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Variational analysis without variational principles

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Contents

Chapter 1

Introduction

Modern variational analysis can be viewed as a further extension of the calculus of variations with focus on optimization of functions relative to various constraints and on sensitivity and stability of optimization-related problems with respect to perturbations.

One of the most characteristic features of modern variational analysis is the intrinsic presence of nonsmoothness, i.e. the necessity to deal with nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings. One reason for the growth of the subject has been, without a doubt, the recognition that nondifferentiable phenomena are more widespread, and play a more important role than smooth ones. Many fundamental objects frequently appearing in the framework of variational analysis (e.g., the distance function, value functions in optimization and control problems, solution maps, etc.) are inevitably nonsmooth and also have set-valued structures requiring the development of new forms of analysis that involve generalized differentiation.

Even the simplest and historically earliest problems of optimal control are intrinsically nonsmooth, in contrast to the classical calculus of variations. Optimal control has always been a major source for applications of advanced methods of variational analysis and generalized differentiation.

Since the discovery of Pontryagin maximum principle, the dawn of optimal control theory, various versions of this result have been established, under different technical assumptions and with different proofs. As early as in 1965, Dubovickii and Miljutin realized the importance of convex approximations of closed sets for obtaining necessary optimality conditions for nonlinear problems in optimization. In a series of papers (cf., for example, the bibliography of [60]), the corresponding proofs are based on theorems for nonseparation of sets.

The classical concept of transversality has been applied successfully as a qualification condition in nonseparation results. Transversality is originally studied in the fields of mathematical analysis and differential topology. Recently, it has proven to be useful in variational analysis as well. As it is stated in [38], the transversality-oriented language is extremely natural and convenient in some parts of variational analysis, including subdifferential calculus and nonsmooth optimization, as well as in proving sufficient conditions for linear convergence of the alternating projections algorithm (cf. [30]).

The classical definition of transversality at an intersection point of two smooth manifolds in a Euclidean space is that the sum of the corresponding tangent spaces at the intersection point is the whole space (cf. [32], [33]).

In order to prove the Pontryagin maximum principle (cf., for example, the bibliography of [60]), Hector Sussmann generalizes the definition of transversality for closed convex cones in $\mathbb{R}^n$: the cones C^A and C^B are transversal if and only if

$$C^A - C^B = \mathbb{R}^n$$

and strongly transversal, if they are transversal and $C^A \cap C^B \neq \{0\}$ (cf. Definitions 3.1 and 3.2 from [60]). In the finite-dimensional case, strong transversality of the approximating cones of the same type (either Clarke or Boltyanski) is a sufficient condition for local nonseparation of sets. The sets A, B containing a point x_0 are said to be locally separated at x_0, if there exists a neighborhood Ω of x_0 so that $\Omega \cap A \cap B = \{x_0\}$. In infinite-dimensional case, strong transversality of the approximating cones of the same type does not imply local nonseparation of sets, as shown by the following example.

Example 1.0.1. *Take the Hilbert cube*

$$A := \{(x_n) \in l_2 : \ |x_n| \leq 1/n\} \subset l_2$$

and a ray $B := \{\lambda y : \ \lambda \geq 0\}$, *where* $y := (1/n^{3/4})_{n=1}^{\infty}$. *We have that the corresponding Clarke tangent cones* $\widehat{T}_A(\mathbf{0}) = l_2$ *and* $\widehat{T}_B(\mathbf{0}) = B$ *are strongly transversal, while the sets A and B are locally separated at 0.*

There are various transversality-type properties reflecting the various needs of the possible applications. In the literature there exist many notions generalizing the classical transversality as well as transversality of cones. Some of them are introduced under different names by different authors, but actually coincide. We refer to [51] for a survey of terminology and comparison of the available concepts. The central ones among them are *transversality* and *subtransversality*. They are also objects of study in the recent book [39]. One of the reasons for that is the close relation to metric regularity and metric subregularity, respectively.

The term subtransversality is recently introduced in [30] in relation to proving linear convergence of the alternating projections algorithm. However, as said earlier, it has been around for more than 20 years, but under different names – see Remark 4 in [51] and the references therein. It is a key assumption for two types of results: linear convergence of sequences generated by projection algorithms and a qualification condition for normal intersection property with respect to the limiting normal cone and a sum rule for the limiting subdifferential. Another quite remarkable feature of subtransversality, was investigated in [9]. It turns out that subtransversality implies a rather general nonseparation result which is crucial for obtaining necessary optimality conditions of Pontryagin maximum principle type (including optimal control problems with infinite-dimensional state space). Moreover, subtransversality is a natural assumption for proving abstract Lagrange multiplier rule.

The following results are taken from [9].

Proposition 1.0.2 (Nonseparation result). *Let A and B be closed subsets of the Banach space X. Let A and B be subtransversal at $x_0 \in A \cap B$ with constants $\delta > 0$ and $K > 0$. Assume that there exist v^A with unit norm which belongs to the Bouligand tangent cone to A at x_0 and v^B with unit norm which belongs to the derivable tangent cone to B at x_0, such that $\|v^A - v^B\| < \frac{1}{K}$. Then A and B cannot be locally separated at x_0.*

Theorem 1.0.3 (Lagrange multiplier rule). *Let us consider the optimization problem*

$$f(x) \to \min \quad \textit{subject to} \quad x \in S \ ,$$

where $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper and S is a closed subset of the Banach space X. Let x_0 be a solution of the above problem. Let $\tilde{C}_{epif}(x_0, f(x_0))$ and $C_S(x_0)$ be closed convex cones, contained in the corresponding Bouligand approximating cones $T_{epif}(x_0, f(x_0))$ and $T_S(x_0)$. Let at least one of them consist of derivable tangent vectors.
(a) If $\tilde{C}_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]$ is not dense in $X \times \mathbb{R}$, then there exists a pair $(\xi, \eta) \in X^* \times \mathbb{R}$ such that

(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) $\langle \xi, v \rangle \ \leq 0$ *for every* $v \in C_S(x_0)$;

(iv) $\langle \xi, w \rangle + \eta s \geq 0$ *for every* $(w, s) \in \tilde{C}_{epif}(x_0, f(x_0))$.

(b) If $\tilde{C}_{epif}(x_0, f(x_0)) - C_S(x_0) \times (-\infty, 0]$ *is dense in* $X \times \mathbb{R}$*, then* $epif$ *and* $S \times (-\infty, f(x_0)]$ *are not subtransversal at* $(x_0, f(x_0))$*.*

Another notion of tranversality - tangential transversality, was introduced recently by Bivas, Krastanov and Ribarska in [9]. The authors arrived to the study of transversality of sets when investigating Pontryagin's type maximum principle for optimal control problems with terminal constraints in infinite dimensional state space.

Besides the aformentioned results, the authors also established intersection rules for tangent cones in Banach spaces and some relations to masiveness of sets. Many questions about tangential transversality remained open (see [9], p. 28).

These results inspired one of the lines of research in the thesis, which is connected to the application of subtransversality and tangential transversality for obtaining necessary optimality conditions in terms of abstract Lagrange multipliers.

The intriguing thing here is to verify the subtransversality assumption in nontrivial cases. Our aim is to find some conditions which are sufficient for subtransversality of two sets. However, the approach we take is proving tangential transversality instead of subtransversality. It happens that usually tangential transversality is easier to verify than subtransversality when the information known concerns the tangential structure of the sets.

We present a general sufficient condition for tangential transversality (cf. Theorem 4.3.1 and Theorem 4.3.2). The underlying idea is that in many cases the uniformness of the local approximation of a closed set can be used instead of some suitable compactness assumption. This is especially important in the infinite-dimensional case.

We motivate the usefulness of the obtained general results by providing some applications. One of them is finding a Lagrange multiplier when one of the sets is the epigraph of a function which is Lipschitz in one of the variables, uniformly with respect to the other.

The main application we obtained, in fact the starting point of this research, was the famous Aubin condition from [15] for the basic problem of the calculus of variations. We formulate an abstract (infinite-dimensional) version of this condition. This abstract version inspired the rest of the results in Chapter 4. We show that if a function (actually its epigraph) satisfies this assumption and the constraint has a specific form (tailored after the constraint in the basic problem of the calculus of variations as an infinite dimensional optimization problem), one can find a Lagrange multiplier. Sure, the proof makes use of our main result. It is worth noting that in our abstract version of Aubin condition, compactness of the operator is not necessary. For our argument, it is sufficient to assume that the image under the operator L of the correcting set is totally bounded in X. In fact, the case when L is the integration operator from $Y = L_1([a,b],\mathbb{R}^n)$ to $X = L_\infty([a,b],\mathbb{R}^n)$ could be important for future applications of our results. Clearly, this operator is bounded but not compact, and it maps weakly compact sets in Y to totally bounded sets in X, thus allowing to use weakly compact sets as "correcting sets". To further motivate our main result, we show that some known sufficient conditions for tangential transversality can be obtained as its particular cases. Namely, we obtain Theorem 4.2.11 (taken from [9]) and Proposition 4.2.13 (taken from [8]) as corollaries of our main result Theorem 4.3.2. Moreover, the well known notion of compactly epi-Lipschitz set is extended for a pair of closed sets (cf. Definition 4.4.7) and is shown that it could also be used as a sufficient condition for tangential transversality. This investigation has been developed in [46], where a more general necessary optimality condition, involving measures of noncompactness, is proved.

Yet another notion of transversality was introduced recently by Drusvyatskiy, Ioffe and Lewis in [30]. It is intermediate between subtransversality and transversality and serves as an important sufficient condition for local linear convergence of alternating projection algorithm for solving finite dimensional nonconvex feasibility problems. It steadily grows in importance and number of researchers extend this transversality concept to more general settings and investigate its primal and dual characterizations. These notions (which some authors call "good arrangements of sets") and the relations between them, have been studied in details. See, e.g., [19],[20], [18], [50], and the literature therein. Still some aspects are not well understood. Indeed, one of the starting points of this investigation was a question of A.Ioffe about finding a metric characterization of intrinsic transversality. In fact, a variety of characterizations of intrinsic transversality in various settings are known (Euclidean, Hilbert, Asplund, Banach and normed linear spaces) but all of them involve the linear structure of the space. The reason is that researchers are mainly concentrated on the dual space. To the best of our knowledge, the first primal characterization of intrinsic transversality is obtained in [61] where the structure of a Hilbert space is assumed in most of the considerations.

These questions, along with the unknown relation between tangential transversality and intrinsic transversality, give rise to another line of research in the thesis.

The result of our study was somewhat surprising: it happened that intrinsic transversality and tangential transversality are "almost" equivalent. Moreover, the relation is very easy to establish, given the characterization of intrinsic transversality via the slope of coupling function due to Ioffe and

Lewis. Thus a primal space characterization of intrinsic transversality has been obtained. We put a significant effort in clarifying the exact relationship of this characterization and the primal characterization of intrinsic transversality obtained by Thao et al. in [61], which they call property $(\mathcal{P})$. We proved that property $(\mathcal{P})$ implies our characterization in general Banach space setting and these properties are equivalent in Hilbert space setting. We would like to emphasize that the property we introduce is simpler (or at least it looks simpler) than the property $(\mathcal{P})$ – less variables are involved.

Establishing the exact relationship between intrinsic transversality and tangential transversality helped us to obtain primal space infinitesimal characterizations and slope characterizations of both transversality and subtransversality close in nature to tangential transversality. Thus, although the definitions and motivations for the four types of transversality properties we consider, are not similarly looking, we obtained characterizations in a unified manner for all of them, which make obvious their close relations on the one handside, and their differences on the other handside. Indeed, it is now obvious that

$$\text{transversality} \implies \begin{array}{c}\text{tangential}\\ \text{transversality}\end{array} \implies \begin{array}{c}\text{intrinsic}\\ \text{transversality}\end{array} \implies \text{subtransversality}$$

and neither implication is invertible. This hierarchy of the properties and of their respective slope characterizations sheds new light on the topic. There have been known primal sufficient conditions and primal necessary conditions for transversality and subtransversality, but no primal characterizations (see [20] and [19]). The relationship of our characterization to these conditions is very similar to the relationship of our characterization of intrinsic transversality to property $(\mathcal{P})$ – we work with less points which makes the situation simpler.

After obtaining characterizations of these transversality concepts in a unified manner, we go on to examine the regularity concepts. We obtain a characterization of metric regularity properties of a set-valued map in terms of transversality properties of sets associated with the graph of the set-valued maps. We show directly that one can transfer from subtransversality to metric subregularity and from transversality to metric regularity. Similar results were already obtained in [21], [22] and [12], but there is no clear statement of such interchangeability. We moreover show proofs of some known primal space characterizations of the regularity concepts, using the already derived characterzations of their transversality counterparts. We also show how one can easily obtain from these results the characterizations of metric regularity via the *first order variation* and the *graphical derivative*.

In the last chapter of the thesis we consider continuity of the optimal value mapping for an abstract optimization problem in metric spaces, where the feasible set varies, i.e. depends on a parameter. Specifically, we deal with the function

$$S_{\text{val}}(p) := \inf\{g(y) \mid y \in D(p)\}.$$

where X and Y are metric spaces, $D : X \rightrightarrows Y$ is a set-valued mapping and $g : Y \to \mathbb{R}$ is a function. The classical Maximum theorem of Berge ([7]) (in the more general setting when X and Y are merely topological spaces) considers the case when g also depends on p and says that when g is continuous (on $X \times Y$) and D is compact-valued and continuous at $\bar{p} \in X$, then S_{val} is continuous at $\bar{p}$. It is widely used in mathematical economics and optimal control.

Another version of this result is due to Berdyshev ([6]) where a so-called t-continuity (which is stronger than the well known Pompeiu-Hausdorff continuity) is required for the mapping D (see Theorem 5.3.1). The result of Berdyshev also shows that when the space is metric and g is uniformly continuous on Y, the Pompeiu-Hausdorff continuity suffices to prove continuity of S_{val}. The corresponding definitions are stated explicitly in the chapter.

Generalizations of the classical Berge theorem, which consider various well-posedness conditions of the function on the constraint set that also guarantee continuity of the value function, can be found in the book of Lucchetti ([54]). Detailed discussion on this topic could be also found in the book by Dontchev and Zolezzi ([28]).

The motivation for our investigations on this topic was Theorem 5 of Chapter IX, Section 1, in [28], which states as follows

Theorem 1.0.4. *Assume that for some point $\bar{p}$ of the topological space X, D is continuous at $\bar{p}$ and g is continuous on $D(\bar{p})$. Then S_{val} is continuous at $\bar{p}$.*

However, in [28] it is not clearly stated what kind of continuity the authors have in mind, and this may lead to a possible confusion. We will show by a counterexample that the theorem is false if the assumed continuity of the mapping D is in the Pompeiu-Hausdorff sense in the case of metric spaces. Note that in [28] the spaces are topological (as in Theorem 1.1) so it is reasonable to assume that a topological definition of continuity is had in mind. Still this is not clearly stated. The main purpose of this chapter of the thesis is to consider this issue (in the case of metric spaces), namely when Theorem 1.0.4 holds and when it does not, and in the latter case, we examine additional assumption, under which it holds. We investigate the interplay between the continuity properties of f and D which would guarantee continuity of S_{val}. In the course of our research, we formulate a continuity assumption depending both on f and D, which we call Relaxed uniform continuity assumption, (**RUCA**). We show that it is sufficient for continuity of S_{val} but is also in some sense necessary. Moreover, we comment on how earlier results fit naturally in our approach.

Throughout the thesis, we have eschewed using variational principles, though some of our results could be obtained in this way, too. However, we prefer to lean more on geometric intuition, which, in our understanding, makes the results and their proofs more natural and well motivated.

The thesis is organized as follows.

The second chapter contains necessary preliminary definitions and results.

The third chapter is divided into six sections. In the first section a primal characterization of subtransversality is obtained. To do it, we prove a technical result (Lemma 3.1.3) allowing to pass from a local inequality to a global one. Its proof essentially appeared in [9] and it is based on transfinite induction. It could have been proved using Ekeland's variational principle like most "rate of descend" results in the literature, but we prefer the transfinite induction because it is really direct to employ and requires very little thought – a simple induction enables the transition to a global property from a local one in a straightforward manner. In our understanding this kind of argument is natural and saves one from the necessity of seeking for the "right" function in every particular case. Moreover, a slope characterization of subtransversality is proved in the first section. The second section is devoted to transversality. A primal characterization of transversality is obtained from the respective characterization of subtransversality. The relation to tangential transversality is clarified emphasizing the fact that in the "uniform" situation of the transversality property the existence of "a positive step" and the existence of "an interval $(0, \delta)$ of possible steps" are equivalent. Using this, two slope characterizations of transversality are obtained. The third section deals with intrinsic transversality. A primal characterization (of purely metric nature) and a slope characterization are readily proved. The exact relation of our approach and the approach of Thao et al. in [61] is established. This is the most technical result in this chapter. We obtain the coincidence of subtransversality and intrinsic transversality in the convex case as an easy consequence. In the fourth section we obtain a characterization of metric regularity via transversality and metric subregularity via subtransversality. In the last two sections we use the primal space characterization of transversality and subtransversality to derive primal space characterizations of metric regularity and metric subregularity.

The fourth chapter contains three sections. It begins with a section on preliminaries, giving detailed definitions and discussion on already existing notions and results we use, mainly concerning uniform tangent sets. The second section contains the main result, an abstract sufficient condition for tangential transversality. It is stated in two forms, the first being more intuitive and easy to grasp, and the second being more general. The third section contains three original applications, along with new proofs of already existing theorems. The first application is concerned with functions which are Lipschitz in one of the variables, uniformly with repsect to the other. After that, a notion of jointly massive sets is introduced, and is used in another sufficient condition. The third application shows that the Aubin condition and compactness of the operator defining the feasible set, are also sufficient for tangential transversality. Finally, all three applications are combined in a theorem providing Lagrange multiplier rule.

The fifth chapter is divided into two sections. The first shows a counterexample to a result whose hypothesis is not enough clarified. After that, an attempt to remedy the situation is provided, making use of the (**RUCA**) as one of the hypothesis. In the second section, a result concerning (**RUCA**) and its relation to the notion of topological continuity of set-valued mappings is proved.

Chapter 2

Preliminaries

2.1 Basic notation

The following notations are used throughout the thesis:

1. if X is a metric space, $\mathbf{B}_r(x_0)$ will denote the open ball centered at x_0 with radius r; the closed ball will be denoted by $\bar{\mathbf{B}}_r(x_0)$.

2. the closure of a set C is denoted by $\bar{C}$

3. the boundary of a set C is denoted be ∂C, i.e. the set of those points in X such that for any $\varepsilon > 0$, $A \cap \mathbf{B}_\varepsilon(x) \neq \varnothing$ and $(X \setminus A) \cap \mathbf{B}_\varepsilon(x) \neq \varnothing$.

4. the convex hull of a set is denoted by *co*.

5. the conical hull of a set C is denoted by cone C, i.e.

$$\text{cone } C = \{rc \mid r > 0,\ c \in C\}.$$

Given a metric space X and a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we define the epigraph of f as follows:

$$\text{epi } f = \{(x, r) \in X \times \mathbb{R} \mid r \geq f(x)\}.$$

We say that such a function is lower-semicontinuous if epi f is a closed subset of $X \times \mathbb{R}$. We say that the function is proper if the set $\{x \in X \mid f(x) < +\infty\}$ is nonempty.

For a subset A of X and $\varepsilon > 0$ we define the $\varepsilon-$neighbourhood of A - A_ε as

$$A_\varepsilon = \bigcup_{x \in A} \mathbf{B}_\varepsilon(x) = \{z \in X \mid \exists x \in A,\ \rho(z, x) < \varepsilon\}.$$

2.2 Set-valued mappings and regularity

Definition 2.2.1. *Let X and Y be metric spaces, and 2^Y be the family of all subsets of Y. A mapping $F : X \to 2^Y \setminus \{\varnothing\}$ is called set-valued mapping. Another notation is $F : X \rightrightarrows Y$.*
Remark: It is possible the domain and the range of F to be topological spaces, which are usually assumed to be Hausdorff. .

The graph of F, denoted by Gr F, is defined by

$$\text{Gr } F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

The inverse map of F, $F^{-1} : Y \rightrightarrows X$ is defined by

$$F^{-1}(y) := \{x \in X \mid y \in F(x)\}, \text{ whenever } y \in Y.$$

We remind the already classical definitions.

Definition 2.2.2. *Let X and Y be metric spaces, $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in$ Gr F. We say that F is (metrically) regular around $(\bar{x}, \bar{y})$ if there exist $K > 0$ and $\delta > 0$ such that for all $x \in \mathbf{B}_\delta(\bar{x})$ and all $y \in \mathbf{B}_\delta(\bar{y})$ the following inequality holds:*

$$d(x, F^{-1}(y)) \leq K d(y, F(x)).$$

Definition 2.2.3. *Let X and Y be metric spaces, $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in$ Gr F. We say that F is (metrically) subregular around $(\bar{x}, \bar{y})$ if there exist $K > 0$ and $\delta > 0$ such that for all $x \in \mathbf{B}_\delta(\bar{x})$ the following inequality holds:*

$$d(x, F^{-1}(\bar{y})) \leq K d(\bar{y}, F(x)).$$

2.3 Slopes and coupling function

For a set A in a metric space X, we denote by $\delta_A : X \to \mathbb{R} \cup \{+\infty\}$ its indicator function

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A \\ +\infty, & \text{otherwise.} \end{cases}$$

Following [23] we introduce two types of slopes.

Definition 2.3.1. *Consider a metric space X, a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ and a point $\bar{x} \in X$ such that $f(\bar{x})$ is finite. The slope of f at $\bar{x}$ is*

$$|\nabla f|(\bar{x}) := \limsup_{x \to \bar{x}} \frac{\max\{f(\bar{x}) - f(x), 0\}}{d(\bar{x}, x)}.$$

The nonlocal slope is

$$|\nabla f|^{\diamond}(\bar{x}) := \sup_{x \neq \bar{x}} \frac{\max\{f(\bar{x}) - f(x), 0\}}{d(\bar{x}, x)}\,.$$

Definition 2.3.2. *For subsets A and B of the metric space X, the so-called "coupling function" $\phi : X \times X \to \mathbb{R} \cup \{+\infty\}$ is defined as*

$$\phi(x, y) = \delta_A(x) + d(x, y) + \delta_B(y)\,.$$

The above definition has been introduced in [29].

