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Region-based theories for space and time

Dynamic relational mereotopology

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Abstract

This dissertation presents a development in the field of *alternative theories of space and time*. Here *alternative* may be considered as a synonym for *region-based* or *point-free* or *Whiteheadean*. The aim is to combine *region-based* spatial theories with temporal theories, while satisfying the following requirements:

- to have a point-free space;
- to apply the point-free approach to time (*moment-free* time);
- to have one integrated spatio-temporal language;
- to use relational languages only.

These requirements are backed by philosophical principles, stated by Alfred Whitehead and others, who have developed alternative theories of space and time, as well as some practical considerations about modeling natural processes and applications in logic.

We will study relational systems, which consist of temporal/dynamic (*stable* and *unstable*) variants of four popular relations from mereotopology: *part-of*, *overlap*, *underlap* and *contact*. A natural semantics will be given for the stable and unstable relations, describing them as dynamic counterparts of the base mereotopological relations. Stable relations are described as ones that always hold (i.e. the relation holds in all moments of time or in all possible situations), while unstable relations may hold sometimes (at least once), but not necessarily always. Thus, *stable* and *unstable* can be considered as simple temporal operators, corresponding to *always* and *sometimes*. Having defined these dynamic relations we will work only with them and exclude all other point-based and moment-based notions from the language.

Stable and unstable mereotopological relations are also defined in the works of Dimitar Vakarelov, but there they are used in stronger systems, with richer language. Thus, the current work may be regarded as a relational generalization of the works of Vakarelov. The choice of weaker language leads to many complications of the techniques and the proofs of the current results.

Our main language will consist of these eight relations - four stable versions of part-of, overlap, underlap and contact and four unstable versions of these relations. We will consider also a sub-language of this main set of relations - the mereological reduct, which consists of six relations (two dynamic variants for each of part-of, overlap and underlap). We will devote attention to this sub-language because it offers a more lightweight systems, that can be used in more cases and with lighter prerequisites. For example, to define the mereotopological relations we need a contact algebra, which corresponds to a topological space. Thus, if we cannot rely on the availability of a topology, then we would not be able to define the mereotopological relations. The relations from the mereological language, however, does not require a topological space, but only a Boolean algebra.

Also in the past developments of region-based spatial theories, mereology has in some cases been considered separately from mereotopology. So, there might be a special interest only in the mereological reduct, while disregarding the purely topological relations. Thus, we will have two sets of results. Every major result that we prove for the language of eight dynamic mereotopological relations, will be coupled (when possible) with a corresponding result for the mereological reduct of the six relations.

These results may be grouped into four main themes. Here follow short summaries for the four main topics. The next two sections contain more details for the results and the structure of their presentation in the current dissertation.

Expressiveness results. These results are about the expressive power of the two languages. It is clear that the mereological language is strictly weaker than the mereotopological one. Also, the mereotopological language is showed to be weaker than the language of dynamic contact algebras and a combination between Linear Temporal Logic and a fragment of RCC.

Representation theory results. Generalizations of Stone's representation theory for Boolean algebras and distributive lattices are developed. This is done for both the mereotopological and the mereological languages.

Axiomatization results. Axiomatizations are given and completeness is proved for a pair of first-order logics. There is a first-order logic for each of the main languages. We also have axiomatizations and completeness for the quantifier-free fragments of these first-order logics and for the two modal logics for the respective languages.

(Un)decidability results. Decidability or undecidability results are presented for the above logics, with exception to the the modal logic of dynamic mereotopological relations. We have decidability results for two reducts of this modal logic, instead.

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Structure

The current dissertation is structured in seven parts. The first part serves as an introduction to region-based/point-free space and to similar approaches to time. Parts II and III contain the technical preliminaries and information about previous works in this field. The next three parts contain the new results, that are presented here. Finally, the last part consists of concluding remarks and a short summary of the dissertation.

Here follow more detailed descriptions for each of the parts.

Part I.

This part presents a short introduction into Whitehead's ideas and also into some following developments about *alternative theories of space and time*. The main principles are the use of *region-based* or *point-free* space, *point-free* or *moment-free* time and of *integrated language* of space and time. Also some considerations about the pros and cons of relational languages are noted. Thus, this part is an introduction into the field, that is studied in this dissertation, and also serves as motivation for the requirements for the developed systems - point-free space, moment-free time and integrated and relational language.

Part II.

This part contains the mathematical preliminaries about theories for space and time, that are needed for the following parts of the dissertation. It also presents a short summary of some earlier works in the field of Whiteheadean theory of space and time and also of other approaches to combined spatio-temporal theories.

The part contains notions and results from the theories of *contact algebras* and the spatial logic $S4_u$ and some fragments of the *Region Connection Calculus*, that can be defined with contact algebras or $S4_u$. A variation of the language of *Linear Temporal Logic*, that has *since* and *until* as base operators, is featured. Finally, some popular techniques for combining spatial and temporal theories are reviewed. They include *fusions* of modal logics, *temporalizations* of logics and a point-free spatio-temporal development - *dynamic contact algebras*. The theory of dynamic contact algebras satisfies almost all of the properties that are required here, with the exception of the requirement for relational language.

Part III.

This part is also a prequel to the main works in the dissertation. It presents results for relational mereological and mereotopological systems. Here it is emphasized to the transition from the full language of contact and Boolean algebras to relational languages. The part also serves as an in-depth review of representation theories for point-free spatial systems. Here we see a relational adaptation of Stone's representation theory for Boolean algebras and the representation theory for contact algebras.

Part IV.

This part presents the formal definitions of stable and unstable mereological and mereotopological relations and of relational structures for these relations. The main languages of the current studies are established - a primary mereotopological language with dynamic relations and a secondary mereological language, which is a sub-language of the first one. Then we study the expressive power of these two languages, in comparison with systems from previous Parts II and III. The main results of the dissertation are the representation theories for dynamic relations - one for each of the two languages. Thus, this part contains the expressiveness results and the representation theory results from the current works.

Part V.

This part presents the axiomatization results from the dissertation. We have two first-order logics - one for the mereotopological language and one for the mereological language. We also consider the quantifier-free fragments of these two logics. The part also contains axiomatization and completeness results for the modal logics of the two studied languages. The completeness of the modal logics goes through the use of p-morphism techniques in order to remove the modally undefinable anti-symmetric condition, which are variants of Segerberg's bulldozer construction.

Part VI.

This part contains the decidability and undecidability results of the logics from Part V. It is shown that several classes of logics, which include the mereological overlap relation or the mereotopological contact relation in their languages, are *hereditary undecidable*. The quantifier-free fragments are decidable and their satisfiability problems are NP-complete problems. The decidability of the modal logic of the main mereotopological language is still an open problem. The decidability for two of its fragments, however, is proved. Also it is shown that the modal logic of the secondary mereological language is decidable.

Part VII.

The final part of the dissertation contains some concluding remarks and a summary of the results and developments. It also summarizes the open problems and future tasks in the field of dynamic relational mereotopology and mereology. They include the task of improving the expressive power of the current languages via expansion with more spatial and temporal means and the task of improving the representation techniques, in order to cope with the increasing difficulty of the expanding relational languages.

The part ends with a detailed list of the results in this dissertation, that represent the contribution of the dissertation. This list is coupled with a list of the papers and conference talks with which these results are reported.

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Part I. Introduction. Region-based space and time.

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This part presents a brief introduction to the field of *region-based* theories of space and time. The main notions in this dissertation - the *stable* and *unstable* mereotopological and mereological relations - represent a development in this field. We want to keep this development in line with Whitehead's ideas about region-based space and time. To achieve this, we will stick to four major requirements: *point-free space*, *point-free time*, one *integrated language* for space and time and, finally, we would like to work only with *relational languages*. Thus, this part will be divided into four sections. Each section will be dedicated to one of these requirements and will present a short introduction into the corresponding topic and will give philosophical and practical motivation for satisfying this requirement.

In the first two sections will be presented the ideas and principles of *point-free* space and time. It may be considered that the development of these ideas originates from Whitehead. In the case of region-based theories of space, we start with the works of Whitehead in the fields of mereology and mereotopology and then continue with the works of de Laguna, Tarski and others. A special attention is given to the correspondences between point-based and point-free definitions of space and to the techniques, used to redefine points in a point-free space (such techniques are customary called *representation theory*). In the case of region-based and point-free time, there are not so many results and the topic remains largely undeveloped. Here we present some of the ideas of Whitehead as a general plan and guideline for work in this field.

The third section discusses ways for obtaining a combined spatio-temporal language from initial separated languages for space and time. A good portion of the content of the section is devoted to a certain formal technique, which is a modification of standard constructions and allows to combine the starting languages into one integrated spatio-temporal language as a result.

Finally, the last section is dedicated to the advantages and disadvantages of working with relational systems. Here we present some philosophical arguments for the use of relational languages, instead of languages with functional operators and constants. Also, some practical considerations are given, that points out that relational languages may be more useful for applications in some fields.

1. Point-free geometry

The development of point-free geometry originates from the works of Whitehead (see [51], [49], [50]). He argues that points are just abstract objects that do not have separate existence in reality. He states the points should not be taken as primitive notions in spatial theories, but rather should be a composite definition in terms of (real) material objects. Thus, in [51] Whitehead proposed a programme to rebuild geometry, so that it is not based on the primitive notions of *point* and *line*, like in classical Euclidean geometry, but on the notion of *region*. The idea is that regions will represent the natural objects that can be observed in reality. This approach to spatial theories is known as *Region Based Theory of Space (RBTS)*. His works are continued by de Laguna [7], Tarski [44] and others. For more about the field of Region Based Theory of Space see [4], [37], [45].

Whitehead shows how points can be defined via classes/sets of regions and through the relationships between the regions in these classes. To achieve that he uses some means from *mereology* as a base in order to build a "point-free" theory of geometry. Mereology is an ontological discipline which can be characterized shortly as a theory of "Parts and Wholes" (see [40] for more about mereology). It deals with regions of space as primary objects and with the possible operations and relations between them. For instance, some of the mereological relations used for the region-based definitions of points are *part-of* and *overlap*. The part-of relations expresses the situation in which one region is entirely contained in other. The overlap signifies that two regions share a common sub-region. Here we will also use another relation - *underlap*. It is dual to overlap and describes when two regions combined do not exhaust the whole space. Tarski, for instance, used a special kind of regions - balls - and mereological means to rebuild Euclidean geometry (see [44]).

Mereology was originally an ontological system. One of the famous mereological ontologies is Leśniewski's ontology. However it does not have a good mathematical definition and is difficult to use. Tarski is the one, that has shown that the mathematical equivalent of mereology are complete Boolean algebras without the zero element (see [40] for details). The choice of Boolean algebras seems natural since they present in some way a point-free analogs of sets. If regions are considered to be elements of the algebra, then mereological relations like part-of, overlap and underlap can be formally defined with Boolean formulae. The philosophical reason to exclude the zero element is that it represents the empty regions, which does not exist. Usually, for practical use, the zero element is added for simplicity of the definitions.

Mereology, however, is a weak system in some sense. It cannot reason about the boundary of a region and, for instance, cannot distinguish between overlap and contact or in some cases between disconnectedness and external connectedness. Thus, an extension of mereology is needed. This extension is called *mereotopology* and is obtained by addition of relations with topological nature.

$$\text{Mereotopology} = \text{Mereology} + \text{topological relations.}$$

The topological relations that are added are usually based on the topological *contact*. The contact relation between two regions means that they share a common point. Since points in mereotopology are not regions contact is stronger than overlap (which states that the regions must share a common sub-region). For instance, it is possible that two regions are not overlapping but are in a contact and in this case they have only boundary points in common but not points from their interiors. With regards to the mathematical definition of mereotopology, we apply the same principle - extend the Boolean algebras with topological relations. Thus, we get that the mathematical equivalent of mereotopology are Contact algebras (see [8], [10], [9], [41]).

Contact algebras = Boolean algebras + contact-based relations.

Mereology and mereotopology are often used to build *region based theories of space*. One of the most popular systems, that is obtained through mereotopology, is the Region Connection Calculus (RCC) introduced by Randell, Cui and Cohn [38]. Two well-known RCC subsystems are the Egenhofer-Franzosa RCC-8 from [11] and its mereological reduct RCC-5. Many logics based on them have been studied - see [21] and [53] for RCC-8 logics and [21] and [19] for RCC-5. See also [2] for more examples for mereotopological theories.

In most cases, in which such point-free spatial systems are developed ([35], [47], [46]), there is a correspondence between the point-free systems and the classical point-based systems. One direction of this correspondence is easy - if we start from point-based environment we can define a point-free environment, as we take the sets of points to be the primary objects. This is usually considered to be the standard/classical definition of point-free systems. In this setting we use the points as primitive objects. The other direction is to show that if we start from a point-free environment, in which there are no points present, then we can redefine/recreate the points in this environment and obtain a point-based system. This technique is often called a *representation theory*. In most cases it is a generalization of Stone's representation theory for Boolean algebras and distributive lattices (see [1]). In these representation theories sets of regions are used to recreate points, which we call *abstract points*. Each abstract point actually represents the set of all regions that should contain this point in the point-based system. Then we redefine the relations between the regions (part-of, overlap, contact, etc.) anew with the abstract points, instead of with the original point-free regions (such re-definitions are also called *characterizations*).

2. Moment-free time

The works of Whitehead also include view on how the theory of time should be approached. His ideas about time are developed mainly in [50], [51]. He states that the theory of time cannot be separated from the theory of space and that time should be defined in the same point-free fashion as space. Thus, we have the notion of *moment-free time*. Whitehead also states that the time should not be defined separately from the space, but rather time and space should be defined simultaneously from one sort of objects, that exhibit both spatial and temporal properties.

Thus, Whitehead presents a development, which is called *Epochal Theory of Time (ETT)*. It this theory we work with regions of time, which are called *epochs*. For instance, with every spatial regions, we can associate a temporal region - the *epoch of existence* of the object. Thus, we could apply mereotopological reasoning to time also. Whitehead talks of spatio-temporal relations, expressing when two objects are *contemporaries* (in [48] this relation is called *temporal contact*). For instance, this could be defined with the statement, that the epochs of existence of the two objects coincide - i.e. the first epoch is part-of the second one and vice versa. Alternatively, we could state that the two epochs overlap.

Unfortunately, the region-based theory of time is not so well developed as the spatial theory. There are not so many results and Whitehead himself did not present such programme, as he did about space. His ideas were presented in a mainly informal philosophical style. Modal logics could be considered as a tool to develop moment-free time, but the current semantics of these logics should be modified. For instance when we use Kripke semantics for the modal logic S4 (in its role as a temporal logic in [18]) or when we use time flow structures for temporal logics (see [22], [23]) we clearly work with the elements from the structure as moments of time. This is true even for interval temporal logics, where we seemingly work with intervals/regions of time, but we use points (of time) as start and end for the intervals and for the validity of formulae.

In the next sections we consider a combination between mereotopological relations and modal semantics, that could produce a formalism for the development of moment-free time.

3. Integrated languages for space and time

According to Whitehead in his studies of alternative theories of space and time, the theory of time cannot be separated from the theory of space and their integrated theory has to be extracted from the existing spatio-temporal relations considered as primitives. Thus, we aim to extend the mereotopological relations with some temporal properties and in this way to obtain such combined spatio-temporal relations.

The usual way to reason about a dynamic space is to observe the space at different moments of time and to take a snapshot of the environment in each time moment. Each such snapshot is a purely spatial system and we can reason about it with a spatial language. Then we can apply some temporal construction (like S5 modality, or *until* and *since* operators) over the time moments. However, in this way we have separated languages for time and space and also we suffer from the fact, that the popular temporal constructions are point-based.

The current idea is a modification of the usual approach. We will enhance the point-free relations and operations from the spatial language in such way, so that every one of them will express both spatial and temporal properties. We will work a relational mereotopological language and for each of its relations we will define two dynamic variants of the base mereotopological relation - a *stable* variant and an *unstable* variant. Each of these dynamic relations will be able to reason not only for one snapshot of the space, but for the whole collection of snapshots in all moments of time at once.

Here is the formal description of this construction. Let I be the set of all considered moments of time and let a and b be two spatial objects (regions). Then every object may exist or may not exist at every time moment (in the second case we will consider that the object is just the empty region). Thus every object can have a different *instance* at every moment. So we will consider that a *history* of an object is the collection of all of its instances. Thus, the histories of a and b will be $a =< a_i : i \in I >$ and $b =< b_i : i \in I >$. Let R be one of the considered spatial relations. Respectively, R has an instance for every time moment i, denoted by R_i. $a_i R_i b_i$ will denote the fact that the objects a and b are in the relation R at moment i. Thus, for every original spatial relation R we define two versions that describe its *stability* in the dynamic environment - *stable* version denoted by $R^\forall$ and *unstable* version denoted by $R^\exists$:

$$\begin{aligned} aR^\forall b &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad \forall i \in I, a_i R_i b_i, \\ aR^\exists b &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad \exists i \in I, a_i R_i b_i. \end{aligned}$$

For instance $a \ \mathsf{C}^\forall \ b$ means that in each moment of time a is in a contact with b, i.e. a and b are always in a contact. Similarly $a \ \mathsf{C}^\exists \ b$ means that a and b are sometimes in a contact. This method is used in [47], [48] and [46] to define dynamic contact algebras via Cartesian products of static contact algebras.

Since we work only with the stable and unstable relations, the time moments are hidden in the construction of the relations. Thus, we cannot use the time moments directly in the language. In this sense, we can say that the developed theories are also "moment-free", as well as "point-free". This, of course, leads to further complications in the representation theory for the dynamic systems that will be defined. Now we have to recreate the moments of time (define abstract moments of time), as well as the spatial points.

Of course the above mentioned construction is one of the simplest ways for building stable and unstable versions of relations. Instead, we could use more complex expression over the set of time moments than just a single quantifier over I. For example, a modal approach could be used (also described in [46]) to define stability and unstability:

$$aR^{\forall}b \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Box(aRb),$$

$$aR^{\exists}b \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Diamond(aRb).$$

In this modal setting the current construction corresponds to the use of S5 modality. This means that we do not have any structured relation between the time moments. If we wish to define dynamic relations corresponding to more complex temporal operators we could use some ordering of the time moments - linear time, branching time, etc (see [48]). In other words, we could use modality different from S5. In the current dissertation only the S5 case is covered.

Similar techniques for temporalizing spatial logics are described in [16], [53] and [20]. The general approach is to apply temporal operators over formulas or terms from the spatial language. In these papers several logics are described by means of this approach. They also represent development in the theories of space and time and most of them have more complex language and more expressive power than the ones presented in this dissertation. The difference is that the temporal and spatial operators in the language of these logics are separated, whereas the language of the stable and unstable mereological relations combines temporal operators and spacial relations into one. Thus, it is closer to Whitehead's idea that the theory of time cannot be separated from the theory of space.

4. Relational language

In this sections we dedicate some attention to the last of the requirements for the current works - to develop only systems with relational languages. The case for language, which is not strictly relational, is studied in [47], [48] and [46]. There, by means of the construction from the previous section, dynamic versions of the language of contact algebras are developed. We have several reasons that motivate the development of the relational case.

First, there are philosophical arguments against the use of functional operators between regions. They state that functions over regions may not always be determined and that in some cases there is no natural justification that the result should exist. So, we will have partial functions in the language. For instance, there is such argument about the union operation between regions. If regions a and b exist, then the union of these regions might not be a valid region itself. For example, regions a and b might be very distant from each other and taking their union as a valid region might be pointless or even incorrect. In this case it is better to work with relations, than to work with partial functions.

There are also arguments, that relational systems are better for models of natural processes and environments. The standard definition of stable and unstable relations is over the Cartesian product of static systems. The vectors of this product are considered to be the dynamic regions. We may argue that not all of these vectors are needed. If we try to model a natural process in which two moving objects a and b participate, then in the current setting we have that these two regions are the vectors of their states at all moments: $a =< a_i : i \in I >$ and $b =< b_i : i \in I >$. Then we might not be interested in vectors that have mixture of a_i and b_i in them. Thus, we may want to exclude such vectors from the Cartesian product. Again, this will lead to partial functions and we will have to resort to relational languages.

Another reason to prefer relational systems is that they are more suitable for Kripke semantics for modal logics (see [21] and [35] for such logics) and also for interpretations for Description logics. Thus, the relational systems may be more applicable, than their counterparts with functions in the language.

Of course, there is a price for working with relational languages. As we will see in Part IV, Section 1, the relational languages are strictly weaker, with respect to expressive power, than the *algebraic* languages (Boolean algebras, contact algebras, dynamic contact algebras). This greatly increases the difficulty of many constructions and proofs. The representation theory for dynamic mereotopological and mereological relations suffers the most. Also, the fact that we cannot work with the Boolean functions, means that we have to use many conditions, to axiomatize the relations, which have to imitate the behavior of the Boolean functions. This is a problem, especially, when we work with modal logics for the dynamic relations, where the proofs for the p-morphism and filtration techniques become very long and tedious.

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Part II. Theories for space and time

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In this part we review some of the spatial and temporal systems and logics, that are already developed for reasoning about space and time. This part might be considered as a technical introduction, that contains the preliminary facts and notions, needed for the remainder of the dissertation. The part is divided into three sections. The first section contains information for purely spatial theories, the second sections is a brief introduction into one of the most popular temporal logics and finally the last section is dedicated to techniques and examples of combinations between spatial and temporal systems, in order to obtain means to reason about dynamic space.

In Section 1 are presented two main formalisms for reasoning about space - contact algebras and the $S4_u$ modal logic. Alongside them are considered several fragments of the famous Region Connection Calculus (RCC), such as RCC-8, RCC-5 and BRCC-8, which can be defined either in contact algebras or in $S4_u$. The most attentions is devoted to the theory of contact algebras. They are very important for the current works in this dissertation, since, as discussed in the previous part, contact and Boolean algebras are the formal mathematical definitions of point-free mereotopology and mereology. Namely, contact and Boolean algebras are used in the following parts to define, first, the static mereotopological and mereological relations and, later, their dynamic counterparts.

Section 2 contains brief description of one variation of the language of Linear Temporal Logic. This language has *until* and *since* as its only temporal operators. The sections presents this language, its formal semantics and also the definability of more temporal modalities, such as *next*, *always* and *sometimes*, with the presented language.

Section 3 is dedicated to some techniques for combining logics to produce formalisms for dynamic systems. I.e. formalisms that can reason for both space and time. The techniques are broadly divided into two main approaches - external approach and internal approach. The external approach corresponds to a point-based style of defining spatio-temporal systems, in which the languages of space and time are separated. I.e. this is a non-Whiteheadean approach. It is represented by the techniques of fusions of modal logics and temporalizations of complete logics. The internal approach, on the other side, is more closer to Whitehead's ideas, as it allows a points-free definitions with one integrated language. Example is given with the notion of dynamic contact algebras.

1. Spatial theories

In this section we present some of the most popular formal systems, that are used for reasoning about space. The first subsection is dedicated to the formal definition of mereotopology - *contact algebras*. It also explores different ways of obtaining contact algebras and their connection to topological spaces. In the next subsection we present the modal logic $\mathrm{S4_u}$, in its role as a spatial logic. Finally, we discuss some popular fragments of the *Region Connection Calculus*, which are definable in contact algebras and in $\mathrm{S4_u}$.

1.1. **Contact algebras.**

Since mereology can be formally defined in Boolean algebras (see [40]) and since mereotopology is just an extension of mereology with topological means/relations

$$\text{Mereotopology} = \text{Mereology} + \text{topological relations},$$

then we can apply the same approach to Boolean algebras, in order to obtain formal definition of mereotopology

$$\text{Contact algebras} = \text{Boolean algebras} + \text{contact relation}.$$

Thus, we give the following general definition for contact algebras, in which they are presented as extensions of Boolean algebras with a binary relations, that satisfies certain properties. Contact algebras are extensively studied in [8], [10], [9], [41].

Definition 1.

Contact algebras *are systems of the sort* $(\underline{B}, \mathsf{C})$*, where* $\underline{B} = (B, 0, 1, ., +, *)$ *is a nondegenerate Boolean algebra ('.' is the Boolean meet, '+' is the Boolean join, and '$*$' is the Boolean complement) and* C *is a binary relation over* B*, satisfying the following conditions:*

$$\begin{aligned} x \mathsf{C} y &\Longrightarrow x \neq 0 \;\&\; y \neq 0, \\ x \mathsf{C} y &\Longrightarrow y \mathsf{C} x, \\ x \mathsf{C} (y+z) &\Longleftrightarrow x \mathsf{C} y \text{ or } x \mathsf{C} z, \\ x.y \neq 0 &\Longrightarrow x \mathsf{C} y. \end{aligned}$$

If we consider the elements of B to be regions, then we see that this definitions treats them as primary objects, and does not suppose that the regions are composed of points. Thus, this can be considered as the general *point-free* definition of contact algebras.

Topological contact algebras.

Standard models for contact algebras are topological spaces. Every topological space generates a contact algebra, which is made of the Boolean algebra of the regular closed sets of the space and the topological contact relation. The construction is the following.

Let $\mathbb{X} = (X, \tau)$ be the topological space, where $\mathbb{C}$ and $\mathbb{I}$ are the closure and interior operators. First, we establish the Boolean part of the contact algebra. This is the Boolean algebra of the regular closed sets in the topological space. The domain of the algebra is:

$$RC(\mathbb{X}) = \{\ x \mid x \subseteq X, x = \mathbb{C}(\mathbb{I}(x))\ \}.$$

For the remaining part of the Boolean algebra $\underline{B} = (RC(\mathbb{X}), 0, 1, ., +, *)$, we have that the constants and the meet, join and complement operations are defined as follows:

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \mathbb{X}, \\ x.y &= \mathbb{C}(\mathbb{I}(x \cap y)), \\ x + y &= x \cup y, \\ x^* &= \mathbb{C}(\mathbb{X} \setminus x). \end{aligned}$$

Finally, the contact relation is defined for arbitrary regular closed sets x and y:

$$x \mathsf{C} y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad x \cap y \neq \emptyset.$$

It is easy to see, that C satisfies the needed conditions from Definition 1. Thus, we prove the following lemma

Lemma 1.

Let $\mathbb{X}$ be a topological space. Then $(\underline{B}, \mathsf{C})$ is a contact algebra, where $\underline{B}$ is the Boolean algebra of the regular closed sets of $\mathbb{X}$ and C is the topological contact in $\mathbb{X}$.

Thus, we have the following type of contact algebras

Definition 2.

A contact algebra $(\underline{B}, \mathsf{C})$ generated by a topological space (as in the above lemma) is called a topological contact algebra*.*

Lemma 1 states that every topological contact algebra is a (general) contact algebra. If we adopt the topological definition of contact algebras to be the *standard point-based* definition and the general Definition 1 to be the *abstract point-free* definition, then we have that from a point-based algebra we can define a point-free algebra. The converse relationship, i.e. that in a point-free system we can redefine the points and, subsequently, a point-based system, is a generalization of Stone's duality between Boolean algebras of sets (sets of points) and generally defined Boolean algebras. The corresponding statement for contact algebras is the following theorem from [8].

Theorem 1.

For every contact algebra $(\underline{B}, \mathsf{C})$ there is a topological contact algebra $(\underline{B}^t, \mathsf{C}^t)$ and an isomorphic embedding from $(\underline{B}, \mathsf{C})$ into $(\underline{B}^t, \mathsf{C}^t)$.

Relational contact algebras.

We will consider one more way of defining standard point-based contact algebras. We will use a relational frame (X, R), where X will be the set of all points. R will be a reflexive and symmetric binary relation over X, which will be interpreted as a *proximity relation* between the points and will help to define the contact. We have that every such frame can generate a contact algebra in the following way

Lemma 2.

Let (X, R) be a frame, where R is a reflexive and symmetric relation. $(\underline{B}, \mathsf{C})$ is a contact algebra, where $\underline{B}$ is the Boolean algebra of the subsets of X and C is defined for $x, y \subseteq X$:

$$(\mathsf{C}_{\text{def}}) \qquad x \mathsf{C} y \quad \stackrel{def}{\longleftrightarrow} \quad \exists a \in x, \exists b \in y, a \mathsf{R} b.$$

Thus, we have a new type of standard point-based contact algebras.

Definition 3.

A contact algebra $(\underline{B}, \mathsf{C})$ *generated by a reflexive and symmetric frame is called a* relational contact algebra.

For the purposes of this dissertation, this definition will prove to be more useful. When we recreate the spacial points in the relational systems in Parts III and IV, it will be easier to recreate the proximity relation between the points, than to recreate the topology over them. Once we have recreated the frame (X, R), then we can define a contact algebra through it and access all mereotopological and mereological relations (see Subsection 1.3) that can be defined in the contact algebra.

Of course, as it was for the topological contact algebras, we have a theorem that proves the duality between the relational contact algebras and the general point-free contact algebras (for details see [9]).

Theorem 2.

For every contact algebra $(\underline{B}, \mathsf{C})$ *there is a relational contact algebra* $(\underline{B}^r, \mathsf{C}^r)$ *and an isomorphic embedding into it.*

1.2. **The $\mathbf{S4_u}$ spatial logic.**

Here we will briefly introduce another spatial system. This is the propositional modal logic S4, extended with a universal modality. It is usually denoted by $\mathrm{S4_u}$. Though it has been initially use for other means, Stone, Tarski and other (see [42], [43]) have discovered that it can be used as a logics for topological spaces, since the S4 box and diamond operators from the modal logic correspond to the interior and closure operators from topological spaces.

The language of $\mathrm{S4_u}$, considered as a spatial logic, is the following (see [20]):

$$\begin{array}{lcl} \tau & ::= & x \mid \tau^* \mid \tau_1.\tau_2 \mid \tau_1 + \tau_2 \mid \mathbb{C}\tau \mid \mathbb{I}\tau, \\ \varphi & ::= & \tau_1 \sqsubseteq \tau_2 \mid \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2, \end{array}$$

where

- x is an object variable;
- *, . and $+$ are the Boolean operators for complement, meet and join;
- $\mathbb{C}$ and $\mathbb{I}$ are the topological closure and interior operators (they correspond to the S4 box and diamond operators);
- $\sqsubseteq$ is a predicate, corresponding to the subset relation;
- $\neg$, & and $\vee$ are the standard Boolean connectives for negation, conjunction and disjunction.

Models for $\mathrm{S4_u}$ in this setting will be pairs $(\mathbb{X}, i)$, where $\mathbb{X}$ is a topological space and i is an interpretation of the object variables as subsets of $\mathbb{X}$. I.e. we have $x^i \subseteq \mathbb{X}$ for every variable x. The interpretation i can be extended for an arbitrary term τ in the following way:

- if τ is τ_1^*, then $\tau^i = \mathbb{X} \setminus \tau_1^i$;
- if τ is $\tau_1.\tau_2$, then $\tau^i = \tau_1^i \cap \tau_2^i$;
- if τ is $\tau_1 + \tau_2$, then $\tau^i = \tau_1^i \cup \tau_2^i$;
- if τ is $\mathbb{C}\tau_1$, then $\tau^i = \mathbb{C}(\tau_1^i)$;
- if τ is $\mathbb{I}\tau_1$, then $\tau^i = \mathbb{I}(\tau_1^i)$.

Finally, if we adopt the standard notations, whether a formula φ is valid in a model $(\mathbb{X}, i)$,

$$(\mathbb{X}, i) \models \varphi \text{ when the formula } \varphi \text{ is valid in the model,}$$
$$(\mathbb{X}, i) \not\models \varphi \text{ when the formula } \varphi \text{ is not valid in the model,}$$

we can complete the interpretation of the language for $(\mathbb{X}, i)$ as follows

- if φ is $\tau_1 \sqsubseteq \tau_2$, then $(\mathbb{X}, i) \models \varphi$ iff $\tau_1^i \subseteq \tau_2^i$);
- if φ is $\neg\varphi_1$, then $(\underline{W}, v) \models \varphi$ iff $(\underline{W}, v) \not\models \varphi_1$;
- if φ is $\varphi_1 \,\&\, \varphi_2$, then $(\underline{W}, v) \models \varphi$ iff $(\underline{W}, v) \models \varphi_1$ and $(\underline{W}, v) \models \varphi_2$;
- if φ is $\varphi_1 \vee \varphi_2$, then $(\underline{W}, v) \models \varphi$ iff $(\underline{W}, v) \models \varphi_1$ or $(\underline{W}, v) \models \varphi_2$.

1.3. **Region Connection Calculus.**

Now we will review some popular formal systems for reasoning about space, that can be defined via contact algebras or via $\mathrm{S4_u}$. There systems are fragments of the Region Connection Calculus (RCC), which is introduced by Randell, Cui and Cohn [38]. For more information about RCC, its fragments and logics about them, see [38], [20], [21], [53], [19].

First, we consider the RCC-8 (see [11]). It is a relational system of eight relations, that represent all possible situations in which two non-empty regions x and y can be (see Figure 1), with regards to being in contact, or overlapping each other, or one of them containing the other, etc. The language of this system is

$$\begin{array}{lll} \varphi & ::= & DC(x,y) \mid EC(x,y) \mid PO(x,y) \mid EQ(x,y) \mid \\ & & TPP(x,y) \mid TPP^{-1}(x,y) \mid NTPP(x,y) \mid NTPP^{-1}(x,y) \mid \\ & & \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2, \end{array}$$

where

- x and y are object variables;
- DC, EC, PO, TPP, TPP^{-1}, $NTPP$, $NTPP^{-1}$ and EQ are the spatial relations between regions;
- $\neg$, $\&$ and $\vee$ are the standard Boolean connectives.

The RCC-8 relations are defined in contact algebras as follows (their definitions in $\mathrm{S4_u}$ can be found in [20]):

$$\begin{array}{rcll}
DC(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & \neg(x \mathrel{\mathsf{C}} y) & \text{disconnected,} \\
EC(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & x \mathrel{\mathsf{C}} y \,\&\, x.y = 0 & \text{extern. connected,} \\
PO(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & \neg(x.y = 0) \,\&\, \neg(x \le y) \,\&\, \neg(y \le x) & \text{partial overlap,} \\
TPP(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & x \le y \,\&\, x \mathrel{\mathsf{C}} y^* \,\&\, \neg(y \le x) & \text{tangential part,} \\
TPP^{-1}(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & y \le x \,\&\, y \mathrel{\mathsf{C}} x^* \,\&\, \neg(x \le y) & \text{tangential part}^{-1}\text{,} \\
NTPP(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & \neg(x \mathrel{\mathsf{C}} y^*) \,\&\, \neg(x = y) & \text{non-tangential part,} \\
NTPP^{-1}(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & \neg(y \mathrel{\mathsf{C}} x^*) \,\&\, \neg(y = x) & \text{non-tang. part}^{-1}\text{,} \\
EQ(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & x = y & \text{equality.}
\end{array}$$

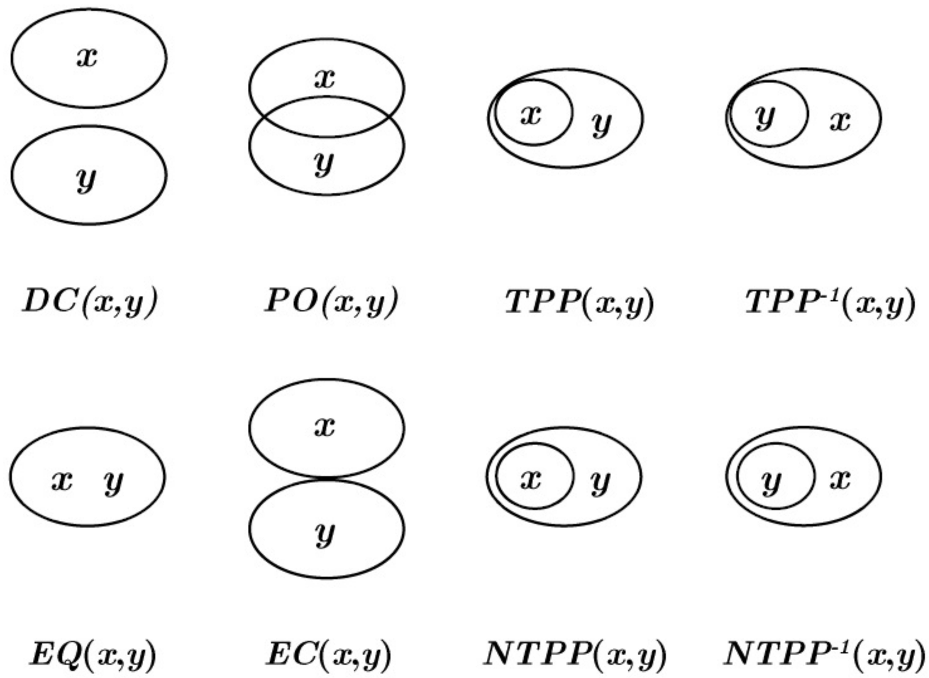

FIGURE 1. Picture of the eight relations of RCC-8.

The mereological reduct of RCC-8 contains only five relations. Thus, it is called RCC-5. Since mereology cannot reason about boundaries or contacts between regions, we have that in RCC-5 we cannot distinguish between external contact and partial overlap (see Figure 2).

The relations of RCC-5 can be defined in contact algebras in the following way:

$$
\begin{array}{rcll}
DC(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & x.y = 0 & \text{disconnected,} \\
PO(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & \neg(x.y = 0) \;\&\; \neg(x \leq y) \;\&\; \neg(y \leq x) & \text{partial overlap,} \\
PP(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & x \leq y \;\&\; \neg(y \leq x) & \text{proper part,} \\
PPI(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & y \leq x \;\&\; \neg(x \leq y) & \text{proper part}^{-1}, \\
EQ(x,y) & \stackrel{\text{def}}{\longleftrightarrow} & x = x & \text{equality.}
\end{array}
$$

We will consider also fragments of RCC, that are not strictly relational (see [20] for more details). For instance, extend RCC-8 with the Boolean operations between spatial variables and the result is the following system

$$\text{BRCC-8} = \text{RCC-8} + \text{Boolean terms.}$$

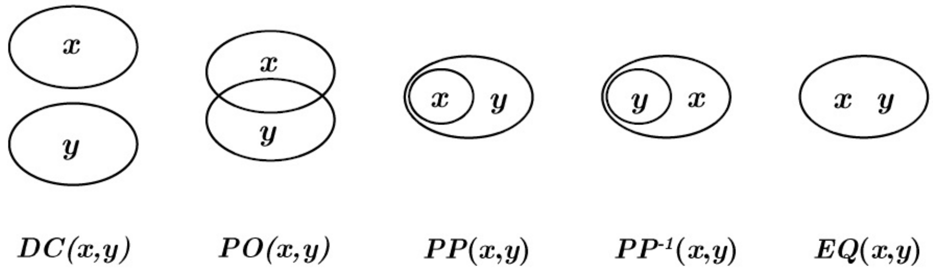

FIGURE 2. Picture of the five relations of RCC-5.

This fragment is introduced in [52] (see also [2]). The language of BRCC-8 is

$$
\begin{aligned}
\tau &::= x \mid \tau^* \mid \tau_1.\tau_2 \mid \tau_1 + \tau_2, \\
\varphi &::= DC(\tau_1,\tau_2) \mid EC(\tau_1,\tau_2) \mid PO(\tau_1,\tau_2) \mid EQ(\tau_1,\tau_2) \mid \\
&\quad\; TPP(\tau_1,\tau_2) \mid TPP^{-1}(\tau_1,\tau_2) \mid NTPP(\tau_1,\tau_2) \mid NTPP^{-1}(\tau_1,\tau_2) \mid \\
&\quad\; \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2.
\end{aligned}
$$

There are also other fragments, that feature more and more topological operators and relations. For instance, RC and $\mathrm{RC}_{\mathrm{max}}$ (see [20], [12]) include the $\mathbb{I}$ and $\mathbb{C}$ operators from $\mathrm{S4_u}$.

2. Temporal theories

In this section we briefly introduce one of the most popular temporal formal systems - the *propositional Linear Temporal Logic (LTL)*. For more extended study of temporal logic see the following references [3], [36], [22] and [23].

Modal logics have often been used for reasoning about time. The modal logic S4 is one such example (see [18]). S5 might also be considered as a temporal logic, although its box and diamond operators are very poor, with respect to expressiveness, temporal operators. Here we will consider a variant of the language of LTL, which contains two main temporal operators - *until* and *since*. They are enough to define the rest of the temporal operators, that are usually included in the language of LTL - *next*, *always*, *sometimes*, *always in the future*, etc. The current language is

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \mathcal{U} \varphi_2 \mid \varphi_1 \mathcal{S} \varphi_2,$$

where

- p is a propositional variable;
- $\neg$, & and $\vee$ are the standard Boolean connectives for negation, conjunction and disjunction;
- $\mathcal{U}$, $\mathcal{S}$ are the until and since operators.

Models of LTL are structures $(T, <)$, called *time flows*, where T is considered to be the set of moments of time and $<$ is a strict linear order over T. The time flows are coupled with a valuation of the propositional variable v, that states which variables are true at which moments of time. Thus for an arbitrary LTL formula φ and a moment of time $t \in T$ we have

- if φ is the propositional variable p, then $((T, <), v), t \models \varphi$ iff $p \in v(t)$;
- if φ is $\neg\varphi_1$, then $((T, <), v), t \models \varphi$ iff $((T, <), v), t \not\models \varphi_1$;
- if φ is $\varphi_1 \,\&\, \varphi_2$, then $((T, <), v), t \models \varphi$ iff $((T, <), v), t \models \varphi_1$ and also $((T, <), v), t \models \varphi_2$;
- if φ is $\varphi_1 \mathcal{U} \varphi_2$, then $((T, <), v), t \models \varphi$ iff there is $s \in T$, such that $t < s$, $((T, <), v), s \models \varphi_2$ and for all $r \in T$, that $t < r < s$, $((T, <), v), r \models \varphi_1$;
- if φ is $\varphi_1 \mathcal{S} \varphi_2$, then $((T, <), v), t \models \varphi$ iff there is $s \in T$, such that $s < t$, $((T, <), v), s \models \varphi_2$ and for all $r \in T$, that $s < r < t$, $((T, <), v), r \models \varphi_1$.

The following temporal operators are definable in terms of *until* and *since* (with the addition of the constant truth symbol $\top$ and the constant false symbol $\bot$):

$$\begin{array}{lcll}
\Diamond_F \varphi & \stackrel{\text{def}}{\longleftrightarrow} & \top \mathcal{U} \varphi & \text{sometimes in the future,} \\
\Diamond_P \varphi & \stackrel{\text{def}}{\longleftrightarrow} & \top \mathcal{S} \varphi & \text{sometimes in the past,} \\
\Box_F \varphi & \stackrel{\text{def}}{\longleftrightarrow} & \neg \Diamond_F \neg \varphi & \text{always in the future,} \\
\Box_P \varphi & \stackrel{\text{def}}{\longleftrightarrow} & \neg \Diamond_P \neg \varphi & \text{always in the past,} \\
\Box_A \varphi & \stackrel{\text{def}}{\longleftrightarrow} & \Box_F \varphi \,\&\, \Box_P \varphi \,\&\, \varphi & \text{always,} \\
\Diamond_A \varphi & \stackrel{\text{def}}{\longleftrightarrow} & \Diamond_F \varphi \text{ or } \Diamond_P \varphi \text{ or } \varphi & \text{sometimes.}
\end{array}$$

Also, in the case of discrete time flows, the *next* operator is defined, as follows:

$$\bigcirc \varphi \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \bot \mathcal{U} \varphi.$$

3. Spatio-temporal theories

This section is devoted to techniques for using spatial logics and temporal logics to produce one combined spatio-temporal system. In [16] Finger and Gabbay discuss that there two main approaches to do this - "external" approach and "internal" approach.

The "external" approach is the usual technique to devise a formal system in order to describe a given dynamic space S. Just take snapshots of S at each moment of time t - S_t. Then use a chosen spatial logic to reason about the snapshots S_t and use a chosen temporal logic to reason about the moments of time. Thus, in this approach we usually have two sorts of objects and two types of variables - spatial and temporal. The valuations for the variables will look like $v(x,t)$, where x is a spatial variable and t is a temporal one. It is evident, that in this approach we work with the moments of time directly and also that we have two separate languages for the space and time. Thus, this approach cannot be considered to be point-free or Whiteheadean.

The "internal" approach can be more useful to build Whiteheadean theories. In this approach we decompose the dynamic space S to components, so that S can be (completely) described by them. Then we consider those components as spatio-temporal objects (possibly dynamic regions) and use some language to describe them. Thus, we could have one integrated language for space and time and each symbol in this language could have both spatial and temporal semantics. Valuations in this language can look like $v(x)$, where we have only one type of variables and x is a spatio-temporal object.

In the following three subsections we will review some examples of the two approaches. The first two subsections will present applications of the external approach from [17] and [16]. The last subsection features examples of the internal approach from [47] and [48].

3.1. **Free combinations of spatial and temporal theories.**

When we try to combine a chosen spatial and temporal logics into one system, the simplest way to do this is to get the *independent join* of the two initial logics. I.e. take the union of the two languages, where each symbol retains its original semantics and formal interpretation. Then, if the two initial logics are truly independent, we could axiomatize the result with the unions of the axioms of the base spatial and temporal logics.

