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Introduction

In the submitted dissertation important qualitative and analytic characteristics of some finite- and infinite-dimensional Hamiltonian systems are studied. Due to this reason the dissertation is divided into two parts.

In the first part we study the integrability in the sense of Liouville of some finite-dimensional Hamiltonian systems. The variables in the studied systems are assumed to be complex and the method for investigation is the same. The approach is rather algebraic. The geometry of these systems is touched partially in the conclusions of Chapter 2 and in Chapter 5. In two recently defended dissertations [92] and [28] a historical background is given on the development of the methods for the investigation of the integrability of Hamiltonian systems. Ibidem information is supplied for the equations with the Painlevé property and for Meijer's functions. We will not repeat them here. The necessary notions, fact and results about the Morales-Ruiz - Ramis theory, as well as their connection with the Differential Galois theory are given in Chapter 1 and they are valid for the all first part. For each of the considered systems, the motivation and reference of the current status of the problem is given.

In the second part we study some important nonlinear partial differential equations. It is known that there is no general theory for the equations with partial derivatives, especially nonlinear. The common feature of the models which we investigate is that most of them are bi-Hamiltonian, and therefore, integrable infinite-dimensional Hamiltonian systems. We enjoy to a certain extend an analogy with finite-dimensional integrable Hamiltonian systems: the first integrals become conservation laws, the notions Poisson brackets, conjugated variables (action-angle variables) are carried on after the corresponding modifications. This gives us some advantage and freedom for carrying by analogy and generalization some results, established for some equations to other equations.

* * *

Contents and main results

Chapter 1 is of auxiliary character. It contains most of the notions and facts necessary for the first part, as well as methods for investigating of the integrability of the Hamiltonian systems with the help of the approach of Ziglin - Morales-Ruiz - Ramis, based on the Differential Galois Theory [70].

Traditionally, a Hamiltonian system is defined with a function H, called Hamiltonian and a system of equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{0.0.1}$$

where (q,p) are usually called canonical variables. We define Poisson brackets of two functions $H(q,p), F(q,p)$ with the formula

$$\{F,H\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}. \tag{0.0.2}$$

Recall that the Poisson bracket has the properties bilinearity, skew-symmetry and satisfies the Jacobi identity. The following identities are immediate

$$\{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}. \tag{0.0.3}$$

Denote $x = (q,p)$ then the Hamiltonian equations (0.0.1) can be written as

$$\dot{x} = J\text{gradH} \quad \text{или} \quad \dot{x} = \{x, H\}, \tag{0.0.4}$$

where $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, E is the unit matrix of dimension n. In some generalizations the antisymmetric matrix J (or equivalently the canonical brackets (0.0.3)) can depend on x. Due to (0.0.4) we can write a Hamiltonian system in the following way

$$\dot{x} = X_H(x). \tag{0.0.5}$$

We say that a Hamiltonian system is integrable in the sense of Liouville if there exist n integrals $F_1 = H, F_2, \ldots, F_n$ in involution $\{F_i, F_j\} = 0$ for every i and j, almost everywhere.

Let $\Psi(t)$ is a non-equilibrium solution of (0.0.5). We can write the variational equation (VE) about this solution

$$\dot{\xi} = DX_H(\Psi(t))\xi. \tag{0.0.6}$$

Using the linear integral $dH(\Psi(t))$ of the (VE) we can reduce the order of (VE) and to obtain the so called normal variational equation (NVE)

$$\dot{\eta} = A(t)\eta. \tag{0.0.7}$$

In order to study the integrability of the Hamiltonian system (0.0.5), Ziglin uses the monodromy group of (NVE). More generally, Morales-Ruiz and Ramis use the differential Galois group of (NVE). Their result is roughly speaking the following: If the Hamiltonian system (0.0.5) is integrable then necessarily the unit component of the differential Galois group of (NVE) is abelian.

The application of the above approach to a concrete Hamiltonian system include the following steps:

1) find a particular solution;
2) write the (VE) and the (NVE);
3) check the identity component of the Galois group G^0 of the (NVE) (or (VE)).

If one obtains that G^0 is not commutative, then it follows that the considered Hamiltonian system is non-integrable. If G^0 is abelian, the the Hamiltonian system is not necessarily integrable. Then the studying of the Galois groups of the higher-order variational equations is needed. As it is seen, only step 2) is easy.

In Chapter 2 [15] we study for integrability the normal forms (of Birkhoff-Gustavson) of the Hamiltonian resonances $1:2:\omega$ truncated to order 3, $\omega = 1, 3, 4$. Recall that the truncated normal forms $\bar{H}$ admit one more first integral – $H_2 = I_1 + 2I_2 + \omega I_3, I_j = q_j^2 + p_j^2$ and $(q_j, p_j), j = 1, 2, 3$ are the canonical variables, so for integrability an additional integral is needed.

The investigation of the dynamics (equilibria, periodic orbits, stability, integrability...) of the Hamiltonian systems admitting the above resonances makes sense, since there exist real systems and also models with such property. To mention only that around 1890 the astronomer Gill discovered an (almost) $1:2:4$ resonance in the orbital motion of the inner Galilean satellites of Jupiter – Io, Europa and Ganimedes. This resonance exerts strong influence on the dynamics of the whole system Jupiter–satellites.

In section 2 the Hamiltonian functions are simplified and take the form correspondingly:

<u>1 : 2 : 3 resonance</u>

$$\bar{H} = a\left[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2\right] + b\left[p_3(p_1p_2 - q_1q_2) + q_3(q_1p_2 + p_1q_2)\right], \quad a, b \geq 0. \tag{0.0.8}$$

The obvious cases for integrability are:
. $b = 0$ with integral I_3,
. $a = 0$ with integral $lI_1 + mI_2 + (l + m)I_3$.

<u>1 : 2 : 4 resonance</u>

$$\bar{H} = a[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2] + b[p_3(p_2^2 - q_2^2) + 2p_2q_2q_3], \quad a, b \geq 0. \tag{0.0.9}$$

The obvious cases for integrability are:
. $b = 0$ with integral I_3,
. $a = 0$ with integral I_1.

Finally, for the case <u>1 : 1 : 2 resonance</u> we take the normal form of H_3 obtained by Duistermaat [24]

$$\bar{H} = H_3 = q_3\left[a(q_1^2 - p_1^2) + b(q_2^2 - p_2^2)\right] + 2p_3\left[ap_1q_1 + bp_2q_2\right], \quad a \geq b \geq 0. \tag{0.0.10}$$

The cases when there exists an additional integral are
. $b = 0$ with integral I_2,
. $a = b$ with integral $G = q_1p_2 - q_2p_1$,
. $a = 2b$ with integral (Duistermaat) $G = (q_1p_2 - p_1q_2)^2(p_2^2 + q_2^2) + 2[\frac{1}{2}q_3(q_2^2 - p_2^2) + p_3q_2p_2]^2$.

Duistermaat [24] proved, using an original approach that the Hamiltonian resonance $1:1:2$ does not admit an additional analytic first integral except in the cases given above. After that Verhulst [100] asked the question about rigorous proof of non-integrability of the resonances $1:2:3$ and $1:2:4$. The next Theorem generalizes slightly the result of Duistermaat and answers the question on integrability of the introduced Hamiltonian resonances.

Theorem 0.0.1. *(Theorem 2.2.1) The systems corresponding to the truncated to order 3 Hamiltonians (0.0.8)-(0.0.10) do not possess additional meromorphic integral, except in the cases listed above, that is, they are not integrable in the Liouvillian sense.*

In Chapter 3 [17] the integrability of some higher-order Painlevé equations is studied. Like the classical Painlevé equations $P_I, \ldots, P_{VI}$, some of these equations admit Hamiltonian formulation, families of rational and transcendent solutions, B́acklund transformations. The motivation for this investigation is the following. In [71] Morales-Ruiz asked the question about the integrability of the classical Painlevé equations as Hamiltonian systems (for the recent development of the study see [93]).

It is natural to extend the question of integrability to the higher-order Painlevé equations. Note that the classification of the equations of Painlevé type is not completed, that is, they are many more in number and few of them have known Hamiltonian formulation.

We first study the following forth-order nonlinear ordinary differential equation

$$w^{(4)} = 5w''(w^2 - w') + 5w(w')^2 - w^5 + (\lambda z + \alpha)w + \gamma, \tag{0.0.11}$$

where λ, α, γ are complex parameters. We assume that $\lambda \neq 0$. This equation is studied from different points of view from several authors. We note Gromak [30] to whom we owe the Hamiltonian formulation, two families of rational solutions and the B́acklund transformations.

Denote $q_1(z) := w(z)$, $\varepsilon^2 = 1$. Then the equation (0.0.11) can be presented as two equivalent Hamiltonian systems with $2 + 1/2$ degrees of freedom and Hamiltonian

$$H_\varepsilon = \frac{1}{2}p_2^2 + \frac{7-9\varepsilon}{12}q_2^3 + p_1q_2 - \frac{1+3\varepsilon}{4}p_1q_1^2 + \frac{3\varepsilon-1}{4}q_2(\lambda z + \alpha) + \left(\gamma + \frac{3\varepsilon-1}{4}\lambda\right) q_1. \tag{0.0.12}$$

We extend in a natural way the above Hamiltonian system to an autonomous one with three degrees of freedom and Hamiltonian $\hat{H}(q_1, q_2, z, p_1, p_2, F) := H_\varepsilon + F$

$$\begin{aligned} \frac{dq_1}{ds} &= q_2 - \frac{3\varepsilon+1}{4}q_1^2, & \frac{dp_1}{ds} &= \frac{1+3\varepsilon}{2}p_1q_1 - \gamma - \frac{3\varepsilon-1}{4}\lambda, \\ \frac{dq_2}{ds} &= p_2, & \frac{dp_2}{ds} &= -p_1 - \frac{7-9\varepsilon}{4}q_2^2 - \frac{3\varepsilon-1}{4}(\lambda z + \alpha), \\ \frac{dz}{ds} &= 1, & \frac{dF}{ds} &= -\lambda\frac{3\varepsilon-1}{4}q_2. \end{aligned} \tag{0.0.13}$$

Out first result is the following

Theorem 0.0.2. *(Theorem 3.1.1) The Hamiltonian system (0.0.13) with parameters $\gamma/\lambda = 3k, \gamma/\lambda = 3k-1, k \in \mathbb{Z}$ is non-integrable in the sense of Liouville by means of rational first integrals.*

As it is noted in Chapter 1, important role in the proof of the non-integrability plays (NVE) and here they are a particular case of the following linear equations

$$\mathrm{D}_{qp}(y) = \left[(-1)^{q-p}x\prod_{j=1}^{p}(\delta + \mu_j) - \prod_{j=1}^{q}(\delta + \nu_j - 1)\right] y = 0, \tag{0.0.14}$$

which are called generalized confluent hypergeometric equations, $\delta = xd/dx$, $0 \le p \le q, \mu_j, \nu_j \in \mathbb{C}, \mu_i - \mu_j \notin \mathbb{Z}$. For these equations, 0 is a regular singular point and ∞ is an irregular singular point when $p < q$. The local Galois group G_0 is subgroup of G_∞, and hence, the global Galois group is $G = G_\infty$. Katz and Gabber [47] have calculated the Galois groups of some classes of linear equations using purely algebraic arguments – global characterization of semi-simple algebras.

In this case (NVE) (after change of the variables, which do not alter the identity component) takes the form

$$\delta\left(\delta - \frac{2}{5} - 1\right)\left(\delta + \frac{1}{5} - 1\right)\left(\delta + \frac{2}{5} - 1\right)u - xu = 0, \tag{0.0.15}$$

i.e., it is a generalized confluent hypergeometric equation of the kind $\mathrm{D}_{40}(y)=0$. For that equation the Katz's theory gives that $G^0=\mathrm{Sp}(4,\mathbb{C})$, which is obviously not abelian, from where follows the non-integrability.

In section 3 we calculate the topological generators of $G=G_\infty$ - the formal monodromy, the exponential torus and the Stokes matrices for the equation (0.0.15) following the approach of Duval, Mitschi [26] and Ramis [82] and obtain that the Galois group is isomorphic to $\mathrm{Sp}(4,\mathbb{C})\rtimes\mathbb{Z}/5\mathbb{Z}$ which correspond to the Katz's result.

A question arises whether the appearance of the generalized confluent hypergeometric equations in the dynamics of the Painlevé equations is incident. It turns out that they are connected also with other higher-order Painlevé equations.

Let us consider the $\mathrm{P_{II}}$-hierarchy whose Hamiltonian structure is found by Mazzocco and Mo [65]

$$\mathrm{P}_{\mathrm{II}}^{(n)}:\left(\frac{d}{dz}+2w\right)\mathcal{L}_n[w'-w^2]+\sum_{l=1}^{n-1}\beta_l\left(\frac{d}{dz}+2w\right)\mathcal{L}_l[w'-w^2]=zw+\alpha_n,\ n\geq 1, \qquad (0.0.16)$$

where $\mathcal{L}_n$ is the operator defined by the recursion relation (the Lenard relation)

$$\frac{d}{dz}\mathcal{L}_{n+1}=\left[\frac{d^3}{dz^3}+4(w'-w^2)\frac{d}{dz}+2(w'-w^2)_z\right]\mathcal{L}_n;\quad \mathcal{L}_0[w'-w^2]=\frac{1}{2} \qquad (0.0.17)$$

and β_l and α_n are arbitrary complex parameters. The first three members of the $\mathrm{P_{II}}$-hierarchy are:

$$\mathrm{P}_{\mathrm{II}}^{(1)}: w''-2w^3=zw+\alpha_1, \qquad (0.0.18)$$

$$\mathrm{P}_{\mathrm{II}}^{(2)}: w^{(4)}-10w(ww''+w'^2)+6w^5+\beta_1(w''-2w^3)=zw+\alpha_2, \qquad (0.0.19)$$

$$\begin{aligned}\mathrm{P}_{\mathrm{II}}^{(3)}: &w^{(6)}-14w^{(4)}w^2-56w^{(3)}w'w+70w''(w^4-w'^2)+140w^3w'^2-42w(w'')^2\\ &-20w^7+\beta_1[w''-2w^3]+\beta_2[w^{(4)}-10w(ww''+w'^2)+6w^5]=zw+\alpha_3. \end{aligned}\qquad (0.0.20)$$

The integrability of $\mathrm{P}_{\mathrm{II}}^{(1)}$ is studied in [72]. The Hamiltonian for $\mathrm{P}_{\mathrm{II}}^{(2)}$ is

$$H^{(2)}=\frac{q_2}{16}+2zp_2-16p_1^2p_2+16p_2^2+\frac{q_1q_2p_2}{8}+\frac{p_1p_2q_2^2}{16}+\frac{\alpha_2(p_1q_2-q_1)}{8}+\beta_1(8p_1-t_1)p_2, \qquad (0.0.21)$$

where $q_j,p_j, j=1,2$ are expressed via w and its derivatives. We extend as usual to an autonomous Hamiltonian system with three degrees of freedom $\hat{H}_2=H^{(2)}+F$

$$\begin{aligned}
q_1' &= -32p_1p_2+\frac{1}{16}p_2q_2^2+\frac{1}{8}q_2\alpha_2+8\beta_1p_2,\\
q_2' &= 2z-16p_1^2+32p_2+\frac{1}{8}q_1q_2+\frac{1}{16}p_1q_2^2+\beta_1(8p_1-\beta_1),\\
p_1' &= -\frac{1}{8}p_2q_2+\frac{1}{8}\alpha_2,\\
p_2' &= -\frac{1}{16}-\frac{1}{8}p_2q_1-\frac{1}{8}p_1p_2q_2-\frac{1}{8}\alpha_2p_1,\\
z' &= 1,\qquad F'=-2p_2.
\end{aligned}\qquad (0.0.22)$$

Similarly the Hamiltonian for $\mathrm{P}_{\mathrm{II}}^{(3)}$ is

$$\begin{aligned}
H^{(3)} &= 64p_1^4-192p_1^2p_2+128p_1p_3+\frac{1}{64}p_3q_3^2-\frac{1}{64}p_1q_2^2+64p_2^2\\
&- \frac{1}{32}q_1q_2+2zp_1+\frac{q_3}{64}-\frac{1}{32}\alpha_3q_3+8\beta_1(p_1^2-p_2)\\
&+ \beta_2(4p_1^2\beta_2-4p_2\beta_2-32p_1^3+64p_1p_2-2p_1\beta_1).
\end{aligned}\qquad (0.0.23)$$

The corresponding autonomous Hamiltonian systems with four degrees of freedom $\hat{H}_3 = H^{(3)} + F$ has the form

$$\begin{aligned}
q_1' &= 256p_1^3 - 384p_1p_2 + 128p_3 - \frac{q_2^2}{64} + 2z + 16p_1\beta_1 + 8p_1\beta_2^2 - 96\beta_2p_1^2 + 64p_2\beta_2 - 2\beta_1\beta_2, \\
q_2' &= -192p_1^2 + 128p_2 - 8\beta_1 - 4\beta_2^2 + 64p_1\beta_2, \\
q_3' &= 128p_1 + \frac{1}{64}q_3^2, \\
p_1' &= \frac{1}{32}q_2, \\
p_2' &= \frac{1}{32}p_1q_2 + \frac{1}{32}q_1, \\
p_3' &= -\frac{1}{32}p_3q_3 - \frac{1}{64} + \frac{1}{32}\alpha_3, \\
z' &= 1, \quad F' = -2p_1.
\end{aligned} \tag{0.0.24}$$

We prove the following result

Theorem 0.0.3. *(Theorem 3.4.1) Suppose that*

(i) $\beta_1 = \alpha_2 = 0$. *Then the Hamiltonian system corresponding to* $\mathrm{P}_{\mathrm{II}}^{(2)}$ *is not integrable by means of rational first integrals;*

(ii) $\beta_1 = \beta_2 = \alpha_3 = 0$. *Then the Hamiltonian system corresponding to* $\mathrm{P}_{\mathrm{II}}^{(3)}$ *is not integrable by means of rational first integrals.*

(NVE) regarding some explicit rational solutions turn out to be generalized confluent hypergeometric equations whose Galois groups we calculate as in section 3 and their identity components are isomorphic to $\mathrm{Sp}(4,\mathbb{C})$ and $\mathrm{Sp}(6,\mathbb{C})$ respectively.

In Chapter 4 [18] we study the integrability of a Hamiltonian system describing the stationary solutions in Bose-Fermi mixtures in one dimensional optical lattice. The model is describing with $N_f + 1$ connected nonlinear Schr'odinger equations

$$\begin{aligned}
i\hbar\frac{\partial\Psi^b}{\partial t} &+ \frac{1}{2m_{\mathrm{B}}}\frac{\partial^2\Psi^b}{\partial x^2} - V\Psi^b - g_{\mathrm{BB}}|\Psi^b|^2\Psi^b - g_{\mathrm{BF}}\rho_f\Psi^b = 0, \\
i\hbar\frac{\partial\Psi_j^f}{\partial t} &+ \frac{1}{2m_{\mathrm{F}}}\frac{\partial^2\Psi_j^f}{\partial x^2} - V\Psi_j^f - g_{\mathrm{BF}}|\Psi^b|^2\Psi_j^f = 0, \quad j = 1,\dots,N_f,
\end{aligned} \tag{0.0.25}$$

where the wave functions Ψ_j^f describe each of N_f fermions and Ψ^b is the wave functions for the bosonic component, $\rho_f = \sum_{i=1}^{N_f} |\Psi_i^f|^2$ and $g_{\mathrm{BB}}, g_{\mathrm{BF}}, m_{\mathrm{F}}, m_{\mathrm{B}}$ are certain physical constants. The potential V is usually of the form $V = V_0 sn^2(\alpha x, \kappa)$, where $sn(\alpha x, \kappa)$ is the Jacobi elliptic function. Here it is taken $V_0 = 0$. We are interested in the stationary solutions of (0.0.25) of the kind

$$\begin{aligned}
\Psi^b(x,t) &= q_0(x)\exp\left(-i\frac{\omega_0}{\hbar}t + i\Theta_0(x) + i\kappa_0\right), \\
\Psi_j^f(x,t) &= q_j(x)\exp\left(-i\frac{\omega_j}{\hbar}t + i\Theta_j(x) + i\kappa_{0,j}\right), \quad j = 1,\dots,N_f,
\end{aligned} \tag{0.0.26}$$

where κ_0, $\kappa_{0,j}$ are constant phases, q_0, q_j and Θ_0, Θ_j are real-valued functions, related by

$$\Theta_0(x) = C_0 \int_0^x \frac{dx'}{q_0^2(x')}, \qquad \Theta_j(x) = C_j \int_0^x \frac{dx'}{q_j^2(x')}, \quad j = 1, \dots, N_f, \tag{0.0.27}$$

C_0, C_j being constants of integration. After substituting (0.0.26) in the equations (0.0.25) and separating the real and imaginary part we get a system of ODE's, which after transformations is reduced to a Hamiltonian one with the Hamiltonian function

$$H = \frac{p_0^2}{2} + \frac{1}{2}\sum_1^{N_f} p_j^2 + \omega_0 q_0^2 + \sum_1^{N_f} \omega_j q_j^2 - g_{\mathrm{BF}} q_0^2 \sum_1^{N_f} q_j^2 - \frac{q_0^4}{2} + \frac{C_0^2}{2q_0^2} + \frac{1}{2}\sum_1^{N_f} \frac{C_j^2}{q_j^2}. \tag{0.0.28}$$

For this Hamiltonian system we consider the cases:

1) $C_0 = 0$, $C_j \neq 0$, $\sum C_j \neq 0$, $\omega_j = \omega^2/2, j = 1, \dots, N_f$;

2) $C_0 \neq 0$, $C_j = 0, j = 1, \dots, N_f$, $g_{\mathrm{BF}} = n(n+1)/2, n \notin \mathbb{Z}$;

3) $C_0 \neq 0$, $C_1 \neq 0, N_f = 1$, g_{BF} sufficiently small.

The following result is proved

Theorem 0.0.4. *(Theorem 4.1.1) For the cases given above, the Hamiltonian system corresponding to (0.0.28) is non-integrable in the Liouville sense unless* $g_{\mathrm{BF}} = 0$.

In other words, the Hamiltonian system under consideration is integrable only when it is separable.

The Case 2 needs explanations. We formulate the following

Conjecture. The Hamiltonian system corresponding to (0.0.28) when $C_0 \neq 0, C_j = 0, j = 1, \dots, N_f$ and $g_{\mathrm{BF}} = n(n+1)/2, n \in \mathbb{Z}$ is non-integrable unless $g_{\mathrm{BF}} = 0$.

This statement is formulated in that way because we have checked it only for $n = 1$ and $n = 2$. Nevertheless, we think that the system corresponding to (0.0.28) is also non-integrable for arbitrary integer $n > 2$.

In Chapter 5 [16] we investigate the Gross-Neveu models in small dimensions. The Gross-Neveu models are Hamiltonian systems, related with the root systems of simple Lie algebras, given by

$$H = \frac{1}{2}(y, y) + \sum_\alpha \exp[(\alpha, x)],$$

where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ are the canonical coordinates in $\mathbb{R}^{2n}$, $(,)$ denotes the standard inner product and α is a root of a simple Lie algebra $\mathfrak{g}$. The sum is extended over the entire root system of $\mathfrak{g}$ or over its appropriate subspace, depending on the model.

Except the Hamiltonian, we have an obvious first integral only in the case of $\mathrm{sl}(n+1)$, namely $\sum y_j = const.$ Hence, the Gross-Neveu model for sl(2) is integrable. It turns out that the model for so(4) is also integrable. The Hamiltonian systems for the remaining cases are non integrable, more precisely the Hamiltonian systems with two or three degrees of freedom were proven to be non-integrable by Horozov [39] with a modification of Ziglin's method while the rest were proven to be non-integrable by Maciejewski et al. [60] with the differential Galois theory approach.

A motivation for this work is a series of papers of Rink [83, 84] who presented the famous Fermi-Pasta-Ulam system as a perturbation of one integrable and KAM non-degenerate system, namely the normal form of order 4 in vicinity of an equilibrium. Non-degenerate in KAM sense integrable system means that the frequency map is a local diffeomorphism.

Our aim is to check wether this fact is true for the Gross-Neveu models.

Theorem 0.0.5. *(Theorem 5.1.1) The Hamiltonian systems, corresponding to the Gross-Neveu models for algebras* $\mathrm{so}(4), \mathrm{so}(5), \mathrm{sp}(4), \mathrm{sl}(3)$ *have Birkhoff-Gustavson normal forms* $\bar{H}^{tr} = H_2 + H_4$ *integrable and non-degenerate in KAM theory sense.*

As a consequence from this fact, we obtain that in a small neighborhood of the equilibrium there are many periodic and quasi-periodic solutions in the Gross-Neveu models.

Unfortunately, such kind of result is not valid in bigger dimensions. We restrict ourselves with considering the models with three degrees of freedom for the sake of simplicity.

Theorem 0.0.6. *(Theorem 5.1.2) The Hamiltonian systems, corresponding to the Gross-Neveu models for algebras* $\mathrm{so}(6) \sim \mathrm{sl}(4), \mathrm{so}(7), \mathrm{sp}(6)$ *have non integrable Birkhoff-Gustavson normal forms* $\bar{H}^{tr} = H_2 + H_4$.

The last result can easily be extended to higher dimensions.

In Chapter 6 [19] we study the integrability of a system five nonlinear ordinary differential equations with symmetries, which arise in the context of fluid mechanics. In order to characterize soliton solutions in incompressible, heavy fluid of finite depth over a flat bottom, Witting [104] introduced a specific series (Witting series). Karabut [48, 49, 50] proved that the problem of summation of this series is reduced to solving a specific system of ordinary differential equations and solved this system in the cases when the equations are three or four.