2.4 Transversality concepts

Assume that A and B are subsets of the normed space X. Consider the function $H_{A,B} : X \times X \to X$ defined as

$$H_{A,B}(x_1, x_2) = \begin{cases} \{x_1 - x_2\}, & x_1 \in A,\ x_2 \in B \\ \varnothing, & else \end{cases} \tag{2.1}$$

Definition 2.4.1. *Let X be a normed space, and A, B be closed subsets of X. Let $\bar{x} \in A \cap B$. Then A and B are called transversal at $\bar{x}$ if $H_{A,B}$ is regular around $((\bar{x}, \bar{x}), \mathbf{0})$.*

Definition 2.4.2. *Let X be a normed space, and A, B be closed subsets of X. Let $\bar{x} \in A \cap B$. Then A and B are called subtransversal at $\bar{x}$ if $H_{A,B}$ is subregular around $((\bar{x}, \bar{x}), \mathbf{0})$.*

These definitions are derived as characterizations in [51], where another definitions of transversality are used. A characterization of transversality derived in [37] (cf. [48]) is

Proposition 2.4.3. *Let A and B be closed subsets of the normed space X. A and B are transversal at $\bar{x} \in A \cap B$, if and only if there exists $K > 0$ and $\delta > 0$ such that*

$$d(x, (A - a) \cap (B - b)) \leq K(d(x, A - a) + d(x, B - b))$$

for all $x \in \bar{\mathbf{B}}_{\delta}(\bar{x})$ and $a, b \in \bar{\mathbf{B}}_{\delta}(\mathbf{0})$.

One observes that only one of the sets could be translated, i.e. we may take $a = 0$ and only vary b. A more general and thorough analysis on this topic is done in [13].

When a and b are fixed to be $\mathbf{0}$ in the last definition, a similar characterization of subtransversality is obtained (cf. [37]):

Proposition 2.4.4. *Let A and B be closed subsets of the Banach space X. A and B are subtransversal at $\bar{x} \in A \cap B$, if and only if there exists $K > 0$ and $\delta > 0$ such that*

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B))$$

for all $x \in \mathbf{B}_\delta(\bar{x})$.

Thus we observe that A and B are transversal at $\bar{x} \in A \cap B$ if and only if the subtransversality inequality holds for $A - a$ and $B - b$ with constant K for all $x \in \bar{\mathbf{B}}_\delta(\bar{x})$ and $a, b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$.

It is worth noting that while the definitions of transversality and subtransversality clearly make use of the linear structure, the characterization of subtransversality, given by Proposition 2.4.4, is purely metric. Thus, one may think of subtransversality as of metric concept with definition in metric spaces given by the characterization in Proposition 2.4.4. However, all chracterizations of transversality use the linear structure.

The notion of tangential transversality is introduced in [9]. The corresponding definition follows.

Definition 2.4.5. *Let A and B be closed subsets of the metric space X. We say that A and B are tangentially transversal at $\bar{x} \in A \cap B$, if there exist $M > 0$, $\delta > 0$ and $\eta > 0$ such that for any two different points $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$, there exist sequences $t_m \searrow 0$, $\{x^A_m\}_{m \geq 1}$ in A and $\{x^B_m\}_{m \geq 1}$ in B such that for all m*

$$d(x^A_m, x^A) \leq t_m M, \quad d(x^B_m, x^B) \leq t_m M, \quad d(x^A_m, x^B_m) \leq d(x^A, x^B) - t_m \eta\,.$$

Clearly, the three constants M, δ, η are redundant. More specifically, one can choose $M = 1$, which changes η, or choose $\eta = 1$ and change M.

Yet another type of transversality, intrinsic transversality, is introduced in [29] and [30]. It is originally considered for finite-dimensional spaces.

Definition 2.4.6. *The closed sets $A, B \subset \mathbb{R}^d$ are intrinsically transversal at the point $\bar{x} \in A \cap B$, if and only if there exist $\delta > 0$ and $\kappa > 0$ such that for all $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A \setminus B$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B \setminus A$ it holds true that*

$$\max \left\{ d\left(\frac{x^A - x^B}{\|x^A - x^B\|},\, N_B\left(x^B\right) \right), d\left(\frac{x^B - x^A}{\|x^B - x^A\|}, N_A\left(x^A\right) \right) \right\} \geq \kappa\,,$$

where $N_D\left(\bar{x}\right)$ is the proximal or limiting normal cone to D at $\bar{x}$.

2.5 Tangent cones

Throughout the thesis, if Y is a Banach space, we will denote by $\mathbf{B}_Y$ $[\overline{\mathbf{B}}_Y]$ its open [closed] unit ball, centered at the origin. The index could be omitted if there is no ambiguity about the space. If S is a closed subset of Y and $y \in S$, we will denote by $T_S(y)$ the Bouligand tangent cone to S at y, i.e.

$$T_S(y) := \left\{ v \in Y : \frac{y_k - y}{\tau_k} \to v \quad \begin{array}{l} \text{for some sequences } y_k \in S, y_k \to y \\ \text{and } \tau_k > 0, \tau_k \to 0 \end{array} \right\};$$

by $G_S(y)$ the derivable tangent cone to S at y, i.e.

$$G_S(y) := \left\{ v \in Y : \frac{\xi(\tau_k) - y}{\tau_k} \to v \quad \begin{array}{c} \text{for some vector-valued function} \\ \xi : [0, \varepsilon] \to S, \xi(0) = y \text{ and for every} \\ \text{choice of a sequence } \tau_k > 0, \tau_k \to 0 \end{array} \right\};$$

and by $\widehat{T}_S(y)$ the Clarke tangent cone to S at y, i.e.

$$\widehat{T}_S(y) := \left\{ v \in Y : \begin{array}{c} \text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \\ \text{such that for every } t \in [0, \delta] \text{ it holds true that} \\ S \cap (y + \delta \overline{\mathbf{B}}) + tv \subset S + t\varepsilon \overline{\mathbf{B}} \end{array} \right\}.$$

Chapter 3

On transversality-type properties

In this chapter we endow the Cartesian product $X \times Y$ of the metric spaces (X, d_X) and (Y, d_Y), with the metric

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

for the sake of simplicity. The particular choice of the metric is relevant only to the constants involved. However, our considerations in this chapter is to derive qualitative results, so that we are not concerned with the constants.

3.1 Primal space characterizations of subtransversality

In this section we obtain primal space characterizations of subtransversality. In the papers [19] and [20] (see Remark 3.5 in [19]) similar conditions are presented. It is proved that these conditions are characterizations (both necessary and sufficient) only in the convex case.

Our approach is to some extent motivated by the considerations in the paper [9]. In it, the notion of tangential transversality (2.4.5) is introduced as a sufficient condition for nonseparation of sets, tangential intersection properties and a Lagrange multiplier rule.

The definition of tangential transversality can be reformulated equivalently.

Proposition 3.1.1. *Let A and B be closed subsets of the metric space X. A and B are tangentially transversal at $\bar{x} \in A \cap B$, if and only if there exist $\delta > 0$ and $\zeta > 0$ such that for any two different points $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$, there exist sequences $\{x_m^A\}_{m \geq 1} \subset A$ and $\{x_m^B\}_{m \geq 1} \subset B$*

converging to x^A *and* x^B *repsectively and such that for all* m

$$d(x_m^A, x_m^B) \le d(x^A, x^B) - \zeta \max\{d(x_m^A, x^A), d(x_m^B, x^B)\}$$

and $\max\{d(x_m^A, x^A), d(x_m^B, x^B)\} > 0$.

Now we introduce a weaker notion. Note that the main difference is that "there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of positive reals tending to zero such that for every t_n belonging to it ..." is replaced by "there exists a positive real θ such that ...". This is indeed a significant difference, as it will be shown later on. The other weakening in the definition, "$\bar{x} \in A \cap B$" to "$A \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$, $B \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$", is for purely technical reasons.

Definition 3.1.2. *Let* A *and* B *be closed subsets of the metric space* X *and* $\bar{x} \in X$. *We say that* A *and* B *have property* $(\mathcal{T})$ *at* $\bar{x}$ *if there exist* $\delta > 0$ *and* $M > 0$ *such that* $A \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$, $B \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$ *and for any* $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *and* $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *with* $x^A \neq x^B$ *there exist* $\theta > 0$, $\hat{x}^A \in A$ *and* $\hat{x}^B \in B$ *such that*

$$d(x^A, \hat{x}^A) \le \theta M \ , \quad d(x^B, \hat{x}^B) \le \theta M \quad \textit{and} \quad d(\hat{x}^A, \hat{x}^B) \le d(x^A, x^B) - \theta \ .$$

Equivalently, A *and* B *have property* $(\mathcal{T})$ *at* $\bar{x}$ *if and only if there exist* $\delta > 0$ *and* $M > 0$ *such that* $A \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$, $B \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$ *and for any* $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *and* $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *with* $x^A \neq x^B$ *there exist* $\hat{x}^A \in A$ *and* $\hat{x}^B \in B$ *such that*

$$d(\hat{x}^A, \hat{x}^B) \le d(x^A, x^B) - \frac{1}{M} \max\{d(x^A, \hat{x}^A), d(x^B, \hat{x}^B)\}$$

and $\max\{d(x^A, \hat{x}^A), d(x^B, \hat{x}^B)\} > 0$.

Note that in this definition we do not require the point $\bar{x}$ to be in the intersection of A and B, only to be sufficiently close to both A and B.

The lemma below is the main technical result, whose direct corollaries will justify the benefits of the above definition. Its proof may seem long, because it essentially contains the proof of the Ekeland variational principle. There are many similar assertions in the literature and their proofs all rely on variational principles (Ekeland variational principle or see e.g. [5], [4], [40] or [14] for alternatives). Our result can be proved using them, but we prefer to prove it using transfinite induction in order to emphasize the usefulness of the method and its geometrical intuition.

Lemma 3.1.3. *Let A and B be closed subsets of the complete metric space X and $\bar{x} \in X$. Let A and B have property $(\mathcal{T})$ at $\bar{x}$ with constants δ and M. Let $x^A \in A$ with $d(x^A, \bar{x}) \leq \dfrac{\delta}{1+2M}$ and $x^B \in B$ with $d(x^B, \bar{x}) \leq \dfrac{\delta}{1+2M}$. Then, there exists $x^{AB} \in A \cap B$ with*

$$d(x^{AB}, x^A) \leq Md(x^A, x^B) \text{ and } d(x^{AB}, x^B) \leq Md(x^A, x^B)\,.$$

Proof. If the points x^A and x^B coincide, the assertion of the theorem is trivial. If $d(x^A, x^B) > 0$, we are going to construct inductively three transfinite sequences indexed by ordinal numbers (cf., for example, § 2 Ordinal numbers of Chapter 1 in [42]). More precisely, we prove that there exist an ordinal number α_0 and transfinite sequences $\{x^A_\alpha\}_{1\leq\alpha\leq\alpha_0} \subset X$, $\{x^B_\alpha\}_{1\leq\alpha\leq\alpha_0} \subset X$, $\{t_\alpha\}_{1\leq\alpha\leq\alpha_0} \subset [0, +\infty)$, such that $x^A_{\alpha_0} = x^B_{\alpha_0}$ and for each $\alpha \in [1, \alpha_0]$ we have that the following properties hold true:

(S0) $x^A_\alpha \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$ and $x^B_\alpha \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$;

(S1) $d(x^A_\alpha, x^B_\alpha) \leq d(x^A, x^B) - t_\alpha$ (and hence t_α is bounded by $d(x^A, x^B)$);

(S2) $d(x^A_\alpha, \bar{x}) \leq d(x^A, \bar{x}) + t_\alpha M$ and $d(x^B_\alpha, \bar{x}) \leq d(x^B, \bar{x}) + t_\alpha M$;

(S3) $d(x^A_\alpha, x^A_\gamma) \leq M\,(t_\alpha - t_\gamma)$ and $d(x^B_\alpha, x^B_\gamma) \leq M\,(t_\alpha - t_\gamma)$ for each $\gamma \leq \alpha$.

We implement our construction using induction on α. The process terminates when $x^A_\alpha = x^B_\alpha$ for some α, and this α is named α_0. We start with $x^A_1 := x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$, $x^B_1 := x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$ and $t_1 = 0$. It is straightforward to verify the inductive assumptions (S1)-(S3) for $\alpha = 1$.

Assume that $x^A_\beta \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$, $x^B_\beta \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$ and t_β are constructed and (S1)-(S3) are true for all ordinals β less than α and the process has not been terminated.

Let us first consider the case when α is a successor ordinal, i.e. $\alpha = \beta + 1$. As $\beta < \alpha_0$ (the process has not been terminated), we have $d(x^A_\beta, x^B_\beta) \neq 0$. Moreover $(S0)$ holds, so we can apply property $(\mathcal{T})$ to obtain $\theta > 0, \hat{x}^A_\beta, \hat{x}^B_\beta$, and we define $t_\alpha := t_\beta + \theta$, $x^A_\alpha := \hat{x}^A_\beta$ and $x^B_\alpha := \hat{x}^B_\beta$. Now we have $x^A_\alpha \in A$, $x^B_\alpha \in B$, $d(x^A_\alpha, x^A_\beta) \leq M\theta$, $d(x^B_\alpha, x^B_\beta) \leq M\theta$ and $d(x^A_\alpha, x^B_\alpha) \leq d(x^A_\beta, x^B_\beta) - \theta$. Using the inductive assumption, we have

$$d(x^A_\alpha, x^B_\alpha) \leq d(x^A_\beta, x^B_\beta) - \theta \leq d(x^A, x^B) - t_\beta - \theta = d(x^A, x^B) - t_\alpha\ .$$

Therefore, (S1) is verified for α.

Now the inequalities $d(x_\alpha^A, x_\beta^A) \le M\theta$, $d(x_\alpha^B, x_\beta^B) \le M\theta$ and the inductive assumption (S2) for β yield

$$d(x_\alpha^A, \bar{x}) \le d(x_\beta^A, \bar{x}) + d(x_\alpha^A, x_\beta^A) \le d(x^A, \bar{x}) + t_\beta M + M\theta = d(x^A, \bar{x}) + t_\alpha M,$$

$$d(x_\alpha^B, \bar{x}) \le d(x_\beta^B, \bar{x}) + d(x_\alpha^A, x_\beta^B) \le d(x^B, \bar{x}) + t_\beta M + M\theta = d(x^B, \bar{x}) + t_\alpha M.$$

Thus (S2) is verified for α. Using the estimate $t_\alpha \le d(x^A, x^B)$ being proved above, the assumption of the lemma and the above inequalities, we obtain

$$d(x_\alpha^A, \bar{x}) \le d(x^A, \bar{x}) + t_\alpha M \le d(x^A, \bar{x}) + M d(x^A, x^B) \le$$

$$d(x^A, \bar{x}) + M(d(x^A, \bar{x}) + d(x^B, \bar{x})) \le \frac{\delta}{1+2M} + M\frac{2\delta}{1+2M} = \delta$$

which means that $x_\alpha^A \in \bar{\mathbf{B}}_\delta(\bar{x})$. Similarly $x_\alpha^B \in \bar{\mathbf{B}}_\delta(\bar{x})$. Thus (S0) holds.

Now we turn our attention to (S3). Let $\gamma \le \alpha$. If $\gamma = \alpha$, (S3) is trivially fulfilled. Now let $\gamma < \alpha$. Then $\gamma \le \beta$ and from the inductive assumption follows

$$d(x_\alpha^A, x_\gamma^A) \le d(x_\beta^A, x_\gamma^A) + d(x_\alpha^A, x_\beta^A) \le M\,(t_\beta - t_\gamma) + M\theta = M\,(t_\alpha - t_\gamma)$$

and in the same way

$$d(x_\alpha^B, x_\gamma^B) \le d(x_\beta^B, x_\gamma^B) + d(x_\alpha^B, x_\beta^B) \le M\,(t_\beta - t_\gamma) + M\theta = M\,(t_\alpha - t_\gamma)\ .$$

We have verified the inductive assumptions (S0)-(S3) for the case of a successor ordinal α.

We next consider the case when α is a limit ordinal number. Let $\beta < \alpha$ be arbitrary. Then $\beta + 1 < \alpha$ too. Since the transfinite process has not stopped at $\beta + 1$, then $d(x_\beta^A, x_\beta^B) > 0$, and hence taking into account (S1) we obtain that $t_\beta < d(x^A, x^B)$. Hence the increasing transfinite sequence $\{t_\beta\}_{1\le\beta<\alpha}$ is bounded, and so it is convergent. We denote $t_\alpha := \lim_{\beta\to\alpha} t_\beta$. Since $d(x_\beta^A, x_\gamma^A) \le (t_\beta - t_\gamma)M$, the transfinite sequence $\{x_\beta^A\}_{1\le\beta<\alpha}$ is fundamental. Hence there exists x_α^A so that $\{x_\beta^A\}_{1\le\beta<\alpha}$ tends to x_α^A as β tends to α with $\beta < \alpha$. In the same way one can prove the existence of x_α^B so that the transfinite sequence $\{x_\beta^B\}_{1\le\beta<\alpha}$ tends to x_α^B as β tends to α. To verify the inductive assumptions (S1)-(S3) for α, one can just take a limit for β tending to α with $\beta < \alpha$ in the same assumptions written for each $\beta < \alpha$. For (S0) one uses that A and B are closed.

We have constructed inductively the transfinite sequences $\{x_\beta^A\}_{\beta\le\alpha} \subset A$, $\{x_\beta^B\}_{\beta\le\alpha} \subset B$ and $\{t_\beta\}_{\beta\le\alpha} \subset [0, +\infty)$. The process will terminate when

$x^A_\alpha = x^B_\alpha$ for some α. Since $d(x^A_\alpha, x^B_\alpha) \le d(x^A, x^B) - t_\alpha$ and the transfinite sequence t_α is strictly increasing, the equality $x^A_\alpha = x^B_\alpha$ will be satisfied for some $\alpha = \alpha_0$ strictly preceding the first uncountable ordinal number. Indeed, the successor ordinals indexing the so constructed transfinite sequences form a countable set (because to every successor ordinal $\alpha+1$ corresponds the open interval $(t_\alpha, t_{\alpha+1}) \subset \mathbb{R}$, these intervals are disjoint and the rational numbers are countably many and dense in $\mathbb{R}$). Therefore, α_0 is countably accessible. On the other handside, assuming the Axiom of countable choice, ω_1 is not countably accessible). Hence our inductive process terminates before ω_1.

Then we put $x^{AB} := x^A_{\alpha_0} = x^B_{\alpha_0} \in A \cap B$ and because of (S1) we have that $t_{\alpha_0} \le d(x^A, x^B)$. Applying (S3) we obtain $d(x^{AB}, x^A) \le M(t_{\alpha_0} - t_1) \le Md(x^A, x^B)$ hence $d(x^{AB}, x^B) \le Md(x^A, x^B)$.

This completes the proof. □

Completeness is crucial in the above lemma. The following theorem is formulated in a way that enables us to use it to obtain primal space characterizations both for subtransversality and transversality.

Theorem 3.1.4. *Let A and B be closed subsets of the complete metric space X and $\bar{x} \in X$. If A and B have property $(\mathcal{T})$ at $\bar{x}$, then there exist $K > 0$ and $\delta > 0$ such that*

$$d(x, A \cap B) \le K(d(x, A) + d(x, B)) \tag{3.1}$$

for all $x \in \bar{\mathbf{B}}_\delta(\bar{x})$.

If there exist $K > 0$ and $\delta > 0$ such that (3.1) holds for all $x \in \bar{\mathbf{B}}_\delta(\bar{x})$, $A \cap \bar{\mathbf{B}}_{\frac{\delta}{4K+10}}(\bar{x}) \neq \varnothing$ and $B \cap \bar{\mathbf{B}}_{\frac{\delta}{4K+10}}(\bar{x}) \neq \varnothing$, then A and B have property $(\mathcal{T})$ at $\bar{x}$.

Proof. Let A and B have property $(\mathcal{T})$ with constants M, δ. Let $\hat{\delta} := \dfrac{\delta}{8(1+2M)}$. Let $x \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{x})$ and choose $\varepsilon \in (0, \hat{\delta})$. Then there exists $x^A \in A$, such that $d(x, x^A) < d(x, A) + \varepsilon$. We have that $d(x, A) \le d(x, \bar{x}) + d(\bar{x}, A) \le \hat{\delta} + \frac{\delta}{2(1+2M)} = 5\hat{\delta}$, so that $d(x, x^A) < 6\hat{\delta}$. Since $d(x, \bar{x}) \le \hat{\delta}$, the triangle inequality implies

$$d(x^A, \bar{x}) < 7\hat{\delta} < \frac{\delta}{1+2M}.$$

Similarly, we find $x^B \in B$, such that $d(x, x^B) < d(x, B) + \varepsilon$ and

$$d(x^B, \bar{x}) < \frac{\delta}{1+2M}.$$

Then x^A and x^B satisfy the requirements in Lemma 3.1.3. Hence, there is $x^{AB} \in A \cap B$, such that

$$d(x^{AB}, x^A) \leq Md(x^A, x^B) \text{ and } d(x^{AB}, x^B) \leq Md(x^A, x^B).$$

We estimate

$$\begin{aligned} d(x, A \cap B) &\leq d(x, x^{AB}) \leq d(x, x^A) + d(x^A, x^{AB}) < d(x, A) + \varepsilon + Md(x^A, x^B) \\ &< d(x, A) + \varepsilon + M(d(x, x^A) + d(x, x^B)) \\ &\leq d(x, A) + \varepsilon + M(d(x, A) + \varepsilon + d(x, B) + \varepsilon) \\ &\leq (M+1)(d(x, A) + d(x, B)) + \varepsilon(1 + 2M) \end{aligned}$$

Letting $\varepsilon \to 0$ proves (3.1) with constants $\hat{\delta}$ and $M + 1$.