The application of this technique to modal logics is studied in [17]. We start from two modal logics $\mathcal{M}_1$ and $\mathcal{M}_2$. Then we combine the languages of $\mathcal{M}_1$ and $\mathcal{M}_2$ without any restrictions. I.e. we can freely apply modal operators from $\mathcal{M}_1$ to formulae containing operators from $\mathcal{M}_2$ and vice versa. This combination of $\mathcal{M}_1$ and $\mathcal{M}_2$ is called *fusion* of $\mathcal{M}_1$ and $\mathcal{M}_2$ and is denoted by $\mathcal{M}_1 \otimes \mathcal{M}_2$. Models for $\mathcal{M}_1 \otimes \mathcal{M}_2$ are Kripke structures $(W, R_1, R_2, \dots R_n, S_1, S_2, \dots S_m)$, where $(W, R_1, R_2, \dots R_n)$ is a model for $\mathcal{M}_1$ and $(W, S_1, S_2, \dots S_m)$ is a model for $\mathcal{M}_2$.

[17] features several results about Kripke completeness, decidability and other properties of fusions. Here are some of them.

Theorem 3.

Let modal logic $\mathcal{M}_1$ be complete over a class of Kripke structures

$$\Sigma_1 = \{\ (W, R_1, R_2, \dots R_n) \mid (W, R_1, R_2, \dots R_n) \textit{ satisfies the axioms of } \mathcal{M}_1\ \},$$

and also $\mathcal{M}_2$ be another modal logic, complete over the class

$$\Sigma_2 = \{\ (W, S_1, S_2, \ldots S_m) \mid (W, S_1, S_2, \ldots S_m) \text{ satisfies the axioms of } \mathcal{M}_2\ \}.$$

Then if both classes Σ_1 and Σ_2 are closed under the formation of disjoint unions and isomorphic copies, the fusion $\mathcal{M}_1 \otimes \mathcal{M}_2$ is complete over the class

$$\Sigma = \{\ (W, R_1, R_2, \ldots R_n, S_1, S_2, \ldots S_m) \mid \text{ satisfying the axioms of } \mathcal{M}_1 \text{ and } \mathcal{M}_2\ \}.$$

Theorem 4.
If $\mathcal{M}_1$ and $\mathcal{M}_2$ are multi-modal logics, having the finite model property, then their fusion $\mathcal{M}_1 \otimes \mathcal{M}_2$ also has the finite model property.

Theorem 5.
If $\mathcal{M}_1$ and $\mathcal{M}_2$ are decidable multi-modal logics, then their fusion $\mathcal{M}_1 \otimes \mathcal{M}_2$ is also a decidable logic.

The prime candidates to apply the fusion technique will be the modal logics LTL (Linear Temporal Logic) and $\mathrm{S4_u}$ (considered as the modal logic of topology). Thus, we obtain a combined spatio-temporal modal logic. The problem with LTL $\otimes$ $\mathrm{S4_u}$ is that in its language we have unrestricted combinations between the spatial and temporal operators. This may not be appropriate in some cases or even some of these combination (applying spatial operator over a temporal one, for instance) may not have any sense. In [17] are describes three fragments of LTL $\otimes$ $\mathrm{S4_u}$ - $\mathrm{ST_0}$, $\mathrm{ST_1}$ and $\mathrm{ST_2}$ - in which are introduced some restriction over the language.

$$\mathrm{ST_0} \subset \mathrm{ST_1} \subset \mathrm{ST_2} \subset \mathrm{LTL} \otimes \mathrm{S4_u}$$

$\mathrm{ST_0}$ is the language of BRCC-8, extended with means from LTL

$$\mathrm{ST_0} = \text{BRCC-8} + \text{since and until operators.}$$

The application of spatial means (operators and relations from BRCC-8), however, are not allowed over temporal operators. The language of $\mathrm{ST_0}$ is

$$\begin{aligned}
\tau \quad &::= \quad x \mid \tau^* \mid \tau_1.\tau_2 \mid \tau_1 + \tau_2,\\
\chi \quad &::= \quad DC(\tau_1,\tau_2) \mid EC(\tau_1,\tau_2) \mid PO(\tau_1,\tau_2) \mid EQ(\tau_1,\tau_2) \mid\\
&\qquad TPP(\tau_1,\tau_2) \mid TPP^{-1}(\tau_1,\tau_2) \mid NTPP(\tau_1,\tau_2) \mid NTPP^{-1}(\tau_1,\tau_2) \mid\\
&\qquad \neg\chi \mid \chi_1 \,\&\, \chi_2 \mid \chi_1 \vee \chi_2,\\
\varphi \quad &::= \quad \chi \mid \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \mathcal{U} \varphi_2 \mid \varphi_1 \mathcal{S} \varphi_2,
\end{aligned}$$

where

- x is a spatial variable;
- *, . and + are the Boolean operators for complement, meet and join;
- $\neg$, & and $\vee$ are the standard Boolean connectives for negation, conjunction and disjunction;
- $\mathcal{U}$, $\mathcal{S}$ are the until and since operators.

Models for $\mathrm{ST_0}$ consists of a structure $(\mathbb{X}, T, <)$ and a valuation v. Here, $\mathbb{X}$ is a topological space, $(T, <)$ is a time flow and v is a valuation, that assigns a subset of the space for each spatial variable x and time moment t,

$$v(x,t) \subseteq \mathbb{X}.$$

The extension of the valuation v to arbitrary term and the definition of validity of a formula for a modal are the same as for BRCC-8 and LTL (see the previous sections).

Languages ST_1 and ST_2 are extensions of ST_0, in which we gradually allow more and more complex spatial terms and, thus, allow application of spatial operators over temporal operators in a special way. In ST_1 we allow term of the sort $\bigcirc\tau$, while in ST_2 we allow spatial terms that include the since and until operators:

$$\mathrm{ST}_1 = \mathrm{ST}_0 + \text{terms } \bigcirc\tau,$$
$$\mathrm{ST}_2 = \mathrm{ST}_1 + \text{terms } \tau_0\mathcal{S}\tau_1,\ \tau_0\mathcal{U}\tau_1.$$

Thus, in ST_1 we are able to express statements like $\bigcirc\bigcirc\cdots\bigcirc\tau$, while in ST_2 we also have $\Box_F\tau, \Diamond_F\tau, \Box_P\tau, \Diamond_P\tau, \Box_A\tau, \Diamond_A\tau$. For instance, for the valuation of $\bigcirc\tau$ terms we have

$$v(\bigcirc\tau, t) = v(\tau, t+1).$$

3.2. **Temporalized logics.**

In [16] we have another technique for producing a combined spatio-temporal system from a given spatial logic. Suppose, we have a complete logic L (L may be modal or first-order or propositional logic) and $\mathcal{TL}_{\mathcal{SU}}$ is the LTL logic with since and until operators, which is complete over some class of time flows T. Then we form a new language, in which as in ST_0 we do not allow operators and relations from L to be applied over the since and until operators. The result is called a *temporalization of L* and is denoted by $\mathcal{TL}_{\mathcal{SU}}(\mathrm{L})$. The language of $\mathcal{TL}_{\mathcal{SU}}(\mathrm{L})$ is

$$\begin{array}{lcl} \chi & ::= & \text{a formula of L}, \\ \varphi & ::= & \chi \mid \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1\mathcal{U}\varphi_2 \mid \varphi_1\mathcal{S}\varphi_2. \end{array}$$

Thus, we have that

$$\mathrm{ST}_0 = \mathcal{TL}_{\mathcal{SU}}(\text{BRCC-8}).$$

For temporalizations of logics we have the following results in [16].

Theorem 6.

$\mathcal{TL}_{\mathcal{SU}}(L)$ is a complete logic.

The axiomatization of $\mathcal{TL}_{\mathcal{SU}}(\mathrm{L})$ is consisted of the axioms of $\mathcal{TL}_{\mathcal{SU}}$, with respect to T. For inference rules we take the following two types of rules:

- the inference rules of $\mathcal{TL}_{\mathcal{SU}}$;
- if φ is a theorem of L, then it is a theorem of $\mathcal{TL}_{\mathcal{SU}}(\mathrm{L})$.

The second type actually contains the axiomatization of L, in some sense.

Theorem 7.

$\mathcal{TL}_{\mathcal{SU}}(L)$ is a conservative extension of L.

Theorem 8.

If L is decidable, then $\mathcal{TL}_{\mathcal{SU}}(L)$ is decidable also.

Also, the separation theorem holds for $\mathcal{TL}_{\mathcal{SU}}(\mathrm{L})$.

3.3. **Dynamic contact algebras.**

In [47] and [48] we have examples of the technique of constructing dynamic relations $R^{\forall}$ and $R^{\exists}$, described in Part I. The definitions may be considered to be examples for the internal approach, since in the resulting language we do not have moments of time and we do not work directly with snapshots of the space $\ldots, S_0, S_1, S_2, \ldots$. Instead of that, we work with objects called *dynamic regions*. These dynamic regions are the components of the dynamic space and all means (relations and operators) from the dynamic languages are defined over them.

Here is the definition of dynamic contact algebras from [47]

Definition 4.

Let $I \neq \emptyset$ *and for every* $i \in I$, $(\underline{B}_i, \mathsf{C}_i)$ *be a contact algebra. Let* $\underline{B}$ *be a subalgebra of* $\prod_{i \in I} \underline{B}_i$ *(since* $\underline{B}_i$ *are Boolean algebras then* $\prod_{i \in I} \underline{B}_i$ *is a Boolean algebra also). Then for* $x, y \in \underline{B}$*:*

$$x \mathrel{\mathsf{c}} y \quad \stackrel{def}{\longleftrightarrow} \quad \forall i \in I, x_i \mathrel{\mathsf{C}_i} y_i \qquad \text{stable contact},$$

$$x \mathrel{\mathsf{C}} y \quad \stackrel{def}{\longleftrightarrow} \quad \exists i \in I, x_i \mathrel{\mathsf{C}_i} y_i \qquad \text{unstable contact}.$$

Then $(\underline{B}, \mathsf{c}, \mathsf{C})$ *is called a* standard dynamic contact algebra.

Here have adopted notations c=$\mathsf{C}^{\forall}$, C=$\mathsf{C}^{\exists}$.

Although, in the definition of dynamic contact algebras we use (static) contact algebras $(\underline{B}_i, \mathsf{C}_i)$, which are snapshot of the space at different moments of time, in the final language of dynamic contact algebras we do not have the time moments explicitly. The objects that we work with are vectors x from $\prod_{i \in I} \underline{B}_i$, which we interpret as dynamic regions. Thus, in the semantics of the first-order theory of dynamic contact algebras we have valuations that for each such dynamic region assign a vector of all states/snapshots of the region at the different moments of time:

$$v(x) = \langle \dots, x_0, x_1, x_2, \dots \rangle.$$

In Part I these vectors are called *histories* of the regions. If we use the style of two-sort valuations from the external approach, we would have that $v(x, i) = x_i \in B_i$.

Since a product of Boolean algebras is also a Boolean algebra, then for the language of standard dynamic contact algebras we have that

$$\begin{aligned} 0 &= \langle \dots, 0_0, 0_1, 0_2, \dots \rangle, \\ 1 &= \langle \dots, 1_0, 1_1, 1_2, \dots \rangle, \\ x.y &= \langle \dots, x_0.y_0, x_1.y_1, x_2.y_2, \dots \rangle, \\ x + y &= \langle \dots, x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots \rangle, \\ x^* &= \langle \dots, x_0^*, x_1^*, x_2^*, \dots \rangle. \end{aligned}$$

Just as in the case of static contact algebras, here we have a general point-free definition for dynamic contact algebras and the respective Stone-duality theorem that states the correspondence between general and standard algebras (see [47]).

Definition 5.

$(\underline{B}, \mathsf{c}, \mathsf{C}) = (B, 0, 1, ., +, *, \mathsf{c}, \mathsf{C})$ *is called a* dynamic contact algebra *if* $(\underline{B}, \mathsf{C})$ *is a contact algebra and* c *is a binary relation satisfying:*

$$\begin{gathered} 1 \mathrel{\mathsf{c}} 1, \\ 0 \mathrel{\overline{\mathsf{c}}} 0, \\ x \mathrel{\mathsf{c}} y \ \&\ z \mathrel{\overline{\mathsf{c}}} t \Longrightarrow x.z^* \mathrel{\mathsf{C}} y.z^* \text{ or } x.t^* \mathrel{\mathsf{C}} y.t^*. \end{gathered}$$

Theorem 9.

For every dynamic contact algebra $(\underline{B}, \mathsf{c}, \mathsf{C})$ *there is an isomorphic standard dynamic contact algebra* $(\underline{B}^s, \mathsf{c}^s, \mathsf{C}^s)$.

Also we may notice that in this definition of dynamic systems there is no ordering of the time. Thus, we can define only the simplest types of temporal constructions

- *always* and *sometimes*. In [48] we have an extended definition of dynamic contact algebras, in which there is an ordering $<$ of the time moments, which helps to define the *temporal contact* (which is an alternative definition of Whitehead's contemporary relation) and *before* relations:

$$\begin{aligned} x \, \mathsf{C}^{\mathsf{T}} \, y &\quad \overset{\text{def}}{\longleftrightarrow} \quad \exists i \in I, x_i \neq 0_i \ \& \ y_i \neq 0_i, \\ x \, \mathsf{B} \, y &\quad \overset{\text{def}}{\longleftrightarrow} \quad \exists i, j \in I, x_i \neq 0_i \ \& \ y_j \neq 0_j \ \& \ i < j. \end{aligned}$$

In [48] it is shown that with these relations we may axiomatize many various properties of the time ordering $<$, such as *right seriality*, *left seriality*, *updirectness*, *downdirectness*, *density*, *reflexivity*, *linearity* and *transitivity*.

♣ ♣ ♣

Part III. Static relational mereotopology and mereology

♣ ♣ ♣

In this part we make the transition from algebraic languages (i.e. languages with functional symbols and constants) to a purely relational languages. As it will be customary in the following parts, we will divide these languages in two major cases - a main mereotopological language, which consists of relations defined in contact algebras (i.e. in topological spaces), and a mereological sub-language of the main language (in which we have omitted the topological part and the relations are defined only in Boolean algebras). In the first two sections we define relational structures for these two languages - structures with mereotopological relations in the first section and structures with mereological relations in the second one. These structures, which we will call static, will serve later as a base for defining the dynamic mereotopological and mereological relations.

In each of the two cases (mereotopological and mereological) we have two definitions of structures for the language. The first definition will be called *standard* and presents the relations as defined with formulae from the contact or Boolean algebras. If we consider the elements (the regions) from these algebras as sets of points, then these formulae actually describe the way that the relations are defined over points as primitives. Thus, these standard definitions will be considered as *point-based*. The second type of definitions is when we consider the structures to be collections of regions and we take the regions as primitive objects. Then we do not express statements about points from these regions, but we describe which first-order conditions, in terms of the relations from the language, must be satisfied by the structures. The second type of definitions will be considered as *point-free*. In each case it is easy to show that the standard structures satisfy the conditions from the second definition. Thus, the second definition is more general definition of the structures.

The last section will present the converse correspondence between the general structures and the standard structures. Such correspondences are called representation theories. They show that for each general structure, there exists a standard structure that is isomorphic to the initial one. Since in the initial structure there are no points, then the points for the corresponding standard structure are recreated as special sets of regions (elements of the general structure). Each such set correspond to a point that should exist in the standard point-based structure. These representation theories are actually generalized variants of Stone's representation theory for Boolean algebras.

1. Mereotopological structures

The mereotopological structures are relational structures, which have four mereotopological relations in their language - part-of $\leq$, overlap O, underlap U and contact C. These structures can be considered to be a relational variant of the contact algebras. Thus, to define a mereotopological structures we will use contact algebras and the different ways to define/construct contact algebras. The following definition is the main definition of mereotopological structures.

Definition 6.

Let $(B, 0, 1, ., +, *, \mathsf{C})$ *be a contact algebra. Then we call the relational structure* $\underline{W} = (W, \leq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *a* (static) standard mereotopological structure *if* $W \neq \emptyset$, $W \subseteq B$, C *is the contact relation and* $\leq$, O *and* U *are binary relations over* W, *defined as follows*

$$x \leq y \quad \stackrel{def}{\longleftrightarrow} \quad x.y* = 0,$$

$$x \,\mathsf{O}\, y \quad \stackrel{def}{\longleftrightarrow} \quad x.y \neq 0,$$

$$x \,\mathsf{U}\, y \quad \stackrel{def}{\longleftrightarrow} \quad x + y \neq 1.$$

Since there are different ways to define contact algebras, thus we have different ways to define standard mereotopological structures. If we use the topological definition of contact algebras (see Definition 2), then we have mereotopological structures generated by topological spaces. If we use the definition of contact algebras through symmetric and reflexive frames (X, R) (see Definition 3) the we might use such frames to define mereotopological structures as well.

Lemma 3.

Let (X, R) *be a reflexive and symmetric relational structure and* $(\underline{B}, \mathsf{C})$ *be the contact algebra defined by it (see Lemma 2) and* $\underline{W} = (W, \leq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be the standard mereotopological structure defined by* $(\underline{B}, \mathsf{C})$. *Then for all* $x, y \in W$ *we have*

$$x \leq y \quad \stackrel{def}{\longleftrightarrow} \quad x \subseteq y,$$

$$x \,\mathsf{O}\, y \quad \stackrel{def}{\longleftrightarrow} \quad x \cap y \neq \emptyset,$$

$$x \,\mathsf{U}\, y \quad \stackrel{def}{\longleftrightarrow} \quad x \cup y \neq W.$$

Proof.

Since $\underline{B}$ is the Boolean algebra of the subsets of X then we have that $x \subseteq X$ and $y \subseteq X$ and in $\underline{B}$ the condition $x \subseteq y$ is equivalent to $x.y* = 0$ (this is the defining condition of $\leq$ in the above Definition 6). Thus, $x \leq y \longleftrightarrow x \subseteq y$ indeed. The same goes for the other two relations. □

The last lemma best presents the *point-based definitions* of relations $\leq$, O and U. I.e. this is the case when we interpret the regions as sets of points. In this setting, then the overlap, for example, is defined for x and y as the case when the two regions have at least one common point. The underlap means the there is at least one point outside of both regions.

In order to axiomatize (define with first-order formulae) the mereotopological structures we must find the conditions, that these structures satisfy. The following lemma deals with this.

Lemma 4.
Let $\underline{W}$ be a standard mereotopological structure. Then $\underline{W}$ satisfies the following first-order conditions

(M1) $$x \leq x$$

(M2) $$x \leq y \;\&\; y \leq z \Longrightarrow x \leq z$$

(M3) $$x \leq y \;\&\; y \leq x \Longrightarrow x = y$$

(M4) $$x \mathsf{O} y \Longrightarrow y \mathsf{O} x$$

(M5) $$x \mathsf{O} y \Longrightarrow x \mathsf{O} x$$

(M6) $$x \mathsf{O} y \;\&\; y \leq z \Longrightarrow x \mathsf{O} z$$

(M7) $$x \mathsf{O} x \text{ or } x \leq y$$

(M8) $$x \mathsf{U} y \Longrightarrow y \mathsf{U} x$$

(M9) $$x \mathsf{U} y \Longrightarrow x \mathsf{U} x$$

(M10) $$x \leq y \;\&\; y \mathsf{U} z \Longrightarrow x \mathsf{U} z$$

(M11) $$y \mathsf{U} y \text{ or } x \leq y$$

(M12) $$x \leq y \text{ or } x \mathsf{O} z \text{ or } y \mathsf{U} z$$

(M13) $$x \mathsf{O} x \text{ or } x \mathsf{U} x$$

(C1) $$x \mathsf{C} y \Longrightarrow y \mathsf{C} x$$

(C2) $$x \mathsf{O} y \Longrightarrow x \mathsf{C} y$$

(C3) $$x \mathsf{C} y \Longrightarrow x \mathsf{O} x$$

(C4) $$x \mathsf{C} y \;\&\; y \leq z \Longrightarrow x \mathsf{C} z$$

Thus, we may give the following general definition for mereotopological structures

Definition 7.
Let $\underline{W} = (W, \leq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ be a relational structure, such that $W \neq \emptyset$. Then we call $\underline{W}$ a (static) mereotopological structure *if it satisfies conditions* (M1) $-$ (M13), (C1) $-$ (C4).

Thus, we have that every standard mereotopological structure is a general mereotopological structure. The converse correspondence between the structures - i.e. that every general mereotopological structure can be represented as a standard one - follows from the representation theory, presented in Section 3.

In order to illustrate the use of (M1) $-$ (M13), (C1) $-$ (C4) we show one consequence from them, which is important later. This consequence will be used as a base for one of the axioms of the dynamic mereotopological structure - (C10) (see Definition 13).

Lemma 5.
The following statement holds in every mereotopological structure

$$z \mathsf{C} t \;\&\; x \overline{\mathsf{U}} y \;\&\; z \overline{\mathsf{O}} y \;\&\; t \overline{\mathsf{O}} x \Longrightarrow x \mathsf{C} y.$$

Proof.
Suppose $z \mathsf{C} t$, $x \overline{\mathsf{U}} y$, $z \overline{\mathsf{O}} y$ and $t \overline{\mathsf{O}} x$.

From $x \mathrel{\overline{\mathsf{U}}} y$ and $z \mathrel{\overline{\mathsf{O}}} y$, by (M12), we get $z \leq x$. From $x \mathrel{\overline{\mathsf{U}}} y$, by (M8), we get $y \mathrel{\overline{\mathsf{U}}} x$. Then, by (M12), from $y \mathrel{\overline{\mathsf{U}}} x$ and $t \mathrel{\overline{\mathsf{O}}} x$ we obtain $t \leq y$.

From $z \mathrel{\mathsf{C}} t$ and $t \leq y$, by (C4), it follows $z \mathrel{\mathsf{C}} y$. By (C1), from $z \mathrel{\mathsf{C}} y$ we have $y \mathrel{\mathsf{C}} z$. Then from $y \mathrel{\mathsf{C}} z$ and $z \leq x$, by (C4) again, we have $y \mathrel{\mathsf{C}} x$. Finally, by (C1), from $y \mathrel{\mathsf{C}} x$ it follows $x \mathrel{\mathsf{C}} y$. □

Duality between the overlap and the underlap.

If we review the Boolean definitions of the overlap and underlap relations from Definition 6

$$x \mathrel{\mathsf{O}} y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad x.y \neq 0 \text{ and}$$

$$x \mathrel{\mathsf{U}} y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad x + y \neq 1,$$

we see that the defining condition of the overlap $x.y \neq 0$ is dual to the definition of the underlap $x + y \neq 1$. This duality can be extended to more complex expressions. If we have a statement S, which is build using only atomic propositions with relations $\leq$, O and U, then we can obtain the dual statement S^d by performing the following tasks:

- replace in S each occurrence of O with U;
- replace in S each occurrence of U with O;
- swap all arguments of $\leq$.

For example if the statement S is

$$x \mathrel{\mathsf{O}} y \;\&\; ((x \leq z \;\&\; z \mathrel{\mathsf{U}} z) \vee (t \leq y \;\&\; t \mathrel{\mathsf{U}} y)),$$

then its dual statement S^d looks like

$$x \mathrel{\mathsf{U}} y \;\&\; ((z \leq x \;\&\; z \mathrel{\mathsf{O}} z) \vee (y \leq t \;\&\; t \mathrel{\mathsf{O}} y)).$$

Thus, we see that we have duality between some of the axioms from (M1) $-$ (M13), (C1) $-$ (C4). For instance, (M4) is dual to (M8), (M5) is dual to (M9), (M6) is dual to (M10) and also (M7) is dual to (M11) (renaming of the variables, when needed).

Thus, we have that this duality correspondence can be done not only for a single statement, but for whole proofs. At each step of the proof we replace the current statement with its dual and we replace the use of axioms (M4), (M5), (M6), (M7), (M8), (M9), (M10) or (M11) with their corresponding dual axiom. This allows us to state the following (informal) principle

Principle (Duality Principle)**.**

If a statement S, about relations $\leq$, O and U, is proved with a proof P, then its dual statement S^d is proved with the dual proof P^d.

This principle allows us to shorten and simplify a lot of proofs in the following sections and parts.

2. Mereological structures

Here we present definitions of structures, such that the relations from their language are purely mereological. In other words all of the relations can be defined only through means of Boolean algebras. Thus, as the mereotopological structures are viewed as a relational alternative to contact algebras, the following structures are a relational alternative to Boolean algebras.

Definition 8.

Let $(B, 0, 1, ., +, *)$ *be a Boolean algebra. Then we call the relational structure* $\underline{W} = (W, \leq, \mathsf{O}, \mathsf{U})$ *a* (static) standard mereological structure *if* $W \neq \emptyset$, $W \subseteq B$ *and* $\leq$, O *and* U *are binary relations over* W, *defined as follows*

$$x \leq y \quad \stackrel{def}{\longleftrightarrow} \quad x.y* = 0,$$

$$x \;\mathsf{O}\; y \quad \stackrel{def}{\longleftrightarrow} \quad x.y \neq 0,$$

$$x \;\mathsf{U}\; y \quad \stackrel{def}{\longleftrightarrow} \quad x + y \neq 1.$$

Since we use only Boolean algebras and we do not need the contact relation, the definition of these structures is more general to that of the mereotopological structures. Thus, we have that the standard mereological structures satisfy only these conditions from Definition 7, that do not concern the contact relation.

Definition 9.

Let $\underline{W} = (W, \leq, \mathsf{O}, \mathsf{U})$ *be a relational structure, such that* $W \neq \emptyset$. *Then we call* $\underline{W}$ *a* (static) mereological structure *if it satisfies conditions* (M1) − (M13).

As in the case of mereotopological structures, here we also have a two-way correspondence between the standard mereological structures and the general structures from the last definition. One direction of this correspondence follows from the fact that every standard mereological structure satisfies (M1) − (M13). The other direction follows from the representation theory in the next section. This correspondence is, in some sense, a relational interpretation of the Stone duality between Boolean algebras and algebras of subsets.

3. Representation theory

The representation theory of mereotopological and mereological structures is a generalization of Stone's technique for distributive lattices and Boolean algebras (see [1]).

For the mereotopological relations it shows that if we define the relations with the conditions from Definition 7 then we can equivalently represent them as defined by a contact algebra, which is generated by a reflexive and symmetric frame (as in Lemma 2). For this purpose, given a mereotopological structure, we must recreate the points from the domain of the frame and the reflexive and symmetric relation R between them. To achieve this we need to define relative variants of the standard set-theoretic notions of *(prime) filters* and *(prime) ideals*. Namely, the prime filters will serve as *abstract points of space*. In essence, an abstract point is the collection of all regions, that should contain this point in the standard definition of regions.

For the mereological relations - take a structure, which is defined as in Definition 9 and redefine the relations from the structure, using the Boolean algebra of the prime filters. Here it is not needed to extend the set of prime filters to a frame with a reflexive and symmetric relation R.

3.1. **Abstract space points.**

Definition 10.

Let $\underline{W}$ be a mereotopological or a mereological structure and let $F \subseteq W$.

- *F is called an* upper set *if for all $x, y \in W$, $x \in F$ and $x \leq y$ imply $y \in F$;*
- *F is called a* filter *if F is an upper set and for all $x, y \in W$, we have that $x \in F$ and $y \in F$ imply $x \mathsf{O} y$;*
- *F is called a* prime filter *if F is a filter and for all $x, y \in W$, we have that $x \notin F$ and $y \notin F$ imply $x \mathsf{U} y$.*

We will also call a prime filter an abstract space point *or just an* abstract point*.*
Let $I \subseteq W$. Then the dual notion of a (prime) ideal is defined as follows:

- *I is called a* lower set *if for all $x, y \in W$, $y \in I$ and $x \leq y$ imply $x \in I$;*
- *I is called an* ideal *if I is a lower set and for all $x, y \in W$, we have that $x \in I$ and $y \in I$ imply $x \mathsf{U} y$;*
- *I is called a* prime ideal *if I is an ideal and for all $x, y \in W$, we have that $x \notin I$ and $y \notin I$ imply $x \mathsf{O} y$.*

Notation.

The collection of the abstract points for a structure $\underline{W}$ will be denoted with $AP(\underline{W})$.

The above definition of (prime) filters and (prime) ideals is used in the representation theory for both the mereotopological and mereological relations. Thus, regions in both mereotopological and mereological systems will be sets of prime filters (i.e. abstract points). The next definition, however, is used only for the mereotopological relations. It is used to recreate the contact relations between regions (i.e. between sets of abstract points).

Definition 11.

Let $\underline{W}$ be a mereotopological structure. We define the relation R over the filters of this structure as follows

$$F \mathsf{R} G \quad \stackrel{def}{\longleftrightarrow} \quad \forall x \in F, \forall y \in G, x \mathsf{C} y.$$

This definition is different from the one given in [35]. The difference is that here R is defined only with C, while in [35] the definition involves also the relations $\widehat{\mathsf{C}}$ and $\ll$. Still, R is a reflexive and symmetric relation. The symmetry follows from the symmetry of C. The reflexivity is checked as follows: take an arbitrary filter F and $x, y \in F$; then from the definition of a filter we have that $x \mathsf{O} y$ and so, by (C2), $x \mathsf{C} y$.

R is a relation over the filters and, ultimately, it is a relation over the prime filters (i.e. the abstract points). Thus $(AP(\underline{W}), \mathsf{R})$ is the needed reflexive and symmetric relational system for the representation.

3.2. **Static characterization.**

Proposition 1 (Static Characterization)**.**

Let $\underline{W}$ be a mereotopological or a mereological structure and $x, y \in W$. Then for relations $\leq$, O and U hold the following statements:

$$(\leq) \qquad x \leq y \quad \longleftrightarrow \quad \forall F \in AP(\underline{W}),\ x \in F \Longrightarrow y \in F;$$

$$(\mathsf{O}) \qquad x \mathsf{O} y \quad \longleftrightarrow \quad \exists F \in AP(\underline{W}),\ x \in F \ \&\ y \in F;$$

$$(\mathsf{U}) \qquad x \mathsf{U} y \quad \longleftrightarrow \quad \exists F \in AP(\underline{W}),\ x \notin F \ \&\ y \notin F.$$

Also if $\underline{W}$ is a mereotopological structure, then for the contact relation C we have

$$(\mathsf{C}) \qquad x \mathsf{C} y \quad \longleftrightarrow \quad \exists F, G \in AP(\underline{W}),\ x \in F \ \&\ y \in G \ \&\ F \mathsf{R} G.$$

Here we see how the point-based definitions of the relations (see Lemma 3) are translated into the terms of abstract points. For example, if x and y were sets of points, the $x \leq y$ holds iff every point from x is also in y, i.e. for every point t we have $t \in x \Longrightarrow t \in y$. Since the abstract points are just the sets of regions, which should contain this point, then the statement $x \in F$ actually means that *the abstract point F is in the region x*. Thus, in this setting $x \in F \Longrightarrow y \in F$ means that *every abstract point F from x is also in y*. I.e. x is part-of y. Similarly, the characterization of the overlap relation states that x and y are in overlap iff they share a common abstract point. For the underlap we have that there must be at least one point, that is outside both x and y. When we characterize the contact we have that the statement $x \in F \ \&\ y \in G \ \&\ F \mathsf{R} G$ means that there is an abstract point F in x and a point G in y and F and G are "close enough", so that x and y are in contact.

This **Static Characterization** is proved by constructing specific prime filters (abstract points) for the different cases. Here we introduce some notations that allow us to build filters and ideals.

Notation.

Let $\underline{W}$ be a mereotopological or a mereological structure and $x \in W$. Then we introduce the following notations

$$[x) = \{\ y \mid y \in W \ \textit{and}\ x \leq y\ \}$$
$$(x] = \{\ y \mid y \in W \ \textit{and}\ y \leq x\ \}$$

$[x)$ is the least upper set containing x and $(x]$ is the least lower set containing x. These properties are proved in [35], as well as the following ones

Lemma 6.
If $\underline{W}$ is a mereotopological or a mereological structure and $x, y \in W$ then

(1) $x \mathsf{O} y \quad \longleftrightarrow \quad [x) \cup [y)$ *is a filter;*
(2) $x \mathsf{U} y \quad \longleftrightarrow \quad (x] \cup (y]$ *is an ideal;*
(3) $x \mathsf{C} y \quad \longleftrightarrow \quad [x) \mathsf{R} [y)$.

If G is an upper set and J is a lower set then

(4) $G \cap J \neq \emptyset \quad \longleftrightarrow \quad \exists x \in G, \exists y \in J, x \leq y$.

If F is a filter and I is an ideal then

(5) $F \cup [x)$ *is a filter* $\quad \longleftrightarrow \quad x \mathsf{O} x$ *and* $\forall y \in F, x \mathsf{O} y$;
(6) $I \cup (x]$ *is an ideal* $\quad \longleftrightarrow \quad x \mathsf{U} x$ *and* $\forall y \in I, x \mathsf{U} y$;
(7) *If $F \cup I = W$ then F is also a prime filter (an abstract point) and I is also a prime ideal;*
(8) *If $\{ A_i \}$ is a non-empty chain of filters (ideals), linearly ordered by set-inclusion, then $\bigcup A_i$ is also a filter (ideal).*

With these constructions we can build specific filters and ideals. But for the characterization we need prime filters (and prime ideals). So here follow a couple of lemmas (**Separation Lemma** and **R-extension Lemma**), that allow us to expand filters and ideals to their prime counterparts. The **Separation Lemma** is used in the characterizations of both mereotopological and mereological relations, while the **R-extension Lemma** is used only for the mereotopological case. The **Separation Lemma** is already proved in [35]. The **R-extension Lemma** is also proved in [35] under the name *ρ-extension Lemma.* But because of the difference in the definitions of R we will have to repeat the proof with the new definition, to make sure that the lemma still holds.

Lemma 7 (Separation Lemma)**.**
Let $\underline{W}$ be a mereotopological or a mereological structure and F' and I' be a filter and an ideal, such that $F' \cap I' = \emptyset$. Then there exist a prime filter (an abstract point) F and a prime ideal I so that

$$F' \subseteq F,\ I' \subseteq I \text{ and } F \cap I = \emptyset.$$

The lemma, given below, is required for the proof of the **R-extension Lemma**.

Lemma 8.
Let $\underline{W}$ be a mereotopological structure and F' and G' be filters and I' and J' be ideals, such that $F' \mathsf{R} G', F' \cap I' = \emptyset$ and $G' \cap J' = \emptyset$. Let x be an element from structure such that $x \notin F'$ and $x \notin I'$. Then at least one of the following two conditions hold:

(a) $\quad F' \cup [x)$ *is a filter,* $(F' \cup [x)) \cap I' = \emptyset$ *and* $(F' \cup [x)) \mathsf{R} G'$

(b) $\quad I' \cup (x]$ *is an ideal and* $F \cap (I' \cup (x]) = \emptyset$

Proof.
Since $x \notin F'$ then for every $y \in F'$ we have $y \nleq x$. Similarly $x \nleq y$ for every $y \in I'$. Then, by Lemma 6 (4), it follows that both $F \cap (x] = \emptyset$ and $[x) \cap I' = \emptyset$ hold. Lets consider several cases:

1) Assume that $x \mathsf{U} x$ and for every $y \in I', x \mathsf{U} y$. Then, by Lemma 6 (6), $I' \cup (x]$ is an ideal and so (b) holds.

If the conditions from the first case are not true, then we have two possibilities: that $x \overline{\mathsf{U}}\, x$ or that there is $y \in I', x \overline{\mathsf{U}}\, y$.

2) $x \overline{\mathsf{U}}\, x$ holds. Then by (M13) $x \mathsf{O} x$. Take arbitrary $z \in F'$. Then $z \mathsf{O} z$. Since $x \overline{\mathsf{U}}\, x$, then by (M11) we get $z \leq x$. Thus from $z \mathsf{O} z$ and $z \leq x$, by (M6), we have $z \mathsf{O} x$. So, by Lemma 6 (5), $F' \cup [x)$ is a filter.
The same reasoning shows that for every $t \in G'$, $t \mathsf{O} x$ and so, by (C2), $t \mathsf{C} x$. Take $v \in [x)$ and $t \in G'$. Then, by (C4), we have $t \mathsf{C} v$ and so $G' \mathsf{R} [x)$. Thus $(F' \cup [x)) \mathsf{R} G'$ and so (a) holds.
3) There is $y \in I'$, such that $x \overline{\mathsf{U}}\, y$. This case is similar to the previous.
$y \in I'$ so $x \nleq y$ and, by (M7), $x \mathsf{O} x$. If $z \in F'$ then, because $F' \cap I' = \emptyset$ and by Lemma 6 (4), $z \nleq y$. Thus from $z \nleq y$ and $x \overline{\mathsf{U}}\, y$, by (M12), we get $z \mathsf{O} x$. Analogously if $t \in G'$ then $t \mathsf{O} x$ and $t \mathsf{C} x$. The rest is the same as in the previous case and finally we have that (a) holds.

□

Lemma 9 (R-extension Lemma)**.**
Let $\underline{W}$ be a mereotopological structure and F' and G' be filters and I' and J' be ideals, such that $F' \mathsf{R} G', F' \cap I' = \emptyset$ and $G' \cap J' = \emptyset$. Then there exist prime filters (abstract points) F and G and prime ideals I and J so that (for the definition of R see Definition 11)

$$F' \subseteq F,\ G' \subseteq G,\ I' \subseteq I,\ J' \subseteq J, F \mathsf{R} G, F \cap I = \emptyset \text{ and } G \cap J = \emptyset.$$

Proof.
Consider the following set of pairs of filters and ideals

$$P = \{\ (F'', I'')\ |F' \subseteq F'', I' \subseteq I'', F'' \cap I'' \neq \emptyset \text{ and } F'' \mathsf{R} G\ \}.$$

If we take a chain of pairs (F_i, I_i) from P, ordered by the condition $F_i \subseteq F_j$ and $I_i \subseteq I_j$ for $i < j$, then, by Lemma 6 (8), the chain has an upper bound in P. Then, by the Zorn's lemma, there is a maximal pair (F, I). If $F \cup I \neq W$ then, by Lemma 8, (F, I) can be extended and is not maximal. So $F \cup I = W$ and, by Lemma 6 (7), F and I are prime filter and ideal.
Because R is symmetric we can swap F with G' and I with J' and repeat the construction to obtain G and J. Then swap again to achieve the final result. □

In order to be able to use these lemmas, first, we have to prepare pairs of non-intersecting filters and ideals. Then we can expand them with either the **Separation Lemma** or the **R-extension Lemma**. We will illustrate the use of this procedure in the proof of the next lemma.

Lemma 10.
Let $\underline{W}$ be a mereotopological or a mereological structure and $x, y \in W$. Then

(1) if $x \nleq y$ then there exists a prime filter (denoted by) $F(x \nleq y)$, containing x but not y;
(2) if $x \mathsf{O} y$ then there exists a prime filter (denoted by $F(x \mathsf{O} y)$) containing x and y;
(3) if $x \mathsf{U} y$ then there exists a prime filter (denoted by $F(x \mathsf{U} y)$) containing neither x nor y;

If $\underline{W}$ is a mereotopological structure then

(4) if $x \mathsf{C} y$ then there exists a pair of prime filters $\langle F, G\rangle$ such that $F \mathsf{R} G$, $x \in F$ and $y \in G$ (notation $\langle F, G\rangle(x \mathsf{C} y)$).

Proof.

(1) We set $F' = [x)$ and $I' = (y]$. Since $x \nleq y$, then by (M7) and (M11) $x \mathsf{O} x$ and $y \mathsf{U} y$. Then by Lemma 6 (1) $[x)$ is a filter and by Lemma 6 (2) $(y]$ is an ideal. Also, by Lemma 6 (4) we have $[x) \cap (y] = \emptyset$. Then we expand F' and I' with the **Separation Lemma** to F and I, having $F \cap I = \emptyset$. Thus, $x \in F' \subseteq F$ and since $y \in I' \subseteq I$ then $y \notin F$. We take F to be $F(x \nleq y)$.
(2) We set $F' = [x) \cup [y)$ and $I' = \emptyset$ and proceed as in the previous case.
(3) Set $F' = \emptyset$ and $I' = (x] \cup (y]$.
(4) Set $F' = [x)$, $G' = [y)$ and $I' = J' = \emptyset$ and use the **R-extension Lemma** to expand F' and G' to F and G. This is the pair $\langle F, G\rangle(x \mathsf{C} y)$.

□

This lemma is actually half of the proof of the **Static Characterization**. Each of the statements there is an equivalence and one of the directions follows from the definition of prime filter of the definition of R, while the other direction follows from Lemma 10.

Thus, we have the following correspondence theorem (proved in [35]) between the mereotopological structures from Definition 7 and the mereotopological structures defined with contact algebras (see Definition 6)

Theorem 10.
For every mereotopological structure $\underline{W}$ there is a contact algebra $(\underline{B}, \mathsf{C})$ and an isomorphic embedding h from $\underline{W}$ to $(\underline{B}, \mathsf{C})$. I.e. h is an injective function from W (the domain of $\underline{W}$) to B (the domain of $(\underline{B}, \mathsf{C})$), which satisfies:

$$\begin{aligned} x \mathsf{C} y &\quad\longleftrightarrow\quad h(x) \mathsf{C} h(y),\\ x \leq y &\quad\longleftrightarrow\quad h(x).h(y)^* = 0,\\ x \mathsf{O} y &\quad\longleftrightarrow\quad h(x).h(y) \neq 0,\\ x \mathsf{U} y &\quad\longleftrightarrow\quad h(x) + h(y) \neq 1. \end{aligned}$$

The key step in the proof of this theorem is the use of the **Static Characterization**. A variant of this theorems states the correspondence between mereological structures and Boolean algebras.

If in the above correspondence theorem we use topological contact algebras (from Definition 2), then we get the following

Theorem 11.
For every mereotopological structure $\underline{W}$ there is a topological *contact algebra $(\underline{B}, \mathsf{C})$ and an isomorphic embedding h from $\underline{W}$ to $(\underline{B}, \mathsf{C})$.*

This theorem shows that the relations from Definition 7 have topological nature indeed. Thus, the application of the term *mereotopological* to them is justified.

3.3. **Notations for specific abstract points.**

In Lemma 10 we see a certain style of denoting the properties of specific abstract points (prime filters) or pairs of abstract points in brackets after them. For instance, in the lemma we have that

- $F(x \nleq y)$ denotes that $x \in F$ & $y \notin F$;
- $F(x \mathsf{O} y)$ denotes that $x \in F$ & $y \in F$;
- $F(x \mathsf{U} y)$ denotes that $x \notin F$ & $y \notin F$;
- $\langle F, G\rangle(x \mathsf{C} y)$ denotes that $x \in F$ & $y \in G$ & $F \mathsf{R} G$.

In these cases the condition, written in the brackets, is actually the condition that guarantees the existence of the abstract point (the pair of abstract points). We will also occasionally use notations for the other possible literal expressions. For example

- $F(x \leq y)$ denotes that $x \in F \Longrightarrow y \notin F$;
- $F(x\ \overline{\mathsf{O}}\ y)$ denotes that $x \notin F \vee y \notin F$;
- $F(x\ \overline{\mathsf{U}}\ y)$ denotes that $x \in F \vee y \in F$;
- $\langle F, G\rangle(x\ \overline{\mathsf{C}}\ y)$ denotes that $x \notin F \vee y \notin G \vee F\ \overline{\mathsf{R}}\ G$.

We may also use multiple conditions in the brackets, to specify that the abstract points have more properties. For example $F(x\ \mathsf{O}\ y, z\ \overline{\mathsf{U}}\ t)$ denotes that $x \in F$ and $y \in F$ (because of $x\ \mathsf{O}\ y$) and also it is the case that $z \in F$ or $t \in F$ (because $z\ \overline{\mathsf{U}}\ t$).

♣ ♣ ♣

Part IV. Dynamic relational mereotopology and mereology

In this part we define the languages of dynamic mereotopological relations and dynamic mereological relations. We study the expressive power of these languages and define relational structures about them. In this part we also present the main result in this dissertation - the representation theory for dynamic mereotopological and mereological relations. This representation theory is instrumental in the completeness proofs for logics for dynamic mereotopological and mereological relations, given in the next part (Part V).

This part is structured in six sections. In the first section we give a formal definition of the stable and unstable mereotopological and mereological relations. Then we study the two languages with respect to expressive power and how they relate to other spatio-temporal languages from Part II.

In the second and third sections we develop the dynamic relational mereotopology. The scheme here is the same, as it was in the static mereotopology. First, we give a standard (point-based) definition of structures with dynamic mereotopological relations. Then we show that they satisfy certain first-order conditions/axioms and through them we give a general (point-free) definition. This is the content of Section 2. Finally, in Section 3 we develop a Stone-like representation theory for the correspondence between these two types of structures. This is a major extension of the static representation theory as we have to recreate in a point-free structure not only the space points, but the time moments as well.

