Epigraphical characterization of uniformly lower regular functions in Hilbert spaces*

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Abstract

We provide a characterization of uniformly lower regular functions defined on a Hilbert space. To this end we introduce and study a property we call epi prox-regularity of an epigraph set which slightly differs from the well-known prox-regularity property of a set.

Key words: prox-regular set, uniformly prox-regular set, primal lower nice function, proximal normal, distance function, metric projection mapping, Hilbert space

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1 Introduction

The concept of a primal lower nice function was introduced by Poliquin in [29] where it was proved that Clarke and proximal subdifferentials of a primal lower nice function on finite-dimensional space coincide. In particular this means that if the definition of primal lower nice property, see (6), is taken with respect to the Clarke subdifferential, this will produce the same class of functions. In [29] Poliquin proved that these functions in $\mathbb{R}^n$ are completely characterized by their Clarke subdifferential. This was the first large class of non-convex lower semicontinuous functions with this property.

The coincidence of proximal and Clarke subdifferentials of a primal lower nice function defined on Hilbert space was proved by Levy, Poliquin and Thibault in [26]. Later Ivanov and Zlateva in [22] showed that Clarke and proximal subdifferential of a primal lower nice function defined on a β smooth Banach space coincide. The result obtained in [22] shows that the class of primal lower nice functions does not depend on what reasonable subdifferential is used in defining the class. Ivanov and Zlateva in [22] suggested "that it is possible to characterize primal lower nice property in terms not involving subdifferentials". As a step in this direction we prove that continuous primal lower nice functions on Hilbert space satisfy a property which does not involve subdifferentials, see Theorem 5.1(i) and Corollary 5.2.

Since the pioneering work of Poliquin [29], primal lower nice functions are studied in a series of publications, see e.g. [30, 31, 21, 8, 35, 27]. These functions are closely related to prox-regular sets, a term due to Poliquin, Rockafellar and Thibault [32]. Indeed, a set in a Hilbert space is prox-regular exactly when its indicator function is primal lower nice, see [32, Proposition 2.1]. The study of prox-regular sets can be traced back to the pioneering work of Federer [17] who introduced them as positively reached sets in $\mathbb{R}^n$. During the years, various names of such sets have been introduced: weakly convex [36] or proximally smooth sets [13] are commonly used in Hilbert spaces; for other names see the survey [14]. Prox-regular sets in Banach spaces are studied in [9, 10, 18, 20, 6, 7, 25] and many others.

Along with the study of prox-regular sets from theoretical point of view, they are intensively studied as involved in the famous Moreau's sweeping processes, see e.g. [2], the survey [28] and the references therein. Various properties of prox-regular sets are established in [1, 3, 4, 5]. More details one can find in the paper [32], the survey [14], the forthcoming book of Thibault [34], as well as, the bibliography therein.

Prox-regularity has been introduced as an important new regularity property in Variational Analysis by Poliquin and Rockafellar in [31], see also Chapter 13F in the monograph of Rockafellar and Wets [33]. They defined the concept for functions and sets and developed the subject in $\mathbb{R}^n$. Numerous significant characterizations of prox-regularity of a closed set C in Hilbert space at point $\overline{x} \in C$ were obtained by Poliquin, Rockafellar and Thibault in [32] in terms of the distance function d_C and metric projection mapping P_C, e.g. d_C being continuously differentiable outside of C on a neighbourhood of $\overline{x}$, or P_C being single-valued and norm-to-weak continuous on this same neighbourhood. On global level, in [32] the authors showed that uniformly prox-regular sets are proximally smooth sets providing new insights on them.

To prove our main result, we introduce and study the epi uniform prox-regularity property of an epigraph set, see Definition 2.3. This notion slightly differs from the usual uniform prox-regularity of an arbitrary set, see Definition 2.1. We choose to work on global level, i.e. with uniform properties of functions and sets involved, but similar results easily can be obtained at local level as well. The properties of epi uniformly prox-regular sets in Hilbert space are also studied, see Section 4. Our main result is Theorem 5.1 where we prove the epigraphical characterization of continuous uniformly lower regular functions on a Hilbert space. It reveals their distant resemblance to convex functions, see Corollary 5.2.

The paper is organized as follows. In the following Section 2 we give some notations, definitions and necessary preliminaries. In Section 3 it is proved that continuous uniformly lower regular functions are exactly those with epi uniformly prox-regular epigraphs, see Theorem 3.1 and Theorem 3.2. In Section 4 are established some basic and important properties of epi uniformly prox-regular sets in $H \times \mathbb{R}$. The proof of our main result, Theorem 5.1, is given in the final Section 5.

2 Preliminaries and notations

Throughout the paper, $(H, \|\cdot\|)$ is a real Hilbert space endowed with the norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ associated to the inner product $\langle \cdot, \cdot \rangle$. The open (resp. closed) ball in H with center x and radius r will be denoted by $B(x, r)$ (resp $B[x, r]$) while the closed unit ball will be denoted by $\mathbb{B}_H$.

For any nonempty subset C of H the *distance function* d_C to C is defined as

$$d_C(x) := \inf_{y \in C} \|x - y\|, \qquad \text{for all } x \in H.$$

The *projection mapping* $P_C : H \rightrightarrows H$ on C is defined by

$$P_C(x) := \{y \in C : d_C(x) = \|x - y\|\} \quad \text{for all } x \in H.$$

Whenever for $x \in H$ the latter set is a singleton, its element is denoted by $p_C(x)$.

The *proximal normal cone* of C at $x \in C$, denoted by $N_C(x)$, is defined as, see [33],

$$N_C(x) := \{p \in H : \exists\, \sigma > 0 \text{ such that } x \in P_C(x + \sigma p)\}.$$

By convention, $N_C(x) = \varnothing$ for all $x \notin C$. It is easy to see that $p \in N_C(x)$, if and only if, there is a real $\sigma > 0$ such that

$$\langle p, x' - x\rangle \le \sigma\|x' - x\|^2, \qquad \text{for all } x' \in C, \tag{1}$$

in which case one says that p is a *proximal normal* to C at x.