For the second part, let (3.1) hold with constants δ and K. Take distinct $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and for $\varepsilon := d(x^A, x^B) > 0$, find $x^{AB} \in A \cap B$ such that

$$d(x^A, x^{AB}) < d(x^A, A \cap B) + \varepsilon \leq Kd(x^A, B) + \varepsilon$$

$$\leq Kd(x^A, x^B) + \varepsilon = (K+1)d(x^A, x^B).$$

Then

$$\begin{aligned} d(x^B, x^{AB}) &\leq d(x^A, x^B) + d(x^A, x^{AB}) \\ &< d(x^A, x^B) + (K+1)d(x^A, x^B) = (K+2)d(x^A, x^B). \end{aligned}$$

Now property $(\mathcal{T})$ follows with $\hat{x}^A = \hat{x}^B = x^{AB}$, $\theta = d(x^A, x^B) > 0$ and $M = K + 2$, because proximity of A and B to $\bar{x}$ is assumed.

□

As a corollary we obtain that property $(\mathcal{T})$ is an equivalent characterization of subtransversality in the presence of completeness.

Corollary 3.1.5. *If $\bar{x} \in A \cap B$, where A and B are closed subsets of the complete metric space X, then A and B have property $(\mathcal{T})$ at $\bar{x}$ if and only if A and B are subtransversal at $\bar{x}$.*

The following proposition is a reformulation of Corollary 3.1.5.

Proposition 3.1.6. *Under completeness of the space X, A and B are subtransversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $\kappa > 0$ such that for all $x \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$, $x \neq y$, it holds*

$$|\nabla \phi|^\diamond(x, y) = \sup_{(u,v) \neq (x,y)} \frac{\max\{\phi(x, y) - \phi(u, v), 0\}}{d((x, y), (u, v))} \geq \kappa.$$

Proof. Assume that the sets A and B are subtransversal. Then according to Corollary 3.1.5 property $(\mathcal{T})$ holds.

Then, there exist $M > 0$, $\delta > 0$ such that for any two different points $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$, there exist $\theta > 0$, $\hat{x}^A \in A$ with $d(x^A, \hat{x}^A) \le \theta M$, and $\hat{x}^B \in B$ with $d(x^B, \hat{x}^B) \le M\theta$, and the following inequality holds true

$$d(\hat{x}^A, \hat{x}^B) \le d(x^A, x^B) - \theta.$$

Clearly, the last inequality yields that $(\hat{x}^A, \hat{x}^B) \neq (x^A, x^B)$. Remind that $\phi(x^A, x^B) = d(x^A, x^B)$ (Definition 2.3.2) since $x^A \in A$ and $x^B \in B$ and similarly $\phi(\hat{x}^A, \hat{x}^B) = d(\hat{x}^A, \hat{x}^B)$. This leads to

$$d(x^A, x^B) - d(\hat{x}^A, \hat{x}^B) \ge \theta \ge \frac{d(x^A, \hat{x}^A) + d(x^B, \hat{x}^B)}{2M}$$

Therefore

$$\frac{\phi(x^A, x^B) - \phi(\hat{x}^A, \hat{x}^B)}{d((x^A, x^B), (\hat{x}^A, \hat{x}^B))} = \frac{d(x^A, x^B) - d(\hat{x}^A, \hat{x}^B)}{d(x^A, \hat{x}^A) + d(x^B, \hat{x}^B)} \ge \frac{1}{2M}.$$

Thus we obtain that

$$|\nabla\phi|^\diamond(x^A, x^B) \ge \frac{1}{2M}$$

for any two different points $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$.

For the reverse direction, we have that for some $\delta > 0$ and $\kappa > 0$ and for any two different points $x \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$

$$|\nabla\phi|^\diamond(x, y) = \sup_{(u,v)\neq(x,y)} \frac{\max\{\phi(x,y) - \phi(u,v), 0\}}{d((x,y),(u,v))} \ge \kappa > 0.$$

So fix $x \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ with $x \neq y$. We obtain that there are u and v such that

$$\frac{\phi(x,y) - \phi(u,v)}{d(x,u) + d(y,v)} \ge \frac{\kappa}{2}.$$

As above, $\phi(x,y) = d(x,y)$. Observe that $\phi(u,v) < \infty$, thus $u \in A$ and $v \in B$. Hence $d(u,v) \le d(x,y) - \theta$ where $\theta = \frac{\kappa}{2}(d(x,u) + d(y,v)) > 0$. Moreover $d(x,u) \le \frac{2}{\kappa}\theta$ and $d(y,v) \le \frac{2}{\kappa}\theta$. Thus we obtain that property $(\mathcal{T})$ holds with constants δ and $M := \frac{2}{\kappa}$. □

3.2 Primal space characterizations of transversality

We continue to obtain primal space characterizations of transversality. A direct consequence of the definition of transversality and Theorem 3.1.4 is a characterization of transversality in terms of "translated" subtransversality.

Proposition 3.2.1. *Let A and B be closed subsets of the Banach space X and $\bar{x} \in A \cap B$. Then A and B are transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$, any $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x}+a)$ and $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x}+b)$ with $x^A - a \neq x^B - b$ there exist $\theta > 0$, $\hat{x}^A \in A$ and $\hat{x}^B \in B$ such that*

$$\|x^A - \hat{x}^A\| \leq \theta M \ , \quad \|x^B - \hat{x}^B\| \leq \theta M \quad \textit{and}$$
$$\|\hat{x}^A - \hat{x}^B - (a-b)\| \leq \|x^A - x^B - (a-b)\| - \theta \ .$$

Proof. Let A and B be transversal at $\bar{x}$ and K and $\hat{\delta}$ be the corresponding constants in the definition. Denote $\delta = \hat{\delta}/(4K+10)$. Then for all $a \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$, the sets $A-a$ and $B-b$ have property $(\mathcal{T})$ with constants δ and $M = K+2$ according to Theorem 3.1.4 and the estimates in its proof. Now let the sets satisfy the above property with constants δ and M. Thus for all $a \in \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\mathbf{0})$ and $b \in \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\mathbf{0})$, the sets $A-a$ and $B-b$ have property $(\mathcal{T})$ with constants δ and M. Then, again Theorem 3.1.4 and the estimates in its proof
implies that

$$d(x,(A-a)\cap(B-b)) \leq (M+1)(d(x,A-a)+d(x,B-b))$$

for all $x \in \bar{\mathbf{B}}_{\frac{\delta}{8(2M+1)}}(\bar{x})$, which is precisely transversality.

□

Another way to prove this proposition is by using the Ioffe criterion for metric regularity, cf. e.g. [14, Proposition 2.2], applied to the mapping $H_{A,B}$.

Strengthening in one of the directions of this proposition gives a characterization of transversality in terms of "translated" tangential transversality.

Proposition 3.2.2. *Let A and B be closed subsets of the Banach space X and $\bar{x} \in A \cap B$. Then A and B are transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$, any $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x}+a)$ and $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x}+b)$ with $x^A - a \neq x^B - b$ there exist $\{x^A_m\}_{m\geq 1} \subset A$, $\{x^B_m\}_{m\geq 1} \subset B$ and $t_m \searrow 0$ such that for each m*

$$\|x^A_m - x^A\| \leq t_m M \ , \quad \|x^B_m - x^B\| \leq t_m M \quad \textit{and}$$
$$\|x^A_m - x^B_m - (a-b)\| \leq \|x^A - x^B - (a-b)\| - t_m \ .$$

Proof. The "if" direction is straightforward from Proposition 3.2.1.
For the converse, let A and B be transversal at $\bar{x}$. This means that the map $H := H_{A,B}$ from (2.1) is regular around $((\bar{x}, \bar{x}), \mathbf{0})$ with constants $\hat{\delta}$ and K. Take $\delta < \hat{\delta}/4$, $a \in \bar{\mathbf{B}}_\delta(\mathbf{0})$, $b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x} + a)$ and $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x} + b)$. Then

$$\begin{aligned}\|x^A - x^B\| &= \|x^A - (\bar{x} + a) - (x^B - (\bar{x} + b)) + a - b\| = \\ &\leq \|x^A - (\bar{x} + a)\| + \|x^B - (\bar{x} + b)\| + \|a\| + \|b\| \leq 4\delta < \hat{\delta}\end{aligned}$$

Define $v = -\dfrac{x^A - x^B - (a - b)}{\|x^A - x^B - (a - b)\|}$ and choose a sequence $t_m \searrow 0$ such that

$$x^A - x^B + t_m v \in \bar{\mathbf{B}}_{\hat{\delta}}(\mathbf{0})$$

and $t_m < \|x^A - x^B - (a - b)\|$. Metric regularity of H implies that

$$d((x^A, x^B), H^{-1}(x^A - x^B + t_m v)) \leq K d(H(x^A, x^B), x^A - x^B + t_m v) \leq K t_m.$$

Since the distance to the empty set is $+\infty$, we have that $H^{-1}(x^A - x^B + t_m v) \neq \varnothing$. For $m \geq 1$ consider $(x^A_m, x^B_m) \in H^{-1}(x^A - x^B + t_m v) \subset A \times B$ such that

$$\|(x^A_m, x^B_m) - (x^A, x^B)\| < d((x^A, x^B), H^{-1}(x^A - x^B + t_m v)) + t_m.$$

Metric regularity once again implies

$$\|(x^A_m, x^B_m) - (x^A, x^B)\| < d((x^A, x^B), H^{-1}(x^A - x^B + t_m v)) + t_m \leq (K + 1)t_m\,.$$

Finally, we have that

$$\begin{aligned}\|x^A_m - x^B_m - (a - b)\| &= \|x^A - x^B + t_m v - (a - b)\| = \\ &= \|x^A - x^B - (a - b)\| - t_m\,.\end{aligned}$$

□

Remark 3.2.3. *In the above proposition we can obtain the (formally) stronger statement that there exists $\lambda > 0$ such that the decreasing property holds for any $t \in (0, \lambda]$ instead of the sequence $\{t_n\}_{n=1}^\infty$ tending to zero from above.*

Remark 3.2.4. *Propositions 3.2.1 and 3.2.2 remain true if we consider translations in only one of the sets, i.e. we may take $a = 0$ and only vary b.*

The following proposition is a reformulation of Proposition 3.2.1 and Proposition 3.2.2 is used for the equivalence of existence of positive local and of positive nonlocal slope.

Proposition 3.2.5. *Under the assumption that X is a Banach space, A and B are transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $\kappa > 0$ such that for all a and b with $\|a\| \leq \delta$ and $\|b\| \leq \delta$ and all $x \in (A-a) \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in (B-b) \cap \bar{\mathbf{B}}_\delta(\bar{x})$ with $x \neq y$ it holds*

$$|\nabla \phi_{a,b}|^\diamond(x,y) = \sup_{(u,v)\neq(x,y)} \frac{\max\{\phi_{a,b}(x,y) - \phi_{a,b}(u,v), 0\}}{\|(x,y)-(u,v)\|} \geq \kappa$$

where $\phi_{a,b}$ denotes the coupling function of $A-a$ and $B-b$ (remind Definition (2.3.2))

Moreover, this is equivalent to the existence of $\hat{\kappa} > 0$ such that for all $x \in (A-a) \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in (B-b) \cap \bar{\mathbf{B}}_\delta(\bar{x})$

$$|\nabla \phi_{a,b}|(x,y) = \limsup_{(u,v)\to(x,y)} \frac{\max\{\phi_{a,b}(x,y) - \phi_{a,b}(u,v), 0\}}{\|(x,y)-(u,v)\|} \geq \hat{\kappa}$$

Proof. The proof of the first part of the proposition is analogous to the proof Proposition 3.1.6.

For the second part, clearly $|\nabla \phi_{a,b}|^\diamond(x,y) \geq |\nabla \phi_{a,b}|(x,y)$, and thus the second slope type property implies the first one.

For the reverse implication, the first part of the Proposition implies that the sets A and B are transversal at $\bar{x} \in A \cap B$ with some constants $\delta > 0$ and $M > 0$. Fix $a \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $x \in (A-a) \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in (B-b) \cap \bar{\mathbf{B}}_\delta(\bar{x})$, $x \neq y$. Recall that Proposition 3.2.2 implies that for $x^A := x + a \in A \cap \bar{\mathbf{B}}_\delta(\bar{x}+a)$ and $x^B := y + b \in B \cap \bar{\mathbf{B}}_\delta(\bar{x}+b)$ (thus $x^A - a \neq x^B - b$) there exist $\{x^A_m\}_{m\geq 1} \subset A$, $\{x^B_m\}_{m\geq 1} \subset B$ and $t_m \searrow 0$ such that

$$\|x^A_m - x^A\| \leq t_m M \ , \quad \|x^B_m - x^B\| \leq t_m M \quad \text{and}$$
$$\|x^A_m - x^B_m - (a-b)\| \leq \|x^A - x^B - (a-b)\| - t_m \ .$$

This is equivalent to

$$\frac{d(x^A - a, x^B - b) - d(x^A_m - a, x^B_m - b)}{t_m} \geq 1 \, .$$

Using that $d(x^A - a, x^A_m - a) \leq M t_m$ and $d(x^B - b, x^B_m - b) \leq M t_m$, we have that

$$\frac{d(x^A - a, x^B - b) - d(x^A_m - a, x^B_m - b)}{d(x^A - a, x^A_m - a) + d(x^B - b, x^B_m - b)} \geq \frac{1}{2M} \, .$$

whence

$$\frac{d(x,y) - d(x^A_m - a, x^B_m - b)}{d(x, x^A_m - a) + d(y, x^B_m - b)} \geq \frac{1}{2M} \, .$$

Since $x \in A - a$ and $y \in B - b$, $\phi_{a,b}(x, y) = d(x, y)$. Similarly $x_m^A - a \in A - a$ and $x_m^B - b \in B - b$, hence $\phi_{a,b}(x_m^A - a, x_m^B - b) = d(x_m^A - a, x_m^B - b)$. Moreover $x_m^A - a \to x,\ x_m^B - b \to y$. This implies that

$$|\nabla \phi_{a,b}|(x, y) = \limsup_{(u,v)\to(x,y)} \frac{\max\{\phi_{a,b}(x, y) - \phi_{a,b}(u, v), 0\}}{\|(x, y) - (u, v)\|} \geq \frac{1}{2M}\,.$$

□

Another way to prove this proposition is by using the Ioffe slope-based criterion for metric regularity from [39] (see also [62]) applied to the mapping $H_{A,B}$.

3.3 Intrinsic transversality - extensions and related notions

In this section we provide a metric characterization of intrinsic transversality. This characterization could be used as a definition of intrinsic transversality in general metric spaces. Moreover, we show that it is almost equivalent to the notion of tangential transversality, via observing a slope type characterization of the latter. Finally we show that the metric characterization we provide is equivalent in Hilbert spaces to a characterization introduced and studied in [61].

Fix a point $\bar{x} \in A \cap B$. First, we provide a characterization of tangential transversality in terms of the slope of the coupling function (remind Definition 2.3.2).

Proposition 3.3.1. *The subsets A and B of the metric space X are tangentially transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $\kappa > 0$ such that for any two different points $x \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ it holds*

$$|\nabla \phi|(x, y) = \limsup_{(u,v)\to(x,y)} \frac{\max\{\phi(x, y) - \phi(u, v), 0\}}{d((x, y), (u, v))} \geq \kappa\,.$$

Proof. The proof is analogous to the proofs of Proposition 3.1.6 and 3.2.5. □

Intrinsic transversality is introduced in [29] and [30] as a sufficient condition for local linear convergence of the alternating projections algorithm in finite dimensions. Drusvyatskiy, Ioffe and Lewis found a characterization of intrinsic transversality in finite dimensional spaces in terms of the slope of the coupling function (cf. Proposition 4.2 in [30]). We use this characterization as a definition of intrinsic transversality in general metric spaces.

Definition 3.3.2. *Let X be a metric space. The closed sets $A, B \subset X$ are intrinsically transversal at the point $\bar{x} \in A \cap B$, if there exist $\delta > 0$ and $\kappa > 0$ such that for all $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A \setminus B$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B \setminus A$ it holds true that*

$$|\nabla \phi|(x^A, x^B) \geq \kappa \,.$$

We continue to observe the "almost" equivalence of intrinsic transversality and tangential transversality. Due to Proposition 3.3.1 we have that the only difference between tangential transversality and intrinsic transversality is that in the original definition of tangential transversality the required condition should hold for all points of A and B (respectively) near the reference point, whereas in intrinsic transversality – only for points in $A \setminus B$ and $B \setminus A$ (respectively). We introduce the following property.

Definition 3.3.3 (Property $(\mathcal{LT})$). *We say that the closed sets A and B satisfy property $(\mathcal{LT})$ at $\bar{x} \in A \cap B$, if there exist $\varepsilon > 0$ and $\theta > 0$ such that for any two different points $x^A \in \bar{\mathbf{B}}_\varepsilon(\bar{x}) \cap A \setminus B$ and $x^B \in \bar{\mathbf{B}}_\varepsilon(\bar{x}) \cap B \setminus A$, there exist sequences $t_m \searrow 0$, $\{x^A_m\}_{m \geq 1} \subset A$ and $\{x^B_m\}_{m \geq 1} \subset B$ such that for all m*

$$d(x^A_m, x^A) \leq t_m, \quad d(x^B_m, x^B) \leq t_m, \quad d(x^A_m, x^B_m) \leq d(x^A, x^B) - t_m \theta \,.$$

The comments above yield the following

Corollary 3.3.4. *The sets A and B are intrinsically transversal at $\bar{x} \in A \cap B$ if and only if they satisfy property $(\mathcal{LT})$ at $\bar{x}$.*

In this way we answer a question of Prof. A. Ioffe about finding a metric characterization of intrinsic transversality.

The following example shows that although the difference is slight, the notion of tangential tranvsersality is stronger than the one of intrinsic transversality.

Example 3.3.5. *Consider the sets in $\mathbb{R}^2$,*

$$A = \{(x, y) \mid y = 3x, \ x \geq 0\} \cup \left\{ \left(\frac{1}{n}, \frac{2}{n} \right) \right\}_{n \geq 1}$$

and

$$B = \{(x, y) \mid y = x, \ x \geq 0\} \cup \left\{ \left(\frac{1}{n}, \frac{2}{n} \right) \right\}_{n \geq 1} .$$

Apparently these two sets are intrinsically transversal at $(0, 0)$, however they are not tangentially transversal, because there are isolated points of the intersection in every neighbourhood of the reference point.

We are also able to answer some of the questions posed in [9]:

1. *Tangential transversality is an intermediate property between transversality and subtransversality. However, the exact relation between this new concept and the established notions of transversality, intrinsic transversality and subtransversality is not clarified yet.*

   This question is now fully answered in the case of complete metric spaces. The characterizations of intrinsic transversality and tangential transversality show that the examples at the end of Section 6 in [30] may be used to prove that tangential transversality is strictly between transversality and subtransversality even in $\mathbb{R}^d$.

2. *It would be useful to find some dual characterization of tangential transversality.*

   The original definition of intrinsic transversality is stated in dual terms (Definition 2.2 in [29] and Definition 3.1 in [30]) –
   Replacing "$x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A \setminus B$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B \setminus A$" with "$x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$, $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$ and $x^A \neq x^B$" we obtain a dual characterization of tangential transversality in finite dimensions.

It is known that intrinsic transversality and subtransverslity coincide for convex sets in finite-dimensional spaces (cf. Proposition 6.1 in [38] or Corollary 3.4 in [50] for an alternative proof). Both proofs exploit the dual characterizations of intrinsic transversality and substransversality. Now we can easily obtain the slightly stronger result

Corollary 3.3.6. *Let X be a Banach space. The closed convex sets $A, B \subset X$ are tangentially transversal at the point $\bar{x} \in A \cap B$, if and only if they are subtransversal at $\bar{x}$.*

Proof. It is enough to check, that if the sets are subtransversal, they are moreover tangentially transversal (Definition 2.4.5). According to the primal characterization obtained in Theorem 3.1.4, subtransversality implies property $(\mathcal{T})$ with some constants δ and M. Let $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$. Then there are $\hat{x}^A \in A$, $\hat{x}^B \in B$ and $\theta > 0$, such that

$$\|x^A - \hat{x}^A\| \leq \theta M \ , \quad \|x^B - \hat{x}^B\| \leq \theta M \quad \text{and} \quad \|\hat{x}^A - \hat{x}^B\| \leq \|x^A - x^B\| - \theta \ .$$

Let $\{r_n\}_{n \geq 1} \subset (0, 1)$ be a sequence tending to 0. Since A is convex, $x_n^A := (1 - r_n)\, x^A + r_n \hat{x}^A \in A$ for all $n \in \mathbb{N}$. Similarly for x_n^B. Then

$$\begin{aligned} &\|x_n^A - x_n^B\| = \big\|(1 - r_n)\,(x^A - x^B) + r_n(\hat{x}^A - \hat{x}^B)\big\| \\ &\leq (1 - r_n)\,\|x^A - x^B\| + r_n(\|x^A - x^B\| - \theta) = \|x^A - x^B\| - t_n \end{aligned}$$

where $t_n = r_n\theta$. Moreover we have

$$\left\|x_n^A - x^A\right\| = r_n \left\|\hat{x}^A - x^A\right\| \leq r_n\theta M = t_n M,$$

and similarly for $x_n^B - x^B$.

□

Thus intrinsic transversality also coincides with tangential transversality and subtransversality in the case of convex sets. This last equivalence is also straight-forward to obtain via function slopes characterizations – using that for convex functions the limiting slope and the nonlocal slope coincide (cf. e.g. Proposition 2.1(vii) in [49]), the result follows from Propositions 3.1.6 and 3.3.1.

In the papers [50] and [61] a generalization of intrinsic transversality to Hilbert spaces is derived. It is based on the normal structure - Definition 2(ii) in [50] and Definition 3 in [61]. Moreover, in paper [61] a so called property $(\mathcal{P})$ is introduced. It is in primal space terms and is shown to be equivalent to the afformentioned extension of intrinsic transversality in Hilbert spaces based on the normal structure (Definition 2(ii) in [50] and Definition 3 in [61]).

In order to state it we need the following notation - for a normed space X,

$$\mathrm{d}(A, B, \Omega) := \inf_{x\in\Omega, a\in A, b\in B} \max\{\|x-a\|, \|x-b\|\}, \qquad \text{for } A, B, \Omega \subset X$$

with the convention that the infimum over the empty set equals infinity.

Here is the corresponding definition.