In section 1 we describe the physical problem and formulate the next in complexity Karabut system of five equations. After some changes of the variables the system takes the form

$$\begin{aligned} \dot{y}_1 &= y_3y_5 - y_2y_4, \\ \dot{y}_2 &= y_4y_1 - y_3y_5, \\ \dot{y}_3 &= y_5y_2 - y_4y_1, \\ \dot{y}_4 &= y_1y_3 - y_5y_2, \\ \dot{y}_5 &= y_2y_4 - y_1y_3. \end{aligned} \tag{0.0.29}$$

Apart from number of discrete symmetries, the above system admits two first integrals

$$I_1 = y_1 + y_2 + y_3 + y_4 + y_5, \quad I_2 = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right). \tag{0.0.30}$$

The system (0.0.29) is obviously non-Hamiltonian and it is not clear whether one can define a suitable Poisson bracket. It is not known if there are more integrals, that is why Karabut studied it numerically.

The next result gives answer to the question of integrability of the above system

Theorem 0.0.7. *(Theorem 6.1.1) The Karabut system is not integrable in non-Hamiltonian sense.*

We recall what is understood under integrable dynamical system in non-Hamiltonian sense.

For the proof of the above Theorem we use a generalization of Ayoul and Zung [7] of the Morales-Ramis Theorem (see Chapter 1 Theorem 1.0.1) in non-Hamiltonian case, which is based on the observation that any dynamical system can be transformed into a Hamiltonian one by a procedure called cotangent lifting.

In this paragraph we recall briefly some notions for the nonlinear evolutionary PDE. We assume that $u(t,x)$ is a real-valued function of two independent variables with a definition domain that depends on the context. That consideration is enough for our purposes (for more general treatment see, for example, Olver [202]).

We are interested in equations of the kind

$$u_t = K(x, u, u_x, \ldots, u^{(n)}). \tag{0.0.31}$$

Since t is separated, we talk about evolutionary equation and also that (0.0.31) defines motion on a infinite-dimensional space of functions u. The conservation law associated to (0.0.31) is an expression of the kind

$$T_t + X_x = 0, \tag{0.0.32}$$

where T, X are functions of t, x, u and of finite number of derivatives of u. T is called density and $-X$ "flux". We talk about local conservation laws when T, X do not depend on t, x explicitly. A typical example is when T, X are polynomials of u and finite number of its derivatives. In this case after integration of (0.0.32) it is seen that the quantity

$$I = \int T dx$$

is constant, $I_t = 0$ or integral. Not all integrals come from conservation laws.

First, we want to make precise the notion Hamiltonian equation for (0.0.31). By an analogy with the finite-dimensional case – formula (0.0.4) given in the beginning, the Hamiltonian $H(x)$ is replaced with a Hamiltonian functional $\mathcal{H}[u] = \int h(u)dx$, the gradient gradH is replaced with the variational derivative $\frac{\delta \mathcal{H}[u]}{\delta u}$ and finally, the antisymmetric matrix J is replaced with an skew-adjoint differential operator $\mathcal{B}$, which may depend on u. Then, we have

$$u_t = \mathcal{B}\frac{\delta \mathcal{H}[u]}{\delta u}. \tag{0.0.33}$$

For a Hamiltonian evolutionary equation (0.0.33), we define a Poisson bracket for two functionals, depending in a bilinear way on their variational derivatives

$$\{\mathcal{P}[u], \mathcal{Q}[u]\} = \int \frac{\delta \mathcal{P}[u]}{\delta u}.\mathcal{B}\frac{\delta \mathcal{Q}[u]}{\delta u}dx. \tag{0.0.34}$$

The skew-symmetry and the Jacobi identity impose restrictions on the operator $\mathcal{B}$ and then we call that $\mathcal{B}$ is a Hamiltonian operator. Two Hamiltonian operators are compatible if their linear combination is again Hamiltonian operator. We will see examples of such operators in a while. From

$$\frac{\partial \mathcal{H}[u]}{\partial t} = \int \frac{\delta \mathcal{H}[u]}{\delta u}\frac{\partial u}{\partial t}dx = \int \frac{\delta \mathcal{H}[u]}{\delta u}.\mathcal{B}\frac{\delta \mathcal{H}[u]}{\delta u}dx = \{\mathcal{H}[u], \mathcal{H}[u]\} = 0$$

it follows that $\mathcal{H}[u]$ is an integral for (0.0.33).

A evolutionary equation is called bi-hamiltonian, if it can be written in hamiltonian form in two different ways

$$u_t = \mathcal{B}_2\frac{\delta \mathcal{H}_1[u]}{\delta u} = \mathcal{B}_1\frac{\delta \mathcal{H}_2[u]}{\delta u}, \tag{0.0.35}$$

where $\mathcal{B}_1, \mathcal{B}_2$ are two hamiltonian operators and $\mathcal{H}_1[u], \mathcal{H}_2[u]$ are the corresponding Hamiltonian functionals. Due to the compatibility of the two Poisson structures a hierarchy of conservation laws can be constructed

$$\mathcal{B}_2 \frac{\delta \mathcal{H}_{n-1}[u]}{\delta u} = \mathcal{B}_1 \frac{\delta \mathcal{H}_n[u]}{\delta u}, \qquad \forall n. \tag{0.0.36}$$

The most celebrated example is the KdV equation. We will see more.

Finally, a question arises naturally:

what is the analogue to the notion of integrability for the hamiltonian evolutionary equation?

For now, there is no universal definition for integrability of the PDE (see for instance, for a discussion Zakharov [227] and Mikhailov [198]).

The availability of enough local conservation laws (or more generally symmetries) can be accepted as an indicator for integrability (sometimes called formal integrability). Therefore, a bi-Hamiltonian equation is considered integrable.

The discoveries of IST and the Lax approach to represent the KdV equation in terms of two commuting operators, known as Lax pair, bring along a more general approach based on the so called zero-curvature representation

$$U_t - V_x + [U, V] = 0$$

or compatibility condition of two linear problems

$$L\Psi = 0, \quad M\Psi = 0, \qquad L = \frac{d}{dx} - U, \quad M = \frac{d}{dt} - V,$$

where $U = U(x, t, \lambda), V = V(x, t, \lambda)$ are square matrices depending on the spectral parameter λ. The equations presented in this way sometimes are called kinematically integrable equations. As we will see in a while, there are also geometrically integrable equations.

In Chapter 7 [145, 146] we study the Dullin-Gottwald-Holm (DGH) equation

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{0.0.37}$$

where $\alpha^2, \gamma, 2\omega$ are certain physical constants. The equation (0.0.37) is derived by using asymptotic expansion in the Hamiltonian of the Euler's equations in the shallow water regime [155]. This equation include into itself two important equations, namely

. when $\alpha = 0$ and $\gamma \neq 0$, it becomes the KdV equation

$$u_t + 2\omega u_x + 3uu_x = -\gamma u_{xxx};$$

. when $\gamma = 0$ and $\alpha = 1$, it becomes the Camassa-Holm equation

$$u_t + 2\omega u_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}.$$

The equation (DGH) is a bi-Hamiltonian one. Denoting $m = u - \alpha^2 u_{xx}$ we have

$$m_t = -\mathcal{B}_2 \frac{\delta H_1[m]}{\delta m} = -\mathcal{B}_1 \frac{\delta H_2[m]}{\delta m}, \tag{0.0.38}$$

where the Hamiltonians are

$$H_1[m] = \frac{1}{2} \int (u^2 + \alpha^2 u_x^2)\, dx = \frac{1}{2} \int mu\, dx, \tag{0.0.39}$$

$$H_2[m] = \frac{1}{2}\int (u^3 + \alpha^2 u u_x^2 + 2\omega u^2 - \gamma u_x^2)\, dx. \quad (0.0.40)$$

We consider solutions in the Schwartz class, denoted here by $S(\mathbb{R})$, so the integration is from $-\infty$ to ∞. The two compatible Hamiltonian operators $\mathcal{B}_1, \mathcal{B}_2$ are (∂ stands for $\partial/\partial x$)

$$\mathcal{B}_2 = \partial m + m\partial + 2\omega\partial + \gamma\partial^3, \quad \mathcal{B}_1 = \partial - \alpha^2\partial^3. \quad (0.0.41)$$

Because the (DGH) equation is bi-Hamiltonian, it has an infinite number of conservation laws H_n, such that

$$\mathcal{B}_2 \frac{\delta H_{n-1}[m]}{\delta m} = \mathcal{B}_1 \frac{\delta H_n[m]}{\delta m} \quad (0.0.42)$$

(in addition to H_1, H_2 we list below only few of them which we need – H_{-1} и H_0).

Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are compatible, any linear combination of these operators defines a Poisson bracket,

$$\{f, h\} = -\int \frac{\delta f}{\delta m}(c_1\mathcal{B}_1 + c_2\mathcal{B}_2)\frac{\delta h}{\delta m} dx$$

with arbitrary c_1 and c_2. In order to simplify calculations, we take $c_1 = \frac{\gamma}{\alpha^2}$, $c_2 = 1$ and denote

$$\Omega = \omega + \frac{\gamma}{2\alpha^2}.$$

In this way we reduce the above bracket to the form

$$\{f, h\}_s = -\int (m + \Omega)\left(\frac{\delta f}{\delta m}\partial\frac{\delta h}{\delta m} - \frac{\delta h}{\delta m}\partial\frac{\delta f}{\delta m}\right) dx. \quad (0.0.43)$$

Then (DGH) can be written as

$$m_t = \{m, \tilde{H}\}_s, \quad (0.0.44)$$

where

$$\tilde{H} := H_1 - \frac{\gamma}{\alpha^2}\left(H_0 - 2\sqrt{\Omega}\, H_{-1}\right) \quad (0.0.45)$$

and

$$H_0[m] = \int m\, dx, \quad H_{-1}[m] = \int (\sqrt{m + \Omega} - \sqrt{\Omega})\, dx. \quad (0.0.46)$$

Note that H_{-1} is a Casimir function for this bracket.

After this preparation, we consider the Inverse Scattering problem, describe how to obtain the conservation laws in terms of scattering data, calculate the Poisson brackets for the scattering data and construct the action-angle variables for the DGH equation. The calculations are analogous to those of the other integrable equations, we follow closely [127].

Finally, we apply a Inverse Scattering method, developed by Constantin, Gerdjikov and Ivanov [125] for the Camassa-Holm equation to the DGH equation. The solutions corresponding to the reflectionless potentials are constructed in terms of scattering data.

In Chapter 8 [149, 147] we study a family of non-evolutionary equations in fluid dynamics, known as Holm-Staley b-family [175]

$$m_t + u m_x + b u_x m = 0, \quad (0.0.47)$$

where $m = u - u_{xx}$, $u(x,t)$ is the fluid velocity, while the constant b is bifurcation parameter for the solution behavior. This equation generalizes two important models in the shallow water dynamics. If $b = 2$ then (0.0.47) becomes the Camassa-Holm equation (CH)

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{0.0.48}$$

If $b = 3$ then the equation (0.0.47 is the Degasperis-Procesi equation (DP)

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{0.0.49}$$

Both the models (CH) and (DP) are integrable and they are the only cases of integrability for the equation (0.0.47).

In this chapter we study the periodic Cauchy problem for the b-family (0.0.47)

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad u(0) = u_0, \quad t \geq 0,\, x \in \mathbb{S}. \tag{0.0.50}$$

The Cauchy problems for (CH) and (DP) in both periodic and non periodic case was studied extensively by many authors. It has been shown that the Cauchy problem for (CH) is locally well-posed in H^s, $s > \frac{3}{2}$ with solutions depending continuously on initial data. The local well-posedness for the equation (0.0.50) is considered, for example in [147].

Sometimes, it is more appropriate to consider other version of well-posedness problem, for example if one requires that the map data - solution is uniformly continuous.

In [173] Himonas and Misiolek showed that for $s \geq 2$ the solution map $u_0 \to u$ for the (CH) equation is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S})$ into $C([0,T], \mathrm{H}^s(\mathbb{S}))$. A key step in the proof of that result is a construction of a sequence of smooth traveling waves. Later Himonas, Kenig and Misiolek extend the above result to the range $3/2 < s < 2$. Their proof significantly explore the conservation of H^1 norm. Note that H^1 is a conservation law for the equation (0.0.47) only for $b = 2$, that is, only for the (CH) equation.

The first result of this chapter is

Theorem 0.0.8. *(Theorem 8.1.1) For any $s \geq 3$, the solution map $u_0 \to u$ for the equation (0.0.50) with $b \neq 0$, is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S})$ into $\mathrm{C}([0, t_0], \mathrm{H}^s(\mathbb{S}))$. More precisely, for each $s \geq 3$ there exist constants $c_{1,2} > 0$ and two sequences of smooth solutions u_n, v_n of the equation (0.0.50) such that for any $t \in [0,1]$*

$$\begin{aligned}
&\sup_n ||\, u_n(t)||_{\mathrm{H}^s} + \sup_n ||v_n(t)||_{\mathrm{H}^s} \leq c_1, \\
&\lim_{n\to\infty} ||\, u_n(0) - v_n(0)||_{\mathrm{H}^s} = 0, \\
&\liminf_n ||\, u_n(t) - v_n(t)||_{\mathrm{H}^s} \geq c_2 \sin\left(\frac{t}{2}\right).
\end{aligned}$$

This approach is not applicable for the case $b = 0$ due to a lack of periodic solutions. Further, we obtain similar result for the Degasperis-Procesi (DP) equation $b = 3$, but in the range $s \geq 2$. Recall that (DP) is integrable.

Theorem 0.0.9. *(Theorem 8.6.1) For any $s \geq 2$, the solution map $u_0 \to u$ for the equation (DP) is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S}))$ into $\mathrm{C}([0,T], \mathrm{H}^s(\mathbb{S})))$. More precisely, for* *each $s \geq 2$ there exist constants $c_{1,2} > 0$ and two sequences of smooth solutions u_n, v_n of the equation (DP) such that for any $t \in [0,1]$*

$$\begin{aligned} &\sup_n || u_n(t)||_{\mathrm{H}^s} + \sup_n ||v_n(t)||_{\mathrm{H}^s} \leq c_1, \\ &\lim_{n\to\infty} || u_n(0) - v_n(0)||_{\mathrm{H}^s} = 0, \\ &\liminf_n || u_n(t) - v_n(t)||_{\mathrm{H}^s} \geq c_2 \sin\left(\frac{t}{2}\right). \end{aligned}$$

B Chapter 9 [150] we study the μCH equation, which was derived recently in [185] as

$$\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, \tag{0.0.51}$$

where $u(x,t)$ is a spatially periodic real-valued function of time variable t and space variable $x \in S^1 = [0,1)$, $\mu(u) = \int_0^1 udx$ denotes its mean. The μCH equation describes the propagation of weakly nonlinear orientation waves in a massive liquid crystal with external magnetic field and self-interaction. In this form the μCH equation appears as the geodesic equation on the diffeomorphism group of the circle corresponding to a natural right invariant Sobolev metric.

By introducing $m = \mathcal{A}u = \mu(u) - u_{xx}$, where $\mathcal{A} := \mu - \partial^2$ is the inertia operator (∂ stands for $\frac{\partial}{\partial x}$), the equation (0.0.51) becomes

$$m_t = -um_x - 2mu_x, \quad m = \mu(u) - u_{xx}. \tag{0.0.52}$$

The μCH equation is closely related to the Camassa-Holm (CH) equation and to the Hunter-Saxton (HS) equation with $\mathcal{A} = -\partial^2$

$$-u_{txx} = 2u_x u_{xx} + uu_{xxx}. \tag{0.0.53}$$

The equation μCH is integrable and bi-Hamiltonian. Let us define the Hamiltonians

$$H_1 = \frac{1}{2}\int umdx, \quad H_2 = \int \left(\mu(u)u^2 + \frac{1}{2}uu_x^2\right) dx. \tag{0.0.54}$$

Then the equation (0.0.52) can be presented as

$$m_t = -\mathcal{B}^1 \frac{\delta H_2}{\delta m} = -\mathcal{B}^2 \frac{\delta H_1}{\delta m}, \tag{0.0.55}$$

where $\mathcal{B}^1 = \partial\mathcal{A} = -\partial^3, \mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators.

In fact, there exists an infinite sequence of conservation laws $H_n[m], n = 0, \pm1, \pm2, \ldots$, such that

$$\mathcal{B}^1 \frac{\delta H_n}{\delta m} = \mathcal{B}^2 \frac{\delta H_{n-1}}{\delta m},$$

the first few of them in the hierarchy are H_2, H_1 given above and

$$H_0 = \int mdx, \quad H_{-1} = \int \sqrt{m}dx, \quad H_{-2} = -\frac{1}{16}\int \frac{m_x^2}{m^{5/2}}dx. \tag{0.0.56}$$

Note that

$$H_0 = \int m dx = \int (\mu(u) - u_{xx}) dx = \mu(u). \tag{0.0.57}$$

Then $\mu(u_t) = 0$ on solutions of the μCH equation – this fact can be seen also if we integrate both sides of the equation (0.0.51) over the circle and use periodicity. This implies that the mean of any solution u is a constant in time, and hence, is completely determined by the initial conditions [185]. This fact is crucial for the calculations. Then the equation (0.0.51) can be written in the form

$$-u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx} \tag{0.0.58}$$

just as it is introduced in [185] under the name μHS equation.

The μCH equation can be written as

$$m_t = -\{m, H_2\}_1 = -\{m, H_1\}_2, \tag{0.0.59}$$

where the two compatible Poisson brackets are

$$\{f, g\}_1 = \int_0^1 \frac{\delta f}{\delta m} \mathcal{B}^1 \frac{\delta g}{\delta m} dx, \quad \{f, g\}_2 = \int_0^1 \frac{\delta f}{\delta m} \mathcal{B}^2 \frac{\delta g}{\delta m} dx. \tag{0.0.60}$$

Note that $H_0 = \int m dx = \int \mu(u) dx = \mu(u)$ is a Casimir for the first bracket and $H_{-1} = \int \sqrt{m} dx$ is a Casimir for the second bracket.

The equation μCH describes pseudo-spherical surface (PSS). We recall the definition (for more details see Chern, Tenenblat, Reyes, Sasaki [120, 206, 209]).

Definition 1. A scalar differential equation $\Xi(x, t, u, u_x, \ldots, u_{x^n t^m}) = 0$ in two independent variables x, t, where $u_{x^n t^m} = \partial^{n+m} u / (\partial x^n \partial t^n)$, is of pseudo-spherical type (or, it describes pseudo-spherical surfaces) if there exist one-forms $\omega^\alpha \neq 0$

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \ldots, u_{x^r t^p})\, dx + f_{\alpha 2}(x, t, u, \ldots, u_{x^s t^q})\, dt,\ \alpha = 1, 2, 3, \tag{0.0.61}$$

whose coefficients $f_{\alpha\beta}$ are smooth functions which depend on x, t and finite number of derivatives of u, such that the 1-forms $\bar{\omega}^\alpha = \omega^\alpha(u(x, t))$ satisfy the structure equations

$$d\bar{\omega}^1 = \bar{\omega}^3 \wedge \bar{\omega}^2,\ d\bar{\omega}^2 = \bar{\omega}^1 \wedge \bar{\omega}^3,\ d\bar{\omega}^3 = \bar{\omega}^1 \wedge \bar{\omega}^2, \tag{0.0.62}$$

whenever $u = u(x, t)$ is a solution of $\Xi = 0$.

Definition 2. An equation $\Xi = 0$ is geometrically integrable if it describes a non-trivial one-parameter family of pseudo-spherical surfaces.

This geometric property turns out to be extremely useful. With its one can recover the Lax pair, (part of) the conservation laws and what is important here: there exists a quadratic pseudo-potential. For the μCH equation the pseudo-potential γ is defined via

$$\gamma_x = \lambda m - \gamma^2, \quad \gamma_t = \left(\frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma \right)_x, \tag{0.0.63}$$

where $\lambda \neq 0$.

Now we examine the nonlocal symmetries of the μCH equation considered as a system of equations for the variables m and u, namely (0.0.52). We search the nonlocal symmetries that preserve the mean of solutions, that is, the integral $\mu(u) = \int u dx$ remains the same constant after the action of any symmetry on a solution. Nonlocal symmetries have been studied rigorously by Krasil'schik and Vinogradov [189]. It is almost impossible to be given in short the corresponding notions and facts. We give the procedure applied to the μCH equation. We define a new nonlocal variable δ via the compatible system of equations

$$\delta_x = \gamma, \quad \delta_t = \frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma. \tag{0.0.64}$$

After that we introduce the potential β determined by the compatible system of equations

$$\beta_x = me^{2\delta}, \quad \beta_t = \left(\frac{\gamma^2}{2\lambda^2} - um\right) e^{2\delta}. \tag{0.0.65}$$

Our first result is the following

Theorem 0.0.10. *(Theorem 9.3.1) The following vector fields are the first-order generalized symmetries for the augmented μCH system (0.0.52), (0.0.63), (0.0.64) u (0.0.65), which preserve the mean of the solutions to the μCH equation (0.0.52)*

$$\begin{aligned} W_1 &= -u_t\frac{\partial}{\partial u} + (m_x u + 2mu_x)\frac{\partial}{\partial m} - \left[\frac{\mu(u)}{2} + \gamma^2\left(u - \frac{1}{2\lambda}\right) - \gamma u_x - \lambda um\right]\frac{\partial}{\partial \gamma} \\ &\quad - \left(\frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma\right)\frac{\partial}{\partial \delta} - \left(\frac{\gamma^2}{2\lambda^2} - um\right)e^{2\delta}\frac{\partial}{\partial \beta}, && (0.0.66) \\ W_2 &= u_x\frac{\partial}{\partial u} + m_x\frac{\partial}{\partial m} + (\lambda m - \gamma^2)\frac{\partial}{\partial \gamma} + \gamma\frac{\partial}{\partial \delta} + me^{2\delta}\frac{\partial}{\partial \beta}, && (0.0.67) \\ W_3 &= \frac{\partial}{\partial \delta} + 2\beta\frac{\partial}{\partial \beta}, && (0.0.68) \\ W_4 &= \frac{\partial}{\partial \beta}, && (0.0.69) \\ W_5 &= \gamma e^{2\delta}\frac{\partial}{\partial u} - \lambda(m_x + 4m\gamma)e^{2\delta}\frac{\partial}{\partial m} - \lambda^2 me^{2\delta}\frac{\partial}{\partial \gamma} - \lambda^2\beta\frac{\partial}{\partial \delta} - (\lambda me^{4\delta} + \lambda^2\beta^2)\frac{\partial}{\partial \beta}. && (0.0.70) \end{aligned}$$

Consequently, these vector fields are nonlocal symmetries of the μCH equation (0.0.52).

Corollary. The five nonlocal symmetries $W_1 - W_5$ generate a Lie algebra $\mathcal{L}$ and their commutators are presented in the Table 1.

Table 1: The commutation table of μCH nonlocal symmetry algebra.

	W_1	W_2	W_3	W_4	W_5
W_1	0	0	0	0	0
W_2	0	0	0	0	0
W_3	0	0	0	$-2W_4$	$2W_5$
W_4	0	0	$2W_4$	0	$-\lambda^2 W_3$
W_5	0	0	$-2W_5$	$\lambda^2 W_3$	0

Remark 1. If we introduce the vector fields $h := -W_3$, $e := \frac{1}{\lambda}W_4$, $f := -\frac{1}{\lambda}W_5$, we find that the commutators $[h, e] = 2e, [h, f] = -2f, [e, f] = h$, i.e. e, f, h generate a copy of $sl(2, \mathbb{R})$.

Therefore, $\mathcal{L}$ is isomorphic to the direct sum of $sl(2,\mathbb{R})$ and the Abelian Lie algebra, generated by W_1 and W_2.

Remark 2. Note that W_1 and W_2 are merely the generators of the shifts with respect to the independent variables – they are $\frac{\partial}{\partial t}$ and $-\frac{\partial}{\partial x}$, respectively.

Usually the symmetries are used for solving (or finding particular solutions) of the system. This is not so simple here. That is why we use them for constructing a Darboux-like transform for the μCH equation.

It may be interesting to see another object connected with the μCH equation. Recall that the so called associated Camassa-Holm (ACH) equation is introduced by Schiff [210]. Let us give by analogy the associated μCH equation. Define

$$p = \sqrt{m}, \quad dy = pdx - pudt, \quad dT = dt \tag{0.0.71}$$

and replace in equation (0.0.52). Note that this change of variables is justified since if $m(0)$ is positive, then $m(x) > 0$ as long as $u(x,t)$ exists (see [185] for the proof). One finds

$$p_T = -p^2 u_y, \quad -p\left(\frac{p_T}{p}\right)_y + \frac{p^2}{2} = \mu(u). \tag{0.0.72}$$

This is the analogue of the ACH equation – the associated μ Camassa-Holm (AμCH) equation. It is not clear yet whether this equation is of use. Nevertheless, our aim is to study nonlocal symmetries of the AμCH equation. First of all, we transform the equations for γ, δ and β (0.0.63), (0.0.64) and (0.0.65) using (0.0.71).