We will use also the *Fréchet normal cone* $N_C^F(x)$ of C at x which consists of all $x^* \in H$ such that for any $\varepsilon > 0$ there exists a neighbourhood U of x such that the inequality $\langle x^*, x' - x\rangle \le \varepsilon\|x' - x\|$ holds for all $x' \in C \cap U$. Since the norm in H is Fréchet differentiable away from the origin, it is not difficult to see that

$$N_C(x) \subseteq N_C^F(x), \quad \forall x \in C, \tag{2}$$

see e.g. [12, Corollary 3.1].

The definition of an uniformly prox-regular set in H is well-known, see e.g. [32, 9, 10]. A nonempty closed subset C of H is *uniformly prox-regular* if there is $r > 0$ such that for any $x \in C$ and $p \in N_C(x) \cap \mathbb{B}_H$ one has

$$\langle p, x' - x\rangle \le \frac{1}{2r}\|x' - x\|^2, \quad \forall x' \in C. \tag{3}$$

It is not difficult to see that it is equivalent to the following

Definition 2.1. *A nonempty closed subset C of H is* uniformly prox-regular *if there is $r > 0$ such that for any $x \in C$ and $p \in N_C(x) \cap \mathbb{B}_H$ one has*

$$\langle p, x' - x\rangle \le \frac{1}{2r}\|x' - x\|^2, \quad \forall x' \in B(x, 2r) \cap C. \tag{4}$$

Indeed, if C is uniformly prox-regular according to Definition 2.1 then (3) holds for some $r > 0$ and any $x \in C$, $p \in N_C(x) \cap \mathbb{B}_H$ for $x' \in B(x, 2r) \cap C$. If $x' \in C$ is such that $\|x' - x\| \geq 2r$, then

$$\langle p, x' - x \rangle \leq \|x' - x\| = \frac{\|x' - x\|^2}{\|x' - x\|} \leq \frac{1}{2r}\|x' - x\|^2,$$

so (3) holds.

If a set $C \subset H$ satisfies Definition 2.1 for some $r > 0$, we will say that C is r prox-regular (omitting "uniformly" for brevity).

For $r \in (0, +\infty]$ one defines the open r-tube of C as the set

$$T_C(r) := \{x \in H : 0 < d_C(x) < r\}.$$

A set $C \subset H$ is r prox-regular exactly when the projection mapping P_C is single-valued and norm-to-weak continuous on $T_C(r)$, see [32, Theorem 4.1].

The space $\overline{H} := H \times \mathbb{R}$ we will consider with the norm $|||(x, r)||| := \sqrt{\|x\|^2 + r^2}$ for $(x, r) \in \overline{H}$. Then $(\overline{H}, ||| \cdot |||)$ is a Hilbert space.

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a function. The *domain* of f is the set $\operatorname{dom} f := \{x \in H : f(x) \in \mathbb{R}\}$ and the *epigraph* of f is the set $\operatorname{epi} f := \{(x, r) \in \overline{H} : r \geq f(x)\}$. The function f is *proper* exactly when $\operatorname{dom} f \neq \varnothing$ and f is lower semicontinuous on H exactly when $\operatorname{epi} f$ is closed in $\overline{H}$.

The *proximal subdifferential* of f at $x \in \operatorname{dom} f$ is defined as the set

$$\partial f(x) := \{p \in H | (p, -1) \text{ is a proximal normal to } \operatorname{epi} f \text{ at } \big(x, f(x)\big)\},$$

while $\partial f(x) = \varnothing$ for $x \notin \operatorname{dom} f$, see e.g. [10, p. 2216]. Obviously,

$$p \in \partial f(x) \iff (p, -1) \in N_{\operatorname{epi} f}\big(x, f(x)\big). \tag{5}$$

After that the concept for a primal lower nice function at a point of its domain was introduced by Poliquin [29], such functions defined on Hilbert space are intensively studied recently, see e.g. [15, 29, 30, 26, 35]. In [9, 10] for a function on uniformly convex Banach space was introduced the J primal lower regular (J-plr in short) concept at a point of its domain, where J stands for the duality mapping. In [23, 24] for a function on a Banach space was studied the s-lower regular concept. For a function on a Hilbert space both J-plr and 1-lower regular concept at a point of its domain coincide with the primal lower nice one.

When the constants involved in the definition of the primal lower nice property are uniform, one speaks about uniform lower nice property.

A proper lower semicontinuous function $f : H \to \mathbb{R}$ is said to be *uniformly primal lower nice* if there exist $\rho > 0$ and $\theta > 0$ such that for any $t \geq \theta$, any $p \in \partial f(x)$ with $\|p\| \leq \rho t$,

$$f(x') \geq f(x) + \langle p, x' - x\rangle - \frac{t}{2}\|x' - x\|^2, \quad \text{for all } x' \in H, \tag{6}$$

see e.g. [10, p. 2226].

From the very definition, it is clear that if f is uniformly primal lower nice with some positive constants ρ, and θ, then it is so for any $\rho' < \rho$ and $\theta' > \theta$. Hence, taking small ρ, and then $\theta = \rho^{-1}$ one comes to the following equivalent definition: a proper lower semicontinuous function $f : H \to \mathbb{R}$ is uniformly primal lower nice if there exists $\rho > 0$ such that for any $t \geq \rho^{-1}$, and any $p \in \partial f(x)$ with $\|p\| \leq \rho t$, (6) holds. When the latter holds for f for some $\rho > 0$ one says that the function f is ρ primal lower nice (omitting "uniformly" for brevity).

It is easy to see that such functions are, for example, the 1-lower regular on the whole space H functions considered in e.g. [23, 24].

We introduce a slightly more general notion named uniform epi lower regularity of a function.

Definition 2.2. *A proper lower semicontinuous function $f : H \to \mathbb{R}$ is said to be uniformly lower regular if there exists $\rho > 0$ such that for any $t \geq \rho^{-1}$, any $p \in \partial f(x)$ with $\|p\| \leq \rho t$,*

$$\alpha' \geq f(x) + \langle p, x' - x\rangle - \frac{t}{2}\|x' - x\|^2, \quad \forall\ (x', \alpha') \in B((x, f(x)), 2\rho) \cap \operatorname{epi} f.$$

If a function f satisfies Definition 2.2, we will say that f is ρ lower regular (again omitting "uniformly"). It is clear that any ρ primal lower nice function is ρ lower regular.