The following two sections deal with the mereological variant of the dynamic mereotopology. As in the static case, here we have a separate language and definitions of mereological structures. This is done in Section 4. Section 5 presents a specialized representation theory for the dynamic mereological structures, which is in some sense better suited for them, than the representation theory for the previous language from Section 3.

The last section states some of the open problems in the development of dynamic relational mereotopology and mereology. These open problems are divided into two major themes. The first one is extending the languages with more spatial relations and more temporal constructions, in order to achieve better expressiveness and applicability. The second theme is about improvement of the representation technique, so that it could cope with the extended languages and the growing complexity of the dependencies between the relations.

1. Dynamic mereotopological and mereological relations

Having established the language, structures and the theory of static mereotopology and mereology in Part III, now we have a base to give a formal definition of the stable and unstable mereological and mereotopological relations. The definition follows the idea from Part I - to use static structures to represent snapshots of the space at each moment of time. Then we use a collection of such snapshots. This will be considered as a history of the space and through such histories we will work with dynamic spaces. We will develop the same notion of a history of a region, in order to work with dynamic regions.

Lets have a collection of static mereotopological structures which are denoted $\underline{W}_i = (W_i, \leq_i, \mathsf{O}_i, \mathsf{U}_i, \mathsf{C}_i)$ (see Definition 7). Each structure will be a snapshot of a given dynamic space at a certain moment of time. Let I be the set of indices of the structures $\underline{W}_i$. Thus, I may be considered as the set of all moments of time, at which we observe the dynamic space. Thus, if we have a region x and we wish to observe its behavior at moment of time i, then we will have to look up the state of this region in the corresponding mereotopological structure $\underline{W}_i$. This state will be denoted by x_i. So similar to the history of the space, which is a collection of static snapshots, we will consider histories of dynamic regions. These histories will be just vectors of their states at each moment of time (i.e. in each static structure):

$$\text{the history of } x = \{\ x_i \mid i \in I, x_i \in W_i\ \}.$$

Thus, we will work with such histories (vectors from $\prod_{i \in I} W_i$) and we will define the dynamic mereotopological and mereological relations over them. The definition is the same as in for the standard dynamic contact algebras from Definition 4 and realized the idea of Part I, to apply S5 modality to a static relation

$$x \ \mathsf{c}\ y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Box(x \ \mathsf{C}\ y) \quad \longleftrightarrow \quad \forall i \in I, x_i \ \mathsf{C}_i\ y_i,$$

$$x \ \mathsf{C}\ y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Diamond(x \ \mathsf{C}\ y) \quad \longleftrightarrow \quad \exists i \in I, x_i \ \mathsf{C}_i\ y_i.$$

Here $\Box$ and $\Diamond$ denote the S5 box and diamond modal operators. The *stable contact* is denoted by c and the *unstable contact* is denoted by C. They correspond to $\mathsf{C}^{\forall}$ and $\mathsf{C}^{\exists}$ from Part I. Note that for the unstable contact we use the same symbol, that is used for the static contact. This will be done for some of the other dynamic relations too. The reasons for that are discussed in the next section.

We will apply the same construction to get dynamic variants of the mereological relations part-of $\leq$, overlap O and underlap U. The *stable part-of* (denoted by the same symbol as the static part-of $\leq$) and the *unstable part-of* $\preceq$ are defined as follows:

$$x \leq y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Box(x \leq y) \quad \longleftrightarrow \quad \forall i \in I, x_i \leq_i y_i,$$

$$x \preceq y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Diamond(x \leq y) \quad \longleftrightarrow \quad \exists i \in I, x_i \leq_i y_i.$$

For the dynamic variants of the overlap we have

$$x \ \mathsf{o}\ y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Box(x \ \mathsf{O}\ y) \quad \longleftrightarrow \quad \forall i \in I, x_i \ \mathsf{O}_i\ y_i,$$

$$x \ \mathsf{O}\ y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Diamond(x \ \mathsf{O}\ y) \quad \longleftrightarrow \quad \exists i \in I, x_i \ \mathsf{O}_i\ y_i,$$

where the *stable overlap* is denoted by o, while the *unstable overlap* shares the same notation as its static counterpart O.

Finally, the *stable underlap* u and the *unstable underlap* U (denoted as the static underlap) are defined:

$$x \mathrel{\mathsf{u}} y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Box(x \mathrel{\mathsf{U}} y) \quad \longleftrightarrow \quad \forall i \in I, x_i \mathrel{\mathsf{U}_i} y_i,$$

$$x \mathrel{\mathsf{U}} y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \Diamond(x \mathrel{\mathsf{U}} y) \quad \longleftrightarrow \quad \exists i \in I, x_i \mathrel{\mathsf{U}_i} y_i.$$

Thus, we will work with two main languages:

- the language of *stable and unstable mereotopological relations*, which consists of $\leq$, o, u, c, $\preceq$, O, U and C;
- the language of *stable and unstable mereological relations*, which consists of $\leq$, o, u, $\preceq$, O and U.

The first language may also be denoted by DMt, for short, and the second language be denoted by DM. If need be, the language of the static mereotopological relations may be denoted by Mt, while the language of the static mereological relations will be just M. We have the following inclusions, with respect to expressive power, for these languages:

$$\mathrm{M} \subseteq \mathrm{Mt}, \mathrm{DM} \subseteq \mathrm{DMt}.$$

Here, for example, $\mathrm{M} \subseteq \mathrm{Mt}$ means, that the statements, that can be expressed with language M, are a subset of these, that can be expressed with Mt.

Since there is a distinct difference between the overlap and the contact (see Lemmas 1 and 3), thus we have that (the expressive power of) the language of static mereology is clearly a proper sublanguage of the static mereotopology. As a corollary, we have the same result for the dynamic languages. We will denote this in the following way

$$\mathrm{M} \subset \mathrm{Mt}, \mathrm{DM} \subset \mathrm{DMt}.$$

The next two subsections are dedicated to the study of the expressive power of DMt and DM. For this purpose we will compare these languages with some of the popular languages, considered in Part II.

1.1. **Comparison with the dynamic contact algebras.**

The dynamic contact algebras were introduced in Part II, Subsection 3.3. We may consider the languages of DMt and DM to be a relational variant of dynamic contact algebras. This is due to the fact that the static base for the dynamic relations are the mereotopological structures, which are relational variants of the contact algebras, which are the static base structures used in the definition of dynamic contact algebras (Definition 4).

Thus, first we will study the expressive power of the static languages M and Mt, with respect to the language of (static) contact algebras. The latter language will be denoted by CA. Since the mereotopological structures are standardly defined with contact algebras (see Definition 6) we have

$$\mathrm{M} \subset \mathrm{Mt} \subseteq \mathrm{CA}.$$

To check the converse inclusion, we will see which operations from the language of CA are definable in the language of Mt and which are not. The contact relations C is present in both languages. For the two constants 0 and 1 and for the complement operation * we have

Lemma 11.

Let $\underline{W} = (W, \leq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a static mereotopological structure. Let* 0 *and* 1 *denote the empty region and the region* W*, respectively. Let* x^* *denotes the complement of* x *with respect to* W*. Then if the structure satisfies the following conditions*

$$\exists x(x \;\mathsf{O}\; x),$$
$$\exists x(x \;\mathsf{U}\; x),$$
$$\forall x \exists y \neg(x \;\mathsf{O}\; y) \;\&\; \neg(x \;\mathsf{U}\; y),$$

we have that the constants 0 *and* 1 *and the complement operation* * *are defined as follows*

$$\begin{aligned} x = 0 &\quad \overset{def}{\longleftrightarrow} \quad \neg(x \;\mathsf{O}\; x), \\ x = 1 &\quad \overset{def}{\longleftrightarrow} \quad \neg(x \;\mathsf{U}\; x), \\ x = y^* &\quad \overset{def}{\longleftrightarrow} \quad \neg(x \;\mathsf{O}\; y) \;\&\; \neg(x \;\mathsf{U}\; y). \end{aligned}$$

Proof.

These definability statements follow from the definitions of O and U from Definition 6. For instance, $\neg(x \;\mathsf{O}\; x)$ iff $x.x = 0$ iff $x = 0$. This guarantees that this x is a unique element from W. The same goes for $x = 1$. Also $\neg(x \;\mathsf{O}\; y) \;\&\; \neg(x \;\mathsf{U}\; y)$ is equivalent to $x.y = 0 \;\&\; x + y = 1$, which is equivalent to $x = y^*$. □

Thus, the Boolean constants 0 and 1 and the complement operation * are definable only in some special mereotopological structures. For instance, if the structure does not satisfy $\exists x(x \;\mathsf{O}\; x)$, which is possible since W can be an arbitrary subset of the domain of a contact algebra (see Definition 6), then there is no element present in W, for which the definition of 0 holds. Thus, 0, 1 and * are only *partially* definable.

The rest of the language of CA is not definable in Mt, also. We will show that the Boolean meet . is not definable. The undefinability of the join operator + will follow from the fact that the complement is (partially) definable.

Lemma 12.

The Boolean meet . is not definable in Mt.

Proof.

We will show that there is a static mereotopological structure and an automorphism on it, that does not preserve the meet operation . over the regions.

Let (X, R) be a frame, where $X = \{\ a, b, c, d, e\ \}$ and R is the universal relation over X. Thus, (X, R) is a reflexive and symmetric frame. Assuming that the elements of X are points, lets have the following regions

$$\begin{aligned} x &= \{\ a, b, c\ \}, \\ y &= \{\ a, d\ \}, \\ z &= \{\ a\ \}, \\ t &= \{\ a, b, e\ \}. \end{aligned}$$

Now let $\underline{W} = (W, \leq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ be the mereotopological structure, where $W = \{\ x, y, z, t\ \}$ and $\leq$, O, U and C are defined as in Lemma 3. It is easy to check that O ,U and C are universal relations over W, while for $\leq$ we have that $u \leq w$ iff $u = w$ or $u = z$.

Consider the bijection h over W, such that it swaps y and t

$$h(x) = x,\ h(y) = t,\ h(z) = z,\ h(t) = y.$$

We have that h preserves the relations $\leq$, O, U and C, i.e.

$$\begin{array}{lcl} x \leq y & \longleftrightarrow & h(x) \leq h(y), \\ x \mathsf{O} y & \longleftrightarrow & h(x) \mathsf{O} h(y), \\ x \mathsf{U} y & \longleftrightarrow & h(x) \mathsf{U} h(y), \\ x \mathsf{C} y & \longleftrightarrow & h(x) \mathsf{C} h(y), \end{array}$$

and so h is an automorphism on $\underline{W}$.

Finally, we have that $x \cap y = z$, but $h(x) \cap h(y) \neq h(z)$. Since the intersection operation over sets corresponds to the meet operator over point-based regions, then we have $x.y = z$ and $h(x).h(y) \neq h(z)$, and therefore the meet . is not definable. $\Box$

Thus, as a result we have the following

Corollary 1.

The languages of static mereotopological and mereological relations are proper sublanguages of the language of contact algebras. I.e.

$$M \subset Mt \subset CA.$$

Now we will compare the dynamic relational languages with the dynamic contact algebras. The language of the dynamic contact algebras will be denoted by DCA. Relations c and C are part of both DMt and DCA (in DCA they are denoted by $\mathsf{C}^\forall$ and $\mathsf{C}^\exists$). For the rest of the relations from DMt we have

Lemma 13.

Relations $\leq$, o ,u, $\preceq$, O and U are definable in DCA. This is done with the following formulae (these formulae can also be found in [47] *in a equivalent form)*

$$\begin{array}{rcl} x \leq y & \stackrel{def}{\longleftrightarrow} & x.y^* = 0, \\ x \mathsf{o} y & \stackrel{def}{\longleftrightarrow} & x.y \mathsf{C}^\forall x.y, \\ x \mathsf{u} y & \stackrel{def}{\longleftrightarrow} & x^*.y^* \mathsf{C}^\forall x^*.y^*, \\ x \preceq y & \stackrel{def}{\longleftrightarrow} & \neg(x.y^* \mathsf{C}^\forall x.y^*), \\ x \mathsf{O} y & \stackrel{def}{\longleftrightarrow} & \neg(x.y = 0), \\ x \mathsf{U} y & \stackrel{def}{\longleftrightarrow} & \neg(x + y = 1). \end{array}$$

Proof.

We will prove the definability for the case of the stable overlap relation o. The remaining proofs are similar.

x and y are dynamic regions from a dynamic contact algebra. So they are vectors of the states of the regions in the static contact algebras, corresponding to the moments of time (see Definition 4). Thus, $x.y$ is also a dynamic region, that corresponds to the vector $\{\ x_i.y_i\ \}$. So $x.y \mathsf{C}^\forall x.y$ is equivalent to $x_i.y_i \mathsf{C}_i x_i.y_i$ for every moment of time i. By (C3), this is equivalent to $x_i.y_i \mathsf{O}_i x_i.y_i$ and thus we have that $x_i.y_i \neq 0$ for every moment of time i. By the definition of the static overlap, we have that this is equivalent to $x_i \mathsf{O}_i y_i$ for every i. This is exactly the definition of $x \mathsf{o} y$. $\Box$

As a result we have the following inclusion

$$\mathrm{DM} \subset \mathrm{DMt} \subseteq \mathrm{DCA}.$$

Regarding the definability of the symbols from DCA in DMt we have the following statement

Lemma 14.
The constants 0 *and* 1 *and the complement operation* * *are (partially) definable in DMt. The meet* . *and join* $+$ *operations are not definable.*

Proof.
For the definability of 0, 1 and * we have the same formulae as in the static case:

$$\begin{aligned} x = 0 &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad \neg(x \ \mathsf{O}\ x), \\ x = 1 &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad \neg(x \ \mathsf{U}\ x), \\ x = y^* &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad \neg(x \ \mathsf{O}\ y)\ \&\ \neg(x \ \mathsf{U}\ y). \end{aligned}$$

Again, these definitions only apply to dynamic structures that satisfy these conditions:

$$\begin{gathered} \exists x(x \ \mathsf{O}\ x), \\ \exists x(x \ \mathsf{U}\ x), \\ \forall x \exists y \neg(x \ \mathsf{O}\ y)\ \&\ \neg(x \ \mathsf{U}\ y). \end{gathered}$$

For the proof of undefinability of . and $+$ we use the automorphism from the static case. We use a collection of a single static mereotopological structure, to define the dynamic relations (in this case we have that the stable part-of coincides with the unstable part-of, the stable overlap coincides with the unstable one and so on ...). Then we use the automorphism h from Lemma 12, and thus we prove the undefinability of . and $+$ operations. $\square$

Finally, we have the end result for the comparison between the dynamic languages.

Corollary 2.
The languages of dynamic mereotopological and mereological relations are proper sublanguages of the language of dynamic contact algebras. I.e.

$$\mathit{DM} \subset \mathit{DMt} \subset \mathit{DCA}.$$

1.2. **Comparison with the temporalized theory of static mereotopology $\mathcal{TL}_{\mathcal{SU}}$(Mt).**
Consider the theory of static mereotopology with language Mt and its temporalized variant $\mathcal{TL}_{\mathcal{SU}}$(Mt) (see Part II, Subsection 3.2). Since the S5 modality is expressible in the language of $\mathcal{TL}_{\mathcal{SU}}$ (see Part II, Section 2) with the formulae for $\Box_A$ and $\Diamond_A$, then the stable and unstable relations can be defined in $\mathcal{TL}_{\mathcal{SU}}$(Mt)

with the following formulae

$$
\begin{aligned}
x \le t &\stackrel{\text{def}}{\longleftrightarrow} \Box(x \le t) \stackrel{\text{def}}{\longleftrightarrow} \Box_A(x \le y),\\
x \,\mathsf{o}\, t &\stackrel{\text{def}}{\longleftrightarrow} \Box(x \,\mathsf{O}\, t) \stackrel{\text{def}}{\longleftrightarrow} \Box_A(x \,\mathsf{O}\, y),\\
x \,\mathsf{u}\, t &\stackrel{\text{def}}{\longleftrightarrow} \Box(x \,\mathsf{U}\, t) \stackrel{\text{def}}{\longleftrightarrow} \Box_A(x \,\mathsf{U}\, y),\\
x \,\mathsf{c}\, t &\stackrel{\text{def}}{\longleftrightarrow} \Box(x \,\mathsf{C}\, t) \stackrel{\text{def}}{\longleftrightarrow} \Box_A(x \,\mathsf{C}\, y),\\
x \preceq t &\stackrel{\text{def}}{\longleftrightarrow} \Diamond(x \le t) \stackrel{\text{def}}{\longleftrightarrow} \Diamond_A(x \le y),\\
x \,\mathsf{O}\, t &\stackrel{\text{def}}{\longleftrightarrow} \Diamond(x \,\mathsf{O}\, t) \stackrel{\text{def}}{\longleftrightarrow} \Diamond_A(x \,\mathsf{O}\, y),\\
x \,\mathsf{U}\, t &\stackrel{\text{def}}{\longleftrightarrow} \Diamond(x \,\mathsf{U}\, t) \stackrel{\text{def}}{\longleftrightarrow} \Diamond_A(x \,\mathsf{U}\, y),\\
x \,\mathsf{C}\, t &\stackrel{\text{def}}{\longleftrightarrow} \Diamond(x \,\mathsf{C}\, t) \stackrel{\text{def}}{\longleftrightarrow} \Diamond_A(x \,\mathsf{C}\, y).
\end{aligned}
$$

Since $\mathcal{S}$ and until $\mathcal{U}$ operators are not expressible with S5 modality. Thus, we have the following result

Proposition 2.

The languages of dynamic mereotopological and mereological relations are proper sublanguages of the temporalized static mereotopology, which in its turn is a proper sublanguage of ST_0 (since $ST_0 = \mathcal{TL}_{\mathcal{SU}}$(BRCC-8); see Part II, Section 2).

$$DM \subset DMt \subset \mathcal{TL}_{\mathcal{SU}}(Mt) \subset ST_0.$$

1.3. **Regarding the fusion of S5 and static mereotopology.**

Since the dynamic relations are defined with application of S5 modality to the static relations, we might be tempted to relate the dynamic relational mereotopology and mereology to the fusion of the S5 modal logic and the modal logic of the static mereotopological relations from [35]. Fusions, however, are not applicable to the Kripke structures for the modal logic of mereotopological relations - the static mereotopological structures. Thus, we cannot use fusions to obtain results for the dynamic theories.

To prove this, we will show that one of the requirements for fusions is not satisfied - the closure under disjoint unions (see Part II, Subsection 3.1).

Lemma 15.

The static mereotopological and mereological structures are not closed under disjoint unions.

Proof.

Take the following two static mereotopological structures $\underline{W}_1 = (W_1, \le_1, \mathsf{O}_1, \mathsf{U}_1, \mathsf{C}_1)$ and $\underline{W}_2 = (W_2, \le_2, \mathsf{O}_2, \mathsf{U}_2, \mathsf{C}_2)$, where

$W_1 = \{\ x\ \}$, such that $x\ \mathsf{O}_1\ x$ does not hold;

$W_2 = \{\ y\ \}$, such that y is a distinct object from x.

Then, when we take $\underline{W}$ to be the disjoint union of $\underline{W}_1$ and $\underline{W}_2$, we have that in $\underline{W}$, $x \le y$ does not hold (because none of $\underline{W}_1$ and $\underline{W}_2$ has any combined information for both x and y). This is a contradiction, however, with (M7) for $\underline{W}$, since $x\ \mathsf{O}\ x$ does not hold in $\underline{W}$ either. Thus, $\underline{W}$ does not satisfy (M7) and therefore is not a static mereotopological structure.

The case for the mereological structures is proved in the same way. $\Box$

2. Dynamic mereotopological structures

Similar to the case of the static mereotopological structures, we will give two types of definitions for structures with dynamic mereotopological relations. The first type will be considered to be standard, in the sense that the space and/or the time will have a point-based definition. The second type will be a general definition, that describes the structures as frames that satisfy certain first-order conditions (like Definition 7).

The standard definition below will use a collection of static structures to define the dynamic mereotopological relations. Each of these structures will be considered as a snapshot of the space in certain moment of time. So there is a one-to-one correspondence between the static structures and the time moments. Thus, the time in this definition is *point-based time*. I.e. we define the dynamic relations by means of the notion of separate moments of time.

Definition 12.

Let I be a nonempty set of moments of time. For every moment $i \in I$, let $\underline{W}_i = (W_i, \leq_i, \mathsf{O}_i, \mathsf{U}_i, \mathsf{C}_i)$ be a static mereotopological structure (see Definition 7). Let $W \subseteq \prod_{i \in I} W_i$, such that $W \neq \emptyset$. Then the stable and unstable mereotopological relations are defined for $x, y \in W$ as follows:

$$
\begin{array}{rclr}
x \leq y & \stackrel{def}{\longleftrightarrow} & \forall i \in I, x_i \leq_i y_i & \text{stable part-of,} \\
x \,\mathsf{o}\, y & \stackrel{def}{\longleftrightarrow} & \forall i \in I, x_i \,\mathsf{O}_i\, y_i & \text{stable overlap,} \\
x \,\mathsf{u}\, y & \stackrel{def}{\longleftrightarrow} & \forall i \in I, x_i \,\mathsf{U}_i\, y_i & \text{stable underlap,} \\
x \,\mathsf{c}\, y & \stackrel{def}{\longleftrightarrow} & \forall i \in I, x_i \,\mathsf{C}_i\, y_i & \text{stable contact,} \\
x \preceq y & \stackrel{def}{\longleftrightarrow} & \exists i \in I, x_i \leq_i y_i & \text{unstable part-of,} \\
x \,\mathsf{O}\, y & \stackrel{def}{\longleftrightarrow} & \exists i \in I, x_i \,\mathsf{O}_i\, y_i & \text{unstable overlap,} \\
x \,\mathsf{U}\, y & \stackrel{def}{\longleftrightarrow} & \exists i \in I, x_i \,\mathsf{U}_i\, y_i & \text{unstable underlap,} \\
x \,\mathsf{C}\, y & \stackrel{def}{\longleftrightarrow} & \exists i \in I, x_i \,\mathsf{C}_i\, y_i & \text{unstable contact.}
\end{array}
$$

We call $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ a standard dynamic mereotopological structure *or just a* standard structure.

Notation.

The complements of the eight stable and unstable mereotopological relations defined above will be denoted by $\not\leq$, $\overline{\mathsf{o}}$, $\overline{\mathsf{u}}$, $\overline{\mathsf{c}}$, $\not\preceq$, $\overline{\mathsf{O}}$, $\overline{\mathsf{U}}$ and $\overline{\mathsf{C}}$.

Using the notations $R^{\forall}$ and $R^{\exists}$ from Part I we have the following correspondence

$$
\leq = \leq^{\forall}, \quad \mathsf{o} = \mathsf{O}^{\forall}, \quad \mathsf{u} = \mathsf{U}^{\forall}, \quad \mathsf{c} = \mathsf{C}^{\forall},
$$
$$
\preceq = \leq^{\exists}, \quad \mathsf{O} = \mathsf{O}^{\exists}, \quad \mathsf{U} = \mathsf{U}^{\exists}, \quad \mathsf{C} = \mathsf{C}^{\exists} .
$$

If we review the construction of the stable and unstable mereotopological relations in the above definition we see that whether the relations hold for a pair $x, y \in W$ depends only on the vectors x and y and on the definitions of the relations in the static structures $\underline{W}_i$. Thus, if we remove some of the vectors from W this does not effect the definition of the relations for the remaining vectors. So, we have the following

Lemma 16.

If $\underline{W}$ is a standard structure and $\underline{W}'$ is a substructure of $\underline{W}$ then $\underline{W}'$ is a standard structure, as well.

On close inspection we also see that the symbols, which are used to denote the stable part-of, the unstable overlap, the unstable underlap and the unstable contact, are also the same symbols that we use for the static mereotopological relations. The reason is that these four dynamic relations ($\leq=\leq^{\forall}$, $\mathsf{O}=\mathsf{O}^{\exists}$, $\mathsf{U}=\mathsf{U}^{\exists}$ and $\mathsf{C}=\mathsf{C}^{\exists}$) satisfy the same first-order conditions as their static counterparts. This is shown in the following

Lemma 17.

Let $\underline{W}$ be a standard structure. Then it satisfies (M1) − (M13), (C1) − (C4) *and the following 23 new conditions*

$$x \preceq x \tag{M14}$$

$$x \leq y \;\&\; y \preceq z \Longrightarrow x \preceq z \tag{M15}$$

$$x \preceq y \;\&\; y \leq z \Longrightarrow x \preceq z \tag{M16}$$

$$x \,\mathsf{o}\, y \Longrightarrow y \,\mathsf{o}\, x \tag{M17}$$

$$x \,\mathsf{o}\, y \Longrightarrow x \,\mathsf{o}\, x \tag{M18}$$

$$x \,\mathsf{o}\, y \;\&\; y \leq z \Longrightarrow x \,\mathsf{o}\, z \tag{M19}$$

$$x \,\mathsf{o}\, y \;\&\; y \preceq z \Longrightarrow x \,\mathsf{O}\, z \tag{M20}$$

$$x \,\mathsf{o}\, x \;\; or \;\; x \preceq y \tag{M21}$$

$$x \,\mathsf{o}\, z \;\; or \;\; y \,\mathsf{U}\, z \;\; or \;\; x \preceq y \tag{M22}$$

$$x \,\mathsf{u}\, y \Longrightarrow y \,\mathsf{u}\, x \tag{M23}$$

$$x \,\mathsf{u}\, y \Longrightarrow x \,\mathsf{u}\, x \tag{M24}$$

$$x \leq y \;\&\; y \,\mathsf{u}\, z \Longrightarrow x \,\mathsf{u}\, z \tag{M25}$$

$$x \preceq y \;\&\; y \,\mathsf{u}\, z \Longrightarrow x \,\mathsf{U}\, z \tag{M26}$$

$$x \,\mathsf{O}\, z \;\; or \;\; y \,\mathsf{u}\, z \;\; or \;\; x \preceq y \tag{M27}$$

$$y \,\mathsf{u}\, y \;\; or \;\; x \preceq y \tag{M28}$$

$$x \,\mathsf{o}\, x \;\; or \;\; x \,\mathsf{U}\, x \tag{M29}$$

$$x \,\mathsf{O}\, x \;\; or \;\; x \,\mathsf{u}\, x \tag{M30}$$

$$x \,\mathsf{c}\, y \Longrightarrow y \,\mathsf{c}\, x \tag{C5}$$

$$x \,\mathsf{o}\, y \Longrightarrow x \,\mathsf{c}\, y \tag{C6}$$

$$x \,\mathsf{c}\, y \Longrightarrow x \,\mathsf{o}\, x \tag{C7}$$

$$x \,\mathsf{c}\, y \;\&\; y \leq z \Longrightarrow x \,\mathsf{c}\, z \tag{C8}$$

$$x \,\mathsf{c}\, y \;\&\; y \preceq z \Longrightarrow x \,\mathsf{C}\, z \tag{C9}$$

$$z \,\mathsf{c}\, t \;\&\; x \,\overline{\mathsf{u}}\, y \;\&\; z \,\overline{\mathsf{O}}\, y \;\& t \,\overline{\mathsf{O}}\, x \Longrightarrow x \,\mathsf{C}\, y \tag{C10}$$

Proof.

The proof is just a straightforward check against Definition 12. We will show it only in the case of (C10). The proof for the other conditions (M1) − (M30), (C1) − (C9) is similar.

Suppose that there is a standard structure $\underline{W}$, which does not satisfy (C10). This means that for $\underline{W}$, $z \ \mathsf{c} \ t$, $x \ \overline{\mathsf{u}} \ y$, $z \ \overline{\mathsf{O}} \ y$, $t \ \overline{\mathsf{O}} \ x$ and $x \ \overline{\mathsf{C}} \ y$ hold. Let $\underline{W}_i$ for $i \in I$ be the static structures from the definition of $\underline{W}$. Then, by Definition 12, $z \ \mathsf{c} \ t$ means that for every $i \in I$ we have $z_i \ \mathsf{C}_i \ t_i$. Similarly, from $z \ \overline{\mathsf{O}} \ y$, $t \ \overline{\mathsf{O}} \ x$ and $x \ \overline{\mathsf{C}} \ y$ we get that for every $i \in I$, $z_i \ \overline{\mathsf{O}}_i \ y_i$, $t_i \ \overline{\mathsf{O}}_i \ x_i$ and $x_i \ \overline{\mathsf{C}}_i \ y_i$. Finally, from $x \ \overline{\mathsf{u}} \ y$ we have that there is $j \in I$ such that $x_j \ \overline{\mathsf{U}}_j \ y_j$. But then for that j we have $z_j \ \mathsf{C}_j \ t_j$, $x_j \ \overline{\mathsf{U}}_j \ y_j$, $z_j \ \overline{\mathsf{O}}_j \ y_j$, $t_j \ \overline{\mathsf{O}}_j \ x_j$ and $x_j \ \overline{\mathsf{C}}_j \ y_j$. This is a contradiction with the condition from Lemma 5.

Similarly, if we suppose that a condition from (M1) − (M30), (C1) − (C9) does not hold for $\underline{W}$, then we will get a contradiction that one of the static structures does not satisfy one of its defining conditions (M1) − (M13), (C1) − (C4) from Definition 7. □

In addition to the main axioms (M1) − (M30), (C1) − (C10) we state some useful consequences.

Lemma 18.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic structure. Then for all* $x, y \in W$*:*

(M≤) $$x \leq y \Longrightarrow x \preceq y$$

(Mo) $$x \ \mathsf{o} \ y \Longrightarrow x \ \mathsf{O} \ y$$

(Mu) $$x \ \mathsf{u} \ y \Longrightarrow x \ \mathsf{U} \ y$$

(Mc) $$x \ \mathsf{c} \ y \Longrightarrow x \ \mathsf{C} \ y$$

Proof.

These propositions follow respectively from axioms (M15), (M20), (M26) and (C9) as each of them is combined with (M14) for y. □

Now, we give a general definition for the dynamic mereological structures in which we do not have separate moments if time. Here we define the dynamic relations over arbitrary primitive objects and instead of describing how the relations are constructed, we describe the general properties, that the relations satisfy.

Definition 13.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a relational structure, such that* $W \neq \emptyset$. *Then we call* $\underline{W}$ *a* dynamic mereotopological structure *(or just* dynamic structure*) if it satisfies conditions* (M1) − (M30)*,* (C1) − (C10)*.*

Thus, the defining conditions for the dynamic structures (M1) − (M30), (C1) − (C10) contain the conditions for the static structures (M1) − (M13), (C1) − (C4) and, by definition, we have the following lemma

Lemma 19.

Every dynamic mereotopological structure is a static mereotopological structure.

Duality amongst the dynamic relations.

The last lemma means that all results for the static structures apply for the dynamic ones, as well. In particular, we may use the duality principle from Section 1 from Part III. In fact, this principle can be extended with the new relations - now we have that o is dual to u. Thus, the duality correspondence between a statement S and its dual S^d is extended in the following way:

- replace in S each occurrence of O with U;

- replace in S each occurrence of U with O;
- replace in S each occurrence of o with u;
- replace in S each occurrence of u with o;
- swap all arguments of $\leq$ and $\preceq$.

Also the duality correspondence between proofs is extended, with new pairs of dual axioms - (M17) and (M23), (M18) and (M24), (M19) and (M25), (M20) and (M26), (M21) and (M28), (M22) and (M27), (M29) and (M30). Thus, we may use the duality principle (in its extended form) for dynamic relations too.

Principle (Duality Principle)**.**

If a statement S, about relations $\leq$, $\preceq$, o, O, u and U, is proved with a proof P, then its dual statement S^d is proved with the dual proof P^d.

3. Representation theory for the dynamic mereotopological relations

We aim to show that every dynamic structure can be represented as (isomorphic to) a standard one. For this purpose we develop another generalization of Stone's technique. Given an arbitrary dynamic structure we will recreate the static spatial structures and the time moments and using them redefine the initial structure through the standard construction from Definition 12.

The static representation from Part III allows us to recreate the static structures. By Lemma 19, we can use the static abstract points (see Definition 10 for the dynamic structures and also all results for the abstract points from Part III apply for the dynamic structures.

To recreate the moments of time we use special sets of space points. Each such set is, in fact, the set of all spatial points that exist at this moment of time. Thus, having recreated all time moments for a given dynamic structure, we can recreate the corresponding mereotopological structures for each time moment and proceed with the standard definition.

3.1. **Abstract time moments.**

The time moments in the representation theory will be sets of abstract space points, which we will use to recreate mereotopological structures. The regions will again be sets of abstract points. To recreate the contact relations we will use reflexive and symmetric relation between the points, which corresponds to R from Definition 11. Thus, the abstract time moments will be reflexive and symmetric relational systems (see Lemmas 2 and 3).

Definition 14.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ be a dynamic mereotopological structure. Let $\mathcal{F} \subseteq AP(\underline{W})$ be such that $\mathcal{F} \neq \emptyset$ and $\mathcal{R}$ be a reflexive and symmetric binary relations over $\mathcal{F}$. We call the relational system $(\mathcal{F}, \mathcal{R})$ an abstract time moment *(or just an* abstract moment*) iff and $(\mathcal{F}, \mathcal{R})$ satisfies the following four conditions for all $x, y \in W$:*

(1) $x \not\preceq y$ implies $\exists F \in \mathcal{F}, x \in F \ \&\ y \notin F$;
(2) $x \,\mathsf{o}\, y$ implies $\exists F \in \mathcal{F}, x \in F \ \&\ y \in F$;
(3) $x \,\mathsf{u}\, y$ implies $\exists F \in \mathcal{F}, x \notin F \ \&\ y \notin F$;
(4) $x \,\mathsf{c}\, y$ implies $\exists F, G \in \mathcal{F}, x \in F \ \&\ y \in G \ \&\ F \,\mathcal{R}\, G$;

and also for all $F, G \in \mathcal{F}$:

(5) $F \,\mathcal{R}\, G$ implies $F \,\mathsf{R}\, G$ (i.e. $\mathcal{R}$ is a subrelation of R).

Notation.

The set of all abstract moments for $\underline{W}$ will be denoted by $AM(\underline{W})$.

Note that conditions $(1) - (4)$ from this definition are conditions for existence of specific abstract points. Also the simplest way to ensure (5) is to take $\mathcal{R}$ to be the restriction of R to $\mathcal{F}$. Thus, given an arbitrary non-empty set of abstract points we can always expand it to an abstract moment in the following way

Lemma 20.

Let $\mathcal{F}'$ be a non-empty set of abstract points for a dynamic structure $\underline{W}$. Then there exists an abstract moment $(\mathcal{F}, \mathcal{R}) \in AM(\underline{W})$, such that $\mathcal{F}' \subseteq \mathcal{F}$.

Proof.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$.

First we start from $\mathcal{F}'$ and add more abstract points to it, in order to satisfy conditions (1), (2), (3) and (4). For instance, for condition (1) if $x, y \in W$ and $x \not\preceq y$ then, by (M$\leq$), this implies $x \not\leq y$ and we add the abstract point $F(x \not\leq y)$ (from Lemma 10) to satisfy the condition for x and y. Similarly, we add abstract points of sorts $F(x \mathsf{O} y)$, $F(x \mathsf{U} y)$ and the points from pairs $\langle F, G\rangle(x \mathsf{C} y)$, using (M$\mathsf{o}$), (M$\mathsf{u}$) and (M$\mathsf{c}$), to satisfy conditions (2), (3) and (4), respectively. Thus, we set $\mathcal{F}$ to be

$$
\begin{aligned}
\mathcal{F} = \mathcal{F}' \cup& \\
&\{ F(x \not\leq y) \mid x, y \in W, x \not\preceq y \} \cup \\
&\{ F(x \mathsf{O} y) \mid x, y \in W, x \mathsf{o} y \} \cup \\
&\{ F(x \mathsf{U} y) \mid x, y \in W, x \mathsf{u} y \} \cup \\
&\{ F \mid x, y \in W, x \mathsf{c} y \text{ and } F \text{ is in } \langle F, G\rangle(x \mathsf{C} y) \} \cup \\
&\{ G \mid x, y \in W, x \mathsf{c} y \text{ and } G \text{ is in } \langle F, G\rangle(x \mathsf{C} y) \}.
\end{aligned}
$$

Finally, set $\mathcal{R}$ to be the restriction of R to $\mathcal{F}$. Thus, we satisfy condition (5). This finishes the construction of $(\mathcal{F}, \mathcal{R})$. □

In the last lemma we have used the notations $F(x \not\leq y)$, $F(x \mathsf{O} y)$, $F(x \mathsf{U} y)$ and $\langle F, G\rangle(x \mathsf{C} y)$ from Part III, Subsection 3.3. We will continue to use these notations, when constructing specific abstract points for the representation of the dynamic structures. We will also use more notations involving the additional dynamic relations. For instance we will use $F(x \preceq y)$ as a synonym for $F(x \leq y)$, $F(x \not\preceq y)$ as a synonym for $F(x \not\leq y)$, $\langle F, G\rangle(x \mathsf{c} y)$ as a synonym for $\langle F, G\rangle(x \mathsf{C} y)$, $\langle F, G\rangle(x \bar{\mathsf{c}} y)$ as a synonym for $\langle F, G\rangle(x \overline{\mathsf{C}} y)$ and so on ...

3.2. **Specific abstract space points.**

We saw in Lemma 20 that when constructing abstract moments we need abstract points with specific properties. In Lemma 20 the featured properties were $x \not\preceq y$, $x \mathsf{o} y$, $x \mathsf{u} y$ and $x \mathsf{c} y$. But to construct more specific abstract moments we will need abstract points satisfying more diverse properties. Thus, we need new sorts of abstract points or pairs of points, besides $F(x \not\leq y)$, $F(x \mathsf{O} y)$, $F(x \mathsf{U} y)$ and $\langle F, G\rangle(x \mathsf{C} y)$. Here is a list of all such points and pairs, that will be used later:

- given that $x \bar{\mathsf{o}} y$ and $z \not\preceq v$, there is an abstract point F, such that $z \in F$, $v \notin F$ and also $x \notin F$ or $y \notin F$; this abstract point will be denoted by $F(x \bar{\mathsf{o}} y, z \not\preceq v)$;
- given that $x \bar{\mathsf{o}} y$ and $z \mathsf{o} v$, there is an abstract point F, such that $z \in F$, $v \in F$ and also $x \notin F$ or $y \notin F$; this abstract point will be denoted by $F(x \bar{\mathsf{o}} y, z \mathsf{o} v)$;
- given that $x \bar{\mathsf{o}} y$ and $z \mathsf{u} v$, there is an abstract point F, such that $z \notin F$, $v \notin F$ and also $x \notin F$ or $y \notin F$; this abstract point will be denoted by $F(x \bar{\mathsf{o}} y, z \mathsf{u} v)$;
- given that $x \bar{\mathsf{o}} y$ and $z \mathsf{c} v$, there is a pair of abstract points $\langle F, G\rangle$, such that $z \in F$, $v \in G$, $F \mathsf{R} G$ and also $x \notin F$ or $y \notin F$ and also $x \notin G$ or $y \notin G$; this pair of abstract points will be denoted by $\langle F, G\rangle(x \bar{\mathsf{o}} y, z \mathsf{c} v)$;

- given that $x \,\bar{\mathsf{u}}\, y$ and $z \not\preceq v$, there is an abstract point F, such that $z \in F$, $v \notin F$ and also $x \in F$ or $y \in F$; this abstract point will be denoted by $F(x \,\bar{\mathsf{u}}\, y, z \not\preceq v)$;
- given that $x \,\bar{\mathsf{u}}\, y$ and $z \,\mathsf{o}\, v$, there is an abstract point F, such that $z \in F$, $v \in F$ and also $x \in F$ or $y \in F$; this abstract point will be denoted by $F(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{o}\, v)$;
- given that $x \,\bar{\mathsf{u}}\, y$ and $z \,\mathsf{u}\, v$, there is an abstract point F, such that $z \notin F$, $v \notin F$ and also $x \in F$ or $y \in F$; this abstract point will be denoted by $F(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{u}\, v)$;
- given that $x \,\bar{\mathsf{u}}\, y$ and $z \,\mathsf{c}\, v$, there is a pair of abstract points $\langle F, G\rangle$, such that $z \in F$, $v \in G$, $F \,\mathsf{R}\, G$ and also $x \in F$ or $y \in F$ and also $x \in G$ or $y \in G$; this pair of abstract points will be denoted by $\langle F, G\rangle(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{c}\, v)$;
- given that $x \preceq y$ and $z \not\preceq v$, there is an abstract point F, such that $z \in F$, $v \notin F$ and also if $x \in F$ then $y \in F$; this abstract point will be denoted by $F(x \preceq y, z \not\preceq v)$;
- given that $x \preceq y$ and $z \,\mathsf{o}\, v$, there is an abstract point F, such that $z \in F$, $v \in F$ and also if $x \in F$ then $y \in F$; this abstract point will be denoted by $F(x \preceq y, z \,\mathsf{o}\, v)$;
- given that $x \preceq y$ and $z \,\mathsf{u}\, v$, there is an abstract point F, such that $z \notin F$, $v \notin F$ and also if $x \in F$ then $y \in F$; this abstract point will be denoted by $F(x \preceq y, z \,\mathsf{u}\, v)$;
- given that $x \preceq y$ and $z \,\mathsf{c}\, v$, there is a pair of abstract points $\langle F, G\rangle$, such that $z \in F$, $v \in G$, $F \,\mathsf{R}\, G$ and also if $x \in F$ then $y \in F$ and also if $x \in G$ then $y \in G$; this pair of abstract points will be denoted by $\langle F, G\rangle(x \preceq y, z \,\mathsf{c}\, v)$;
- given that $x \,\bar{\mathsf{c}}\, y$ and $z \,\mathsf{c}\, v$, there is a pair of abstract points $\langle F, G\rangle$, such that $z \in F$, $v \in G$, $F \,\mathsf{R}\, G$ and also $x \notin F$ or $y \notin G$ and also $x \notin G$ or $y \notin F$; this pair of abstract points will be denoted by $\langle F, G\rangle(x \,\bar{\mathsf{c}}\, y, z \,\mathsf{c}\, v)$.

Now we prove that all these abstract points and pairs of points really do exist, provided that the conditions for them are true. The scheme here is the same as in Lemma 10. First we will prepare some non-intersecting filters and ideals. Then we will extend them to prime filters (abstract points) and prime ideals either with the **Separation Lemma** or with the **R-extension Lemma**. If we wish the abstract points to contain some elements then we will have to ensure that the initial filers contain them as well. Respectively, if we wish that the abstract points do not contain given elements then we will put those elements in the initial ideals.

Lemma 21 (Abstract point $F(x \,\bar{\mathsf{o}}\, y, z \not\preceq v)$).
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \,\bar{\mathsf{o}}\, y$ *and* $z \not\preceq v$*. Then there exists an abstract point* F *such that*

$$z \in F,\ v \notin F \text{ and also } x \notin F \text{ or } y \notin F.$$

Proof.
We will use the **Separation Lemma** to produce the needed abstract point F. We will start from filter F' and ideal I' such that $F' \cap I' = \emptyset$, $z \in F'$, $v \in I'$ and also $x \in I'$ or $y \in I'$. Thus, when we expand F' and I' to F and I with the **Separation Lemma** we will have that $z \in F$ because $F' \subseteq F$. Also because $I' \subseteq I$ and $F \cap I = \emptyset$ we will have that $v \notin F$ and also $x \notin F$ or $y \notin F$.

It is given that $z \not\preceq v$ and so, by (M21), $z \,\mathsf{o}\, z$. By (Mo) this implies $z \,\mathsf{O}\, z$. So, by (1) from Lemma 6, $F' = [z)$ is a filter. Also, since $z \not\preceq v$, by (M$\leq$), $z \nleq v$ and so it follows that $[z) \cap (v] = \emptyset$. Suppose otherwise, i.e. that $[z) \cap (v] \neq \emptyset$. Then, from Lemma 6 (4), there are $s \in [z)$ and $t \in (v]$ so that $s \leq t$. $s \in [z)$ implies $z \leq s$ and $t \in (v]$ gives $t \leq v$. Then from $z \leq s$, $s \leq t$ and $t \leq v$, by several application of (M2), we get the contradiction $z \leq v$. Thus, $[z) \cap (v] = \emptyset$.

An appropriate ideal I' should contain both x and v or contain both y and v. To find it let us consider the following three cases:

case 1: Suppose $y \,\overline{\mathsf{U}}\, v$. Then let $I' = (x] \cup (v]$.

By (M8), we have $v \,\overline{\mathsf{U}}\, y$ and by this and $x \,\overline{\mathsf{o}}\, y$, from (M22), it follows that $x \preceq v$. From (M28) and $z \not\preceq v$ we have $v \,\mathsf{u}\, v$. Now, apply (M26) for $v \,\mathsf{u}\, v$ and $x \preceq v$ to get that $x \,\mathsf{U}\, v$. By Lemma 6 (2), I' is an ideal.