Proposition 0.0.1. *(Proposition 9.4.1) The AμCH equation (0.0.72) admits a pseudo-potential γ and potentials δ, β determined by the compatible equations, respectively*

$$\gamma_y = \frac{\lambda p}{2} - \frac{\gamma^2}{p}, \quad \gamma_T = \frac{\mu(u)}{2} - \frac{\gamma^2}{2\lambda} - p\gamma u_y, \tag{0.0.73}$$

$$\delta_y = \frac{\gamma}{p}, \quad \delta_T = \frac{pu_y}{2} + \frac{\gamma}{2\lambda}, \tag{0.0.74}$$

$$\beta_y = \frac{p}{2}e^{2\delta}, \quad \beta_T = \frac{\gamma^2}{2\lambda^2}e^{2\delta}. \tag{0.0.75}$$

Our next result is

Theorem 0.0.11. *(Theorem 9.4.1) The following vector fields are first order generalized symmetries for the augmented AμCH system (0.0.72)-(0.0.75)*

$$W_1 = u_T\frac{\partial}{\partial u} - p^2 u_y\frac{\partial}{\partial p} + \left(\frac{\mu(u)}{2} - \frac{\gamma^2}{2\lambda} - p\gamma u_y\right)\frac{\partial}{\partial \gamma} + \left(\frac{pu_y}{2} + \frac{\gamma}{2\lambda}\right)\frac{\partial}{\partial \delta} + \frac{\gamma^2}{2\lambda^2}e^{2\delta}\frac{\partial}{\partial \beta}, \tag{0.0.76}$$

$$W_2 = u_y\frac{\partial}{\partial u} + p_y\frac{\partial}{\partial p} + \left(\frac{\lambda p}{2} - \frac{\gamma^2}{p}\right)\frac{\partial}{\partial \gamma} + \frac{\gamma}{p}\frac{\partial}{\partial \delta} + \frac{p}{2}e^{2\delta}\frac{\partial}{\partial \beta}, \tag{0.0.77}$$

$$W_3 = \frac{1}{2}\frac{\partial}{\partial \delta} + \beta\frac{\partial}{\partial \beta}, \tag{0.0.78}$$

$$W_4 = \frac{\partial}{\partial \beta}, \tag{0.0.79}$$

$$W_5 = -(\gamma + \lambda p u_y)e^{2\delta}\frac{\partial}{\partial u} + 2\lambda p\gamma e^{2\delta}\frac{\partial}{\partial p} + \lambda\gamma^2 e^{2\delta}\frac{\partial}{\partial \gamma} + (\lambda^2\beta - \lambda\gamma e^{2\delta})\frac{\partial}{\partial \delta} + \lambda^2\beta^2\frac{\partial}{\partial \beta}. \tag{0.0.80}$$

Therefore, these vector fields are nonlocal symmetries for the AμCH equation (0.0.72).

Note that the symmetry algebra $\mathcal{L}$ is again isomorphic to a direct sum of $sl(2,\mathbb{R})$ and abelian algebra generated by W_1 and W_2, which are equivalent to $-\frac{\partial}{\partial T}, -\frac{\partial}{\partial y}$, respectively. One can find a Darboux transform in an analogous way. However, here we are able to find one-parameter solution of AμCH (0.0.72).

In Chapter 10 [148, 151] we study two problems, related with the introduced in the previous Chapter equations and notions. The motivation for the consideration of the first problem is as follows. Recently a new equation of sixth order named KdV6 was derived in [183]. After some rescaling this equation can be presented as the following system

$$\begin{aligned} u_t &= 6uu_x + u_{xxx} - w_x, \\ w_{xxx} &+ 4uw_x + 2u_x w = 0. \end{aligned} \tag{0.0.81}$$

This system gives a perturbation to the KdV equation ($w = 0$) and since the constraint on w is differential, this is a nonholonomic deformation.

Kuperschmidt [191] suggested a general construction applicable to any bi - Hamiltonian system providing a nonholonomic perturbation on it. This perturbation is conjectured to preserve integrability. In the case of KdV6, the system (0.0.81) can be converted into

$$\begin{aligned} u_t = \mathcal{B}^1 \tfrac{\delta H_{n+1}}{\delta u} - \mathcal{B}^1(w) = \mathcal{B}^2 \tfrac{\delta H_n}{\delta u} - \mathcal{B}^1(w), \\ \mathcal{B}^2(w) = 0, \end{aligned} \tag{0.0.82}$$

where $\mathcal{B}^1 = \partial, \mathcal{B}^2 = \partial^3 + 2(m\partial + \partial m)$ are the two standard Hamiltonian operators of the KdV hierarhy and

$$H_1 = \int u dx, \quad H_2 = \frac{1}{2}\int u^2 dx, \ldots. \tag{0.0.83}$$

In the same article Kuperscmidt verifies the integrability of KdV6, as well as the integrability of such nonholonomic deformations for some representative cases: the classical long - wave equation, the Toda lattice (both continuous and discrete), and Euler top.

In fact, Kersten et al. [186] prove that the Kuperscmidt deformation of every bi - Hamiltonian equation is again bi - Hamiltonian system and every hierarchy of conservation laws of the original bi-Hamiltonian system gives rise to a hierarchy of conservation laws of the Kuperschmidt deformation.

It is natural to think that maybe there exists a general link in a sense:

Conjecture. The Kuperschmidt deformation of geometrically integrable system is again geometrically integrable.

We haven't succeeded in establishing such a link, that is why we restrict ourselves with some relevant examples. We show by a direct construction of the 1-forms that the equation CH and the equation μCH have geometrically integrable Kuperschmidt deformations. We show that KdV6 equation and two - component CH system [144] are also geometrically integrable.

Then, we consider again the μCH equation introduced earlier (0.0.58). We construct canonically conjugated variables with respect to the both Poisson brackets (0.0.60) following Penskoi [204], where the conjugated variables for the periodic CH equation are obtained. Recall that the equation (0.0.58) can be expressed as a condition of compatibility between

$$\psi_{xx} = -\lambda m \psi \tag{0.0.84}$$

and

$$\psi_t = -(\frac{1}{2\lambda} + u)\psi_x + \frac{1}{2}u_x\psi, \tag{0.0.85}$$

that is, $(\psi_{xx})_t = (\psi_t)_{xx}$, where λ is a spectral parameter.

Consider the spectral problem (0.0.84). Recall that $u(x+1) = u(x)$ and $m(x+1) = m(x)$.

Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be a fundamental system of solutions of (0.0.84), subjected to the normalization

$$\begin{aligned} y_1(0,\lambda) &= 1, \qquad y_1'(0,\lambda) = 0, \\ y_2(0,\lambda) &= 0, \qquad y_2'(0,\lambda) = 1. \end{aligned}$$

Every solution ψ of (0.0.84) can be expressed as a linear combination of $y_{1,2}$:

$$\psi(x,\lambda) = \psi(0,\lambda)y_1(x,\lambda) + \psi'(0,\lambda)y_2(x,\lambda). \tag{0.0.86}$$

Then we have the formula

$$\begin{pmatrix} \psi(x,\lambda) \\ \psi'(x,\lambda) \end{pmatrix} = \begin{pmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ y_1'(x,\lambda) & y_2'(x,\lambda) \end{pmatrix} \begin{pmatrix} \psi(0,\lambda) \\ \psi'(0,\lambda) \end{pmatrix} \tag{0.0.87}$$

and denote the matrix in the last formula with $U(x,\lambda)$. From the definition of $y_{1,2}$ we have that $\det U(x,\lambda) = Wr(y_1,y_2) = Wr(0) = 1$. Let us define also the discriminant

$$\Delta(\lambda) = \frac{1}{2}\mathrm{tr}\, U(1,\lambda) = \frac{1}{2}\left(y_1(1,\lambda) + y_2'(1,\lambda)\right). \tag{0.0.88}$$

We first consider (0.0.84) conditioned by the periodic boundary conditions

$$\psi(0) = \psi(1), \quad \psi'(0) = \psi'(1).$$

There exists an infinite sequence of eigenvalues

$$\lambda_0^+ < \lambda_1^+ \le \lambda_2^+ < \lambda_3^+ \ldots, \quad \lambda_n^+ \to \infty \text{ as } n \to \infty.$$

Next we consider the antiperiodic eigenvalue problem, that is, the boundary conditions for (0.0.84) are of the form

$$\psi(1) = -\psi(0), \quad \psi'(1) = -\psi'(0).$$

The corresponding sequence of eigenvalues is

$$\lambda_1^- \le \lambda_2^- < \lambda_3^- \le \lambda_4^- \ldots, \quad \lambda_n^- \to \infty \text{ as } n \to \infty.$$

The quantities $\lambda_m^\pm$ are the roots of$\Delta(\lambda) = \pm 1$, λ_0^+ is always simple. It is known that

$$\lambda_0^+ < \lambda_1^- \le \lambda_2^- < \lambda_1^+ \le \lambda_2^+ < \lambda_3^- \le \lambda_4^- < \lambda_3^+ \ldots.$$

The intervals

$$(\lambda_0^+, \lambda_1^-), (\lambda_2^-, \lambda_1^+), (\lambda_2^+, \lambda_3^-), \ldots$$

are called the intervals of stability. Similarly we can name the other intervals as intervals of instability or gaps. Some of the intervals of instability may disappear, but this one $(-\infty, \lambda_0)$ is always present.

Trivial arguments show that in our case $\lambda_0^+ = 0$ and for $\lambda \in (-\infty, 0)$ the solutions of (0.0.84) are unbounded.

Recall that a solution of (0.0.84) is said to be a Floquet solution if there exists a number ρ called a Floquet multiplier satisfying

$$\psi(x+1,\lambda) = \rho\psi(x,\lambda).$$

It is straightforward from (0.0.87) that a Floquet solution and the corresponding ρ are an eigenvector and an eigenvalue of $U(x,\lambda)$. Hence, ρ is obtained from

$$\rho^2 - 2\Delta(\lambda)\rho + 1 = 0. \qquad (0.0.89)$$

Now let us consider the auxiliary eigenvalues μ_j defined as solutions of the equation $y_2(1,\mu_j) = 0$. Since $m(x)$ is periodic, $y_2(x+1,\mu_j)$ is a solution of (0.0.84) for $\lambda = \mu_j$. Due to (0.0.86) we have

$$y_2(x+1,\mu_j) = y_2'(1,\mu_j)y_2(x,\mu_j),$$

that is, $y_2(x,\mu_j)$ is a Floquet solution with $\rho_j = y_2'(1,\mu_j)$. So, we have a root of (0.0.89) for $\lambda = \mu_j$ namely ρ_j. The other root is $\tilde{\rho}_j = \frac{1}{\rho_j}$. Denote by $y(x,\mu_j)$ the corresponding to $\tilde{\rho}_j$ Floquet solution

$$y(x+1,\mu_j) = \tilde{\rho}_j y(x,\mu_j). \qquad (0.0.90)$$

Since y and y_2 are linearly independent, we normalize y by $y(0,\mu_j) = 1$. It is not difficult to be shown that the points of "auxiliary spectrum" μ_j lie in the intervals of instability. Since $\mu_j \neq 0$ we can define the following variables $f_j = -\frac{\ln|\rho_j|}{\mu_j^2}$ and $g_j = -\frac{\ln|\rho_j|}{\mu_j^3}$. Our result is the following

Theorem 0.0.12. *(Theorem 10.3.1) The variables*

a) μ_i *and* $f_j = -\frac{\ln|\rho_j|}{\mu_j^2}$ *are conjugate with respect to the bracket* $\{,\}_2$*;*

b) μ_i *and* $g_j = -\frac{\ln|\rho_j|}{\mu_j^3}$ *are conjugate with respect to the bracket* $\{,\}_1$*.*

Part I

Non-integrability of Some Finite-Dimensional Hamiltonian Systems

Chapter 1

Theoretical backgrounds

In this chapter we recall some notions and facts about integrability of Hamiltonian systems in the real and complex domain, the Ziglin–Morales-Ruiz–Ramis theory and their relations with differential Galois groups of linear equations.

First, we consider the real case and more specifically its geometry.

Let (M^{2n}, ω) be a $2n$ dimensional real symplectic manifold and let H be a Hamiltonian function on M^{2n} defining the corresponding Hamiltonian system

$$\dot{x} = X_H(x). \tag{1.0.1}$$

An Hamiltonian system is integrable if there exist n independent integrals $F_1 = H, F_2, \ldots, F_n$ in involution, namely $\{F_i, F_j\} = 0$ for all i and j, where $\{,\}$ is the Poison bracket [5]. On a neighborhood U of the connected compact level sets of the integrals $M_c = \{F_j = c_j, j = 1, \ldots, n\}$ by Liouville - Arnold theorem one can introduce a special set of symplectic coordinates, I_j, φ_j, called action - angle variables. Then, the integrals $F_1 = H, F_2, \ldots, F_n$ are functions of action variables only and the flow of X_H is described by the canonical equations

$$\dot{I}_j = 0, \quad \dot{\varphi}_j = \frac{\partial H}{\partial I_j}, \quad j = 1, \ldots, n. \tag{1.0.2}$$

Therefore, near M_c, the phase space is foliated with X_{F_i} invariant tori over which the flow of X_H is quasi - periodic with frequencies $(\omega_1(I), \ldots, \omega_n(I)) = (\frac{\partial H}{\partial I_1}, \ldots, \frac{\partial H}{\partial I_n})$.

The map

$$(I_1, I_2, \ldots, I_n) \to (\frac{\partial H}{\partial I_1}, \frac{\partial H}{\partial I_2}, \ldots, \frac{\partial H}{\partial I_n}) \tag{1.0.3}$$

is called frequency map.

Now, consider a small perturbation of an integrable Hamiltonian H_0

$$H = H_0(I) + \varepsilon H_1(I, \varphi), \quad \varepsilon << 1.$$

A natural question is whether this small perturbation destroy the quasi - periodic motions of the unperturbed system. KAM - theory [53, 4, 78] gives conditions for the integrable system H_0 which ensures the survival of the most of the invariant tori. One condition, usually called Kolmogorov's condition, is that the frequency map should be a local diffeomorphism, that is

$$\det\left(\frac{\partial^2 H_0}{\partial I_i \partial I_j}\right) \neq 0 \tag{1.0.4}$$

on an open and dense subset of U. We should note that the measure of the surviving tori decreases with the increase of both perturbation and the measure of the set where above Hessian is too close to zero.

Another condition of this type is the so called Arnold - Moser condition of isoenergetical non - degeneracy. Let us fix an energy level $H_0 = h_0$. Define the following map

$$F_{h_0} : I \to (\omega_1(I) : \omega_2(I) : \ldots : \omega_n(I)) \tag{1.0.5}$$

form the $(n-1)$ dimensional variety $H_0^{-1}(h_0)$ into projective space $\mathbb{P}^{n-1}$. Then the system is isoenergetically non - degenerate if the map F_{h_0} is a homeomorphism. Analytically this is equivalent to non - vanishing of the following determinant

$$D_1 = \begin{pmatrix} \partial^2 H_0/\partial I^2 & \partial H_0/\partial I \\ \partial H_0/\partial I & 0 \end{pmatrix}. \tag{1.0.6}$$

Of course, again the measure of the surviving tori depends on the measure of the set where the determinant D_1 is too close to zero.

Let us turn now to the complex case. Consider a Hamiltonian system

$$\dot{x} = X_H(x), \quad t \in \mathbb{C}, \quad x \in M \tag{1.0.7}$$

corresponding to an analytic Hamiltonian H, defined on the complex $2n$-dimensional manifold M. Again we say that a Hamiltonian system with n degrees of freedom is integrable in the sense of Liouville if there exist n (almost everywhere) independent first integrals in involution. Note that in general, we do not have that nice geometry as in the real case.

Suppose the system (1.0.7) has a non-equilibrium solution $\Psi(t)$. Denote by Γ its phase curve. We can write the equation in variation (VE) along this solution

$$\dot{\xi} = DX_H(\Psi(t))\xi, \quad \xi \in T_\Gamma M. \tag{1.0.8}$$

Further, consider the normal bundle of Γ, $F := T_\Gamma M/TM$ and let $\pi : T_\Gamma M \to F$ be the natural projection. The equation (1.0.8) induces an equation on F

$$\dot{\eta} = \pi_*(DX_H(\Psi(t))(\pi^{-1}\eta), \quad \eta \in F. \tag{1.0.9}$$

which is called the normal variational equation (NVE) along Γ. The (NVE) (1.0.9) admits a first integral dH, linear on the fibers of F. The level set $F_p := \{\eta \in F | dH(\eta) = p\}, p \in \mathbb{C}$, is $(2n-2)$-dimensional affine bundle over Γ. We shall call F_p the reduced phase space of (1.0.9) and the restriction of the (NVE) on F_p is called the reduced normal variational equation.

Then the main result of the Ziglin–Morales-Ruiz–Ramis [70] theory is:

Theorem 1.0.1. *Suppose that the Hamiltonian system (1.0.7) has n meromorphic first integrals in involution. Then the identity component G^0 of the Galois group of the variational equation is abelian.*

Next we consider a linear non-autonomous system

$$y' = A(x)y, \quad y \in \mathbb{C}^n, \tag{1.0.10}$$

or equivalently a linear homogeneous differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0, \tag{1.0.11}$$

with $x \in \mathbb{CP}^1$ (which is enough for our purposes) and $A \in \mathrm{gl}(n, \mathbb{C}(x))$, ($a_j(x) \in \mathbb{C}(x)$). Let $S := \{x_1, \ldots, x_s\}$ be the set of singular points of (1.0.10) (or (1.0.11)) and let $Y(x)$ be a fundamental solution of (1.0.10) (or (1.0.11)) at $x_0 \in \mathbb{C} \setminus S$. By the existence theorem there is a fundamental solution $Y(x)$, analytic in a vicinity of x_0. The continuation of $Y(x)$ along a nontrivial loop on $\mathbb{CP}^1$ defines a linear automorphism of the vector space of all solutions analytic in the neighborhood of x_0, called the monodromy transformation. Analytically this transformation can be presented in the following way. The linear automorphism Δ_γ, associated with a loop $\gamma \in \pi_1(\mathbb{CP}^1 \setminus S, x_0)$ corresponds to multiplication of $Y(x)$ from the right by a constant matrix M_γ, called monodromy matrix

$$\Delta_\gamma Y(x) = Y(x) M_\gamma.$$

The set of these matrices form the monodromy group. Equivalently, the monodromy group can be defined as a group of automorphisms of the solution space [96, 110].

We may attach another object to the system (1.0.10) (or (1.0.11)) - a differential Galois group. A differential field K is a field with a derivation $\partial ='$, i.e. an additive mapping satisfying Leibnitz rule. A differential automorphism of K is an automorphism commuting with the derivation.

The coefficient field in (1.0.10) (and (1.0.11)) is $K = \mathbb{C}(x)$. Let y_{ij} be the elements of the fundamental matrix $Y(x)$. Let $L(y_{ij})$ be the extension of K generated by K and y_{ij} – a differential field. This extension is called a Picard-Vessiot extension. Similarly to classical Galois Theory we define the Galois group $G := Gal_K(L) = Gal(L/K)$ to be the group of all differential automorphisms of L leaving the elements of K fixed. The Galois group is, in fact, an algebraic group. It has a unique connected component G^0 which contains the identity and which is a normal subgroup of finite index. The Galois group G can be represented as an algebraic linear subgroup of $\mathrm{GL}(n, \mathbb{C})$ by

$$\sigma(Y(x)) = Y(x) R_\sigma,$$

where $\sigma \in G$ and $R_\sigma \in \mathrm{GL}(n, \mathbb{C})$.

We can do the same locally at $a \in \mathbb{CP}^1$, replacing $\mathbb{C}(x)$ by the field of germs of meromorphic functions at a. In this way we can speak of a local differential Galois group G_a of (1.0.10) at $a \in \mathbb{CP}^1$, defined in the same way for Picard-Vessiot extensions of the field $\mathbb{C}\{x-a\}[(x-a)^{-1}]$.

Let K be a differential field with an algebraically closed subfield of constants. An extension L/K is called a Liouville extension of K if $Const(K) = Const(L)$ and there exists a tower of extensions

$$K = L_0 \subset L_1 \subset \ldots \subset L_n = L,$$

such that for $i = 1, \ldots, n$ $L_i = L_{i-1}(\alpha_i)$ and one of the following holds: either

1) $\alpha_i' \in L_{i-1}$: we say that α_i is an integral of an element of L_{i-1}; or

2) $\alpha_i \neq 0$ $\alpha_i'/\alpha_i \in L_{i-1}$: we say α_i is an exponential of an integral of an element of L_{i-1}; or

3) α_i is algebraic over L_{i-1}.

It can be proven that L is a Liouville extension of K if and only if the identity component G^0 of $G = Gal(L/K)$ is a solvable subgroup.

Remark 1. Suppose the Picard-Vessiot extension is defined only by adjunction of quadratures

$$L = K(\int f_1, \ldots, \int f_k),$$

where $f_1, \ldots, f_k \in K$. Then the Galois group is isomorphic to an additive group $\mathbb{C}_+^s$ for some $s \leq k$, where $\mathbb{C}_+$ denotes the additive group over $\mathbb{C}$. Note that the only algebraic subgroups of $\mathbb{C}_+$ are itself and $\{0\}$ - the trivial group. Moreover, $Gal(L/K)$ is represented as a subgroup of unipotent matrices contained in $GL(2s,\mathbb{C})$. In particular, $Gal(L/K)$ is connected and L/K is a purely transcendental extension [70].

One should note that by its definition the monodromy group is contained in the differential Galois group of the corresponding system.

We say that two linear systems $y' = A(x)y$ and $z' = B(x)z$ are K-equivalent if the latter is obtained from the first by a K-linear change $y = Pz, P \in \mathrm{GL}(n,\mathbb{C})$ and $B = P^{-1}AP - P^{-1}P'$.

Next, we review some facts from the theory of linear systems with singularities. We call a singular point x_i *regular* if any of the solutions of (1.0.10) (or of (1.0.11)) has at most polynomial growth in arbitrary sector with a vertex at x_i. Otherwise the singular point is called *irregular*.

We say that the system (1.0.10) has a singularity of the Fuchs type at x_i if $A(x)$ has a simple pole at $x = x_i$. For the equation (1.0.11) the Fuchs type singularity at x_i means that the functions $(x-x_i)^j a_j(x)$ are holomorphic in a neighborhood of x_i.

If the system (1.0.10) has a singularity of the Fuchs type, then this singularity is regular. The converse is not true. However, for the equation (1.0.11) the regular singularities coincide with the singularities of the Fuchs type.

Theorem 1.0.2. *(Schlesinger [89]) For a system with only regular singular points, the differential Galois group coincides with the Zariski closure in* $\mathrm{GL}(n,\mathbb{C})$ *of the monodromy group.*

Assume that $x_1 = 0$ is a regular singular point for (1.0.10). We can rewrite the system (1.0.10) locally near $x_1 = 0$ as

$$x\dot{Y} = A(x)Y, \tag{1.0.12}$$

where $A(x)$ is holomorphic at $x = 0$.

The change $Y = P(x)Z$, where $P(x)$ is holomorphic at $x = 0$ brings the system (1.0.12) to the form

$$z\dot{Z} = B(x)Z, \quad \text{where} \quad B(x) = P^{-1}(x)A(x)P(x) - tP^{-1}(x)\dot{P}(x).$$

In order to find the fundamental solution and the monodromy around the singularity we can determine $P(x)$ in such a way that $B(x)$ is as simple as possible. Let

$$A(x) = \sum_{s=0}^{\infty} A_s x^s, \quad P(x) = \sum_{s=0}^{\infty} P_s x^s, \quad B(x) = \sum_{s=0}^{\infty} B_s x^s.$$

We can take $P_0 = E$ and $B_0 = A_0$. If the eigenvalues of A_0 do not differ by a positive integer a theorem from [42, 102] asserts that we can obtain all $B_s = 0, s \geq 1$. So, the system (1.0.12) takes the form

$$x\dot{Z} = A(0)Z,$$

which is solved as $Z = \exp(A_0 \log x) = x^{A_0}$. Hence, $Y = P(x)x^{A_0}$. For more details, one can consult [42, 102].

Now we briefly recall the Ramis description of the local Galois group of (1.0.10) at 0 which we assume to be an irregular singularity. To the end of the section we follow mainly Mitschi [68] pp. 368–370 and [69] pp. 153–159.

Let $K = \mathbb{C}\{x\}[x^{-1}](\widehat{K} = \mathbb{C}[[x]][x^{-1}])$ be the field of convergent Laurent series near 0 (field of formal Laurent series), $K_t = \mathbb{C}\{t\}[t^{-1}](\widehat{K} = \mathbb{C}[[t]][t^{-1}])$ are the same objects with respect to the variable t and $A \in \mathrm{gl}(n, K)$. It is known from the classical theory that there exists a formal fundamental solution to (1.0.11) :

$$\widehat{Y}(t) = \widehat{H}(t)x^L e^{Q(t)}, \tag{1.0.13}$$

where $t^\sigma = x\,(\sigma \in \mathbb{N}^*)$, $L = Mat(n, \mathbb{C})$, $\widehat{H} \in \mathrm{GL}(n, \widehat{K}_t)$ and $Q = \mathrm{diag}(q_1, \ldots, q_n)$, $q_i \in t^{-1}\mathbb{C}[\frac{1}{t}], i = 1, \ldots, n$. The integer σ is called ramification degree at 0. Denote also $\zeta = e^{2\pi i/\sigma}$.

First we recall the formal invariants of (1.0.11). The change of variable $x \to xe^{2\pi i}$ commutes with the derivation, so it defines an element $\widehat{m} \in G$, the formal monodromy ($t \to t\zeta$ commutes with the corresponding derivation). Relative to $\widehat{Y}$, the automorphism $\widehat{m}$ can be represented by a matrix $\widehat{M}$:

$$\widehat{Y}(t\zeta) = \widehat{Y}(t)\widehat{M}. \tag{1.0.14}$$

By definition the exponential torus $\mathcal{T}$ of (1.0.10) relative to $\widehat{Y}$ is the group of the differential $\widehat{K}_t$-automorphisms of the differential extension

$$\widehat{K}_t(e^Q) = \widehat{K}_t(e^{q_1}, e^{q_2}, \ldots, e^{q_n}) \quad \text{of} \quad \widehat{K}_t.$$

$\mathcal{T}$ is isomorphic to $(\mathbb{C}^*)^l$, where l is the rank of $\mathbb{Z}$-module generated by the q_i,s.

The matrix $\widehat{M}$, clearly invariant by $\widehat{K}$-equivalence is a formal invariant of (1.0.11). The same thing applies to the exponential torus $\mathcal{T}$.

Let $V_d(\alpha)$ be an open sector in $\mathbb{C}^* \setminus \{0\}$ with its vertex at 0:

$$V_d(\alpha) = \{x \in \mathbb{C}^* | 0 < |x| < R, d - \frac{\alpha}{2} < \arg(x) < d + \frac{\alpha}{2}\}$$

and let f be a holomorphic function on $V_d(\alpha)$. We say that f is asymptotic to $\hat{f} = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$ on $V_d(\alpha)$ (in Poincaré sense) if, for every closed subsector $W \subset V_d(\alpha)$ there exists a positive constant $M_{W,n}$, such that for every $x \in W$

$$|x|^{-n}\Big|f(x) - \sum_{m=0}^{n-1} a_m x^m\Big| \leq M_{W,n}$$

for every n. We write $f \sim \hat{f}$ on $V_d(\alpha)$.