Definition 3.3.7 (Property $(\mathcal{P})$)**.** *A pair of closed sets $\{A, B\}$ is said to satisfy property $(\mathcal{P})$ at a point $\bar{x} \in A \cap B$ if there are numbers $\alpha \in (0,1)$ and $\varepsilon > 0$ such that for any $a \in (A \setminus B) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$, $b \in (B \setminus A) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$ and $x \in \bar{\mathbf{B}}_\varepsilon(\bar{x})$ with $\|x-a\| = \|x-b\|$ and number $\delta > 0$, there exists $\rho \in (0,\delta)$ satisfying*

$$\mathrm{d}\left(A \cap \bar{\mathbf{B}}_\lambda(a), B \cap \bar{\mathbf{B}}_\lambda(b), \bar{\mathbf{B}}_\rho(x)\right) + \alpha\rho \leq \|x-a\|, \text{ where } \lambda := (\alpha + 1/\sqrt{\varepsilon})\rho$$

It is clearly equivalent to the following sequential form

Definition 3.3.8 (Property $(\mathcal{P}')$)**.** *A pair of closed sets $\{A, B\}$ is said to satisfy property $(\mathcal{P}')$ at a point $\bar{x} \in A \cap B$ if there are numbers $\alpha \in (0,1)$ and $\varepsilon > 0$ such that for any $a \in (A \setminus B) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$, $b \in (B \setminus A) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$ and $x \in \bar{\mathbf{B}}_\varepsilon(\bar{x})$ with $\|x-a\| = \|x-b\|$ there exists a sequence $s_n \searrow 0$, satisfying*

$$\mathrm{d}\left(A \cap \bar{\mathbf{B}}_{\lambda_n}(a), B \cap \bar{\mathbf{B}}_{\lambda_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right) + \alpha s_n \leq \|x-a\|, \text{ where } \lambda_n := (\alpha + 1/\sqrt{\varepsilon})s_n$$

for large enough n.

The following two theorems show that in general normed spaces property $(\mathcal{P})$ implies property $(\mathcal{LT})$, while in Hilbert spaces they are equivalent.

Theorem 3.3.9. *Let X be a normed space, A and B be closed subsets of X and $\bar{x} \in A \cap B$. Assume that A and B satisfy property $(\mathcal{P})$ at $\bar{x}$. Then they satisfy property $(\mathcal{LT})$ at $\bar{x}$.*

Proof. Let A and B satisfy property $(\mathcal{P}')$ with constants ε and α. Fix any $a \in (A \setminus B) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$, $b \in (B \setminus A) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$ and set $x = \dfrac{a+b}{2} \in \bar{\mathbf{B}}_\varepsilon(\bar{x})$. Propety $(\mathcal{P}')$ implies that there exists a sequence $s_n \searrow 0$ such that

$$\mathrm{d}\left(A \cap \bar{\mathbf{B}}_{\lambda_n}(a), B \cap \bar{\mathbf{B}}_{\lambda_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right) + \alpha s_n \leq \frac{1}{2}\|a-b\|, \text{ where } \lambda_n := (\alpha + 1/\sqrt{\varepsilon}) s_n$$

There exist $a_n \in A \cap \bar{\mathbf{B}}_{\lambda_n}(a)$, $b_n \in B \cap \bar{\mathbf{B}}_{\lambda_n}(B)$ and $x_n \in \bar{\mathbf{B}}_{s_n}(x)$ such that

$$\|x_n - a_n\| \leq \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{\lambda_n}(a), B \cap \bar{\mathbf{B}}_{\lambda_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right) + \frac{1}{2}\alpha s_n,$$

$$\|x_n - b_n\| \leq \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{\lambda_n}(a), B \cap \bar{\mathbf{B}}_{\lambda_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right) + \frac{1}{2}\alpha s_n.$$

Summing the latter two inequalities we obtain

$$\begin{aligned} \|x_n - a_n\| + \|x_n - b_n\| &\leq 2\mathrm{d}\left(A \cap \bar{\mathbf{B}}_{\lambda_n}(a), B \cap \bar{\mathbf{B}}_{\lambda_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right) + \alpha s_n \\ &\leq \|a-b\| - 2\alpha s_n + \alpha s_n = \|a-b\| - \alpha s_n \end{aligned}$$

The triangle inequality implies

$$\|a_n - b_n\| \leq \|x_n - a_n\| + \|x_n - b_n\| \leq \|a-b\| - \alpha s_n$$

Setting $t_n := \lambda_n$ we obtain

$$\|a_n - a\| \leq t_n, \quad \|b_n - b\| \leq t_n, \quad \|a_n - b_n\| \leq \|a-b\| - t_n \frac{\alpha}{\alpha + 1/\sqrt{\varepsilon}}.$$

Thus property $(\mathcal{LT})$ holds with ε and $\theta := \dfrac{\alpha}{\alpha + 1/\sqrt{\varepsilon}}$.

□

Theorem 3.3.10. *Let X be a Hilbert space, A and B be closed subsets of X and $\bar{x} \in A \cap B$. Assume that A and B satisfy property $(\mathcal{LT})$ at $\bar{x}$. Then they satisfy property $(\mathcal{P})$ at $\bar{x}$.*

Proof. Let A and B satisfy property $(\mathcal{LT})$ with constants ε and θ. We may assume $\varepsilon < 1$. We shall check that A and B satisfy property $(\mathcal{P}')$ with the same ε and α to be specified later. To this end fix any $a \in (A \setminus B) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$, $b \in (B \setminus A) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$ and $x \in \bar{\mathbf{B}}_\varepsilon(\bar{x})$ with $\|x - a\| = \|x - b\|$. Denote $v := x - (a+b)/2$. The equation $\|x - a\| = \|x - b\|$ implies $(v, a - b) = 0$. Moreover $\|x - a\| = \|(a-b)/2 + v\|$. Denote $\psi = \dfrac{2\|v\|}{\|a-b\|}$. We distinguish three cases based on the value of ψ - $\psi > \dfrac{\theta}{5}$, $\psi \in \left(0, \dfrac{\theta}{5}\right]$, $\psi = 0$.

First case, $\psi > \dfrac{\theta}{5}$. Set $s_n = \dfrac{\|v\|}{n}$. Then putting $x_n := \dfrac{a+b}{2} + \dfrac{n-1}{n} v$ we obtain $x_n \in \bar{\mathbf{B}}_{s_n}(x)$, $\|x_n - a\| = \|x_n - b\|$ and

$$\frac{1}{s_n}\left(\|x-a\| - \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{s_n}(a), B \cap \bar{\mathbf{B}}_{s_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right)\right) \geq \frac{1}{s_n}(\|x-a\| - \|x_n - a\|) =$$

$$\frac{1}{s_n}\left(\left\|\frac{a-b}{2} + v\right\| - \left\|\frac{a-b}{2} + \frac{n-1}{n} v\right\|\right) =$$

$$\frac{2n-1}{n} \frac{\|v\|}{\left\|\frac{a-b}{2} + v\right\| + \left\|\frac{a-b}{2} + \frac{n-1}{n} v\right\|} \geq \frac{2n-1}{n} \frac{\psi\|v\|}{2\sqrt{\psi^2+1}\|v\|} \geq$$

$$\geq \frac{\psi}{2\sqrt{\psi^2+1}} =: f(\psi).$$

Observe that f is increasing in $[0, \infty)$, so that

$$\frac{1}{s_n}\left(\|x-a\| - \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{s_n}(a), B \cap \bar{\mathbf{B}}_{s_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right)\right) \geq f\left(\frac{\theta}{5}\right).$$

Second case, $\psi \in \left(0, \dfrac{\theta}{5}\right]$. Then $v \neq 0$. Since A and B satisfy property $(\mathcal{LT})$ there exist sequences $t_n \searrow 0$, $\{a_n\}_{n\geq 1} \subset (A \setminus B)$ and $\{b_n\}_{n \geq 1} \subset (B \setminus A)$ such that for all n

$$\|a_n - a\| \leq t_n, \quad \|b_n - b\| \leq t_n, \quad \|a_n - b_n\| \leq \|a - b\| - t_n\theta. \tag{3.2}$$

Denote

$$u_n = \frac{a+b}{2} + v - \frac{a_n + b_n}{2} \quad \text{and} \quad w_n = \frac{a_n - b_n}{\|a_n - b_n\|}.$$

Observe that

$$|(u_n, w_n)| \leq \left|\left(\frac{a + b - a_n - b_n}{2}, \frac{a_n - b_n}{\|a_n - b_n\|}\right)\right| + \left|\frac{1}{\|a_n - b_n\|}(v, (a - a_n) - (b - b_n))\right|$$

$$\leq t_n\left(1+\frac{2\|v\|}{\|a_n-b_n\|}\right)\leq t_n\left(1+2\psi\right)$$

for large enough n. Thus $(u_n, w_n) \to 0$. Clearly $u_n \to v$. Hence for large enough n holds $\|u_n\|^2 - (u_n, w_n) \geq \|v\|^2/2$. For these n define

$$\gamma_n := 1-\sqrt{\frac{(1+2\psi)^2t_n^2-(u_n,w_n)^2}{\|u_n\|^2-(u_n,w_n)^2}}$$

and

$$v_n := \gamma_n(u_n - (u_n, w_n)w_n).$$

Then $\gamma_n \in (0, 1)$ for large enough n (and actually $\gamma_n \to 1$ since the numerator in the root tends to 0, and the denominator stays away from 0). It is easy to see that $(v_n, w_n) = 0$ so that $(v_n, a_n - b_n) = 0$. Next we observe that

$$\|v_n-u_n\|^2=\gamma_n^2(\|u_n\|^2-(u_n,w_n)^2)-2\gamma_n(\|u_n\|^2-(u_n,w_n)^2)+\|u_n\|^2=$$

$$=(\gamma_n-1)^2(\|u_n\|^2-(u_n,w_n)^2)+(u_n,w_n)^2=(1+2\psi)^2t_n^2.$$

Thus $\|v_n - u_n\| = (1 + 2\psi)t_n$. Set $s_n = (1 + 2\psi)t_n \geq t_n$. Set $x_n := \dfrac{a_n + b_n}{2} + v_n$. Since $\|x - x_n\| = \|v_n - u_n\|$, we have $x_n \in \bar{\mathbf{B}}_{s_n}(x)$ and $\|x_n - a_n\| = \|x_n - b_n\|$. Moreover, using (3.2), $a_n \in A \cap \bar{\mathbf{B}}_{s_n}(a)$, $b_n \in B \cap \bar{\mathbf{B}}_{s_n}(b)$. Thus

$$\frac{1}{s_n}(\|x-a\|-\mathrm{d}\left(A\cap\bar{\mathbf{B}}_{s_n}(a),B\cap\bar{\mathbf{B}}_{s_n}(b),\bar{\mathbf{B}}_{s_n}(x)\right))\geq\frac{1}{s_n}(\|x-a\|-\|x_n-a_n\|)=$$

$$\frac{1}{(1+2\psi)t_n}\left(\left\|\frac{a-b}{2}+v\right\|-\left\|\frac{a_n-b_n}{2}+v_n\right\|\right)=$$

$$\frac{1}{(1+2\psi)t_n}\frac{(\left\|\frac{a-b}{2}\right\|-\left\|\frac{a_n-b_n}{2}\right\|)(\left\|\frac{a-b}{2}\right\|+\left\|\frac{a_n-b_n}{2}\right\|)}{\left\|\frac{a-b}{2}+v\right\|+\left\|\frac{a_n-b_n}{2}+v_n\right\|}+\frac{1}{(1+2\psi)t_n}\frac{\|v\|^2-\|v_n\|^2}{\left\|\frac{a-b}{2}+v\right\|+\left\|\frac{a_n-b_n}{2}+v_n\right\|}.$$

For the first summand, using (3.2), we obtain

$$\frac{1}{(1+2\psi)t_n}\frac{(\left\|\frac{a-b}{2}\right\|-\left\|\frac{a_n-b_n}{2}\right\|)(\left\|\frac{a-b}{2}\right\|+\left\|\frac{a_n-b_n}{2}\right\|)}{\left\|\frac{a-b}{2}+v\right\|+\left\|\frac{a_n-b_n}{2}+v_n\right\|}\geq\frac{\theta}{2(1+2\psi)}\frac{\frac{8}{5}\frac{\|a-b\|}{2}}{\frac{12}{5}\left\|\frac{a-b}{2}+v\right\|}$$

$$=\frac{\theta}{3}\frac{1}{(1+2\psi)\sqrt{\psi^2+1}}.$$

Here we have used that since $\dfrac{a_n-b_n}{2}+v_n \xrightarrow[n\to\infty]{} \dfrac{a-b}{2}+v$, for large enough n holds $\dfrac{7}{5}\left\|\dfrac{a-b}{2}+v\right\| \geq \left\|\dfrac{a_n-b_n}{2}+v_n\right\|$ and $\dfrac{a_n-b_n}{2} \xrightarrow[n\to\infty]{} \dfrac{a-b}{2}$ so that

for large enough n holds $\left\|\dfrac{a_n-b_n}{2}\right\| \geq \dfrac{3}{5}\left\|\dfrac{a-b}{2}\right\|$.
For the second summand,

$$\|v\|^2-\|v_n\|^2=(1-\gamma_n^2)\|v\|^2+\gamma_n^2(\|v\|^2-\|u_n\|^2)+\gamma_n^2(u_n,w_n)^2\geq$$

$$\geq \gamma_n^2(\|v\|^2-\|u_n\|^2)=-\gamma_n^2\left((a+b-a_n-b_n,v)+\left\|\frac{a+b-a_n-b_n}{2}\right\|^2\right)$$

$$\geq -\gamma_n^2(a+b-a_n-b_n,v)-\gamma_n^2t_n^2\geq -2\gamma_n^2t_n\|v\|-\gamma_n^2t_n^2$$

since $\left\|\dfrac{a+b-a_n-b_n}{2}\right\|^2 \leq t_n^2$. Thus

$$\frac{1}{(1+2\psi)t_n}\frac{\|v\|^2-\|v_n\|^2}{\left\|\frac{a-b}{2}+v\right\|+\left\|\frac{a_n-b_n}{2}+v_n\right\|}\geq -\frac{\gamma_n^2}{1+2\psi}\left(\frac{2\|v\|+t_n}{\left\|\frac{a-b}{2}+v\right\|+\left\|\frac{a_n-b_n}{2}+v_n\right\|}\right)$$

$$\geq -\frac{\gamma_n^2}{1+2\psi}\frac{2\|v\|}{\frac{8}{5}\left\|\frac{a-b}{2}+v\right\|}-t_n\frac{\gamma_n^2}{1+2\psi}\frac{1}{\frac{8}{5}\left\|\frac{a-b}{2}+v\right\|}$$

$$\geq -\frac{5}{4}\frac{\psi}{(1+2\psi)\sqrt{\psi^2+1}}-t_n\frac{\gamma_n^2}{1+2\psi}\frac{1}{\frac{8}{5}\left\|\frac{a-b}{2}+v\right\|}\geq -\frac{4}{3}\frac{\psi}{(1+2\psi)\sqrt{\psi^2+1}},$$

for large enough n, since $t_n \searrow 0$. Consequently,

$$\frac{1}{(1+2\psi)t_n}\left(\left\|\frac{a-b}{2}+v\right\|-\left\|\frac{a_n-b_n}{2}+v_n\right\|\right)\geq \frac{\theta-4\psi}{3(1+2\psi)\sqrt{\psi^2+1}}=:g(\psi).$$

Observe that g is decreasing in $\left[0,\dfrac{\theta}{5}\right]$. Indeed, this could be seen as follows: evaluate

$$g'(\psi)=\frac{8\psi^3-(4\psi^2+\psi+2)\,\theta-4}{3(2\psi+1)^2\,(\psi^2+1)^{3/2}}.$$

Clearly the denominator is positive.
Let us denote $h(\psi)=8\psi^3-(4\psi^2+\psi+2)\,\theta-4$. We have $h(0)=-2\theta-4<0$ and $h\left(\dfrac{\theta}{5}\right)=\dfrac{1}{125}\left(-12\theta^3-25\theta^2-250\theta-500\right)<0$. Let $\psi_0\geq 0$ be such that $h(\psi_0)=0$. Thus $\theta=\dfrac{4\,(2\psi_0^3-1)}{4\psi_0^2+\psi_0+2}$. Observe that

$$5\psi_0-\theta=5\psi_0-\frac{4\,(2\psi_0^3-1)}{4\psi_0^2+\psi_0+2}=\frac{12\psi_0^3+5\psi_0^2+10\psi_0+4}{4\psi_0^2+\psi_0+2}>0.$$

Thus $\psi_0 > \frac{\theta}{5}$. Hence h is nonzero on $\left[0, \frac{\theta}{5}\right]$. Moreover h is negative at the endpoints of that interval, and is also continuous. Hence h attains only negative values at $\left[0, \frac{\theta}{5}\right]$, therefore g' is negative as well and thus g is decreasing on this interval.

Back to the proof, we obtain

$$\frac{1}{s_n}(\|x-a\| - \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{s_n}(a), B \cap \bar{\mathbf{B}}_{s_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right)) \geq g(\psi) \geq g\left(\frac{\theta}{5}\right) > 0.$$

Third case, $\psi = 0$. Then $v = 0$. Observe that

$$\left\|\frac{a+b}{2} - \frac{a_n+b_n}{2}\right\| \leq t_n.$$

Set $s_n = t_n$. Then $x_n := \frac{a_n+b_n}{2} \in \bar{\mathbf{B}}_{s_n}(x)$, $\|x_n - a_n\| = \|x_n - b_n\|$ and as before $a_n \in A \cap \bar{\mathbf{B}}_{s_n}(a)$, $b_n \in B \cap \bar{\mathbf{B}}_{s_n}(b)$. Thus

$$\frac{1}{s_n}(\|x-a\| - \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{s_n}(a), B \cap \bar{\mathbf{B}}_{s_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right)) \geq \frac{1}{s_n}(\|x-a\| - \|x_n - a_n\|) =$$

$$\frac{1}{t_n}\left(\left\|\frac{a+b}{2} - a\right\| - \left\|\frac{a_n+b_n}{2} - a_n\right\|\right) = \frac{\|a-b\| - \|a_n - b_n\|}{2t_n} \geq \frac{\theta}{2}.$$

We conclude

$$\frac{1}{s_n}(\|x-a\| - \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{s_n}(a), B \cap \bar{\mathbf{B}}_{s_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right)) \geq \tag{3.3}$$

$$\min\left\{\frac{\theta}{2}, f\left(\frac{\theta}{5}\right), g\left(\frac{\theta}{5}\right)\right\} > 0.$$

Set $\alpha := \min\left\{\frac{\theta}{2}, f\left(\frac{\theta}{5}\right), g\left(\frac{\theta}{5}\right)\right\}$ and $\lambda_n := \left(\alpha + \frac{1}{\sqrt{\varepsilon}}\right) s_n$. Observe that $\lambda_n > s_n$ since $\varepsilon < 1$. Thus, using (3.3) we obtain

$$\frac{1}{s_n}(\|x-a\| - \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{\lambda_n}(a), B \cap \bar{\mathbf{B}}_{\lambda_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right)) \geq$$

$$\frac{1}{s_n}(\|x-a\| - \mathrm{d}\left(A \cap \bar{\mathbf{B}}_{s_n}(a), B \cap \bar{\mathbf{B}}_{s_n}(b), \bar{\mathbf{B}}_{s_n}(x)\right)) \geq \alpha.$$

Finally, property $(\mathcal{P}')$ holds with ε and α. □

3.4 Basic relations between (sub)transversality and (sub)regularity

The next theorem shows that regularity and subregularity could be characterized in terms of transversality and subtransversality. The same sets as in the formulations below appear in the papers [21](Theorem 5.2), [22] (Theorem 4.2) and [12] (Theorem 4), but the equivalence with (sub)regularity is not explicitly stated.

Theorem 3.4.1. *Let $F : X \rightrightarrows Y$ be a set-valued mapping between the metric spaces X and Y, and $(\bar{x}, \bar{y}) \in Gr\ F$. Define the sets $A := Gr\ F$ and $B := X \times \{\bar{y}\}$. Then F is subregular at $(\bar{x}, \bar{y})$ if and only if A and B are subtransversal at $(\bar{x}, \bar{y})$.*

Proof. Let the sets be subtransversal, that is there are $\delta > 0$ and $K_1 > 0$ such that

$$d((x, y), A \cap B) \leq K_1(d((x, y), A) + d((x, y), B))$$

for all $(x, y) \in \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$. Observe that $A \cap B = \{(\hat{x}, \bar{y}) \mid \hat{x} \in F^{-1}(\bar{y})\}$. Let $x \in \bar{\mathbf{B}}_\delta(\bar{x})$. Then

$$d((x, \bar{y}), A \cap B) = d(x, F^{-1}(\bar{y}))\,.$$

On the other hand $d((x, \bar{y}), A) \leq d(\bar{y}, F(x))$ and $d((x, \bar{y}), B) = 0$, whence subtransversality implies

$$d(x, F^{-1}(\bar{y})) \leq K_1 d(\bar{y}, F(x)),$$

hence F is subregular at $(\bar{x}, \bar{y})$ with constants K_1 and δ.

For the reverse direction, let F be subregular at $(\bar{x}, \bar{y})$, that is there are $\delta > 0$ and $K_2 > 0$ such that

$$d(x, F^{-1}(\bar{y})) \leq K_2 d(\bar{y}, F(x))\,.$$

Take $(x, y) \in \bar{\mathbf{B}}_{\delta/3}((\bar{x}, \bar{y}))$ and $\varepsilon \in (0, \delta/3)$. Observe that $d((x, y), B) = d(y, \bar{y})$. Let $(x', y') \in A$ be such that $d(x, x') + d(y, y') \leq d((x, y), A) + \varepsilon$. Note that

$$\begin{aligned} d(x', \bar{x}) &\leq d((x', y'), (\bar{x}, \bar{y})) \leq d((x', y'), (x, y)) + d((x, y), (\bar{x}, \bar{y})) \\ &\leq d((x, y), A) + \varepsilon + d((x, y), (\bar{x}, \bar{y})) \leq \varepsilon + 2d((x, y), (\bar{x}, \bar{y})) \leq \delta. \end{aligned}$$

Then

$$\begin{aligned}
d((x,y),A\cap B) &= d(x,F^{-1}(\bar y)) + d(y,\bar y) \le d(x',F^{-1}(\bar y)) + d(x,x') + d(y,\bar y)\\
&\le K_2 d(\bar y, F(x')) + d(x,x') + d(y,\bar y) \le K_2 d(y',\bar y) + d(x,x') + d(y,\bar y)\\
&\le K_2 d(\bar y, y) + K_2 d(y,y') + d(x,x') + d(y,\bar y)\\
&\le (K_2+1)d((x,y),B) + (K_2+1)d((x,y),A) + (K_2+1)\varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$ proves subtransversality with constants $K_1 = K_2 + 1$ and $\delta/3$. $\square$

Corollary 3.4.2. *Let $F : X \rightrightarrows Y$, X and Y be metric spaces, and $(\bar x,\bar y) \in Gr\ F$ as above. Define the sets $A := Gr\ F$ and $B_y := X \times \{y\}$. Then F is regular at $(\bar x,\bar y)$ if and only if there are constants $\delta > 0$ and $K > 0$ such that for any $(x,y) \in \bar{\mathbf{B}}_\delta((\bar x,\bar y))$ and any $\hat y \in \bar{\mathbf{B}}_\delta(\bar y)$*

$$d((x,y),A\cap B_{\hat y}) \le K(d((x,y),A) + d((x,y),B_{\hat y}))\,. \tag{3.4}$$

If in addition X and Y are normed spaces, then this is also equivalent to A and $B := B_{\bar y}$ being transversal at $(\bar x,\bar y)$.