From $x \preceq v$ and $z \not\preceq v$, by (M15), we have $z \nleq x$. Thus, again by Lemma 6 (4) and (M2), we have $[z) \cap (x] = \emptyset$. Combine this with $[z) \cap (v] = \emptyset$ and we get $F' \cap I' = \emptyset$.

case 2: Suppose $x \,\overline{\mathsf{U}}\, v$. Then just like in the previous case we prove that $y \,\mathsf{U}\, v$ and $z \nleq y$. So $I' = (y] \cup (v]$ is an ideal such that $F' \cap I' = \emptyset$.

case 3: $x \,\mathsf{U}\, v$ and $y \,\mathsf{U}\, v$. Here both $(x] \cup (v]$ and $(y] \cup (v]$ are ideals so we can choose between them. Suppose both $z \leq x$ and $z \leq y$. Then from $x \,\overline{\mathsf{o}}\, y$, by two applications of (M19), we get $z \,\overline{\mathsf{o}}\, z$. Then, by (M21), we get $z \preceq v$ which is a contradiction. So $z \nleq x$ or $z \nleq y$ and, thus, in each case we may choose $I' = (x] \cup (v]$ or $I' = (y] \cup (v]$ and always have $F' \cap I' = \emptyset$.

$\square$

In the proofs of the next lemmas we will only give the initial filters and ideals F', G', I' and J'. In every case the proof can be finished similarly by applying the **Separation Lemma** or the **R-extension Lemma** to get the final F and G.

Lemma 22 (Abstract point $F(x \,\overline{\mathsf{o}}\, y, z \,\mathsf{o}\, v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \,\overline{\mathsf{o}}\, y$ *and* $z \,\mathsf{o}\, v$*. Then there exists an abstract point* F *such that*

$$z \in F,\ v \in F \text{ and also } x \notin F \text{ or } y \notin F.$$

Proof.
Let $F' = [z) \cup [v)$. Clearly F' contains z and v. From $z \,\mathsf{o}\, v$, by (Mo), $z \,\mathsf{O}\, v$. Then, by Lemma 6 (1), F' is a filter. To find an appropriate ideal I' let consider the following two cases:

case 1: $z \nleq x$ and $v \nleq x$. So let $I' = (x]$.

If $x \,\overline{\mathsf{U}}\, x$ then, by (M11), we get $v \leq x$ which is impossible. So, $x \,\mathsf{U}\, x$ and thus, I' is an ideal (by (2) from Lemma 6) containing x.

Now suppose $F' \cap I' \neq \emptyset$. Then $[z) \cap (x] \neq \emptyset$ or $[v) \cap (x] \neq \emptyset$ which, by Lemma 6 (4), gives that there are $s \in F'$, $t \in I'$ such that $s \leq t$. By the definitions of $[z)$, $[v)$ and $(x]$ and by applications of (M2) we get $z \leq x$ or $v \leq x$. This is a contradiction with the premises $z \nleq x$ and $v \nleq x$. Thus, $F' \cap I' = \emptyset$.

case 2: $z \leq x$ or $v \leq x$. We will show that $z \nleq y$ and $v \nleq y$. Thus we set $I' = (y]$ and proceed with the proof as in the previous case.

If $z \leq x$ then from $z \mathsf{o} v$, by (M17) and (M19), we have $v \mathsf{o} x$. From $z \mathsf{o} v$, by (M18), it follows that $z \mathsf{o} z$ and, again by (M19), we have $z \mathsf{o} x$ as well. Similarly, in the case of $v \leq x$ we prove $z \mathsf{o} x$ and $v \mathsf{o} x$. So, by (M17), we have $x \mathsf{o} z$ and $x \mathsf{o} v$.

Now, if $z \leq y$ or $v \leq y$ we would have $x \mathsf{o} y$ (using (M19) in either case). This is contradiction with the initial condition $x \overline{\mathsf{o}} y$. Thus, $z \nleq y$ and $v \nleq y$.

$\square$

Lemma 23 (Abstract point $F(x \overline{\mathsf{o}} y, z \mathsf{u} v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \overline{\mathsf{o}} y$ *and* $z \mathsf{u} v$*. Then there exists an abstract point* F *such that*

$$z \notin F,\ v \notin F \text{ and also } x \notin F \text{ or } y \notin F.$$

Proof.
Let the filter be $F' = \emptyset$. So we have $F' \cap I' = \emptyset$ for any I'. $z \mathsf{u} v$, be (Mu), implies $z \mathsf{U} v$. Thus, by Lemma 6 (2), $(z] \cap (v]$ is an ideal. Now, consider these three cases:

case 1: $x \mathsf{U} z$ and $x \mathsf{U} v$. We will show that $I' = (z] \cap (v] \cap (x]$ is an ideal. According to Lemma 6 (6), we have to show that $x \mathsf{U} x$ (this follows from $x \mathsf{U} z$, by (M9)) and $x \mathsf{U} t$ for every $t \in (z] \cap (v]$.

Take $t \in (z] \cap (v]$. Then $t \in (z]$ or $t \in (v]$. We will check only $t \in (z]$ as the other case is similar. $t \in (z]$ gives $t \leq z$ and since $x \mathsf{U} z$, by (M8) and (M10), we have $t \mathsf{U} x$. Again, by (M8), obtain $x \mathsf{U} t$.

case 2: $x \overline{\mathsf{U}} z$. From $x \overline{\mathsf{o}} y$ and $x \overline{\mathsf{U}} z$, by (M17), (M8) and (M22), we have $y \preceq z$. Add $z \mathsf{u} v$ and, by (M26), we get $y \mathsf{U} v$. From $z \mathsf{u} v$, by (M24), follows $z \mathsf{u} z$. Since $x \overline{\mathsf{U}} z$ and $z \mathsf{u} z$ then, by (M26), $x \not\preceq z$. From $x \overline{\mathsf{o}} y$ and $x \not\preceq z$, by (M22), we have $z \mathsf{U} y$. Finally, by (M8), get $y \mathsf{U} z$.

Thus, from $y \mathsf{U} z$ and $y \mathsf{U} v$ we proceed as in the previous case and set $I' = (z] \cap (v] \cap (y]$.

case 3: $x \overline{\mathsf{U}} v$. In this case as in the previous one we show $y \mathsf{U} z$ and $y \mathsf{U} v$, take $I' = (z] \cap (v] \cap (y]$ and continue with the proof as in the first case.

$\square$

Lemma 24 (Abstract points $\langle F, G\rangle(x \overline{\mathsf{o}} y, z \mathsf{c} v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \overline{\mathsf{o}} y$ *and* $z \mathsf{c} v$*. Then there is a pair of abstract points* $\langle F, G\rangle$ *such that*

$$z \in F,\ v \in G,\ F \mathsf{R} G \text{ and also } x \notin F \text{ or } y \notin F \text{ and also } x \notin G \text{ or } y \notin G.$$

Proof.
We will build filters F' and G' such that $z \in F'$, $v \in G'$ and $F' \mathsf{R} G'$. We will also build ideals I' and J' such that $F' \cap I' = \emptyset$, $G' \cap J' = \emptyset$ and also $x \in I'$ or $y \in I'$ and also $x \in J'$ or $y \in J'$. Then we apply the **R-extension Lemma** to get the needed F and G.

From $z \mathsf{c}\, v$, by (C7), it follows that $z \mathsf{o}\, z$ and, by (Mo), we have $z \mathsf{O}\, z$. So, by Lemma 6 (1), $F' = [z)$ is a filter. Now consider the two possible cases - $z \not\leq x$ or $z \leq x$.

case 1: $z \not\leq x$. By (M11), also $x \mathsf{U}\, x$. So, by Lemma 6 (2), $I' = (x]$ is an ideal and, because $z \not\leq x$, by Lemma 6 (4), $F' \cap I' = \emptyset$.

case 2: $z \leq x$. If $z \leq y$ then from $z \mathsf{o}\, z$, by applications of (C8) and (C5), we get $x \mathsf{c}\, y$. So $z \not\leq y$ and, thus, $y \mathsf{U}\, y$ and as in the previous case we get $I' = (y]$ is an ideal and $F' \cap I' = \emptyset$.

Thus, in either case we have an ideal I' such that $F' \cap I' = \emptyset$ and $x \in I'$ or $y \in I'$.

Similarly, we build the filter $G' = [v)$ and the ideal J' such that $G' \cap J' = \emptyset$ and $x \in J'$ or $y \in J'$. Since, $z \mathsf{c}\, v$, by Lemma 6 (3), then $F' \mathsf{R}\, G'$. □

Lemma 25 (Abstract point $F(x\, \overline{\mathsf{u}}\, y, z \not\preceq v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x\, \overline{\mathsf{u}}\, y$ *and* $z \not\preceq v$*. Then there exists an abstract point* F *such that*

$$z \in F,\ v \notin F \text{ and also } x \in F \text{ or } y \in F.$$

Proof.
The proof of this lemma is dual to the proof of Lemma 21 and so we will only give a sketch of the proof. The details can be retrieved from the proof of Lemma 21 by applying the **Duality Principle**.

From $z \not\preceq v$ and so, by (M28), (Mu) and (M$\leq$), $v \mathsf{U}\, v$ and $z \not\leq v$. Thus, $I' = (v]$ is an ideal and $[z) \cap (v] = \emptyset$.

To find the filter F' we check the following cases:

case 1: If $y\, \overline{\mathsf{O}}\, z$ then (by (M27), (M21) and (M20)) $x \mathsf{O}\, z$ and $x \not\leq v$. So $F' = [x) \cup [z)$ is a filter such that $F' \cap I' = \emptyset$.

case 2: If $x\, \overline{\mathsf{O}}\, z$ then, similarly, $y \mathsf{O}\, z$ and $y \not\leq v$ and $F' = [y) \cup [z)$.

case 3: $x \mathsf{O}\, z$ and $y \mathsf{O}\, z$. In this case we show that $x \not\leq v$ or $y \not\leq v$ (use (M25) and (M28)) and we may choose $F' = [x) \cup [z)$ or $F' = [y) \cup [z)$ accordingly. □

Lemma 26 (Abstract point $F(x\, \overline{\mathsf{u}}\, y, z \mathsf{o}\, v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x\, \overline{\mathsf{u}}\, y$ *and* $z \mathsf{o}\, v$*. Then there exists an abstract point* F *such that*

$$z \in F,\ v \in F \text{ and also } x \in F \text{ or } y \in F.$$

Proof.
The proof of this lemma is dual to the proof of Lemma 23 (see **Duality Principle**).

Take the ideal $I' = \emptyset$ and choose as filter either $F' = [z) \cap [v) \cap [x)$ or alternatively $F' = [z) \cap [v) \cap [y)$, according to the following cases:

case 1: $x \mathsf{O}\, z$ and $x \mathsf{O}\, v$. Choose $F' = [z) \cap [v) \cap [x)$. Here use Lemma 6 (5) and (M4), (M5) and (M6) to prove that F' is a filter.

case 2: $x \overline{\mathsf{O}}\, z$. Prove $y \mathrel{\mathsf{O}} z$ and $y \mathrel{\mathsf{O}} v$ and proceed as in the previous case with $F' = [z) \cap [v) \cap [y)$. Here use (M18), (M20), (M23) and (M27).

case 3: $x \overline{\mathsf{O}}\, v$. The same as the second case. Set $F' = [z) \cap [v) \cap [y)$.

□

Lemma 27 (Abstract point $F(x \,\overline{\mathsf{u}}\, y, z \mathrel{\mathsf{u}} v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ be a dynamic mereotopological structure and $x, y, z, v \in W$ such that $x \,\overline{\mathsf{u}}\, y$ and $z \mathrel{\mathsf{u}} v$. Then there exists an abstract point F such that

$$z \notin F,\ v \notin F \text{ and also } x \in F \text{ or } y \in F.$$

Proof.
The proof of this lemma is dual to the proof of Lemma 22 (see **Duality Principle**).

Let $I' = (z] \cup (v]$. From $z \mathrel{\mathsf{u}} v$ we have that I' is an ideal. Now, consider these two cases:

case 1: $x \not\leq z$ and $x \not\leq v$. From this we have $x \mathrel{\mathsf{O}} x$ and, thus, let $F' = [x)$ be the filter. From the premises $x \not\leq z$ and $x \not\leq v$ we also have that $F' \cap I' \neq \emptyset$.

case 2: $x \leq z$ or $x \leq v$. We show that $x \mathrel{\mathsf{u}} z$ and $x \mathrel{\mathsf{u}} v$ (by (M23), (M24), (M25)) and as a result $y \not\leq z$ and $y \not\leq v$. In this case we set $F' = [y)$ and proceed with the proof as in the previous case.

□

Lemma 28 (Abstract points $\langle F, G\rangle(x \,\overline{\mathsf{u}}\, y, z \mathrel{\mathsf{c}} v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ be a dynamic mereotopological structure and $x, y, z, v \in W$ such that $x \,\overline{\mathsf{u}}\, y$ and $z \mathrel{\mathsf{c}} v$. Then there is a pair of abstract points $\langle F, G\rangle$ such that

$$z \in F,\ v \in G,\ F \mathrel{\mathsf{R}} G \text{ and also } x \in F \text{ or } y \in F \text{ and also } x \in G \text{ or } y \in G.$$

Proof.
Take the ideals $I' = J' = \emptyset$. Thus, $F' \cap I' = \emptyset$ and $G' \cap J' = \emptyset$ are guaranteed. We will show that we can set the filter F' to be either $F' = [z) \cup [x)$ or $F' = [z) \cup [y)$ and set the second filter either $G' = [v) \cup [x)$ or $G' = [v) \cup [y)$ and in every case to have that $F' \mathrel{\mathsf{R}} G'$.

Now consider several possible cases

case 1: $z \mathrel{\mathsf{O}} x$ and $v \mathrel{\mathsf{O}} x$. Then, by Lemma 6 (1), $F' = [z) \cup [x)$ and $G' = [v) \cup [x)$ are filters. By (C2), $z \mathrel{\mathsf{O}} x$ and $v \mathrel{\mathsf{O}} x$ imply $z \mathrel{\mathsf{C}} x$ and $v \mathrel{\mathsf{C}} x$. $z \mathrel{\mathsf{c}} v$ implies $z \mathrel{\mathsf{C}} v$. By (M4), (M5) and (C2), $z \mathrel{\mathsf{O}} x$ implies $x \mathrel{\mathsf{C}} x$. Thus, by Lemma 6 (3), we have $[z) \mathrel{\mathsf{R}} [v)$, $[z) \mathrel{\mathsf{R}} [x)$, $[v) \mathrel{\mathsf{R}} [x)$ and $[x) \mathrel{\mathsf{R}} [x)$ and so $F' \mathrel{\mathsf{R}} G'$.

case 2: $z \mathrel{\mathsf{O}} y$ and $t \mathrel{\mathsf{O}} y$. This case is similar. Set $F' = [z) \cup [y)$ and $G' = [v) \cup [y)$.

If the first two cases are not true then we have that $z \overline{\mathsf{O}}\, x$ or $v \overline{\mathsf{O}}\, x$ and also $z \overline{\mathsf{O}}\, y$ or $v \overline{\mathsf{O}}\, y$. These are four cases. We will show, however, that $z \overline{\mathsf{O}}\, x$ implies $z \mathrel{\mathsf{O}} y$ and $v \overline{\mathsf{O}}\, y$ implies $v \mathrel{\mathsf{O}} x$. Thus, the possible cases are really two: one in which $z \overline{\mathsf{O}}\, x$ and $v \overline{\mathsf{O}}\, y$ and one for $z \overline{\mathsf{O}}\, y$ and $v \overline{\mathsf{O}}\, x$. By (C3), $z \mathrel{\mathsf{c}} v$ implies $z \mathrel{\mathsf{o}} z$.

By (M20), $z \mathsf{o} z$ and $z \overline{\mathsf{O}} x$ imply $z \not\preceq x$. Then, by (M27), $z \not\preceq x$ and $x \overline{\mathsf{u}} y$ imply $z \mathsf{O} y$. Similarly, $v \overline{\mathsf{O}} y$ implies $v \mathsf{O} x$.

case 3: $z \overline{\mathsf{O}} x$ and $v \overline{\mathsf{O}} y$. They imply both $z \mathsf{O} y$ and $v \mathsf{O} x$. Thus, take $F' = [z) \cup [y)$ and $G' = [v) \cup [x)$. $z \mathsf{c} v$ implies $z \mathsf{C} v$. By (M27), $z \overline{\mathsf{O}} x$ and $x \overline{\mathsf{u}} y$ imply $z \preceq y$. Similarly we have $v \preceq x$. Combine these results with $z \mathsf{c} v$ and, by (C9), we get $z \mathsf{C} x$ and $v \mathsf{C} y$. Thus, by Lemma 6 (3), we have $[z) \mathsf{R} [v)$, $[z) \mathsf{R} [x)$ and $[v) \mathsf{R} [y)$. From $z \mathsf{c} v$, $x \overline{\mathsf{u}} y$, $z \overline{\mathsf{O}} x$ and $v \overline{\mathsf{O}} y$, by (C10), we get $x \mathsf{C} y$. So $[x) \mathsf{R} [y)$ and finally $F' \mathsf{R} G'$.

case 4: $z \overline{\mathsf{O}} y$ and $v \overline{\mathsf{O}} x$. This case is proved the same way as the previous one. Set $F' = [z) \cup [x)$ and $G' = [v) \cup [y)$.

□

Lemma 29 (Abstract point $F(x \preceq y, z \not\preceq v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \preceq y$ *and* $z \not\preceq v$. *Then there exists an abstract point* F *such that*

$$z \in F,\ v \notin F \text{ and also if } x \in F \text{ then } y \in F.$$

Proof.
$z \not\preceq v$, by (M21) and (M28), implies $z \mathsf{o} z$ and $v \mathsf{u} v$. Then, by (Mo) and (Mu), it follows that $z \mathsf{O} z$ and $v \mathsf{U} v$. Also, by (M$\leq$), $z \not\preceq v$ gives $z \not\leq v$. This implies, by Lemma 6 (4), $[z) \cap (v] = \emptyset$.

Now, consider several cases:

case 1: $y \mathsf{O} z$ and $y \not\leq v$. $y \mathsf{O} z$, by Lemma 6 (1), implies that $F' = [y) \cup [z)$ is a filter. By Lemma 6 (2), from $v \mathsf{u} v$ we have that $I' = (v]$ is an ideal.

From $y \not\leq v$, by Lemma 6 (4), we get $[y) \cap (v] = \emptyset$ and since $[z) \cap (v] = \emptyset$ then $F' \cap I' = \emptyset$.

case 2: $y \leq v$. From it and $v \mathsf{u} v$, by (M25), we have $y \mathsf{u} v$. From $y \mathsf{u} v$ and $x \preceq y$, by (M26), get $x \mathsf{U} v$. Thus, by Lemma 6 (2), $I' = (x] \cup (v]$ is an ideal. $z \mathsf{O} z$, by Lemma 6 (1), ensures that $F' = [z)$ is a filter.

$x \preceq y$ and $y \leq v$, by (M16), imply $x \preceq v$. Then $x \preceq v$ and $z \not\preceq v$, by (M15), give $z \not\leq x$. Then, by Lemma 6 (4), gives $[z) \cap (x] = \emptyset$. Take also $[z) \cap (v] = \emptyset$ and, thus, $F' \cap I' = \emptyset$.

case 3: $y \overline{\mathsf{O}} z$. By (M4), we get $z \overline{\mathsf{O}} y$. From $z \overline{\mathsf{O}} y$ and $z \not\preceq v$, by (M27) and (M23), we have $y \mathsf{u} v$. $x \preceq y$ and $y \mathsf{u} v$, by (M26), imply $x \mathsf{U} v$. This gives, by Lemma 6 (2), that $I' = (x] \cup (v]$ is an ideal. By Lemma 6 (1), since $z \mathsf{O} z$ then $F' = [z)$ is a filter.

$x \preceq y$ and $y \overline{\mathsf{O}} z$, by (M4) and (M20), imply $z \overline{\mathsf{o}} x$. From $z \mathsf{o} z$ and $z \overline{\mathsf{o}} x$, by (M19), we get $z \not\leq x$. This, by Lemma 6 (4), gives $[z) \cap (x] = \emptyset$. Combine this with $[z) \cap (v] = \emptyset$ and obtain $F' \cap I' = \emptyset$.

In every case we have $x \notin F'$ or $y \in F'$ which means that $x \in F'$ implies $y \in F'$ and this will hold for F also. □

Lemma 30 (Abstract point $F(x \preceq y, z \mathsf{o} v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \preceq y$ *and* $z \mathsf{o} v$. *Then there exists an abstract point* F

such that

$$z \in F,\ v \in F \text{ and also if } x \in F \text{ then } y \in F.$$

Proof.

By (Mo), $z \mathrel{\mathsf{o}} v$ implies $z \mathrel{\mathsf{O}} v$. Then, by Lemma 6 (1), $[z) \cup [v)$ is a filter.

Lets split the proof in three cases:

case 1: $y \mathrel{\mathsf{O}} z$ and $y \mathrel{\mathsf{O}} v$. We will show that $F' = [z) \cup [v) \cup [y)$ is a filter and, thus, take the ideal $I' = \emptyset$. To prove that F' is a filter we will use Lemma 6 (5).

From $y \mathrel{\mathsf{O}} z$, by (M5), we have $y \mathrel{\mathsf{O}} y$. Now, take $t \in [z) \cup [v)$. We check only when $t \in [z)$, since the other case $t \in [v)$ is proved the same. $t \in [z)$ gives $z \leq t$ and from $y \mathrel{\mathsf{O}} z$, by (M6), we get $y \mathrel{\mathsf{O}} t$.

case 2: $y \mathrel{\overline{\mathsf{O}}} v$. Combine $y \mathrel{\overline{\mathsf{O}}} v$ with $z \mathrel{\mathsf{o}} v$ and, by (M4), (M17) and (M20), we have $z \not\preceq y$. By (M28), this implies $y \mathrel{\mathsf{u}} y$. From $y \mathrel{\mathsf{u}} y$ and $x \preceq y$, by (M26), we get $x \mathrel{\mathsf{U}} y$ and, by (M9), it follows $x \mathrel{\mathsf{U}} x$. Thus we have, by Lemma 6 (2), that $I' = (x]$ is an ideal.

Now, take $F' = [z) \cup [v)$ as the filter. We will show that $F' \cap I' = \emptyset$. From $x \preceq y$ and $z \not\preceq y$, by (M15), we have $z \nleq x$. So, by Lemma 6 (4), $[z) \cap (x] = \emptyset$. $z \mathrel{\mathsf{o}} v$, by (M18), implies $v \mathrel{\mathsf{o}} v$. From $y \mathrel{\overline{\mathsf{O}}} v$ and $v \mathrel{\mathsf{o}} v$, by (M4) and (M20), follows that $v \not\preceq y$. So, by (M15), from it and $x \preceq y$ we have that $v \nleq x$. This gives $[v) \cap (x] = \emptyset$.

case 3: $y \mathrel{\overline{\mathsf{O}}} z$. This case is similar to the previous one. Get again $x \mathrel{\mathsf{U}} x$ and use $F' = [z) \cup [v)$ and $I' = (x]$.

Again in every case we have $x \notin F'$ or $y \in F'$. □

Lemma 31 (Abstract point $F(x \preceq y, z \mathrel{\mathsf{u}} v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \preceq y$ *and* $z \mathrel{\mathsf{u}} v$*. Then there exists an abstract point* F *such that*

$$z \notin F,\ v \notin F \text{ and also if } x \in F \text{ then } y \in F.$$

Proof.

This lemma is proved similarly to the previous Lemma 30. The differences can be obtained by application of the **Duality Principle**.

By (Mu), $z \mathrel{\mathsf{u}} v$ implies $z \mathrel{\mathsf{U}} v$ and, by Lemma 6 (2), $(z] \cup (v]$ is an ideal. Now, consider the following cases:

case 1: $x \mathrel{\mathsf{U}} z$ and $x \mathrel{\mathsf{U}} v$. In this case we show that $I' = (x] \cup (z] \cup (v]$ (use Lemma 6 (6)). For filter take $F' = \emptyset$.

case 2: $x \mathrel{\overline{\mathsf{U}}} v$. Here, prove $y \mathrel{\mathsf{O}} y$, $y \nleq z$ and $y \nleq v$ (use (M5), (M8), (M10), (M21), (M23)). This gives that $F' = [y)$ is a filter and $F' \cap I' = \emptyset$ for the ideal $I' = (z] \cup (v]$.

case 3: $x \mathrel{\overline{\mathsf{U}}} z$. This case is the same as the previous one. Take $F' = [y)$ and $I' = (z] \cup (v]$.

□

Lemma 32 (Abstract points $\langle F, G\rangle(x \preceq y, z \mathsf{c}\, v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x \preceq y$ *and* $z \mathsf{c}\, v$*. Then there is a pair of abstract points* $\langle F, G\rangle$ *such that*

$z \in F$*,* $v \in G$*,* $F \mathsf{R}\, G$ *and also if* $x \in F$ *then* $y \in F$ *and also if* $x \in G$ *then* $y \in G$*.*

Proof.
From $z \mathsf{c}\, v$, by (C7) and (C5), we have $z \mathsf{o}\, z$ and $v \mathsf{o}\, v$ and, by (Mo), $z \mathsf{O}\, z$ and $v \mathsf{O}\, v$. Consider the four cases about $z \leq x$ and $v \leq x$:

case 1: $z \not\leq x$ and $v \not\leq x$. So $[z) \cap (x] = \emptyset$ and $[v) \cap (x] = \emptyset$. By (M11), $x \mathsf{U}\, x$ and we take filters $F' = [z)$ and $G' = [v)$ and ideals $I' = J' = (x]$. From $z \mathsf{c}\, v$, by (Mc), $z \mathsf{C}\, v$ and so, by Lemma 6 (3), $F' \mathsf{R}\, G'$.

case 2: $z \not\leq x$ and $v \leq x$. $z \not\leq x$, by (M11), implies $x \mathsf{U}\, x$ and take filter $F' = [z)$ and ideal $I' = (x]$. From $v \leq x$ and $x \preceq y$, by (M15), $v \preceq y$. Then add $v \mathsf{o}\, v$ and, by (M20), we get $v \mathsf{O}\, y$. Thus take the filter $G' = [v) \cup [y)$ and the ideal $J' = \emptyset$.

Finally, from $z \mathsf{c}\, v$ and $v \preceq y$, by (C9), we have $z \mathsf{C}\, y$ and thus, by Lemma 6 (3), $F' \mathsf{R}\, [y)$. Because $z \mathsf{c}\, v$ implies $z \mathsf{C}\, v$, then $F' \mathsf{R}\, [v)$. So $F' \mathsf{R}\, G'$.

case 3: $z \leq x$ and $v \not\leq x$. This case is the same as the previous one. Take filters $F' = [z) \cup [y)$ and $G' = [v)$ and ideals $I' = \emptyset$ and $J' = (x]$.

case 4: $z \leq x$ and $v \leq x$. Similarly to the previous two cases we get $z \mathsf{O}\, y$, $v \mathsf{O}\, y$, $z \mathsf{C}\, y$ and $v \mathsf{C}\, y$. From $z \mathsf{O}\, y$, by (M4), (M5) and (C2), we have $y \mathsf{C}\, y$. Thus, take filters $F' = [z) \cup [y)$ and $G' = [t) \cup [y)$ and ideals $I' = J' = \emptyset$. Because $z \mathsf{C}\, y$, $v \mathsf{C}\, y$ and $y \mathsf{C}\, y$ we have $F' \mathsf{R}\, G'$.

In every case we have $x \notin F'$ or $y \in F'$ which means that $x \in F'$ implies $y \in F'$. The same goes for G'. □

Lemma 33 (Abstract points $\langle F, G\rangle(x\, \bar{\mathsf{c}}\, y, z \mathsf{c}\, v)$)**.**
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a dynamic mereotopological structure and* $x, y, z, v \in W$ *such that* $x\, \bar{\mathsf{c}}\, y$ *and* $z \mathsf{c}\, v$*. Then there is a pair of abstract points* $\langle F, G\rangle$ *such that*

$z \in F$*,* $v \in G$*,* $F \mathsf{R}\, G$ *and also* $x \notin F$ *or* $y \notin G$ *and also* $x \notin G$ *or* $y \notin F$*.*

Proof.
Suppose $z \leq x$ and $v \leq y$. Then from $z \mathsf{c}\, v$, by applications of (C8) and (C5), we get $x \mathsf{c}\, y$. It is the same if $z \leq y$ and $v \leq x$. So $z \not\leq x$ or $v \not\leq y$ and also $z \not\leq y$ or $v \not\leq x$. Thus we have four cases

case 1: $z \not\leq x$ and $v \not\leq x$. By (M11), $z \not\leq x$ implies $x \mathsf{U}\, x$. By (C7) and (C5), $z \mathsf{c}\, v$ implies $z \mathsf{o}\, z$ and $v \mathsf{o}\, v$ and, by (Mo), we have $z \mathsf{O}\, z$ and $v \mathsf{O}\, v$. Thus take $F' = [z)$, $G' = [v)$ and $I' = J' = (x]$.

case 2: $v \not\leq y$ and $z \not\leq y$. This case is the same as the previous one with the difference $I' = J' = (y]$.

case 3: $z \not\leq x$ and $z \not\leq y$. By (C5), (C7) and (C6), $z \mathsf{c}\, v$ implies $v \mathsf{c}\, v$. If $v \leq x$ and $v \leq y$ then, by (C8) and (C5), we get $x \mathsf{c}\, y$. So $v \not\leq x$ or $v \not\leq y$ and thus we reduce this case to the first two cases.

case 4: $v \not\leq y$ and $v \not\leq x$. This case is the same as the previous one.

$\square$

3.3. **Dynamic characterization.**

Having the notions of abstract space point and abstract time moment we can characterize (represent) the dynamic mereotopological relations as follows

Proposition 3 (Dynamic Characterization)**.**

For every dynamic structure $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *and for all* $x, y \in W$ *the following statements hold:*

$(\leq)$ $\quad x \leq y \quad \longleftrightarrow \quad \forall(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F;$

(o) $\quad x \,\mathsf{o}\, y \quad \longleftrightarrow \quad \forall(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \in F \,\&\, y \in F;$

(u) $\quad x \,\mathsf{u}\, y \quad \longleftrightarrow \quad \forall(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \notin F \,\&\, y \notin F;$

(c) $\quad x \,\mathsf{c}\, y \quad \longleftrightarrow \quad \forall(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F, G \in \mathcal{F}, x \in F \,\&\, y \in G \,\&\, F \,\mathcal{R}\, G;$

$(\preceq)$ $\quad x \preceq y \quad \longleftrightarrow \quad \exists(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F;$

(O) $\quad x \,\mathsf{O}\, y \quad \longleftrightarrow \quad \exists(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \in F \,\&\, y \in F;$

(U) $\quad x \,\mathsf{U}\, y \quad \longleftrightarrow \quad \exists(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \notin F \,\&\, y \notin F;$

(C) $\quad x \,\mathsf{C}\, y \quad \longleftrightarrow \quad \exists(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F, G \in \mathcal{F}, x \in F \,\&\, y \in G \,\&\, F \,\mathcal{R}\, G.$

Notice that the **Dynamic Characterization** is just a natural combination of the quantification over the moments of time I from Definition 12 (but this time over the set of abstract moments of time $AM(\underline{W})$) and the conditions from the **Static Characterization**.

Characterization (Characterization of $\leq$)**.**

$$x \leq y \quad \longleftrightarrow \quad \forall(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F$$

Proof.

($\longrightarrow$): If $x \leq y$ and $(\mathcal{F}, \mathcal{R})$ is an abstract moment and $F \in \mathcal{F}$ then, because F is an abstract point and $\underline{W}$ is also a static structure (see Lemma 19), the **Static Characterization** $(\leq)$ shows that $x \in F$ implies $y \in F$.

($\longleftarrow$): Suppose that the right-hand side of the condition holds but $x \not\leq y$. We will show that $x \not\leq y$ implies that there is an abstract moment, that is a counter-example for the right-hand side condition and, thus, leading to contradiction. By the **Static Characterization** $x \not\leq y$ implies that there is an abstract point F such that $x \in F$ and $y \notin F$. Then set $\mathcal{F}' = \{\ F\ \}$ and obtain $(\mathcal{F}, \mathcal{R})$ from $\mathcal{F}'$, as in Lemma 20. Thus, $(\mathcal{F}, \mathcal{R})$ is the needed abstract moment that serves as a counter-example.

$\square$

Characterization (Characterization of o)**.**

$$x \,\mathsf{o}\, y \quad \longleftrightarrow \quad \forall(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \in F \,\&\, y \in F$$

Proof.

($\longrightarrow$): If $x \,\mathsf{o}\, y$ and $(\mathcal{F}, \mathcal{R})$ is an arbitrary abstract moment then, by Definition 14 (2), there is $F \in \mathcal{F}$ such that $x \in F$ and $y \in F$.

($\longleftarrow$): We will prove that if $x \,\bar{\mathsf{o}}\, y$ holds, then there is an abstract moment, which is a counter-example for the right-hand side condition. Consider the following set of abstract points

$$\begin{aligned}
\mathcal{F}' = \quad & \{\ F(x \,\bar{\mathsf{o}}\, y, z \not\preceq v) \mid z, v \in W, z \not\preceq v\ \} \cup \\
& \{\ F(x \,\bar{\mathsf{o}}\, y, z \,\mathsf{o}\, v) \mid z, v \in W, z \,\mathsf{o}\, v\ \} \cup \\
& \{\ F(x \,\bar{\mathsf{o}}\, y, z \,\mathsf{u}\, v) \mid z, v \in W, z \,\mathsf{u}\, v\ \} \cup \\
& \{\ F \mid z, v \in W, z \,\mathsf{c}\, v \text{ and } F \text{ is in } \langle F, G\rangle(x \,\bar{\mathsf{o}}\, y, z \,\mathsf{c}\, v)\ \} \cup \\
& \{\ G \mid z, v \in W, z \,\mathsf{c}\, v \text{ and } G \text{ is in } \langle F, G\rangle(x \,\bar{\mathsf{o}}\, y, z \,\mathsf{c}\, v)\ \}.
\end{aligned}$$

Thus, $\mathcal{F}'$ satisfies conditions (1), (2), (3) and (4) from Definition 14. Also, by the constructions of the abstract points in $\mathcal{F}'$, for all $F \in \mathcal{F}'$ we have $x \notin F$ or $y \notin F$.

We have to ensure that the set of abstract points is non-empty. If $\mathcal{F}' \neq \emptyset$ then let $\mathcal{F} = \mathcal{F}'$. Otherwise, if $\mathcal{F}' = \emptyset$ then it follows, for example, that $z \,\bar{\mathsf{o}}\, v$ for all $z, v \in W$ and more importantly $x \,\bar{\mathsf{o}}\, x$. Then, by (M29), $x \,\mathsf{U}\, x$. So let $\mathcal{F} = \{\ F(x \,\mathsf{U}\, x)\ \}$. We have that $x \notin F(x \,\mathsf{U}\, x)$ and so in both cases we get that for all $F \in \mathcal{F}$, $x \notin F$ or $y \notin F$.

Finally, we take $(\mathcal{F}, \mathcal{R})$ in which $\mathcal{R}$ is the restriction of R to $\mathcal{F}$. This abstract moment is the needed counter-example.

□

Characterization (Characterization of u)**.**

$$x \,\mathsf{u}\, y \quad \longleftrightarrow \quad \forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \notin F \,\&\, y \notin F$$

Proof.

($\longrightarrow$): If $x \,\mathsf{u}\, y$ and $(\mathcal{F}, \mathcal{R})$ is an abstract moment, by Definition 14 (3), there is $F \in \mathcal{F}$ such that $x \notin F$ and $y \notin F$.

($\longleftarrow$): We will show that $x \,\bar{\mathsf{u}}\, y$ implies that there is a counter-example for the right-hand side condition. Consider the set

$$\begin{aligned}
\mathcal{F}' = \quad & \{\ F(x \,\bar{\mathsf{u}}\, y, z \not\preceq v) \mid z, v \in W, z \not\preceq v\ \} \cup \\
& \{\ F(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{o}\, v) \mid z, v \in W, z \,\mathsf{o}\, v\ \} \cup \\
& \{\ F(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{u}\, v) \mid z, v \in W, z \,\mathsf{u}\, v\ \} \cup \\
& \{\ F \mid z, v \in W, z \,\mathsf{c}\, v \text{ and } F \text{ is in } \langle F, G\rangle(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{c}\, v)\ \} \cup \\
& \{\ G \mid z, v \in W, z \,\mathsf{c}\, v \text{ and } G \text{ is in } \langle F, G\rangle(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{c}\, v)\ \}.
\end{aligned}$$

$\mathcal{F}'$ satisfies conditions (1), (2), (3) and (4) from Definition 14 and for all $F \in \mathcal{F}'$, $x \in F$ or $y \in F$. If $\mathcal{F}' \neq \emptyset$ take $\mathcal{F} = \mathcal{F}'$. Otherwise, we have $z \,\bar{\mathsf{u}}\, v$ for all $z, v \in W$ and, thus, $x \,\bar{\mathsf{u}}\, x$. Then, by (M30), $x \,\mathsf{O}\, x$ and take $\mathcal{F} = \{\ F(x \,\mathsf{O}\, x)\ \}$.

Finally, take $(\mathcal{F}, \mathcal{R})$ in which $\mathcal{R}$ is the restriction of R to $\mathcal{F}$. This is the counter-example.

□

Characterization (Characterization of c)**.**

$$x \,\mathsf{c}\, y \quad \longleftrightarrow \quad \forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \exists F, G \in \mathcal{F}, x \in F \,\&\, y \in G \,\&\, F \,\mathcal{R}\, G$$

Proof.

($\longrightarrow$): Let $x \mathsf{c} y$ and $(\mathcal{F}, \mathcal{R}) \in AM(\underline{W})$. Then the needed condition follows from Definition 14 (4).

($\longleftarrow$): We will prove that $x \bar{\mathsf{c}} y$ implies that there is a counter-example of the right-hand side.

$x \bar{\mathsf{c}} y$ implies $x \bar{\mathsf{o}} y$, by (C6). Thus, by the characterization of o from the **Dynamic Characterization**, there is an abstract moment $(\mathcal{F}', \mathcal{R}')$ such that for every $F \in \mathcal{F}'$, $x \notin F$ or $y \notin F$. Then take

$$\mathcal{F} = \mathcal{F}' \cup \{ F \mid z, v \in W, z \mathsf{c} v \text{ and } F \text{ is in } \langle F, G\rangle(x \bar{\mathsf{c}} y, z \mathsf{c} v) \}$$

and for the relation take $\mathcal{R}''$ to be restriction of R to $\mathcal{F}$ and set

$$\mathcal{R}{=}\mathcal{R}'' \setminus \{ \langle F, G\rangle, \langle G, F\rangle \mid x \in F, y \in G, F \mathsf{R} G \}.$$

In essence, we remove the pairs from $\mathcal{R}''$ that would satisfy the the right-hand side condition. Thus, if $(\mathcal{F}, \mathcal{R})$ is an abstract moment it will be a counter-example.

By construction $\mathcal{R}$ is symmetric. By Lemma 33, for every abstract point H from a pair of the sort $\langle F, G\rangle(x \bar{\mathsf{c}} y, z \mathsf{c} t)$ we have $x \notin H$ or $y \notin H$ and so this is true for every abstract point in $\mathcal{F}$. Thus all pairs $\langle F, F\rangle$ remain in $\mathcal{R}$ and it is reflexive.

Because $\mathcal{F}'$ satisfies conditions (1), (2) and (3) from Definition 14 so does $\mathcal{F}$ as well. To check (4) lets take arbitrary $z, v \in$ such that $z \mathsf{c} v$. Then there is the pair $\langle F, G\rangle(x \bar{\mathsf{c}} y, z \mathsf{c} v)$ such that $z \in F$, $v \in G$, $F \mathsf{R} G$, and $F, G \in \mathcal{F}$. Since $x \notin F$ or $y \notin G$ and also $x \notin G$ or $y \notin F$ then neither $\langle F, G\rangle$ nor $\langle G, F\rangle$ have been removed from $\mathcal{R}''$ when forming $\mathcal{R}$. Thus $F \mathcal{R} G$ and (4) is satisfied also.

$\square$

Characterization (Characterization of $\preceq$)**.**

$$x \preceq y \quad \longleftrightarrow \quad \exists(\mathcal{F}, \mathcal{R}) \in AM(\underline{W}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F$$

Proof.

($\longrightarrow$): Let $x \preceq y$. We will construct the needed abstract moment for the right-hand side condition in the following way

$$\begin{aligned}
\mathcal{F}' = \quad & \{ F(x \preceq y, z \not\preceq v) \mid z, v \in W, z \not\preceq v \} \cup \\
& \{ F(x \preceq y, z \mathsf{o} v) \mid z, v \in W, z \mathsf{o} v \} \cup \\
& \{ F(x \preceq y, z \mathsf{u} v) \mid z, v \in W, z \mathsf{u} v \} \cup \\
& \{ F \mid z, v \in W, z \mathsf{c} v \text{ and } F \text{ is in } \langle F, G\rangle(x \preceq y, z \mathsf{c} v) \} \cup \\
& \{ G \mid z, v \in W, z \mathsf{c} v \text{ and } G \text{ is in } \langle F, G\rangle(x \preceq y, z \mathsf{c} v) \}.
\end{aligned}$$

If $\mathcal{F}' \neq \emptyset$ then let $\mathcal{F} = \mathcal{F}'$. If $\mathcal{F}' = \emptyset$ then $z \bar{\mathsf{o}} v$ for all $z, v \in W$ and so $x \bar{\mathsf{o}} x$. By (M29), $x \mathsf{U} x$ and in this case take $\mathcal{F} = \{ F(x \mathsf{U} x) \}$. Thus, for all $F \in \mathcal{F}$, $x \in F$ implies $y \in F$.

$\mathcal{F}'$ satisfies Definition 14 (1), (2), (3) and (4). To complete the proof take $(\mathcal{F}, \mathcal{R})$ with $\mathcal{R}$ to be the restriction of R to $\mathcal{F}$.

($\longleftarrow$): Suppose that the right-hand side condition holds but still $x \not\preceq y$. Then, by Definition 14 (1), there is $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$. This contradicts that for all $F \in \mathcal{F}$, $x \in F$ implies $y \in F$. So $x \preceq y$.

□

Characterization (Characterization of O)**.**

$$x \mathsf{O} y \quad \longleftrightarrow \quad \exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \in F \;\&\; y \in F$$

Proof.

($\longrightarrow$): If $x \mathsf{O} y$ then by the **Static Characterization** we have that there is $F \in AP(\underline{W})$ such that $x \in F$ and $y \in F$. Then set $\mathcal{F}' = \{ F \}$ and we get the needed $(\mathcal{F},\mathcal{R})$ by Lemma 20.

($\longleftarrow$): Suppose there are $(\mathcal{F},\mathcal{R}) \in AM(\underline{W})$ and $F \in \mathcal{F}$ such that $x \in F$ and $y \in F$. Since F is an abstract point then, by the **Static Characterization**, we have $x \mathsf{O} y$.

□

Characterization (Characterization of U)**.**

$$x \mathsf{U} y \quad \longleftrightarrow \quad \exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W}), \exists F \in \mathcal{F}, x \notin F \;\&\; y \notin F$$

Proof.

($\longrightarrow$): If $x \mathsf{U} y$ then by the **Static Characterization** there is $F \in AP(\underline{W})$ such that $x \notin F$ and $y \notin F$. Obtain $(\mathcal{F},\mathcal{R})$ from $\mathcal{F}' = \{ F \}$, by Lemma 20.

($\longleftarrow$): Suppose there are $(\mathcal{F},\mathcal{R}) \in AM(\underline{W})$ and $F \in \mathcal{F}$ such that $x \notin F$ and $y \notin F$. By the **Static Characterization**, we have $x \mathsf{U} y$.

□

Characterization (Characterization of C)**.**

$$x \mathsf{C} y \quad \longleftrightarrow \quad \exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W}), \exists F, G \in \mathcal{F}, x \in F \;\&\; y \in G \;\&\; F \mathcal{R} G$$

Proof.

($\longrightarrow$): By the **Static Characterization**, $x \mathsf{C} y$ implies that there are abstract points F and G so that $x \in F$, $y \in G$ and $F \mathsf{R} G$. Then, by Lemma 20, expand the set $\mathcal{F}' = \{ F, G \}$ to the abstract moment $(\mathcal{F},\mathcal{R})$. Since $\mathcal{R}$ is just the restriction of R to $\mathcal{F}$ then we also have that $F \mathcal{R} G$.

($\longleftarrow$): $F \mathcal{R} G$ implies $F \mathsf{R} G$. So $x \mathsf{C} y$ follows from $x \in F$ and $y \in G$, by the definition of R (Definition 11).

□

3.4. **Representation theorems.**

Having proved the **Dynamic Characterization** we can now proceed with proving the crucial representation theorems. First, we will prove that for arbitrary dynamic structure we can construct a standard structure and embed the initial structure into in. Then we use the fact that the standard structures are closed with respect to substructures and we can restrict the constructed embedding structure such that it is isomorphic to the initial dynamic structure. This constitutes the **Representation Theorem**.

