sum_{i=1}^{s} \alpha_i = 1$, and elements $u_i \in U$ and $y_i \in \mathbb{R}^k$, $i = 1, \ldots, s$, such that $M(y_i^T,\ \mathbf{0}^T)^T \in L \cap \rho B$, $\gamma_i \|h(y_i, \mathbf{0}, u_i)\| = 1$ and

$$\mathbf{0} = \sum_{i=1}^{s} \alpha_i \gamma_i h(y_i, \mathbf{0}, u_i).$$

From here we obtain the existence of $\tilde{y} \in \mathbb{R}^k$ such that

$$\begin{aligned} \begin{pmatrix} \tilde{y} \\ \mathbf{0} \end{pmatrix} &= \sum_{i=1}^{s} \alpha_i \gamma_i \begin{pmatrix} g(y_i, \mathbf{0}, u_i) \\ h(y_i, \mathbf{0}, u_i) \end{pmatrix} = \\ &= \sum_{i=1}^{s} \alpha_i \gamma_i M^{-1} f\left(M\begin{pmatrix} y_i \\ \mathbf{0} \end{pmatrix}, u_i\right) = \\ &= \sum_{i=1}^{s} \alpha_i \gamma_i M^{-1} f\left(l_i, u_i\right), \end{aligned}$$

where $l_i := M(y_i^T,\ \mathbf{0}^T)^T \in L \cap \rho\bar{\mathbf{B}}$. The last equality implies that

$$L \ni \tilde{l} := M \begin{pmatrix} \tilde{y} \\ \mathbf{0} \end{pmatrix} = \sum_{i=1}^{s} \alpha_i \gamma_i f\left(l_i, u_i\right).$$

Because $\alpha_1\gamma_1 > 0$, the above written equality implies that

$$-f\left(l_1, u_1\right) = \frac{1}{\alpha_1\gamma_1} \sum_{i=2}^{s} \alpha_i \gamma_i f\left(l_i, u_i\right) + \bar{l} \tag{5.7}$$

belongs to cone $(f(L \cap \rho\bar{\mathbf{B}}, U) \cup L)$, where $\bar{l} := -\tilde{l}/(\alpha_1\gamma_1)$ and

$$f(L \cap \rho\bar{\mathbf{B}}, U) := \left\{ f(x, u) : \ x \in L \cap \rho\bar{\mathbf{B}}, u \in U \right\}.$$

Let us assume that $f\left(l_1, u_1\right) \in L$. Then the definition of the matrix M (cf., also (5.4)) implies that $M^{-1} f\left(l_1, u_1\right) = (\hat{y}, 0)$ for some $\hat{y} \in \mathbb{R}^k$. But this contradicts the equality, $M^{-1} f\left(l_1, u_1\right) = (g(y_1, \mathbf{0}, u_1), h(y_1, \mathbf{0}, u_1))$ with $h(y_1, \mathbf{0}, u_1) \neq \mathbf{0}$. The obtained contradiction shows that $f\left(l_1, u_1\right) \notin L$.

Clearly, $f\left(l_1, u_1\right) \in$ cone $(\ f(L \cap \rho\bar{\mathbf{B}}, U) \ )$. The inclusion (5.7) implies that $-f\left(l_1, u_1\right) \in$ cone $(\ f(L \cap \rho\bar{\mathbf{B}}, U) \cup L)$. Because $f\left(l_1, u_1\right) \notin L$, we obtain a contradiction with the Assumption A4. Hence

$$\mathbf{0} \notin \ \text{co} \ \Big(\big\{ \lambda \ h(y, \mathbf{0}, u) | \ \lambda \geq 0, u \in U \ \text{and} \ \ y \in R^k, \\ \text{such that} \ \ M \begin{pmatrix} y \\ 0 \end{pmatrix} \in L \cap \rho\bar{\mathbf{B}} \big\} \cap S_{R^{n-k}} \Big).$$

Then the Separability theorem implies the existence of a nonvanishing vector ξ and a real $\varepsilon > 0$ such that

$$\left\langle \xi, \frac{h(y, \mathbf{0}, u)}{\|h(y, \mathbf{0}, u)\|} \right\rangle \geq \varepsilon \tag{5.8}$$

for each $h(y, \mathbf{0}, u) \neq \mathbf{0}$ corresponding to some $u \in U$ and $y \in \mathbb{R}^k$ and satisfying the relation $M(y^T, \ 0^T)^T \in L \cap \rho\bar{\mathbf{B}}$.