Let us restrict ourselves to the case when all non-zero expressions $(q_i - q_j)$ have the same degree, that is, $(q_i - q_j) = (\lambda_i - \lambda_j)t^{-k}$, $i, j = 1, \ldots, n$, $k \in \mathbb{N}^*$. By the classical theory (Sibuya [91], Wasow [102], Martinet, Ramis [64]) we have the following result:

Theorem 1.0.3. *For the system (1.0.10) with a formal solution (1.0.13), there exists an actual solution $Y = Hx^L e^Q$, where $H \in \mathrm{GL}(n, \mathbb{C}\{t\})$ has asymptotic expansion $\widehat{H}$ ($H \sim \widehat{H}$ and $Y \sim \widehat{Y}$) in any open angular sector with opening $\pi/(k\sigma)$.*

We want to extend the solution Y to sectors with opening greater than $\pi/(k\sigma)$. For this purpose we define:

- a Stokes ray as a direction where, for some $i, j = 1, \ldots, n$, one has $\mathrm{Re}[q_i(t) - q_j(t)] = 0$.
- a singular ray is a direction of maximal decay for some $exp(q_i - q_j)$, i.e., a bisecting ray of a maximal sector where $\mathrm{Re}[q_i(t) - q_j(t)] < 0$.

Let d be a singular direction for (1.0.10) at $x = 0$, let d^+ and d^- be nearby directions with arguments $d^+ = d + \varepsilon, d^- = d - \varepsilon$. Then $V^\pm = V_{d^\pm}(\pi/(k\sigma))$ are two overlaping sectors containing d. Let Y^- and Y^+ be actual solutions of (1.0.10), such that $Y^- \sim \widehat{Y}$ in V^- and $Y^+ \sim \widehat{Y}$ in V^+. Hence, we have two actual solutions Y^-, Y^+ over $V_d(\pi/(k\sigma))$ (by analytic continuation to this sector, $\varepsilon \to 0$). Then there exists $S_d \in \mathrm{GL}(n, \mathbb{C})$, such that

$$Y^- = Y^+ S_d.$$

The matrix S_d is called Stokes matrix (or multiplier) with respect to d and $\widehat{Y}$. The Stokes matrices are unipotent. Moreover, they are invariant under K-equivalence, that is, they are analytic invariants for (1.0.10) (see also Balser, Jurkat, Lutz [9]).

The actual solutions Y are usually obtained by summation procedure of $\widehat{H}$ along non-singular directions in maximal sectors. We will not recall here the summation theory developed by Ramis (see, for instance [64, 96] for more details) because we don't need it.

Finally, we have a theorem that generalize the Schlesinger's result for the Fuchsian case.

Theorem 1.0.4. *(Ramis) With respect to the formal solution (1.0.13) the analytic Galois group of (1.0.11) at 0 is the Zariski closure in* $\mathrm{GL}(n, \mathbb{C})$ *of the subgroup generated by the formal monodromy* $\widehat{M}$*, the exponential torus* $\mathcal{T}$ *and the Stokes matrices* S_d *for all singular rays.*

Now, let G_a be the local Galois groups of (1.0.10), $a \in S$. All G_a can be simultaneously identified with closed subgroups of G and the following result holds (Mitschi [68] Prop. 1.3):

The global Galois group G *is topologically generated in* $\mathrm{GL}(n, \mathbb{C})$ *by the subgroups* G_a*, for all* $a \in S$*.*

Once it is proven, that G^0 is not abelian, the respective Hamiltonian system is non-integrable in the Liouville sense. Note that the fact that G^0 is abelian doesn't imply necessarily integrability of the corresponding Hamiltonian system. Thus, one needs other obstructions to the integrability. A method based on the higher variational equations has been introduced in [70] and Theorem 1.0.1 has been extended in [75]. Before formulating this result let us give an idea of higher variational equations. For the system (1.0.7) with a particular solution $\Psi(t)$ we put

$$x = \Psi(t) + \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + \ldots + \varepsilon^k\xi^{(k)} + \ldots, \tag{1.0.15}$$

where ε is a formal small parameter. Substituting the above expression into Eq. (1.0.7) and comparing terms with the same order in ε we obtain the following chain of linear non-homogeneous equations

$$\dot{\xi}^{(k)} = A(t)\xi^{(k)} + f_k(\xi^{(1)}, \ldots, \xi^{(k-1)}), \quad k = 1, 2, \ldots, \tag{1.0.16}$$

where $A(t) = DX_H(\Psi(t))$ and $f_1 \equiv 0$. The equation (1.0.16) is called k-th variational equation (VE_k). Let $X(t)$ be the fundamental matrix of (VE_1)

$$\dot{X} = A(t)X.$$

Then the solutions of $(\mathrm{VE}_k), k > 1$ can be found by

$$\xi^{(k)} = X(t)c(t), \tag{1.0.17}$$

where $c(t)$ is a solution of

$$\dot{c} = X^{-1}(t) f_k. \tag{1.0.18}$$

Although (VE_k) are not actually homogeneous equations, they can be put in that frame, and therefore, one can define successive extensions $K \subset L_1 \subset L_2 \subset \ldots \subset L_k$, where L_k is the extension obtained by adjoining the solutions of (VE_k). Correspondingly one can define the Galois groups $Gal(L_1/K), \ldots, Gal(L_k/K)$. The following result is proven in [75].

Theorem 1.0.5. *If the Hamiltonian system (1.0.7) is integrable in Liouville sense then the identity component of every Galois group $Gal(L_k/K)$ is abelian.*

Note that we apply Theorem 1.0.5 in the situation when the identity component of the Galois group $Gal(L_1/K)$ is abelian. This means that the first variational equation is solvable. Once we have the solution of (VE_1), then the solutions of (VE_k) can be found by the method of variations of constants as explained above. Hence, the Galois groups $Gal(L_k/K)$ are solvable. One possible way to show that some of them is not commutative is to find a logarithmic term in the corresponding solution (see detailed descriptions and explanations in [70, 75, 76]).

Finally, we recall a perturbational technique which is still related to the Differential Galois approach. Let M_0 be a two-dimensional complex analytic symplectic manifold, $H_0(q,p)$ be a holomorphic Hamiltonian and X_{H_0} be the corresponding Hamiltonian vector field. Assume that the system

$$\dot{q} = H_{0,p}, \quad \dot{p} = -H_{0,q} \tag{1.0.19}$$

has a hyperbolic equilibrium (q_0, p_0). Then the system (1.0.19) has a separatrix

$$\Gamma_0 : (q_0(t), p_0(t)), \lim_{t\to\infty} q_0(t) = q_0\, , \lim_{t\to\infty} p_0(t) = p_0. \tag{1.0.20}$$

The functions $q_0(t), p_0(t)$ are meromorphic in $t \in \mathbb{C}$. Let

$$H(q,p,t,\varepsilon) = H_0(q,p) + \varepsilon H_1(q,p,t) + \ldots \tag{1.0.21}$$

be a meromorphic small (complex) perturbation of H_0 satisfying $H_1(q,p,t+\omega) = H_1(q,p,t)$ with a period $\omega \in \mathbb{C}$. This function H is defined over $M = M_0 \times F_\omega$, $F_\omega = \mathbb{C}/\omega\mathbb{Z}$. We can write the Hamiltonian system defined by $H(q,p,\varphi)$ over M as

$$\dot{q} = H_p, \quad \dot{p} = -H_q, \quad \dot{\varphi} = 1, \quad (q,p,\varphi) \in M. \tag{1.0.22}$$

When $\varepsilon = 0$ the system (1.0.22) reduces to

$$\dot{q} = H_{0,p}, \quad \dot{p} = -H_{0,q}, \quad \dot{\varphi} = 1, \quad (q,p,\varphi) \in M. \tag{1.0.23}$$

The unperturbed system (1.0.23) has a hyperbolic ω-periodic orbit $\Pi_0 := (q_0, p_0, \varphi = t(\mod \omega))$. It is well known that for small $|\varepsilon|$ the perturbed system (1.0.22) has also an ω-periodic orbit $\Pi_\varepsilon := (q(t,\varepsilon), p(t,\varepsilon), \varphi = t - t_0(\mod \omega))$, such that $(q(t,0), p(t,0)) = (q_0, p_0)$.

We define the (stable) complex separatrix Λ_ε^+ of the system (1.0.22) as the set of integral curves of (1.0.22) asymptotic to Π_ε as $t \to \infty$. For fixed ε, it is a two-dimensional complex surface. This separatrix can have transverse self-intersections.

Remark 2. Recall that in the real case the separatrices can not have transverse self-intersections. Such intersections can occur between stable and unstable separatrices. For real Hamiltonian systems, the existence of such transverse orbits is considered as a source of chaotic behavior and is an obstruction to existence of an analytic first integral.

Ziglin [108] proved that for complex Hamiltonian systems, the existence of transverse self-intersections for separatrices is also an obstruction to the integrability.

The unperturbed separatrix is given by $\Lambda_0^+ = \Gamma_0 \times F_\omega$. It is foliated by the one-parameter family of integral curves

$$\Gamma_{t_0} : (q_0(t), p_0(t), t - t_0), \tag{1.0.24}$$

$t_0 \in F_\omega$ being the parameter. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed path in the complex plane with $\gamma(0) = \gamma(1) \in \mathbb{R} \subset \mathbb{C}$. The following function on F_ω

$$d(t_0) := \int_\gamma \{H_0, H_1\}(q_0(t), p_0(t), t - t_0)dt \tag{1.0.25}$$

is usually called Poincaré-Arnold-Melnikov integral. Here $\{,\}$ is the Poisson bracket. Then the following result is valid:

Theorem 1.0.6. *(Ziglin) If the function $d(t_0)$ has a simple zero, then for sufficiently small $|\varepsilon| \neq 0$, the separatrix Λ_ε^+ has a transversal self-intersection and the system (1.0.22) has no additional holomorphic first integral.*

It appears that there is a relation between Theorem 1.0.1 and Theorem 1.0.6. Morales-Ruiz [74] proved that, under certain assumptions, the Ziglin's condition about the Poincaré-Arnold-Melnikov integral can be interpreted by the fact that the Galois group of the perturbed variational equation along the integral curve Γ_0 is non-abelian. In other words, if Poincaré-Arnold-Melnikov integral $d(t_0)$ is not identically zero, the Galois group of the perturbed variational equation is not abelian and the system is not integrable by means of meromorphic first integrals.

Chapter 2

Non-integrability of first order resonances in Hamiltonian systems in three degrees of freedom

The normal forms of the Hamiltonian $1 : 2 : \omega$ resonances to degree three for $\omega = 1, 3, 4$ are studied for integrability. We prove that these systems are non-integrable except for the discrete values of the parameters which are well known. We use the Ziglin-Morales-Ramis method based on the differential Galois theory. The results of this chapter are published in [15].

2.1 Introduction

In this chapter we study integrability of genuine first (or primary) order resonances in Hamiltonian systems with three degrees of freedom. The term genuine first order resonance is introduced by van der Aa and Sanders [98] for the Hamiltonian systems with the resonant ratios $1 : 2 : \omega, \quad \omega = 1, 2$, 3 , 4, since these are the only cases for which two independent resonance relations appear at order three.

Let us recall briefly some facts about normal forms of Hamiltonian systems with three degrees of freedom. Most of the results are valid for systems with arbitrary degrees of freedom. Consider the Hamiltonian system with a Hamiltonian $H(x, y)$. In the neighborhood of an equilibrium $(x, y) = (0, 0)$ we have the following expansion of H

$$H = H_2 + H_3 + H_4 + \dots,$$
$$H_2 = \sum \omega_j (x_j^2 + y_j^2), \quad \omega_j > 0.$$

We consider only the case when H_2 is a positively defined quadratic form. The frequency $\omega = (\omega_1, \omega_2, \omega_3)$ is said to be in resonance if there exists a vector $k = (k_1, k_2, k_3)$, $k_j \in \mathbb{Z}, j = 1, 2, 3$, such that $(\omega, k) = \sum k_j \omega_j = 0$, where $|\,k| = \sum |\,k_j|$ is the order of resonance. We usually speak about a strong resonance when $|\,k| \leq 4$. One can introduce the term full resonance, which means that there are two linearly independent vectors k and k', satisfying $(\omega, k) = (\omega, k') = 0$. Using a canonical transformation close to the identity (in fact a series of canonical transformations), H simplifies. The simplified Hamiltonian in the non-resonant case is called a Birkhoff normal form. When resonances appear, the corresponding normal form is called a Birkhoff-Gustavson normal form. In order to detect the behaviour in a small vicinity of the fixed point, instead of the Hamiltonian H one considers the normal form truncated to some order

$$\bar{H} = H_2 + \ldots + H_m.$$

It is known that the truncated to any order Birkhoff normal form is integrable [6]. The truncated Birkhoff-Gustavson normal form has at least two integrals - H_2 and $\bar{H}$. Hence the truncated normal forms of Hamiltonians with two degrees of freedom are integrable. It is natural to ask about the integrability of the truncated Birkhoff-Gustavson normal forms for Hamiltonian systems with more degrees of freedom. It is known that if the frequency $\omega = (\omega_1, \omega_2, \omega_3)$ is not fully resonant, one can always find one more independent integral, see Arnold et al. [6]. Therefore, one needs to assume that the frequency ω is fully resonant, in which case the truncated normal form is not automatically integrable.

In order to obtain estimates of the approximation by normalization in a neighborhood of an equilibrium point we introduce $x = \varepsilon\tilde{x}$, $y = \varepsilon\tilde{y}$. Here ε is a small positive parameter and ε^2 is a measure for the energy relative to the equilibrium energy. Then, dividing by ε^2 and removing tildes we get

$$\bar{H} = H_2 + \varepsilon H_3 + \ldots + \varepsilon^{m-2} H_m.$$

Provided that $\omega_j > 0$ it is proven in [100] that $\bar{H}$ is an integral for the original system with error $O(\varepsilon^{m-1})$ and H_2 is an integral for the original system with error $O(\varepsilon)$ for the whole time interval. If we are given an additional independent integral, then it is an integral for the original Hamiltonian system with error $O(\varepsilon^{m-2})$ on the time scale $1/\varepsilon$. The exact integrals for the normal form $\bar{H}$ are approximate integrals for the original system, i.e. if the normal form is integrable then the original system is near integrable in the above sense.

The truncated normal forms up to order 3 of $1:2:\omega$ resonances are considered by van der Aa [97], where the periodic orbits and their stability are studied. The author also proves that no independent additional quadratic or cubic first integral can exist.

The integrability of Hamiltonian $1:1:2$ resonance is considered by Duistermaat [24]. He simplifies considerably the normal form up to order 3 and isolates several cases of integrability (see Section 2). Further, it appears that on the hypersurface $H_3 = 0$ all solutions are periodic. The period function of these solutions turn out to be infinitely branched, which excludes the existence of an additional analytic first integral.

The Hamiltonian $1:2:2$ resonance is known to be asymptotically integrable, i.e. the normal form up to order 3 has three independent integrals. For the potential problem this result has been presented by Martinet et al [63], for truncated to order 3 normal form the proof has been given by van der Aa [97], and finally this result has been extended by van der Aa and Verhulst [99] to the systems with more degrees of freedom.

The integrability of Hamiltonian $1:2:3$ resonance has been studied for the most part numerically. Ford, [27], studied the truncated to order 3 normal form and showed that the dynamics of the system exhibits widespread stochastic behavior. In [41], Hoveijn and Verhulst studied the normal forms up to degree 3 and 4. Numerical calculations carried out there suggest that the system is non-integrable. In the normal form to degree 4 the Devaney-Šilnikov theory is applied to prove the existence of a horseshoe map in the dynamics of the system which imply non-integrability.

Integrability of Hamiltonian $1:2:4$ resonance is considered by Verhulst [100] in the presence of some discrete symmetries. Moreover, in [100] Verhulst asked the question for rigorous proofs of non-integrability of $1:2:3$ and $1:2:4$ Hamiltonian resonances.

Finally, let us mention the paper of Haller and Wiggins, [34], where they consider a special class of resonances in Hamiltonian systems with three degrees of freedom in which the truncated to order 3 normal forms are integrable and the detailed description of the dynamics of these resonances is given. Then these integrable truncations are perturbed with forth order resonant terms and the existence of families of 3-tori and whiskered 2-tori with nearby chaotic dynamics is proven.

The chater is organized as follows. In Section 2 we simplify the Hamiltonians and formulate the main result. We do not consider here the truncated normal form in $1:2:2$ resonance since it is integrable. Section 3 is devoted to the proof of the main Theorem.

2.2 Simplifications and statement of the main result

In this section we simplify the truncated normal forms of the Hamiltonians in first order resonances. Then we formulate our main result.

With the so called action - angle variables

$$q_j = \sqrt{2I_j}\sin\phi_j, \quad p_j = \sqrt{2I_j}\cos\phi_j, \quad j = 1,2,3 \tag{2.2.1}$$

the normal form of the Hamiltonian $1:2:3$ resonance up to order three reads [97]

$$\begin{aligned}\bar{H} &= H_2 + H_3 = I_1 + 2I_2 + 3I_3 \\ &+ \varepsilon\{2aI_1\sqrt{2I_2}\cos(2\phi_1 - \phi_2 - c) + 2b\sqrt{2I_1I_2I_3}\cos(\phi_1 + \phi_2 - \phi_3 - d)\}.\end{aligned} \tag{2.2.2}$$

Since $\bar{H}$ is a truncated normal form we have two independent integrals $\bar{H}$ and H_2. The obvious cases of integrability (with the existence of an additional integral) are:

1. $b = 0$ with integral I_3,

2. $a = 0$ with integral $lI_1 + mI_2 + (l+m)I_3$.

We can simplify $\bar{H}$ by using a time-dependent canonical transformation (see Ford [27])

$$\begin{array}{ll} I_j \to I'_j, & \phi_1 \to \phi'_1 + t + \delta_1, \\ \bar{H} \to \bar{H}', & \phi_2 \to \phi'_2 + 2t + \delta_2, \\ & \phi_3 \to \phi'_3 + 3t + \delta_3. \end{array} \tag{2.2.3}$$

Here $\delta_1, \delta_2, \delta_3$ represent rigid rotations on the coordinate space by which the parameters c and d (inessential for the integrability) are eliminated. Then $\bar{H}'$ becomes

$$\bar{H}' = \varepsilon[2aI'_1\sqrt{2I'_2}\cos(2\phi'_1 - \phi'_2) + 2b\sqrt{2I'_1I'_2I'_3}\cos(\phi'_1 + \phi'_2 - \phi'_3)],$$

where a and b are real numbers. Note that the change (2.2.3) allows us to reduce the dynamics to the study of the slow variables I'_j, ϕ'_j. Apparently, the parameter ε does not influence the integrability, we remove it and we do the same for the other cases. We drop the primes for simplicity and return to cartesian coordinates in which $\bar{H}$ has the form

$$\bar{H} = a\left[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2\right] + b\left[p_3(p_1p_2 - q_1q_2) + q_3(q_1p_2 + p_1q_2)\right]. \tag{2.2.4}$$

We can assume also that $a, b \geq 0$ since supposing that $b < 0$, then by a canonical change $q_3 \to -q_3, p_3 \to -p_3$ the case is reduced to $b > 0$. Similarly if $a < 0$, then by the canonical change $p_1 \to -p_1, q_1 \to -q_1, p_2 \to -p_2, q_2 \to -q_2$ we reduce the consideration to the case $a > 0$. The equations of motion corresponding to $\bar{H}$ (2.2.4) are

$$\begin{array}{ll} \dot{q}_1 = 2a(p_1p_2 + q_1q_2) + b(p_2p_3 + q_2q_3), & \dot{p}_1 = 2a(p_2q_1 - p_1q_2) + b(q_2p_3 - p_2q_3), \\ \dot{q}_2 = a(p_1^2 - q_1^2) + b(p_1p_3 + q_1q_3), & \dot{p}_2 = -2ap_1q_1 + b(q_1p_3 - p_1q_3), \\ \dot{q}_3 = b(p_1p_2 - q_1q_2), & \dot{p}_3 = -b(q_1p_2 + p_1q_2). \end{array} \tag{2.2.5}$$

Next, the normal form of the Hamiltonian $1:2:4$ resonance to degree 3 is given by

$$\bar{H} = I_1 + 2I_2 + 4I_3 + 2aI_1\sqrt{2I_2}\cos(2\phi_1 - \phi_2 - c) + 2bI_2\sqrt{2I_3}\cos(2\phi_2 - \phi_3 - d). \tag{2.2.6}$$

The normal form (2.2.6) has two independent first integrals: $\bar{H}$ and $H_2 = I_1 + 2I_2 + 4I_3$. The obvious cases of integrability here are

1. $b = 0$ with integral I_3,

2. $a = 0$ with integral I_1.

To simplify the Hamiltonian $\bar{H}$ (2.2.6) we can perform a similar to (2.2.3) time-dependent canonical transformation and the resulting Hamiltonian in cartesian coordinates is

$$\bar{H} = a[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2] + b[p_3(p_2^2 - q_2^2) + 2p_2q_2q_3]. \tag{2.2.7}$$

We can also assume $a, b \geq 0$. Then the corresponding equations of motion take the form

$$\begin{array}{ll} \dot{q}_1 = 2a(q_1q_2 + p_1p_2), & \dot{p}_1 = 2a(p_2q_1 - p_1q_2), \\ \dot{q}_2 = a(p_1^2 - q_1^2) + 2b(p_2p_3 + q_2q_3), & \dot{p}_2 = -2ap_1q_1 + 2b(q_2p_3 - p_2q_3), \\ \dot{q}_3 = b(p_2^2 - q_2^2), & \dot{p}_3 = -2bp_2q_2. \end{array} \tag{2.2.8}$$

Finally, for the case $1:1:2$ we take the form of H_3 obtained by Duistermaat [24]:

$$\bar{H} = H_3 = q_3\left[a(q_1^2 - p_1^2) + b(q_2^2 - p_2^2)\right] + 2p_3\left[ap_1q_1 + bp_2q_2\right]. \tag{2.2.9}$$

Here $a \geq b \geq 0$ and $H_2 = q_1^2 + p_1^2 + q_2^2 + p_2^2 + 2(q_3^2 + p_3^2)$ is the other independent first integral. The cases when there exists an additional integral are

1. $b = 0$ with integral $G = p_2^2 + q_2^2$,

2. $a = b$ with integral $G = q_1p_2 - p_1q_2$,

3. $a = 2b$ with integral $G = (q_1p_2 - p_1q_2)^2(p_2^2 + q_2^2) + 2[\frac{1}{2}q_3(q_2^2 - p_2^2) + p_3q_2p_2]^2$.

In the case $a = 2b$ the additional integral was found by Duistermaat [24].

The equations of motion governed by the Hamiltonian $\bar{H}$ (2.2.9) are

$$\begin{array}{ll} \dot{q}_1 = 2a(q_1p_3 - p_1q_3), & \dot{p}_1 = -2a(q_1q_3 + p_1p_3), \\ \dot{q}_2 = 2b(q_2p_3 - p_2q_3), & \dot{p}_2 = -2b(q_2q_3 + p_2p_3), \\ \dot{q}_3 = 2aq_1p_1 + 2bq_2p_2, & \dot{p}_3 = a(p_1^2 - q_1^2) + b(p_2^2 - q_2^2). \end{array} \tag{2.2.10}$$

In what follows we consider complexified systems, i.e. the variables $(q_j(t), p_j(t))$ are in $\mathbb{C}^6$ with standard symplectic structure, $t \in \mathbb{C}$, but we keep a and b real.

Our main result is the following:

Theorem 2.2.1. *The systems (2.2.5), (2.2.8) and (2.2.10) do not admit additional meromorphic first integral except in the cases listed above. That is, the truncated Hamiltonians up to order 3 in resonances* $1:2:3$, $1:2:4$ *and* $1:1:2$ *are not integrable in the Liouvillian sense.*

This theorem will be proven in the next section. We start the proof with the system at $1:2:3$ resonance because we think that the proof is more complicated. The cases $1:2:4$ and $1:2:1$ are treated similarly but differently from the previous one.

2.3 Proof of the main Theorem

In this section we prove the Theorem 2.2.1. In each of the cases we choose a particular solution and study the Galois group of the linearized system. In the cases of $1:2:4$ and $1:2:1$ resonances the study of the monodromy group is sufficient to conclude non - integrability.

The case of $1:2:3$ resonance is more difficult. First, we find a relatively simple particular solution and the information of the monodromy group of the corresponding linearized system is enough to resolve all the cases but $a=b$. For this case we will find another particular solution. The Galois group of the corresponding (NVE) is again abelian. Then we use the approach from Morales-Ruis, Simó and Simón [77] and get information for the Galois group of (VE) which is shown to be non-commutative.