Proof. Observe that in the first part of the proof above, we never made explicit use of the fact that $(\bar x,\bar y) \in A\cap B$. Pick $\hat y \in \bar{\mathbf{B}}_\delta(\bar y)$. The inequality (3.4) is satisfied with $B_{\hat y}$ instead of B, so that, according to Theorem 3.4.1, we arrive at $d(x,F^{-1}(\hat y)) \le Kd(\hat y,F(x))$ for all $x \in \bar{\mathbf{B}}_\delta(\bar x)$. Thus, we obtain regularity at $(\bar x,\bar y)$.

For the other direction, again take $\hat y \in \bar{\mathbf{B}}_\delta(\bar y)$. Since $d(x',F^{-1}(\hat y)) \le Kd(\hat y,F(x'))$, for x' near $\bar x$, as in the proof of Theorem 3.4.1, we obtain

$$d((x,y),A\cap B_{\hat y}) \le (K+1)(d((x,y),A) + d((x,y),B_{\hat y})),$$

for all $(x,y) \in \bar{\mathbf{B}}_{\delta/3}((\bar x,\bar y))$.

If the spaces are normed, then $B_{\hat y} = B + (x,\hat y - \bar y)$ for any x. Thus the inequality (3.4) is the inequality defining transversality. $\square$

3.5 Primal space characterizations of subregularity

We are going to use the characterization of subtransversality from 3.1. The lemma below is our main technical tool, allowing to pass from a local inequality to a global one. It is a slight generalization of Lemma 3.1.3, and the proof of the latter could be easily adapted to obtain a proof of the former.

Lemma 3.5.1. *Let A and B be closed subsets of the complete metric space X and $\bar{x} \in X$. Let $f : X \times X \to [0, +\infty)$ be lower semicontinuous such that $f(x, y) \le d(x, y)$ for all $x, y \in X$. Assume that there exist $\delta > 0$ and $M > 0$ such that for any $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ with $f(x^A, x^B) \neq 0$, there are $\theta > 0$, $\hat{x}^A \in A$ and $\hat{x}^B \in B$, such that $d(x^A, \hat{x}^A) \le M\theta$, $d(x^B, \hat{x}^B) \le M\theta$ and*

$$f(\hat{x}^A, \hat{x}^B) \le f(x^A, x^B) - \theta.$$

Fix $x^A \in A \cap \bar{\mathbf{B}}_{\frac{\delta}{1+2M}}(\bar{x})$ and $x^B \in B \cap \bar{\mathbf{B}}_{\frac{\delta}{1+2M}}(\bar{x})$. Then there exist $\tilde{x}^A \in A$ and $\tilde{x}^B \in B$, such that $f(\tilde{x}^A, \tilde{x}^B) = 0$, $d(\tilde{x}^A, x^A) \le M f(x^A, x^B)$ and $d(\tilde{x}^B, x^B) \le M f(x^A, x^B)$.

The next theorem is a primal space characterization of subregularity (cf. Theorem 2.58 in [39] or Corollaries 5.8 and 5.9 in [49]). We prove it in a new way based on the earlier results.

Theorem 3.5.2. *Let $F : X \rightrightarrows Y$ be with closed graph and $(\bar{x}, \bar{y}) \in Gr\ F$, where X and Y are complete metric spaces. Then F is subregular at $(\bar{x}, \bar{y}) \in Gr\ F$ if and only if there exist constants $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in Gr\ F \cap \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$, there is $(\hat{x}, \hat{y}) \in Gr\ F \setminus \{(x, y)\}$, such that*

$$d(\hat{y}, \bar{y}) \le d(y, \bar{y}) - \tau d((x, y), (\hat{x}, \hat{y}))$$

Proof. According to Theorem 3.4.1, F is subregular at $(\bar{x}, \bar{y})$ if and only if the sets $A :=$ Gr F and $B := X \times \{\bar{y}\}$ are subtransversal at that point. Assume that they are subtransversal. Then, according to Proposition 3.1.4, *property* $(\mathcal{T})$ holds with some constants δ and M. Take $(x, y) \in A \cap \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$. Then $(x, \bar{y}) \in B \cap \bar{B}_\delta(\bar{x}, \bar{y})$ and thus there exist $(\hat{x}, \hat{y}) \in A$ and $(\hat{x}_B, \bar{y}) \in B$ such that

$$d((\hat{x}, \hat{y}), (\hat{x}_B, \bar{y})) \le d((x, y), (x, \bar{y})) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\}$$

and $\max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} > 0$. If we assume that $(\hat{x}, \hat{y}) = (x, y)$, then in particular $\hat{y} = y$. Thus

$$d((x, y), (x, \bar{y})) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} < d(y, \bar{y})$$

and

$$d(y, \bar{y}) \le d((\hat{x}, \hat{y}), (\hat{x}_B, \bar{y})).$$

This contradicts the earlier inequality, hence $(\hat{x}, \hat{y}) \neq (x, y)$.

From here we obtain

$$
\begin{aligned}
d(\hat{y},\bar{y}) \le d((\hat{x},\hat{y}),(\hat{x}_B,\bar{y})) &\le d((x,y),(x,\bar{y})) - \frac{1}{M}\max\{d((x,y),(\hat{x},\hat{y})), d(x,\hat{x}_B)\} \\
&\le d(y,\bar{y}) - \tau d((x,y),(\hat{x},\hat{y}))\,,
\end{aligned}
$$

for $\tau := \dfrac{1}{M}$.

Now assume that there exist constants $\delta > 0$ and $\tau > 0$ such that for all $(x,y) \in \mathrm{Gr}\ F \cap \bar{\mathbf{B}}_\delta((\bar{x},\bar{y}))$, there is $(\hat{x},\hat{y}) \in \mathrm{Gr}\ F$, such that

$$
d(\hat{y},\bar{y}) \le d(y,\bar{y}) - \tau d((x,y),(\hat{x},\hat{y}))\,.
$$

Let the function $f : (X\times Y)\times(X\times Y) \to [0,+\infty)$ be given by $f((x_1,y_1),(x_2,y_2)) = d(y_1,y_2)$. Let $(x,y) \in A \cap \bar{\mathbf{B}}_\delta((\bar{x},\bar{y}))$ and $(x_B,\bar{y}) \in B$ with $d(x_B,\bar{x}) \le \delta$ be arbitrary. Then, there exists a point $(\hat{x},\hat{y}) \in A$, satisfying the inequality. For the points $(\hat{x},\hat{y}) \in A$ and $(x_B,\bar{y}) \in B$ we estimate

$$
\begin{aligned}
f((\hat{x},\hat{y}),(x_B,\bar{y})) &= d(\hat{y},\bar{y}) \le d(y,\bar{y}) - \tau d((x,y),(\hat{x},\hat{y})) \\
&= f((x,y),(x_B,\bar{y})) - \tau \max\{d((x,y),(\hat{x},\hat{y})), d((x_B,\bar{y}),(x_B,\bar{y}))\}\,,
\end{aligned}
$$

which means that we can apply Lemma 3.5.1 for A and B at $(\bar{x},\bar{y})$ with function f and constants δ and $M := \dfrac{1}{\tau}$ if the starting points are sufficiently close to $(\bar{x},\bar{y})$.

Let $\hat{\delta} = \frac{\tau}{(\tau+2)(\tau+1)}\delta$ and take $x \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{x})$. If $d(\bar{y},F(x)) \ge \tau\hat{\delta}$, then

$$
\frac{1}{\tau} d(\bar{y},F(x)) \ge \hat{\delta} \ge d(x,\bar{x}) \ge d(x,F^{-1}(\bar{y}))
$$

Otherwise, take $\varepsilon \in \Big(0, \tau\hat{\delta} - d(\bar{y},F(x))\Big)$. Take $y \in F(x)$ for which

$$
d(y,\bar{y}) \le d(\bar{y},F(x)) + \varepsilon \le \tau\hat{\delta} < \frac{\tau}{\tau+2}\delta.
$$

For $M = 1/\tau$, we have $d(x,\bar{x}) \le \frac{\delta}{1+2M}$ and $d(y,\bar{y}) \le \frac{\delta}{1+2M}$.

Applying Lemma 3.5.1 to $(x,y) \in A$ and $(x,\bar{y}) \in B$, it follows that there exist points $(\tilde{x},\tilde{y}) \in A$ and $(\tilde{x}_B,\bar{y}) \in B$ such that $f((\tilde{x},\tilde{y}),(\tilde{x}_B,\bar{y})) = 0$, hence $\tilde{y} = \bar{y}$, and $d((\tilde{x},\bar{y}),(x,y)) \le Mf((x,y),(x,\bar{y})) = Md(y,\bar{y})$. Using that $\tilde{x} \in F^{-1}(\bar{y})$ and the choice of y, we obtain that

$$
\begin{aligned}
d(x,F^{-1}(\bar{y})) &\le d(x,\tilde{x}) \le d((\tilde{x},\bar{y}),(x,y)) \le Md(y,\bar{y}) \\
&\le Md(\bar{y},F(x)) + M\varepsilon
\end{aligned}
$$

Letting $\varepsilon \to 0$, we obtain $d(x,F^{-1}(\bar{y})) \le Md(\bar{y},F(x))$ for all $x \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{x})$. We have verified that F is subregular at $(\bar{x},\bar{y})$ by definition. □

3.6 Primal space characterizations of transversality and regularity

We use again the previously obtained characterizations of transversality in section 3.2. The following theorem is a classical "rate of descent" characterization (cf. Theorem 2.50 in [39] or Theorem 7 in [41]).

Theorem 3.6.1. *Let $F : X \rightrightarrows Y$ be with closed graph and $(\bar{x}, \bar{y}) \in Gr\, F$, where X and Y are complete metric spaces. Then F is regular at $(\bar{x}, \bar{y}) \in Gr\, F$ if and only if there exist $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in Gr\, F \cap \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$ and all $v \in \bar{\mathbf{B}}_\delta(\bar{y})$, there is $(\hat{x}, \hat{y}) \in Gr\, F \setminus \{(x, y)\}$, such that*

$$d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y}))\,.$$

Proof. According to Corollary 3.4.2, F is regular at $(\bar{x}, \bar{y})$ if and only if there are constants $\delta > 0$ and $K > 0$ such that for any $(x, y) \in \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$ and any $v \in \bar{\mathbf{B}}_\delta(\bar{y})$ it holds

$$d((x, y), A \cap B_v) \leq K(d((x, y), A) + d((x, y), B_v))\,,$$

where $B_v := X \times \{v\}$.

Let F be regular at $(\bar{x}, \bar{y})$. Fix $\hat{\delta} := \frac{\delta}{4K+10}$, $(x, y) \in A \cap \bar{\mathbf{B}}_{\hat{\delta}}((\bar{x}, \bar{y}))$ and $v \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{y})$. According to Proposition 3.1.4, A and B_v have *property* ($\mathcal{T}$) at $(\bar{x}, \bar{y})$ with constants $\hat{\delta}$ and M. Hence, there exist $(\hat{x}, \hat{y}) \in A$ and $(\hat{x}^B, v) \in B_v$ such that

$$d((\hat{x}, \hat{y}), (\hat{x}^B, v)) \leq d((x, y), (x, v)) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}^B)\}$$

and $\max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} > 0$. If we assume that $(\hat{x}, \hat{y}) = (x, y)$, then in particular $\hat{y} = y$. Thus

$$d((x, y), (x, v)) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} < d(y, v)$$

and

$$d(y, v) \leq d((\hat{x}, \hat{y}), (\hat{x}_B, v)).$$

This contradicts the earlier inequality, hence $(\hat{x}, \hat{y}) \neq (x, y)$. From here we obtain

$$d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y}))\,,$$

for $\tau := \dfrac{1}{M}$.

Now, assume that there exist constants $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in \text{Gr } F \cap \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$ and all $v \in \bar{\mathbf{B}}_\delta(\bar{y})$, there is $(\hat{x}, \hat{y}) \in \text{Gr } F \setminus \{(x, y)\}$, such that

$$d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y})) .$$

Let the function $f : (X \times Y) \times (X \times Y) \to [0, +\infty)$ be given by $f((x_1, y_1), (x_2, y_2)) = d(y_1, y_2)$. Let us fix $(x, y) \in A \cap \bar{\mathbf{B}}_{\delta/2}((\bar{x}, \bar{y}))$, $v \in \bar{\mathbf{B}}_{\delta/2}(\bar{y})$ and $(x^B, v) \in B_v$ with $d(x_B, \bar{x}) \leq \delta/2$. Then, there exists a point $(\hat{x}, \hat{y}) \in A$, satisfying the above inequality.

For the points $(\hat{x}, \hat{y}) \in A$ and $(x_B, v) \in B_v$ we estimate

$$\begin{aligned} f((\hat{x}, \hat{y}), (x_B, v)) &= d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y})) \\ &= f((x, y), (x_B, v)) - \tau \max\{d((x, y), (\hat{x}, \hat{y})), d((x_B, v), (x_B, v))\} , \end{aligned}$$

which means that we can apply Lemma 3.5.1 for A and B_v at $(\bar{x}, \bar{y})$ with function f and constants $\delta/2$ and $M := \dfrac{1}{\tau}$ if the starting points are sufficiently close to $(\bar{x}, \bar{y})$.

Let $\hat{\delta} := \frac{\tau}{4(\tau+2)(\tau+1)}\delta$ and take $v \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{y})$ and $x \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{x})$. Applying Lemma 3.5.1 for $(\bar{x}, \bar{y}) \in A$ and $(\bar{x}, v) \in B_v$ we arrive at a point $x_v \in F^{-1}(v)$ such that $d(x_v, \bar{x}) \leq M d(\bar{y}, v) \leq \hat{\delta}/\tau$

If $d(v, F(x)) \geq \hat{\delta}(1 + \tau)$, then

$$\frac{1}{\tau} d(v, F(x)) \geq \hat{\delta} + \frac{\hat{\delta}}{\tau} \geq d(x, \bar{x}) + d(x_v, \bar{x}) \geq d(x, x_v) \geq d(x, F^{-1}(v))$$

Otherwise, take $\varepsilon \in \Big(0, \hat{\delta}(\tau + 1) - d(\bar{y}, F(x))\Big)$. Take $y \in F(x)$ for which

$$d(y, v) \leq d(v, F(x)) + \varepsilon \leq \hat{\delta}(\tau + 1) \leq \frac{\tau}{4(\tau + 2)}\delta.$$

Recall that $M = 1/\tau$, hence $d(x, \bar{x}) \leq \frac{\delta/2}{1+2M}$ and $d(y, \bar{y}) \leq d(y, v) + d(v, \bar{y}) \leq \frac{\tau}{2(\tau+2)}\delta = \frac{\delta/2}{1+2M}$.

Applying Lemma 3.5.1 to $(x, y) \in A$ and $(x, v) \in B_v$, it follows that there exist points $(\tilde{x}, \tilde{y}) \in A$ and $(\tilde{x}_B, v) \in B_v$ such that $f((\tilde{x}, \tilde{y}), (\tilde{x}_B, \bar{y})) = 0$, hence $\tilde{y} = v$, and $d((\tilde{x}, v), (x, y)) \leq M f((x, y), (x, v)) = M d(y, v)$. Using that $\tilde{x} \in F^{-1}(v)$ and the choice of y, we obtain that

$$\begin{aligned} d(x, F^{-1}(v)) &\leq d(x, \tilde{x}) \leq d((\tilde{x}, v), (x, y)) \leq M d(y, v) \\ &\leq M d(v, F(x)) + M\varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain $d(x, F^{-1}(v)) \leq M d(v, F(x))$ for all $x \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{x})$ and $v \in \bar{\mathbf{B}}_{\hat{\delta}}(\bar{y})$. We have verified that F is regular at $(\bar{x}, \bar{y})$ by definition.

□

Using the above theorem, we establish a characterization of metric regularity of a map $F : X \rightrightarrows Y$, X – complete metric space and Y – Banach space, using its *first order (contingent) variation* $F^{(1)}(x, y)$. This is first done in [31] (see also Theorem 4.13 and Remark 4.14(c) in [3] for a proof in Banach spaces or [41] for an alternative proof). Given $(x, y) \in \mathrm{Gr}F$, define $F^{(1)} : X \times Y \rightrightarrows Y$ by

$$F^{(1)}(x, y) := \limsup_{t \to 0_+} \frac{F(\bar{\mathbf{B}}_t(x)) - y}{t},$$

where lim sup stands for the Kuratowski limit superior of sets. Equivalently, $v \in F^{(1)}(x, y)$ exactly when there exist sequences $t_n \to 0_+$, $v_n \to v$ and $(x_n, y_n) \in \mathrm{Gr}\ F$ such that $d(x_n, x) \le t_n$ and $y_n = y + t_n v_n$.

Our proof is done via a sequential characterization of metric regularity, which we have not seen stated anywhere in the literature.

Corollary 3.6.2. *Let us consider $F : X \rightrightarrows Y$ with closed graph, where X is a complete metric space and Y is a Banach space. Then, the following are equivalent*

(i) F is regular at $(\bar{x}, \bar{y}) \in Gr\ F$

(ii) there exist $\delta > 0$ and $r > 0$ such that

$$\mathbf{B}_r(\mathbf{0}) \subset F^{(1)}(x, y) \text{ for all } (x, y) \in \bar{\mathbf{B}}_\delta(\bar{x}, \bar{y}) \cap Gr\ F$$

(iii) there exist $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in Gr\ F \cap \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$ and all $\hat{y} \in \bar{\mathbf{B}}_\delta(\bar{y})$, there is a sequence $\{(x_n, y_n)\}_{n \ge 1} \subset Gr\ F \setminus \{(x, y)\}$ converging to (x, y) such that for all n it holds

$$\|y_n - \hat{y}\| \le \|y - \hat{y}\| - \tau d((x_n, y_n), (x, y)).$$

Proof. We have that (iii) implies (i) by Theorem 3.6.1.

Next, we will show that (i) implies (ii). Let F be regular at $(\bar{x}, \bar{y}) \in \mathrm{Gr}\ F$. By definition there exist $K > 0$ and $\delta > 0$ such that for all $x \in \mathbf{B}_\delta(\bar{x})$ and all $y \in \mathbf{B}_\delta(\bar{y})$ the following inequality holds:

$$d(x, F^{-1}(y)) \le K d(y, F(x)).$$

Fix arbitrary $(x, y) \in \bar{\mathbf{B}}_{\frac{\delta}{2}}((\bar{x}, \bar{y})) \cap \mathrm{Gr}\ F$, $v \in Y$ with $\|v\| < \dfrac{1}{K} =: r$ and a sequence $t_n \to 0_+$ such that $y_n := y + t_n v \in \mathbf{B}_\delta(\bar{y})$. Then, there exist $\varepsilon > 0$

such that $\|v\| \leq \dfrac{1-\varepsilon}{K}$. Moreover, for every $n \in \mathbb{N}$ there exists $x_n \in F^{-1}(y_n)$ such that $d(x, x_n) \leq d(x, F^{-1}(y_n)) + \varepsilon t_n$. Thus

$$\begin{aligned} d(x, x_n) &\leq d(x, F^{-1}(y_n)) + \varepsilon t_n \leq K d(y_n, F(x)) + \varepsilon t_n \\ &\leq K\|y_n - y\| + \varepsilon t_n \leq (1 - \varepsilon + \varepsilon) t_n = t_n . \end{aligned}$$

Having that $(x_n, y_n) = (x_n, y + t_n v) \in \text{Gr } F$, $v \in F^{(1)}(x, y)$ by definition. We have shown that (ii) holds, if F is regular at $(\bar{x}, \bar{y})$.

It remains to prove that (ii) implies (iii). Assume that (ii) holds. Let $(x, y) \in \bar{\mathbf{B}}_\delta(\bar{x}, \bar{y}) \cap \text{Gr } F$ and $\hat{y} \in \bar{\mathbf{B}}_\delta(\bar{y})$ be arbitrary. Let us denote $v := \rho \frac{\hat{y}-y}{\|\hat{y}-y\|}$ for some $\rho \in (0, r)$. Then, $v \in Y$ with $\|v\| = \rho$ and due to (ii) there exist sequences $t_n \to 0_+$, $v_n \to v$ and $(x_n, y_n) \in \text{Gr } F$ such that $d(x_n, x) \leq t_n$ and $y_n = y + t_n v_n$. Since $\rho t_n < 1$ for n – large enough, we estimate

$$\|y_n - \hat{y}\| = \left\| y - \hat{y} + t_n \rho \frac{\hat{y} - y}{\|\hat{y} - y\|} \right\| = \|y - \hat{y}\| - t_n \rho \, .$$

Moreover, we have that $t_n \geq d(x_n, x) > \frac{d(x_n, x)}{\rho+1}$ and $t_n = \frac{\|y_n - y\|}{\|v_n\|} \geq \frac{\|y_n - y\|}{\rho+1}$ for n – large enough. Therefore

$$\|y_n - \hat{y}\| \leq \|y - \hat{y}\| - \tau d((x_n, y_n), (x, y)) \, ,$$

where $\tau := \frac{\rho}{2(\rho+1)}$.