Theorem 12 (Embedding theorem)**.**

Let $\underline{W}^d = (W^d, \leq^d, \mathsf{o}^d, \mathsf{u}^d, \mathsf{c}^d, \preceq^d, \mathsf{O}^d, \mathsf{U}^d, \mathsf{C}^d)$ *be a dynamic structure. Then there exists a standard structure* $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *and an isomorphic embedding* h *from* $\underline{W}^d$ *to* $\underline{W}$.

Proof.

We start from $\underline{W}^d$. Since the abstract moments are reflexive and symmetric relational systems (see Definition 14), then for every abstract moment for $\underline{W}^d$ we will build a new static mereotopological structure and then we will take the standard dynamic structure defined with these static structures as in Definition 12.

Let $\mathcal{M} = (\mathcal{F}, \mathcal{R}) \in AM(\underline{W}^d)$ be an arbitrary abstract moment. Take the contact algebra $(\underline{B}, \mathsf{C})$ defined by $\mathcal{M}$ (see Lemma 2) and then take the static mereotopological structure defined by $(\underline{B}, \mathsf{C})$. This static mereotopological structure will be denoted by $\underline{W}_{\mathcal{M}} = (W_{\mathcal{M}}, \leq_{\mathcal{M}}, \mathsf{O}_{\mathcal{M}}, \mathsf{U}_{\mathcal{M}}, \mathsf{C}_{\mathcal{M}})$. We also define a function $f_{\mathcal{M}}$ from W^d to the power set of $\mathcal{F}$ such that for $x \in W^d$

$$f_{\mathcal{M}}(x) = \{ \ F \mid F \in \mathcal{F} \text{ and } x \in F \ \}.$$

The meaning of $f_{\mathcal{M}}(x)$ is that these are all abstract points from $\mathcal{F}$ that should be elements of the region x in the standard representation of $\underline{W}^d$.

Now, take $I = AM(\underline{W}^d)$ to be the set of moments of time and take all static structures $\underline{W}_{\mathcal{M}}$ for $\mathcal{M} \in I$. From the static structures define the standard structure $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ as in Definition 12. We will show that there is an isomorphic embedding h from $\underline{W}^d$ to $\underline{W}$.

For $x \in W^d$ the value $h(x)$ is a vector from $\prod_{\mathcal{M} \in AM(\underline{W}^d)} W_{\mathcal{M}}$. Thus, we will define h as we set each coordinate of the vector $h(x)$ for an arbitrary $x \in W^d$ in the following way:

$$(h(x))_{\mathcal{M}} = f_{\mathcal{M}}(x).$$

To prove that h is an isomorphic embedding from $\underline{W}^d$ to $\underline{W}$ first we will show that it preserves the relations. I.e. for all $x, y \in W^d$

$$\begin{array}{rcl}
x \leq^d y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \leq h(y) \text{ holds in } \underline{W}, \\
x \ \mathsf{o}^d \ y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \ \mathsf{o} \ h(y) \text{ holds in } \underline{W}, \\
x \ \mathsf{u}^d \ y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \ \mathsf{u} \ h(y) \text{ holds in } \underline{W}, \\
x \ \mathsf{c}^d \ y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \ \mathsf{c} \ h(y) \text{ holds in } \underline{W}, \\
x \preceq^d y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \preceq h(y) \text{ holds in } \underline{W}, \\
x \ \mathsf{O}^d \ y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \ \mathsf{O} \ h(y) \text{ holds in } \underline{W}, \\
x \ \mathsf{U}^d \ y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \ \mathsf{U} \ h(y) \text{ holds in } \underline{W}, \\
x \ \mathsf{C}^d \ y \text{ holds in } \underline{W}^d & \longleftrightarrow & h(x) \ \mathsf{C} \ h(y) \text{ holds in } \underline{W}.
\end{array}$$

For relation $\leq$ this is proved in the following way

$x \leq^d y$ iff, by the **Dynamic Characterization** $(\leq)$,

$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W}^d), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F$ iff

$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W}^d), \forall F \in \mathcal{F}, F \in f_{(\mathcal{F},\mathcal{R})}(x) \Longrightarrow F \in f_{(\mathcal{F},\mathcal{R})}(y)$ iff

$\forall \mathcal{M} \in AM(\underline{W}^d), f_{\mathcal{M}}(x) \subseteq f_{\mathcal{M}}(y)$ iff

$\forall \mathcal{M} \in AM(\underline{W}^d), (h(x))_{\mathcal{M}} \subseteq (h(y))_{\mathcal{M}}$ iff, by Lemma 3,

$\forall \mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \leq_{\mathcal{M}} (h(y))_{\mathcal{M}}$ iff, by Definition 12,
$h(x) \leq h(y)$.

For the other relations the proofs are as follows

$x \, \mathsf{o}^d \, y$ iff, by the **Dynamic Characterization** (o),
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, x \in F \;\&\; y \in F$ iff
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, F \in f_{(\mathcal{F},\mathcal{R})}(x) \;\&\; F \in f_{(\mathcal{F},\mathcal{R})}(y)$ iff
$\forall \mathcal{M} \in AM(\underline{W^d}), f_{\mathcal{M}}(x) \cap f_{\mathcal{M}}(y) \neq \emptyset$ iff
$\forall \mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \cap (h(y))_{\mathcal{M}} \neq \emptyset$ iff, by Lemma 3,
$\forall \mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \, \mathsf{O}_{\mathcal{M}} \, (h(y))_{\mathcal{M}}$ iff, by Definition 12,
$h(x) \, \mathsf{o} \, h(y)$.

$x \, \mathsf{u}^d \, y$ iff, by the **Dynamic Characterization** (u),
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, x \notin F \;\&\; y \notin F$ iff
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, F \notin f_{(\mathcal{F},\mathcal{R})}(x) \;\&\; F \notin f_{(\mathcal{F},\mathcal{R})}(y)$ iff
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), f_{(\mathcal{F},\mathcal{R})}(x) \cup f_{(\mathcal{F},\mathcal{R})}(y) \neq \mathcal{F}$ iff
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), (h(x))_{(\mathcal{F},\mathcal{R})} \cup (h(y))_{(\mathcal{F},\mathcal{R})} \neq \mathcal{F}$ iff, by Lemma 3,
$\forall \mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \, \mathsf{U}_{\mathcal{M}} \, (h(y))_{\mathcal{M}}$ iff, by Definition 12,
$h(x) \, \mathsf{u} \, h(y)$.

$x \, \mathsf{c}^d \, y$ iff, by the **Dynamic Characterization** (c),
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \exists F, G \in \mathcal{F}, x \in F \;\&\; y \in G \;\&\; F \, \mathcal{R} \, G$ iff
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \exists F, G \in \mathcal{F}, F \in f_{(\mathcal{F},\mathcal{R})}(x) \;\&\; G \in f_{(\mathcal{F},\mathcal{R})}(y) \;\&\; F \, \mathcal{R} \, G$ iff
$\forall (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \exists F, G \in \mathcal{F}, F \in (h(x))_{\mathcal{M}} \;\&\; G \in (h(y))_{\mathcal{M}} \;\&\; F \, \mathcal{R} \, G$
iff, by Lemma 2 ($\mathsf{C}_{\mathrm{def}}$),
$\forall \mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \, \mathsf{C}_{\mathcal{M}} \, (h(y))_{\mathcal{M}}$ iff, by Definition 12,
$h(x) \, \mathsf{c} \, h(y)$.

$x \preceq^d y$ iff, by the **Dynamic Characterization** ($\preceq$),
$\exists (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F$ iff
$\exists (\mathcal{F}, \mathcal{R}) \in AM(\underline{W^d}), \forall F \in \mathcal{F}, F \in f_{(\mathcal{F},\mathcal{R})}(x) \Longrightarrow F \in f_{(\mathcal{F},\mathcal{R})}(y)$ iff
$\exists \mathcal{M} \in AM(\underline{W^d}), f_{\mathcal{M}}(x) \subseteq f_{\mathcal{M}}(y)$ iff
$\exists \mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \subseteq (h(y))_{\mathcal{M}}$ iff, by Lemma 3,
$\exists \mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \leq_{\mathcal{M}} (h(y))_{\mathcal{M}}$ iff, by Definition 12,
$h(x) \preceq h(y)$.

$x \ \mathsf{O}^d \ y$ iff, by the **Dynamic Characterization** (O),

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, x \in F \ \& \ y \in F$ iff

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, F \in f_{(\mathcal{F},\mathcal{R})}(x) \ \& \ F \in f_{(\mathcal{F},\mathcal{R})}(y)$ iff

$\exists\mathcal{M} \in AM(\underline{W^d}), f_{\mathcal{M}}(x) \cap f_{\mathcal{M}}(y) \neq \emptyset$ iff

$\exists\mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \cap (h(y))_{\mathcal{M}} \neq \emptyset$ iff, by Lemma 3,

$\exists\mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \ \mathsf{O}_{\mathcal{M}} \ (h(y))_{\mathcal{M}}$ iff, by Definition 12,

$h(x) \ \mathsf{O} \ h(y)$.

$x \ \mathsf{U}^d \ y$ iff, by the **Dynamic Characterization** (U),

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, x \notin F \ \& \ y \notin F$ iff

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), \exists F \in \mathcal{F}, F \notin f_{(\mathcal{F},\mathcal{R})}(x) \ \& \ F \notin f_{(\mathcal{F},\mathcal{R})}(y)$ iff

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), f_{(\mathcal{F},\mathcal{R})}(x) \cup f_{(\mathcal{F},\mathcal{R})}(y) \neq \mathcal{F}$ iff

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), (h(x))_{(\mathcal{F},\mathcal{R})} \cup (h(y))_{(\mathcal{F},\mathcal{R})} \neq \mathcal{F}$ iff, by Lemma 3,

$\exists\mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \ \mathsf{U}_{\mathcal{M}} \ (h(y))_{\mathcal{M}}$ iff, by Definition 12,

$h(x) \ \mathsf{U} \ h(y)$.

$x \ \mathsf{C}^d \ y$ iff, by the **Dynamic Characterization** (C),

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), \exists F, G \in \mathcal{F}, x \in F \ \& \ y \in G \ \& \ F \ \mathcal{R} \ G$ iff

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), \exists F, G \in \mathcal{F}, F \in f_{(\mathcal{F},\mathcal{R})}(x) \ \& \ G \in f_{(\mathcal{F},\mathcal{R})}(y) \ \& \ F \ \mathcal{R} \ G$ iff

$\exists(\mathcal{F},\mathcal{R}) \in AM(\underline{W^d}), \exists F, G \in \mathcal{F}, F \in (h(x))_{\mathcal{M}} \ \& \ G \in (h(y))_{\mathcal{M}} \ \& \ F \ \mathcal{R} \ G$

iff, by Lemma 2 ($\mathsf{C}_{\mathrm{def}}$),

$\exists\mathcal{M} \in AM(\underline{W^d}), (h(x))_{\mathcal{M}} \ \mathsf{C}_{\mathcal{M}} \ (h(y))_{\mathcal{M}}$ iff, by Definition 12,

$h(x) \ \mathsf{C} \ h(y)$.

Finally, we show that h is injective. Let $x, y \in W^d$ such that $x \neq y$. Then, from (M3), we have $x \nleq y$ or $y \nleq x$. Assume that $x \nleq y$. Then, from the **Dynamic Characterization** ($\leq$), it follows that there is $\mathcal{M} = (\mathcal{F},\mathcal{R}) \in AM(\underline{W^d})$ and $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$. This is equivalent to $F \in f_{\mathcal{M}}(x)$ and $F \notin f_{\mathcal{M}}(y)$). Thus, $f_{\mathcal{M}}(x) \neq f_{\mathcal{M}}(y)$, i.e. $(h(x))_{\mathcal{M}} \neq (h(y))_{\mathcal{M}}$, for the abstract moment $\mathcal{M}$. Then $h(x) \neq h(y)$, since the vectors $h(x)$ and $h(y)$ are different in their $\mathcal{M}$-th coordinates. The other case in which $y \nleq x$ is checked in the same way. □

Now, we prove the main representation theorem.

Theorem 13 (Representation Theorem)**.**
Let $\underline{W^d}$ be a dynamic structure. Then there exists a standard structure $\underline{W}$ isomorphic to $\underline{W^d}$.

Proof.

By the previous theorem we have that there is a standard structure $\underline{W}^s$ and an isomorphic embedding h from $\underline{W}^d$ to $\underline{W}^s$. Then take $\underline{W}$ to be the substructure of $\underline{W}^s$ such that the domain of $\underline{W}$ is exactly the range of h. By Lemma 16, $\underline{W}$ is a standard structure too. But now h is a bijection with respect to the domain of $\underline{W}$ and, thus, $\underline{W}$ is isomorphic to the initial structure $\underline{W}^d$. $\square$

4. Dynamic mereological structures

This section is dedicated to relational structures for the language of dynamic mereological relations. We will use the static mereological structures as snapshots of the space at the different moments of time. Then we define the dynamic relations over those structures, instead of over the mereotopological ones. Thus, we have the following dynamic relations

$$\leq = \leq^{\forall}, \quad \mathsf{o} = \mathsf{O}^{\forall}, \quad \mathsf{u} = \mathsf{U}^{\forall}, \quad \preceq = \leq^{\exists}, \quad \mathsf{O} = \mathsf{O}^{\exists}, \quad \mathsf{U} = \mathsf{U}^{\exists},$$

which do not contain the dynamic contacts c and C. Here is the standard, point-based, definition of dynamic structures in this setting

Definition 15.
Let I be a nonempty set of moments of time. For every moment $i \in I$, let $\underline{W}_i = (W_i, \leq_i, \mathsf{O}_i, \mathsf{U}_i)$ be a static mereological structure (see Definition 9). Let $W \subseteq \prod_{i \in I} W_i$, such that $W \neq \emptyset$. Then the stable and unstable mereological relations are defined for $x, y \in W$ as follows:

$$x \leq y \quad \overset{def}{\longleftrightarrow} \quad \forall i \in I, x_i \leq_i y_i \qquad \text{stable part-of,}$$

$$x \,\mathsf{o}\, y \quad \overset{def}{\longleftrightarrow} \quad \forall i \in I, x_i \,\mathsf{O}_i\, y_i \qquad \text{stable overlap,}$$

$$x \,\mathsf{u}\, y \quad \overset{def}{\longleftrightarrow} \quad \forall i \in I, x_i \,\mathsf{U}_i\, y_i \qquad \text{stable underlap,}$$

$$x \preceq y \quad \overset{def}{\longleftrightarrow} \quad \exists i \in I, x_i \leq_i y_i \qquad \text{unstable part-of,}$$

$$x \,\mathsf{O}\, y \quad \overset{def}{\longleftrightarrow} \quad \exists i \in I, x_i \,\mathsf{O}_i\, y_i \qquad \text{unstable overlap,}$$

$$x \,\mathsf{U}\, y \quad \overset{def}{\longleftrightarrow} \quad \exists i \in I, x_i \,\mathsf{U}_i\, y_i \qquad \text{unstable underlap.}$$

We call $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ a standard dynamic mereological structure*.*

We have that the standard dynamic mereological structures satisfy those of the first-order conditions from Definition 13, which do not contain c and C. The proof is the same as in Lemma 17. Thus, we have the corresponding general point-free definition

Definition 16.
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ be a relational structure, such that $W \neq \emptyset$. Then we call $\underline{W}$ a dynamic mereological structure *if it satisfies conditions* (M1) $-$ (M30)*.*

We have that every standard dynamic mereological structure is a general dynamic mereological structure. The opposite correspondence follows from the representation theory in the next section.

5. Representation theory for the dynamic mereological relations

When we have to represent an arbitrary dynamic mereological structure as a standard one, we can use the mereotopological representation theory from Section 3 without any changes. For instance, take a dynamic mereological structure $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ and extend it by definition with relations c and C in the following way

$$\begin{aligned} x \,\mathsf{c}\, y \quad &\longleftrightarrow \quad x \,\mathsf{o}\, y, \\ x \,\mathsf{C}\, y \quad &\longleftrightarrow \quad x \,\mathsf{O}\, y. \end{aligned}$$

The newly acquired structure $\underline{W}^d = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ is a dynamic mereotopological structure and we can apply the **Representation Theorem** to get an isomorphic standard dynamic mereotopological structure $\underline{W}^s$. Finally, we just take the reduction of $\underline{W}^s$ to the language of dynamic mereological relations. This reduct is isomorphic to the initial structure $\underline{W}$.

However, in this section we will briefly present a new representation theory, which is a slight variation of the representation theory from Section 3. The reason to do this is to avoid some anomalies that can happen, when we apply the mereotopological representation theory to dynamic mereological structures. For example, the new definition of abstract time moments that we will use is the following

Definition 17.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ *be a general dynamic mereological structure. Let* $\mathcal{F} \subseteq AP(\underline{W})$ *be such that* $\mathcal{F} \neq \emptyset$. *We call* $\mathcal{F}$ *a* mereological abstract time moment *iff and* $\mathcal{F}$ *satisfies the following three conditions for all* $x, y \in W$*:*

(1) $x \not\preceq y$ *implies* $\exists F \in \mathcal{F}, x \in F \,\&\, y \notin F$*;*
(2) $x \,\mathsf{o}\, y$ *implies* $\exists F \in \mathcal{F}, x \in F \,\&\, y \in F$*;*
(3) $x \,\mathsf{u}\, y$ *implies* $\exists F \in \mathcal{F}, x \notin F \,\&\, y \notin F$*.*

This is a variation of Definition 14, from which we have omitted all statements for the dynamic contacts c and C. The statement, that we want to avoid in particular is (5), which states that we must have a relations $\mathcal{R}$ between the abstract points in $\mathcal{F}$, which is a subrelation of R. Note that for a given set of abstract points $\mathcal{F}$, which satisfies the rest of the conditions from Definition 14, condition (5) means that we can have multiple abstract time moments for this set $\mathcal{F}$. This means that if we use the initial definition for abstract time moments (Definition 14), instead of the one given here, then we could have much more abstract time moments than needed. For instance, if we start from a standard dynamic mereological structure $\underline{W}$ and consider it as a general dynamic mereological structure and apply the mereotopological representation theory for it, then the result could have more abstract time moments than the ones that were in the initial structure $\underline{W}$.

Thus, we will use this new definition for abstract time moments, in order to have more accurate representation theory. The notion of *abstract space points* will remain the same from the previous representation theory.

Notation.

The set of all mereological abstract moments for $\underline{W}$ *will be denoted by* $MAM(\underline{W})$*.*

We also have the corresponding statement, that every non-empty set of abstract points can be expanded to a mereological abstract time moment.

Lemma 34.

Let $\mathcal{F}'$ be a non-empty set of abstract points for a dynamic mereological structure $\underline{W}$. Then there exists a mereological abstract moment $\mathcal{F} \in MAM(\underline{W})$, such that $\mathcal{F}' \subseteq \mathcal{F}$.

Proof.
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$. Let $\mathcal{F}$ be

$$\begin{aligned} \mathcal{F} = \mathcal{F}' \cup & \\ & \{\ F(x \not\leq y) \mid x, y \in W, x \not\preceq y\ \} \cup \\ & \{\ F(x\ \mathsf{O}\ y) \mid x, y \in W, x\ \mathsf{o}\ y\ \} \cup \\ & \{\ F(x\ \mathsf{U}\ y) \mid x, y \in W, x\ \mathsf{u}\ y\ \}. \end{aligned}$$

The rest of the proof, that $\mathcal{F}$ is a mereological abstract moment, is the same as in Lemma 20. □

Thus, with the new means we have the following characterization of the dynamic mereological relations $\leq$, o, u, $\preceq$, O and U.

Proposition 4 (Dynamic Characterization)**.**
For every dynamic mereological structure $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ *and for all* $x, y \in W$ *the following statements hold:*

$(\leq)$ $\quad x \leq y \quad \longleftrightarrow \quad \forall \mathcal{F} \in MAM(\underline{W}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F;$

(o) $\quad x\ \mathsf{o}\ y \quad \longleftrightarrow \quad \forall \mathcal{F} \in MAM(\underline{W}), \exists F \in \mathcal{F}, x \in F\ \&\ y \in F;$

(u) $\quad x\ \mathsf{u}\ y \quad \longleftrightarrow \quad \forall \mathcal{F} \in MAM(\underline{W}), \exists F \in \mathcal{F}, x \notin F\ \&\ y \notin F;$

$(\preceq)$ $\quad x \preceq y \quad \longleftrightarrow \quad \exists \mathcal{F} \in MAM(\underline{W}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F;$

(O) $\quad x\ \mathsf{O}\ y \quad \longleftrightarrow \quad \exists \mathcal{F} \in MAM(\underline{W}), \exists F \in \mathcal{F}, x \in F\ \&\ y \in F;$

(U) $\quad x\ \mathsf{U}\ y \quad \longleftrightarrow \quad \exists \mathcal{F} \in MAM(\underline{W}), \exists F \in \mathcal{F}, x \notin F\ \&\ y \notin F.$

The proofs for the characterizations of $\leq$, O and U is the same as it was for the mereotopological case (see the proof of Proposition 3). For the other three relations we have minor changes.

Characterization (Characterization of o)**.**

$$x\ \mathsf{o}\ y \quad \longleftrightarrow \quad \forall \mathcal{F} \in MAM(\underline{W}), \exists F \in \mathcal{F}, x \in F\ \&\ y \in F$$

Proof.

($\longrightarrow$): This direction follows from Definition 17 (2).

($\longleftarrow$): We prove that from $x\ \bar{\mathsf{o}}\ y$ follows a counter-example for the right-hand side condition. The counter-example $\mathcal{F}$ is build as follows. Let

$$\begin{aligned} \mathcal{F}' = \quad & \{\ F(x\ \bar{\mathsf{o}}\ y, z \not\preceq v) \mid z, v \in W, z \not\preceq v\ \} \cup \\ & \{\ F(x\ \bar{\mathsf{o}}\ y, z\ \mathsf{o}\ v) \mid z, v \in W, z\ \mathsf{o}\ v\ \} \cup \\ & \{\ F(x\ \bar{\mathsf{o}}\ y, z\ \mathsf{u}\ v) \mid z, v \in W, z\ \mathsf{u}\ v\ \}. \end{aligned}$$

If $\mathcal{F}' \neq \emptyset$ then set $\mathcal{F} = \mathcal{F}'$. Otherwise, let $\mathcal{F} = \{\ F(x\ \mathsf{U}\ x)\ \}$.

□

Characterization (Characterization of u)**.**

$$x\ \mathsf{u}\ y \quad \longleftrightarrow \quad \forall \mathcal{F} \in MAM(\underline{W}), \exists F \in \mathcal{F}, x \notin F\ \&\ y \notin F$$

Proof.

($\longrightarrow$): This direction follows from Definition 17 (3).

($\longleftarrow$): $x \,\bar{\mathsf{u}}\, y$ implies that there is a counter-example $\mathcal{F}$.

$$\begin{aligned}\mathcal{F}' = \quad & \{\ F(x \,\bar{\mathsf{u}}\, y, z \not\leq v) \mid z, v \in W, z \not\leq v\ \} \cup \\ & \{\ F(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{o}\, v) \mid z, v \in W, z \,\mathsf{o}\, v\ \} \cup \\ & \{\ F(x \,\bar{\mathsf{u}}\, y, z \,\mathsf{u}\, v) \mid z, v \in W, z \,\mathsf{u}\, v\ \}.\end{aligned}$$

If $\mathcal{F}' \neq \emptyset$ then $\mathcal{F} = \mathcal{F}'$, else $\mathcal{F} = \{\ F(x \,\mathsf{O}\, x)\ \}$.

□

Characterization (Characterization of $\preceq$)**.**

$$x \preceq y \quad \longleftrightarrow \quad \exists \mathcal{F} \in MAM(\underline{W}), \forall F \in \mathcal{F}, x \in F \Longrightarrow y \in F$$

Proof.

($\longrightarrow$): The needed mereological abstract time moment is

$$\begin{aligned}\mathcal{F}' = \quad & \{\ F(x \preceq y, z \not\leq v) \mid z, v \in W, z \not\leq v\ \} \cup \\ & \{\ F(x \preceq y, z \,\mathsf{o}\, v) \mid z, v \in W, z \,\mathsf{o}\, v\ \} \cup \\ & \{\ F(x \preceq y, z \,\mathsf{u}\, v) \mid z, v \in W, z \,\mathsf{u}\, v\ \}.\end{aligned}$$

If $\mathcal{F}' \neq \emptyset$ then $\mathcal{F} = \mathcal{F}'$, else $\mathcal{F} = \{\ F(x \,\mathsf{U}\, x)\ \}$.

($\longleftarrow$): Follows from Definition 17 (1).

□

Finally, we have the same pair of representation theorems, as we had for the dynamic mereotopological relations (see Theorems 12 and 13). The current variations are proved in the same way, using the characterization of the dynamic mereological relations.

Theorem 14 (Embedding theorem)**.**
Let $\underline{W}^d = (W^d, \leq^d, \mathsf{o}^d, \mathsf{u}^d, \preceq^d, \mathsf{O}^d, \mathsf{U}^d)$ be a dynamic mereological structure. Then there exists a standard dynamic mereological structure $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ and an isomorphic embedding h from $\underline{W}^d$ to $\underline{W}$.

Theorem 15 (Representation theorem)**.**
Let $\underline{W}^d$ be a dynamic mereological structure. Then there exists a standard dynamic mereological structure $\underline{W}$ isomorphic to $\underline{W}^d$.

6. Open problems and future work

There are still many open problems and room for future development of the dynamic relational mereotopology and mereology. Two main themes may be established amongst them.

Extension of the language.

First and foremost, we have the task of adding more relations to the languages, that we have so far. We have two possible directions in which to advance. The first is to add more spatial relations. We may add relations in order to be able to express these operations from the language of contact algebras, that are undefinable at the moment (the Boolean meet and join operations - see Section 1). Another possible extension is to add more topological relations, in order to capture topological operations such as closure and interior (and thus to close the gap to the language of $S4_u$, for instance).

The second direction in extending the language is to add more temporal constructions. Natural candidates for this are temporal operations like *since* and *until*. They will help to catch up to LTL, with respect to expressive power. Other options are some of the temporal relations considered by Whitehead, like *before* or *temporal contact* (see [48]). They can be defined in the standard definition of dynamic mereotopological structures. The semantics of the temporal contact is that two objects exist at the same moment of time (i.e. their times of existence are in contact):

$$x \mathsf{C}^{\mathsf{T}} y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \exists i \in I, x_i \neq 0_i \ \&\ y_i \neq 0_i.$$

For the before relations we will need some ordering of the time moments. We say that one dynamic object is before another, if there is a moment of time, at which the first object exists, and this moment is before another, in which the second object exists. Here is the formal definition

$$x \mathsf{B} y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \exists i, j \in I, x_i \neq 0_i \ \&\ y_j \neq 0_j \ \&\ i < j.$$

Of course in every case we will maintain the requirements for the language - point-free space, moment-free time, only relations in the language and integrated spatio-temporal language. A possible development is also to change the construction with which the spatial relations and the temporal operators are combined. We could use constructions from interval temporal logic or even representing the time as a mereotopological structure/space and use some kind of multi-dimensional mereotopology.

A general representation theory.

Finally, there is the question of improving the representation theory in general. The current problem is that in each new system with new mereotopological or mereological language we have to do the whole construction of space points and time moments again. For example, the definitions of abstract time moments here - Definitions 14 and Definition 17 - depend highly one the relations that are in the language.

Also in [35] the mereotopological relations that are used are $\leq$, O, $\widehat{\mathsf{O}}$ (this is an alternative notation for the underlap relation), C, $\widehat{\mathsf{C}}$ (dual contact) and $\ll$ (internal part-of). The abstract space points there are the same as it is here (see Definition 10). The difference is in the definition of the relation R between abstract space points (see Definition 11). The current definition uses only C, while the definition in [35] uses C, $\widehat{\mathsf{C}}$ and $\ll$:

$$\begin{aligned} F\,\mathsf{R}\,G \quad \overset{\text{def}}{\longleftrightarrow} \quad & \forall x \in F, \forall y \in G, x\,\mathsf{C}\,y && \& \\ & \forall x \in W \setminus F, \forall y \in W \setminus G, x\,\widehat{\mathsf{C}}\,y && \& \\ & \forall x \in F, \forall y \in W \setminus G, x \not\ll y && \& \\ & \forall x \in W \setminus F, \forall y \in G, y \not\ll x. \end{aligned}$$

So, if we want to add dynamic variants of $\widehat{\mathsf{C}}$ or $\ll$ to the language of dynamic relational mereotopology, we will have to change the definition of R accordingly and we will have to redo a lot of the proofs in the representation theory.

Thus, we see that for every new set of relations we have to change the definitions regarding the abstract objects, that are reconstructed in the representation theory, and then we have to do some of the proofs in the representation theory again. And the more relations that we add the harder these tasks become. This makes the extension of the current languages with new spatial and temporal means much more difficult.

Thus, there is a need for more general representation theory, that will settle this problem for large enough class of mereotopological and mereological relational languages in one go. The idea is to have one common representation theory (probably based on a model-theoretic approach) for all types of relational structures $(W, \mathcal{R}_1, \mathcal{R}_2, \dots \mathcal{R}_k)$, where $\mathcal{R}_i$ are (dynamic) mereotopological relations.

Here are a few examples of this general representation theory with simpler static mereological languages. Consider relations part-of $\leq$ and overlap O and the three possible mereological languages that we have with them. Lets recall the point-definitions of $\leq$ and O (when the regions are interpreted as sets of points)

$$\begin{aligned} x \leq y \quad & \overset{\text{def}}{\longleftrightarrow} \quad x.y* = 0 \qquad \longleftrightarrow \qquad \text{for all points } a,\ a \in x \Longrightarrow a \in y, \\ x\,\mathsf{O}\,y \quad & \overset{\text{def}}{\longleftrightarrow} \quad x.y \neq 0 \qquad \longleftrightarrow \qquad \text{there is point } a,\ a \in x\ \&\ a \in y. \end{aligned}$$

Then for every of the three possible languages we will have a different definition of an abstract space point

Definition 18 (Abstract points for language $\leq$)**.**
If $(W, \leq)$ is a $\leq$-structure then we define the notion of $\leq$-point *as the subset of W that satisfies*

$$F \textit{ is a } \leq\text{-point} \quad \longleftrightarrow \quad \textit{for all } x, y \in W,\ x \in F \textit{ and } x \leq y \textit{ imply } y \in F.$$

Definition 19 (Abstract points for language O)**.**
If (W, O) is a O-structure then we define O-point *as follows*

$$F \textit{ is a } \mathsf{O}\text{-point} \quad \longleftrightarrow \quad \textit{for all } x, y \in W,\ x \in F \textit{ and } y \in F \textit{ imply } x\,\mathsf{O}\,y.$$

Definition 20 (Abstract points for language $\leq, \mathsf{O}$)**.**
If $(W, \leq, \mathsf{O})$ is a $\leq, \mathsf{O}$-structure then we define $\leq, \mathsf{O}$-point *as follows*

$$\begin{aligned} F \textit{ is a } \leq, \mathsf{O}\text{-point} \quad \longleftrightarrow \quad & \textit{for all } x, y \in W,\ x \in F \textit{ and } x \leq y \textit{ imply } y \in F \quad \& \\ & \textit{for all } x, y \in W,\ x \in F \textit{ and } y \in F \textit{ imply } x\,\mathsf{O}\,y. \end{aligned}$$

Then, the following characterizations are easy to be proved. In every case we use only those conditions from (M1) $-$ (M30), (C1) $-$ (C9) that concern the relations from the respective language.

Characterization (Characterization of $\leq$)**.**
For every $\leq$*-structure* $(W, \leq)$*, that satisfies* (M1) $-$ (M3)*, and for all* $x, y \in W$ *we have that the following equivalence holds*

$$x \leq y \quad \longleftrightarrow \quad \textit{for all } \leq\textit{-points } F,\ x \in F \Longrightarrow y \in F.$$

Characterization (Characterization of O)**.**
For every O*-structure* (W, O)*, that satisfies* (M4) *and* (M5)*, and for all* $x, y \in W$ *we have that the following equivalence holds*

$$x \mathsf{O} y \quad \longleftrightarrow \quad \textit{there is a } \mathsf{O}\textit{-point } F,\ x \in F \ \&\ y \in F.$$

Characterization (Characterization of $\leq, \mathsf{O}$)**.**
For every $\leq, \mathsf{O}$*-structure* $(W, \leq, \mathsf{O})$*, that satisfies* (M1) $-$ (M7)*, and for every* $x, y \in W$ *we have that the following equivalences hold*

$$x \leq y \quad \longleftrightarrow \quad \textit{for all } \leq, \mathsf{O}\textit{-points } F,\ x \in F \Longrightarrow y \in F,$$
$$x \mathsf{O} y \quad \longleftrightarrow \quad \textit{there is a } \leq, \mathsf{O}\textit{-point } F,\ x \in F \ \&\ y \in F.$$

In each case the characterizations of the relations are exactly the same, as the formulae for the point-definition. Only that the points are replaced with abstract points ($\leq$-points and O-points and $\leq, \mathsf{O}$-points). This is the essential requirement for the representation theory that allows to take an arbitrary relational structure, that satisfies the axioms, and isomorphically represent it in the standard point-based definition. The important thing to notice is that each relation with its point-based defining formula corresponds to a specific condition for the definition of abstract points. Thus, the set of relations in the language completely determines the conditions for the abstract points. This correspondence is the core principle in the general representation theory to come.

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Part V. Logics for dynamic mereotopological relations

♣ ♣ ♣

In this part we study the logics about stable and unstable mereotopological and mereological relations. The part is divided in two sections. The first one is dedicated to the first-order logics, which have dynamic mereotopological and mereological structures as interpretations for the language. The second section is for the modal logics of those structures, considered as Kripke frames.

Section 1 describes two first-order logics - one for the dynamic mereotopological relations and one for the dynamic mereological relations. Complete axiomatizations are presented for these logics. The representation theorems from the previous part play a key role in the completeness proofs. In the next part - Part VI - we will show that these logics, amongst others, are hereditary undecidable. Thus, we study the quantifier-free fragments of those logics in order to obtain decidable alternatives to the full first-order logics. Completeness results for the two quantifier-free fragments - the mereotopological one and the mereological one - are also presented in this section.

In Section 2 we study the problems of completeness of the modal logics, generated by the different types of dynamic structures. The axiomatizations are provided via modal formulae, that define the first-order conditions from the general definitions of dynamic structures (see Definitions 13 and 16). There is a problem in this scheme. Condition (M3) is an anti-symmetric condition and, thus, is modally undefinable. We deal with this trough a p-morphism technique, that is a variation of Segerberg's bulldozer technique. The rest is a standard completeness proof with Sahlqvist modal formulae for axioms.

1. First-order logics

The language of these logics is the first-order language with no functional symbols, no constants and the predicate symbols consisting of the dynamic mereotopological or mereological relations and equality. The standard dynamic structures from Definitions 12 and 15 will serve as models for the logics. The section is structured in four subsections. The first two are about the logic of the eight stable and unstable mereotopological relations - one subsection for the full first-order logic and one for its quantifier-free fragment. The third subsection is for the first-order theory of the six dynamic mereological relations and the last subsection presents the quantifier-free fragment of this logic.

1.1. **The full theory of the dynamic mereotopological relations.**

The formulae in this first-order theory are

$$\begin{aligned} \varphi \quad ::= \quad & x \leq y \mid x \mathsf{o}\, y \mid x \mathsf{u}\, y \mid x \mathsf{c}\, y \mid \\ & x \preceq y \mid x \mathsf{O}\, y \mid x \mathsf{U}\, y \mid x \mathsf{C}\, y \mid x = y \mid \\ & \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \Longrightarrow \varphi_2 \mid \varphi_1 \Longleftrightarrow \varphi_2 \mid \\ & \forall x \varphi \mid \exists x \varphi, \end{aligned}$$

where

- x and y are object variables;
- $\leq$, o, u, c, $\preceq$, O, U and C are the dynamic mereotopological relations and $=$ is the equality relation;
- $\neg$, $\&$, $\vee$, $\Longrightarrow$ and $\Longleftrightarrow$ are the standard Boolean connectives - negation, conjunction, disjunction, implication and equivalence.

The models will be denoted by $(\underline{W}, v)$ where $\underline{W}$ is a standard structure and v is a valuation, which is a mapping from the variables into the universe of the structure. With v_x^a we denote the modification of a given valuation

$$\begin{aligned} & v_x^a(x) = a, \\ & v_x^a(y) = v(y) \text{ for every other variable } y, \end{aligned}$$

where v is an arbitrary valuation and a is an arbitrary element from the universe of the structure $\underline{W}$. Thus, for a model $(\underline{W}, v)$ we will write

$$\begin{aligned} & (\underline{W}, v) \models \varphi \text{ when the formula } \varphi \text{ is } \textit{valid} \text{ in this model,} \\ & (\underline{W}, v) \not\models \varphi \text{ when the formula } \varphi \text{ is not valid in this model.} \end{aligned}$$

The validity of φ in the model is defined with the standard first-order semantics

- if φ is $x \leq y$, then $(\underline{W}, v) \models \varphi$ iff $v(x) \leq v(y)$ holds in $\underline{W}$;
- if φ is one of the remaining atomic formulae - $x \mathsf{o}\, y$, $x \mathsf{u}\, y$, $x \mathsf{c}\, y$, $x \preceq y$, $x \mathsf{O}\, y$, $x \mathsf{U}\, y$, $x \mathsf{C}\, y$ or $x = y$, then the validity is determined as in the previous case - query the structure if the predicate holds for $v(x)$ and $v(y)$;
- if φ is $\neg\varphi_1$, then $(\underline{W}, v) \models \varphi$ iff $(\underline{W}, v) \not\models \varphi_1$;
- if φ is $\varphi_1 \,\&\, \varphi_2$, then $(\underline{W}, v) \models \varphi$ iff $(\underline{W}, v) \models \varphi_1$ and $(\underline{W}, v) \models \varphi_2$;
- if φ is $\varphi_1 \vee \varphi_2$ or $\varphi_1 \Longrightarrow \varphi_2$ or $\varphi_1 \Longleftrightarrow \varphi_2$, the validity is determined accordingly;

- for the validity of φ when $\forall x\varphi_1$ or $\exists x\varphi_1$ we have the standard first-order semantics:

$$(\underline{W}, v) \models \forall x\varphi_1 \text{ iff for every } a \in W,\ (\underline{W}, v_x^a) \models \varphi_1;$$
$$(\underline{W}, v) \models \exists x\varphi_1 \text{ iff there is } a \in W,\ (\underline{W}, v_x^a) \models \varphi_1.$$

Completeness.

The completeness of the logic is proved with the help of the **Representation Theorem**. We use the following axiom schemes:

- the axiom schemes of the minimal first-order logic for the current language;
- the universal closures of conditions (M1) − (M30), (C1) − (C10).

The inference rules in the logic are

(Modus Ponens) $$\frac{\varphi \to \psi, \varphi}{\psi}$$

(Universal generalization) $$\frac{\varphi}{\forall x\varphi}$$

Theorem 16.
The first-order logic of dynamic mereotopological relations is complete with respect to (M1) − (M30)*,* (C1) − (C10)*.*

Proof.
Let φ be a formula of the logic, which is not a theorem. We will show that there is a standard structure in which φ is not valid.

By the Lindenbaum's lemma there is a maximal consistent set Γ, that does not contain φ. Lets consider the canonical structure $\underline{W}'$ for Γ, produced by the Henkin construction. Thus, φ is not valid in $\underline{W}'$. Since (M1) − (M30), (C1) − (C10) are axioms of the logic, then all of them are included in Γ and are valid in $\underline{W}'$. So, by Definition 13, $\underline{W}'$ is a dynamic structure. Then, by the **Representation Theorem**, there is a standard structure $\underline{W}$ which is isomorphic to $\underline{W}'$ and thus φ is not valid in $\underline{W}$. □

1.2. **Quantifier-free fragment of the theory of the dynamic mereotopological relations.**

The representation theory gives the completeness of the first-order logic of stable and unstable mereotopological relations but unfortunately the full first-order theory is hereditary undecidable as it will be shown in the next part. So, we consider the quantifier-free fragment of this theory. This is the logic of dynamic mereotopological relations without use of quantifiers. The language of this fragment consists of object variables, predicate symbols for every one of the six dynamic relations, predicate for equality and the propositional connectives:

$$\begin{array}{lll} \varphi & ::= & x \leq y \mid x\ \mathsf{o}\ y \mid x\ \mathsf{u}\ y \mid x\ \mathsf{c}\ y \mid \\ & & x \preceq y \mid x\ \mathsf{O}\ y \mid x\ \mathsf{U}\ y \mid x\ \mathsf{C}\ y \mid x = y \mid \\ & & \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \Longrightarrow \varphi_2 \mid \varphi_1 \Longleftrightarrow \varphi_2. \end{array}$$

The models of the fragment will be denoted by $(\underline{W}, v)$ where $\underline{W}$ is a standard structure and v is valuation of the variables. The validity of formulae is defined in the same way as for the full first-order logic, omitting the last case for quantified formulae.

Completeness.

We will show that this fragment has a complete axiomatization. This axiomatization consists of

- the Boolean tautologies (the axioms of the classical propositional logic);
- conditions (M1) − (M30), (C1) − (C10), taken as open formulae;
- the axioms describing = as an equivalence relation and the following axiom

(M=) $$x = y \Longrightarrow x \leq y.$$

In fact the last portion of axioms, combined with some of (M1) − (M30), (C1) − (C10), are enough to obtain the rest of the equality axioms for the mereotopological relations:

($\leq$=) $$x = x' \ \&\ y = y' \Longrightarrow x \leq y \text{ iff } x' \leq y';$$

(o=) $$x = x' \ \&\ y = y' \Longrightarrow x \ \mathsf{o}\ y \text{ iff } x' \ \mathsf{o}\ y';$$

(u=) $$x = x' \ \&\ y = y' \Longrightarrow x \ \mathsf{u}\ y \text{ iff } x' \ \mathsf{u}\ y';$$

(c=) $$x = x' \ \&\ y = y' \Longrightarrow x \ \mathsf{c}\ y \text{ iff } x' \ \mathsf{c}\ y';$$

($\preceq$=) $$x = x' \ \&\ y = y' \Longrightarrow x \preceq y \text{ iff } x' \preceq y';$$

(O=) $$x = x' \ \&\ y = y' \Longrightarrow x \ \mathsf{O}\ y \text{ iff } x' \ \mathsf{O}\ y';$$

(U=) $$x = x' \ \&\ y = y' \Longrightarrow x \ \mathsf{U}\ y \text{ iff } x' \ \mathsf{U}\ y';$$

(C=) $$x = x' \ \&\ y = y' \Longrightarrow x \ \mathsf{C}\ y \text{ iff } x' \ \mathsf{C}\ y'.$$

Modus Ponens will be the only inference rule.

To prove the completeness we use a canonical model construction which is in a sense a simplified version of the Henkin canonical construction for the first-order logic. Here we will give only a brief description of the construction and omit most of the proofs since they are standard for such Henkin-style constructions. More detailed proofs for similar systems can be found in [2] as well as in [47].

Theorem 17.

The quantifier-free fragment of the first-order logic of dynamic mereotopological relations is complete with respect to (M1) − (M30)*,* (C1) − (C10)*.*

Proof.

Let Γ be a maximal theory (maximal consistent set) in the logic. Then we define the following equivalence relation over the object variables

$$x \equiv y \Longleftrightarrow x = y \in \Gamma.$$

The equivalence class for every object variable x will be denoted by $[x]$. Then the structure in the canonical model $\underline{W}^C = (W^C, \leq^C, \mathsf{o}^C, \mathsf{u}^C, \mathsf{c}^C, \preceq^C, \mathsf{O}^C, \mathsf{U}^C, \mathsf{C}^C)$ is defined as follows:

$$W^C = \{\ [x] \mid x \text{ is an object variable }\},$$

$$[x]\ R^C\ [y] \quad \longleftrightarrow \quad x\ R\ y \in \Gamma, \text{ for all } R \in \{\ \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C}\ \}.$$

It is easy to prove, using the equality axioms ($\leq$=), (o=), (u=), (c=), ($\preceq$=), (O=), (U=) and (C=), that $\equiv$ is a congruence relation depending on Γ. This ensures the correctness of the definitions of the relations in the canonical structure.

Now we will show that $\underline{W}^C$ is an abstract structure. This is done by showing that it satisfies conditions (M1) − (M30), (C1) − (C10). To prove this for (M1) suppose that $\underline{W}^C$ does not satisfy the condition. Then there is $[x] \in W^C$ such that $[x] \not\leq^C [x]$. Then by the definition of $\leq^C$ it follows that $x \leq x \notin \Gamma$. But this is impossible because $x \leq x$ is an axiom and Γ is a theory. The satisfiability of the other 39 conditions is shown identically.