2.3.1 1 : 2 : 3 resonance

Let $b>0$, denote $\mu = a/b, \mu \in [0,\infty)$ and let $s = -\frac{\mu+\sqrt{\mu^2-1}}{\sqrt{2}}$ be a root of the equation $2s^2+2\sqrt{2}s\mu+1=0$. Then, the system (2.2.5) has a particular solution

$$\begin{aligned} q_1 &= \frac{\sqrt{2}sF}{\sqrt{2s^2+3}}\frac{1}{\cosh(bsFt)}, & q_2 &= \frac{F}{\sqrt{2}}\tanh(bsFt), q_3 = \frac{F}{\sqrt{2s^2+3}}\frac{1}{\cosh(bsFt)}, \\ p_1 &= p_2 = p_3 = 0, & & q_1^2+2q_2^2+3q_3^2 = F^2. \end{aligned} \tag{2.3.11}$$

Let $dq_j = \xi_j, dp_j = \eta_j$, $j=1,2,3$. Then, $d\bar{H}$ does not depend on ξ_1, ξ_2, ξ_3

$$d\bar{H} = (2aq_1q_2 + bq_2q_3)\eta_1 + (bq_1q_3 - aq_1^2)\eta_2 - bq_1q_2\eta_3.$$

We take $\xi_j, j=1,2,3$ as coordinates in the reduced space and write (NVE) in them

$$\begin{aligned} \dot{\xi}_1 &= 2aq_2\xi_1 + (2aq_1+bq_3)\xi_2 + bq_2\xi_3, \\ \dot{\xi}_2 &= (-2aq_1+bq_3)\xi_1 + bq_1\xi_3, \\ \dot{\xi}_3 &= -bq_2\xi_1 - bq_1\xi_2. \end{aligned} \tag{2.3.12}$$

The system (2.3.12) admits an integral

$$dH_2 = q_1\xi_1 + 2q_2\xi_2 + 3q_3\xi_3 = g = const. \tag{2.3.13}$$

First, we take $g = 0$ (which will resolve most cases) and let us express ξ_3 from (2.3.13). Then (NVE) (2.3.12) takes the form

$$\dot{Y} = A(t)Y, \tag{2.3.14}$$

where $Y = (\xi_1, \xi_2)^t$, $A(t) = (a_{ij},\ i, j = 1, 2)$ and

$$a_{11} = \frac{b}{3}(6\mu - \sqrt{2}s)q_2,\ a_{12} = \frac{b}{3}\frac{(3 - 4s^2)q_3^2 - F^2}{q_3},$$
$$a_{21} = \frac{2b}{3}(2s^2 + 3)q_3,\ a_{22} = -\frac{2\sqrt{2}bs}{3}q_2.$$

The system (2.3.14) can be reduced to a single equation with respect to, say, ξ_2

$$\ddot{\xi}_2 + (bF)(2s - \sqrt{2}\mu)\tanh(bFst)\dot{\xi}_2 + (bf)^2[2s^2 + \tanh^2(bFst)]\xi_2 = 0. \tag{2.3.15}$$

Denote $k = \frac{\mu}{\sqrt{2}s}$. When $\mu \in [1, \infty)$, k takes values in $[-1, -1/2]$. The equation (2.3.15) can be brought to a hypergeometric equation after the change $x = \frac{1}{2}(\tanh(bFst) + 1)$ of independent variable x

$$\frac{d^2\xi_2}{dx^2} + k\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{d\xi_2}{dx} - \left[k\left(\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + 2\frac{k+1}{x(x-1)}\right]\xi_2 = 0. \tag{2.3.16}$$

The equation (2.3.16) is called algebraic form of the equation (2.3.15). It is known [70], that although the differential Galois group is changed by such a transformation, the identity component is preserved.

One can easily obtain the exponents of the singular points, namely

$$(\alpha, \alpha') = (1, -k),\ (\beta, \beta') = (-2, 2k + 1),\ (\gamma, \gamma') = (1, -k). \tag{2.3.17}$$

Let the exponent differences be

$$\hat{\lambda} = \alpha - \alpha' = 1 + k, \quad \hat{\mu} = \beta - \beta' = -3 - 2k, \quad \hat{\nu} = \gamma - \gamma' = 1 + k.$$

The number $\hat{\lambda} + \hat{\mu} + \hat{\nu} = -1$ is an odd integer. According to Kimura theorem [52] the identity component of the equation (2.3.16) is solvable. In order to find when this identity component is abelian, we use the results for the monodromy group of the hypergeometric equation. According to Iwasaki et al. [46], p.86 we have the following reducible cases.

1) $\hat{\lambda} \notin \mathbb{Z}, \quad \hat{\nu} \notin \mathbb{Z}, \quad \hat{\mu} \notin \mathbb{Z}$

We see that in this case k is not an integer. Recall that $a, b \geq 0$, so $\mu = a/b \in [0, \infty)$. In particular, when $\mu \in (0, 1)$, k is a complex number and when $\mu > 1$, k belongs to $(-1, -1/2)$. Then the monodromy matrices, corresponding to windings around 0 and 1 are

$$M_0 = \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha'} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} e^{2\pi i\gamma} & 1 \\ 0 & e^{2\pi i\gamma'} \end{pmatrix},$$

which clearly do not commute, since

$$M_0M_1M_0^{-1}M_1^{-1} = \begin{pmatrix} 1 & \frac{1-\kappa}{\kappa^2} \\ 0 & 1 \end{pmatrix} \text{ with } \kappa = e^{-2\pi ik}.$$

2) $\underline{\hat{\lambda} \notin \mathbb{Z}, \quad \hat{\nu} \notin \mathbb{Z}, \quad \hat{\mu} \in \mathbb{Z}}$

In this case $2k \in \mathbb{Z}$ and hence $k = -1/2$, i.e. $\mu = \infty$ or $b = 0$. The corresponding monodromy matrices commute, but we know that this is an integrable case.

3) $\underline{\hat{\lambda} \in \mathbb{Z}, \quad \hat{\nu} \in \mathbb{Z}, \quad \hat{\mu} \in \mathbb{Z}}$

In this case $k \in \mathbb{Z}$. The first possibility is $k = 0$, i.e. $\mu = 0$ or $a = 0$ - which is an integrable case. The second possibility is $k = -1$, i.e. $\mu = 1$ or $a = b$. In this case the monodromy group is commutative.

To summarize so far: The hypergeometric equation (2.3.16) is Fuchsian, its monodromy group topologically generates the Galois group which turns out to be solvable but not abelian except in the cases $a, b = 0$ and $a = b$. So, the Hamiltonian $1 : 2 : 3$ resonance is not integrable due to Morales-Ramis theorem (Theorem 1.0.1) when $a \neq 0, b \neq 0$ and $a \neq b$.

Remark 2. We give a direct proof that the Galois group of (VE) (and in particular of (NVE)) along the solution (2.3.11), when $a = b$, is commutative. In this case $\mu = 1, s = 1/\sqrt{2}$. From the previous calculations we saw that F can be chosen arbitrary, so we take $F = -\sqrt{2}$. Then (VE) takes the form

$$\begin{aligned}
\dot{\xi}_1 &= -2\tanh t\xi_1 + \frac{\sqrt{2}}{2\cosh t}\xi_2 - \tanh t\xi_3, & \dot{\eta}_1 &= 2\tanh t\,\eta_1 + \frac{3\sqrt{2}}{2\cosh t}\eta_2 - \tanh t\eta_3, \\
\dot{\xi}_2 &= -\frac{3\sqrt{2}}{2\cosh t}\xi_1 + \frac{\sqrt{2}}{2\cosh t}\xi_3, & \dot{\eta}_2 &= -\frac{\sqrt{2}}{2\cosh t}\eta_1 + \frac{\sqrt{2}}{2\cosh t}\eta_3 \\
\dot{\xi}_3 &= \tanh t\,\xi_1 - \frac{\sqrt{2}}{2\cosh t}\,\xi_2, & \dot{\eta}_3 &= \tanh t\,\eta_1 - \frac{\sqrt{2}}{2\cosh t}\,\eta_2.
\end{aligned} \tag{2.3.18}$$

Note that the subsystem with respect to ξ_1, ξ_2, ξ_3 forms the (NVE). Denote the fundamental matrix of (2.3.18) with Φ. It has the form

$$\Phi = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix},$$

where Φ_1 is the fundamental matrix of the subsystem with respect to ξ_1, ξ_2, ξ_3, and Φ_2 is the fundamental matrix of the subsystem with respect to η_1, η_2, η_3. One can verify that

$$\Phi_1 = \begin{pmatrix} -\frac{1}{5}\frac{1}{\cosh t} & -\frac{\sinh t}{\cosh^2 t} & \frac{4}{5}\frac{1}{\cosh t} - t\frac{\sinh t}{\cosh^2 t} \\ \frac{4\sqrt{2}}{5}\frac{\sinh t}{\cosh t} & -\frac{\sqrt{2}}{\cosh^2 t} & -\frac{\sqrt{2}}{5}\frac{\sinh t}{\cosh t} - t\frac{\sqrt{2}}{\cosh^2 t} \\ \frac{1}{\cosh t} & \frac{\sinh t}{\cosh^2 t} & t\frac{\sinh t}{\cosh^2 t} \end{pmatrix}$$

and

$$\Phi_2 = \begin{pmatrix} -\frac{1}{3}\frac{1}{\cosh t} & \cosh t & \sinh t + t(4\cosh t + \frac{1}{\cosh t}) \\ \frac{2\sqrt{2}}{3}\frac{\sinh t}{\cosh t} & 0 & 2\sqrt{2}(1 - t\tanh t) \\ \frac{1}{\cosh t} & \cosh t & -3\sinh t + t(4\cosh t - \frac{3}{\cosh t}) \end{pmatrix}.$$

The coefficient field of the system is the field of hyperbolic functions K. By adjoining the solutions of the system, in fact we adjoin a single quadrature $t = \int 1dt$, we obtain an extension $L_1 = K(t = \int 1dt)$.

Let $\sigma \in G = Gal(L_1/K)$. Then $\sigma\Phi = \begin{pmatrix} \sigma\Phi_1 & 0 \\ 0 & \sigma\Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$, where

$$\sigma\Phi_1 = \Phi_1 M_1 = \Phi_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma\Phi_2 = \Phi_2 M_2 = \Phi_2 \begin{pmatrix} 1 & 0 & -3\alpha \\ 0 & 1 & 4\alpha \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly $G = \left\{ \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad \alpha \in \mathbb{C} \right\}$ is a commutative group.

So, in order to study the integrability in the case $a = b$ one can proceed with higher variational equations along this solution. Instead, we will take the path explained in the beginning of the section.

Without loss of generality we assume that $a = b = 1$. The equations of motion admit another particular solution

$$q_1 = -q_3 = \frac{u}{\sqrt{2}}, \quad q_2 = \frac{\dot{u}}{u}, \quad p_1 = p_3 = \frac{i}{\sqrt{2}u}, \quad p_2 = 0, \tag{2.3.19}$$

where u satisfies the so called lemniscatic curve

$$\Gamma : \dot{u}^2 = 1 - u^4. \tag{2.3.20}$$

Let $\xi_j = dq_j, \eta_j = dp_j, j = 1, 2, 3$. Denote $z = (\xi, \eta)^T$. Then the variational equations along the solution (2.3.19) read

$$\dot{z} = Az, \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix}, \tag{2.3.21}$$

where

$$A_1 = \begin{pmatrix} 2q_2 & q_1 & q_2 \\ -3q_1 & 0 & q_1 \\ -q_2 & -q_1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 3p_1 & 0 \\ 3p_1 & 0 & p_1 \\ 0 & p_1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -p_1 & 0 \\ -p_1 & 0 & -p_1 \\ 0 & -p_1 & 0 \end{pmatrix}.$$

Taking advantage of independent first integrals of (VE) $d\bar{H}$ and dH_2 we construct a symplectic matrix P of the form $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (see for details [67], pp. 36–38 and [70], pp. 75–77), where

$$a = \begin{pmatrix} \frac{2p_1^2 - 3q_1^2}{4p_1q_1q_2} & 0 & -\frac{3}{4p_1q_2} \\ 0 & -\frac{1}{q_2} & 0 \\ -\frac{2p_1^2 + q_1^2}{4p_1q_1q_2} & 0 & -\frac{1}{4p_1q_2} \end{pmatrix}, \quad b = \begin{pmatrix} -q_2 & p_1 & q_1q_2 \\ 2q_1 & 0 & 2(p_1^2 - q_1^2) \\ q_2 & 3p_1 & -q_1q_2 \end{pmatrix},$$

$$c = \begin{pmatrix} \frac{1}{4q_2} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3}{4q_2} & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & -q_1 & -p_1q_2 \\ 0 & -2q_2 & 0 \\ 0 & 3q_1 & -p_1q_2 \end{pmatrix}.$$

We perform the change of variables in (2.3.21) $z = Py$ to obtain

$$\dot{y} = By = P^{-1}(AP - \dot{P})y, \tag{2.3.22}$$

where $B = \begin{pmatrix} B_1 & 0 \\ B_3 & -B_1^T \end{pmatrix}$ and

$$B_1 = \begin{pmatrix} \frac{\dot{q}_2 - q_2^2}{q_2} & 0 & -\frac{q_1}{q_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} \frac{2(q_1^2 - p_1^2)}{p_1 q_1 q_2} & 0 & \frac{1}{p_1 q_2} \\ 0 & 0 & -\frac{1}{2q_2^2} \\ \frac{1}{p_1 q_2} & -\frac{1}{2q_2^2} & 0 \end{pmatrix}.$$

Next we put $y = Jw$, where $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$, and $\dot{w} = Cw$, $C = -JBJ$. Finally, we exchange the variables w_1 and w_3. Keeping the same notations for the variables w and the matrix C we reach the system

$$\dot{w} = Cw, \tag{2.3.23}$$

where $C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_4 \end{pmatrix}$ and

$$C_1 = \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}u^2\dot{u}}{1-u^4} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2\dot{u}}{u(1-u^4)} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \frac{\sqrt{2}iu^2\dot{u}}{1-u^4} & \frac{u^2}{2(1-u^4)} & 0 \\ 0 & 0 & \frac{u^2}{2(1-u^4)} \\ \frac{2i(u^4+1)\dot{u}}{u(1-u^4)} & 0 & \frac{\sqrt{2}iu^2\dot{u}}{1-u^4} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -\frac{2\dot{u}}{u(1-u^4)} & 0 & -\frac{\sqrt{2}u^2\dot{u}}{1-u^4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We are interested in the fundamental matrix of the system (2.3.23). Let it be in the form

$$\Psi = \begin{pmatrix} \Phi_1 & Q \\ 0 & \Phi_2 \end{pmatrix}. \tag{2.3.24}$$

The solution of the lower right block of (2.3.23) is

$$w_4 = \frac{\sqrt{1-u^4}}{u^2}(w_4^0 - I_1 w_6^0),\; w_5 = w_5^0,\; w_6 = w_6^0, \tag{2.3.25}$$

where

$$I_1 = \sqrt{2}\int \frac{u^4}{(1-u^4)^{3/2}}du. \tag{2.3.26}$$

Then the fundamental matrix corresponding to the lower right block is

$$\Phi_2 = \begin{pmatrix} \frac{\sqrt{1-u^4}}{u^2} & 0 & -\frac{\sqrt{1-u^4}}{u^2}I_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3.27}$$

The solution of the upper left block of (2.3.23) is

$$w_1 = I_1 w_3^0,\ w_2 = w_2^0,\ w_3 = \frac{u^2}{\sqrt{1-u^4}} w_3^0. \tag{2.3.28}$$

The corresponding fundamental matrix is

$$\Phi_1 = \begin{pmatrix} 1 & 0 & I_1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{u^2}{\sqrt{1-u^4}} \end{pmatrix}. \tag{2.3.29}$$

At this place we take a look at I_1

$$I_1 = \frac{\sqrt{2}}{2}\left(\frac{u}{\sqrt{1-u^4}} - \int^u \frac{dv}{\sqrt{1-v^4}}\right).$$

The integral in the above expression is

$$t = \int^u \frac{dv}{\sqrt{1-v^4}} = \int 1 dt,$$

namely the inverse function of $u(t)$, defined by (2.3.20). The dependance of t on u is transcendent. Hence, by adjoining the integral I_1 (or equivalently t) to the field of elliptic functions $K = \mathcal{M}(\Gamma)$ we get a transcendental extension

$$K \subset L_1 = K(I_1). \tag{2.3.30}$$

Then $Gal(L_1|K) \cong \mathbb{C}_+$ and it acts on Φ_1 and Φ_2 in the following way

$$\sigma\Phi_1 = \Phi_1 M_1 = \Phi_1 \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma\Phi_2 = \Phi_2 M_2 = \Phi_2 \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3.31}$$

with $\alpha \neq 0$ complex number.

Remark 3. As we have seen from the previous computations, we may take the equations for w_4 and w_6 in the lower right block for the normal variational equations (NVE). However, one can infer from (2.3.31) that the corresponding Galois group is commutative, i.e. we can not conclude from here non integrability.

In order to compute the block Q in (2.3.24), we consider the system

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}^{\cdot} = C_1 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + C_2 \begin{pmatrix} w_4 \\ w_5 \\ w_6 \end{pmatrix}. \tag{2.3.32}$$

The solution is obtained by applying the variation of constants formula [43]. Then

$$Q(t) = \Phi_1(t) \int V(t)dt, \quad V(t) = \Phi_1^{-1} C_2 \Phi_2(t). \tag{2.3.33}$$

Hence, the fundamental matrix of the system (2.3.23) has the form

$$\Psi = \begin{pmatrix} \Phi_1 & \Phi_1 \int V(t)dt \\ 0 & \Phi_2 \end{pmatrix}. \tag{2.3.34}$$

Adjoining the matrix integral from (2.3.34) we obtain another extension

$$K \subset L_1 \subset L_2 = L_1(\int v_{ij}dt). \tag{2.3.35}$$

In what follows the number of quadratures will be specified. Since each of the above extensions in the chain (2.3.35) is a result of adjoining quadratures, the extension L_2/K is a Liouville extension that is, $G = Gal(L_2/K)$ is a solvable group and also G^0 - its identity component. Our aim is to prove that G^0 is not commutative.

We identify $\sigma \in G$ with the corresponding matrices R_σ. Using block notation we have

$$\sigma\Psi = \Psi R_\sigma, \quad R_\sigma = \begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix}.$$

More precisely,

$$\begin{aligned} \sigma\Psi = \begin{pmatrix} \sigma\Phi_1 & \sigma\Phi_1 \int V(t)dt \\ 0 & \sigma\Phi_2 \end{pmatrix} &= \begin{pmatrix} \Phi_1 & \Phi_1 \int V(t)dt \\ 0 & \Phi_2 \end{pmatrix} \begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \\ &= \begin{pmatrix} \Phi_1 M_1 & \Phi_1 M_3 + \Phi_1 \int V(t)dt M_2 \\ 0 & \Phi_2 M_2 \end{pmatrix} \end{aligned} \tag{2.3.36}$$

Applying the identity $\partial \circ \sigma = \sigma \circ \partial$ on $\int V$ and integrating the obtained equation, we get

$$\sigma \int \Phi_1^{-1} C_2 \Phi_2 dt = \int \sigma\Phi_1^{-1} C_2 \Phi_2 dt + M, \quad M \in Mat(3, \mathbb{C}). \tag{2.3.37}$$

Noting that $\sigma C_2 = C_2$, we have

$$\sigma(\Phi_1^{-1} C_2 \Phi_2) = \sigma\Phi_1^{-1} C_2 \sigma\Phi_2 = M_1^{-1}\Phi_1^{-1} C_2 \Phi_2 M_2$$

from where

$$\sigma \int \Phi_1^{-1} C_2 \Phi_2 dt = \int \sigma\Phi_1^{-1} C_2 \Phi_2 dt = M_1^{-1} \int \Phi_1^{-1} C_2 \Phi_2 dt M_2 + M. \tag{2.3.38}$$

This expression allows us to write (2.3.36) as

$$\sigma(\Psi) = \begin{pmatrix} \Phi_1 M_1 & \Phi_1[M_1 M + \int \Phi_1^{-1} C_2 \Phi_2 dt M_2] \\ 0 & \Phi_2 M_2 \end{pmatrix}$$

which together with (2.3.37) gives $M_3 = M_1 M$. In order to know the matrix M more specifically we calculate $\Phi_1^{-1} C_2 \Phi_2$.

$$\Phi_1^{-1} C_2 \Phi_2 = \begin{pmatrix} \sqrt{2}i - 2iI_1 \frac{(u^4+1)\dot{u}}{u^5} & \frac{u^2}{2(1-u^4)} & -I_1(2\sqrt{2}i - 2iI_1 \frac{(u^4+1)\dot{u}}{u^5}) \\ 0 & 0 & \frac{u^2}{2(1-u^4)} \\ \frac{2i(u^4+1)\dot{u}}{u^5} & 0 & \sqrt{2}i - 2iI_1 \frac{(u^4+1)\dot{u}}{u^5} \end{pmatrix}.$$

Then $\int V(t)dt = \int \Phi_1^{-1} C_2 \Phi_2 =$

$$\begin{pmatrix} \int \sqrt{2}i - 2iI_1 \frac{(u^4+1)\dot{u}}{u^5} dt & \int \frac{u^2}{2(1-u^4)} dt & \int -I_1(2\sqrt{2}i - 2iI_1 \frac{(u^4+1)\dot{u}}{u^5})dt \\ 1 & 1 & \int \frac{u^2}{2(1-u^4)} dt \\ \int \frac{2i(u^4+1)\dot{u}}{u^5} dt & 1 & \int \sqrt{2}i - 2iI_1 \frac{(u^4+1)\dot{u}}{u^5} dt \end{pmatrix}.$$

Hence, the extension L_2 in (2.3.35) is obtained adjoining four quadratures

$$L_2 = L_1(\int v_{11}, \int v_{12}, \int v_{13}, \int v_{31}).$$

The matrix M in (2.3.37) is in the form

$$M = \begin{pmatrix} \beta & \gamma & \delta \\ 0 & 0 & \gamma \\ \nu & 0 & \beta \end{pmatrix}, \quad \beta, \gamma, \nu, \delta \in \mathbb{C}.$$

Then for the matrix $M_3 = M_1 M$ we obtain

$$M_3 = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & \gamma & \delta \\ 0 & 0 & \gamma \\ \nu & 0 & \beta \end{pmatrix} = \begin{pmatrix} \beta + \alpha\nu & \gamma & \delta + \alpha\beta \\ 0 & 0 & \gamma \\ \nu & 0 & \beta \end{pmatrix}.$$

This gives us that $G = Gal(L_2/K)$ is represented by the unipotent matrices R_σ and $G^0 = G$ (see Remark 1)

$$G^0 = \left\{ \begin{pmatrix} 1 & 0 & \alpha & \beta + \alpha\nu & \gamma & \delta + \alpha\beta \\ 0 & 1 & 0 & 0 & 0 & \gamma \\ 0 & 0 & 1 & \nu & 0 & \beta \\ 0 & 0 & 0 & 1 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha \neq 0, \beta, \gamma, \nu, \delta \in \mathbb{C} \right\}. \tag{2.3.39}$$

Recall that our aim is to prove that G^0 is not commutative. To do this without going into a detailed study of $Gal(L_2/K)$ it appears that we need to show (as one possibility) that $\nu \neq 0$. The complex number ν corresponds to adjoining the integral $\int v_{31} dt$ which is

$$\int v_{31} dt = 2i \int \frac{u^4+1}{u^5} \dot{u} dt = 2i \left(\ln u - \frac{1}{4u^4} \right).$$

The presence of the logarithmic term gives that $\nu \neq 0$.

Now, let R_1 and R_2 be any two matrices from (2.3.39). Then their commutator is $R_1 R_2 R_1^{-1} R_2^{-1} = \begin{pmatrix} Id & D \\ 0 & Id \end{pmatrix}$, where

$$D = \begin{pmatrix} \alpha_1\nu_2 - \alpha_2\nu_1 & 0 & 2\alpha_1\alpha_2(\nu_2 - \nu_1) + 2(\alpha_1\beta_2 - \alpha_2\beta_1) + \alpha_1^2\nu_2 - \alpha_2^2\nu_1 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_1\nu_2 - \alpha_2\nu_1 \end{pmatrix}$$

which clearly is not the zero matrix since α_j and ν_j, $j = 1, 2$ are not trivial, which we have already seen. Therefore G^0 is not a commutative group and from the Morales-Ramis theorem (Theorem 1.0.1) we can conclude that the Hamiltonian system with $a = b$ is not integrable. This completes the proof of this part of the Theorem 2.2.1.

2.3.2 1 : 2 : 4 resonance

Let $b > 0$. Recall that the equations (2.2.8) have first integrals $\bar{H} = h$ and $H_2 = F^2$. On the level $F = 0$, we have the following particular solution

$$p_1 = q_1 = 0, \quad q_3 = \frac{i}{\sqrt{2}} p_2, \quad p_3 = \frac{i}{\sqrt{2}} q_2, \qquad (2.3.40)$$
$$q_2 = \frac{C}{3} \frac{1}{\wp(\tau)}, \quad p_2 = -\frac{1}{2} \frac{\wp'(\tau)}{\wp(\tau)},$$

where $C = -\frac{ih\sqrt{2}}{b} \neq 0, \tau = ib\sqrt{2}t$ and $\wp$ is the Weirstrass elliptic function satisfying

$$\Gamma : \wp'^2 = 4\wp^3 - g_3, \quad g_3 = -\frac{4C^2}{27}. \qquad (2.3.41)$$

Let $dq_j = \xi_j, dp_j = \eta_j, j = 1, 2, 3$. Then $d\bar{H}$ does not depend on ξ_1, η_1 **since**

$$d\bar{H} = b[2(p_2 q_3 - q_2 p_3)\xi_2 + 2p_2 q_2 \xi_3 + 2(q_2 q_3 + p_2 p_3)\eta_2 + (p_2^2 - q_2^2)\eta_3].$$

The (NVE) along the particular solution (2.3.40) is written in ξ_1, η_1 variables. Take $' = d/d\tau$ and denote $\mu = a/b$. Thus (NVE) takes the form

$$Y' = A(\tau)Y, \qquad (2.3.42)$$

where $Y = (\xi_1, \eta_1)^t$, $A(\tau) = (a_{ij},\, i, j = 1, 2)$ and

$$a_{11} = -a_{22} = -\sqrt{2}\mu i \frac{C}{3\wp(\tau)}, \quad a_{12} = a_{21} = \sqrt{2}\mu i \frac{\wp'(\tau)}{2\wp(\tau)}.$$

The system (2.3.42) has three regular singular points in the parallelogram of the periods of $\wp$, namely $\tau = 0$ and $\tau = \tau_j$ - the zeros of $\wp(\tau)$ (note that $\tau_2 = -\tau_1$). Hence, it is Fuchsian. In a neighborhood of $\tau = 0$, the NVE can be presented as

$$\tau Y' = A^0(\tau)Y, \qquad (2.3.43)$$

where $Y = (\xi_1, \eta_1)^t$ and $A^0(\tau)$ is holomorphic at $\tau = 0$. As it was pointed out in section 3, if $A^0_0 = A^0(0)$ has no eigenvalues which differ from each other by positive integers, we can bring the system (2.3.43) to the form

$$\tau Z' = A^0_0 Z. \qquad (2.3.44)$$

Here A^0_0 is the matrix

$$A^0_0 = \begin{pmatrix} 0 & -i\sqrt{2}\mu \\ -i\sqrt{2}\mu & 0 \end{pmatrix}$$

with eigenvalues $\lambda_{1,2} = \pm i\sqrt{2}\mu$ which clearly do not differ by a positive integer. Then the local monodromy around $\tau = 0$ is $M_0 = e^{2\pi i A_0^0}$.