A non-empty closed set $C \subset \overline{H}$ will be called *epigraph set* if $C \equiv \operatorname{epi} f$ for a proper lower semicontinuous function $f : H \to \mathbb{R} \cup \{+\infty\}$.

For an epigraph set in $\overline{H}$ we introduce the notion of epi uniform prox-regularity which slightly differs from the well-known uniform prox-regularity of a set in $\overline{H}$.

Definition 2.3. *Let C be an epigraph set in $\overline{H}$. One says that C is uniformly epi prox-regular if there is $r > 0$ such that for any $(x, \alpha) \in C$, and $(q, \eta) \in N_C\big(x, \alpha\big) \cap \mathbb{B}_{\overline{H}}$ one has*

$$(7) \quad \big\langle (q, \eta), (x' - x, \alpha' - \alpha) \big\rangle \leq \frac{1}{2r} \|x' - x\|^2, \quad \forall (x', \alpha') \in B((x, \alpha), 2r) \cap C.$$

If an epigraph set C satisfies Definition 2.3, we will say that C is epi r prox-regular (omitting "uniformly").

From the very definitions it is clear that if an epigraph set $C \subset \overline{H}$ is epi r prox-regular according to Definition 2.3, then C is r prox-regular in $\overline{H}$ according to Definition 2.1. Indeed, if $(x, \alpha) \in C$, and $(q, \eta) \in N_C\big(x, \alpha\big) \cap \mathbb{B}_{\overline{H}}$ from (7) it follows that for all $(x', \alpha') \in B((x, \alpha), 2r) \cap C$,

$$\big\langle (q, \eta), (x' - x, \alpha' - \alpha) \big\rangle \leq \frac{1}{2r} \|x' - x\|^2,$$

so

$$\big\langle (q, \eta), (x' - x, \alpha' - \alpha) \big\rangle \leq \frac{1}{2r} \|x, -x\|^2 \leq \frac{1}{2r} |||(x' - x, \alpha' - \alpha)|||^2,$$

which is (4) in $\overline{H} = H \times \mathbb{R}$.

From uniform prox-regularity of an epigraph set it does not hold in general that it is epi uniformly prox-regular.

Before proceeding with the rest of the paper, let us note that the uniform results obtain in the rest of the paper have their local counterparts that could be proven in the same manner.

3 Epigraphical characterization of epi uniformly lower regular functions

First we will prove that if $f : H \to \mathbb{R}$ is a continuous uniformly lower regular function, then $\operatorname{epi} f$ is an epi uniformly prox-regular set in $\overline{H}$. The proof follows the lines of the proofs of [10, Propositions 4.1 and 4.4] where J-plr functions are considered.

Theorem 3.1. *If $f : H \to \mathbb{R}$ is a continuous ρ lower regular function, then $C \equiv \operatorname{epi} f$ is an epi ρ prox-regular set in $\overline{H}$.*

Proof. Let $(x, \alpha) \in C$ and $(x^*, -\lambda) \in N_C(x, \alpha) \cap \mathbb{B}_{\overline{H}}$. We will consider the following two cases:

CASE 1. $\lambda > 0$.

In this case it is clear that $\alpha = f(x)$.

Since $N_C(x, f(x))$ is a cone, we have that $\big(\frac{x^*}{\lambda}, -1\big) \in N_C\big(x, f(x)\big)$. From (5) it holds that $\frac{x^*}{\lambda} \in \partial f(x)$. As $\|x^*\| \leq 1$, it follows that $\big\|\frac{x^*}{\lambda}\big\| \leq \frac{1}{\lambda}$.

Let us take $t = \frac{1}{\lambda\rho}$. So, $\big\|\frac{x^*}{\lambda}\big\| \leq t\rho$.

Since f is a ρ lower regular function, $\frac{x^*}{\lambda} \in \partial f(x)$, $t \geq \frac{1}{\rho}$, and $\big\|\frac{x^*}{\lambda}\big\| \leq t\rho$, we get that for all $(x', \alpha') \in B((x, f(x)), 2\rho) \cap \operatorname{epi} f$,

$$\alpha' \geq f(x) + \langle \frac{x^*}{\lambda}, x' - x \rangle - \frac{t}{2}\|x' - x\|^2.$$

Multiplying by $\lambda > 0$ and using that $f(x) = \alpha$ we obtain,

$$0 \geq \lambda(\alpha - \alpha') + \langle x^*, x' - x \rangle - \frac{\lambda t}{2}\|x' - x\|^2.$$

Equivalently,

$$\langle (x^*, -\lambda), (x' - x, \alpha' - \alpha) \rangle \leq \frac{1}{2\rho}\|x' - x\|^2, \quad \forall\, (x', \alpha') \in B((x, f(x)), 2\rho) \cap \operatorname{epi} f.$$

CASE 2. $\lambda = 0$. In this case we have that $(x^*, 0) \in N_C\big(x, f(x)\big)$. By the inclusion (2), we have that $(x^*, 0) \in N_C^F\big(x, f(x)\big)$.

Using the approximation result of Ioffe [19, p. 190], we can find sequences $\{\lambda_n\}$, $\{u_n\}$, $\{u_n^*\}$ such that $\lambda_n > 0$ and as n tends to infinity, $\lambda_n \searrow 0$, $(u_n^*, -\lambda_n) \in N_C^F\big(u_n, f(u_n)\big)$ and

$$\big(u_n, f(u_n)\big) \to \big(x, f(x)\big); \tag{8}$$

$$|||(u_n^*, -\lambda_n) - (x^*, 0)||| \to 0. \tag{9}$$

Further, we use the approximation result in [10, Proposition 3.1] to find sequences $(x_n, \alpha_n) \in C$ and $(y_n^*, -\mu_n) \in N_C\big(x_n, \alpha_n\big)$ such that

$$|||(x_n, \alpha_n) - \big(u_n, f(u_n)\big)||| < \frac{\lambda_n}{2}; \tag{10}$$

$$|||(y_n^*, -\mu_n) - (u_n^*, -\lambda_n)||| < \frac{\lambda_n}{2}. \tag{11}$$

From (11) it follows that $|\lambda_n - \mu_n| < \frac{\lambda_n}{2}$, hence $\frac{\lambda_n}{2} < \mu_n < \frac{3\lambda_n}{2}$, so $\mu_n > 0$ and $\mu_n \searrow 0$.