Let T be an arbitrary real from the interval $(0, T_0)$. Because

$$z(t) = \int_0^t h(y(s), z(s), u(s)) \mathrm{d}s, \ \ t \in [0, T],$$

we obtain

$$z(t) = \int_0^t h(y(s), z(s), u(s)) \mathrm{d}s =$$

$$= \int_0^t h(y(s), z(s), u(s)) - h(y(s), \mathbf{0}, u(s)) \mathrm{d}s +$$

$$+\int_0^t h(y(s),\mathbf{0},u(s))\mathrm{d}s.$$

According to (5.6) we obtain that

$$\|z(t)\| = \left\| \int_0^t h(y(s),z(s),u(s)) - h(y(s),\mathbf{0},u(s))\mathrm{d}s + \right.$$

$$\left. + \int_0^t h(y(s),\mathbf{0},u(s))\mathrm{d}s \right\| \leq$$

$$\leq \int_0^t \|h(y(s),z(s),u(s)) - h(y(s),\mathbf{0},u(s))\|\mathrm{d}s +$$

$$+\int_0^t \|h(y(s),\mathbf{0},u(s))\|\mathrm{d}s \leq$$

$$\leq \tilde{K}\int_0^t \|z(s)\|\mathrm{d}s + \int_0^T \|h(y(s),\mathbf{0},u(s))\|\mathrm{d}s.$$

Applying the Gronwall inequality, we obtain that

$$\|z(t)\| \leq e^{\tilde{K}t}\int_0^T \|h(y(s),\mathbf{0},u(s))\|\mathrm{d}s,\ t\in[0,T]. \tag{5.9}$$

Then $\langle \xi, z(t)\rangle = \langle \xi, \int_0^t h(y(s),z(s),u(s))\mathrm{d}s\rangle =$

$$= \langle \xi, \int_0^t (h(y(s),z(s),u(s)) - h(y(s),\mathbf{0},u(s)))\mathrm{d}s\rangle +$$

$$+\langle \xi, \int_0^t h(y(s),0,u(s))\mathrm{d}s\rangle$$

We estimate each of the addends. Applying (5.9), we obtain that:

$$\left| \int_0^t \langle \xi, h(y(s),z(s),u(s)) - h(y(s),\mathbf{0},u(s))\rangle \mathrm{d}s \right| \leq$$

$$\leq \int_0^t \|\xi\|.\|h(y(s),z(s),u(s)) - h(y(s),\mathbf{0},u(s))\|\mathrm{d}s \leq$$

$$\leq \|\xi\| \int_0^t \tilde{K}\|z(s)\|\mathrm{d}s \leq$$

$$\leq \tilde{K}\|\xi\| \int_0^t e^{\tilde{K}s} \int_0^T \|h(y(\sigma),\mathbf{0},u(\sigma))\|\mathrm{d}\sigma\mathrm{d}s \leq$$

$$\leq \tilde{K}\|\xi\|te^{\tilde{K}t} \int_0^T \|h(y(\sigma),\mathbf{0},u(\sigma))\|\mathrm{d}\sigma.$$

We set $\Theta := \{s \in [0,T_0] : \ h(y(s),\mathbf{0},u(s)) \neq 0\}$. Then

$$\int_0^t \langle \xi, h(y(s),\mathbf{0},u(s))\mathrm{d}s\rangle = \int_\Theta \langle \xi, h(y(s),\mathbf{0},u(s))\mathrm{d}s\rangle =$$

$$= \int_\Theta \left\langle \xi, \frac{h(y(s),\mathbf{0},u(s))}{\|h(y(s),\mathbf{0},u(s))\|}\|h(y(s),\mathbf{0},u(s))\| \right\rangle \mathrm{d}s \geq$$

$$\geq \varepsilon \int_\Theta \|h(y(s),\mathbf{0},u(s))\|\mathrm{d}s = \varepsilon \int_0^t \|h(y(s),\mathbf{0},u(s))\|\mathrm{d}s$$

The last inequality follows from (5.8). Using the above written estimates, we obtain that

$$\langle \xi, z(T)\rangle = \int_0^T \langle \xi, h(y(s),\mathbf{0},u(s))\mathrm{d}s\rangle +$$

$$+ \int_0^T \langle \xi, h(y(s),z(s),u(s)) - h(y(s),\mathbf{0},u(s))\rangle \mathrm{d}s \geq$$

$$\geq \int_0^T \langle \xi, h(y(s), \mathbf{0}, u(s))\mathrm{d}s \rangle$$

$$- \left| \int_0^T \langle \xi, h(y(s), z(s), u(s)) - h(y(s), \mathbf{0}, u(s))) \rangle \mathrm{d}s \right| \geq$$