Similarly, in a neighborhood of $\tau = \tau_j$, we have

$$(\tau - \tau_j)Y' = A^j(\tau - \tau_j)Y, \tag{2.3.45}$$

where $A^j(\tau - \tau_j)$ are holomorphic at $\tau = \tau_j$. Denote for short $c_j = \frac{C}{3\wp'(\tau_j)}$ $(c_2 = -c_1)$. Then A_0^j are

$$A_0^j = \begin{pmatrix} -i\sqrt{2}\mu c_j & \frac{i\mu}{\sqrt{2}} \\ \frac{i\mu}{\sqrt{2}} & i\sqrt{2}\mu c_j \end{pmatrix}$$

with eigenvalues $\lambda_{1,2} = \pm i\sqrt{2}\mu$. Hence, by the above arguments the local monodromy matrices are $M_j = e^{2\pi i A_0^j}$.

Then it is a straightforward computation that $M_0 M_1 M_0^{-1} M_1^{-1} \neq Id$, if $\mu \neq 0$. To see this, we put $\kappa = \exp(2\sqrt{2}\pi\mu) \geq 1$ and $w = 2(c_1 + 1)$. We set also $U_1 = \kappa + 1/\kappa$, $U_2 = \kappa - 1/\kappa$, $U_3 = w^2\kappa + 1/\kappa$, $U_4 = w^2/\kappa + \kappa$. Then

$$M_0 M_1 M_0^{-1} M_1^{-1} = \frac{1}{4(w^2+1)^2} \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix},$$

where

$$\begin{aligned}
W_1 &= U_1^2 U_3 U_4 - U_2^2 U_4^2 + w^2 U_2^2 (U_2^2 - U_1^2) - w U_2^2 U_1 (U_3 - U_4), \\
W_2 &= U_2 (w U_2^2 - U_1 U_3)(U_3 - U_4), \\
W_3 &= -U_2 (w U_2^2 - U_1 U_4)(U_3 - U_4), \\
W_4 &= U_1^2 U_3 U_4 - U_2^2 U_3^2 + w^2 U_2^2 (U_2^2 - U_3^2) + w U_2^2 U_1 (U_3 - U_4).
\end{aligned}$$

It is immediate that when $\kappa = 1$ $(\mu = 0)$ the above commutator is the identity and $U_3 - U_4 = 0, U_2 = 0$ if and only if $\kappa = 1$. Assume that $W_2 = 0$ for some $\kappa \neq 1$. Then $wU_2^2 = U_1 U_3$ has to be satisfied. Substituting this expression in W_3 we get $W_3 = -U_2 U_1 (U_3 - U_4)^2$ which is not equal to zero by the assumption. Hence, the above commutator can not be the identity for any $\kappa \neq 1$.

Thus the monodromy group is not abelian. Since (2.3.42) is a Fuchsian equation, as we mentioned before, its monodromy group topologically generates the Galois group. It is clear that the monodromy matrices M_0, M_j are in the identity component. Therefore, the identity component of the Galois group of (2.3.42) is not abelian. From the Morales - Ramis theorem (Theorem 1.0.1) we can conclude that the Hamiltonian resonance 1 : 2 : 4 is not integrable except in the obvious cases. This finishes the proof of this part of Theorem 2.2.1.

2.3.3 1 : 1 : 2 resonance

The case of 1 : 1 : 2 resonance is treated in similar way to the previous one.

Recall that $a \geq b \geq 0$ in $\bar{H}$ (2.2.9). Let $b > 0$. On the level $H_2 = 0$ we have the following particular solution

$$\begin{aligned}
&p_2 = q_2 = 0, \quad q_1 = -i\sqrt{2}p_3, \quad p_1 = -i\sqrt{2}q_3, \\
&q_3 = -\frac{C}{3}\frac{1}{\wp(\tau)}, \quad p_3 = \frac{1}{2}\frac{\wp'(\tau)}{\wp(\tau)},
\end{aligned} \tag{2.3.46}$$

where $C = h/(2a) \neq 0, \tau = 2at,' = d/d\tau$ and $\wp$ satisfies (2.3.41).

The (NVE) along this particular solution are written in (ξ_2, η_2) variables and denoting $\mu = b/a \leq 1$ we get

$$Y' = A(\tau)Y, \tag{2.3.47}$$

where $Y = (\xi_2, \eta_2)^t$, $A(\tau) = (a_{ij},\ i,j = 1,2)$ and

$$a_{11} = -a_{22} = \frac{\mu}{2}\frac{\wp'(\tau)}{\wp(\tau)}, \quad a_{21} = a_{12} = \frac{\mu C}{3}\frac{1}{\wp(\tau)}.$$

The system (2.3.47) is Fuchsian with singular points 0 and $\tau_{1,2}$ - the zeros of $\wp$ in the its parallelogram of periods $(\tau_2 = -\tau_1)$.

In a neighborhood of $\tau = 0$, the (NVE) can be written as

$$\tau Y' = A^0(\tau)Y, \tag{2.3.48}$$

where A^0 is holomorphic at $\tau = 0$, $Y = (\xi_2, \eta_2)^t$. The residue matrix is $A_0^0 = diag(\mu, -\mu)$ (recall that $\mu \in [0,1]$). The difference of the eigenvalues 2μ is a positive integer only when $\mu = 1/2$ or $\mu = 1$. In these cases, as we know, we have an additional integral. So, when $\mu \neq 1/2$, 1, the system can be transformed to

$$\tau Z' = A_0^0 Z. \tag{2.3.49}$$

Hence, the local monodromy around $\tau = 0$ is $M_0 = e^{2\pi i A_0^0}$.

Similarly, in a neighborhood of $\tau = \tau_j$, we have

$$(\tau - \tau_j)Y' = A^j(\tau - \tau_j)Y, \tag{2.3.50}$$

where $A^j(\tau - \tau_j)$ are holomorphic at $\tau = \tau_j$. Denote for short $c_j = \frac{C}{3\wp'(\tau_j)}$ $(c_2 = -c_1)$. Then, $A_0^j = A^j(0)$ are

$$A_0^j = \begin{pmatrix} \frac{\mu}{2} & \mu c_j \\ \mu c_j & -\frac{\mu}{2} \end{pmatrix}$$

with eigenvalues $\lambda_{1,2} = \pm\mu$. By the above arguments we can transform the system (2.3.50) to the form

$$(\tau - \tau_j)Z' = A_0^j Z, \tag{2.3.51}$$

with the local monodromies $M_j = e^{2\pi i A_0^j}$. The straightforward computations give that $M_0 M_1 M_0^{-1} M_1^{-1} \neq Id$, if $\mu \neq 0, 1/2, 1$, but these cases correspond to integrable Hamiltonians. Indeed, using that $c_j^2 = 3/4$ and denoting this time $\kappa = \exp(2\pi i\mu)$, $U_2 = \kappa - 1/\kappa$, $U_3 = 3\kappa + 1/\kappa$ and $U_4 = 3/\kappa + \kappa$. we obtain

$$M_0 M_1 M_0^{-1} M_1^{-1} = \frac{1}{16}\begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix},$$

where

$$W_1 = U_3 U_4 - 3U_2^2\kappa^2, \qquad W_2 = 2c_1 U_2 U_3(\kappa^2 - 1),$$

$$W_3 = 2c_1 U_2 U_4\left(\frac{1}{\kappa^2} - 1\right), \qquad W_4 = U_3 U_4 - 3\frac{U_2^2}{\kappa^2}.$$

It is clear that the above commutator is the identity only when $\kappa = \pm 1$.

Hence, if $\mu \neq 0, 1/2, 1$ the monodromy group is not abelian and by the same arguments as at the end of the previous section, the identity component of the Galois group of (2.3.47) is not abelian. From Morales-Ramis theorem (Theorem 1.0.1) we can conclude that the Hamiltonian 1 : 1 : 2 resonance is not integrable by means of meromorphic integrals. This finishes the proof of this part and together with it that of the Theorem 2.2.1.

2.4 Conclusions

In this paper the integrability of genuine first order resonances in Hamiltonian systems with three degrees of freedom is studied. Unfortunately, no new integrable case appears. A theorem (Theorem 2.2.1) is proven, which says that the resonances 1 : 2 : 1, 1 : 2 : 3 and 1 : 2 : 4 are generically not integrable except in the cases found earlier. The provided rigorous proofs of the cases 1 : 2 : 3 and 1 : 2 : 4 answer the question of Verhulst [100]. Our main tool is the Ziglin - Morales - Ramis theory based on Differential Galois Theory. This approach affords slightly to enforce the theorem of Duistermaat [24] for 1 : 2 : 1 resonance to the class of meromorphic first integrals. In the case 1 : 2 : 4 and 1 : 2 : 1 we study the monodromy group of the corresponding linearized system to establish that the identity component of the Galois group is not abelian. In the case 1 : 2 : 3 we study the monodromy group as well as the Galois group to conclude non-integrability.

As always the question arises about the meaning of non-integrability to the dynamics.

In [75] p. 876 Morales-Ruiz, Ramis and Simó formulate a problem which can be explained as follows: Assume a complex analytical Hamiltonian is proved to be non-integrable by the methods of their theory (explained in Chapter 1).

Is it true that some transversal homoclinic orbit to an invariant object exists?

Recall the Hamiltonian 1 : 2 : 3 resonance considered in section 2.3. A detailed analysis of the orbits on the manifold $H_2 = 1, H_3 = 0$ carried out by Hoveijn and Verhulst in [41, 101] shows that the homoclinic orbits of the complex unstable stationary points ($a < b$) are not transverse. The stable and unstable manifolds are equal: the homoclinic orbits form a single-parameter family.

Our solution (2.3.11) is a heteroclinic one and belongs to the level $H_2 = F^2, H_3 = 0$ (as we mentioned before F is unessential provided $F \neq 0$). Theorem 2.2.1 says that the Hamiltonian 1 : 2 : 3 resonance is not integrable when $a < b$ in a neighborhood of (2.3.11). So, there is no transversal homoclinic orbit in a neighborhood of (2.3.11). Of course, such an orbit could exist elsewhere or could be related to another invariant object like a hyperbolic (or partially hyperbolic) periodic orbit or an invariant torus.

Chapter 3

Non-integrability of Some Higher-Order Painlevé Equations in the Sense of Liouville

In this chapter we study the equation

$$w^{(4)} = 5w''(w^2 - w') + 5w(w')^2 - w^5 + (\lambda z + \alpha)w + \gamma,$$

which is one of the higher-order Painlevé equations (i.e. equations in the polynomial class having the Painlevé property). Like the classical Painlevé equations, this equation admits a Hamiltonian formulation, Bäcklund transformations and families of rational and special functions.

We prove that this equation considered as a Hamiltonian system with parameters $\gamma/\lambda = 3k, \gamma/\lambda = 3k-1, k \in \mathbb{Z}$, is not integrable in Liouville sense by means of rational first integrals. To do that we use the Ziglin - Morales-Ruiz - Ramis approach.

Then we study the integrability of the second and third members of the $\mathrm{P_{II}}$-hierarchy. Again as in the previous case it turns out that the normal variational equations are particular cases of the generalized confluent hypergeometric equations whose differential Galois groups are non-commutative and hence, they are obstructions to integrability. The results of this chapter are published in [17].

3.1 Introduction

The Painlevé property for a system of differential equations is the property that its general solution is without movable critical points.

Let us given a nonlinear differential equation

$$F(x, y, y', \dots, y^{(n)}) = 0, \qquad x, y \in \mathbb{CP}^1,$$

where F is a polynomial with respect to $y, y', \dots, y^{(n)}$. Let $y = \varphi(x)$ be a solution of the above equation, which usually turns out to be a multivalued holomorphic function.

A *critical point* is a point of ramification of the solution $y = \varphi(x)$. The critical point is called *movable* if its position depends on the solution φ, that is, on the constants of integration. For example, the first order equations without movable critical points are the linear equations and the Riccati equations (see [29] and [110]). Equations with this property are called equations of Painlevé type.

In the beginning of 20th century Painlevé and Gambier investigated this property for second-order ordinary differential equations. They proved that there are fifty equations possessing the Painlevé property. Among them six equations turned out to be new at that time, now called classical Painlevé equations. Although derived in a pure mathematical way, the six Painlevé equations have appeared in many physical applications: in the description of nonlinear waves, in statistical mechanics, in the theory of quantum gravity, in topological field theory, in plasma physics, in the theory of random matrix models and so on.

The classical Painlevé equations have many remarkable properties, in particular they admit a Hamiltonian formulation. In [71] Morales-Ruiz asked the question about the integrability as Hamiltonian systems of classical Painlevé equations which have particular rational solutions. This question was answered affirmatively for P_{II} with values of the parameter $\alpha = n \in \mathbb{Z}$. For the recent development in the study of the non-integrability of the other Painlevé equations we refer to Stoyanova [93]. In a very recent paper Żołądek and Filipuk [224] have proved that the classical Painlevé equations do not admit a first integral that can be expressed in terms of elementary functions, except for some known cases of P_{III} and P_V.

It is natural to extend the question of integrability to the higher-order Painlevé equations. Consider the following fourth-order nonlinear ordinary differential equation

$$w^{(4)} = 5w''(w^2 - w') + 5w(w')^2 - w^5 + (\lambda z + \alpha)w + \gamma, \tag{3.1.1}$$

where λ, α, γ are complex parameters.

This equation appears as a group-invariant reduction of the modified Kaup-Kupershmidt (or Sawada-Kotera) equation, see for instance Hone [36], Kudryashov [55]. Then it appears as equation F-XVIII in the classification made by Cosgrove [21] of all fourth- and fifth-order equations with Painlevé property. It is also studied by V. Gromak [30] from different points of view. It is proven in [55], that (3.1.1) with $\lambda = 1, \alpha = 0$ has no polynomial first integral.

Like the classical Painlevé equations, this equation admits a Hamiltonian formulation, Bäcklund transformations and families of rational and special functions. For instance:

- when $\lambda = 0, \gamma \neq 0$ it is solved in terms of hyperelliptic functions;
- when $\lambda = 0, \gamma = 0$ it is solved via elliptic functions;
- when $\gamma = -\lambda/2$, $w(z)$ can be expressed in terms of two Painlevé I solutions.

Further, we assume that $\lambda \neq 0$.

The equation (3.1.1) possesses two families of rational solutions:

I) $\gamma/\lambda = 3k, k \in \mathbb{Z}$

$$k = 0,\ w = 0; \quad k = 1,\ w_{(1)} = -\frac{3\lambda}{\alpha + \lambda z};\ etc. \tag{3.1.2}$$

II) $\gamma/\lambda = 3k - 1, k \in \mathbb{Z}$

$$k = 0,\ w = \frac{\lambda}{\alpha + \lambda z}; \quad k = 1,\ w_{(1)} = -\frac{2\lambda}{\alpha + \lambda z};\ etc. \tag{3.1.3}$$

In fact, these two families are the only rational solutions of (3.1.1) (Gromak [30] Theorem 3).

Denote $q_1(z) := w(z)$, $\varepsilon^2 = 1$. Then the equation (3.1.1) can be presented as two equivalent $2 + 1/2$ degrees of freedom Hamiltonian systems with

$$H_\varepsilon = \frac{1}{2}p_2^2 + \frac{7-9\varepsilon}{12}q_2^3 + p_1 q_2 - \frac{1+3\varepsilon}{4}p_1 q_1^2 + \frac{3\varepsilon-1}{4}q_2(\lambda z + \alpha) + \left(\gamma + \frac{3\varepsilon-1}{4}\lambda\right) q_1. \tag{3.1.4}$$

The corresponding system of equations is ($' = d/dz$):

$$\begin{aligned} q_1' &= q_2 - \frac{3\varepsilon+1}{4}q_1^2, \quad p_1' = \frac{1+3\varepsilon}{2}p_1 q_1 - \gamma - \frac{3\varepsilon-1}{4}\lambda \\ q_2' &= p_2, \qquad\qquad\quad\; p_2' = -p_1 - \frac{7-9\varepsilon}{4}q_2^2 - \frac{3\varepsilon-1}{4}(\lambda z + \alpha). \end{aligned} \tag{3.1.5}$$

There exist Bäcklund transformations (see Gromak [30], p. 1022) T_1, T_2 and $T := T_2 T_1$ for the equation (3.1.1) acting on the parameters in the following way :

$$T_1(\lambda) = \lambda, \quad T_1(\alpha) = \alpha, \quad T_1(\gamma) = -\gamma - \lambda, \tag{3.1.6}$$

$$T_2(\lambda) = \lambda, \quad T_2(\alpha) = \alpha, \quad T_2(\gamma) = -\gamma + 2\lambda, \tag{3.1.7}$$

$$T(\gamma) = \gamma + 3\lambda, \qquad T^{-1}(\gamma) = \gamma - 3\lambda. \tag{3.1.8}$$

Gromak has shown that these Bäcklund transformations are birational. It is easy to see that they are canonical also.

We can extend in a natural way the Hamiltonian system (3.1.5) to a three degrees of freedom autonomous system by denoting $\hat{H}(q_1, q_2, z, p_1, p_2, F) := H_\varepsilon + F$. Then we have

$$\begin{aligned} \frac{dq_1}{ds} &= q_2 - \frac{3\varepsilon+1}{4}q_1^2, \quad \frac{dp_1}{ds} = \frac{1+3\varepsilon}{2}p_1 q_1 - \gamma - \frac{3\varepsilon-1}{4}\lambda, \\ \frac{dq_2}{ds} &= p_2, \qquad\qquad\quad\; \frac{dp_2}{ds} = -p_1 - \frac{7-9\varepsilon}{4}q_2^2 - \frac{3\varepsilon-1}{4}(\lambda z + \alpha), \\ \frac{dz}{ds} &= 1, \qquad\qquad\quad\;\; \frac{dF}{ds} = -\lambda\frac{3\varepsilon-1}{4}q_2. \end{aligned} \tag{3.1.9}$$

Our first result is the following

Theorem 3.1.1. *The Hamiltonian system (3.1.9) with parameters $\gamma/\lambda = 3k, \gamma/\lambda = 3k-1, k \in \mathbb{Z}$ is not integrable in the Liouville sense by means of rational first integrals.*

We use as a tool the Ziglin-Morales-Ruiz-Ramis Theorem. We obtain a particular solution of (3.1.9), write the normal variational equation and study its differential Galois group which appears to be large.

The chapter is organized as follows. In [47] N. Katz and O. Gabber have calculated the Galois groups of some classes of linear equations using purely algebraic arguments– global characterization of semisimple algebras. In section 3 we apply their result to prove Theorem 3.1.1. In section 4 we recover Katz's result about the Galois group for our particular linear equation by giving its topological generators. It turns out that the linear equations which appear here are the confluent generalized hypergeometric equations. We use the approach of Duval and Mitschi [26] for the calculation of the formal monodromy, exponential torus and Stokes matrices in the corresponding case.

In fact, the confluent generalized hypergeometric equations have appeared also along the study of other higher-order Painlevé equations. In section 5 we prove that the second and the third members of the P_{II}-hierarchy are non-integrable in the Hamiltonian context for some particular values of the parameters. Again the differential Galois groups of confluent generalized hypergeometric equations are obstructions to integrability. We conjecture that the higher members in the P_{II}-hierarchy satisfy the same property.

3.2 Proof of Theorem 3.1.1

Consider first the family of rational solutions (I). Take $\gamma/\lambda = 0$ or $k = 0$ and $w = 0$. Then it is straightforward to be seen that

$$w = q_1 = 0, \quad q_2 = 0, \quad p_2 = 0, \quad p_1 = \frac{1-3\varepsilon}{4}(\lambda s + \alpha), \quad z = s, \quad F = F_0 = const. \tag{3.2.10}$$

is a particular solution.

Let $\xi_j = dq_j, \eta_j = dp_j, j = 1, 2, \xi_3 = ds, \eta_3 = dF$ be the variations. The variational equation (VE) along the solution (3.2.10) takes the form

$$\begin{aligned} \xi_1' &= \xi_2, \qquad \eta_1' = \frac{1+3\varepsilon}{2}p_1\xi_1, \\ \xi_2' &= \eta_2, \qquad \eta_2' = -\eta_1 + \frac{1-3\varepsilon}{4}\lambda\xi_3, \\ \xi_3' &= 0, \qquad \eta_3' = -\lambda\frac{3\varepsilon-1}{4}\xi_2. \end{aligned} \tag{3.2.11}$$

Then the normal variational equation (NVE) is

$$\begin{aligned} \xi_1' &= \xi_2, \quad \eta_1' = \frac{1+3\varepsilon}{2}p_1\xi_1, \\ \xi_2' &= \eta_2, \quad \eta_2' = -\eta_1. \end{aligned} \tag{3.2.12}$$

Reducing (3.2.12) to a single equation yields

$$\xi_1^{(4)} = (\lambda z + \alpha)\xi_1. \tag{3.2.13}$$

After setting $z = 1/\tau$ we obtain

$$y^{(4)} + \frac{12}{\tau}y^{(3)} + \frac{36}{\tau^2}y^{(2)} + \frac{24}{\tau^3}y' - \frac{\lambda + \alpha\tau}{\tau^9}y = 0$$

from where $\tau = 0$ or $z = \infty$ is an irregular singular point. After changing the independent variable $z \to \lambda z + \alpha$ we get ($\partial = d/dz$)

$$L_1\xi_1 = 0, \quad L_1 = \partial^4 + az, \quad a := -1/\lambda^4. \tag{3.2.14}$$

The operator L_1 is usually called an Airy type operator. It is irreducible and it is obviously invariant under $\partial \to -\partial$, hence, L_1 is selfdual in the terminology of Katz [47]. In the same paper Katz [47] has found that the identity component of the Galois group of $L_1\xi_1 = 0$ is $G^0 = \mathrm{Sp}(4, \mathbb{C})$ which is clearly non-commutative. Therefore, by Theorem 1.0.1 the Hamiltonian system (3.1.9) is not integrable in a neighborhood of the particular solution (3.2.10).

Further we consider the second family of rational solutions (II). Take $\gamma/\lambda = -1$ or $k = 0$ and $w = \frac{1}{z+\alpha/\lambda}$. Then we have the following particular solution to (3.1.9)

$$\begin{aligned} w = q_1 &= \frac{1}{s+\frac{\alpha}{\lambda}}, \quad q_2 = \frac{3}{4}\frac{\varepsilon-1}{(s+\frac{\alpha}{\lambda})^2}, \quad p_2 = -\frac{3}{2}\frac{\varepsilon-1}{(s+\frac{\alpha}{\lambda})^3}, \\ p_1 &= \frac{1-3\varepsilon}{4}(\lambda s + \alpha), \quad z = s, \quad F = \frac{3\lambda}{4}\frac{1-\varepsilon}{s+\frac{\alpha}{\lambda}} + F_0. \end{aligned} \tag{3.2.15}$$

The (VE) along this solution is

$$\begin{aligned}\xi_1' &= \xi_2 - \frac{1+3\varepsilon}{2}q_1\xi_1, \quad \eta_1' = \frac{1+3\varepsilon}{2}(p_1\xi_1 + q_1\eta_1),\\ \xi_2' &= \eta_2, \qquad\qquad\qquad \eta_2' = -\eta_1 + \frac{1-3\varepsilon}{4}\lambda\xi_3 + \frac{9\varepsilon-7}{2}q_2\xi_2,\\ \xi_3' &= 0, \qquad\qquad\qquad\ \ \eta_3' = -\lambda\frac{3\varepsilon-1}{4}\xi_2.\end{aligned} \tag{3.2.16}$$

Then the (NVE) becomes

$$\begin{aligned}\xi_1' &= \xi_2 - \frac{1+3\varepsilon}{2}q_1\xi_1, \quad \eta_1' = \frac{1+3\varepsilon}{2}(p_1\xi_1 + q_1\eta_1),\\ \xi_2' &= \eta_2, \quad \eta_2' = -\eta_1 + \frac{9\varepsilon-7}{2}q_2\xi_2.\end{aligned} \tag{3.2.17}$$

Again reducing (3.2.17) to a single equation gives

$$\xi_1^{(4)} - \frac{10}{(z+\frac{\alpha}{\lambda})^2}\,\xi_1'' + \frac{20}{(z+\frac{\alpha}{\lambda})^3}\,\xi_1' - \left[\frac{20}{(z+\frac{\alpha}{\lambda})^4} + \lambda\left(z+\frac{\alpha}{\lambda}\right)\right]\xi_1 = 0. \tag{3.2.18}$$

We make some transformations in order to put this equation in appropriate form. First we shift the independent variable $z+\frac{\alpha}{\lambda} \to z$. Then we make a change $x = \frac{\lambda z^5}{5^4}$, which is a finite branched covering map $\mathbb{CP}^1 \to \mathbb{CP}^1$. In general, the differential Galois group is changed under such a transformation, but the identity component remains unchanged (see [70] p. 28). As a result we get

$$\xi_1^{(4)} + \frac{24}{5x}\,\xi_1^{(3)} + \frac{86}{25x^2}\,\xi_1'' + \frac{4}{125x^3}\,\xi_1' - \left(\frac{4}{125x^4} + \frac{1}{x^3}\right)\xi_1 = 0.$$

Finally, we put $u := x^{2/5}\xi_1$ from where denoting $\delta = xd/dx$ we obtain

$$L_2u := \delta\left(\delta - \frac{2}{5} - 1\right)\left(\delta + \frac{1}{5} - 1\right)\left(\delta + \frac{2}{5} - 1\right)u - xu = 0. \tag{3.2.19}$$

The operator L_2 in (3.2.19) is a particular case of so-called Kloosterman operators

$$\mathrm{Kl} = \prod_1^n(\delta - a_i) + \lambda x \tag{3.2.20}$$

with $n = 4, \lambda = -1, a_1 = 0, a_2 = 7/5, a_3 = 4/5, a_4 = 3/5$. Katz has found that (see [47] Theorem 4.5.3, pp. 59-60) the identity component of the Galois group of $L_2u = 0$ is $G^0 = \mathrm{Sp}(4,\mathbb{C})$ which is noncommutative. Hence, by Theorem 1.0.1 the Hamiltonian system (3.1.9) is not integrable in a neighborhood of the particular solution (3.2.15).