Finally the canonical model is completed by defining the valuation:

$$v^C(x) = [x].$$

The usual statement that a formula is theorem of the logic iff it is valid in all canonical models is proved in the standard way. The combination of this statement, the fact that every canonical structure is an abstract structure and the **Representation Theorem** give the completeness of the fragment: a formula is a theorem iff it is true in all standard structures. The proof is the same as for the full first-order theory. □

1.3. **The full theory of the dynamic mereological relations.**

The case with the dynamic mereological relations is similar. The languages for the logics are almost the same. The only difference is that we do not have the dynamic contact relations c and C. For the first-order theory we have the following formulae

$$\begin{array}{rcl} \varphi & ::= & x \leq y \mid x \,\mathsf{o}\, y \mid x \,\mathsf{u}\, y \mid \\ & & x \preceq y \mid x \,\mathsf{O}\, y \mid x \,\mathsf{U}\, y \mid x = y \mid \\ & & \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \Longrightarrow \varphi_2 \mid \varphi_1 \Longleftrightarrow \varphi_2 \mid \\ & & \forall x\varphi \mid \exists x\varphi. \end{array}$$

The models for the logic are of the form $(\underline{W}, v)$, where $\underline{W}$ is a standard dynamic mereological structure.

Completeness.

We use the axiomatization for the first-order logic for dynamic mereotopological relations, without the axioms for the dynamic contacts. The current axiomatization is:

- the axiom schemes of the minimal first-order logic for the current language;
- the universal closures of conditions (M1) $-$ (M30);
- *Modus Ponens* and *Universal generalization* as inference rules.

The representation theory for dynamic mereological relations from Part IV, Section 5 gives the completeness of the theory in the same way as for the previous case.

Theorem 18.

The first-order logic of dynamic mereological relations is complete with respect to (M1) $-$ (M30).

1.4. **Quantifier-free fragment of the theory of the dynamic mereological relations.**

For the quantifier-free fragment of this theory we have the same situation - drop relations c and C, drop the axioms for them (C1) $-$ (C10) and use the corresponding mereological representation theory. The language in this fragment is

$$
\begin{aligned}
\varphi \quad ::= \quad & x \leq y \;|\; x \,\mathsf{o}\, y \;|\; x \,\mathsf{u}\, y \;| \\
& x \preceq y \;|\; x \,\mathsf{O}\, y \;|\; x \,\mathsf{U}\, y \;|\; x = y \;| \\
& \neg\varphi \;|\; \varphi_1 \,\&\, \varphi_2 \;|\; \varphi_1 \vee \varphi_2 \;|\; \varphi_1 \Longrightarrow \varphi_2 \;|\; \varphi_1 \Longleftrightarrow \varphi_2.
\end{aligned}
$$

Completeness.

The axiomatization of the fragment consists of

- the Boolean tautologies (the axioms of the classical propositional logic);
- the open formulae (M1) − (M30);
- the axioms for the equivalence relation = and (M=);
- inference rule *Modus Ponens*.

Theorem 19.

The quantifier-free fragment of the first-order logic of dynamic mereological relations is complete with respect to (M1) − (M30).

Proof.

The canonical model $((W^C, \leq^C, \mathsf{o}^C, \mathsf{u}^C, \preceq^C, \mathsf{O}^C, \mathsf{U}^C), v^C)$ is constructed in the same way, as in Theorem 17. $\square$

2. Modal logics

In this section we introduce the poly-modal logics based on the dynamic mereotopological and mereological relations. It the first case it will have modal box operators for the eight stable and unstable mereotopological relations, for the converse relations of $\leq$ and $\preceq$ (these are added for better definability), as well as for the universal relation denoted by A: $[\leq]$, $[\geq]$, $[\mathsf{o}]$, $[\mathsf{u}]$, $[\mathsf{c}]$, $[\preceq]$, $[\succeq]$, $[\mathsf{O}]$, $[\mathsf{U}]$, $[\mathsf{C}]$ and $[\mathsf{A}]$. In the second case - the mereological case - we have all of the above modalities, excepts for $[\mathsf{c}]$ and $[\mathsf{C}]$. The dual diamond operators $\langle R\rangle$ are defined in the usual way, $\neg[R]\neg$. The rest of the language consists of propositional variables and the Boolean connectives. The semantics for the logics is Kripke semantics, based on the dynamic structures from Definitions 12, 13, 15 and 16.

The first three subsections of this section are about the mereotopological modal logic. They present, respectively

- the Kripke semantics of the logic;
- the p-morphism technique, that deals with the undefinable anti-symmetric condition;
- the axiomatization and the completeness proof.

The last subsection is for the mereological variant of the modal logic.

2.1. **Kripke semantics for the modal logic.**

We use Kripke semantics over dynamic mereotopological structures. The formulae in the logic are

$$\begin{aligned}\varphi \quad ::= \quad & p \mid \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \Longrightarrow \varphi_2 \mid \varphi_1 \Longleftrightarrow \varphi_2 \mid \\ & [\leq]\varphi \mid \langle\leq\rangle\varphi \mid [\geq]\varphi \mid \langle\geq\rangle\varphi \mid [\mathsf{o}]\varphi \mid \langle\mathsf{o}\rangle\varphi \mid [\mathsf{u}]\varphi \mid \langle\mathsf{u}\rangle\varphi \mid \\ & [\mathsf{c}]\varphi \mid \langle\mathsf{c}\rangle\varphi \mid [\preceq]\varphi \mid \langle\preceq\rangle\varphi \mid [\succeq]\varphi \mid \langle\succeq\rangle\varphi \mid [\mathsf{O}]\varphi \mid \langle\mathsf{O}\rangle\varphi \mid \\ & [\mathsf{U}]\varphi \mid \langle\mathsf{U}\rangle\varphi \mid [\mathsf{C}]\varphi \mid \langle\mathsf{C}\rangle\varphi \mid [\mathsf{A}]\varphi \mid \langle\mathsf{A}\rangle\varphi,\end{aligned}$$

where

- p is a propositional variable;
- $\neg$, $\&$, $\vee$, $\Longrightarrow$ and $\Longleftrightarrow$ are the standard Boolean connectives - negation, conjunction, disjunction, implication and equivalence;
- $[\leq]$, $\langle\leq\rangle$, $[\geq]$, $\langle\geq\rangle$, $[\mathsf{o}]$, $\langle\mathsf{o}\rangle$, $[\mathsf{u}]$, $\langle\mathsf{u}\rangle$, $[\mathsf{c}]$, $\langle\mathsf{c}\rangle$, $[\preceq]$, $\langle\preceq\rangle$, $[\succeq]$, $\langle\succeq\rangle$, $[\mathsf{O}]$, $\langle\mathsf{O}\rangle$, $[\mathsf{U}]$, $\langle\mathsf{U}\rangle$, $[\mathsf{C}]$, $\langle\mathsf{C}\rangle$, $[\mathsf{A}]$ and $\langle\mathsf{A}\rangle$ are the modal operators.

We use the standard notation for models $(\underline{W}, v)$, where $\underline{W}$ is a structure and v is a valuation that maps to each element x from the universe of the structure the subset of variables, that are true in this element. The truth in a given element $x \in W$ from a given model $(\underline{W}, v)$ for arbitrary modal formula φ is denoted as follows

$$(\underline{W}, v), x \models \varphi \text{ when the formula } \varphi \text{ is } \textit{true} \text{ in } x \text{ for this model,}$$

$$(\underline{W}, v), x \not\models \varphi \text{ when the formula } \varphi \text{ is } \textit{false} \text{ in } x \text{ for this model.}$$

The truth of φ is expanded from the truth of propositional variables in the standard Kripke fashion

- if φ is the propositional variable p, then $(\underline{W}, v), x \models \varphi$ iff $p \in v(x)$;
- if φ is $\neg\varphi_1$, then $(\underline{W}, v), x \models \varphi$ iff $(\underline{W}, v), x \not\models \varphi_1$;
- if φ is $\varphi_1 \,\&\, \varphi_2$, then $(\underline{W}, v), x \models \varphi$ iff $(\underline{W}, v), x \models \varphi_1$ and $(\underline{W}, v), x \models \varphi_2$;

- if φ is $\varphi_1 \vee \varphi_2$ or $\varphi_1 \Longrightarrow \varphi_2$ or $\varphi_1 \Longleftrightarrow \varphi_2$, the truth is determined accordingly;
- for the truth of φ when $[R]\varphi_1$ or $\langle R\rangle\varphi_1$, for any of the eleven modalities $\leq$, $\geq$, o, u, c, $\preceq$, $\succeq$, O, U, C or A, we have the standard Kripke semantics:

$$(\underline{W}, v), x \models [R]\varphi_1 \text{ iff } \text{ for all } y \in W,\ x\ R\ y \text{ implies } (\underline{W}, v), y \models \varphi_1;$$

$$(\underline{W}, v), x \models \langle R\rangle\varphi_1 \text{ iff } \text{ there is } y \in W, \text{ such that } x\ R\ y \text{ and } (\underline{W}, v), y \models \varphi_1.$$

We also adopt the usual notions of validity of a formula in a model and in a structure, denoted with $(\underline{W}, v) \models \varphi$ and $\underline{W} \models \varphi$ respectively.

$$(\underline{W}, v) \models \varphi \text{ iff } (\underline{W}, v), x \models \varphi \text{ for every } x \in W;$$

$$\underline{W} \models \varphi \text{ iff } (\underline{W}, v) \models \varphi \text{ for every possible valuation for this structure.}$$

Since we use dynamic mereotopological structures for models of the logic, we will have to prepare some useful notations, in order to distinguish which structures do we use.

Notation.
The class of all standard structures (see Definition 12) will be denoted by Σ_{std}.

Notation.
The class of all dynamic structures (see Definition 13) will be denoted by Σ_{dyn}.

We also use the notation for the logic of a particular class of structures. This is the set of all modal formulae that are valid in every structure from the class. We denote this in the following way

Notation.
The logic of the class of structures Σ *is*

$$\mathcal{L}(\Sigma) = \{\ \varphi \mid \underline{W} \models \varphi \textit{ for every } \underline{W} \in \Sigma\ \}.$$

Thus, when we use standard structures to obtain Kripke semantics for the modal logic we will say that we use *standard semantics* for the logic. In this case we work with the logic of the standard structures - $\mathcal{L}(\Sigma_{\mathrm{std}})$. Accordingly, when we use the generally defined dynamic structures we have *dynamic semantics* and the logic of the dynamic structures - $\mathcal{L}(\Sigma_{\mathrm{dyn}})$.

Since the **Representation Theorem** essentially states that, with respect to isomorphism, Σ_{std} is the same class as Σ_{dyn}, then we have the same relation between their corresponding logics.

Proposition 5.
The logic of the standard structures is equal to the logic of the dynamic structures:

$$\mathcal{L}(\Sigma_{std}) = \mathcal{L}(\Sigma_{dyn}).$$

Proof.
From Lemma 17 we have $\Sigma_{\mathrm{std}} \subseteq \Sigma_{\mathrm{dyn}}$. Thus, $\mathcal{L}(\Sigma_{\mathrm{dyn}}) \subseteq \mathcal{L}(\Sigma_{\mathrm{std}})$.

Suppose that the opposite inclusion does not hold. I.e. $\mathcal{L}(\Sigma_{\mathrm{std}}) \nsubseteq \mathcal{L}(\Sigma_{\mathrm{dyn}})$. Then there is a formula α such that:

$$\alpha \in \mathcal{L}(\Sigma_{\mathrm{std}}) \text{ but } \alpha \notin \mathcal{L}(\Sigma_{\mathrm{dyn}}).$$

This means that there is a structure $\underline{W} \in \Sigma_{\mathrm{dyn}}$, in which α is not valid. I.e. there is a valuation v in $\underline{W}$ and an element x from the universe of $\underline{W}$ such that:

$$(\underline{W}, v), x \not\models \alpha.$$

From the **Representation Theorem** we know that there is $\underline{W'} \in \Sigma_{\text{std}}$ which is isomorphic to $\underline{W}$ with the isomorphic mapping h. Then we define the valuation v' in $\underline{W'}$ such that

$$p \in v'(h(y)) \quad \longleftrightarrow \quad p \in v(y)$$

for every propositional variable p and every element y from the universe of $\underline{W}$. Then we get that:

$$(\underline{W'}, v'), h(x) \not\models \alpha.$$

Thus, we have that $\alpha \notin \mathcal{L}(\Sigma_{\text{std}})$ which is contradiction.

Then $\mathcal{L}(\Sigma_{\text{std}}) \subseteq \mathcal{L}(\Sigma_{\text{dyn}})$ and so $\mathcal{L}(\Sigma_{\text{std}}) = \mathcal{L}(\Sigma_{\text{dyn}})$. □

Thus, we have equality between the logics of Σ_{std} and Σ_{dyn}. So we can use the defining conditions of Σ_{dyn} - (M1) $-$ (M30), (C1) $-$ (C10) - to obtain axiomatization for the modal logic of the dynamic structures and this will give us an axiomatization for $\mathcal{L}(\Sigma_{\text{std}})$ as well.

Almost all of (M1) $-$ (M30), (C1) $-$ (C10) are modally definable with Sahlqvist formulae. The problem is that (M3) (the anti-symmetric condition) is not. To mend this, we will replace (M3) with three other conditions, which are modally definable. This will produce a new set of first-order conditions for the models and new class of structures and semantics for the modal logic. Will show that the new modal logic coincides with $\mathcal{L}(\Sigma_{\text{std}})$ and $\mathcal{L}(\Sigma_{\text{dyn}})$.

2.2. **Generalized semantics.**

To deal with (M3) we replace it with three new conditions:

(M3$'$) $\quad x \, \overline{\mathsf{O}} \, x$ and $y \leq x \Longrightarrow x = y$

(M3$''$) $\quad x \, \overline{\mathsf{U}} \, x$ and $x \leq y \Longrightarrow x = y$

(M3$'''$) $\quad z \, \overline{\mathsf{O}} \, x$ and $z \, \overline{\mathsf{U}} \, y$ and $y \leq x \Longrightarrow x = y$

They are consequences of (M1) $-$ (M30), (C1) $-$ (C10). This is a more complex variation of Segerberg's bulldozer construction from [39] (similar technique is used in [35]).

Thus, we have a new set of conditions - (M1), (M2), (M3$'$) $-$ (M3$'''$), (M4) $-$ (M30), (C1) $-$ (C10). This set describes a greater, more general class of structures than Σ_{dyn}. Thus, the new kind of structures will be called *generalized structures*. This gives a new modal logic - the modal logic of the generalized structures, with a new semantics - *generalized semantics*.

Using the p-morphism techniques we will prove that the modal logic of the new class of structures is the same as the modal logics of Σ_{std} and Σ_{dyn}.

Definition 21.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ *be a relational structure such that* $W \neq \emptyset$ *and* $\leq$, o, u, c, $\preceq$, O, U *and* C *be binary relations over* W*. Then* $\underline{W}$ *is called a* generalized dynamic mereotopological structure *(or just a* generalized structure*) if it satisfies conditions* (M1), (M2), (M3$'$) $-$ (M3$'''$), (M4) $-$ (M30), (C1) $-$ (C10).

Notation.

The class of all generalized structures will be denoted by Σ_{gen}.

Notation.

The logic of the generalized structures is $\mathcal{L}(\Sigma_{gen})$.

Now we have to prove that the modal logic of the new class of structures is the same as the modal logics of the previously defined classes. Part of it is obvious by the fact that $\Sigma_{\text{dyn}} \subseteq \Sigma_{\text{gen}}$. The nontrivial part will be done by showing that there is a p-morphism from the structures from Σ_{dyn} to the structures from Σ_{gen}.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ be a generalized structure ($\underline{W} \in \Sigma_{\text{gen}}$).

Definition 22.

Let $\equiv$ be a binary relation over W such that for all $x, y \in W$

$$x \equiv y \quad \longleftrightarrow \quad x \leq y \textit{ and } y \leq x.$$

$\equiv$ is an equivalence relation. $[x]$ will denote the equivalence class of $x \in W$ with respect to $\equiv$. We say that $[x]$ is degenerate *if it is a singleton i.e.:*

$$[x] \textit{ is } \text{degenerate} \quad \longleftrightarrow \quad [x] = \{x\}$$

Let $\ll$ be a well ordering of W. Then the structure $\underline{W}'$ is built as follows:

Let $\mathbb{Z}$ be the set of integers and ω is such that $\omega \notin \mathbb{Z}$. We will define one function $f_i : W \longrightarrow W \times (\mathbb{Z} \bigcup \{\ \omega\ \})$ for every integer i

$$f_i(x) = \begin{cases} (x, \omega), & \text{if } [x] \text{ is degenerate} \\ (x, i), & \text{otherwise} \end{cases}$$

for every $i \in \mathbb{Z}$ and for every $x \in W$.

By $\underline{W}'$ we denote the structure $(W', \leq', \mathsf{o}', \mathsf{u}', \mathsf{c}', \preceq', \mathsf{O}', \mathsf{U}', \mathsf{C}')$ such that:

$$W' = \{\ f_i(x) \mid i \in \mathbb{Z}, x \in W\ \}$$

the relation $\leq'$ is defined for every $i, j \in \mathbb{Z}, x, y \in W$:

- if $x \not\equiv y$ or $[x]$ is degenerate or $[y]$ is degenerate: $f_i(x) \leq' f_j(y)$ iff $x \leq y$;
- otherwise: $f_i(x) \leq' f_j(y)$ iff $i < j$ or ($i = j$ and $x \ll y$);

while the relations o', u', c', $\preceq'$, O', U' and C' are defined:

$$f_i(x)\ \mathsf{o}'\ f_j(y) \text{ iff } x\ \mathsf{o}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$
$$f_i(x)\ \mathsf{u}'\ f_j(y) \text{ iff } x\ \mathsf{u}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$
$$f_i(x)\ \mathsf{c}'\ f_j(y) \text{ iff } x\ \mathsf{c}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$
$$f_i(x) \preceq' f_j(y) \text{ iff } x \preceq y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$
$$f_i(x)\ \mathsf{O}'\ f_j(y) \text{ iff } x\ \mathsf{O}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$
$$f_i(x)\ \mathsf{U}'\ f_j(y) \text{ iff } x\ \mathsf{U}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$
$$f_i(x)\ \mathsf{C}'\ f_j(y) \text{ iff } x\ \mathsf{C}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W.$$

Proposition 6.

Let $\underline{W} \in \Sigma_{gen}$ and $\underline{W}'$ is obtained from $\underline{W}$ by the above construction. Then the following two propositions are true:

(1) $\underline{W}'$ is a dynamic structure (i.e. $\underline{W}' \in \Sigma_{dyn}$);
(2) there exists a p-morphism from $\underline{W}'$ onto $\underline{W}$.

Proof.

(1) To prove that $\underline{W}' \in \Sigma_{\text{dyn}}$ we have to verify that $\underline{W}'$ satisfies all axioms (M1) – (M30), (C1) – (C10). This requires a long and tedious verification for each axiom. Here we present a proof for several of them, noting that the rest are done in similar way.

To check (M1) consider $x' \in W'$. There are $x \in W$ and $i \in \mathbb{Z}$ such that $x' = f_i(x)$. Let us see the definition of $\leq'$. If $[x]$ is degenerate then $x' \leq' x'$ follows from $x \leq x$ ((M1) is true for $\underline{W}$). Otherwise since $i = i$ and $x \ll x$ then by definition we have $x' \leq' x'$.

Now we prove (M3). Let $x', y' \in W'$ be such that $x' \leq' y'$ and $y' \leq' x'$. There are $x, y \in W$ and $i, j \in \mathbb{Z}$ such that $f_i(x) = x'$ and $f_j(y) = y'$. If $x \not\equiv y$ then $x \not\leq y$ or $y \not\leq x$ and then by the definition of $\leq'$ it follows that $f_i(x) \not\leq' f_j(y)$ or $f_j(y) \not\leq' f_i(x)$. This however is contradiction and thus $x \equiv y$ (that means $[x] = [y]$). Now let us see the cases from the definition of $\leq'$

- if $[x]$ is degenerate then $x = y$ and $[y]$ is also degenerate. Then we have $x' = f_i(x) = (x, \omega) = (y, \omega) = f_j(y) = y'$.
- if $[x]$ is not degenerate (and also $[y]$) then by the definition of $\leq'$ and the initial conditions $x' \leq' y'$ and $y' \leq' x'$ it follows that $i = j$ and $x \ll y$ and $y \ll x$. Since $\ll$ is well ordering then $x = y$ and hence $x' = f_i(x) = (x, i) = (y, j) = f_j(y) = y'$.

To prove (M19) assume that $x' \mathsf{o}' y'$ and $y' \leq' z'$ for some $x', y', z' \in W'$. That means that x', y', z' are images of elements from W. I.e. we have $x' = f_i(x)$, $y' = f_j(y)$ and $z' = f_k(z)$ for some $x, y, z \in W$ and $i, j, k \in \mathbb{Z}$. Then by the construction of o' we have $x \mathsf{o} y$ and in either case of the construction of $\leq'$ we have $y \leq z$. So by (M19) applied for $\underline{W}$ we have $x \mathsf{o} z$. Then again by the construction of o' it follows that $f_i(x) \mathsf{o}' f_k(z)$ i.e. $x' \mathsf{o}' z'$.

Satisfiability of the rest of the conditions that have two premises and one conclusion (like (M6), (M10), (M26), etc.) is done in the same way. This also applies for the conditions that are disjunctions. For example, (M22) is shown as we prove that $x' \overline{\mathsf{o}}' z'$ and $y' \overline{\mathsf{U}}' z' \Longrightarrow x' \preceq' y'$.

The same goes even for the most complex of the conditions - (C10). Now, suppose that

$$z' \mathsf{c}' t' \;\&\; x' \overline{\mathsf{u}}' y' \;\&\; z' \overline{\mathsf{O}}' y' \;\&\; t' \overline{\mathsf{O}}' x'$$

holds in $\underline{W}'$. From this we get that there are $x, y, z, t \in W$ such that $x' = f_i(x)$, $y' = f_j(y)$, $z' = f_k(z)$ and $t' = f_l(t)$ and

$$z \mathsf{c} t \;\&\; x \overline{\mathsf{u}} y \;\&\; z \overline{\mathsf{O}} y \;\&\; t \overline{\mathsf{O}} x$$

in $\underline{W}$. By (C10), for $\underline{W}$ we get $x \mathsf{C} y$ and by the construction of C' we have the needed result $x' \mathsf{C}' y'$ in $\underline{W}'$.

(2) Let P be a mapping from W' to W such that: $P(f_i(x)) = x$ for every element $f_i(x)$ of W'. We shall prove that P is p-morphism from $\underline{W}'$ onto $\underline{W}$. To achieve that we must verify that:

(R1) For every $x', y' \in W'$ the following conditions hold:

$$x' \leq' y' \text{ implies } P(x') \leq P(y');$$
$$x' \mathsf{o}' y' \text{ implies } P(x') \mathsf{o} P(y');$$
$$x' \mathsf{u}' y' \text{ implies } P(x') \mathsf{u} P(y');$$
$$x' \mathsf{c}' y' \text{ implies } P(x') \mathsf{c} P(y');$$
$$x' \preceq' y' \text{ implies } P(x') \preceq P(y');$$
$$x' \mathsf{O}' y' \text{ implies } P(x') \mathsf{O} P(y');$$
$$x' \mathsf{U}' y' \text{ implies } P(x') \mathsf{U} P(y');$$
$$x' \mathsf{C}' y' \text{ implies } P(x') \mathsf{C} P(y').$$

(R2) For every $x' \in W', y \in W$ the following conditions hold:

$$P(x') \leq y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \leq' y';$$
$$P(x') \mathsf{o}\, y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \mathsf{o}' y';$$
$$P(x') \mathsf{u}\, y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \mathsf{u}' y';$$
$$P(x') \mathsf{c}\, y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \mathsf{c}' y';$$
$$P(x') \preceq y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \preceq' y';$$
$$P(x') \mathsf{O}\, y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \mathsf{O}' y';$$
$$P(x') \mathsf{U}\, y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \mathsf{U}' y';$$
$$P(x') \mathsf{C}\, y \text{ implies that there is } y' \in W' \text{ such that } P(y') = y \text{ and } x' \mathsf{C}' y'.$$

The proof of (R1) for all relations follows straight from the construction of $\leq'$, o', u', c', $\preceq'$, O', U' and C'. (R2) is true for o', u', c', $\preceq'$, O', U' and C' for $y' = f_i(y)$ for any $i \in \mathbb{Z}$. As for (R2) for $\leq'$ provided that $x' = f_i(x)$ for some $x \in W$ and $i \in \mathbb{Z}$ then (R2) holds if we define y':

- if $x \not\equiv y$ or $[x]$ is degenerate or $[y]$ is degenerate: $y' = f_j(y)$ for any $j \in \mathbb{Z}$;
- otherwise: $y' = f_j(y)$ for any $j > i$.

□

Now we can prove the essential result about the modal logics of the classes.

Proposition 7.

The modal logic of the class of dynamic structures coincides with the modal logic of the generalized structures.

$$\mathcal{L}(\Sigma_{dyn}) = \mathcal{L}(\Sigma_{gen})$$

Proof.

From the fact that $\Sigma_{\text{dyn}} \subseteq \Sigma_{\text{gen}}$. Thus, $\mathcal{L}(\Sigma_{\text{gen}}) \subseteq \mathcal{L}(\Sigma_{\text{dyn}})$.

Suppose that the opposite inclusion does not hold. I.e. $\mathcal{L}(\Sigma_{\text{dyn}}) \nsubseteq \mathcal{L}(\Sigma_{\text{gen}})$. Then there is a formula α such that:

$$\alpha \in \mathcal{L}(\Sigma_{\text{dyn}}) \text{ but } \alpha \notin \mathcal{L}(\Sigma_{\text{gen}}).$$

This means that there is a structure $\underline{W} \in \Sigma_{\text{gen}}$, in which α is not valid. I.e. there is a valuation v in $\underline{W}$ and an element x from the universe of $\underline{W}$ such that:

$$(\underline{W}, v), x \not\models \alpha.$$

From Proposition 6 we know that there is $\underline{W}' \in \Sigma_{\mathrm{dyn}}$ and a p-morphism P from $\underline{W}'$ onto $\underline{W}$. Then we define the valuation v' in $\underline{W}'$ such that

$$p \in v'(y') \quad \longleftrightarrow \quad p \in v(P(y'))$$

for every propositional variable p and every element y' from the universe of $\underline{W}'$. Then we get that there is x' from $\underline{W}'$ such that $P(x') = x$ and:

$$(\underline{W}', v'), x' \not\models \alpha.$$

Thus, we have that $\alpha \notin \mathcal{L}(\Sigma_{\mathrm{dyn}})$ which is contradiction.

Then $\mathcal{L}(\Sigma_{\mathrm{dyn}}) \subseteq \mathcal{L}(\Sigma_{\mathrm{gen}})$ and so $\mathcal{L}(\Sigma_{\mathrm{dyn}}) = \mathcal{L}(\Sigma_{\mathrm{gen}})$. $\square$

2.3. **Completeness of the modal logic of dynamic mereotopological relations.**

The axiomatization of the modal logic $\mathcal{L}(\Sigma_{\mathrm{gen}})$ consists of the modal formulae that define the first-order conditions for generalized structures. The conditions (M1), (M2), (M3′) – (M3‴), (M4) – (M30), (C1) – (C10) are modally definable with Sahlqvist formulae. Since the completeness with respect to the generalized structures is proven by means of generated canonical models and since, by Propositions 5 and 7, $\mathcal{L}(\Sigma_{\mathrm{std}}) = \mathcal{L}(\Sigma_{\mathrm{dyn}}) = \mathcal{L}(\Sigma_{\mathrm{dyn}})$ this implies the completeness of the axiomatization for Σ_{std} and Σ_{dyn} as well.

Axiom schemes.

We start from the axioms of the minimal multi-modal logic with modalities $[\leq]$, $[\geq]$, $[\mathsf{o}]$, $[\mathsf{u}]$, $[\mathsf{c}]$, $[\preceq]$, $[\succeq]$, $[\mathsf{O}]$, $[\mathsf{U}]$, $[\mathsf{C}]$ and $[\mathsf{A}]$:

- the Boolean tautologies (the axioms of the classical propositional logic);
- the distribution axiom for each of the modalities

(K[≤]) $[\leq](\alpha \Longrightarrow \beta) \Longrightarrow ([\leq]\alpha \Longrightarrow [\leq]\beta)$

(K[≥]) $[\geq](\alpha \Longrightarrow \beta) \Longrightarrow ([\geq]\alpha \Longrightarrow [\geq]\beta)$

(K[o]) $[\mathsf{o}](\alpha \Longrightarrow \beta) \Longrightarrow ([\mathsf{o}]\alpha \Longrightarrow [\mathsf{o}]\beta)$

(K[u]) $[\mathsf{u}](\alpha \Longrightarrow \beta) \Longrightarrow ([\mathsf{u}]\alpha \Longrightarrow [\mathsf{u}]\beta)$

(K[c]) $[\mathsf{c}](\alpha \Longrightarrow \beta) \Longrightarrow ([\mathsf{c}]\alpha \Longrightarrow [\mathsf{c}]\beta)$

(K[⪯]) $[\preceq](\alpha \Longrightarrow \beta) \Longrightarrow ([\preceq]\alpha \Longrightarrow [\preceq]\beta)$

(K[⪰]) $[\succeq](\alpha \Longrightarrow \beta) \Longrightarrow ([\succeq]\alpha \Longrightarrow [\succeq]\beta)$

(K[O]) $[\mathsf{O}](\alpha \Longrightarrow \beta) \Longrightarrow ([\mathsf{O}]\alpha \Longrightarrow [\mathsf{O}]\beta)$

(K[U]) $[\mathsf{U}](\alpha \Longrightarrow \beta) \Longrightarrow ([\mathsf{U}]\alpha \Longrightarrow [\mathsf{U}]\beta)$

(K[C]) $[\mathsf{C}](\alpha \Longrightarrow \beta) \Longrightarrow ([\mathsf{C}]\alpha \Longrightarrow [\mathsf{C}]\beta)$

(K[A]) $[\mathsf{A}](\alpha \Longrightarrow \beta) \Longrightarrow ([\mathsf{A}]\alpha \Longrightarrow [\mathsf{A}]\beta)$

Now, we add axioms for each of the conditions from Definition 21. They are just the modal formulae that define the 42 conditions and are denoted with the same notations - (M1), (M2), (M3′) – (M3‴), (M4) – (M30), (C1) – (C10). All of the following formulae are Sahlqvist, provided that α, β and γ are propositional variables.

(M1) $[\leq]\alpha \Longrightarrow \alpha$

(M2) $[\leq]\alpha \Longrightarrow [\leq][\leq]\alpha$

(M3′) $\langle\leq\rangle([\mathsf{O}]\alpha \mathbin{\&} \neg\alpha \mathbin{\&} \beta) \Longrightarrow \beta$

(M3″) $\langle\geq\rangle([\mathsf{U}]\alpha \mathbin{\&} \neg\alpha \mathbin{\&} \beta) \Longrightarrow \beta$

(M3‴) $\langle\mathsf{A}\rangle(\beta \mathbin{\&} \neg\gamma \mathbin{\&} \langle\leq\rangle(\alpha \mathbin{\&} \gamma)) \Longrightarrow \langle\mathsf{O}\rangle\alpha \vee \langle\mathsf{U}\rangle\beta$

(M4) $\langle\mathsf{O}\rangle[\mathsf{O}]\alpha \Longrightarrow \alpha$

(M5) $\langle\mathsf{O}\rangle\top \mathbin{\&} [\mathsf{O}]\alpha \Longrightarrow \alpha$

(M6) $[\mathsf{O}]\alpha \Longrightarrow [\mathsf{O}][\leq]\alpha$

(M7) $[\mathsf{O}]\alpha \mathbin{\&} [\leq]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta$

(M8) $\langle\mathsf{U}\rangle[\mathsf{U}]\alpha \Longrightarrow \alpha$

(M9) $\langle\mathsf{U}\rangle\top \mathbin{\&} [\mathsf{U}]\alpha \Longrightarrow \alpha$

(M10) $[\mathsf{U}]\alpha \Longrightarrow [\leq][\mathsf{U}]\alpha$

(M11) $[\mathsf{U}]\alpha \mathbin{\&} [\geq]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta$

(M12) $[\leq]\alpha \mathbin{\&} [\mathsf{O}]\beta \mathbin{\&} \langle\mathsf{A}\rangle([\mathsf{U}]\gamma \mathbin{\&} \neg\alpha) \Longrightarrow [\mathsf{A}](\beta \vee \gamma)$

(M13) $([\mathsf{O}]\alpha \Longrightarrow \alpha) \vee ([\mathsf{U}]\beta \Longrightarrow \beta)$

(M14) $[\preceq]\alpha \Longrightarrow \alpha$

(M15) $[\preceq]\alpha \Longrightarrow [\leq][\preceq]\alpha$

(M16) $[\preceq]\alpha \Longrightarrow [\preceq][\leq]\alpha$

(M17) $\langle\mathsf{o}\rangle[\mathsf{o}]\alpha \Longrightarrow \alpha$

(M18) $\langle\mathsf{o}\rangle\top \mathbin{\&} [\mathsf{o}]\alpha \Longrightarrow \alpha$

(M19) $[\mathsf{o}]\alpha \Longrightarrow [\mathsf{o}][\leq]\alpha$

(M20) $[\mathsf{O}]\alpha \Longrightarrow [\mathsf{o}][\preceq]\alpha$

(M21) $[\mathsf{o}]\alpha \mathbin{\&} [\preceq]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta$

(M22) $[\preceq]\alpha \mathbin{\&} [\mathsf{o}]\beta \mathbin{\&} \langle\mathsf{A}\rangle([\mathsf{U}]\gamma \mathbin{\&} \neg\alpha) \Longrightarrow [\mathsf{A}](\beta \vee \gamma)$

(M23) $\langle\mathsf{u}\rangle[\mathsf{u}]\alpha \Longrightarrow \alpha$

(M24) $\langle\mathsf{u}\rangle\top \mathbin{\&} [\mathsf{u}]\alpha \Longrightarrow \alpha$

(M25) $[\mathsf{u}]\alpha \Longrightarrow [\leq][\mathsf{u}]\alpha$

(M26) $[\mathsf{U}]\alpha \Longrightarrow [\preceq][\mathsf{u}]\alpha$

(M27) $[\preceq]\alpha \mathbin{\&} [\mathsf{O}]\beta \mathbin{\&} \langle\mathsf{A}\rangle([\mathsf{u}]\gamma \mathbin{\&} \neg\alpha) \Longrightarrow [\mathsf{A}](\beta \vee \gamma)$

(M28) $[\mathsf{u}]\alpha \mathbin{\&} [\succeq]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta$

(M29) $([\mathsf{o}]\alpha \Longrightarrow \alpha) \vee ([\mathsf{U}]\beta \Longrightarrow \beta)$

(M30) $([\mathsf{O}]\alpha \Longrightarrow \alpha) \vee ([\mathsf{u}]\beta \Longrightarrow \beta)$

(C1) $\langle\mathsf{C}\rangle[\mathsf{C}]\alpha \Longrightarrow \alpha$

(C2) $[\mathsf{C}]\alpha \Longrightarrow [\mathsf{O}]\alpha$

(C3) $\langle\mathsf{C}\rangle\top \mathbin{\&} [\mathsf{O}]\alpha \Longrightarrow \alpha$

(C4) $[\mathsf{C}]\alpha \Longrightarrow [\mathsf{C}][\leq]\alpha$

(C5) $\langle\mathsf{c}\rangle[\mathsf{c}]\alpha \Longrightarrow \alpha$

(C6) $[\mathsf{c}]\alpha \Longrightarrow [\mathsf{o}]\alpha$

(C7) $\langle\mathsf{c}\rangle\top \mathbin{\&} [\mathsf{o}]\alpha \Longrightarrow \alpha$

(C8) $[\mathsf{c}]\alpha \Longrightarrow [\mathsf{c}][\leq]\alpha$

(C9) $[\mathsf{C}]\alpha \Longrightarrow [\mathsf{c}][\preceq]\alpha$

(C10) $\alpha \;\&\; \langle\mathsf{c}\rangle\beta \;\&\; \langle\mathsf{A}\rangle([\mathsf{O}]\neg\alpha \;\&\; \gamma) \Longrightarrow [\mathsf{A}](\langle\mathsf{O}\rangle\beta \vee \langle\mathsf{u}\rangle\gamma \vee \langle\mathsf{C}\rangle\gamma)$

Here $\top$ is the truth constant or a fixed Boolean tautology.

We have also axioms for the universal relation A which define it as an equivalence relation that contains all other relations

(A1) $[\mathsf{A}]\alpha \Longrightarrow \alpha$

(A2) $\langle\mathsf{A}\rangle[\mathsf{A}]\alpha \Longrightarrow \alpha$

(A3) $[\mathsf{A}]\alpha \Longrightarrow [\mathsf{A}][\mathsf{A}]\alpha$

(A4) $[\mathsf{A}]\alpha \Longrightarrow [\leq]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\geq]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\mathsf{o}]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\mathsf{u}]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\mathsf{c}]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\preceq]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\succeq]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\mathsf{O}]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\mathsf{U}]\alpha$

$[\mathsf{A}]\alpha \Longrightarrow [\mathsf{C}]\alpha$

Finally we have axioms for modalities $\geq$ and $\succeq$ which define them as the converse relations of $\leq$ and $\preceq$:

($\geq$1) $\langle\leq\rangle[\geq]\alpha \Longrightarrow \alpha$

($\succeq$1) $\langle\preceq\rangle[\succeq]\alpha \Longrightarrow \alpha$

($\geq$2) $\langle\geq\rangle[\leq]\alpha \Longrightarrow \alpha$

($\succeq$2) $\langle\succeq\rangle[\preceq]\alpha \Longrightarrow \alpha$

Inference rules:

(Modus Ponens) $\dfrac{\alpha \Longrightarrow \beta, \alpha}{\beta}$

(Necessitation) $\dfrac{\alpha}{[R]\alpha}$, for each $R \in \{\, \leq, \geq, \mathsf{o}, \mathsf{u}, \preceq, \succeq, \mathsf{O}, \mathsf{U}, \mathsf{A} \,\}$

Most of the first-order conditions (M1), (M2), (M3′) − (M3‴), (M4) − (M30), (C1) − (C10) are standard universal conditions for reflexivity, symmetricity, transitivity or minor variations or combinations of such properties, which are proved with small modifications of the standard proofs (see [5], [18] for details and more examples).

$$
\begin{array}{rll}
[R]\alpha \Longrightarrow \alpha & \text{defines} & x\ R\ x, \\
[R]\alpha \Longrightarrow [R][R]\alpha & \text{defines} & x\ R\ y\ \&\ y\ R\ z \Longrightarrow x\ R\ z, \\
\langle R\rangle[R]\alpha \Longrightarrow \alpha & \text{defines} & x\ R\ y \Longrightarrow y\ R\ x, \\
[S]\alpha \Longrightarrow [R]\alpha & \text{defines} & x\ R\ y \Longrightarrow x\ S\ y, \\
[T]\alpha \Longrightarrow [R][S]\alpha & \text{defines} & x\ R\ y\ \&\ y\ S\ z \Longrightarrow x\ T\ z, \\
([R]\alpha \Longrightarrow \alpha) \vee ([S]\beta \Longrightarrow \beta) & \text{defines} & x\ R\ x \text{ or } x\ S\ x, \\
\langle R\rangle[R^{-1}]\alpha \Longrightarrow \alpha & \text{defines} & x\ R\ y \Longrightarrow y\ R^{-1}\ x, \\
\langle R^{-1}\rangle[R]\alpha \Longrightarrow \alpha & \text{defines} & x\ R^{-1}\ y \Longrightarrow y\ R\ x.
\end{array}
$$

These definability statements cover conditions (M1), (M2), (M4), (M6), (M8), (M10), (M13), (C1), (C2), (C4), (M14) – (M17), (M19), (M20), (M23), (M25), (M26), (M29), (M30), (C5), (C6), (C8) and (C9). Thus, we have that these conditions are modally defined by their corresponding modal axioms. For the remaining, more complex, conditions we will present the following definability statements and prove a selected few of them as representative examples. Some of these definabilities are used also in [35].

The definability of conditions (M7), (M11), (M21) and (M28) is proved with the following

Lemma 35.

$$
\begin{array}{rll}
[R]\alpha\ \&\ [S]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta & \textit{defines} & x\ R\ x \textit{ or } x\ S\ y, \\
[R]\alpha\ \&\ [S^{-1}]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta & \textit{defines} & y\ R\ y \textit{ or } x\ S\ y.
\end{array}
$$

Proof.

We will show a proof only for the first definability statement. The proof for the second one is the same as we transform $y\ R\ y$ or $x\ S\ y$ into $y\ R\ y$ or $y\ S^{-1}\ x$.

($\longrightarrow$) Let $\underline{W}$ be any generalized structure, such that

$$\underline{W} \models [R]\alpha\ \&\ [S]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta.$$

Lets suppose that there are $x, y \in W$, such that $x\ \overline{R}\ x$ and $x\ \overline{S}\ y$.

Then suppose that α and β are propositional variables. We define a special valuation v in the following way

$$
\begin{array}{l}
\alpha \notin v(x) \text{ and } \alpha \in v(t) \text{ for every } t \in W \text{ such that } t \neq x; \\
\beta \notin v(y) \text{ and } \beta \in v(t) \text{ for every } t \in W \text{ such that } t \neq y.
\end{array}
$$

If $x\ R\ t$, because $x\ \overline{R}\ x$, then $t \neq x$. So, by the definition of v we have $(\underline{W}, v), t \models \alpha$ and, thus, $(\underline{W}, v), x \models [R]\alpha$. Similarly, we show that $(\underline{W}, v), x \models [S]\beta$. Since $\alpha \notin v(x)$ and $\beta \notin v(y)$, then we have that $(\underline{W}, v), x \not\models \alpha \vee [\mathsf{A}]\beta$. Thus, we get that

$$(\underline{W}, v), x \not\models [R]\alpha\ \&\ [S]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta,$$

which is a contradiction with the fact that

$$\underline{W} \models [R]\alpha\ \&\ [S]\beta \Longrightarrow \alpha \vee [\mathsf{A}]\beta.$$

($\longleftarrow$) Suppose that $(\underline{W}, v)$ is a model of the logic, such that $x\ R\ x$ or $x\ S\ y$ holds for $\underline{W}$ and also x be an arbitrary element from the universe of $\underline{W}$, such that

$$(\underline{W}, v), x \models [R]\alpha \ \&\ [S]\beta.$$

If $x\ R\ x$ we have that from $(\underline{W}, v), x \models [R]\alpha$ it follows $(\underline{W}, v), x \models \alpha$. If $x\ \overline{R}\ x$ then we have $x\ S\ y$ for all $y \in W$. So, from $(\underline{W}, v), x \models [S]\beta$ we have that $(\underline{W}, v), x \models [\mathsf{A}]\beta$.

Thus, in every case we have that

$$(\underline{W}, v), x \models \alpha \vee [\mathsf{A}]\beta.$$

□

The definability of (M12), (M22) and (M27) is proved with this lemma.

Lemma 36.

$[R]\alpha \ \&\ [S]\beta \ \&\ \langle\mathsf{A}\rangle([T]\gamma \ \&\ \neg\alpha) \Longrightarrow [\mathsf{A}](\beta \vee \gamma)$ *defines* $x\ R\ y$ *or* $x\ S\ z$ *or* $y\ T\ z$.

Proof.

($\longrightarrow$) Let $\underline{W}$ be a generalized structure, such that

$$\underline{W} \models [R]\alpha \ \&\ [S]\beta \ \&\ \langle\mathsf{A}\rangle([T]\gamma \ \&\ \neg\alpha) \Longrightarrow [\mathsf{A}](\beta \vee \gamma).$$

Lets suppose that there are $x, y, z \in W$, such that $x\ \overline{R}\ y$, $x\ \overline{S}\ z$ and $y\ \overline{T}\ z$. Then take α, β and γ to be propositional variables and build the valuation v as follows

$$\alpha \notin v(y) \text{ and } \alpha \in v(t) \text{ for every } t \in W \text{ such that } t \neq y;$$
$$\beta \notin v(z) \text{ and } \beta \in v(t) \text{ for every } t \in W \text{ such that } t \neq z;$$
$$\gamma \notin v(z) \text{ and } \gamma \in v(t) \text{ for every } t \in W \text{ such that } t \neq z.$$

Thus, we can show that $(\underline{W}, v), x \models [R]\alpha$, $(\underline{W}, v), x \models [S]\beta$ and also that $(\underline{W}, v), y \models [T]\gamma \ \&\ \neg\alpha$, from which we get $(\underline{W}, v), x \models \langle\mathsf{A}\rangle([T]\gamma \ \&\ \neg\alpha)$. We also have $(\underline{W}, v), z \models \neg\beta \ \&\ \neg\gamma$, i.e. $(\underline{W}, v), z \not\models \beta \vee \gamma$. Thus, we get $(\underline{W}, v), x \not\models [\mathsf{A}](\beta \vee \gamma)$. This is a contradiction with

$$\underline{W} \models [R]\alpha \ \&\ [S]\beta \ \&\ \langle\mathsf{A}\rangle([T]\gamma \ \&\ \neg\alpha) \Longrightarrow [\mathsf{A}](\beta \vee \gamma).$$

($\longleftarrow$) Suppose that $(\underline{W}, v)$ is a model, such that $x\ R\ y$ or $x\ S\ z$ or $y\ T\ z$ holds and $x \in W$ so that

$$(\underline{W}, v), x \models [R]\alpha,$$
$$(\underline{W}, v), x \models [S]\beta,$$
$$(\underline{W}, v), x \models \langle\mathsf{A}\rangle([T]\gamma \ \&\ \neg\alpha).$$

From the last statement we have that there is an element $y \in W$, such that $(\underline{W}, v), y \models [T]\gamma$ and $(\underline{W}, v), y \models \neg\alpha$.