Since $\mu_n > 0$, $f(x_n) = \alpha_n$. So, $(y_n^*, -\mu_n) \in N_C\big(x_n, f(x_n)\big)$.

Let us denote $x_n^* := \dfrac{y_n^*}{\mu_n}$. Hence $(x_n^*, -1) \in N_C\big(x_n, f(x_n)\big)$. Therefore, $x_n^* \in \partial f(x_n)$, see (5).

We will show that

$$x_n \to x, \quad f(x_n) \to f(x), \quad \mu_n x_n^* \to x^* \text{ as } n \to \infty. \tag{12}$$

From the triangle inequality,

$$\|x_n - x\| \le \|x_n - u_n\| + \|u_n - x\|$$

and $x_n \to x$ from (10) and (8).

From $f(x_n) = \alpha_n$ and the triangle inequality,

$$|f(x_n) - f(x)| = |\alpha_n - f(x)| \le |\alpha_n - f(u_n)| + |f(u_n) - f(x)|,$$

and $f(x_n) \to f(x)$ because of (10) and (8).

As $\mu_n x_n^* = y_n^*$, from the triangle inequality we have

$$\|\mu_n x_n^* - x^*\| = \|y_n^* - x^*\| \le \|y_n^* - u_n^*\| + \|u_n^* - x^*\|$$

and $\mu_n x_n^* \to x^*$ using (11) and (9).

Let us assume for a while that $x^* \ne 0$ and let us denote

$$t_n := \max\left\{\frac{1}{\rho\mu_n}, \frac{\|x_n^*\|}{\rho\|x^*\|}\right\}.$$

Obviously when n goes to infinity,

$$\mu_n t_n = \max\left\{\frac{1}{\rho}, \frac{\mu_n\|x_n^*\|}{\rho\|x^*\|}\right\} \to \frac{1}{\rho}. \tag{13}$$

Let $(x', \alpha') \in B((x, f(x)), 2\rho) \cap \operatorname{epi} f$ be arbitrary. Then for sufficiently large n, $(x', \alpha') \in B((x_n, f(x_n)), 2\rho) \cap \operatorname{epi} f$, and $t_n \ge \rho^{-1}$. Since f is a ρ lower regular function and $x_n^* \in \partial f(x_n)$ with $\|x_n^*\| \le t_n\rho$, we have that

$$\alpha' \ge f(x_n) + \langle x_n^*, x' - x_n\rangle - \frac{t_n}{2}\|x' - x_n\|^2.$$

Multiplying by $\mu_n > 0$ we get

$$0 \geq \mu_n(f(x_n) - \alpha') + \langle \mu_n x_n^*, x' - x_n \rangle - \frac{\mu_n t_n}{2} \|x' - x_n\|^2.$$

Now letting n tend to infinity and using (12), and (13) we obtain

$$0 \geq \langle x^*, x' - x \rangle - \frac{1}{2\rho} \|x' - x\|^2.$$

Since the latter obviously holds for $x^* = 0$, the proof is completed. $\square$

Now we will prove the converse, i.e. that if $C \equiv \operatorname{epi} f$ is an epi uniformly prox-regular set in $\overline{H}$, then f is an uniformly lower regular function on H.

Theorem 3.2. *If the epigraph set $C \equiv \operatorname{epi} f$ in $\overline{H}$ is epi r prox-regular, then the corresponding $f : H \to \mathbb{R} \cup \{+\infty\}$ is a ρ lower regular function for $\rho = r/\sqrt{2}$.*

Proof. Let $p \in \partial f(x)$ be such that $\|p\| \leq \rho t$ for some $t \geq \rho^{-1}$. From (5) we have that $(p, -1) \in N_C\big(x, f(x)\big)$, hence

$$\frac{1}{\sqrt{\|p\|^2 + 1}} (p, -1) \in N_C\big(x, f(x)\big) \cap \mathbb{B}_{\overline{H}}.$$

As the set C is epi r prox-regular and $\rho < r$, for all $(x', \alpha') \in B((x, f(x)), 2\rho) \cap C$,

$$\frac{1}{\sqrt{\|p\|^2 + 1}} \left\langle (p, -1), \big(x' - x, \alpha' - f(x)\big) \right\rangle \leq \frac{1}{2r} \|x' - x\|^2,$$

hence,

$$\langle p, x' - x \rangle + f(x) - \alpha' \leq \frac{\sqrt{\|p\|^2 + 1}}{2r} \|x' - x\|^2.$$

Therefore,

$$\alpha' \geq f(x) + \langle p, x' - x \rangle - \frac{\sqrt{\|p\|^2 + 1}}{2r} \|x' - x\|^2. \tag{14}$$

Using that $\|p\| \leq t\rho$ and that $\frac{1}{t} \leq \rho$, we get

$$\|p\|^2 + 1 \leq t^2 \rho^2 + 1 = t^2 \left(\rho^2 + \frac{1}{t^2} \right) \leq 2t^2 \rho^2 = r^2 t^2,$$

hence,

$$-\frac{\sqrt{\|p\|^2+1}}{r} \geq -\frac{tr}{r} = -t. \tag{15}$$

From (14), and (15) it follows that

$$\alpha' \geq f(x) + \langle p, x'-x\rangle - \frac{t}{2}\|x'-x\|^2,\ \forall (x',\alpha') \in B((x,f(x)),2\rho)\cap C,$$

which means that f is a ρ lower regular function. $\square$

It is worth noticing here that for a continuous function $f : H \to \mathbb{R}$ being uniformly lower regular is equivalent to its epigraph being a uniformly epi prox-regular set.

4 Properties of epi uniformly prox-regular sets

The following characteristic property of a r prox-regular set set C in a Hilbert space H is well-known: for any $a, b \in C$ with $\|a-b\| < 2r$ and any $\lambda \in (0,1)$ for $x_\lambda := \lambda a + (1-\lambda)b$ there exists $u_\lambda \in C$ such that

$$\|x_\lambda - u_\lambda\| \leq \varphi(\lambda),$$

where $\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}$, see the papers of J.-P. Vial [36], G. E. Ivanov [20, Lemma 4.2], and the book of L. Thibault [34, Proposition 15.41]. By using different arguments, we also established the latter in [25, Theorem 1]. We will use here arguments in the line of our paper [25] to show that epi uniformly prox regular set in $\overline{H}$ possesses a similar property as well.