To finish the proof note that having the variable $q_1 = w$ we can obtain the other phase variables from the equations (3.1.5). Recall that the equation (3.1.1) has rational solutions only for $\gamma/\lambda = 3k$ and $\gamma/\lambda = 3k-1$, $k \in \mathbb{Z}$ and therefore, the Hamiltonian system (3.1.9) has particular rational solutions for these values of the parameters.

For the first family (I) $\gamma/\lambda = 3k, k \in \mathbb{Z}$ we can relate the solution $w = 0$ for $\gamma/\lambda = 0 (k = 0)$ and the corresponding rational solution $w_{(k)}$ for $\gamma/\lambda = 3k$ via the Bäcklund transformation $T^k, k \in \mathbb{Z}$ (3.1.8). Since these transformations acting on the phase coordinates are birational (and canonical), the non-integrability of the Hamiltonian system (3.1.9) for $\gamma/\lambda = 0 (k = 0)$ by means of rational first integrals implies the non-integrability of the corresponding Hamiltonian systems for $\gamma/\lambda = 3k$, k is any integer. Applying the same arguments to the rational solutions of the second family (II), we conclude the non-integrability of the Hamiltonian systems for $\gamma/\lambda = 3k-1, k \in \mathbb{Z}$. This ends the proof of Theorem 3.1.1. ■

3.3 Stokes matrices

In this section we explicitly compute the differential Galois group of (3.2.19) using the approach taken by Duval and Mitschi [26, 68, 69] based on obtaining the topological generators of the Galois group, namely the formal and analytical invariants of the equation. We focus only on the equation (3.2.19) because the other equation (3.2.14) after the change of the independent variable $x = z^5/(5^4\lambda^4)$ becomes

$$\delta\left(\delta+\frac{2}{5}-1\right)\left(\delta+\frac{3}{5}-1\right)\left(\delta+\frac{4}{5}-1\right)\xi_1 - x\xi_1 = 0,$$

which is of similar kind as (3.2.19).

The following equation

$$\mathrm{D}_{qp}(y) = \left[(-1)^{q-p}x\prod_{j=1}^{p}(\delta+\mu_j) - \prod_{j=1}^{q}(\delta+\nu_j-1)\right]y = 0, \tag{3.3.21}$$

is called generalized confluent hypergeometric equation since it generalizes the classical confluent Kummer equation. Here $\delta = xd/dx$, $0 \le p \le q, \mu_j, \nu_j \in \mathbb{C}, \mu_i - \mu_j \notin \mathbb{Z}$. For this equation the point 0 is a regular singularity and ∞ is an irregular singularity, assuming $p < q$. For such kind of equations the local Galois group G_0 is a subgroup of G_∞, so the global Galois group is $G = G_\infty$. In what follows we need some notations:

1) $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$;

2) $\langle\underline{\alpha}\rangle_m = \prod_{j=1}^n \alpha_j(\alpha_j+1)\dots(\alpha_j+m-1)$;

3) For $a \in \mathbb{C}^p, b \in (\mathbb{C} \setminus \mathbb{Z}^-)^q$, let :

$$ {}_pF_q(\underline{a};\underline{b}\,|x) = \sum_{n\ge 0}\frac{\langle\underline{a}\rangle_n}{\langle\underline{b}\rangle_n}\frac{x^n}{n!} \tag{3.3.22}$$

be the generalized hypergeometric series and

$$G_{pq}^{m\,n}\left(x\,\middle|\,\frac{\underline{a}}{\underline{b}}\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^q\Gamma(a_j-s)}x^s d\,s, \tag{3.3.23}$$

be the Meijer G-function [66] and C is a suitable path in the complex plane. It is proven that ${}_pF_q$ can be expressed as a G-function ;

4) Let $\widetilde{\mathbb{C}}$ be the Riemann surface of the logarithm, $l_1, l_2 \in \mathbb{R} : l_1 < l_2$. By a sector we understand the following set:

$$\theta(l_1, l_2) := \{x \in \widetilde{\mathbb{C}} |\arg x \in (l_1, l_2)\};$$

5) $\sigma = q - p, \quad \zeta = e^{2\pi i/\sigma}, \quad \lambda = \frac{1}{2}(\sigma + 1) + \sum_{j=1}^{p} \mu_j - \sum_{j=1}^{q} \nu_j$.

If $(\nu_j - \mu_k) \notin \mathbb{Z}$ for $j = 1, \ldots, q$ and $k = 1, \ldots, p$, there exists a basis of solutions to (3.3.21) near $x = 0$ given in terms of $G^{1\,p}_{p\,q}$ or ${}_pF_{q-1}$. Similarly, in a neighborhood of $x = \infty$ there exists a fundamental system of solutions expressed in terms of $G^{q\,1}_{p\,q}$ and $G^{q\,0}_{p\,q}$ (see Meijer, Luke [66, 58]). We specialize them for our particular case.

Duval and Mitschi have calculated the formal invariants (formal monodromy and exponential torus) and the analytic invariants (Stokes matrices) for all families of the equations (3.3.21) assuming $p \geq 1$. Their calculations can be adapted to the case $p = 0$ in which we are. It is obvious that the Kloosterman equation is nothing but a D_{q0} type equation. Note that these equations are generically irreducible over $\mathbb{C}(x)$ (see Beukers, Brownawell, Heckmann [12] p. 297 and Duval, Mitschi [26] p. 41 for the proof).

In the rest of the section we carry out detailed calculations in our particular case for the reader's convenience.

Let us rewrite L_2 from (3.2.19) in the form

$$L_2 = (\delta + \nu_1 - 1)(\delta + \nu_2 - 1)(\delta + \nu_3 - 1)(\delta + \nu_4 - 1) - x \tag{3.3.24}$$

which is of type $\mathrm{D}_{4\,0}$ with $\nu_1 = 1, \nu_2 = -2/5, \nu_3 = 1/5, \nu_4 = 2/5$.

A fundamental system of solutions near $x = 0$ of $L_2\xi_1 = 0$ is

$$\left\{ x^{7/5}{}_0F_3\left(; \frac{8}{5}, \frac{9}{5}, \frac{12}{5} \middle| x\right), x^{3/5}{}_0F_3\left(; \frac{1}{5}, \frac{4}{5}, \frac{8}{5} \middle| x\right), {}_0F_3\left(; -\frac{2}{5}, \frac{1}{5}, \frac{2}{5} \middle| x\right), x^{4/5}{}_0F_3\left(; \frac{2}{5}, \frac{6}{5}, \frac{9}{5} \middle| x\right) \right\}. \tag{3.3.25}$$

Then the monodromy M_0 around $x = 0$ becomes

$$M_0 = \mathrm{diag}(e^{2\pi i 2/5}, e^{2\pi i 3/5}, 1, e^{2\pi i 4/5}). \tag{3.3.26}$$

Since 0 is regular singular, the monodromy generates the local Galois group G_0.

Recall that in our case we have $\sigma = 4, \zeta = i, \lambda = 5/2 - 6/5$. We prefer using letters ζ and λ instead of their particular values.

Let us turn to the description of the local Galois group G_∞. To define a fundamental system near $x = \infty$ we need one more function. Let C be a path in the complex plane, connecting $-i\infty$ and $i\infty$ and enclosing the points $n - \nu_j, j = 1, \ldots, 4$; $n \in \mathbb{N}$. The following function

$$\mathbb{G}_0(x) := G^{4\,0}_{0\,4}\left(x \,\middle|_{\underline{\nu}}\right) = \frac{1}{2\pi i}\int_C \Gamma(1 - \underline{\nu} - s)x^s ds \tag{3.3.27}$$

is a solution of $L_2\xi_1 = 0$, holomorphic in $\theta(-2\pi, 2\pi)$. The analytic continuation of $\mathbb{G}_0$ in the sector $\theta(-5\pi, 5\pi)$ admits the following asymptotic expansion at $x \to \infty$ (Propositions 1.2, 1.3 Duval, Mitschi [26] or Luke [58])

$$e^{-4x^{1/4}} x^{\lambda/4}\Theta(x), \tag{3.3.28}$$

where Θ is a formal series in $x^{-1/4}$. It is straightforward that

$$\mathbb{G}_0(xe^{-2\pi i h}), \quad h \in \mathbb{Z} \tag{3.3.29}$$

are also solutions of the same equation. In order to get a fundamental solution near $x = \infty$, one needs to pick four of them in our case. Next, we need a particular version of Meijer's Expansion Theorem which expresses every G-function as a finite sum of the functions $\mathbb{G}_0(xe^{-2\pi i h})$.

Proposition 3.3.1. *(see [26] Proposition 1.5). Let $x \in \theta(3\pi, 5\pi)$. Then the following identity holds*

$$\mathbb{G}_0(x) = A_1 \mathbb{G}_0(xe^{-2\pi i}) + A_2 \mathbb{G}_0(xe^{-4\pi i}) + B_0 \mathbb{G}_0(xe^{-8\pi i}) + B_1 \mathbb{G}_0(xe^{-6\pi i}), \tag{3.3.30}$$

where

$$A_h = -\frac{d^h}{dx^h}\left(\langle 1 - xe^{-2\pi i \underline{\nu}}\rangle_1\right)_{x=0}, \quad B_h = e^{2\pi i \lambda}\frac{d^h}{dx^h}\left(\langle 1 - xe^{2\pi i \underline{\nu}}\rangle_1\right)_{x=0}. \tag{3.3.31}$$

The formal solutions of D_{qp} at ∞ are known [12, 58] and can be verified by computer in our particular case. It is more convenient from now on to use a new variable $t = x^{1/4}$. Denote by $\underline{L}_2$ the equation obtained after the change of variable. Then we have

$$\begin{aligned} \underline{\Theta}_{-1}(t) &= e^{-4\zeta^{-1}t} t^{\lambda} \underline{\Theta}(\zeta^{-1}t), \quad & \underline{\Theta}_0(t) &= e^{-4\zeta^{0}t} t^{\lambda} \underline{\Theta}(\zeta^{0}t), \\ \underline{\Theta}_{1}(t) &= e^{-4\zeta^{1}t} t^{\lambda} \underline{\Theta}(\zeta^{1}t), \quad & \underline{\Theta}_2(t) &= e^{-4\zeta^{2}t} t^{\lambda} \underline{\Theta}(\zeta^{2}t), \end{aligned} \tag{3.3.32}$$

where $\underline{\Theta}(t) \in \mathbb{C}[[t^{-1}]]$. We denote the basis of formal solutions at ∞ of $\underline{L}_2$ by

$$\underline{\Sigma}(t) = \{\underline{\Theta}_{-1}(t), \underline{\Theta}_0(t), \underline{\Theta}_1(t), \underline{\Theta}_2(t)\}.$$

In this basis the formal monodromy is $\underline{\Sigma}(\zeta t) = \underline{\Sigma}(t)\widehat{M}_\infty$:

$$\widehat{M}_\infty = e^{2\pi i \lambda/4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.3.33}$$

Since in our case

$$q_{-1}(t) = -4\zeta^{-1}t,\; q_0(t) = -4\zeta^{0}t,\; q_1(t) = -4\zeta^{1}t,\; q_2(t) = -4\zeta^{2}t, \tag{3.3.34}$$

the exponential torus is $\mathcal{T} \cong (\mathbb{C}^*)^2$ and can be presented by

$$\mathcal{T} = \{\mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})\}, \quad t_1, t_2 \in \mathbb{C}^*. \tag{3.3.35}$$

The Stokes rays can easily calculated from (3.3.34) to be

$$\arg t = n\frac{\pi}{4}, \quad n = 0, \ldots, 7. \tag{3.3.36}$$

Similarly the singular rays d_s, i.e., the rays bisecting the sectors $\mathrm{Re}(q_i(t) - q_j(t)) < 0$ turn out to be the same as (3.3.36).

Let us define the sectors

$$\theta_n = \theta\left(-\frac{\pi}{2} + \frac{n-1}{4}\pi, \frac{\pi}{2} + \frac{n}{4}\pi\right), \quad n = 0, \ldots, 7. \tag{3.3.37}$$

The following Proposition is proven by Ramis [82] for the general confluent hypergeometric equation D_{qp}. We reformulate it for our particular case.

Proposition 3.3.2. *For every sector $\theta_n, n \in [0, 1, 2, \ldots, 7]$, there exists a unique basis of solutions $\Sigma_n(t)$ of $\underline{L}_2$ in θ_n with asymptotic expansion $\underline{\Sigma}(t)$ at ∞.*

This solution corresponds to "summation" of $\underline{\Sigma}(t)$ along a direction in the sector $\theta\left(\frac{n-1}{4}\pi, \frac{n}{4}\pi\right)$. As we will see in the sequel we won't need summation because there exist fundamental systems of actual solutions in θ_n.

In these notations the Stokes matrix corresponding to the singular ray $\frac{n}{4}\pi$, $n \in [0, 1, \ldots, 7]$ is defined via

$$\Sigma_n(t) = \Sigma_{n+1}(t) S_n, \qquad t \in \theta_n \bigcap \theta_{n+1}. \tag{3.3.38}$$

Proposition 3.3.3. *Suppose n and n' belong to $[0, 1, \ldots, 7]$ and $n' - n = 2$. Then*

$$\Sigma_{n'}(\zeta t) = \Sigma_n(t) \widehat{M}_\infty.$$

Proof. If $t \in \theta_n$, then $\zeta t \in \theta_{n'}$ and $\Sigma_{n'}(\zeta t)\widehat{M}_\infty^{-1}$ is a basis of solutions to L_2 in θ_n which admits the asymptotic expansion $\underline{\Sigma}_n(t)$. The uniqueness from Proposition 3.3.2 gives the desired result.

□

Proposition 3.3.4. *Let $n \in [0, 1, \ldots, 7]$ and $n = 2m + r$. Then*

$$S_n = \widehat{M}_\infty^{-m} S_r \widehat{M}_\infty^m.$$

Proof. From the relation (3.3.38) after changing variables we obtain $\Sigma_n(\zeta t) = \Sigma_{n+1}(\zeta t) S_n$. Proposition 3.3.3 gives that

$$\Sigma_{n-2}(t)\widehat{M}_\infty = \Sigma_{n+1-2}(t)\widehat{M}_\infty S_n.$$

This procedure repeated m times yields

$$\Sigma_r(t)\widehat{M}_\infty^m = \Sigma_{r+1}(t)\widehat{M}_\infty^m S_n.$$

But by definition we have $\Sigma_r(t) = \Sigma_{r+1}(t) S_r$ from where the result is immediate.

□

This Proposition reduces the calculation of the Stokes matrices to S_0 and $S_1 := S_{\pi/4}$ only.

Proposition 3.3.5. *The function $V(t) = \mathbb{G}_0(t^4)$ is asymptotic to $\underline{\Theta}_0$ in $\theta(-\frac{3\pi}{4}, \frac{5\pi}{4})$ when $t \to \infty$.*

This is a reformulation of (3.3.27) and (3.3.28) in terms of the variable t.

□

Proposition 3.3.6. *If $t \in \theta(\frac{3\pi}{4}, \frac{5\pi}{4})$, the following identity holds*

$$V(t) - e^{2\pi i\lambda} V(te^{-2\pi i}) = A_1 V(t\zeta^{-1}) + A_2 V(t\zeta^{-2}) + B_1 V(te^{-2\pi i}\zeta). \tag{3.3.39}$$

This is a version of the formula (3.3.30) in terms of the variable t ($B_0 = e^{2\pi i\lambda}, \zeta = e^{2\pi i/4}$).

□

Next, we find fundamental systems of actual solutions near ∞ only in $\theta_0, \theta_1, \theta_2$ since we need only S_0 and S_1.

Proposition 3.3.7. *(see Proposition 4.8 of [26]) Let $n = 0, 1, 2$. The following sets of solutions form fundamental systems of actual solutions in θ_n :*

$$\Sigma_n(t) := \{Y_{n,j}(t),\ j \in \{-1, 0, 1, 2\}\},$$

where

$$\begin{aligned} Y_{n,-1}(t) &= \zeta^{\lambda}V(t\zeta^{-1}), & n=0,1,2, \quad (3.3.40)\\ Y_{n,0}(t) &= V(t), & n=0,1,2, \quad (3.3.41)\\ Y_{n,1}(t) &= \zeta^{-\lambda}V(t\zeta), & n=0,1. \quad (3.3.42) \end{aligned}$$

$$Y_{0,2}(t) = \begin{cases} \zeta^{-2\lambda}\big[V(t\zeta^2) - A_1V(t\zeta)\big], & t\in\theta(-\frac{3\pi}{4},\frac{\pi}{4}),\\ \zeta^{-2\lambda}\big[e^{2\pi i\lambda}V(te^{-2\pi i}\zeta^2) + A_2V(t) + B_1V(t\zeta^{-1})\big], & t\in\theta(-\frac{\pi}{4},\frac{\pi}{2}). \end{cases} \quad (3.3.43)$$

$$Y_{1,2}(t) = \begin{cases} \zeta^{-2\lambda}\big[V(t\zeta^2) - A_1V(t\zeta) - A_2V(t)\big], \; t\in\theta(-\frac{\pi}{2},\frac{\pi}{4}),\\ \zeta^{-2\lambda}\big[e^{2\pi i\lambda}V(te^{-2\pi i}\zeta^2) + B_1V(t\zeta^{-1})\big], \; t\in\theta(-\frac{\pi}{4},\frac{3\pi}{4}). \end{cases} \quad (3.3.44)$$

$$Y_{2,1}(t) = \begin{cases} \zeta^{-\lambda}\big[V(t\zeta) - A_1V(t)\big], \; t\in\theta(-\frac{\pi}{4},\frac{3\pi}{4}),\\ \zeta^{-\lambda}\big[e^{2\pi i\lambda}V(te^{-2\pi i}\zeta) + A_2V(t\zeta^{-1}) + B_1V(t\zeta^{-2})\big], \; t\in\theta(\frac{\pi}{4},\pi). \end{cases} \quad (3.3.45)$$

$$Y_{2,2}(t) = \begin{cases} \zeta^{-2\lambda}\big[V(t\zeta^2) - A_1V(t\zeta) - A_2V(t) - B_1V(t\zeta^{-1})\big], \; t\in\theta(-\frac{\pi}{4},\frac{\pi}{4}),\\ \zeta^{-2\lambda}e^{2\pi i\lambda}V(te^{-2\pi i}\zeta^2), & t\in\theta(-\frac{\pi}{4},\pi). \end{cases} \quad (3.3.46)$$

Proof. Rewriting the formula (3.3.29) in terms of t gives us solutions of $\underline{L}_2$

$$V(t\zeta^{-h}), h\in\mathbb{Z}.$$

Using Proposition 3.3.6 we combine some of them in order to obtain proper asymptotic in θ_n. It remains to verify the validity of the formulas (3.3.43), (3.3.44), (3.3.45), (3.3.46) in the intersection of their definition intervals. We check only (3.3.43) since the rest are treated in the same way. So, we need to verify that

$$V(t\zeta^2) - A_1V(t\zeta) = e^{2\pi i\lambda}V(te^{-2\pi i}\zeta^2) + A_2V(t) + B_1V(t\zeta^{-1})$$

is valid in $\theta(-\frac{\pi}{4},\frac{\pi}{4})$. Using (3.3.39) from Proposition 3.3.6 ($\zeta^4 = e^{2\pi i}$) and making the change $t\to t\zeta^2$ gives the needed result.

□

Denote by E_{ij} the square matrix with elements 1 at i,j place and zeroes elsewhere.

Proposition 3.3.8. *(see Theorem 5.2 [26]) The Stokes matrices S_0 and S_1 are given by the following formulas :*

$$\begin{aligned} S_0 &= \mathbb{I} + \zeta^{-2\lambda}A_2E_{24}, \\ S_1 &= \mathbb{I} + \zeta^{-\lambda}A_1E_{23} + e^{-i\pi\lambda}\zeta^{-\lambda}B_1E_{14}. \end{aligned} \quad (3.3.47)$$

Proof. By definition $\Sigma_n(t) = \Sigma_{n+1}(t)S_n$, $t\in\theta_n\bigcap\theta_{n+1}$. For $t\in\theta_0\bigcap\theta_1 = \theta(-\frac{\pi}{2},\frac{\pi}{2})$ from Proposition 3.3.7 we get

$$\begin{aligned} Y_{0,-1}(t) &= Y_{1,-1}(t), \quad & Y_{0,0}(t) &= Y_{1,0}(t),\\ Y_{0,1}(t) &= Y_{1,1}(t), \quad & Y_{0,2}(t) &= Y_{1,2}(t) + \zeta^{-2\lambda}A_2Y_{1,0}(t). \end{aligned}$$

Then

$$\{Y_{0,-1}(t), Y_{0,0}(t), Y_{0,1}(t), Y_{0,2}(t)\} = \{Y_{1,-1}(t), Y_{1,0}(t), Y_{1,1}(t), Y_{1,2}(t)\}S_0$$

gives S_0. Similarly, for $t \in \theta_1 \bigcap \theta_2 = \theta(-\frac{\pi}{4}, \frac{3\pi}{4})$ we have

$$\begin{array}{ll} Y_{1,-1}(t) = Y_{2,-1}(t), & Y_{1,0}(t) = Y_{2,0}(t), \\ Y_{1,1}(t) = Y_{2,1}(t) + \zeta^{-\lambda} A_1 Y_{2,0}, & Y_{1,2}(t) = Y_{2,2}(t) + \zeta^{-2\lambda} B_1 \zeta^{-\lambda} Y_{2,-1}(t). \end{array}$$

Then

$$\{Y_{1,-1}(t), Y_{1,0}(t), Y_{1,1}(t), Y_{1,2}(t)\} = \{Y_{2,-1}(t), Y_{2,0}(t), Y_{2,1}(t), Y_{2,2}(t)\} S_1$$

gives S_1 since $\zeta^{-2\lambda} = e^{-\pi i \lambda}$.

□

From the above Proposition we know the Stokes matrices S_0 and S_1. In our case easy calculations give that

$$S_0 = \mathbb{I} + aE_{24}, \quad S_1 = \mathbb{I} + bE_{23} + cE_{14}, \tag{3.3.48}$$

where $a = 2i$, $c = i\zeta^{-\lambda} e^{2\pi i/5}$, $b = ic$.

Then Proposition 3.3.4 gives the other Stokes matrices obtained from S_0, S_1 :

$$\begin{array}{c} S_2 = \mathbb{I} + aE_{13},\ S_4 = \mathbb{I} + aE_{42},\ S_6 = \mathbb{I} + aE_{31}, \\ S_3 = \mathbb{I} + cE_{12} + icE_{43},\ S_5 = \mathbb{I} + cE_{41} + icE_{32},\ S_7 = \mathbb{I} + cE_{34} + icE_{21}. \end{array} \tag{3.3.49}$$

Now consider the (connected) subgroup topologically generated by the Stokes matrices and the exponential torus $G_s = \{S_j, \mathcal{T}\}$ which is normal in the Galois group G_∞ [68]. Hence, by Theorem 1.0.4, G_∞ is topologically generated by G_s and $\widehat{M}_\infty$. Let $\mathcal{G}_s$ be the Lie algebra of G_s ($\mathcal{G}_s \subset \mathrm{sl}(4, \mathbb{C})$).

To compute G_∞ we first determine the Lie algebra $\mathcal{G}_s$, then the corresponding connected subgroup G_s of G_∞ and after that we describe the action of $\widehat{M}_\infty$ on G_s (see [68]).

We will show that $\mathcal{G}_s \cong \mathrm{sp}(4, \mathbb{C})$. Denote $s_j \in \mathcal{G}_s$ such that $S_j = \exp s_j$. We have that $[s_2, s_6] = a^2 d_1, d_1 = E_{11} - E_{33}$ and $[s_0, s_4] = a^2 d_2, d_2 = E_{22} - E_{44}$. Then the Lie algebra $\mathcal{G}_s$ admits the following basis :

$$\mathcal{B} := \{s_0, s_2, s_4, s_6, d_1, d_2, s_1, s_3, s_5, s_7\}. \tag{3.3.50}$$

Hence, $\mathcal{G}_s$ consists of all matrices V such that $V^T J_1 + J_1 V = 0$, where J_1 is the following skew-symmetric matrix

$$J_1 = \begin{pmatrix} 0 & 0 & i\beta & 0 \\ 0 & 0 & 0 & \beta \\ -i\beta & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \beta^4 = -1.$$

Therefore, we get that $G_s \cong \mathrm{Sp}(4, \mathbb{C})$ (and $G^0 = \mathrm{Sp}(4, \mathbb{C})$).

Furthermore, $(\widehat{M}_\infty^k)^T J_1 \widehat{M}_\infty^k \neq J_1, k = 1, 2, 3, 4$, i.e., $\widehat{M}_\infty^k \notin G_s$, but $\widehat{M}_\infty^5$ already belongs to G_s. Hence, the formal monodromy generates a finite group G_M isomorphic to $\mathbb{Z}/5\mathbb{Z}$ acting nontrivially on G_s. This gives that G_∞ (and therefore G) is isomorphic to a non-trivial semidirect product $\mathrm{Sp}(4, \mathbb{C}) \rtimes \mathbb{Z}/5\mathbb{Z}$.