Now, let we have an arbitrary $z \in W$. $x\ R\ y$ cannot be true, because in this case we would have that $(\underline{W}, v), y \models \alpha$. So $x\ \overline{R}\ y$. Thus either $x\ S\ z$ or $y\ T\ z$. If $x\ S\ z$, then from $(\underline{W}, v), x \models [S]\beta$ we have that $(\underline{W}, v), z \models \beta$. Otherwise, if $y\ T\ z$ then from $(\underline{W}, v), y \models [T]\gamma$ we have that $(\underline{W}, v), z \models \gamma$. In either case we get

$$(\underline{W}, v), z \models \beta \vee \gamma,$$

and thus we conclude that

$$(\underline{W}, v), x \models [\mathsf{A}](\beta \vee \gamma).$$

□

The proof for the following lemma is similar.

Lemma 37.

$$\langle R\rangle T \mathbin{\&} [R]\alpha \Longrightarrow \alpha \quad \textit{defines} \quad x\ R\ y \Longrightarrow x\ R\ x,$$
$$\langle R\rangle T \mathbin{\&} [S]\alpha \Longrightarrow \alpha \quad \textit{defines} \quad x\ R\ y \Longrightarrow x\ S\ x.$$

This lemma show the definability of (M5), (M9), (C3), (M18), (M24) and (C7). The definability of (M3′) − (M3‴) is also shown in the same way (see also [35]). Finally for the definability of (C10) we will use an equivalent form of this condition

$$z\ \mathsf{c}\ t \Longrightarrow y\ \mathsf{O}\ z \text{ or } x\ \mathsf{O}\ t \text{ or } x\ \mathsf{u}\ y \text{ or } x\ \mathsf{C}\ y.$$

Notice that the atomic propositions $z\ \mathsf{O}\ y$ and $t\ \mathsf{O}\ x$ from the original form of (C10) have been reversed - $y\ \mathsf{O}\ z$ and $x\ \mathsf{O}\ t$. So, this modal axiom actually defined a first-order condition, that is slightly different from (C10). This is not a problem, however, since the relation O is symmetric. Thus, the condition that is defined by the modal axiom is equivalent to (C10), with respect to the other first-order conditions. So the structures that are axiomatized with the collection of all modal axioms are indeed the generalized structures.

Lemma 38.

$$\alpha \mathbin{\&} \langle \mathsf{c}\rangle\beta \mathbin{\&} \langle \mathsf{A}\rangle([\mathsf{O}]\neg\alpha \mathbin{\&} \gamma) \Longrightarrow [\mathsf{A}](\langle \mathsf{O}\rangle\beta \vee \langle \mathsf{u}\rangle\gamma \vee \langle \mathsf{C}\rangle\gamma)$$

defines

$$z\ \mathsf{c}\ t \Longrightarrow y\ \mathsf{O}\ z \textit{ or } x\ \mathsf{O}\ t \textit{ or } x\ \mathsf{u}\ y \textit{ or } x\ \mathsf{C}\ y.$$

Proof.

(⟶) Suppose that $\underline{W}$ is a generalized structure, so that

$$\underline{W} \models \alpha \mathbin{\&} \langle \mathsf{c}\rangle\beta \mathbin{\&} \langle \mathsf{A}\rangle([\mathsf{O}]\neg\alpha \mathbin{\&} \gamma) \Longrightarrow [\mathsf{A}](\langle \mathsf{O}\rangle\beta \vee \langle \mathsf{u}\rangle\gamma \vee \langle \mathsf{C}\rangle\gamma).$$

Suppose also, that there are $x, y, z, t \in W$, such that $z\ \mathsf{c}\ t$, $y\ \overline{\mathsf{O}}\ z$, $x\ \overline{\mathsf{O}}\ t$, $x\ \overline{\mathsf{u}}\ y$ and $x\ \overline{\mathsf{C}}\ y$. Then if α, β and γ are propositional variables, we build the following valuation v as follows

$\alpha \in v(z)$ and $\alpha \notin v(u)$ for every $u \in W$ such that $u \neq z$;

$\beta \in v(t)$ and $\beta \notin v(u)$ for every $u \in W$ such that $u \neq t$;

$\gamma \in v(y)$ and $\gamma \notin v(u)$ for every $u \in W$ such that $u \neq y$.

Thus we have that $(\underline{W}, v), z \models \alpha$ and, since $z\ \mathsf{c}\ t$, also $(\underline{W}, v), z \models \langle \mathsf{c}\rangle\beta$. Since α is true only in z and $y\ \overline{\mathsf{O}}\ z$ then $(\underline{W}, v), y \models [\mathsf{O}]\neg\alpha$ and thus $(\underline{W}, v), z \models \langle \mathsf{A}\rangle([\mathsf{O}]\neg\alpha \mathbin{\&} \gamma)$.

Finally, from $x\ \overline{\mathsf{O}}\ t$, $x\ \overline{\mathsf{u}}\ y$ and $x\ \overline{\mathsf{C}}\ y$ and the fact that β is true only in t and γ is true only in y we have that $(\underline{W}, v), x \not\models \langle \mathsf{O}\rangle\beta$, $(\underline{W}, v), x \not\models \langle \mathsf{u}\rangle\gamma$ and $(\underline{W}, v), x \not\models \langle \mathsf{C}\rangle\gamma$. Thus we get that

$$(\underline{W}, v), z \not\models [\mathsf{A}](\langle \mathsf{O}\rangle\beta \vee \langle \mathsf{u}\rangle\gamma \vee \langle \mathsf{C}\rangle\gamma).$$

This is a contradiction with the original assumption.

($\longleftarrow$) Let $(\underline{W}, v)$ be a model, that satisfies the equivalent form of (C10). Let $z \in W$ so that

$$(\underline{W}, v), z \models \alpha,$$
$$(\underline{W}, v), z \models \langle \mathsf{c} \rangle \beta,$$
$$(\underline{W}, v), z \models \langle \mathsf{A} \rangle ([\mathsf{O}] \neg \alpha \ \& \ \gamma).$$

From $(\underline{W}, v), z \models \langle \mathsf{c} \rangle \beta$ we have that there is $t \in W$, such that $z \ \mathsf{c} \ t$ and $(\underline{W}, v), t \models \beta$. From $(\underline{W}, v), z \models \langle \mathsf{A} \rangle ([\mathsf{O}] \neg \alpha \ \& \ \gamma)$ follows that there is $y \in W$ and also $(\underline{W}, v), y \models [\mathsf{O}] \neg \alpha$ and $(\underline{W}, v), y \models \gamma$. From $(\underline{W}, v), y \models [\mathsf{O}] \neg \alpha$ and $(\underline{W}, v), z \models \alpha$ we get that $y \ \overline{\mathsf{O}} \ z$. Lets take and arbitrary element $x \in W$. From $z \ \mathsf{c} \ t$ and $y \ \overline{\mathsf{O}} \ z$, by (C10), we have $x \ \mathsf{O} \ t$ or $x \ \mathsf{u} \ y$ or $x \ \mathsf{C} \ y$. Thus, we have three cases:

case 1: Suppose that $x \ \mathsf{O} \ t$ holds. Then, from $(\underline{W}, v), t \models \beta$, we have $(\underline{W}, v), x \models \langle \mathsf{O} \rangle \beta$.

case 2: The case when $x \ \mathsf{u} \ y$ is similar. Then $(\underline{W}, v), x \models \langle \mathsf{u} \rangle \gamma$ holds.

case 3: In this case from $x \ \mathsf{C} \ y$ follows $(\underline{W}, v), x \models \langle \mathsf{C} \rangle \gamma$.

So in every case we have that $(\underline{W}, v), x \models \langle \mathsf{O} \rangle \beta \vee \langle \mathsf{u} \rangle \gamma \vee \langle \mathsf{C} \rangle \gamma$, . Since this is for an arbitrary x, we get

$$(\underline{W}, v), z \models [\mathsf{A}](\langle \mathsf{O} \rangle \beta \vee \langle \mathsf{u} \rangle \gamma \vee \langle \mathsf{C} \rangle \gamma),$$

which completes the proof.

□

Theorem 20 (Completeness theorem for the modal logics)**.**
The following conditions are equivalent for any formula α:

(1) α is a theorem of the modal logic (is derivable from the axiomatization);
(2) α is valid in all generalized dynamic structures;
(3) α is valid in all dynamic structures;
(4) α is valid in all standard structures.

Proof.
(1) $\longrightarrow$ (2) is a straightforward check of the soundness of the axiom schemes and the inference rules while (2) $\longrightarrow$ (1) is proved through use of standard techniques of generated canonical models (see [5], [6]).

We take a formula α, that is valid in all generalized structure, but suppose that α is not a theorem. Then, there is a maximal theory Γ (a maximal consistent set) that contains $\neg\alpha$. We construct the canonical model $(\underline{W}^C, v^C)$ from Γ, in the standard way. We have that

$$(\underline{W}^C, v^C) \models \neg\alpha, \text{ i.e. } (\underline{W}^C, v^C) \not\models \alpha.$$

Because of the first-order conditions, that are defined by the axioms, we have that $\underline{W}^C$ satisfies (M1), (M2), (M3$'$) $-$ (M3$'''$), (M4) $-$ (M30), (C1) $-$ (C10). The only problem is that A^C is not the universal relation in $\underline{W}^C$, but only an equivalence relation, that contains the other relations. This is corrected as we take a generated sub-model of $(\underline{W}^C, v^C)$ - the model $(\underline{W}', v')$. Now, $(\underline{W}', v')$ is a model of the logic and $\underline{W}'$ is a generalized structure. Thus

$$(\underline{W}', v') \not\models \alpha,$$

which is a contradiction.

(2) $\longleftrightarrow$ (3) follows from Proposition 7.
(3) $\longleftrightarrow$ (4) follows from Proposition 5. $\square$

2.4. **The modal logic of dynamic mereological relations.**

As it was in the case of first-order logics, here we have the mereological variant of the modal logics. Again, the language exclude the dynamic contacts c and C and their corresponding modalities and defining conditions. The formulae in the new language are

$$\begin{array}{rcl}\varphi & ::= & p \mid \neg\varphi \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \Longrightarrow \varphi_2 \mid \varphi_1 \Longleftrightarrow \varphi_2 \mid \\ & & [\leq]\varphi \mid \langle\leq\rangle\varphi \mid [\geq]\varphi \mid \langle\geq\rangle\varphi \mid [\mathsf{o}]\varphi \mid \langle\mathsf{o}\rangle\varphi \mid [\mathsf{u}]\varphi \mid \langle\mathsf{u}\rangle\varphi \mid \\ & & [\preceq]\varphi \mid \langle\preceq\rangle\varphi \mid [\succeq]\varphi \mid \langle\succeq\rangle\varphi \mid [\mathsf{O}]\varphi \mid \langle\mathsf{O}\rangle\varphi \mid [\mathsf{U}]\varphi \mid \langle\mathsf{U}\rangle\varphi \mid \\ & & [\mathsf{A}]\varphi \mid \langle\mathsf{A}\rangle\varphi.\end{array}$$

We use Kripke semantics, based on the standard dynamic mereological structures and the abstractly defined dynamic mereological structures (see Definitions 15 and 16). We have the corresponding modal logics for each class of structures - the standard dynamic mereological modal logic and the dynamic mereological modal logic. The mereological representation theory from Part IV, Section 5 ensures the following

Proposition 8.
The modal logic of the standard dynamic mereological structures is equal to the modal logic of the dynamic mereological structures.

Here we also have generalized dynamic structures, that will produce modal logic with Kripke semantics based on then, which is easier to axiomatize. These structures satisfy the following definition

Definition 23.
Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ *be a relational structure such that* $W \neq \emptyset$ *and* $\leq$, o, u, $\preceq$, O *and* U *be binary relations over* W. *Then* $\underline{W}$ *is called a* generalized dynamic mereological structure *if it satisfies conditions* (M1), (M2), (M3$'$) $-$ (M3$'''$), (M4) $-$ (M30).

The corresponding p-morphism statement that relates the modal logic of the new structures to the previous ones is as follows

Proposition 9.
Let $\underline{W}$ *be a generalized dynamic mereological structure. Then there is a relational structures* $\underline{W}'$, *such that:*

(1) $\underline{W}'$ *is a dynamic mereological structure;*
(2) there exists a p-morphism from $\underline{W}'$ *onto* $\underline{W}$.

Proof.
We use the same variant of Segerberg's bulldozer construction. Let $\underline{W}'$ be the structure $(W', \leq', \mathsf{o}', \mathsf{u}', \preceq', \mathsf{O}', \mathsf{U}')$ and let $\ll$ be a well ordering of W. Let $\mathbb{Z}$ be the set of integers and $\omega \notin \mathbb{Z}$. Then we define the functions f_i:

$$f_i(x) = \begin{cases} (x, \omega), & \text{if } [x] \text{ is degenerate} \\ (x, i), & \text{otherwise} \end{cases}$$

for every $i \in \mathbb{Z}$ and for every $x \in W$.

Finally the universe W' of the new structure and the dynamic mereological relations in it are defined as follows:

$$W' = \{ f_i(x) \mid i \in \mathbb{Z}, x \in W \}$$

For every $i, j \in \mathbb{Z}, x, y \in W$ the relation $\leq'$ is defined:

- if $x \not\equiv y$ or $[x]$ is degenerate or $[y]$ is degenerate: $f_i(x) \leq' f_j(y)$ iff $x \leq y$;
- otherwise: $f_i(x) \leq' f_j(y)$ iff $i < j$ or ($i = j$ and $x \ll y$);

while the relations o', u', $\preceq'$, O' and U' are defined:

$$f_i(x)\ \mathsf{o}'\ f_j(y) \text{ iff } x\ \mathsf{o}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$

$$f_i(x)\ \mathsf{u}'\ f_j(y) \text{ iff } x\ \mathsf{u}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$

$$f_i(x) \preceq' f_j(y) \text{ iff } x \preceq y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$

$$f_i(x)\ \mathsf{O}'\ f_j(y) \text{ iff } x\ \mathsf{O}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W;$$

$$f_i(x)\ \mathsf{U}'\ f_j(y) \text{ iff } x\ \mathsf{U}\ y \text{ for } i, j \in \mathbb{Z}, x, y \in W.$$

□

Thus we have the following result

Proposition 10.

The modal logic of the generalized dynamic mereological structures is equal to the modal logic of the dynamic mereological structures.

Completeness.

The axiomatization of the modal logic of generalized dynamic mereological consists of the following components:

- the Boolean tautologies;
- distribution axioms (K[$\leq$]), (K[$\geq$]), (K[o]), (K[u]), (K[$\preceq$]), (K[$\succeq$]), (K[O]), (K[U]) and (K[A]);
- the modal formulae for (M1), (M2), (M3$'$) $-$ (M3$'''$), (M4) $-$ (M30);
- the axioms for the universal modality - $\mathsf{A}1$, $\mathsf{A}2$, $\mathsf{A}3$ and $\mathsf{A}4$;
- the axioms for the inverse modalities - $\geq$1, $\succeq$1, $\geq$2 and $\succeq$2;
- inference rules *Modus Ponens* and *Necessitation*.

Finally, we get the completeness of the dynamic mereological modal logics. The proof of this theorem is the same as the proof for Theorem 20.

Theorem 21 (Completeness theorem for the modal logics)**.**

The following conditions are equivalent for any formula α:

(1) α is a theorem of the modal logic;
(2) α is valid in all generalized dynamic mereological structures;
(3) α is valid in all dynamic mereological structures;
(4) α is valid in all standard dynamic mereological structures.

Part VI. Decidability and undecidability of logics for dynamic mereotopological relations

Here we will address the problem of decidability of the logics, described in the previous part - Part V. We will also give complexity results for some of the logics - for the fragments of the first-order theories. We show that the satisfiability problems of these fragments are NP-complete. Complexity results for the other decidable logics, however, are not present yet and are subject to future work.

In Section 1 we prove that the full first-order theories of dynamic mereotopological and mereological relations are *hereditary undecidable*. Hereditary undecidable means that the theory itself is undecidable and also all of its subtheories in the same language are undecidable. In fact, we give a general construction, that proves undecidability not only for these two theories, but for a whole array of theories which have the overlap relations or the contact relations in their languages. For some of these theories undecidability was proved already, but in most cases it is only undecidability and not hereditary undecidability. Thus, these results may be considered to be an extension of the previous undecidability results.

Since the full first-order theories are undecidable then it makes sense to look for decidable alternatives. We will consider several decidable fragments of these theories. In Section 2 we review the quantifier-free fragments and show that they are decidable and their satisfiability problems are NP-complete. The same complexity is proved for very similar temporalized spatial logic in [20]. There the box and diamond operators from LTL are applied to formulae from the language RC for reasoning about regions from topological spaces. We can relate the satisfiability problem of the current fragments to the satisfiability problem of the logics from [20].

Also in Section 3 we show decidability for two fragments of the modal logic of the dynamic mereotopological relational and decidability for the mereological modal logic. These modal logics can also be considered as fragments of the full first-order theories (as modal logic in general can be considered as a fragment of first-order logic). The results in this section are also improvement of previous results about decidability of modal logics for mereological relations. For example, modal logics based on RCC-5 are proved to be undecidable in [21].

Finally, we conclude the part with a section, dedicated to the unsolved problems and future work about decidability and complexity of the current logics.

1. Hereditary undecidability of the overlap and the contact

We use the method described in [13], [14] and [15] to prove hereditary undecidability for the described theories. The method begins from the fact that the first-order theory of the class of finite structures with single symmetric and irreflexive relation $\Sigma^{\text{fin}}_{\text{sym,irref}}$ is hereditary undecidable. To prove hereditary undecidability for a theory T we will show that $\Sigma^{\text{fin}}_{\text{sym,irref}}$ is *relatively elementary interpretable* in the class of models of T.

By definition $\Sigma^{\text{fin}}_{\text{sym,irref}}$ is relatively elementary interpretable in the models of T if there exist formulae $\alpha(a)$, $\beta(a,b)$, $\gamma(a,b)$ in the first-order language of the theory T, such that for every structure $\underline{M} \in \Sigma^{\text{fin}}_{\text{sym,irref}}$ there exists a model $\underline{W}$ of T with universe W, so that the following conditions are true:

(1) the set $L = \{ a \in W \mid \alpha(a) \text{ is true in } \underline{W} \}$ is non-empty;
(2) if $\gamma(a,b)$ interprets the relation S in the language of T, then $\beta(a,b)$ defines a congruence η in L with respect to the interpretation of S;
(3) if $\underline{L}$ is a structure in the language of $\Sigma^{\text{fin}}_{\text{sym,irref}}$ defined by the universe L and the interpretation of S by $\gamma(a,b)$, then $\underline{M}$ is isomorphic to the factor model $\underline{L}_{/\eta}$.

Here S denotes the symmetric and irreflexive relation from the language of $\Sigma^{\text{fin}}_{\text{sym,irref}}$.

We will show that $\Sigma^{\text{fin}}_{\text{sym,irref}}$ is relatively elementary interpretable in the models of any theory T, such that its models contain certain kind of structures.

1.1. **Construction for structures with mereological relations.**

First we show the condition about the models of theories for mereological and mereotopological relations. We will use the mereological relation overlap to interpret the relation S. With this condition we aim to prove hereditary undecidability for theories which have overlap as part of their language.

The formulae required for the relatively elementary interpretation will be:

$$\begin{aligned}
\alpha(a) &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad a = a,\\
\beta(a,b) &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad a = b,\\
\gamma(a,b) &\quad \stackrel{\text{def}}{\longleftrightarrow} \quad a \neq b \;\&\; a \mathsf{O} b.
\end{aligned}$$

Let $\underline{M} = (M, S)$ be any structure from $\Sigma^{\text{fin}}_{\text{sym,irref}}$.

Let h be a mapping from M to $\mathcal{P}(M \times M)$, which is defined:

$$h(x) = \{ \langle x, x\rangle \} \cup \{ \langle x, y\rangle \mid x \; S \; y \} \cup \{ \langle y, x\rangle \mid x \; S \; y \}.$$

Note that the only pair in $h(x)$ for which the first and the second elements are equal is $\langle x, x\rangle$ itself.

Let $W = \{ h(x) \mid x \in M \}$. Lets define the binary relation O for $\forall a, b \in W$: $a \mathsf{O} b \Longleftrightarrow a \cap b \neq \emptyset$. We will show that (1), (2) and (3) apply for $\underline{W} = (W, \mathsf{O})$:

(1) since $M \neq \emptyset$ then $L = W \neq \emptyset$;
(2) the formula $a = b$ defines a trivial congruence in L (and thus $\underline{L}_{/\eta} = \underline{L}$);
(3) finally we have to prove that $\underline{M}$ is isomorphic to $\underline{L}$. Clearly h is a bijection between M and L, because every $h(x)$ contains the pair $\langle x, x\rangle$ which is unique to it. So it suffices to show that $x \; S \; y$ iff $\gamma(h(x), h(y))$ is true in $\underline{W}$.

($\longrightarrow$) Suppose $x\ S\ y$. Since S is irreflexive then $x \neq y$ and so $h(x) \neq h(y)$. Since S is symmetric then by construction $\langle x, y\rangle \in h(x)$, $\langle y, x\rangle \in h(x)$, $\langle x, y\rangle \in h(y)$ and $\langle y, x\rangle \in h(y)$. So $h(x) \cap h(y) \neq \emptyset$ and again by construction $h(x)\ \mathsf{O}\ h(y)$. Thus, $\gamma(h(x), h(y))$ is true in $\underline{W}$.

($\longleftarrow$) Now suppose $\gamma(h(x), h(y))$ is true in $\underline{W}$. So $h(x) \neq h(y)$ and thus $x \neq y$. From $h(x)\ \mathsf{O}\ h(y)$ it follows that $h(x)$ and $h(y)$ have a common pair that can be either $\langle x, y\rangle$ or $\langle y, x\rangle$. In case $\langle y, x\rangle$ is this common pair then by the construction we have that $\langle x, y\rangle$ is also a common pair. Given that $x \neq y$ then by the definition of h it follows that $x\ S\ y$.

Note that if we add more relations to the language of $\underline{W}$ it does not change the result, as long as we do not change the construction of the universe W or the semantics of the relation O. Thus, we can apply this construction in the following cases.

The theory of RCC-5.

We just have to add the Boolean ordering $\leq$ to the language of (W, O) and all five relations from RCC-5 are definable with $\leq$ and O (see Part II, Section 1.3). This gives appropriate models for RCC-5. Thus, we have

Proposition 11.
$\Sigma^{fin}_{sym,irref}$ *is relatively elementary interpretable in the class of models of RCC-5.*

Proof.
Take a model $\underline{M} \in \Sigma^{\mathrm{fin}}_{\mathrm{sym,irref}}$. Then construct (W, O) from it and extend (W, O) to $(W, \leq, \mathsf{O})$. Then, using the definitions from Part II, Section 1.3, obtain an extension by definitions of $(W, \leq, \mathsf{O})$, which is a model of the RCC-5 theory. This is the model $\underline{L}$ which corresponds to $\underline{M}$. $\square$

Corollary 3.
The first-order theory of RCC-5 is hereditary undecidable.

The theory of static mereological relations.

This is the theory of static mereological structures $(W, \leq, \mathsf{O}, \mathsf{U})$ from [35] (see also Part III). The language of this theory contains the overlap relation, as well as relations part-of and underlap, which are defined independently.

Proposition 12.
$\Sigma^{fin}_{sym,irref}$ *is relatively elementary interpretable in the class of static mereological structures.*

Proof.
Construct (W, O) from $\underline{M} \in \Sigma^{\mathrm{fin}}_{\mathrm{sym,irref}}$ and extend it to a model $(W, \leq, \mathsf{O}, \mathsf{U})$ by adding two new relations - the Boolean ordering $\leq$ and the underlap relation U. This is the needed model. $\square$

Corollary 4.
The first-order theory of static mereological relations is hereditary undecidable.

The theory of static mereotopological relations.

This is the theory of static structures $(W, \leq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ (see Part III, Definition 6) with two more relations - the *dual contact* and *internal part-of*. These structures are studied in [35], where they are denoted as $(W, \leq, \mathsf{O}, \widehat{\mathsf{O}}, \mathsf{C}, \widehat{\mathsf{C}}, \ll)$. $\widehat{\mathsf{O}}$ is alternative notation for the underlap relation. $\widehat{\mathsf{C}}$ and $\ll$ are the dual contact and the internal part-of.

Proposition 13.
$\Sigma^{fin}_{sym,irref}$ *is relatively elementary interpretable in the class of static mereotopological structures.*

Proof.
Extend (W, O) to a model $(W, \leq, \mathsf{O}, \widehat{\mathsf{O}})$ by adding $\leq$ and $\widehat{\mathsf{O}}$. Then define the remaining relations as follows

$$\begin{aligned} x \,\mathsf{C}\, y &\overset{\text{def}}{\longleftrightarrow} x \,\mathsf{O}\, y, \\ x \,\widehat{\mathsf{C}}\, y &\overset{\text{def}}{\longleftrightarrow} x \,\widehat{\mathsf{O}}\, y, \\ x \ll y &\overset{\text{def}}{\longleftrightarrow} x \leq y. \end{aligned}$$

I.e. $\mathsf{C} = \mathsf{O}$, $\widehat{\mathsf{C}} = \widehat{\mathsf{O}}$ and $\ll\, = \,\leq$. The result $(W, \leq, \mathsf{O}, \widehat{\mathsf{O}}, \mathsf{C}, \widehat{\mathsf{C}}, \ll)$ is the required target model. $\square$

Corollary 5.
The first-order theory of static mereotopological relations is hereditary undecidable.

The theory of dynamic mereotopological relations.

This is the first-order theory of the standard dynamic structures, presented in Part V, Subsection 1.1.

Proposition 14.
$\Sigma^{fin}_{sym,irref}$ *is relatively elementary interpretable in the class of standard structures.*

Proof.
Extend (W, O) to $(W, \leq, \mathsf{O}, \mathsf{U})$ and then extend the second model to the final model - the standard structure $(W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$. The new relations in the language are defined $\preceq\, = \,\leq$, $\mathsf{o} = \mathsf{c} = \mathsf{C} = \mathsf{O}$ and $\mathsf{u} = \mathsf{U}$. $\square$

Corollary 6.
The first-order theory of dynamic mereotopological relations is hereditary undecidable.

The theory of dynamic mereological relations.

This is the first-order theory of the standard dynamic mereological structures, presented in Part V, Subsection 1.3.

Proposition 15.
$\Sigma^{fin}_{sym,irref}$ *is relatively elementary interpretable in the class of standard dynamic mereological structures.*

Proof.
Extend (W, O) first to $(W, \leq, \mathsf{O}, \mathsf{U})$ and then to $(W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ by defining $\preceq = \leq$, $\mathsf{o} = \mathsf{O}$ and $\mathsf{u} = \mathsf{U}$. $\square$

Corollary 7.
The first-order theory of dynamic mereological relations is hereditary undecidable.

1.2. **Construction for structures with contact relations.**

To prove hereditary undecidability for theories that use contact algebras as models, however, we cannot use the above construction. That is because the contact algebras require their universe to be a Boolean algebra, while the universe W from the previous case is not. So we show another construction for contact algebras that interprets the relation S with the contact relation C.

Let the formulae for the interpretation be:

$$\begin{aligned} \alpha(a) &\stackrel{\text{def}}{\longleftrightarrow} a \neq 0 \ \&\ (\forall b)(b \neq 0 \ \&\ b \leq a \to b = a), \\ \beta(a, b) &\stackrel{\text{def}}{\longleftrightarrow} a = b, \\ \gamma(a, b) &\stackrel{\text{def}}{\longleftrightarrow} a \neq b \ \&\ a \mathrel{\mathsf{C}} b. \end{aligned}$$

Here we use the fact that $\leq$ is definable in the language of contact algebras.

Let $\underline{M} = (M, S)$ be a structure from $\Sigma^{\text{fin}}_{\text{sym,irref}}$ and (M, S') is a frame produced from (M, S) by adding reflexivity to S: $S' = S \cup \{ \langle x, x \rangle \mid x \in M \}$.

Thus let B be the Boolean algebra $B = (\mathcal{P}(M), \emptyset, M, \cap, \cup, \setminus)$ and define C as follows: $a \mathrel{\mathsf{C}} b \Longleftrightarrow \exists x \in a, \exists y \in b : x \mathrel{S'} y$, for $\forall a, b \subseteq M$. Thus (B, C) is a contact algebra. It remains to prove that (1), (2) and (3) apply for (B, C):

(1) $\alpha(a)$ actually defines the singletons in $\mathcal{P}(M)$. So $L = \{ \{x\} \mid x \in M \}$ is non-empty;
(2) $\beta(a, b)$ defines trivial congruence in L and $\underline{L}_{/\eta} = \underline{L}$;
(3) let h be the bijective mapping from M to L: $h(x) = \{x\}$. We have to show that $x \mathrel{S} y$ iff $\gamma(h(x), h(y))$ is true in (B, C). If $x \mathrel{S} y$ then $x \neq y$ and so $h(x) \neq h(y)$. Also $h(x) \mathrel{\mathsf{C}} h(y)$ is true by the definition of C. So $\gamma(h(x), h(y))$ is true in (B, C). The converse statement is also guaranteed by the definition of C: if $h(x) \mathrel{\mathsf{C}} h(y)$ then $x \mathrel{S'} y$ and if $h(x) \neq h(y)$ then $x \neq y$ and this combined with $x \mathrel{S'} y$ gives that $x \mathrel{S} y$.

We apply this second construction for three first-order theories.

The theory of contact algebras.

Proposition 16.
$\Sigma^{fin}_{sym,irref}$ *is relatively elementary interpretable in the class of contact algebras.*

Proof.
We just use the above construction without any change. $\square$

Corollary 8.
The first-order theory of contact algebras is hereditary undecidable.

The theory of dynamic contact algebras.

As showed in [47] the dynamic contact algebras $(B, \mathsf{C}^{\forall}, \mathsf{C}^{\exists})$ can be obtained through products of static contact algebras. Thus,

Proposition 17.
$\Sigma^{fin}_{sym,irref}$ is relatively elementary interpretable in the class of dynamic contact algebras.

Proof.
Start from $\underline{M} \in \Sigma^{\mathrm{fin}}_{\mathrm{sym,irref}}$ and obtain (B, C) with the construction. Then obtain $(B, \mathsf{C}^{\forall}, \mathsf{C}^{\exists})$ as the product of this single contact algebra (i.e. $\mathsf{C}^{\forall} = \mathsf{C}^{\exists} = \mathsf{C}$). Thus, $(B, \mathsf{C}^{\forall}, \mathsf{C}^{\exists})$ is the needed model of the theory of dynamic contact algebras. □

Corollary 9.
The first-order theory of dynamic contact algebras is hereditary undecidable.

The theory of RCC-8.

The RCC-8 relations are definable in the language of contact algebras (see Part II, Section 1.3).

Proposition 18.
$\Sigma^{fin}_{sym,irref}$ is relatively elementary interpretable in the class of models of RCC-8.

Proof.
Construct a contact algebra (B, C) from $\underline{M} \in \Sigma^{\mathrm{fin}}_{\mathrm{sym,irref}}$. Then define with the formulae from Part II, Section 1.3 a model for RCC-8. This model preserves the contact relation C and, thus, the interpretability result holds. □

Corollary 10.
The first-order theory of RCC-8 is hereditary undecidable.

2. Decidability of the quantifier-free fragments

Since, in the last section we showed, that the full first-order theory of dynamic mereotopological relations (Part V, Subsection 1.2) and the full first-order theory of dynamic mereological relations (Part V, Subsection 1.4) are hereditary undecidable we will consider the quantifier-free fragments of this theories. We will show that these fragments are decidable and that their satisfiability problems are NP-complete problems. These results will be proved by showing that the fragments have the *polysize model property*. That means that if we have a model of the fragment $(\underline{W}, v)$ and a formula α, then there is a finite model of the fragment $(\underline{W'}, v')$ such that

$$(\underline{W}, v) \models \alpha \text{ iff } (\underline{W'}, v') \models \alpha.$$

2.1. The mereotopological quantifier-free fragment.

Proposition 19.

The quantifier-free fragment of the first-order logic of dynamic mereotopological relations has the polysize model property.

Proof.

Let $(\underline{W}, v)$ and α be an arbitrary model and formula. We will build a finite model $(\underline{W'}, v')$ that will preserve the truth of the formula.

Let $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$. Let $Var(\alpha)$ denote the set of all object variables in α. Then let $\underline{W'}$ be the substructure of $\underline{W}$ determined by the values of the variables from $Var(\alpha)$ with respect to v:

$$W' = \{\ v(x) \mid x \in Var(\alpha)\ \};$$

$$\begin{array}{rcl}
v(x) \leq' v(y) & \longleftrightarrow & v(x) \leq v(y), \text{ for every } x \in Var(\alpha); \\
v(x)\ \mathsf{o}'\ v(y) & \longleftrightarrow & v(x)\ \mathsf{o}\ v(y), \text{ for every } x \in Var(\alpha); \\
v(x)\ \mathsf{u}'\ v(y) & \longleftrightarrow & v(x)\ \mathsf{u}\ v(y), \text{ for every } x \in Var(\alpha); \\
v(x)\ \mathsf{c}'\ v(y) & \longleftrightarrow & v(x)\ \mathsf{c}\ v(y), \text{ for every } x \in Var(\alpha); \\
v(x) \preceq' v(y) & \longleftrightarrow & v(x) \preceq v(y), \text{ for every } x \in Var(\alpha); \\
v(x)\ \mathsf{O}'\ v(y) & \longleftrightarrow & v(x)\ \mathsf{O}\ v(y), \text{ for every } x \in Var(\alpha); \\
v(x)\ \mathsf{U}'\ v(y) & \longleftrightarrow & v(x)\ \mathsf{U}\ v(y), \text{ for every } x \in Var(\alpha); \\
v(x)\ \mathsf{C}'\ v(y) & \longleftrightarrow & v(x)\ \mathsf{C}\ v(y), \text{ for every } x \in Var(\alpha).
\end{array}$$

The size of $\underline{W'}$ is limited by the size of α. Then let v' be just an arbitrary valuation in W' that preserves the same values of the variables from $Var(\alpha)$ under v:

$$v'(x) = v(x) \text{ for every } x \in Var(\alpha).$$

Since there are no quantifiers in the language, α is just a Boolean combination of atomic formulae of the kind $x \leq y$, $x\ \mathsf{o}\ y$, $x\ \mathsf{u}\ y$, $x\ \mathsf{c}\ y$, $x \preceq y$, $x\ \mathsf{O}\ y$, $x\ \mathsf{U}\ y$ or $x\ \mathsf{C}\ y$. Then it is clear that its truth depends only on the valuation of the variables in it. And because the new model contains the valuations of all variables from $Var(\alpha)$, it follows that

$$(\underline{W}, v) \models \alpha \text{ iff } (\underline{W'}, v') \models \alpha.$$

$\square$

From this proposition we get the following two corollaries.

Corollary 11.

The quantifier-free fragment of the first-order logic of dynamic mereotopological relations is decidable.

Corollary 12.

The satisfiability problem of the quantifier-free fragment of the first-order logic of dynamic mereotopological relations is NP-complete.

2.2. **The mereological quantifier-free fragment.**

For this fragment the situation is virtually the same. The difference is that we do not have relations c and C in the language. The semantics for the rest of the predicates and Boolean connectives remains unchanged. Thus, we have the same decidability and satisfiability results.

Proposition 20.

The quantifier-free fragment of the first-order logic of dynamic mereological relations has the polysize model property.

Proof.

The proof is the same as for the previous fragment. □

Thus, we obtain the results for this fragment as well.

Corollary 13.

The quantifier-free fragment of the first-order logic of dynamic mereological relations is decidable.

Corollary 14.

The satisfiability problem of the quantifier-free fragment of the first-order logic of dynamic mereological relations is NP-complete.

3. Decidability of the modal logics

Here we show the decidability of two fragments of the modal logic of dynamic mereotopological relations (from Part V, Section 2.1). The first fragment is obtained by dropping the stable contact c from the system. The second fragment is obtained when we take out the unstable part-of $\preceq$. The decidability is proved by showing that the fragments have the strong form of the finite model property. This is done by proving that if we use the Kripke semantics defined through the generalized structures then the logic of the fragment admits filtration. These proofs are located in the first two subsections - the first subsection is for the fragment without c and the second is for the one without $\preceq$.

The last subsection shows the decidability of the modal logic of dynamic mereological relations (see Part V, Section 2.4). This logic can also be considered as a fragment of the mereotopological modal logic - this will be the fragment without dynamic contact modalities c and C. The proof is also via application of a filtration over the fragment.

3.1. Reduct of the modal logic of mereotopological dynamic relations without c.

Here we consider the fragment of the modal logic of dynamic mereotopological relations, which does not contain the c modality. I.e. the modal formulae from this fragment does not contain the modal operators $[\mathsf{c}]$ and $\langle\mathsf{c}\rangle$. Thus, we consider the models of this fragment to be structures of the form $(W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$, which satisfy those conditions from Definition 21, which does not concert c - (M1), (M2), (M3′) − (M3‴), (M4) − (M30), (C1) − (C4). We will consider these structures to be the *generalized semantics* of the fragment.

Proposition 21.

The fragment of the modal logic without c *admits filtration with respect to its generalized semantics.*

Proof.

Let $(\underline{W}, v)$ be a model of the fragment, where $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U}, \mathsf{C})$ satisfies (M1), (M2), (M3′) − (M3‴), (M4) − (M30), (C1) − (C4). We will build a finite structure $\underline{W}' = (W', \leq', \mathsf{o}', \mathsf{u}', \preceq', \mathsf{O}', \mathsf{U}', \mathsf{C}')$, which satisfies the same conditions, and a corresponding model $(\underline{W}', v')$, such that the following filtration conditions are satisfied for all $x, y \in W$:

(R1)
$$x \leq y \text{ implies } f(x) \leq' f(y),$$
$$x\ \mathsf{o}\ y \text{ implies } f(x)\ \mathsf{o}'\ f(y),$$
$$x\ \mathsf{u}\ y \text{ implies } f(x)\ \mathsf{u}'\ f(y),$$
$$x \preceq y \text{ implies } f(x) \preceq' f(y),$$
$$x\ \mathsf{O}\ y \text{ implies } f(x)\ \mathsf{O}'\ f(y),$$
$$x\ \mathsf{U}\ y \text{ implies } f(x)\ \mathsf{U}'\ f(y),$$
$$x\ \mathsf{C}\ y \text{ implies } f(x)\ \mathsf{C}'\ f(y),$$

(R2) $f(x) \leq' f(y)$, $[\leq]\alpha \in \Gamma$ and $(\underline{W}, v), x \models [\leq]\alpha$ imply $(\underline{W}, v), y \models \alpha$,

$f(x)\ \mathsf{o}'\ f(y)$, $[\mathsf{o}]\alpha \in \Gamma$ and $(\underline{W}, v), x \models [\mathsf{o}]\alpha$ imply $(\underline{W}, v), y \models \alpha$,

$f(x)\ \mathsf{u}'\ f(y)$, $[\mathsf{u}]\alpha \in \Gamma$ and $(\underline{W}, v), x \models [\mathsf{u}]\alpha$ imply $(\underline{W}, v), y \models \alpha$,

$f(x) \preceq' f(y)$, $[\preceq]\alpha \in \Gamma$ and $(\underline{W}, v), x \models [\preceq]\alpha$ imply $(\underline{W}, v), y \models \alpha$,

$f(x)\ \mathsf{O}'\ f(y)$, $[\mathsf{O}]\alpha \in \Gamma$ and $(\underline{W}, v), x \models [\mathsf{O}]\alpha$ imply $(\underline{W}, v), y \models \alpha$,

$f(x)\ \mathsf{U}'\ f(y)$, $[\mathsf{U}]\alpha \in \Gamma$ and $(\underline{W}, v), x \models [\mathsf{U}]\alpha$ imply $(\underline{W}, v), y \models \alpha$,

$f(x)\ \mathsf{C}'\ f(y)$, $[\mathsf{C}]\alpha \in \Gamma$ and $(\underline{W}, v), x \models [\mathsf{C}]\alpha$ imply $(\underline{W}, v), y \models \alpha$.

Here f is the filtration mapping and Γ is the set of formulae for which the filtration is done.

Let Γ be a finite set of modal formulae that satisfies the following conditions:

(1) Γ is closed under sub-formulae;
(2) $\langle \mathsf{o} \rangle \top \in \Gamma$, $\langle \mathsf{u} \rangle \top \in \Gamma$, $\langle \mathsf{O} \rangle \top \in \Gamma$, $\langle \mathsf{U} \rangle \top \in \Gamma$ and $\langle \mathsf{C} \rangle \top \in \Gamma$, where $\top$ is an arbitrary fixed tautology in the fragment;
(3) if $[R]\alpha \in \Gamma$ for any of the modalities $[\leq]$, $[\geq]$, $[\mathsf{o}]$, $[\mathsf{u}]$, $[\preceq]$, $[\succeq]$, $[\mathsf{O}]$, $[\mathsf{U}]$ or $[\mathsf{C}]$, then $[R]\alpha \in \Gamma$ for all modalities of the logic (i.e. $\Gamma \cap MOD(\alpha) \neq \emptyset$ implies $MOD(\alpha) \subseteq \Gamma$, where $MOD(\alpha)$ is the following set of formulae $\{ [\leq]\alpha, [\geq]\alpha, [\mathsf{o}]\alpha, [\mathsf{u}]\alpha, [\preceq]\alpha, [\succeq]\alpha, [\mathsf{O}]\alpha, [\mathsf{U}]\alpha, [\mathsf{C}]\alpha \}$).

The important consequence is that if we have any finite set of formulae Γ_0 and we obtain Γ by closing Γ_0 under the above three conditions, then the resulting set Γ will also be finite (usually this is done with Γ_0 consisting of a single formula).

So the new model $(\underline{W}', v')$ is built in the following way:

Let $\simeq$ be a binary relation over W such that, for all $x, y \in W$,

$$x \simeq y \quad \text{iff} \quad \text{for all } \alpha \in \Gamma,\ (\underline{W}, v), x \models \alpha \longleftrightarrow (\underline{W}, v), y \models \alpha.$$

Let $[x]$ denote the equivalence class of x with respect to $\simeq$. The filtration mapping and the universum of the structure are defined standardly:

$$f(x) = [x], \quad W' = \{ f(x) \mid x \in W \}.$$

Note that W' is finite because Γ is finite. It has at most $2^{size(\Gamma)}$ elements.

Each of the seven relations in $\underline{W}'$ is defined for every two classes $[x]$ and $[y]$ as a conjunction of several conditions about x, y and certain formulae from Γ.