The proof is complete. $\square$

Next, we establish the relation between the metric regularity of a map $F : X \rightrightarrows Y$, X and Y – Banach spaces, and its *graphical (contingent) derivative* $DF(x|y)$. Given $(x, y) \in \text{Gr}F$, define $DF(x|y) : X \rightrightarrows Y$ as the map, whose graph is the (Bouligand) tangent cone $T_{\text{Gr}F}(x, y)$, i.e.

$$v \in DF(x|y)(u) \Leftrightarrow (u, v) \in T_{\text{Gr}F}(x, y) \, .$$

Corollary 3.6.3 (cf. Theorem 1.2 in [25] and Theorem 4.13 and Remark 4.14(b) in [3])**.** *Let $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in Gr\, F$, where X and Y are Banach spaces. Assume there exist $\delta > 0$ and $K > 0$ such that for any $(x, y) \in GrF$ with $\|x - \bar{x}\| \leq \delta$, $0 < \|y - \bar{y}\| \leq \delta$ and any $v \in Y$, $\|v\| = 1$, it holds*

$$\inf \left\{ \|u\| \;\middle|\; v \in DF(x|y)(u) \right\} \leq K \, .$$

Then F is regular at $(\bar{x}, \bar{y}) \in Gr\, F$. The reverse direction is also true when X is finite-dimensional.

Proof. For the first part, we have that for every $\varepsilon > 0$, every $(x, y) \in \mathrm{Gr}F$ with $\|x - \bar{x}\| \le \delta$, $0 < \|y - \bar{y}\| \le \delta$ and for every $v \in Y$ with $\|v\| = 1$ there is $(u, v) \in T_{\mathrm{Gr}\,F}(x, y)$ such that $\|u\| < K + \varepsilon$. From the definition of Bouligand tangent cone, there exist sequences $u_n \to u$, $v_n \to v$ and a sequence of positive t_n tending to zero, such that $(x+t_n u_n, y+t_n v_n) \in \mathrm{Gr}\ F$. Let us fix an arbitrary $\lambda \in (0, 1]$. We have that $\tau_n := \frac{t_n(K+\varepsilon)}{\lambda} \to 0$ and $(x + t_n u_n, y + t_n v_n) = (x + \tau_n \frac{\lambda u_n}{K+\varepsilon}, y + \tau_n \frac{\lambda v_n}{K+\varepsilon}) \in \mathrm{Gr}\ F$. Without loss of generality we can assume that $\|u_n\| \le K+\varepsilon$ for n – large enough. Taking into account that $d(x, x + \tau_n \frac{\lambda u_n}{K+\varepsilon}) \le \lambda \tau_n \le \tau_n$, we obtain that $\frac{\lambda v}{K+\varepsilon} \in F^{(1)}(x, y)$. Since the unit vector $v \in Y$, $\lambda \in (0, 1]$ and $\varepsilon > 0$ are arbitrary and $\|\frac{\lambda v}{K+\varepsilon}\| = \frac{\lambda}{K+\varepsilon}$, we obtain that $\mathbf{B}_{\frac{1}{K}}(\mathbf{0}) \subset F^{(1)}(x, y)$. Then, F is regular at $(\bar{x}, \bar{y})$ due to Corollary 3.6.2.

For the converse, let X be finite-dimensional and F be regular at $(\bar{x}, \bar{y})$ with constants δ and K. Let $(x, y) \in \mathrm{Gr}F$ with $\|x - \bar{x}\| \le \delta$ and $0 < \|y - \bar{y}\| \le \delta$, $\varepsilon \in (0, \frac{1}{K})$ and $v \in Y$ with $\|v\| = 1$ be arbitrary. Then, we have that $w := (\frac{1}{K} - \varepsilon)v \in F^{(1)}(x, y)$ due to Corollary 3.6.2. That is, there exist sequences $t_n \to 0_+$, $w_n \to w$ and $(x_n, y_n) \in \mathrm{Gr}\ F$ such that $\|x_n - x\| \le t_n$ and $y_n = y + t_n w_n$. Moreover, since X is finite-dimensional, we have $x_n = x + t_n p_n$, where $\|p_n\| \le 1$ which implies $p_n \longrightarrow_{n\to\infty} p$ (up to a subsequence, labeled in the same way) and $(p, w) \in T_{\mathrm{Gr}\ F}$. Hence $(\frac{p}{\|w\|}, \frac{w}{\|w\|}) = (\frac{p}{\frac{1}{K}-\varepsilon}, v) \in T_{\mathrm{Gr}\ F}$ and $\|\frac{p}{\frac{1}{K}-\varepsilon}\| \le \frac{1}{\frac{1}{K}-\varepsilon}$. We have obtained that for any $v \in Y$, $\|v\| = 1$, it holds

$$\inf\left\{\|u\| \;\middle|\; v \in DF(x|y)(u)\right\} \le K\,.$$

The proof is complete. □

Chapter 4

Sufficient condition for tangential transversality

4.1 Introduction

In this chapter we will apply the following simplified version of the Lagrange multiplier rule stated earlier, Theorem 1.0.3. The simplification is regarding the cones involved - we will use only Clarke tangent cones. Here is the corresponding theorem.

Theorem 4.1.1 (Lagrange multiplier rule)**.** *Let us consider the optimization problem*

$$f(x) \to \min \quad \textit{subject to} \quad x \in S \ ,$$

where $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous and proper and* S *is a closed subset of the Banach space* X*. Let* x_0 *be a solution of the above problem. Let* $\widehat{T}_{epif}(x_0, f(x_0))$ *and* $\widehat{T}_S(x_0)$ *be the Clarke tangent cones to epi* f *and* S *respectively. Then one of the following alternatives hold*
(a) If $\widehat{T}_{epif}(x_0, f(x_0)) - \widehat{T}_S(x_0) \times (-\infty, 0]$ *is not dense in* $X \times \mathbb{R}$*, then there exists a pair* $(\xi, \eta) \in X^* \times \mathbb{R}$ *such that*

(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$*;*

(ii) $\eta \in \{0, 1\}$*;*

(iii) $\langle \xi, v \rangle \ \leq 0$ *for every* $v \in \widehat{T}_S(x_0)$*;*

(iv) $\langle \xi, w \rangle + \eta s \geq 0$ *for every* $(w, s) \in \widehat{T}_{epif}(x_0, f(x_0))$*.*

(b) If $\widehat{T}_{epif}(x_0, f(x_0)) - \widehat{T}_S(x_0) \times (-\infty, 0]$ *is dense in* $X \times \mathbb{R}$*, then* $epif$ *and* $S \times (-\infty, f(x_0)]$ *are not subtransversal at* $(x_0, f(x_0))$*.*

4.2 Preliminaries

Here we state some preliminary results and definitions. The following definitions are from [44]:

Definition 4.2.1. *Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ one can find $\lambda > 0$ for which $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is nonempty for each $t \in [0, \lambda]$.*

Definition 4.2.2. *Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a sequence uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ one can find a sequence of positive reals $t_m \to 0$ for which $S \cap (x + t_m(v + \varepsilon\overline{\mathbf{B}}))$ is nonempty for each positive integer m.*

The next theorem is the main result from [10].

Theorem 4.2.3. *Let S be a closed subset of X and x_0 belong to S. The following are equivalent*

1. *$D_S(x_0)$ is a uniform tangent set to S at the point x_0*
2. *$D_S(x_0)$ is a sequence uniform tangent set to S at the point x_0*
3. *for each $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ the set $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is nonempty for each $t \in [0, \lambda]$.*

The basic properties of uniform tangent sets are gathered in the next proposition taken from [8]:

Proposition 4.2.4. *Let S be a closed subset of X and let $x_0 \in S$. Let $D_S(x_0)$ be a uniform tangent set to S at the point x_0. Then, the following hold true:*

1. *$D \subset \widehat{T}_A(x_0)$;*
2. *the set $c\,D_S(x_0)$ is a uniform tangent set to S at x_0 for each fixed constant $c > 0$;*
3. *if $D'_S(x_0) \subset D_S(x_0)$, then $D'_S(x_0)$ is a uniform tangent set to S at x_0;*
4. *if $D'_S(x_0)$ is another uniform tangent set to S at x_0, then $D_S(x_0) \cup D'_S(x_0)$ is a uniform tangent set to S at x_0;*

5. *the convex closed closure* $\overline{co}\, D_S(x_0)$ *of* $D_S(x_0)$ *is a uniform tangent set to* S *at* x_0

6. *if* S *is convex, then* $(S - x_0) \cap M\overline{\mathbf{B}}$ *is a uniform tangent set to* S *at* x_0 *for every* $M > 0$.

Remark 4.2.5. *From points* 2., 4. *and* 5. *follows that when* D *and* D' *are uniform tangent sets to* A *at the point* x_0*, their sum,* $D + D'$*, is also an uniform tangent set to* A *at* x_0*, because* $D + D' \subseteq 2co(D \cup D')$.

The following result gives a sufficient condition for a subset of the Clarke tangent cone to be a uniform tangent set.

Theorem 4.2.6 ([57])**.** *Let* A *be closed subset of the Banach space* X *and* $x_0 \in A$*. If* D *is compact subset of* $\hat{T}_A(x_0)$*, then* D *is uniform tangent set.*

In the very particular (yet useful) case when D is a finite subset of $\hat{T}_A(x_0)$, D is a uniform tangent set. Having in mind that in finite-dimensional spaces the closed bounded sets are compact, we obtain

Corollary 4.2.7. *Let* A *be closed subset of* $\mathbb{R}^n$ *and* $x_0 \in A$*. Then* $\bar{\boldsymbol{B}} \cap \hat{T}_A(x_0)$ *is uniform tangent set to* A *at* x_0*. It also generates the Clarke tangent cone* $\hat{T}_A(x_0)$.

Thus, in finite dimensional spaces, the problem of existence of uniform tangent set which generates the corresponding Clarke tangent cone, is solved. However, in infinite-dimensional spaces the closed balls are not compact. Indeed, a result similar to those in Corollary 4.2.7 does not hold in general, as observed in the following

Example 4.2.8. *Let* A *be the Hilbert cube in* ℓ_2*, defined in Example 1.0.1. Then* $\hat{T}_A(\boldsymbol{0}) = \ell_2$*. The set* D *consisting of standard unit vectors* $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$*,* $n \geq 1$*, is a subset of* $\bar{\boldsymbol{B}} \cap \hat{T}_A(\boldsymbol{0})$*, but it is not uniform tangent set. Indeed, assume on the contrary that for* $\eta = \frac{1}{2}$ *there exists* $\delta > 0$ *such that*

$$A \cap \delta\bar{\boldsymbol{B}} + te_n \subset A + \frac{t}{2}\bar{\boldsymbol{B}}$$

holds for every $t \in [0, \delta]$ *and every* $n \geq 1$*. In particular, for all* $n \geq 1$ *it is true that* $\delta e_n \in A + \frac{\delta}{2}\bar{\boldsymbol{B}}$*. Comparing the* n*-th position of the vectors, we obtain that for all* n *holds* $\delta < \frac{1}{n} + \frac{\delta}{2}$*, which is clearly impossible, since* $\delta > 0$*. Thus* D *is not a uniform tangent set. Hence* $\bar{\boldsymbol{B}} \cap \hat{T}_A(x_0)$ *cannot be a uniform tangent set as well, since* $D \subset \bar{\boldsymbol{B}} \cap \hat{T}_A(x_0)$.

The following result identifies potentially infinite-dimensional situations in which the Clarke tangent cone could be generated by uniform tangent set. More results in this direction could be found in [8].

Theorem 4.2.9. *[10] Let A be closed subset of the Banach space X and $x_0 \in A$. Let $\hat{T}_A(x_0)$ be separable set. Then there exists uniform tangent set D to A at x_0, which generates $\hat{T}_A(x_0)$.*

Next we remind the classical concept of compactly epi-Lipschitz sets in Banach spaces. It was introduced by J.M. Borwein and H.M. Strojwas in 1985 in [11] and it includes all finite-dimensional and all epi-Lipschitsian sets in Banach spaces. Since then, it has been an important notion in nonsmooth analysis and has been frequently used in qualification conditions for obtaining normal intersection properties and calculus rules concerning limiting Fréchet cones and subdifferentials (in Asplund spaces, cf. [55] and [56]) and G-cones and G-subdifferentials (in general Banach spaces, cf. [43] and [38]). Compactly epi-Lipschitz sets are called *massive* in [39]. Here is the corresponding

Definition 4.2.10. *Let A be a closed subset of the Banach space X and $x_0 \in A$. We say that A is compactly epi-Lipschitz (massive) at x_0, if there exist $\varepsilon > 0$, $\delta > 0$ and a compact set $K \subset X$, such that for all $t \in [0, \delta]$ the following inclusion holds true*

$$A \cap (x_0 + \delta\overline{\mathbf{B}}) + \varepsilon\overline{\mathbf{B}} \subset A + tK \ .$$

Next we formulate two sufficient conditions for tangential transversality which can be obtained from the main result of this chapter in a unified manner. Theorem 4.2.11 is taken from [8] and Proposition 4.2.13 can be found in [9].

Theorem 4.2.11. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Let A be massive (or compactly epi-Lipschitz) and $\widehat{T}_A(x_0) - \widehat{T}_B(x_0)$ be dense in X. Then A and B are tangentially transversal at x_0.*

Definition 4.2.12. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. We say that A and B are strongly tangentially transversal at x_0 if there exist $D_A(x_0)$ – uniform tangent set to A at the point x_0, $D_B(x_0)$ – uniform tangent set to the set B at the point x_0 and $\rho > 0$ such that*

$$\rho\overline{\mathbf{B}} \subset \overline{co}\,(D_A(x_0) - D_B(x_0))$$

Proposition 4.2.13 ([8]). *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. If A and B are strongly tangentially transversal at x_0, then A and B are tangentially transversal at x_0.*

We will need the following theorem.

Theorem 4.2.14 ([8]). *Let A and B be closed subsets of the Banach space X and $x_0 \in A \cap B$. Let D_A and D_B be uniform tangent sets (to A and B respectively) at x_0, which generate the respective Clarke tangent cones - $\hat{T}_A(x_0)$ and $\hat{T}_B(x_0)$. Let $D_A - D_B$ has nonempty interior and $\hat{T}_A(x_0) - \hat{T}_B(x_0)$ be dense in X. Then A and B are strongly tangentially transversal.*

4.3 Main result

In the following two theorems we formulate our main result. The first is a simpler version which motivate the general:

Theorem 4.3.1. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$ and:*

(i) there exist bounded "ball covering" sets M_A, M_B such that $M_A - M_B$ is dense in $\varepsilon \overline{\boldsymbol{B}}$ and "correcting" sets U_A, U_B such that

$$A \cap (x_0 + \delta \overline{\boldsymbol{B}}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta \overline{\boldsymbol{B}}) + tM_B \subset B + tU_B$$

whenever $t \in [0, \delta]$;

(ii) there exist uniform tangent sets D_A (to A at x_0) and D_B (to B at x_0) such that $D_A - D_B$ is dense in $U_A - U_B$.

Then A and B are tangentially transversal at x_0.

If, in the above theorem, $M_B = U_B = \{\mathbf{0}\}$, our assumption would be reduced to the definition of A being compactly epi-Lipschitz, but with the difference that the compact set is replaced by a difference of two uniform tangent sets. Moreover, the "massiveness-like" property is split between the sets. The above theorem is a direct consequence of its "quantified" version below. For the statement of the next result we will need the notion of ε-density: we say that a set A is ε-dense in the set B, if for all $v \in B$ there is $u \in A$ such that $\|v - u\| < \varepsilon$.

Theorem 4.3.2. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$, $q_1 > 0$, $q_2 > 0$, such that $q_1 + q_2 < 1$ and:*

(i) there exist bounded "ball covering" sets M_A and M_B such that $M_A - M_B$ is εq_1-dense in $\varepsilon\overline{\boldsymbol{B}}$ and "correcting" sets U_A, U_B such that

$$A \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_B \subset B + tU_B$$

whenever $t \in [0, \delta]$;

(ii) there exist two bounded sets D_A and D_B such that $D_A - D_B$ is εq_2-dense in $U_A - U_B$ and they are "η-uniform" with $\eta := (1 - q_1 - q_2)/3$, i.e. for each $t \in [0, \delta]$

$$A \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tD_A \subset A + t\eta\overline{\boldsymbol{B}} \text{ and } B \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tD_B \subset B + t\eta\overline{\boldsymbol{B}}.$$

Then A and B are tangentially transversal at x_0.

Proof. From (ii) we see that $U_A - U_B$ is bounded, and hence each of the sets U_A and U_B is bounded as well. We set

$$N := \sup\{\|u\| \bigm| u \in U_A \cup U_B \cup D_A \cup D_B \cup M_A \cup M_B\} \text{ and } \bar{\delta} := \frac{\delta}{1 + 2N}.$$

Let $x^A \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap A$ and $x^B \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap B$, $x_A \neq x_B$ be arbitrary. Let us fix $t \in \left(0, \min\left\{\bar{\delta}, \dfrac{\|x^A - x^B\|}{\varepsilon}\right\}\right)$ and let us set

$$v := -\frac{x^A - x^B}{\|x^A - x^B\|}.$$

Then $\|\varepsilon v\| = \varepsilon$. According to (i), there exist $m_A \in M_A$ and $m_B \in M_B$, such that $\|\varepsilon v - (m_A - m_B)\| \leq \varepsilon q_1$. Since $0 < t < \bar{\delta} \leq \delta$ and

$$x^A \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap A \subset (x_0 + \delta\overline{\mathbf{B}}) \cap A,\ x^B \in (x_0 + \bar{\delta}\overline{\mathbf{B}}) \cap B \subset (x_0 + \delta\overline{\mathbf{B}}) \cap B,$$

there exist $u_A \in U_A$ and $u_B \in U_B$ such that

$$\widetilde{x}^A := x^A + t(m_A - u_A) \in A \text{ and } \widetilde{x}^B := x^B + t(m_B - u_B) \in B.$$

Because $u_A \in U_A$, $u_B \in U_B$ and $D_A - D_B$ is εq_2-dense in $U_A - U_B$, then $\|(d^A - d^B) - (u_A - u_B)\| \leq \varepsilon q_2$ for some $d^A \in D_A$ and $d^B \in D_B$.

We estimate

$$\|\widetilde{x}^A - x_0\| \leq \|x^A - x_0\| + t\|m_A - u_A\| \leq \bar{\delta} + \bar{\delta}(\|m_A\| + \|u_A\|\|) \leq \bar{\delta}(1 + 2N) = \delta.$$

Therefore there exists $w_A \in \eta\overline{\mathbf{B}}$ such that $\widetilde{x}^A + t(d^A - w_A) \in A$ and we obtain

$$\widetilde{x}^A + t(d^A - w_A) = x^A + t(m_A - u_A + d^A - w_A) \in A$$

and

$$\|m_A - u_A + d^A - w_A\| \leq \|m_A\| + \|u_A\| + \|d^A\| + \|w_A\| \leq 3N + \eta.$$

Analogously we obtain $w_B \in \eta\overline{\mathbf{B}}$ such that $x^B + t(m_B - u_B + d^B - w_B) \in B$ and $\|m_B - u_B + d^B - w_B\| \leq 3N + \eta$.

Hence

$$\|(x^A + t(m_A - u_A + d^A - w_A)) - (x^B + t(m_B - u_B + d^B - w_B))\|$$

$$= \left\| x^A - x^B - t\varepsilon \frac{x^A - x^B}{\|x^A - x^B\|} - t(\varepsilon v - (m_A - m_B)) - tw_A + tw_B + \right.$$

$$\left. + t((d^A - d^B) - (u_A - u_B)) \right\| \leq$$

$$\leq \|x^A - x^B\| - t\varepsilon + t(\varepsilon q_1 + \eta + \eta + \varepsilon q_2) = \|x^A - x^B\| - t\eta.$$

This verifies the tangential transversality of the sets A and B at the point x_0 with constants $3N + \eta > 0$, $\bar{\delta} > 0$ and $\eta > 0$.

□

4.4 Applications of the main result

Corollary 4.4.1. *Theorem 4.2.13 as a corollary of Theorem 4.3.1.*

Proof. Let D_A and D_B are uniform tangent sets to A and B respectively, at x_0 and for some $\rho > 0$ holds

$$\rho\bar{\mathbf{B}} \subset \overline{co}(D_A - D_B).$$

Taking into account the properties of uniform tangent set (Theorem 5.3.4), we may assume that $D_A - D_B$ is dense in $\rho\bar{\mathbf{B}}$. Let $M_A = U_A := D_A$ and $M_B = U_B := D_B$. Let $\delta > 0$ is arbitrary and $\varepsilon = \rho$. In this way we can apply Theorem 4.3.1, hence A and B are tangentially transversal. □

Lemma 4.4.2. *Let X and Y are Banach spaces and let $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be proper and lower-semicontinuous. Let $L : Y \to X$ be continuous linear operator. Denote*

$$S = \{(x, y) \in X \times Y \mid x = Ly\}.$$

Let $(\bar{x}, \bar{y}) \in S$ be such that there exist $\bar{\delta} > 0$ and $K > 0$, such that for all $y \in \bar{y} + \bar{\delta}\bar{\boldsymbol{B}}_Y$ and for all $x' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$, $x'' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$ holds

$$|f(x', y) - f(x'', y)| \leq K\|x' - x''\|.$$

Then there exists uniform tangent set D to epif at $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$, such that $D - S \cap B_{X\times Y} \times [-1, 0]$ has nonempty interior in $X \times Y \times \mathbb{R}$.

Proof. Define
$$D := \left\{ (v, \mathbf{0}, d) \;\middle|\; d \in (0, \bar{\delta}),\ \|v\| < \frac{d}{K} \right\}.$$
We will show that D is uniform tangent set to epi$f$ at $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$. Denote $C = \text{epi} f$. Fix arbitrary $\varepsilon > 0$. We should find $\delta > 0$, such that for all $(v, \mathbf{0}, d) \in D$, for all $(x, y, r) \in ((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})) + \delta B_{X\times Y\times\mathbb{R}}) \cap C$ and for all $t \in [0, \delta]$ holds
$$((x, y, r) + t\,((v, \mathbf{0}, d) + \varepsilon B_{X\times Y\times\mathbb{R}})) \cap C \neq \emptyset.$$
Let $\delta = \min\left\{\frac{\bar{\delta}}{2}, \frac{\bar{\delta}}{2M}\right\}$, where $M = \sup_{v\in D} \|v\|$ (finite, since D is bounded). Let $(x, y, r) \in ((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})) + \delta B_{X\times Y\times\mathbb{R}}) \cap C$. Then $\bar{x} + \delta\bar{\mathbf{B}}_X$, $y \in \bar{y} + \delta\bar{\mathbf{B}}_Y$ Pë
$$\|x + tv - \bar{x}\| \le \|x - \bar{x}\| + \frac{\bar{\delta}}{2M} M \le \bar{\delta},$$
hence $x + tv \in \bar{x} + \bar{\delta}\bar{\mathbf{B}}_X$. In this way
$$f(x + tv, y) \le f(x, y) + K\|tv\| \le r + td$$
(we utilized the Lipschitz condition on f with respect to the first variable, $\|v\| < d/K$ and $f(x, y) \le r$), hence $(x, y, r) + t(v, \mathbf{0}, d) \in C$.