Theorem 4.1. *Let $C \subset \overline{H}$ be epi r prox-regular. Let $(a,\alpha),(b,\beta) \in C$ be such that $|||(a,\alpha)-(b,\beta)||| < 2r$.*

Then for any $\lambda \in [0,1]$ for $(x_\lambda,\gamma_\lambda)$, where $x_\lambda := \lambda a + (1-\lambda)b$, and $\gamma_\lambda := \lambda\alpha + (1-\lambda)\beta$, there exists $(u_\lambda,\xi_\lambda) \in C$ such that

$$d_C(x_\lambda,\gamma_\lambda) = |||(x_\lambda,\gamma_\lambda)-(u_\lambda,\xi_\lambda)||| \leq \varphi(\lambda), \tag{16}$$

where $\varphi(\lambda) = r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}$.

Proof. Take an arbitrary $\lambda \in [0,1]$ and consider the corresponding to it $(x_\lambda, \gamma_\lambda)$. We fix λ and further we will omit it from the index. If $(x,\gamma) \in C$ then (16) holds for $(u,\xi) = (x,\gamma)$. Now, consider the case $(x,\gamma) \notin C$. Since $(x,\gamma) \in T_C(r)$ and the set C is prox-regular, there is unique $(u,\xi) \in C$ such that $(u,\xi) = p_C(x,\gamma)$. Denote $(p,\eta) := (x-u, \gamma-\xi)$. So, $|||(p,\eta)||| \neq 0$, $\|p\| \le |||(p,\eta)||| < r$ and

$$u = \lambda a + (1-\lambda)b - p, \quad \xi = \gamma - \eta. \tag{17}$$

As $(u,\xi) \in P_C(x,\gamma)$, it holds that $0 \neq (p,\eta) \in N_C(u,\xi)$. Hence, $\eta \le 0$ or, equivalently, $\gamma \le \xi$.

Since $\dfrac{(p,\eta)}{|||p,\eta)|||} \in N_C(u,\xi) \cap \mathbb{B}_{\overline{H}}$ and C is an epi r prox-regular set, we have that for all $(x',\alpha') \in C$ such that $|||(x',\alpha') - (u,\xi)||| < 2r$ it holds that

$$\frac{1}{|||(p,\eta)|||} \left\langle (p,\eta), (x',\alpha') - (u,\xi) \right\rangle \le \frac{1}{2r}\|x'-u\|^2. \tag{18}$$

Since

$$\begin{aligned} |||(a,\alpha) - (u,\xi)||| &\le |||(a,\alpha) - (x,\gamma)||| + |||(x,\gamma) - (u,\xi)||| \\ &= (1-\lambda)|||(a,\alpha) - (b,\beta)||| + |||(x,\gamma) - (u,\xi)||| \\ &\le (1-\lambda)|||(a,\alpha) - (b,\beta)||| + |||(x,\gamma) - (b,\beta)||| \\ &= |||(a,\alpha) - (b,\beta)||| < 2r \end{aligned}$$

we can put $(x',\alpha') = (a,\alpha)$ in (18) to get

$$\langle p, a-u\rangle + \eta(\alpha - \xi) \le \frac{|||(p,\eta)|||}{2r}\|a-u\|^2.$$

Using the expressions for u and ξ from (17) in the latter, we obtain that

$$\big\langle p, p + (1-\lambda)(a-b)\big\rangle + \eta(\alpha - \gamma + \eta) \le$$

$$\frac{|||(p,\eta)|||}{2r}\|p + (1-\lambda)(a-b)\|^2 = \tag{19}$$

$$\frac{|||(p,\eta)|||}{2r}\left(\|p\|^2 + 2(1-\lambda)\langle p, a-b\rangle + (1-\lambda)^2\|a-b\|^2\right).$$

Analogously, as

$$\begin{aligned} |||(b,\beta) - (u,\xi)||| &\le |||(b,\beta) - (x,\gamma)||| + |||(x,\gamma) - (u,\xi)||| \\ &= \lambda|||(a,\alpha) - (b,\beta)||| + |||(x,\gamma) - (u,\xi)||| \\ &\le \lambda|||(a,\alpha) - (b,\beta)||| + |||(x,\gamma) - (a,\alpha)||| \\ &= |||(a,\alpha) - (b,\beta)||| < 2r \end{aligned}$$

we can put $(x', \alpha') = (b, \beta)$ in (18) to obtain

$$\big\langle p, p + \lambda(b-a)\big\rangle + \eta(\beta - \gamma + \eta) \leq \frac{|||(p,\eta)|||}{2r}\left(\|p\|^2 + 2\lambda\langle p, b-a\rangle + \lambda^2\|a-b\|^2\right). \tag{20}$$

Multiplying (19) by λ, (20) by $(1-\lambda)$ and adding them we obtain

$$\langle p, p\rangle + \eta(\eta + (\lambda\alpha + (1-\lambda)\beta - \gamma)) \leq \frac{|||(p,\eta)|||}{2r}\left(\|p\|^2 + \lambda(1-\lambda)\|a-b\|^2\right).$$

Since $\eta \leq 0$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$, the latter yields

$$\|p\|^2 + \eta^2 \leq \frac{|||(p,\eta)|||}{2r}\left(\|p\|^2 + \lambda(1-\lambda)\|a-b\|^2\right).$$

Hence,

$$2r|||(p,\eta)||| \leq \|p\|^2 + \lambda(1-\lambda)\|a-b\|^2. \tag{21}$$

As $\|p\| \leq |||(p,\eta)|||$ from the latter it holds that the quadratic inequality

$$t^2 - 2rt + \lambda(1-\lambda)\|a-b\|^2 \geq 0. \tag{22}$$

is satisfied by $|||(p,\eta)|||$ as well as by $\|p\|$.