Relation $\leq'$ is defined in $\underline{W}'$ as follows:
$[x] \leq' [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{O} \rangle \top \text{ implies } (\underline{W}, v), y \models \langle \mathsf{O} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{U} \rangle \top \text{ implies } (\underline{W}, v), x \models \langle \mathsf{U} \rangle \top,$$
$$(\underline{W}, v), x \models \langle \mathsf{o} \rangle \top \text{ implies } (\underline{W}, v), y \models \langle \mathsf{o} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{u} \rangle \top \text{ implies } (\underline{W}, v), x \models \langle \mathsf{u} \rangle \top,$$

and also for all $[\leq]\alpha \in \Gamma$,

$$
\begin{aligned}
&(\underline{W}, v), x \models [\leq]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,\\
&(\underline{W}, v), y \models [\geq]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha,\\
&(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\mathsf{O}]\alpha,\\
&(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\mathsf{U}]\alpha,\\
&(\underline{W}, v), x \models [\preceq]\alpha \text{ implies } (\underline{W}, v), y \models [\preceq]\alpha,\\
&(\underline{W}, v), y \models [\succeq]\alpha \text{ implies } (\underline{W}, v), x \models [\succeq]\alpha,\\
&(\underline{W}, v), y \models [\mathsf{o}]\alpha \text{ implies } (\underline{W}, v), x \models [\mathsf{o}]\alpha,\\
&(\underline{W}, v), x \models [\mathsf{u}]\alpha \text{ implies } (\underline{W}, v), y \models [\mathsf{u}]\alpha,\\
&(\underline{W}, v), y \models [\mathsf{C}]\alpha \text{ implies } (\underline{W}, v), x \models [\mathsf{C}]\alpha.
\end{aligned}
$$

Relation O' is defined:
$[x]\ \mathsf{O}'\ [y]$ holds iff the following conditions are met

$$
\begin{aligned}
&(\underline{W}, v), x \models \langle \mathsf{O} \rangle \top,\\
&(\underline{W}, v), y \models \langle \mathsf{O} \rangle \top
\end{aligned}
$$

and for all $[\mathsf{O}]\alpha \in \Gamma$,

$$
\begin{aligned}
&(\underline{W}, v), x \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,\\
&(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\leq]\alpha.
\end{aligned}
$$

Relation U' is defined:
$[x]\ \mathsf{U}'\ [y]$ holds iff the following conditions are met

$$
\begin{aligned}
&(\underline{W}, v), x \models \langle \mathsf{U} \rangle \top,\\
&(\underline{W}, v), y \models \langle \mathsf{U} \rangle \top
\end{aligned}
$$

and for all $[\mathsf{U}]\alpha \in \Gamma$,

$$
\begin{aligned}
&(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\geq]\alpha,\\
&(\underline{W}, v), y \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha.
\end{aligned}
$$

Relation $\preceq'$ is defined:
$[x] \preceq' [y]$ holds iff the following conditions are met

$$
\begin{aligned}
&(\underline{W}, v), x \models \langle \mathsf{o} \rangle \top \text{ implies } (\underline{W}, v), y \models \langle \mathsf{O} \rangle \top,\\
&(\underline{W}, v), y \models \langle \mathsf{u} \rangle \top \text{ implies } (\underline{W}, v), x \models \langle \mathsf{U} \rangle \top,
\end{aligned}
$$

and for all $[\preceq]\alpha \in \Gamma$,

$$
\begin{aligned}
&(\underline{W}, v), x \models [\preceq]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,\\
&(\underline{W}, v), y \models [\succeq]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha,\\
&(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\mathsf{o}]\alpha,\\
&(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\mathsf{u}]\alpha.
\end{aligned}
$$

Relation o' is defined:

$[x]\ \mathsf{o}'\ [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{o} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{o} \rangle \top$$

and for all $[\mathsf{o}]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\mathsf{o}]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{o}]\alpha \text{ implies } (\underline{W}, v), x \models [\leq]\alpha,$$
$$(\underline{W}, v), x \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), y \models [\preceq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\preceq]\alpha.$$

Relation u' is defined:

$[x]\ \mathsf{u}'\ [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{u} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{u} \rangle \top$$

and for all $[\mathsf{u}]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\mathsf{u}]\alpha \text{ implies } (\underline{W}, v), y \models [\geq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{u}]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha,$$
$$(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\succeq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), x \models [\succeq]\alpha.$$

Relation C' is defined:

$[x]\ \mathsf{C}'\ [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{O} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{O} \rangle \top$$

and for all $[\mathsf{C}]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\mathsf{C}]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{C}]\alpha \text{ implies } (\underline{W}, v), x \models [\leq]\alpha.$$

The valuation v' is defined for every variable p as follows $p \in v'([x]) \longleftrightarrow p \in v(x)$.

We now have to prove (R1) and (R2) and the fact that $\underline{W}'$ is generalized structure for the fragment. This is a long proof considering the fact that (R1) and (R2) must be proved for each of the ten modalities of the fragment and then all 36 conditions (M1), (M2), (M3$'$) $-$ (M3$'''$), (M4) $-$ (M30), (C1) $-$ (C4) have to be proved for $\underline{W}'$. But every single statement is actually easily proven through straightforward check of the definitions of the relations in $\underline{W}'$. The difficulties in the overall proof come only from the long definitions and from the large number of conditions that have to be checked. So here follow the proofs of some of these statements, noting that the rest can be proven in the same way.

For example, we show (R1) and (R2) for the relation O'.

To prove (R1) suppose that $x\ \mathsf{O}\ y$. To show that $[x]\ \mathsf{O}'\ [y]$ we have to prove for any $[\mathsf{O}]\alpha \in \Gamma$ the following four conditions from the definition of O':

(1) To prove that $(\underline{W}, v), x \models [\mathsf{O}]\alpha$ implies $(\underline{W}, v), y \models [\leq]\alpha$, suppose that $(\underline{W}, v), x \models [\mathsf{O}]\alpha$ and $y \leq z$. Then, by (M6) for $\underline{W}$, we get $x \mathsf{O} z$ and since $(\underline{W}, v), x \models [\mathsf{O}]\alpha$ it follows that $(\underline{W}, v), z \models \alpha$. This shows that $(\underline{W}, v), y \models [\leq]\alpha$.
(2) The second condition, $(\underline{W}, v), y \models [\mathsf{O}]\alpha$ implies $(\underline{W}, v), x \models [\leq]\alpha$, is proved in the same way as the first one.
(3) $x \mathsf{O} y$ and the fact $(\underline{W}, v), y \models \top$ shows the third condition. I.e. that $(\underline{W}, v), x \models \langle\mathsf{O}\rangle\top$.
(4) The last condition is proved in the same way, as the third.

In order to show (R2) suppose that $[x] \mathsf{O}' [y]$ and $[\mathsf{O}]\alpha \in \Gamma$ such that we have $(\underline{W}, v), x \models [\mathsf{O}]\alpha$. Then by the first condition of the definition of O' we have $(\underline{W}, v), y \models [\leq]\alpha$ and since $y \leq y$, by (M1) for $\underline{W}$, we have $(\underline{W}, v), y \models \alpha$.

And finally here are some representable proofs for some of the conditions for $\underline{W}'$ being a generalized structure.

Proof of (M1): By (M1) applied for $\underline{W}$, we have $x \leq x$ and from (R1) it follows that $[x] \leq' [x]$. So (M1) is true for $\underline{W}'$ as well.

Proof of (M2): Let $[x] \leq' [y]$ and $[y] \leq' [z]$. We have to show the 12 implications from the definition of $\leq'$ to prove $[x] \leq' [z]$. Let us show, for example, the first one: $(\underline{W}, v), x \models [\leq]\alpha$ implies $(\underline{W}, v), z \models [\leq]\alpha$. Because $[x] \leq' [y]$ from the same first implication for $[x]$ and $[y]$ we have: $(\underline{W}, v), x \models [\leq]\alpha$ implies $(\underline{W}, v), y \models [\leq]\alpha$. And the same reasoning about $[y] \leq' [z]$ gives that $(\underline{W}, v), y \models [\leq]\alpha$ implies that we have $(\underline{W}, v), z \models [\leq]\alpha$. So from these two implications combined follow the implication that we must prove.

Proof of (M3‴): Suppose $[z] \overline{\mathsf{O}}' [x]$, $[z] \overline{\mathsf{U}}' [y]$ and $[y] \leq' [x]$. Suppose that $[x] \neq [y]$ and as a result we have $x \neq y$. From $[z] \overline{\mathsf{O}}' [x]$ and $[z] \overline{\mathsf{U}}' [y]$, by (R1), it follows that $z \overline{\mathsf{O}} x$ and $z \overline{\mathsf{U}} y$. Then, considering (M3‴) for $\underline{W}$, we see that $y \not\leq x$. Then, from (M12) for $\underline{W}$, it follows that $y \mathsf{O} z$ or $x \mathsf{U} z$. We can see that in each case we have a contradiction:

- from $y \mathsf{O} z$, by (R1), it follows $[y] \mathsf{O}' [z]$ and then, by (M4) for $\underline{W}'$, we have $[z] \mathsf{O}' [y]$. $[z] \mathsf{O}' [y]$ and $[y] \leq' [x]$, from (M6) for $\underline{W}'$, imply $[z] \mathsf{O}' [x]$, which is contradiction with $[z] \overline{\mathsf{O}}' [x]$. We have to note of course that (M4) and (M6) for $\underline{W}'$ are proved independently from (M3‴).
- if $x \mathsf{U} z$ by applying similar reasoning we get the contradiction $[z] \mathsf{U}' [y]$.

Since in each case we get contradiction then it follows that $[x] = [y]$ which shows that (M3‴) is true for $\underline{W}'$.

Proof of (M8): We have to prove that $[x] \mathsf{U}' [y]$ implies $[y] \mathsf{U}' [x]$. The symmetry of U' follows directly from the fact that the definition of U' is symmetric with respect to x and y.

$\square$

Corollary 15.

The fragment of the modal logic of dynamic mereotopological relations without the c *modality has the strong form of the finite model property.*

Corollary 16.

The fragment of the modal logic of dynamic mereotopological relations without the c *modality is decidable.*

3.2. **Reduct of the modal logic of mereotopological dynamic relations without $\preceq$.**

This is the fragment of the logic with both contacts, but without the unstable part-of $\preceq$. The models here are relational structures $(W, \leq, \mathsf{o}, \mathsf{u}, \mathsf{c}, \mathsf{O}, \mathsf{U}, \mathsf{C})$, satisfying conditions (M1), (M2), (M3$'$) – (M3$'''$), (M4) – (M13), (M17) – (M19), (M23) – (M25), (M29), (M30), (C1) – (C8), (C10). Kripke semantics about those structures will constitute the *generalized semantics* of the fragment.

Proposition 22.
The fragment of the modal logic without $\preceq$ admits filtration with respect to its generalized semantics.

Proof.
We start from a model $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ and a finite set of formulae Γ and for Γ we have the following requirements:

(1) Γ is closed under sub-formulae;
(2) if $\alpha \in \Gamma$ and α does not start with $[\leq]$ (i.e. α is not in the form of $[\leq]\beta$) then $[\leq]\alpha \in \Gamma$ and $[\leq]\neg\alpha \in \Gamma$.

The filtered model $(\underline{W}', v')$ is defined as follows:

The usual equivalence relation $\simeq$ is defined for all $x, y \in W$

$$x \simeq y \quad \text{iff} \quad \text{for all } \alpha \in \Gamma,\ (\underline{W}, v), x \models \alpha \longleftrightarrow (\underline{W}, v), y \models \alpha.$$

Then we define the filtration mapping and the universe of the new structure

$$f(x) = [x], \quad W' = \{\ f(x) \mid x \in W\ \}.$$

Relation $\leq'$ in the filtered model is defined with the standard condition for S4 modality (see [5], [18]):

$$[x] \leq' [y] \text{ iff for all } [\leq]\alpha \in \Gamma,\ (\underline{W}, v), x \models [\leq]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha.$$

The rest of the structure is defined as follows for every $[x], [y] \in W'$:

$$[x]\ \mathsf{O}'\ [y] \text{ iff } \exists z, t \in W,\ [z] \leq' [x],\ [t] \leq' [y] \text{ and } z\ \mathsf{O}\ t;$$
$$[x]\ \mathsf{U}'\ [y] \text{ iff } \exists z, t \in W,\ [x] \leq' [z],\ [y] \leq' [t] \text{ and } z\ \mathsf{U}\ t;$$
$$[x]\ \mathsf{o}'\ [y] \text{ iff } \exists z, t \in W,\ [z] \leq' [x],\ [t] \leq' [y] \text{ and } z\ \mathsf{o}\ t;$$
$$[x]\ \mathsf{u}'\ [y] \text{ iff } \exists z, t \in W,\ [x] \leq' [z],\ [y] \leq' [t] \text{ and } z\ \mathsf{u}\ t;$$
$$[x]\ \mathsf{c}'\ [y] \text{ iff } \exists z, t \in W,\ [z] \leq' [x],\ [t] \leq' [y] \text{ and } z\ \mathsf{c}\ t;$$
$$[x]\ \mathsf{C}'\ [y] \text{ iff } \exists z, t \in W,\ [z] \leq' [x],\ [t] \leq' [y] \text{ and } z\ \mathsf{C}\ t.$$

Now we have to prove that $\underline{W}'$ satisfies conditions (M1), (M2), (M3$'$) – (M3$'''$), (M4) – (M13), (M17) – (M19), (M23) – (M25), (M29), (M30), (C1) – (C8), (C10) and the filtration conditions

(R1)
$$x \le y \text{ implies } f(x) \le' f(y),$$
$$x \mathsf{o}\, y \text{ implies } f(x)\, \mathsf{o}'\, f(y),$$
$$x \mathsf{u}\, y \text{ implies } f(x)\, \mathsf{u}'\, f(y),$$
$$x \mathsf{c}\, y \text{ implies } f(x)\, \mathsf{c}'\, f(y),$$
$$x \mathsf{O}\, y \text{ implies } f(x)\, \mathsf{O}'\, f(y),$$
$$x \mathsf{U}\, y \text{ implies } f(x)\, \mathsf{U}'\, f(y),$$
$$x \mathsf{C}\, y \text{ implies } f(x)\, \mathsf{C}'\, f(y),$$

(R2)
$$f(x) \le' f(y),\ [\le]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\le]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,$$
$$f(x)\, \mathsf{o}'\, f(y),\ [\mathsf{o}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{o}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,$$
$$f(x)\, \mathsf{u}'\, f(y),\ [\mathsf{u}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{u}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,$$
$$f(x)\, \mathsf{c}'\, f(y),\ [\mathsf{c}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{c}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,$$
$$f(x)\, \mathsf{O}'\, f(y),\ [\mathsf{O}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{O}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,$$
$$f(x)\, \mathsf{U}'\, f(y),\ [\mathsf{U}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,$$
$$f(x)\, \mathsf{C}'\, f(y),\ [\mathsf{C}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{C}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha.$$

As the definition of $\le'$ is the standard S4 filtration, then we have that (R1) and (R2), as well as (M1)and (M2), hold for $\underline{W}'$ ([5], [18]). (R1) for the other six relations is proved easily. For example, for o' we have the following proof. Suppose that $x \,\mathsf{o}\, y$. Then, by (M1) for $\underline{W}'$ we have that $[x] \le' [x]$ and $[y] \le' [y]$. Thus, by the definition of o' we have that $[x]\, \mathsf{o}'\, [y]$.

When proving (R2) we will again show in details only the case for o'. The proofs for the other five relations are similar. In these proofs we may use (R2) for $\le'$. This is correct, because (R2) for $\le'$ is proved independently from (R2) for the other relations. To prove it for o', suppose $[x]\, \mathsf{o}'\, [y]$. We will show that

$$(\underline{W}, v), y \not\models \alpha \text{ implies } (\underline{W}, v), x \not\models [\mathsf{o}]\alpha.$$

From $[x]\, \mathsf{o}'\, [y]$, by the definition of o', we have that there are z and t, such that $[z] \le' [x]$, $[t] \le' [y]$ and $z \,\mathsf{o}\, t$. If $(\underline{W}, v), y \not\models \alpha$, then from $[t] \le' [y]$, by (R2) for $\le'$, it follows that $(\underline{W}, v), t \not\models [\le]\alpha$. Thus, there is $w \in W$, such that $t \le w$ and $(\underline{W}, v), w \not\models \alpha$. We will show that $(\underline{W}, v), z \models [\le]\neg[\mathsf{o}]\alpha$. Take any $u \in W$, such that $z \le u$. From $z \,\mathsf{o}\, t$, $t \le w$ and $z \le u$, by (M6) for $\underline{W}$, we get $u \,\mathsf{o}\, w$. Since $(\underline{W}, v), w \not\models \alpha$, then $(\underline{W}, v), u \models \neg[\mathsf{o}]\alpha$. Thus, $(\underline{W}, v), z \models [\le]\neg[\mathsf{o}]\alpha$ indeed. Then from $[z] \le' [x]$, by (R2) for $\le'$, we get $(\underline{W}, v), x \models \neg[\mathsf{o}]\alpha$ (we have that $\neg[\mathsf{o}]\alpha \in \Gamma$). I.e. $(\underline{W}, v), x \not\models [\mathsf{o}]\alpha$.

Most of the conditions (M1), (M2), (M3′) – (M3‴), (M4) – (M13), (M17) – (M19), (M23) – (M25), (M29), (M30), (C1) – (C8), (C10) are proved easily. For example, the symmetricity of o', u', c', O', U' and C' follows from the symmetricity of their definitions and from the symmetricity of the original o,u,c,O,U and C. All conditions, that are disjunctions are proved like (M12), for instance. Suppose that in $\underline{W}'$ we have

$$[x] \not\le' [y] \text{ and } [x]\, \overline{\mathsf{O}}'\, [z] \text{ and } [y]\, \overline{\mathsf{U}}'\, [z].$$

Then, from (R1) for $\leq'$, O' and U', we obtain that in $\underline{W}$ holds

$$x \not\leq y \text{ and } x\ \overline{\mathsf{O}}\ z \text{ and } y\ \overline{\mathsf{U}}\ z,$$

which is a contradiction with (M12) for $\underline{W}$.

If we have any of the other axioms with $\leq'$, the proof is similar to that for (M6). Suppose that $[x]\ \mathsf{O}'\ [y]$ and $[y] \leq' [z]$. By the definition of O', we have that there are t and w, such that $t\ \mathsf{O}\ w$, $[t] \leq' [x]$ and $[w] \leq' [y]$. From $[w] \leq' [y]$ and $[y] \leq' [z]$, by (M2) for $\underline{W}'$, we get that $[w] \leq' [z]$. Thus, we have $t\ \mathsf{O}\ w$, $[t] \leq' [x]$ and $[w] \leq' [z]$. Again, by the definition of O', this gives us $[x]\ \mathsf{O}'\ [z]$.

Thus, only (M3$'$) – (M3$'''$) and (C10). We will show representative proofs for (M3$'$) and (C10). To prove (M3$'$) suppose $[x]\ \overline{\mathsf{O}}'\ [x]$ and $[y] \leq' [x]$. By (R1) for O', we have that $x\ \overline{\mathsf{O}}\ x$. From $[x]\ \overline{\mathsf{O}}'\ [x]$ and $[y] \leq' [x]$, by the definition of O', it follows that $y\ \overline{\mathsf{O}}\ y$, as well. Then, by (M7) for $\underline{W}$, we get $y \leq x$. From $x\ \overline{\mathsf{O}}\ x$ and $y \leq x$, by (M3$'$), we have $x = y$ and, finally, $[x] = [y]$.

For (C10) we will show that its equivalent form

$$[z]\ \mathsf{c}'\ [t] \Longrightarrow [x]\ \mathsf{u}'\ [y] \text{ or } [z]\ \mathsf{O}'\ [y] \text{ or } [t]\ \mathsf{O}'\ [x] \text{ or } [x]\ \mathsf{C}'\ [y]$$

holds. Suppose $[z]\ \mathsf{c}'\ [t]$. Then, by definition, there are u and w, such that $u\ \mathsf{c}\ w$, $[u] \leq' [z]$ and $[w] \leq' [t]$. Then from $u\ \mathsf{c}\ w$, by (C10) for $\underline{W}$, we have

$$x\ \mathsf{u}\ y \text{ or } u\ \mathsf{O}\ y \text{ or } w\ \mathsf{O}\ x \text{ or } x\ \mathsf{C}\ y.$$

Then, by the definitions of u', O' and C', we have that

$$[x]\ \mathsf{u}'\ [y] \text{ or } [u]\ \mathsf{O}'\ [y] \text{ or } [w]\ \mathsf{O}'\ [x] \text{ or } [x]\ \mathsf{C}'\ [y].$$

Since $[u] \leq' [z]$ and $[w] \leq' [t]$ then, by (M6) for $\underline{W}'$, from $[u]\ \mathsf{O}'\ [y]$ follows $[z]\ \mathsf{O}'\ [y]$ and from $[w]\ \mathsf{O}'\ [x]$ it follows that $[t]\ \mathsf{O}'\ [x]$. Thus, we get the needed

$$[x]\ \mathsf{u}'\ [y] \text{ or } [z]\ \mathsf{O}'\ [y] \text{ or } [t]\ \mathsf{O}'\ [x] \text{ or } [x]\ \mathsf{C}'\ [y].$$

$\square$

Corollary 17.

The fragment of the modal logic of dynamic mereotopological relations without the $\preceq$ modality has the strong form of the finite model property.

Corollary 18.

The fragment of the modal logic of dynamic mereotopological relations without the $\preceq$ modality is decidable.

3.3. **The modal logic of dynamic mereological relations.**

The filtration construction here is the same as the one in the first subsection - the one for the fragment without c. The only difference is that here we drop the C modality too. Models of this fragment/logic are generalized dynamic mereological structures (see Definition 23). With respect to these models we will show that the logic admits filtration. Since we have proved in Part V, Subsection 1.4 that the modal logic of the generalized structures is the same as the modal logics of the dynamic mereological structures and the standard dynamic mereological structures, then we may consider that we have decidability for these two modal logics as well.

Proposition 23.

The the modal logic of dynamic mereological relations admits filtration with respect to its generalized semantics.

Proof.

Let $(\underline{W}, v)$ be a model of the fragment, where $\underline{W} = (W, \leq, \mathsf{o}, \mathsf{u}, \preceq, \mathsf{O}, \mathsf{U})$ is a generalized dynamic mereological structure. We will build a finite generalized structure $\underline{W}' = (W', \leq', \mathsf{o}', \mathsf{u}', \preceq', \mathsf{O}', \mathsf{U}')$ and the model $(\underline{W}', v')$, such that for all $x, y \in W$:

(R1)
$$
\begin{gathered}
x \leq y \text{ implies } f(x) \leq' f(y),\\
x \;\mathsf{o}\; y \text{ implies } f(x) \;\mathsf{o}'\; f(y),\\
x \;\mathsf{u}\; y \text{ implies } f(x) \;\mathsf{u}'\; f(y),\\
x \preceq y \text{ implies } f(x) \preceq' f(y),\\
x \;\mathsf{O}\; y \text{ implies } f(x) \;\mathsf{O}'\; f(y),\\
x \;\mathsf{U}\; y \text{ implies } f(x) \;\mathsf{U}'\; f(y),
\end{gathered}
$$

(R2)
$$
\begin{gathered}
f(x) \leq' f(y),\ [\leq]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\leq]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,\\
f(x) \;\mathsf{o}'\; f(y),\ [\mathsf{o}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{o}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,\\
f(x) \;\mathsf{u}'\; f(y),\ [\mathsf{u}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{u}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,\\
f(x) \preceq' f(y),\ [\preceq]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\preceq]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,\\
f(x) \;\mathsf{O}'\; f(y),\ [\mathsf{O}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{O}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha,\\
f(x) \;\mathsf{U}'\; f(y),\ [\mathsf{U}]\alpha \in \Gamma \text{ and } (\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ imply } (\underline{W}, v), y \models \alpha.
\end{gathered}
$$

Here f is the filtration mapping and Γ is the set of formulae for which the filtration is done.

If Γ is a finite set of modal formulae that satisfies the following conditions

(1) Γ is closed under sub-formulae;
(2) $\langle \mathsf{o} \rangle \top \in \Gamma$, $\langle \mathsf{u} \rangle \top \in \Gamma$, $\langle \mathsf{O} \rangle \top \in \Gamma$ and $\langle \mathsf{U} \rangle \top \in \Gamma$, where $\top$ is a fixed tautology;
(3) $\Gamma \cap MOD(\alpha) \neq \emptyset$ implies $MOD(\alpha) \subseteq \Gamma$, where $MOD(\alpha)$ is the set of formulae $\{\ [\leq]\alpha, [\geq]\alpha, [\mathsf{o}]\alpha, [\mathsf{u}]\alpha, [\preceq]\alpha, [\succeq]\alpha, [\mathsf{O}]\alpha, [\mathsf{U}]\alpha\ \}$);

then the model $(\underline{W}', v')$ and the filtration mapping are constructed as follows:

$$
f(x) = [x], \quad W' = \{\ f(x) \mid x \in W\ \}.
$$

where $[x]$ are the equivalence classes with respect to

$$
x \simeq y \quad \text{iff} \quad \text{for all } \alpha \in \Gamma,\ (\underline{W}, v), x \models \alpha \longleftrightarrow (\underline{W}, v), y \models \alpha,
$$

and the relations in $\underline{W}'$ are defined:

Relation $\leq'$ is defined in $\underline{W}'$ as follows:
$[x] \leq' [y]$ holds iff the following conditions are met

$$
\begin{gathered}
(\underline{W}, v), x \models \langle \mathsf{O} \rangle \top \text{ implies } (\underline{W}, v), y \models \langle \mathsf{O} \rangle \top,\\
(\underline{W}, v), y \models \langle \mathsf{U} \rangle \top \text{ implies } (\underline{W}, v), x \models \langle \mathsf{U} \rangle \top,\\
(\underline{W}, v), x \models \langle \mathsf{o} \rangle \top \text{ implies } (\underline{W}, v), y \models \langle \mathsf{o} \rangle \top,\\
(\underline{W}, v), y \models \langle \mathsf{u} \rangle \top \text{ implies } (\underline{W}, v), x \models \langle \mathsf{u} \rangle \top,
\end{gathered}
$$

and also for all $[\leq]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\leq]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,$$
$$(\underline{W}, v), y \models [\geq]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\mathsf{O}]\alpha,$$
$$(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\mathsf{U}]\alpha,$$
$$(\underline{W}, v), x \models [\preceq]\alpha \text{ implies } (\underline{W}, v), y \models [\preceq]\alpha,$$
$$(\underline{W}, v), y \models [\succeq]\alpha \text{ implies } (\underline{W}, v), x \models [\succeq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{o}]\alpha \text{ implies } (\underline{W}, v), x \models [\mathsf{o}]\alpha,$$
$$(\underline{W}, v), x \models [\mathsf{u}]\alpha \text{ implies } (\underline{W}, v), y \models [\mathsf{u}]\alpha.$$

Relation O' is defined:
$[x]\ \mathsf{O}'\ [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{O} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{O} \rangle \top$$

and for all $[\mathsf{O}]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\leq]\alpha.$$

Relation U' is defined:
$[x]\ \mathsf{U}'\ [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{U} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{U} \rangle \top$$

and for all $[\mathsf{U}]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\geq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha.$$

Relation $\preceq'$ is defined:
$[x] \preceq' [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{o} \rangle \top \text{ implies } (\underline{W}, v), y \models \langle \mathsf{O} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{u} \rangle \top \text{ implies } (\underline{W}, v), x \models \langle \mathsf{U} \rangle \top,$$

and for all $[\preceq]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\preceq]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,$$
$$(\underline{W}, v), y \models [\succeq]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\mathsf{o}]\alpha,$$
$$(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\mathsf{u}]\alpha.$$

Relation o' is defined:
$[x]\ \mathsf{o}'\ [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{o} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{o} \rangle \top$$

and for all $[\mathsf{o}]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\mathsf{o}]\alpha \text{ implies } (\underline{W}, v), y \models [\leq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{o}]\alpha \text{ implies } (\underline{W}, v), x \models [\leq]\alpha,$$
$$(\underline{W}, v), x \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), y \models [\preceq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{O}]\alpha \text{ implies } (\underline{W}, v), x \models [\preceq]\alpha.$$

Relation u' is defined:

$[x]\ \mathsf{u}'\ [y]$ holds iff the following conditions are met

$$(\underline{W}, v), x \models \langle \mathsf{u} \rangle \top,$$
$$(\underline{W}, v), y \models \langle \mathsf{u} \rangle \top$$

and for all $[\mathsf{u}]\alpha \in \Gamma$,

$$(\underline{W}, v), x \models [\mathsf{u}]\alpha \text{ implies } (\underline{W}, v), y \models [\geq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{u}]\alpha \text{ implies } (\underline{W}, v), x \models [\geq]\alpha,$$
$$(\underline{W}, v), x \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), y \models [\succeq]\alpha,$$
$$(\underline{W}, v), y \models [\mathsf{U}]\alpha \text{ implies } (\underline{W}, v), x \models [\succeq]\alpha.$$

The valuation v' is defined for every variable p as follows $p \in v'([x]) \longleftrightarrow p \in v(x)$.

The rest of the proof is as in Proposition 21. □

Corollary 19.

The modal logic of dynamic mereological relations has the strong form of the finite model property.

Corollary 20.

The modal logic of dynamic mereological relations is decidable.

4. Open problems and future work

With regards to the first order theory - we have established the undecidability of the full theories and the decidability and complexity of the quantifier-free fragments. A future task in this line of work is to find other proper first-order fragments, which are decidable and to study their complexity.

With regards to the modal logic - we have decidability of some fragments, but the decidability of the full modal logic (with 8 relations and 11 modalities in total) is still an open problem. The two filtrations for the fragment without c and for the fragment without $\preceq$ have met with some difficulties in achieving this goal. For instance, the problem that occurs with the first filtration is that the resulting filtered model cannot satisfy a certain axiom of the stable contact. This is axiom

$$z \mathbin{\mathsf{c}} t \,\&\, x \mathbin{\overline{\mathsf{u}}} y \,\&\, z \mathbin{\overline{\mathsf{O}}} y \,\&\, t \mathbin{\overline{\mathsf{O}}} x \Longrightarrow x \mathbin{\mathsf{C}} y \tag{C10}$$

The problem is that with the current definitions of the relations in the filtered model, the proof for this axioms dissolves into checking more than 200 separate cases, some of which do not give the needed result. The difficulties in the second filtration come from the complex interactions between the orders $\leq$ and $\preceq$ via axioms

$$x \leq y \,\&\, y \preceq z \Longrightarrow x \preceq z \tag{M15}$$

$$x \preceq y \,\&\, y \leq z \Longrightarrow x \preceq z \tag{M16}$$

and between $\leq$ and $\preceq$ and the other modalities via similar axioms.

These problems will get only worse if the mereotopological and mereological systems are extended with more spatial relations and temporal constructions, like *since*, *until*, *before* and others. This poses the need for more general approach for proving decidability for modalities for relations defined over contact algebras. The main requirement for this approach is to avoid the long and messy check for filtration conditions for every new system.

Thus, the decidability of the full modal logic (and of modal logics for any extended systems) remains an open problem. Also future tasks are determining the complexity of the modal logic (if it is decidable) and of any of the fragments, for which decidability is already proved.

Part VII. Conclusion

In conclusion we give a brief summary of the notions and results, that are presented in this dissertation. They contribute to the development of Whiteheadean integrated theory of space and time. The main features of the studied spatio-temporal systems are that they are: *region-based* (i.e. they are *point-free*), *relational* and have *integrated spatio-temporal language* (i.e. the languages about time and space are not separated). Here we also summarize the open problems that remain to be solved. Namely, these open problems lay the foundations for the future developments of the current works.

The results of the study of dynamic relational mereotopology and mereology are grouped in three parts - Parts IV, V and VI. The major results are included in Part IV. There we give formal definitions of stable and unstable mereotopological and mereological relations. The definitions of stable and unstable relations, present them as simple temporal variants of the base relations. Stable relations hold in all moments of time, while unstable relations hold in some of the moments. Thus, in this setting *stable* and *unstable* are seen as synonyms of *always* and *sometimes*. We work with two main languages:

- a mereotopological language, consisting of dynamic relations $\leq$, o, u, c, $\preceq$, O, U and C;
- a mereological language, consisting of $\leq$, o, u, $\preceq$, O and U.

Then we establish the structures, that feature such relations. We have two types of structures. The ones from the first type are called *standard structures* and are considered as the *point-based* definitions for the dynamic relations. In this definition we use collections of mereotopological structures, which are relational alternatives to topological spaces, for the first language and we use mereological structures, which are relational alternatives to Boolean algebras, for the second language. Thus, while the dynamic mereotopological language and structures are the primary notions, that we study here, we also consider the mereological sublanguage, which is more lightweight and allows the application of the current results even in simpler cases, when we cannot depend on the availability of topological spaces. The second type of structures that we define represent the *point-free* definition of the dynamic relations. In this *general* definition the dynamic relations are considered to be just

	Axiomatization	Decidability	Complexity
First-order DMt	Finite	Hereditary undecidable	
Quantifier-free DMt	Finite	Decidable	NP-complete
Modal DMt	Finite	?	
Modal DMt w/o c	-	Decidable	?
Modal DMt w/o $\preceq$	-	Decidable	?
First-order DM	Finite	Hereditary undecidable	
Quantifier-free DM	Finite	Decidable	NP-complete
Modal DM	Finite	Decidable	?

Table 1. Results about logics of dynamic mereotopological and mereological relations.

relations between arbitrary abstract objects, which do not need to be composed of points.

The main result about the above languages and structures is the representation theory, that gives a dual (in the sense of Stone-duality) correspondence between the standard and the general definitions of the dynamic relations. The main results are Theorems 12, 13, 14 and 15. They collectively state that every general, point-free mereotopological/mereological structure can be isomorphically represented as s standard, point-based structure. This representation theory is the main result in the current works and is a crucial instrument in understanding the nature (and also in the axiomatization) of the dynamic mereotopological and mereological relations and in relating this alternative point of view to space and time to the classical point-based spacial and temporal theories.

In Parts V and VI we study the axiomatization and the decidability of first-order and modal logics about the dynamic relations. The standard and general structures serve as interpretations or Kripke structures for these logics. Mainly through the use of the representation theory (with the help of p-morphisms in the modal case) we have proved completeness for these logics. We have that the full first-order theories are *hereditary undecidable* and that some of their fragments (quantifier-free fragments, modal logics, reducts of the modal logics) are decidable. These results are summarized in Table 1, where we adopt the abbreviations DMt and DM, for the dynamic mereotopological and mereological relational systems, from Part IV, Section 1.

A full list of the results, that represent the contribution of this dissertation, is given in the next section. The results are grouped into the four main result themes - *expressiveness results*, *representation theory results*, *axiomatization and completeness results* and *decidability and undecidability results*.

The open problems about dynamic relational mereotopology and mereology are described in details in Part IV, Section 6 and Part VI, Section 4. They can be divided into three major topics. Each of these topics represents a possible development of the current studies.

The first development is to extend the languages with both spacial and temporal means. As it is shown in Part IV, Section 1, the current relational systems are weaker than other formal systems, that reason about space and time (e.g. dynamic contact algebras, fusions of spatial and temporal logics, temporalized logics). Thus, we have to add more combined spatio-temporal relations to the language, in order to catch up, with respect to expressive power, to systems like contact algebras, $S4_u$, LTL and combinations of them. Another future task in this topic is to try alternative temporal constructions. The obvious candidates for extending the temporal side of the language are the *since* and *until* operators from LTL, which are in some sense moment-based (defined over point-based time). We could use a different approach to time and to apply a more region-based temporal construction, instead. Using interval temporal logics or even regarding time as a mereotopological/mereological structure are some of the possibilities.

The second development is to continue the study of the logics of dynamic mereotopological and mereological relations. The problem of the decidability of the full modal logic of dynamic mereotopological relations (DMt) is still unanswered. Also the complexity of the decidable modal logics and their reductions is unknown. Another line of inquiry in this development is to look for other useful decidable fragments of the first-order theories, besides the quantifier-free fragments.

Finally, another possible development of mereotopology and mereology, in general, is to try to devise a generic representation theory for mereotopological and mereological relations. This generic representation theory should be able to show the customary Stone-duality for large enough language of relations and to guarantee the representation for any of its sublanguages. Such generic (maybe set-theoretic) techniques may prove to be very useful in the extension of the current languages. In this case we would not have to modify the definitions of abstract space points and time moments and to repeat the proofs for every addition of new relations.

Contribution

Here we list the main new developments and proved results from this dissertation (see Part IV, Part V and Part VI). They represent the contribution of the current works to the fields of *alternative* and *region-based* theories of space and time. Most of the major results will come in pairs - a result for the primary mereotopological language, which is coupled (if possible) with a corresponding result for the secondary mereological sub-language.

These results are grouped into four main topics - *expressiveness results*, results about *representation theories*, *axiomatization and completeness results* about logics for the two main languages and *decidability or undecidability results* about those logics. Here follow more detailed descriptions and references to those results.

Expressiveness results.

These results come from the study of the expressive power of the two languages. They are included in Part IV. We have that the mereological sub-language is a strict sub-language (i.e. this means that its expressive power is strictly less than the expressive power) of the primary language. Then the primary mereotopological language is compared with the language of dynamic contact algebras and with the temporalized static mereotopological language (see [16]) and the spatio-temporal language ST_0 (see [17]). The main results from this topic are:

- Corollary 2, page 44 - this is the comparison between the mereological and mereotopological languages and the language of dynamic contact algebras;
- Proposition 2, page 45 - this is the comparison between the two main languages and the temporalized mereotopology and ST_0.

Representation theory results.

This topic covers the representation theories for the two studied languages. These representation theories are generalizations of Stone's representation theory for Boolean algebras and distributive lattices. The main results are (a pair of) dynamic *characterizations* and (two sets of) *representation theorems*. They can be found in Part IV.

- Proposition 3, page 60 - this proposition shows how dynamic mereotopological relations can be characterized with *abstract space points*;
- Theorem 12, page 63, Theorem 13, page 66 - these are the theorems, that show that point-free defined dynamic mereotopological relations can be represented in the standard point-based way;
- Proposition 4, page 70 - the dynamic mereological relations, characterized with *abstract space points*;
- Theorem 14, page 71, Theorem 15, page 71 - prove that point-free defined dynamic mereological relations can be represented with points.

Axiomatization results.

These results are contained in Part V. They show axiomatizations and completeness proofs for logics, which language is consisted of the dynamic relations. First we study the two first-order logics about the mereotopological language and the mereological language. Then we study the quantifier-free fragments of these two first-order logics. Finally, the modal logics of the two relational languages are considered. The results are as follows:

- Theorem 16, page 77 - this theorem gives the completeness of the first-order logic of dynamic mereotopological relations;
- Theorem 17, page 78 - this shows the completeness of the quantifier-free fragment of the first-order logic of dynamic mereotopological relations;
- Theorem 18, page 79 - the completeness of the first-order logic of dynamic mereological relations;
- Theorem 19, page 80 - the completeness of the quantifier-free fragment of the mereological first-order logic;
- Theorem 20, page 93 - the completeness of the mereotopological modal logic;
- Theorem 21, page 95 - the completeness of the mereological modal logic.

(Un)decidability results.

Part VI features the results from the last topic. They are about the decidability and undecidability of the logics from the previous part. We have that the full first-order logics and also other first-order logics that has the overlap and contact relations are *hereditary undecidable*. We have that the quantifier-free fragments are decidable and their satisfiability problems are NP-complete. For the modal logics, we have that two reducts of the modal logic of dynamic mereotopological relations are decidable and also that the dynamic mereological modal logic is decidable. Here is a list of the results:

- Corollary 6, page 99 - this states the hereditary undecidability of the first-order logic of dynamic mereotopological relations;
- Corollary 7, page 100 - this states the hereditary undecidability of the first-order logic of dynamic mereological relations;
- Corollary 3, page 98, Corollary 4, page 98, Corollary 5, page 99, Corollary 8, page 100, Corollary 9, page 101, Corollary 10, page 101 - hereditary undecidability for other first-order logics about overlap and contact;
- Corollary 11, page 103, Corollary 12, page 103 - decidability and complexity results for the quantifier-free fragment of the first-order logic of dynamic mereotopological relations;
- Corollary 13, page 103, Corollary 14, page 103 - decidability and complexity results for the mereological quantifier-free fragment;
- Corollary 16, page 108, Corollary 18, page 111 - decidability results for two reducts of the modal logic of dynamic mereotopological relations;
- Corollary 20, page 114 - decidability of the dynamic mereological modal logic.

Publications

The results and developments from this dissertation are published in several peer reviewed papers (in scientific journals and conference proceedings) and are also presented at numerous international conference and workshop venues and at a FMI science session at the Sofia University. Here follow the lists of these publications.

Refereed (peer reviewed) papers.

The peer reviewed papers consist of two major journal articles and of four shorter publications in conference proceedings - in three consecutive *Panhellenic Logic Symposiums* and in *Advances in Modal Logic (2012).* The journal articles are published in *Central European Journal of Mathematics* (with impact factor 0.44 for year 2011) and in a special issue of *Logic and Logical Philosophy*, which is dedicated to point-free geometry and topology.

***An axiomatization of dynamic ontology of stable and unstable mereological relations*, with Dimitar Vakarelov, Proceedings of 7-th Panhellenic Logic Symposium (reference [34]).**

This paper presents the stable and unstable mereological relations (Definition 15) and their representation theory (Proposition 4, Theorem 14, Theorem 15).

***Undecidability of logics for mereological and mereotopological relations*, Proceedings of 8-th Panhellenic Logic Symposium (reference [27]).**

This paper is about the hereditary undecidability results about first-order logics with overlap and contact relations (Corollary 3, Corollary 4, Corollary 5, Corollary 6, Corollary 7, Corollary 8, Corollary 9, Corollary 10).

***Logics for stable and unstable mereological relations*, Central European Journal of Mathematics 9 (2011) (reference [26]).**

This paper presents the stable and unstable mereological relations (Definition 15), the representation theory about them (Proposition 4, Theorem 14, Theorem 15) and results about first-order and modal logics about these relations (Theorem 18, Theorem 19, Corollary 7, Corollary 13, Corollary 14, Theorem 21, Corollary 20).

***Dynamic relational mereotopology: A modal logic for stable and unstable relations*, Proceedings of Advances in Modal Logic 2012 (reference [29]).**

This paper presents the stable and unstable mereotopological relations (Definition 12), their representation theory and the completeness of their modal logic (Proposition 3, Theorem 12, Theorem 13, Theorem 20).

***Dynamic relational mereotopology: Logics for stable and unstable relations*, Logic and Logical Philosophy 22 (2013) (reference [33]).**

This article contains the development of stable and unstable mereotopological relations (Definition 12), their representation theory and the completeness and decidability and undecidability results about their first-order logic (Proposition 3, Theorem 12, Theorem 13, Theorem 16, Theorem 17, Corollary 6, Corollary 11, Corollary 12).

***Decidability of modal logics for dynamic contact relations*, Proceedings of 9-th Panhellenic Logic Symposium (reference [31]).**

This paper presents decidability results for two reducts of the modal logic of dynamic mereotopological relations (Corollary 16, Corollary 18).

Citations.

Two of the above papers are cited in other publications. Here is the list of this citations.

An axiomatization of dynamic ontology of stable and unstable mereological relations (reference [34]*) is cited in:*

- Dimitar Vakarelov, **Dynamic Mereotopology: A Point-free Theory of Changing Regions. I. Stable and unstable mereotopological relations**, Fundamenta Informaticae 100 (2010) 1-4, pp. 159-180 (reference [47]).

Logics for stable and unstable mereological relations from CEJM (reference [26]*) is cited in:*

- Dimitar Vakarelov, **Dynamic Mereotopology II: Axiomatizing some Whiteheadean Type Space-time Logics**, Advances in Modal Logic 9, pp. 538-558 (reference [48]).

Reports at scientific conferences and workshops.

The development of dynamic mereological and mereotopological relations and results about them are presented in short talks at nine international conferences in the field of mathematical logic (four times at the *Logic Colloquium*, at three *Panhellenic Logic Symposiums*, once at the *Advances in Modal Logic* and once at a *MASSEE* International Congress on Mathematics), at two workshops on logics for space and time and at a science report session at the Faculty of Mathematics and Informatics of the Sofia University.

7-th Panhellenic Logic Symposium, Patras, Greece, July 15-19, 2009 (reference [34]).

The representation theory of the dynamic mereological relations is presented.

MASSEE International Congress on Mathematics, Ohrid, Macedonia, September 16-20, 2009 (reference [24]).

The representation theory of the dynamic mereological relations and the results about their first-order and modal logics are presented.

Logic Colloquium 2010, Paris, France, July 25-31, 2010 (reference [25]).

The report presents the results about the representation theory and the first-order and modal logics of the dynamic mereological relations.

8-th Panhellenic Logic Symposium, Ioannina, Greece, July 04-08, 2011 (reference [27]).

This report is about the hereditary undecidability of first-order logics with overlap and contact relations in their language.

Workshop on Logics for Space and Time, Guilechica, Bulgaria, June 23-26, 2011.

The presented results consist of the representation theory, the completeness, decidability and undecidability of logics about stable and unstable mereological relations.

Logic Colloquium 2011, Barcelona, Spain, July 11-16, 2011 (reference [28]**).**

This report is about the hereditary undecidability of first-order logics about overlap and contact.

Logic Colloquium 2012, Manchester, United Kingdom, July 12-18, 2012 (reference [30]**).**

This report presents the dynamic mereotopological relations, their representation theory and the completeness, decidability and undecidability results for their first-order logics.

Advances in Modal Logic 2012, Copenhagen, Denmark, August 22-25, 2012 (reference [29]**).**

The representation theory and the completeness of the modal logic of stable and unstable mereotopological relations are presented.

Sofia University, FMI science session 2013, Sofia, Bulgaria, March 16, 2013.

Completeness, decidability and undecidability results about first-order and modal logics for dynamic mereological and mereotopological relations are reported.

9-th Panhellenic Logic Symposium, Athens, Greece, July 15-18, 2013 (reference [31]**).**

This report presents the decidability of two reducts of the modal logic of dynamic mereotopological relations.

Logic Colloquium 2013, Evora, Portugal, July 22-27 2013 (reference [32]**).**

This report is about the decidability of two reducts of the modal logic of dynamic mereotopological relations.

Second Workshop on Logics for Space and Time, Sofia, Bulgaria, February 14-15, 2014.

The development of the stable and unstable mereological and mereotopological relations is presented. The reported results consist of the completeness, decidability and undecidability results about logics for these relations.

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