We will show that thus constructed uniform tangent set D does the work for our claim. Since L is continuous, it has finite norm. Clearly $(\mathbf{0}, \mathbf{0}, -1/2) \in S \cap B_{X\times Y} \times [-1, 0]$. We know that D is open in $X \times \{\mathbf{0}\} \times \mathbb{R}$. Let $(x, \mathbf{0}, r) \in D$ and $\varepsilon_1 > 0$ be such that, $(x, \mathbf{0}, r) + \varepsilon_1 B_{X\times\{\mathbf{0}\}\times\mathbb{R}} \subseteq D$. We claim that $z := (x, \mathbf{0}, r) - (\mathbf{0}, \mathbf{0}, -1/2) = (x, \mathbf{0}, r + 1/2)$ lies in the interior of $D - S \cap B_{X\times Y} \times [-1, 0]$, i.e. there exists $\varepsilon > 0$, such that $z + \varepsilon B_{X\times Y\times\mathbb{R}} \subseteq D - S \cap B_{X\times Y} \times [-1, 0]$. This would finish the proof. Let $\varepsilon = \min\left\{\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2\|L\|}, \frac{1}{\|L\|}, 1\right\}$ and take arbitrary
$$z' = (x', y', r' + 1/2) \in z + \varepsilon B_{X\times Y\times\mathbb{R}}.$$
Thus $\|x' - x\| < \varepsilon_1/2$ and $\|y'\| < \frac{\varepsilon_1}{2\|L\|}$, hence
$$\|x' + L(-y') - x\| \le \|x' - x\| + \|Ly'\| < \frac{\varepsilon_1}{2} + \|L\|\frac{\varepsilon_1}{2\|L\|} = \varepsilon_1.$$
Moreover $|r' - r| = |r' + 1/2 - (r + 1/2)| < \varepsilon_1$.
In this way $z_1 := (x' + L(-y'), \mathbf{0}, r') \in D$.
On the other hand, since $\|y'\| < \min\left\{1, \frac{1}{\|L\|}\right\}$, we have $\|Ly'\| < 1$. Thus $(L(-y'), -y') \in S \cap B_{X\times Y}$. Therefore $z_2 := (L(-y'), -y', -1/2) \in S \cap B_{X\times Y} \times [-1, 0]$. It remains to observe that $z' = z_1 - z_2$. $\square$

Theorem 4.4.3. *Let X and Y be Banach space and let $f : X \times Y \to \mathbb{R}$ be proper lower-semicontinuous function. Let $L : Y \to X$ be continuous linear operator and*

$$S = \{(Ly, y) \mid y \in Y\}.$$

Let $(\bar{x}, \bar{y}) \in S$ be such that there exists $\bar{\delta} > 0$ and $K > 0$, such that for all $y \in \bar{y} + \bar{\delta}\bar{\boldsymbol{B}}_Y$ and for all $x' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$, $x'' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$ holds

$$|f(x', y) - f(x'', y)| \leq K\|x' - x''\|.$$

Moreover, let $\widehat{T}_{epi\ f}((\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))) - \widehat{T}_{S\times(-\infty, f(\bar{x},\bar{y})]}((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})))$ be dense in $X \times Y \times \mathbb{R}$ and there exists an uniform tangent set D_g which generates $\widehat{T}_{epi\ f}((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})))$. Then epi f and $S \times (-\infty, f(\bar{x}, \bar{y})]$ are tangentially transversal.

Proof. For brevity denote $A :=$ epi f and $B = S \times (-\infty, f(\bar{x}, \bar{y})]$ and $z = (\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$. According to Lemma 4.4.2, there exists uniform tangent set D to A at the point z, such that $D - S \cap B_{X\times Y} \times [-1, 0]$ has nonempty interior. According to Theorem 4.2.4 (ii), $D_A := D \cup D_g$ is an uniform tangent set. On the other hand, since S is a vector space, then $D_B := S \cap B_{X\times Y} \times [-1, 0]$ is an uniform tangent set to B at z. We know that $\widehat{T}_A(z) - \widehat{T}_B(z)$ is dense in $X \times Y \times \mathbb{R}$. In this way, the assumptions of Theorem 4.2.14. We obtain that the sets A and B are tangentially transversal. □

Remark 4.4.4. *If in the above theorem X and Y are separable Banach spaces, then the existence of D_g follows directly from Proposition 4.2.9.*

Below we formulate an abstract (infinite-dimensional) version of the well-known Aubin condition from [15] for the basic problem of the calculus of variations:

Definition 4.4.5. *Let X and Y be Banach spaces and $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which has finite value at $(\overline{x}, \overline{y}) \in X \times Y$. It is said that f satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ iff there exist positive reals $\overline{\delta} > 0$ and $K > 0$ such that for every $t \in [0, \overline{\delta}]$ the following inclusion holds true:*

$$\operatorname{epi} f \cap \left((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X\times Y\times R}\right) + t\left(\overline{\mathbf{B}}_X, \mathbf{0}, 0\right) \subset$$

$$\subset \operatorname{epi} f + t\left(\mathbf{0}, K \cdot \overline{\mathbf{B}}_Y, K[-1, 1]\right).$$

The next theorem is the main motivation of our research:

Theorem 4.4.6. *Let X and Y be Banach spaces and $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$. Let $L : Y \longrightarrow X$ be a compact linear operator and $S := \{(Ly, y) : y \in Y\}$. We assume that*

$$\widehat{T}_{\operatorname{epi} f}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) - S \times (-\infty, 0]$$

is dense in $X \times Y \times R$. Then $\operatorname{epi} f$ and $S \times (-\infty, f(\overline{x}, \overline{y})]$ are tangentially transversal at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$.

Proof. We are going to use our main result for the sets $A := \operatorname{epi} f$ and $B := S \times (-\infty, f(\overline{x}, \overline{y})]$. The ball covering sets are

$$M_A := \left(\overline{\mathbf{B}}_X, \mathbf{0}, [-1, 1]\right) \text{ and } M_B := \{(Ly, y, 0) : \|y\| \le 1\},$$

i.e. $M_A - M_B \supset \varepsilon \overline{\mathbf{B}}_{X \times Y \times R}$, where $\varepsilon = \dfrac{1}{1 + \|L\|}$. Indeed, let $(x, y, r) \in \varepsilon \overline{\mathbf{B}}_{X \times Y \times R}$ be arbitrary. Then

$$(x, y, r) = (x - Ly, \mathbf{0}, r) - (-Ly, -y, 0)$$

and $(L(-y), -y, 0) \in M_B$, because $\|y\| \le \varepsilon \le 1$. Moreover, $(x - Ly, \mathbf{0}, r) \in M_A$ because $|r| \le \varepsilon \le 1$ and

$$\|x - Ly\| \le \|x\| + \|L\| \cdot \|y\| \le (1 + \|L\|)\varepsilon = 1.$$

The correcting sets are

$$U_A := \left(\mathbf{0}, K \cdot \overline{\mathbf{B}}_Y, (1 + K)[-1, 1]\right) \text{ and } U_B := \{(\mathbf{0}, \mathbf{0}, 0)\}.$$

The Aubin condition implies

$$A \cap \left((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X \times Y \times R}\right) + tM_A \subset A + tU_A \text{ for every } t \in [0, \overline{\delta}].$$

The fact that S is a vector space implies that

$$B \cap \left((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X \times Y \times R}\right) + tM_B \subset B + tU_B \text{ for every } t \in [0, \overline{\delta}].$$

Now we have to cover $U_A - U_B = U_A$ (with some accuracy $\eta > 0$) by the difference of two uniform tangent sets to A and B, respectively. To this end, we fix an arbitrary $\eta > 0$ (sufficiently small) and we consider the set

$$C := \left(K \cdot L\left(\overline{\mathbf{B}}_Y\right), \mathbf{0}, (1 + K)[-1, 1]\right).$$

The fact that L is a compact linear operator implies that C is totally bounded. Now the density of $\widehat{T}_{\mathrm{epi}\, f}(\overline{x},\overline{y},f(\overline{x},\overline{y})) - S \times (-\infty, 0]$ implies the existence of a finite η-net for C consisting of elements of this dense difference. Therefore there exist finite sets $F \subset \widehat{T}_{\mathrm{epi}\, f}(\overline{x},\overline{y},f(\overline{x},\overline{y}))$ and $G \subset S \times (-\infty, 0]$ such that

$$(F - G) + \eta\overline{\mathbf{B}}_{X\times Y\times R} \supset C \ .$$

Let $(\mathbf{0}, y, r)$ be an arbitrary element of U_A. Then

$$(\mathbf{0}, y, r) = (Ly, y, 0) + (-Ly, \mathbf{0}, r) = (-Ly, \mathbf{0}, r) - (-Ly, -y, 0) \ .$$

Therefore

$$U_A \subset C - \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right) \ .$$

Using above inclusion and the η-net for C we obtain that

$$U_A \subset (F - G) + \eta\overline{\mathbf{B}}_{X\times Y\times R} - \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right) =$$

$$= F - \left(G + \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right)\right) + \eta\overline{\mathbf{B}}_{X\times Y\times R} \ .$$

It remains to notice that $D_A := F$ is a uniform tangent set to A at $(\overline{x},\overline{y},f(\overline{x},\overline{y}))$ (it is a finite subset of the respective Clarke tangent cone) and $D_B := G + \left(S \cap \left(K \cdot \max\{1, \|L\|\} \cdot \overline{\mathbf{B}}_{X\times Y}\right), 0\right)$ is a uniform tangent set to B at $(\overline{x},\overline{y},f(\overline{x},\overline{y}))$. Therefore, the assumptions of Theorem 4.3.2 hold true, and hence the sets A and B are tangentially transversal. □

Now we turn to an extension of the notion of compactly epi-Lipschitz (massive) sets (cf. Definition 4.2.10). The main difference with respect to the classical one is that we speak about "massiveness" of two sets as a pair.

Definition 4.4.7. *Let A and B be closed subsets of the Banach space X and $x_0 \in A \cap B$. We say that A and B are jointly massive at x_0 if there exist $\varepsilon > 0$, $\bar{\delta} > 0$, bounded sets $M_A \subset X$, $M_B \subset X$ and a compact set $K \subset X$ such that:*
(i) $\varepsilon\overline{\boldsymbol{B}}_X \subset \overline{M_A - M_B}$;
(ii) $A \cap (x_0 + \overline{\delta\boldsymbol{B}}) + tM_A \subset A + tK$ and $B \cap (x_0 + \overline{\delta\boldsymbol{B}}) + tM_B \subset B + tK$ whenever $t \in [0, \bar{\delta}]$.

Clearly if the sets A and B are closed, $x_0 \in A \cap B$ and A is massive at x_0, then A and B are jointly massive at x_0. Indeed, let $A \cap (x_0 + \overline{\delta\mathbf{B}}) + t\varepsilon\overline{\mathbf{B}}_X \subset A + tK$ for some compact set K. Setting $M_A := \varepsilon\overline{\mathbf{B}}_X$, $M_B := \{\mathbf{0}\}$ and $\tilde{K} := K \cup \{\mathbf{0}\}$, the above written definition holds true because $\tilde{K}$ is compact as well.

The next assertion is a direct generalization of Theorem 4.3 of [9].

Proposition 4.4.8. *Let A and B be jointly massive at x_0 and $\widehat{T}_A(x_0) - \widehat{T}_B(x_0)$ be dense in X. Then A and B are tangentially transversal at x_0.*

Proof. Let ε and $\bar{\delta}$ are the constants from *(i)* in Definition 4.4.7. We put $U_A = U_B = K$. From (i) we see that for each $q_1 > 0$ the assumption (i) of Theorem 4.3.2 holds true. We fix arbitrary $q_1 > 0$ and $q_2 > 0$ with $q_1+q_2 < 1$. Because $K - K$ is compact, there is a finite $\varepsilon q_2/2$-net for $K - K$, i.e. a set $F = \{z_1, \ldots, z_n\}$, such that for any $z \in K - K$ there is $i \in \{1, \ldots, n\}$ with $\|z_i - z\| \leq \varepsilon q_2/2$. Since $\widehat{T}_A(x_0) - \widehat{T}_B(x_0)$ is dense in X, it is dense in F. Thus for any $i \in \{1, \ldots, n\}$, one finds $d_i^A \in \widehat{T}_A(x_0)$ and $d_i^B \in \widehat{T}_B(x_0)$, such that $\|d_i^A - d_i^B - z_i\| \leq \varepsilon q_2/2$. So for any $z \in K - K$, there is $i \in \{1, \ldots, n\}$, such that

$$\|d_i^A - d_i^B - z\| \leq \|d_i^A - d_i^B - z_i\| + \|z_i - z\| < \varepsilon q_2/2 + \varepsilon q_2/2 = \varepsilon q_2.$$

Therefore, if $D_A := \{d_i^A\}_{i=1}^n$ and $D_B := \{d_i^B\}_{i=1}^n$, then $D_A - D_B$ is εq_2-dense in $U_A - U_B$. As the set D_A (D_B, respectively) is a finite subset of $\widehat{T}_A(x_0)$ (of $\widehat{T}_B(x_0)$, respectively), it is a uniform tangent set to A (to B, respectively) at x_0. In particular, for $\eta = \varepsilon(1 - q_1 - q_2)/3 > 0$ there exits $\widetilde{\delta}$ such that for each $t \in [0, \widetilde{\delta}]$

$$A \cap (x_0 + \widetilde{\delta}\overline{\mathbf{B}}) + tD_A \subset A + t\eta\overline{\mathbf{B}} \text{ and } B \cap (x_0 + \widetilde{\delta}\overline{\mathbf{B}}) + tD_B \subset B + t\eta\overline{\mathbf{B}}.$$

Now (i) and (ii) from Theorem 4.3.2 are fulfilled with ε and $\delta = \min\{\bar{\delta}, \widetilde{\delta}\}$. □

The obtained sufficient conditions could be applied with Theorem 4.1.1, to obtain Lagrange multipliers in different situations, as summarized in the following

Theorem 4.4.9. *Let X and Y be Banach spaces. We consider the optimization problem*

$$f(x, y) \to \min \quad \textit{subject to} \quad (x, y) \in S \ ,$$

where $f : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, proper and S is a closed subset of $X \times Y$. Let one of the following three conditions be satisfied:

1. *X and Y are separable, $S = \{(Ly, y) \mid y \in Y\}$, where L is a continuous linear operator and there exist $\bar{\delta} > 0$ and $K > 0$, such that for all $y \in \bar{y} + \bar{\delta}\bar{\boldsymbol{B}}_Y$ and all $x' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$, $x'' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$ holds $|f(x', y) - f(x'', y)| \leq K\|x' - x''\|$*

2. *epi f and $S \times (-\infty, f(\bar{x}, \bar{y})]$ are jointly massive at $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$.*

3. $S = \{(Ly, y) \mid y \in Y\}$, *where* L *is a compact linear operator and* f *satisfies the Aubin condition at* $(\bar{x}, \bar{y})$.

If 1. or 3. holds, then there exists a triple $(\xi, \eta, \zeta) \in X^* \times Y^* \times \mathbb{R}$ *such that*

(i) $(\xi, \eta, \zeta) \neq (\mathbf{0}, \mathbf{0}, 0)$;

(ii) $\zeta \in \{0, 1\}$;

(iii) $\langle \xi, Ly \rangle + \langle \eta, y \rangle = 0$ *for every* $y \in Y$;

(iv) $\langle \xi, u \rangle + \langle \eta, v \rangle + \zeta s \geq 0$ *for every* $(u, v, s) \in \widehat{T}_{epif}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$.

If 2. holds, then all of the above also hold for some triple $(\xi, \eta, \zeta) \in X^* \times Y^* \times \mathbb{R}$, *except for (iii) which is replaced by* $\langle \xi, u \rangle + \langle \eta, v \rangle \leq 0$ *for every* $(u, v) \in \widehat{T}_S(\overline{x}, \overline{y})$.

Proof. Denote for brevity $A = \text{epi } f$, $B = S \times (-\infty, f(\bar{x}, \bar{y})]$ and $z = (\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$. For any of the three additional conditions apply Theorem 4.1.1 and some of the obtained sufficient conditions for tangential transversality.
Assume that alternative b) of Theorem 4.1.1 holds. Then $\hat{T}_A(z) - \hat{T}_B(z)$ is dense in $X \times Y \times \mathbb{R}$.
If 1. holds, we arrive at the setting of Theorem 4.4.3, hence A and B are tangentially transversal.
If 2. holds, we arrive at the setting of Corollary 4.4.8, hence A and B are tangentially transversal.
If 3. holds, we arrive at the setting of Theorem 4.4.6, hence A and B are tangentially transversal.
In any of the cases we obtain that A and B are tangentially transversal. This contradicts the conclusion of alternative b). Thus alternative a) remains, and consequently we have Lagrange multipliers. □

Chapter 5

On continuity of optimal value map

5.1 Preliminaries

We will assume that S_{val} assumes only finite values. Throughout the chapter, all the topological spaces involved will be metric spaces with the property that every open ball is connected (clearly this is the case for normed vector spaces). We will need the following

Lemma 5.1.1. *Let X be a metric space, such that every open ball in it is connected. Let A be a closed subset of X and $x \in X \setminus A$. Let $c > 0$ be such that $d(x, A) < c$. Then there exists $y \in \partial A$ such that $d(x, y) < c$.*

Proof. Consider $\mathbf{B}_c(x)$. If $\mathbf{B}_c(x) \cap \partial A = \varnothing$, then $X \setminus A \cap \mathbf{B}_c(x)$ and $A \cap \mathbf{B}_c(x)$ are two nonempty nonintersecting open sets, whose union is $\mathbf{B}_c(x)$. This is a contradiction, since $\mathbf{B}_c(x)$ is connected. □

Let (X, ρ) be a metric space. We denote $\mathbf{B}_r(x) = \{z \mid \rho(x, z) < r\}$ and $\bar{\mathbf{B}}_r(x) = \{z \mid \rho(x, z) \leq r\}$. For a subset A of X and $\varepsilon > 0$ we define

$$A_\varepsilon = \bigcup_{x \in A} \mathbf{B}_\varepsilon(x) = \{z \in X \mid \exists x \in A,\ \rho(z, x) < \varepsilon\}.$$

We will consider set-valued maps with closed values only.

We introduce the two continuity (semi-continuity) notions considered in this chapter.

Definition 5.1.2. *Two notions of upper semicontinuity.*

- *Topological upper semicontinuity (t-usc).* $F : X \rightrightarrows Y$ *is t-usc at* $\bar{x} \in X$ *if for any open* U *containing* $F(\bar{x})$*, there exists an open neighbourhood* V *of* $\bar{x}$ *such that* $F(x) \subseteq U$ *for all* $x \in V$*.*

- *Pompeiu-Hausdorff upper semicontinuity (h-usc).* $F : X \rightrightarrows Y$ *is h-usc at* $\bar{x} \in X$ *if for any* $\varepsilon > 0$*, there exists an open neighbourhood* V *of* $\bar{x}$ *such that* $F(x) \subseteq F(\bar{x})_\varepsilon$ *for all* $x \in V$*.*

Clearly t-usc implies h-usc. However, the reverse implication might not hold, since in general there are open sets U containing $F(\bar{x})$ which do not contain a set of the form $F(\bar{x})_\varepsilon$. However, both notions coincide when $F(\bar{x})$ is compact as observed in [2], [6], [27]. Now we state the corresponding definitions of lower semicontinuity.

Definition 5.1.3. *Two notions of lower semicontinuity.*

- *Topological lower semicontinuity (t-lsc).* $F : X \rightrightarrows Y$ *is t-lsc at* $\bar{x} \in X$ *if for any open* U *such that* $U \cap F(\bar{x}) \neq \varnothing$*, there exists an open neighbourhood* V *of* $\bar{x}$ *such that* $U \cap F(x) \neq \varnothing$ *for all* $x \in V$*.*

- *Pompeiu-Hausdorff lower semicontinuity (h-lsc).* $F : X \rightrightarrows Y$ *is h-lsc at* $\bar{x} \in X$ *if for any* $\varepsilon > 0$*, there exists an open neighbourhood* V *of* $\bar{x}$ *such that* $F(\bar{x}) \subseteq F(x)_\varepsilon$ *for all* $x \in V$*.*

In general, h-lsc implies t-lsc, and they are equivalent when the value of the reference point is compact. A mapping is t-continuous (h-continuous) if it is both t-usc and t-lsc (h-usc and h-lsc).

There are a number of concepts of continuity of set-valued mappings that are usually tied with corresponding concepts of convergence of sequences of sets; among them the popular Kuratowski-Painleve continuity ([27]), based on the notion of set convergence introduced by Painleve and elaborated by Kuratowski. A good reference for convergence of sets is the survey by Sonntag and Zalinescu [59].

5.2 The counterexample and a remedy

We begin with a counterexample to Theorem 1.0.4 if continuity is in the Pompeiu-Hausdorff sense.