Since $\|a-b\| < 2r$, and $\lambda \in [0,1]$,

$$D := 4r^2 - 4\lambda(1-\lambda)\|a-b\|^2 > 0$$

and any t satisfying (22) should be such that $t \leq t_1$ or $t \geq t_2$, where

$$t_1 := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}, \quad t_2 := r + \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Since $t_2 \geq r > \|p\|$, we have that $\|p\| \leq t_1$, which reads

$$\|p\| \leq r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} = \varphi(\lambda).$$

Using the latter in (21) we obtain that $|||(p,\eta)||| \leq \varphi(\lambda)$, which is (16) and the proof is completed. □

Let us note here that if we have used only the prox-regularity of C, the estimate would be

$$d_C(x_\lambda, \gamma_\lambda) = |||(x_\lambda, \gamma_\lambda) - (u, \xi)||| \le r - \sqrt{r^2 - \lambda(1-\lambda)|||(a, \alpha) - (b, \beta)|||^2},$$

which, because of $|||(a, \alpha) - (b, \beta)||| \ge \|a - b\|$, is weaker than the estimate (16) we obtained.

Theorem 4.2. *Let $C \subset \overline{H}$ be an epigraph set. Then the following are equivalent:*
(a) *C is epi r prox-regular;*
(b) *For any $(a, \alpha), (b, \beta) \in C$ such that $|||(a, \alpha) - (b, \beta)||| < 2r$, it holds that*

$$d_C\big(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta\big) \le r - \sqrt{r^2 - \lambda(1-\lambda)\|a - b\|^2};$$

(c) *For any $(a, \alpha), (b, \beta) \in C$ such that $|||(a, \alpha) - (b, \beta)||| < 2r$, it holds that*

$$d_C\big(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta\big) \le \frac{1}{2r}\min\{\lambda, 1-\lambda\}\|a - b\|^2. \tag{23}$$

Proof. The implication (a)⇒(b) is established in Theorem 4.1.

Let (b) holds. For proving (b)⇒(c) it is enough to show that if $\|a - b\| < 2r$ and $\lambda \in [0, 1]$, then

$$r - \sqrt{r^2 - \lambda(1-\lambda)\|a - b\|^2} \le \frac{1}{2r}\min\{\lambda, 1-\lambda\}\|a - b\|^2.$$

To this end we consider two cases.

Case 1. $\min\{\lambda, 1-\lambda\} = \lambda$.

In this case $\lambda \le \frac{1}{2}$ and we have to show that

$$r - \sqrt{r^2 - \lambda(1-\lambda)\|a - b\|^2} \le \frac{\lambda}{2r}\|a - b\|^2,$$

or

$$r - \frac{\lambda}{2r}\|a - b\|^2 \le \sqrt{r^2 - \lambda(1-\lambda)\|a - b\|^2}.$$

As the left hand side of the inequality is greater than zero, it is equivalent to show that

$$\left(r - \frac{\lambda}{2r}\|a - b\|^2\right)^2 \le r^2 - \lambda(1-\lambda)\|a - b\|^2,$$

or

$$r^2 - \lambda\|a-b\|^2 + \frac{\lambda^2}{4r^2}\|a-b\|^4 \le r^2 - \lambda(1-\lambda)\|a-b\|^2,$$

$$\big(\lambda(1-\lambda) - \lambda\big)\|a-b\|^2 + \frac{\lambda^2}{4r^2}\|a-b\|^4 \le 0,$$

$$\lambda^2\|a-b\|^2\left(\frac{\|a-b\|^2}{4r^2} - 1\right) \le 0.$$

The last inequality holds since $\|a-b\| < 2r$.

CASE 2. $\min\{\lambda, 1-\lambda\} = 1-\lambda$.

In this case $\lambda \ge \frac{1}{2}$ and we have to show that

$$r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2} \le \frac{1-\lambda}{2r}\|a-b\|^2,$$

or equivalently

$$r - \frac{1-\lambda}{2r}\|a-b\|^2 \le \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Since the left hand side of the inequality is positive, it is equivalent to show that

$$\left(r - \frac{1-\lambda}{2r}\|a-b\|^2\right)^2 \le r^2 - \lambda(1-\lambda)\|a-b\|^2,$$

or

$$r^2 - (1-\lambda)\|a-b\|^2 + \frac{(1-\lambda)^2}{4r^2}\|a-b\|^4 \le r^2 - \lambda(1-\lambda)\|a-b\|^2,$$

$$\big(\lambda(1-\lambda) - (1-\lambda)\big)\|a-b\|^2 + \frac{(1-\lambda)^2}{4r^2}\|a-b\|^4 \le 0,$$

$$(1-\lambda)^2\|a-b\|^2\left(\frac{\|a-b\|^2}{4r^2} - 1\right) \le 0,$$

which holds as $\|a-b\| < 2r$. The proof of (b)$\Rightarrow$(c) is completed.

Let (c) holds. Let $y = (a,\alpha)$, $x = (b,\beta) \in C$ be such that $|||x-y||| < 2r$, and $v = (q,\eta) \in N_C(x) \cap \mathbb{B}_{\overline{H}}$.

We consider $(H \times \mathbb{R}) \times \mathbb{R}$ with the norm $||||\cdot||||$ defined as $||||(z,\alpha)|||| := \sqrt{|||z|||^2 + |\alpha|^2}$. Hence, $(\overline{H} \times \mathbb{R}, ||||\cdot||||)$ is a Hilbert space.

As $v \in N_C(x) \cap \mathbb{B}_{\overline{H}}$, $v \in \partial d_C(x)$. Hence from (5) we have that $(v, -1) \in N_{\operatorname{epi} d_C}(x, d_C(x)) \subset \overline{H} \times \mathbb{R}$ and then by (1) there exist some $\sigma > 0$ such that

$$\langle (v,-1)\,,(x'-x,\alpha'-d_C(x))\rangle \leq \sigma |||| (x'-x,\alpha'-d_C(x))||||^2, \quad \forall (x',\alpha') \in \operatorname{epi} d_C.$$

Since $x \in C$, $d_C(x) = 0$, and then

$$\langle (v,-1)\,,(x'-x,\alpha')\rangle \leq \sigma |||| (x'-x,\alpha')||||^2, \quad \forall (x',\alpha') \in \operatorname{epi} d_C.$$

For $\lambda \in [0,1]$ consider the point $z_\lambda := (\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta)$.