$$\geq \varepsilon \int_0^T \|h(y(s), \mathbf{0}, u(s))\| \mathrm{d}s$$

$$-\tilde{K}\|\xi\|Te^{\tilde{K}T} \int_0^T \|h(y(s), \mathbf{0}, u(s))\| \mathrm{d}s =$$

$$= \left(\varepsilon - \tilde{K}\|\xi\|Te^{\tilde{K}T}\right) \int_0^T \|h(y(s), \mathbf{0}, u(s))\| \mathrm{d}s \geq 0$$

for each sufficiently small $T > 0$. It follows from here that the control system Σ_n is not STLC at the origin. $\diamond$

Chapter 6

Conclusion

6.1 Main contributions

The main contributions in the thesis due to the author are:

1. Extension of the Sussman's approach to neutralize brackets. It's provided a new sufficient condition for small-time local controllability for polynomial right-hand side which is homogeneous of second degree using neutralization of "bad" bracket with the same weight.
2. It is developed a new method for constructing new elements of the set of tangent vector fields to the reachable set of a polynomial control system whose drift term is a homogeneous vestor field of second degree.
3. A new look on the "weight" is considered. It's developed a nonstandard and complicated method to construct a "weight" of a Lie bracket.
4. A new necessary condition for small-time local controllability is obtained under natural assumptions.

6.2 Publications related to the thesis

1. M. I. Krastanov, M. N. Nikolova, A necessary condition for small-time local controllability, Automatica,2020, ISSN (online):0005-1098, Ref web of Science, IF: 5.944 (2020), Web of Science Quartile: Q1 (10/63 Automation & ControlSystems, JCR-2020)

2. M. I. Krastanov, M. N. Nikolova, A sufficient condition for small-time controllability of a polynomial control system, Comptes rendus de l'Academie bulgare des Sciences, 2020, vol:73, issue:12, pages: 1638-1649, ISSN(print):1310-1331, ISSN(online):2367–5535, Ref Web of Science, IF: 0.378 (2020), Web of Science Quartile: Q4 (71/72 Multidisciplinary Sciences, JCR-2020)

3. M. I. Krastanov, M. N. Nikolova, On the small-time local controllability, Systems & Control Letters, 2023, vol:177, ISSN:0167-6911, EISSN:1872-7956, Ref Web of Science, IF: 2.742 (2021), Web of Science Quartile: Q2 (38/87 Operations research & management science, JCR-2021)

6.3 Approbation of the thesis

The results from the thesis have been presented in the following talks:

1. "A sufficient condition for small-time controllability of a polynomial control systems", Seminar on Optimization, Faculty of Mathematics and Informatics, Sofia University, 3 November 2020 (based on a joint work with M. Krastanov)

2. "Approximations of control affine systems", Seminar on Optimization, Faculty of Mathematics and Informatics, Sofia University, 10 May 2021 (based on a paper of H. Hermes)

3. "On small-time local controllability", The 13th International Conference on Large-Scale Scientific Computations LSSC 2021, June 7 - 11, 2021, Sozopol, Bulgaria, (based on a joint work with M. Krastanov)

4. "A sufficient condition for small-time controllability" Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University, 26 March 2022 (based on a joint work with M. Krastanov)

5. "A sufficient condition for small-time controllability" Seminar on Optimization, Faculty of Mathematics and Informatics, Sofia University, 6 June 2022 (based on a joint work with M. Krastanov)

6. "High-order small-time local controllability" Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University, 25 March 2023 (based on a joint work with M. Krastanov)

7. "High-order small-time local controllability", The 14th International Conference on Large-Scale Scientific Computations LSSC 2023, June 5 - 9, 2023, Sozopol, Bulgaria, (based on a joint work with M. Krastanov)

8. ”On small-time local controllability”, 16-th International Workshop on Well-Posedness of Optimization Problems and Related Topics , July 3 - 7, 2023, Borovets, Bulgaria, (based on a joint work with M. Krastanov)

Noticed citations:

1. U. Boscain, D. Cannarsa, V. Franceschi, M. Sigalotti, Local controllability does imply global controllability,(2021) arxiv preprint, arXiv:2110.06631
2. D Cannarsa, Surfaces in three-dimensional contact sub-Riemannian manifolds and controllability of nonlinear ODEs (2021), PhD Thesis, https://theses.hal.science/tel-04053226/

6.4 Declaration of originality

The author declares that the thesis contains original results obtained by him or in cooperation with his coauthors. The usage of results of other scientists is accompanied by suitable citations.

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