Counterexample 5.2.1. *Let* $X = \mathbb{R}$*,* $Y = \mathbb{R}^2$*. Consider*

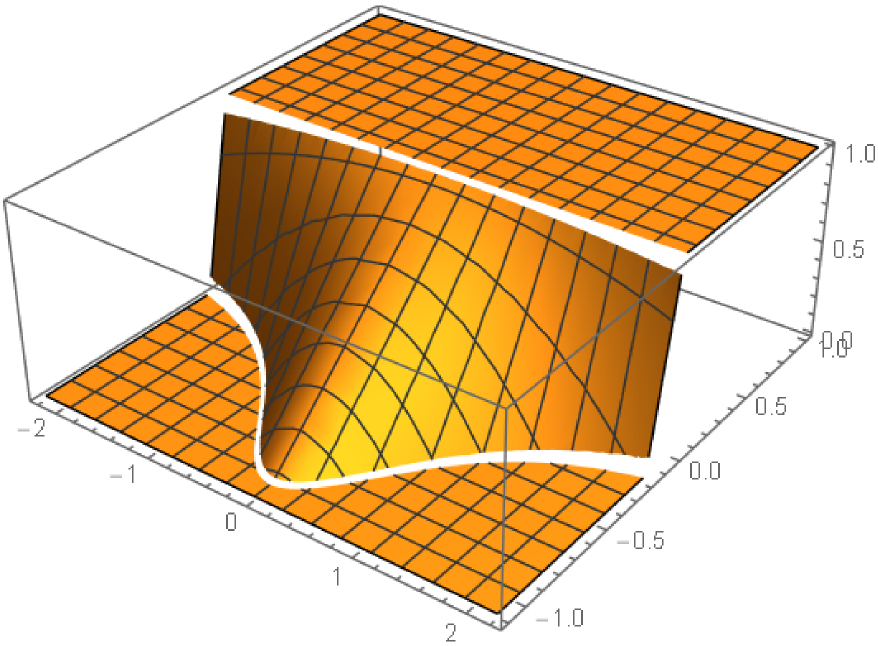
$$D(p) = \{(x, y) \in \mathbb{R}^2 \mid y \geq -|p|\}$$

and

$$g(x,y) = \begin{cases} 0, & y \le -\frac{1}{1+x^2} \\ (1+x^2)y+1, & -\frac{1}{1+x^2} < y < 0 \\ 1, & y \ge 0 \end{cases}$$

Then D is continuous at $\bar{p} = 0$, g is continuous on all of $\mathbb{R}^2$ (in the sense of Pompeiu-Hausdorff), $S_{val}(0) = 1$, while for $p \neq 0$, $S_{val}(p) = 0$.

Here is a picture of the graph of the function



It is evident, that the conclusion fails, because the function g is arbitrary steep around $\partial D(0)$. To circumvent this possibility, we introduce a "relaxed" uniform continuity around $\partial D(\bar{p})$.

According to the authors knowledge the following definition is new.

Definition 5.2.2. *Let $F : X \rightrightarrows Y$ be a set-valued map and $f : Y \to \mathbb{R}$. We say that the couple (F, f) satisfies the relaxed uniform continuity assumption $(\boldsymbol{RUCA})$ at $\bar{x}$ if*

$(\boldsymbol{RUCA})$ | *for any $\varepsilon > 0$ there is $\delta > 0$, such that for all x with $\rho(x, \bar{x}) < \delta$, if $y \in \partial F(\bar{x})$ and $z \in F(x) \setminus F(\bar{x})$ satisfy $\rho(y, z) < \delta$, then $|f(y) - f(z)| < \varepsilon$*

Theorem 5.2.3. *Assume that for some $\bar{p} \in X$, D is h-continuous at $\bar{p}$ and g is continuous on $D(\bar{p})$. Assume moreover that the couple (D, g) satisfies (**RUCA**) at $\bar{p}$. Then S_{val} is continuous at $\bar{p}$.*

Proof. Upper semicontinuity. Let $p_k \to \bar{p}$ and $\varepsilon > 0$. Take $x \in D(\bar{p})$ such that $g(x) < S_{\text{val}}(\bar{p}) + \varepsilon/2$ (assuming $S_{\text{val}}(\bar{p}) > -\infty$). Since g is continuous at x, then there exists $\delta > 0$ such that for all y with $\rho(y, x) < \delta$ one has $|g(x) - g(y)| < \varepsilon/2$. For large enough k, $d(x, D(p_k)) < \delta$ due to h-lsc of D, so that for all such k, there is some $x_k \in D(p_k)$, such that $\rho(x, x_k) < \delta$. And then

$$S_{\text{val}}(p_k) \leq g(x_k) < g(x) + \varepsilon/2 < S_{\text{val}}(\bar{p}) + \varepsilon.$$

Lower semicontinuity. Assume that $\liminf_{p \to \bar{p}} S_{\text{val}}(p) < S_{\text{val}}(\bar{p})$. Thus there exists a sequence $p_k \to \bar{p}$ and $\varepsilon > 0$, such that $S_{\text{val}}(p_k) < S_{\text{val}}(\bar{p}) - 2\varepsilon$. For each k, consider $x_k \in D(p_k)$ such that $g(x_k) \leq S_{\text{val}}(p_k) + \varepsilon$. If for any of these k, $x_k \in D(\bar{p})$, we readily have a contradiction, since then

$$g(x_k) \leq S_{\text{val}}(p_k) + \varepsilon < S_{\text{val}}(\bar{p}) - \varepsilon < S_{\text{val}}(\bar{p}) = \inf\{g(x) \mid x \in D(\bar{p})\}.$$

If $x_k \notin D(\bar{p})$ for every k, then h-usc of D at $\bar{p}$ and Lemma 5.1.1 imply that for each k there exists $\tilde{x}_k \in \partial D(\bar{p})$, such that $\lim_{k \to \infty} \rho(x_k, \tilde{x}_k) = 0$. Applying (**RUCA**) with ε we obtain that there exists $\delta > 0$ such that for all $x \in \partial D(\bar{p})$ and $y \in D(p_k) \setminus D(\bar{p})$ for large enough k, with $\rho(x, y) < \delta$ one has $|g(x) - g(y)| < \varepsilon$. Thus, for large enough k, $|g(x_k) - g(\tilde{x}_k)| < \varepsilon$. This, combined with the choice of ε, gives us

$$g(\tilde{x}_k) < g(x_k) + \varepsilon \leq S_{\text{val}}(p_k) + 2\varepsilon < S_{\text{val}}(\bar{p}),$$

which contradicts the definition of S_{val}.

□

It is easy to observe that the pair (D, g) from Counterexample 5.2.1 does not satisfy (**RUCA**). From this theorem we obtain the following corollary, which could also be derived as a special case of the theorem of Berge, since, as noted, when $D(\bar{p})$ is compact, h-continuity is equivalent to t-continuity.

Corollary 5.2.4. *Let X be a complete metric space. Assume that for some $\bar{p} \in X$, D is h-continuous at $\bar{p}$, g is continuous on $D(\bar{p})$ and $D(\bar{p})$ is totally bounded (bounded if X is finite dimensional normed vector space). Then S_{val} is continuous at $\bar{p}$.*

Proof. When $D(\bar{p})$ is totally bounded, it is compact (since closed by assumption). Then (**RUCA**) is satisfied automatically.

□

5.3 Involving the measure of noncompactness

We turn our attention next to the case of t-continuity of the feasibility mapping. As noted earlier, the following theorem follows from a result of Berdyshev [6] (and is essentially equivalent to it in the case of metric spaces).

Theorem 5.3.1. *Assume that for some $\bar{p} \in X$, D is t-continuous at $\bar{p}$ and g is continuous on $D(\bar{p})$. Then S_{val} is continuous at $\bar{p}$.*

We will present a proof of Theorem 5.3.1 at the end of the chapter.

Another result following from Berdyshev's work is the following

Theorem 5.3.2. *Assume that for some $\bar{p} \in X$, D is h-continuous at $\bar{p}$ and g is uniformly continuous on $D(\bar{p})$. Then S_{val} is continuous at $\bar{p}$.*

Here is a proof along the lines of our results so far.

Proof ot Theorem 5.3.2. Clearly, uniform continuity of g implies (**RUCA**) for the pair (D, g). Thus we may apply Theorem 5.2.3. □

Recall that the ball measure of noncompactness is defined as follows

$$\alpha(A) = \inf\{r > 0 \mid \text{there exist finitely many balls of radius } r \text{ which cover } A\}.$$

It is evident that A is totally bounded if and only if $\alpha(A) = 0$.

The following lemma and the equivalence of (i) and (ii) in Proposition 5.3.4 are close to some observations in [24], [28], [53].

Lemma 5.3.3. *Let A and $\{A_k\}_{k\geq 1}$ be closed subsets of X. Assume that for any open set U which contains A, there exists $N \in \mathbb{N}$ such that $A_k \subset U$ for any $k > N$. Then*

$$\lim_{n\to\infty} \alpha\left(\bigcup_{k=n}^{\infty} A_k \setminus A\right) = 0. \tag{5.1}$$

In particular, any sequence $\{a_n\}_{n\geq 1}$ with $a_n \in A_n \setminus A$, has a convergent subsequence.

Proof. For brevity denote $B_n = \bigcup_{k=n}^{\infty} A_k \setminus A$. Assume that (5.1) is violated. Clearly $\alpha(B_n)$ is decreasing, so there exists $\varepsilon > 0$ such that for all n, $\alpha(B_n) > \varepsilon$. Now we construct inductively a sequence $\{x_n\}_{n\geq 1}$ with $x_n \in B_n$ and $\rho(x_n, x_m) > \varepsilon$ for $n \neq m$. Let $x_1 \in B_1$. Then there exists $x_2 \in B_2$ with $\rho(x_2, x_1) > \varepsilon$. Indeed, if this was not true, then $\bar{\mathbf{B}}_\varepsilon(x_1)$ would be a finite ε cover of B_2 - but this is impossible, since $\alpha(B_2) > \varepsilon$. Assume that the first n elements of the sequence are constructed: that is we are given $x_1, x_2, \ldots, x_n$, with $x_i \in B_i$ and $\rho(x_i, x_j) > \varepsilon$. Then there is $x_{n+1} \in B_{n+1}$ with $\rho(x_{n+1}, x_i) > \varepsilon$ for $i = 1, 2, \ldots, n$. If this was not the case, then $\bigcup_{i=1}^{n} \bar{\mathbf{B}}_\varepsilon(x_i)$ would be a finite ε cover of B_{n+1} - contradiction as before. For $n \in \mathbb{N}$, $x_n \notin A$. Consider $V := \bigcup_{n=1}^{\infty} \{x_n\}$. Since $\rho(x_n, x_m) > \varepsilon$ for $n \neq m$, we see that V is closed. Hence $U := X \setminus V$ is open, contains A and for any N, there exists $n > N$ with $A_n \not\subseteq U$. This contradicts the assumed property for the sequence A_n.

For the second part, we see that $a_n \in B_d$ for $n \geq d$. Since the measure of noncompactness does not depend on finitely many elements of a set, we have

$$0 \leq \alpha(\{y_n\}_{n\geq 1}) = \alpha(\{y_n\}_{n\geq d}) \leq \alpha(B_d) \xrightarrow{d\to\infty} 0.$$

Hence $\alpha(\{y_n\}_{n\geq 1}) = 0$.

□

Clearly (**RUCA**) for (F, f) at $\bar{x}$ is a property depending on both the set-valued map F and the real-valued function f. However, in some cases, strong properties of only one of the objects ensures (**RUCA**) independently of the other object. For example, if the function f is uniformly continuous on the whole of Y, (**RUCA**) is satisfied independently of the properties of the set-valued map F - i.e. for any map F. On the other hand, as in Corollary 5.2.4, if F is h-usc at $\bar{x}$ and $F(\bar{x})$ is totally bounded, then (**RUCA**) is satisfied for any function f which is continuous on $F(\bar{x})$. The following Proposition clarifies when such a situation is present. It shows that if F is h-usc at $\bar{x}$, then (**RUCA**) for (F, f) at $\bar{x}$ holds for any function f continuous on $F(\bar{x})$ if and only if F is t-usc.

Proposition 5.3.4. *Let $F : X \rightrightarrows Y$ and $\bar{x} \in X$. The following are equivalent*

(i) F is t-usc at $\bar{x}$;

(ii) F is h-usc at $\bar{x}$ and for every $\varepsilon > 0$ there exists an open neighbourhood V of $\bar{x}$ such that

$$\alpha\left(\bigcup_{x\in V} F(x) \setminus F(\bar{x})\right) < \varepsilon;$$

*(iii) F is h-usc at $\bar{x} \in X$ and for any function $f : Y \to \mathbb{R}$ which is continuous on $F(\bar{x})$, the couple (F, f) satisfies (**RUCA**) at $\bar{x}$.*

Proof. $(ii) \Rightarrow (i)$. Assume on the contrary, that F is not t-usc at $\bar{x}$. This means that there exists an open set U which contains $F(\bar{x})$ such that for any $n \in \mathbb{N}$ there exists $x_n \in X$ with $\rho(x_n, \bar{x}) < \dfrac{1}{n}$ and $y_n \in F(x_n)$, such that $y_n \notin U$. Consider $A = F(\bar{x})$ and $A_n := F(x_n)$. Then, according to the second part of (ii),

$$\lim_{n\to\infty} \alpha \left(\bigcup_{k=n}^{\infty} A_k \setminus A \right) = 0,$$

and, as in the proof of Lemma 5.3.3, since $y_n \in A_n \setminus A$, we may extract a convergent subsequence (without relabelling). Then there exists $\hat{y}$, such that $y_n \to \hat{y}$. Since $y_n \in F(x_n)$ where $x_n \to \bar{x}$, F is h-usc at $\bar{x}$ and $F(\bar{x})$ is closed, we have $\hat{y} \in F(\bar{x})$. Thus $y_n \to \hat{y}$, $y_n \notin U$ and $\hat{y} \in U$. But this is impossible, since U is open.

$(i) \Rightarrow (iii)$. Clearly F is h-usc and assume on the contrary, that for some function $f : Y \to \mathbb{R}$ which is continuous on $F(\bar{x})$, (**RUCA**) is not satisfied for (F, f). This means that there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exists x_n with $\rho(x_n, \bar{x}) < \dfrac{1}{n}$, $y_n \in \partial F(\bar{x})$, $z_n \in F(x_n) \setminus F(\bar{x})$, such that $\rho(y_n, z_n) < \dfrac{1}{n}$ and $|f(y_n) - f(z_n)| \geq \varepsilon$. Consider $A = F(\bar{x})$ and $A_n := F(x_n)$. Since F is t-usc, the conditions of Lemma 5.3.3 are satisfied and since $z_n \in A_n \setminus A$, we may extract convergent subsequence (without relabelling). Since $\rho(y_n, z_n) < \dfrac{1}{n}$, the sequence $\{y_n\}_{n\geq 1}$ is also convergent. Since $\partial F(\bar{x})$ is closed, there exists $\hat{y} \in \partial F(\bar{x})$ such that $y_n \to \hat{y}$. As $\rho(y_n, z_n) < \dfrac{1}{n}$, $z_n \to \hat{y}$ as well. Since f is continuous at $\hat{y}$, there is $\delta > 0$ such that for all $v \in \bar{\mathbf{B}}_\delta(\hat{y})$ one has $|f(\hat{y}) - f(v)| < \dfrac{\varepsilon}{2}$. Thus, for large enough n

$$\varepsilon \leq |f(y_n) - f(z_n)| \leq |f(\hat{y}) - f(y_n)| + |f(\hat{y}) - f(z_n)| < \varepsilon,$$

contradiction.

$(iii) \Rightarrow (ii)$. Assume that (ii) does not hold. Since F is h-usc, there exists $\varepsilon > 0$ such that

$$\alpha \left(\bigcup_{x \in \mathbf{B}_{1/n}(\bar{x})} F(x) \setminus F(\bar{x}) \right) \geq \varepsilon.$$

for any n. We set $A = F(\bar{x})$ and $A_n := \bigcup_{x \in \mathbf{B}_{1/n}(\bar{x})} F(x)$. Then, as in the proof of Lemma 5.3.3, we obtain a sequence $\{z_n\}_{n\geq 1}$ with $z_n \in A_n \setminus A$ and

$\rho(z_n, z_m) > \varepsilon$ for $n \neq m$. Thus $V := \bigcup_{n=1}^{\infty}\{z_n\}$ is a closed set, disjoint from A (which is also closed). According to Urysohn's lemma, there exists a continuous function $g : Y \to \mathbb{R}$ such that $g_{|A} = 1$, $g_{|V} = 0$. Now we apply (**RUCA**) for the couple (F, g) at $\bar{x}$ with $\varepsilon = 1/2$. Then there exists $\delta > 0$, such that for all x with $\rho(x, \bar{x}) < \delta$, $y \in \partial F(\bar{x})$ and $z \in F(x) \setminus F(\bar{x})$, such that $\rho(y, z) < \delta$, holds $|g(y) - g(z)| < 1/2$. By the construction of z_n, we can find $x_n \in \mathbf{B}_{1/n}(\bar{x})$ such that $z_n \in F(x_n)$. As F is h-usc at $\bar{x}$, $d(z_n, F(\bar{x})) \to 0$. Thus for large enough n, $d(z_n, F(\bar{x})) < \delta$. Then, according to Lemma 5.1.1, there exists $y_n \in \partial F(\bar{x})$ with $\rho(y_n, z_n) < \delta$. It remains to take n such that $\frac{1}{n} < \delta$ (so that $\rho(x_n, \bar{x}) < \delta$). Then the conditions in (**RUCA**) are satisfied for $x = x_n$, $y = y_n$, $z = z_n$. However $g(y_n) - g(z_n) = 1 - 0 > 1/2$ - contradiction.

□

If Y is finite dimensional normed vector space, then there exists an open neighbourhood V of $\bar{x}$ such that

$$\alpha\left(\bigcup_{x \in V} F(x) \setminus F(\bar{x})\right) = 0.$$

This is due to the fact, that in finite-dimensional spaces, the measure of noncompactness attains only two values - 0 and $+\infty$. However, in infinite dimensions, we cannot hope for a result of the form "there exists an open neighbourhood V of $\bar{x}$ such that $\alpha(F(x) \setminus F(\bar{x})) = 0$ for all $x \in V$" as shown by the following

Example 5.3.5. *Let $F : \mathbb{R} \rightrightarrows Y$, where Y is infinite dimensional normed vector space, be defined by $F(t) = \bar{\mathbf{B}}_{|t|}(\mathbf{0})$. Then F is t-usc at 0, but $F(t) \setminus F(0) = \bar{\mathbf{B}}_{|t|}(\mathbf{0}) \setminus \{\mathbf{0}\}$ is not totally bounded for any $t \neq 0$.*

Proof ot Theorem 5.3.1. Since $(i) \Rightarrow (iii)$ in Proposition 5.3.4, we may apply Theorem 5.2.3. □

Chapter 6

Conclusion

6.1 Main contributions

These are the main accomplishments in the thesis due to the author:

1. A general sufficient condition for tangential transversality is obtained. It is shown that it has as special cases some known sufficient conditions for tangential transversality

2. The general condition for tangential transversality is applied to derive tangential transversality of the feasible set of a minimization problem and the epigraph of the function in interest, at a given reference point. More specifically, three different scenarios are considered: the function satisfies Lipschitz condition with respect to the first variable, uniformly in the second, the feasible set is the graph of a continuous linear operator, and there exists a uniform tangent set generating the Clarke tangent cone to the epigraph at the reference point; The function satisfies the Aubin condition at the reference point and the feasible set is the graph of a compact linear operator; The graph and the feasible set are jointly massive at the referrence point. In each of the three cases, we used the obtained tangential transversality to derive a Lagrange multiplier rule if the reference point is a solution to the minimization problem.

3. Characterization of subtransversality, transversality and intrinsic transversality are obtained in the spirit of the original definition of tangential transversality, i.e. primal space characterizations. The question of the relation between all these notions is fully answered. A characterization of transversality in terms of "translated" tangential transversality is derived.

4. Extension of intrinsic transversality to infinite-dimensional spaces is proposed. It is shown to be implied by a previously proposed extension ([61]), and is proved that both coincide in the case of Hilbert spaces.

5. It is clearly stated and proved that transversality and subtransversality could be used as a characterization of metric regularity and metric subregularity. This is later used to obtain new proofs of the well-known primal space characterizations of the regularity concepts. We use sequential primal space characterization of metric regularity to provide new proof of the characterization of regularity via the first order variation and via the graphical derivative.

6. The optimal value map associated with a minimization problem whose feasible set depends on a parameter is considered. It is provided a counterexample to a probable interpretation of a result concerning the continuity of such map. We propose an additional assumption (**RUCA**) under which we could prove continuity. We go on to show that (**RUCA**) is in some sense necessary to obtain continuity of the map: (**RUCA**) is satisfied for all functions in interest if and only if the multivalued map which defines the feasible set is topologically continuous.

6.2 Publications related to the thesis

1. Apostolov, S.; Krastanov, M.; Ribarska, N. (2020) *"Sufficient Condition for Tangential Transversality"*, Journal of Convex Analysis 27, 19-30

2. Apostolov, S. (2021) *"On continuity of optimal value map"*, Comptes rendus de l'Academie bulgare des Sciences, Vol 74, No4, pp 506-513

3. Apostolov, S.; Bivas, M.; Ribarska, N. (2022) *"Characterizations of Some Transversality-Type Properties"*. Set-Valued and Variational Analysis. https://doi.org/10.1007/s11228-022-00633-4

4. Apostolov, S.; Bivas, M. *Characterizations of metric (sub)regularity via (sub)transversality*, submitted.

6.3 Approbation of the thesis

The results from the thesis have been presented in the following talks:

1. *"Sufficient conditions for tangential transversality"*, 47th Winter School in Abstract Analysis, Svratka, Czech Republic, 2019,

https://www2.karlin.mff.cuni.cz/ lhota/ (based on a joint work with Mikhail Krastanov and Nadezhda Ribarska)

2. *"Intrinsic transversality and tangential transversality"*, 15-th International Workshop on Well-Posedness of Optimization Problems and Related Topics, June 28 - July 2, 2021, Borovets, Bulgaria, http://www.math.bas.bg/ bio/WP21/ (based on a joint work with Mira Bivas and Nadezhda Ribarska)

3. *"Intrinsic transversality and tangential transversality"*, The 13th International Conference on Large-Scale Scientific Computations LSSC 2021, June 7 - 11, 2021, Sozopol, Bulgaria (based on a joint work with Mira Bivas and Nadezhda Ribarska)

4. *"Intrinsic transversality and tangential transversality"*, Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University, 27 March 2021 (based on a joint work with Mira Bivas and Nadezhda Ribarska)

5. *"On continuity of optimal value map"*, Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University, 26 March 2022

6.4 Declaration of originality

The author declares that the thesis contains original results obtained by him or in cooperation with his coauthors. The usage of results of other scientists is accompanied by suitable citations.

6.5 Acknowledgements

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