For the point $(x',\alpha') = (z_\lambda, d_C(z_\lambda))$ the last inequality gives

$$\langle v, z_\lambda - x\rangle - d_C(z_\lambda) \leq \sigma \left(|||z_\lambda - x|||^2 + d_C^2(z_\lambda)\right).$$

Hence,

$$\begin{array}{rcl} \langle v, z_\lambda - x\rangle & \leq & d_C(z_\lambda) + \sigma |||z_\lambda - x|||^2 + \sigma d_C^2(z_\lambda) \\ & \leq & d_C(z_\lambda) + \sigma |||z_\lambda - x|||^2 + \sigma |||z_\lambda - x|||^2 \\ & = & d_C(z_\lambda) + 2\sigma |||z_\lambda - x|||^2. \end{array} \tag{24}$$

So, for $\lambda < 1/2$, we have

$$\begin{array}{rcl} \langle v, \lambda(a-b,\alpha-\beta)\rangle & = & \langle v, z_\lambda - x\rangle \quad \text{from (24)} \\ & \leq & d_C(z_\lambda) + 2\sigma |||z_\lambda - x|||^2 \\ & = & d_C(z_\lambda) + 2\sigma\lambda^2 |||(b-a,\beta-\alpha)|||^2 \quad \text{from (23)} \\ & \leq & \dfrac{\lambda}{2r}\|a-b\|^2 + 2\sigma\lambda^2 |||(b-a,\beta-\alpha)|||^2. \end{array}$$

Dividing the last inequality by $\lambda > 0$ we get

$$\langle v, (a-b,\alpha-\beta)\rangle \leq \frac{1}{2r}\|a-b\|^2 + 2\sigma\lambda |||(b-a,\beta-\alpha)|||^2.$$

Now letting λ to zero and using that $v = (q,\eta)$ we obtain

$$\langle (q,\eta), (a-b,\alpha-\beta)\rangle \leq \frac{1}{2r}\|a-b\|^2,$$

which entails (a). □

5 Main result

Theorem 5.1. *Let $f : H \to \mathbb{R}$ be a continuous function. If f is a r lower regular function, then*
(i) *for any (a, α), $(b, \beta) \in \operatorname{epi} f$ such that $|||(a, \alpha) - (b, \beta)||| < 2r$ and any $\lambda \in [0, 1]$ there is $(u_\lambda, \xi_\lambda) \in \operatorname{epi} f$ such that*

$$(25) \qquad \|u_\lambda - (\lambda a + (1 - \lambda)b)\|^2 + |\xi_\lambda - (\lambda\alpha + (1 - \lambda)\beta)|^2 \le \varphi^2(\lambda),$$

where $\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1 - \lambda)\|a - b\|^2}$.

Conversely, if (i) *holds, then f is a ρ lower regular for $\rho = \frac{r}{\sqrt{2}}$.*

Proof. Let f be a r lower regular function. According to Theorem 3.1, the set $C \equiv \operatorname{epi} f$ is epi r prox-regular in $\overline{H}$. Applying Theorem 4.1 to the set C and the points (a, α) and (b, β) in C we have that for any $\lambda \in [0, 1]$ for $(\lambda a + (1 - \lambda)b, \lambda\alpha + (1 - \lambda)\beta)$, there exists $(u_\lambda, \xi_\lambda) \in C$ such that $|||(\lambda a + (1 - \lambda)b, \lambda\alpha + (1 - \lambda)\beta) - (u_\lambda, \xi_\lambda)||| \le \varphi(\lambda)$, which is (25) and (i) holds.

It is clear that either

$$f(u_\lambda) \le \lambda\alpha + (1 - \lambda)\beta,$$

or

$$\lambda\alpha + (1 - \lambda)\beta < f(u_\lambda) \le \xi_\lambda \le \lambda\alpha + (1 - \lambda)\beta + \varphi(\lambda).$$

Let now (i) holds for f. This is equivalent to the feature that the epigraph set $C \equiv \operatorname{epi} f$ satisfies the condition of Theorem 4.2(b) and, therefore, it is an epi r prox-regular set. Then Theorem 3.2 ensures that f is a ρ lower regular function for $\rho = \frac{r}{\sqrt{2}}$. □

Taking $\alpha = f(a)$, $\beta = f(b)$ in Theorem 5.1 we obtain

Corollary 5.2. *If $f : H \to \mathbb{R}$ is a continuous ρ lower regular function, then for any $a, b \in \operatorname{dom} f$ such that $|||(a, f(a)) - (b, f(b))||| < 2r$ and any $\lambda \in [0, 1]$ there is $u \in \operatorname{dom} f \cap B[\lambda a + (1 - \lambda)b), \varphi(\lambda)]$ such that either*

$$f(u) \le \lambda f(a) + (1 - \lambda)f(b),$$

or

$$\lambda f(a) + (1 - \lambda)f(b) < f(u) \le \lambda f(a) + (1 - \lambda)f(b) + \varphi(\lambda),$$

where $\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1 - \lambda)\|a - b\|^2}$. In particular,

$$\inf_{B[\lambda a + (1-\lambda)b), \varphi(\lambda)]} f \le \lambda f(a) + (1 - \lambda)f(b) + \varphi(\lambda).$$

Primal lower nice functions were introduced as generalizations of convex functions. Since their former definition involves a subdifferential, they could be considered as, in some sense, a dual generalization. However, a closer look at the last result shows that primal lower nice functions are in fact a good primal generalization of the convex functions defined on Hilbert spaces.

References

[1] S. Adly, F. Nacry and L. Thibault, *Preservation of prox-regularity of sets with applications to constrained optimization*, SIAM J. Optim., 26 (2016), 448–473.

[2] S. Adly, F. Nacry and L. Thibault, *Discontinuous sweeping process with prox-regular sets*, ESAIM: COCV, 23 (2017), No 4, 1293–1329.

[3] S. Adly, F. Nacry and L. Thibault, *Prox-regularity approach to generalized equations and image projection*, ESAIM: COCV, 24 (2018), 677–708.

[4] S. Adly, F. Nacry and Lionel Thibault, *Prox-regular sets and Legendre-Fenchel transform related to separation properties*, Optimization, 71 (2022) Issue 7, 2097–2129.

[5] S. Adly, F. Nacry and L. Thibault, *New metric properties for prox-regular sets*, Mathematical Programming, 189 (2021) No 1, 7–36.

[6] M. V. Balashov and G. E. Ivanov, *Weakly convex and proximally smooth sets in Banach spaces*, Izv. Math., 73 (2009), 455–499.

[7] F. Bernard and L. Thibault, *Prox-regular functions and sets in Banach spaces*, Set-Valued Analysis, 12 (2004), 25–47.

[8] F. Bernard, L. Thibault and D. Zagrodny, *Integration of primal lower nice functions in Hilbert spaces*, J. Optimization Theory Appl., 124 (2005) No 3, 561–579.

[9] F. Bernard, L. Thibault and N. Zlateva, *Characterizations of prox-regular sets in uniformly convex Banach spaces*, J. Convex Anal., 13 (2006), 525–559.

[10] F. Bernard, L. Thibault and N. Zlateva, *Prox-regular sets and epigraphs in uniformly convex Banach spaces: various regularities and other properties*, Trans. Amer. Math. Soc., 363 (2011), 2211–2247.

[11] J. M. Borwein and J. R. Giles, *The proxima normal formula in Banach space*, Trans. Amer. Math. Soc., 302 (1987), 371–381.

[12] J. M. Borwein and H. M. Strójwas, *Proximal analysis and boundaries of closed sets in Banach space. I. Theory*, Canad. J. Math., 38 (1986) No 2, 428–472.

[13] F. H. Clarke, R. J. Stern and P. R. Wolenski, *Proximal smoothness and the lower-C^2 property*, J. Convex Anal., 2 (1995), 117–144.

[14] G. Colombo and L. Thibault, *Prox-regular sets and applications*, In: Handbook of nonconvex analysis and applications, (2010) Int. Press, Somerville, MA, 99–182.

[15] C. Combari, A. Elhilali Alaoui, A. Levy, R. A. Poliquin and L. Thibault, *Convex composite functions in Banach spaces and the primal-lower-nice property*, Proc. Amer. Math. Soc., 126 (1998), 3701–3708.

[16] R. Correa, A. Jofré and L. Thibault, *Characterization of lower semicontinuoud convex fuctions*, Proc. Amer. Math. Soc., 116 (1992), 61–72.

[17] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc., 93 (1959), 418–491.

[18] V. Goncharov and G. E. Ivanov, *Strong and Weak Convexity of Closed Sets in a Hilbert Space*, Operations Research, Engineering, and Cyber Security, Springer Optimization and Its Application series, 113 (2017), 259–297.

[19] A. D. Ioffe, *Proximal analysis and approximate subdifferentials*, J. London Math. Soc., 41 (1990), 175–192.

[20] G. E. Ivanov, *Weak convexity in the sense of Vial and Efimov-Stechkin*, Izv. Ross. Akad. Nauk Ser. Mat., 69 (2005), 35–60 (in Russian); English tanslation in Izv. Math., 69 (2005), 1113–1135.

[21] M. Ivanov and N. Zlateva, *On primal lower-nice property*, Compt. rend. Acad. bulg. Sci., 54 (2001) No 11, 5–10.

[22] M. Ivanov and N. Zlateva, *Subdifferential characterization of primal lower-nice functions on smooth Banach spaces*, Compt. rend. Acad. bulg. Sci., 57 (2004) No 5, 13–18.

[23] I. Kecis and L. Thibault, *Subdifferential characterization of s-lower regular functions*, Applicable Anal., 94 (2015) Issue 1, 85–98.

[24] I. Kecis and L. Thibault, *Moreau envelopes of s-lower regular functions*, Nonlinear Anal., 127 (2015), 157–181.

[25] M. Konstantinov and N. Zlateva, *Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces*, J. Appl. Anal., (2023) https://doi.org/10.1515/jaa-2022-1010.

[26] A. Levy, R. A. Poliquin and L. Thibault, *Partial extension of Attouch's theorem with applications to proto-derivatives of subgradient mappings*, Trans. Amer. Math. Soc., 347 (1995), 1269–1294.

[27] M. Mazade and L. Thibault, *Primal lower nice functions and their Moreau envelopes*, In: Computational and Analytical Mathematics: In Honor of Jonathan Borwein's 60-th Birthday, David H. Bailey, Heinz H. Bauschke, Peter Borwein, Frank Garvan, Michel Théra, Jon D. Vanderwer, Henry Wolkowicz Editors, Springer Proceedings in Mathematics & Statistics, 50 (2013), 521–553.

[28] F. Nacry and L. Thibault, *Regularization of sweeping process: old and new*, Pure and Applied Functional Analysis, 4 (2019), 59–117.

[29] R. A. Poliquin, *Integration of subdifferentials of nonconvex functions*, Nonlinear Anal. Th. Meth. Appl., 17 (1991), 385–398.

[30] R. A. Poliquin, *An extension of Attouch's Theorem and its application to second-order epidifferentiation of convexly composite functions*, Trans. Amer. Math. Soc., 332 (1992), 861–874.

[31] R. A. Poliquin and R. T. Rockafellar, *Prox-regular functions in variational analysis*, Trans. Amer. Math. Soc., 348 (1996), 1805–1838.

[32] R. A. Poliquin, R. T. Rockafellar and L. Thibault, *Local differentiability of distance functions*, Trans. Amer. Math. Soc., 352 (2000) No 11, 5231–5249.

[33] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis, Grundlehren der Mathematischen Wissenschaften*, 317 (1998), Springer, New York.

[34] L. Thibault, *Unilateral Variational Analysis in Banach Spaces. Part I: General Theory. Part II: Special Classes of Functions and Sets*, (2023), World Scientific, ISBN: 978-981-125-816-9.

[35] L. Thibault and D. Zagrodny, *Integration of subdifferentials of lower semicontinuous functions*, J. Math. Anal. Appl., 189 (1995), 33–58.

[36] J.-P. Vial, *Strong and weak convexity of sets and functions*, Math. Oper. Res., 8 (1983), 231–259.