3.4 Non-integrability of the second and third members of the Painlevé II - hierarchy

We have proved that some particular fourth-order Painlevé equation is non-integrable in the Liouville sense for the set of parameters $\gamma/\lambda = 3k$ and $\gamma/\lambda = 3k - 1, k \in \mathbb{Z}$. It turns out that the normal variational equations along certain rational solutions are well-known generalized hypergeometric equations whose differential Galois groups can be found. Since generically these groups are large, the non-integrability comes from the Morales-Ruiz–Ramis theorem. In particular, our result implies that the Hamiltonian system corresponding to the considered fourth-order Painlevé equation does not possesses another rational first integral except the Hamiltonian. To see that we assume the contrary: there exists a rational first integral $I_1 \neq H$. Then I_1 gives rise to a rational integral of (NVE). This mean that (NVE) is reducible over $\mathbb{C}(x)$. But this is not the case for D_{q0} - contradiction.

We briefly mention an interesting relation concerning the linear equations that have appeared in this paper. Let X be a smooth complex projective Fano variety. One can define quantum differential equations on X (see e.g. [33, 22] and the references there for details). When the quantum equation is a linear ordinary differential equation Cruz Morales and van der Put [22] confirm Dubrovin's conjecture that the Gram matrix of X coincides with the Stokes matrix of the quantum differential equation (up to certain equivalence). It appears that for $X = \mathbb{P}^{n-1}$ the quantum differential operator is the Airy type operator $\delta^n - z$, and for the weighted projective spaces $\mathbb{P}(w_0, \dots, w_n)$ the quantum differential operator is of Kloosterman type or D_{q0} for certain q. The classical Stokes matrices are then computed for these operators using "multisummation" and the "monodromy identity" (see [22] for details).

It is interesting to note that generalized hypergeometric functions and generalized confluent hypergeometric equations are also related with other Painlevé equations. For instance, the classical dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-s)}{s} ds,$$

whose nontrivial monodromy plays an essential role in proving the non-integrability of some Painlevé VI equations studied by Horozov and Stoyanova [40] p. 626, is related to the generalized hypergeometric function as

$$\mathrm{Li}_2(z) = z\,{}_3F_2(1,1,1;2,2|z).$$

The polylogarithms Li_k have similar representations.

Let us turn our attention to other higher-order Painlevé equations which admit a Hamiltonian formulation. Consider the $\mathrm{P_{II}}$-hierarchy which is given by (see [65] and the references there)

$$\mathrm{P}_{\mathrm{II}}^{(n)} : \left(\frac{d}{dz} + 2w\right) \mathcal{L}_n[w' - w^2] + \sum_{l=1}^{n-1} \beta_l \left(\frac{d}{dz} + 2w\right) \mathcal{L}_l[w' - w^2] = zw + \alpha_n, \quad n \geq 1, \tag{3.4.51}$$

where $\mathcal{L}_n$ is the operator defined by the recursion relation (the Lenard relation)

$$\frac{d}{dz}\mathcal{L}_{n+1} = \left[\frac{d^3}{dz^3} + 4(w' - w^2)\frac{d}{dz} + 2(w' - w^2)_z\right] \mathcal{L}_n; \quad \mathcal{L}_0[w' - w^2] = \frac{1}{2} \tag{3.4.52}$$

and β_l and α_n are arbitrary complex parameters. (Denoting for short $u := u(z) := w' - w^2$, one gets consecutively $\mathcal{L}_1[u] = u, \mathcal{L}_2[u] = u'' + 3u^2, \mathcal{L}_3[u] = u^{(4)} + 10uu'' + 5(u')^2 + 10u^3$ and so on). A particular member of (3.4.51) is a nonlinear ODE of order $2n$, $n \geq 1$. Some authors consider all β_l to be trivial. The first three members of the $\mathrm{P_{II}}$-hierarchy are :

$$\mathrm{P}_{\mathrm{II}}^{(1)} : w'' - 2w^3 = zw + \alpha_1, \tag{3.4.53}$$

$$\mathrm{P}_{\mathrm{II}}^{(2)} : w^{(4)} - 10w(ww'' + w'^2) + 6w^5 + \beta_1(w'' - 2w^3) = zw + \alpha_2, \tag{3.4.54}$$

$$\begin{aligned}\mathrm{P}_{\mathrm{II}}^{(3)} : {}& w^{(6)} - 14w^{(4)}w^2 - 56w^{(3)}w'w + 70w''(w^4 - w'^2) + 140w^3w'^2 - 42w(w'')^2 \\ & -20w^7 + \beta_1[w'' - 2w^3] + \beta_2[w^{(4)} - 10w(ww'' + w'^2) + 6w^5] = zw + \alpha_3.\end{aligned} \tag{3.4.55}$$

The equation $\mathrm{P}_{\mathrm{II}}^{(2)}$ appears in Cosgrove [21], p. 58 as F-XVII.

We are interested in the integrability of the Hamiltonian systems corresponding to these equations. The Hamiltonian for $\mathrm{P}_{\mathrm{II}}^{(1)}$ was known long ago from Okamoto [79] (and also for the other classical Painlevé equations). The Hamiltonian structure for the P_{II}-hierarchy was found by Mazzocco and Mo [65]. We study the Liouville integrability of the Hamiltonian systems corresponding to the first three members of the P_{II}-hierarchy, which are "manageable".

Consider first the Hamiltonian for $\mathrm{P}_{\mathrm{II}}^{(1)}$, namely

$$H^{(1)} = 4p^2 + \frac{1}{4}q + \frac{1}{4}pq^2 + 2pz - \frac{1}{2}q\alpha_1, \tag{3.4.56}$$

where $q = 4w$, $p = \frac{1}{2}(w' - w^2 - \frac{z}{2})$. Extending (3.4.56) in a natural way to a two degrees of freedom autonomous Hamiltonian system $\hat{H}_1 = H^{(1)} + F$, one finds ($' = d/ds$) the corresponding equations

$$\begin{aligned} q' &= 8p + \frac{1}{4}q^2 + 2z, \qquad z' = 1, \\ p' &= -\frac{1}{4} - \frac{1}{2}pq + \frac{1}{2}\alpha_1, \quad F' = -2p. \end{aligned} \tag{3.4.57}$$

The system (3.4.57) admits the following particular solution when $\alpha_1 = 0$

$$q = 0, \quad p = -\frac{1}{4}s, \quad z = s, \quad F = \frac{s^2}{4} + F_0. \tag{3.4.58}$$

The (NVE) along (3.4.58) is

$$\xi_1'' = z\xi_1. \tag{3.4.59}$$

This is the Airy equation whose Galois group is $G = \mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Sp}(2,\mathbb{C})$. The Hamiltonian system (3.4.57) is therefore non-integrable with rational first integrals, but we know that from Morales-Ruiz [72].

Next we consider the Hamiltonian for $\mathrm{P}_{\mathrm{II}}^{(2)}$:

$$H^{(2)} = \frac{q_2}{16} + 2zp_2 - 16p_1^2p_2 + 16p_2^2 + \frac{q_1q_2p_2}{8} + \frac{p_1p_2q_2^2}{16} + \frac{\alpha_2(p_1q_2 - q_1)}{8} + \beta_1(8p_1 - t_1)p_2, \tag{3.4.60}$$

where $q_j, p_j, j = 1, 2$ are expressible via w and its derivatives. Extending as usual to a three degrees of freedom autonomous Hamiltonian system $\hat{H}_2 = H^{(2)} + F$ we get

$$\begin{aligned} q_1' &= -32p_1p_2 + \frac{1}{16}p_2q_2^2 + \frac{1}{8}q_2\alpha_2 + 8\beta_1p_2, \\ q_2' &= 2z - 16p_1^2 + 32p_2 + \frac{1}{8}q_1q_2 + \frac{1}{16}p_1q_2^2 + \beta_1(8p_1 - \beta_1), \\ p_1' &= -\frac{1}{8}p_2q_2 + \frac{1}{8}\alpha_2, \\ p_2' &= -\frac{1}{16} - \frac{1}{8}p_2q_1 - \frac{1}{8}p_1p_2q_2 - \frac{1}{8}\alpha_2p_1, \\ z' &= 1, \qquad F' = -2p_2. \end{aligned} \tag{3.4.61}$$

The system (3.4.61) admits the following particular solution when $\alpha_2 = 0$:

$$q_1 = q_2 = 0, \quad p_1 = \frac{\beta_1}{4}, \quad p_2 = -\frac{s}{16}, \quad z = s, \quad F = \frac{s^2}{16} + F_0. \tag{3.4.62}$$

The (NVE) along the solution (3.4.62) reduced to a single linear equation is

$$\xi_1^{(4)} - \frac{5}{s}\xi_1^{(3)} + \left(\frac{12}{s^2} - \frac{\beta_1 s}{16}\right)\xi_1'' + \left(\frac{\beta_1}{16} - \frac{12}{s^3}\right)\xi_1' - \frac{s^3}{256}\xi_1 = 0. \qquad (3.4.63)$$

Here we take the case $\beta_1 = 0$ which is simpler. After introducing the new independent variable $z = s^7/(2^8 7^4)$ the above equation becomes

$$\delta\left(\delta + \frac{2}{7} - 1\right)\left(\delta + \frac{3}{7} - 1\right)\left(\delta + \frac{5}{7} - 1\right)\xi_1 - z\xi_1 = 0, \qquad (3.4.64)$$

which is an equation of type $D_{4\,0}\,\xi_1 = 0$ with $\nu_1 = 1, \nu_2 = 2/7, \nu_3 = 3/7, \nu_4 = 5/7$. In a similar way as in Section 4 (or referring to Katz [47]) one obtains that the identity component of the Galois group of (3.4.64) is $G^0 = \mathrm{Sp}(4, \mathbb{C})$, which is not commutative. Hence, the Hamiltonian system corresponding to the higher-order Painlevé equation $\mathrm{P}_{\mathrm{II}}^{(2)}$ is not integrable in the Liouville sense.

Finally, let us write the Hamiltonian for $\mathrm{P}_{\mathrm{II}}^{(3)}$.

$$\begin{aligned} H^{(3)} &= 64p_1^4 - 192p_1^2p_2 + 128p_1p_3 + \frac{1}{64}p_3q_3^2 - \frac{1}{64}p_1q_2^2 + 64p_2^2 \\ &- \frac{1}{32}q_1q_2 + 2zp_1 + \frac{q_3}{64} - \frac{1}{32}\alpha_3q_3 + 8\beta_1(p_1^2 - p_2) \\ &+ \beta_2(4p_1^2\beta_2 - 4p_2\beta_2 - 32p_1^3 + 64p_1p_2 - 2p_1\beta_1). \end{aligned} \qquad (3.4.65)$$

Extending as usual to a four degrees of freedom autonomous Hamiltonian system $\hat{H}_3 = H^{(3)} + F$, we obtain

$$\begin{aligned} q_1' &= 256p_1^3 - 384p_1p_2 + 128p_3 - \frac{q_2^2}{64} + 2z + 16p_1\beta_1 + 8p_1\beta_2^2 - 96\beta_2p_1^2 + 64p_2\beta_2 - 2\beta_1\beta_2, \\ q_2' &= -192p_1^2 + 128p_2 - 8\beta_1 - 4\beta_2^2 + 64p_1\beta_2, \\ q_3' &= 128p_1 + \frac{1}{64}q_3^2, \\ p_1' &= \frac{1}{32}q_2, \\ p_2' &= \frac{1}{32}p_1q_2 + \frac{1}{32}q_1, \\ p_3' &= -\frac{1}{32}p_3q_3 - \frac{1}{64} + \frac{1}{32}\alpha_3, \\ z' &= 1, \quad F' = -2p_1. \end{aligned} \qquad (3.4.66)$$

Here we consider only the case $\beta_1 = \beta_2 = 0$. When $\alpha_3 = 0$ the system (3.4.66) admits the following particular solution

$$q_1 = q_2 = q_3 = 0, \quad p_1 = p_2 = 0, \quad p_3 = -\frac{s}{64}, \quad z = s, \quad F = F_0 = const. \qquad (3.4.67)$$

The (NVE) along the solution (3.4.67), reduced to a single linear equation becomes

$$\xi_1^{(6)} - \frac{4}{s}\xi_1^{(5)} + \frac{12}{s^2}\xi_1^{(4)} - \frac{24}{s^3}\xi_1^{(3)} + \frac{24}{s^4}\xi_1'' - s\xi_1 = 0. \qquad (3.4.68)$$

After changing the independent variable by $x = s^7/7^6$ we obtain

$$\delta\left(\delta+\frac{1}{7}-1\right)\left(\delta+\frac{2}{7}-1\right)\left(\delta+\frac{3}{7}-1\right)\left(\delta+\frac{4}{7}-1\right)\left(\delta+\frac{6}{7}-1\right)\xi_1 - x\xi_1 = 0, \tag{3.4.69}$$

which is an equation of type $D_{60}\,\xi_1 = 0$ with $\nu_1 = 1, \nu_2 = 1/7, \nu_3 = 2/7, \nu_4 = 3/7, \nu_5 = 4/7, \nu_6 = 6/7$. We proceed in a similar way as in Section 4. Here

$$\sigma = q - p = 6, \quad \zeta = e^{2\pi i/6}, \quad \lambda = \frac{1}{2} - \frac{2}{7}. \tag{3.4.70}$$

The local monodromy at 0 is clear. It is more convenient to use a new variable $t = x^{1/6}$. The basis of the formal solutions is straightforward (see for instance [12, 26]) and the formal monodromy is

$$\widehat{M}_\infty = e^{2\pi i\lambda/6}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.4.71}$$

The exponential torus is again $\mathcal{T} \cong (\mathbb{C}^*)^2$

$$\mathcal{T} = \{\mathrm{diag}(t_1, t_2, t_1^{-1}t_2, t_1^{-1}, t_2^{-1}, t_2^{-1}t_1)\}, \quad t_1, t_2 \in \mathbb{C}^*. \tag{3.4.72}$$

The Stokes rays and the singular rays are

$$\arg t = n\frac{\pi}{6}, \quad n = 0, \ldots, 11.$$

We define the sectors $\theta_n = \theta\left(-\frac{\pi}{2} + \frac{n-1}{6}\pi, \frac{\pi}{2} + \frac{n}{6}\pi\right), \quad n = 0, \ldots, 11$. Again we need only S_0 and $S_1 := S_{\pi/6}$ in order to obtain all Stokes matrices. The Stokes matrices S_0 and S_1 are given by (see Theorem 5.2 [26])

$$S_0 = \mathbb{I} + aE_{45} + bE_{36} + cE_{21}, \quad S_1 = \mathbb{I} + dE_{35} + fE_{26}, \tag{3.4.73}$$

where $a = \zeta^{-\lambda}A_1, b = \zeta^{-3\lambda}A_3\, c = \zeta^{\lambda}e^{-2\pi i\lambda}B_1,\ d = \zeta^{-2\lambda}A_2, f = \zeta^{-\lambda}e^{-\pi i\lambda}B_2$. Using (3.4.70) and (3.3.31) we get

$$f = i\zeta^{-\lambda}e^{2\pi i 2/7},\ d = f^2,\ c = \frac{i}{f},\ b = i,\ a = if\ (f^3 = -1).$$

The matrices S_0 and S_1 together with Proposition 3.3.4 for $n = 0, \ldots, 11$ give the rest of the Stokes matrices

$$\begin{aligned} S_2 &= \mathbb{I} + ifE_{34} + iE_{25} + \frac{i}{f}E_{16},\ S_3 = \mathbb{I} + f^2E_{24} + fE_{15},\ S_4 = \mathbb{I} + ifE_{23} + iE_{14} + \frac{i}{f}E_{65},\\ S_5 &= \mathbb{I} + f^2E_{13} + fE_{64},\ S_6 = \mathbb{I} + ifE_{12} + iE_{63} + \frac{i}{f}E_{54},\ S_7 = \mathbb{I} + f^2E_{62} + fE_{53},\\ S_8 &= \mathbb{I} + ifE_{61} + iE_{52} + \frac{i}{f}E_{43},\ S_9 = \mathbb{I} + f^2E_{51} + fE_{42},\\ S_{10} &= \mathbb{I} + ifE_{56} + iE_{41} + \frac{i}{f}E_{32},\ S_{11} = \mathbb{I} + f^2E_{46} + fE_{31}. \end{aligned} \tag{3.4.74}$$

Now consider again the (connected) subgroup topologically generated by the Stokes matrices and the exponential torus $G_s = \{S_j, \mathcal{T}\}$. Let $\mathcal{G}_s$ be the Lie algebra of G_s ($\mathcal{G}_s \subset \mathrm{sl}(6, \mathbb{C})$).

Proposition 3.4.1. $\mathcal{G}_s \cong \mathrm{sp}(6,\mathbb{C})$.

Proof. Denote again $s_j \in \mathcal{G}_s$ such that $S_j = \exp s_j, j = 0,\ldots,11$ and $\tau_1 = E_{11} - E_{33} - E_{44} + E_{66}$ and $\tau_2 = E_{22} + E_{33} - E_{55} - E_{66}$ which belong to Lie $\mathcal{T}$. Direct calculations yield

$$[s_0, s_3] = 2iE_{25}, \quad [s_0, s_9] = -2iE_{41}, \quad [s_6, s_9] = 2iE_{52},$$

$$[s_2, s_{11}] = -2iE_{36}, \quad [s_2, s_5] = 2iE_{14}, \quad [s_4, s_7] = 2iE_{63},$$

and hence, $E_{14}, E_{41}, E_{25}, E_{52}, E_{36}, E_{63} \in \mathcal{G}_s$.

Additionally we have that the elements

$$B_1 := E_{11} - E_{44} = [s_8, s_2] - \tau_2, B_2 := E_{22} - E_{55} = [s_9, s_3] - B_1, B_3 := E_{33} - E_{66} = B_1 - \tau_1,$$

and

$$B_4 := \frac{1}{f}E_{12} - E_{54} = \frac{i}{f^2}\left([s_6, \tau_2] - 2iE_{63}\right), B_5 := \frac{1}{f}E_{23} - E_{65} = -if^2\left([\tau_1, s_4] - 2iE_{14}\right),$$

$$B_6 := fE_{21} - E_{45} = -if^2\left([\tau_2, s_0] - 2iE_{36}\right), B_7 := fE_{32} - E_{56} = -if^2\left([s_{10}, \tau_1] - 2iE_{41}\right),$$

$$B_8 := \frac{1}{f^2}E_{16} + E_{34} = \frac{i}{f}\left([\tau_2, s_2] - 2iE_{25}\right), B_9 := \frac{1}{f^2}E_{61} + E_{43} = -if\left([s_8, \tau_2] - 2iE_{52}\right)$$

also belong to $\mathcal{G}_s$.

Then the Lie algebra $\mathcal{G}_s$ admits the following basis :

$$\mathcal{B} := \{E_{14}, E_{41}, E_{25}, E_{52}, E_{36}, E_{63}, s_1, s_3, s_5, s_7, s_9, s_{11}, B_j, j = 1,\ldots,9\}. \tag{3.4.75}$$

Hence, $\mathcal{G}_s$ consists of all matrices V such that $V^T J_1 + J_1 V = 0$, where J_1 is the following skew-symmetric matrix

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & f^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -f^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -f & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

from where we get the desired result.

□

Therefore, we get that $G_s \cong \mathrm{Sp}(6,\mathbb{C})$ (and $G^0 = \mathrm{Sp}(6,\mathbb{C})$).
Furthermore, $(\widehat{M_\infty^k})^T J_1 \widehat{M_\infty^k} \neq J_1, k = 1,\ldots,6$, i.e., $\widehat{M_\infty^k} \notin G_s$, but $\widehat{M_\infty^7}$ already belongs to G_s. Hence, the formal monodromy generates a finite group G_M isomorphic to $\mathbb{Z}/7\mathbb{Z}$ acting nontrivially on G_s. In this way we obtain that G_∞ (and therefore G) is isomorphic to a non-trivial semidirect product $\mathrm{Sp}(6,\mathbb{C}) \rtimes \mathbb{Z}/7\mathbb{Z}$.

Hence, the Hamiltonian system corresponding to the higher-order Painlevé equation $\mathrm{P}_{\mathrm{II}}^{(3)}$ is not integrable in the Liouville sense. Summarizing we have the following

Theorem 3.4.1. *Suppose that*

(i) $\beta_1 = \alpha_2 = 0$. Then the Hamiltonian system corresponding to $\mathrm{P}_{\mathrm{II}}^{(2)}$ is not integrable by means of rational first integrals;

(ii) $\beta_1 = \beta_2 = \alpha_3 = 0$. Then the Hamiltonian system corresponding to $\mathrm{P}_{\mathrm{II}}^{(3)}$ is not integrable by means of rational first integrals.

The study of the other members of the P_{II}-hierarchy is technically involved. However, we think that the (NVE) along certain nontrivial solutions reduced to single equations are of the type $D_{q\,0}\,\xi = 0$ with q even. Since the identity components of their differential Galois groups are $\mathrm{Sp}(q, \mathbb{C})$, the autonomous Hamiltonian systems corresponding to these equations are non-integrable.

The result of Theorem 3.4.1 can be extended to the entire orbits of parameters using Bäcklund transformations and other special solutions recently found by Sakka [86, 87]. This issue will be addressed elsewhere.

Chapter 4

On the integrability of a system describing the stationary solutions in Bose-Fermi mixtures

We study the integrability of a Hamiltonian system describing the stationary solutions in Bose–Fermi mixtures in one dimensional optical lattices. We prove that the system is integrable in the Liouville sense only when it is separable in three generic cases. The proof is based on the Differential Galois approach and the Ziglin-Morales-Ramis method. The results of this chapter are published in [18].

4.1 Introduction

In this chapter we study the integrability of the system that comes from the time dependent mean field equations of Bose–Fermi mixture (BFM) in one dimensional optical lattices. The interest in BFM arises after the discovery of Bose–Einstein Condensates (BEC) in 1995 and the desire to understand strongly interacting and strongly correlated systems, with applications in solid state physics, nuclear physics, astrophysics, quantum computing and nanotechnologies. For more detailed physical background of BFM we refer to [51, 88, 10, 14, 54] and the literature therein.

At mean field approximation we consider the following $N_f + 1$ coupled nonlinear Schrödinger equations

$$i\hbar\frac{\partial\Psi^b}{\partial t} + \frac{1}{2m_{\rm B}}\frac{\partial^2\Psi^b}{\partial x^2} - V\Psi^b - g_{\rm BB}|\Psi^b|^2\Psi^b - g_{\rm BF}\rho_f\Psi^b = 0, \tag{4.1.1}$$

$$i\hbar\frac{\partial\Psi_j^f}{\partial t} + \frac{1}{2m_{\rm F}}\frac{\partial^2\Psi_j^f}{\partial x^2} - V\Psi_j^f - g_{\rm BF}|\Psi^b|^2\Psi_j^f = 0, \quad j = 1, \dots, N_f, \tag{4.1.2}$$

where the wavefunctions Ψ_j^f describe each of N_f fermions and Ψ^b is the wavefunction for the bosonic component, $\rho_f = \sum\limits_{i=1}^{N_f} |\Psi_i^f|^2$ and $g_{\rm BB}, g_{\rm BF}, m_{\rm F}, m_{\rm B}$ are certain physical constants. In particular, $g_{\rm BB}$ and $g_{\rm BF}$ are related with the s-wave collisions for boson-boson and boson-fermion interactions, respectively. The potential V is usually of the form $V = V_0 sn^2(\alpha x, \kappa)$, where $sn(\alpha x, \kappa)$ is the Jacobi elliptic sine function. In this paper we take $V_0 = 0$ as in [10].

We are interested in the stationary solutions to the system (4.1.1), (4.1.2) of the kind

$$\Psi^b(x,t) \quad = q_0(x)\exp\left(-i\frac{\omega_0}{\hbar}t + i\Theta_0(x) + i\kappa_0\right), \tag{4.1.3}$$

$$\Psi_j^f(x,t) \quad = q_j(x)\exp\left(-i\frac{\omega_j}{\hbar}t + i\Theta_j(x) + i\kappa_{0,j}\right), \quad j = 1,\dots,N_f, \tag{4.1.4}$$

where κ_0, $\kappa_{0,j}$ are constant phases, q_0, q_j and Θ_0, Θ_j are real-valued functions related by

$$\Theta_0(x) = C_0\int_0^x \frac{dx'}{q_0^2(x')}, \qquad \Theta_j(x) = C_j\int_0^x \frac{dx'}{q_j^2(x')}, \quad j = 1,\dots,N_f \tag{4.1.5}$$

C_0, C_j, being constants of integration. After substituting (4.1.3), (4.1.4) in the equations (4.1.1), (4.1.2) and separating the real and imaginary part we get

$$\frac{1}{2m_{\rm B}}q_0^3 q_{0xx} - g_{\rm BB}q_0^6 - g_{\rm BF}\left(\sum_{i=1}^{N_f} q_i^2\right)q_0^4 + \omega_0 q_0^4 = \frac{C_0^2}{2m_{\rm B}}, \tag{4.1.6}$$

$$\frac{1}{2m_{\rm F}}q_j^3 q_{jxx} - g_{\rm BF}q_0^2 q_j^4 + \omega_j q_j^4 = \frac{C_j^2}{2m_{\rm F}}, \quad j = 1,\dots,N_f.$$

Kostov et al. [54] have found plenty of particular (quasiperiodic, periodic and soliton) solutions to the system (4.1.6) and therefore, stationary solutions to the system (4.1.1), (4.1.2). It is natural to ask whether we can obtain more, that is, for what set of the parameters the system (4.1.6) has enough first integrals to be integrable. Note that when $g_{\rm BF} = 0$ the equations separate, i.e., the system is solvable.

Before giving our main result let first get rid of the inessential (for integrability) parameters. In what follows we assume that the parameters $\omega_0, \omega_j, m_{\rm F}, m_{\rm B}, g_{\rm BB}$ are positive since they have an origin from physics, and $C_0, C_j, g_{\rm BF}$ are arbitrary real parameters. We put $q_0 = \beta\tilde{q}_0, q_j = \alpha\tilde{q}_j, x = \gamma\tilde{x}$. Then we choose $\alpha = \sqrt{m_{\rm F}}, \beta = \sqrt{m_{\rm B}}, \gamma = 1/(m_{\rm B}\sqrt{g_{\rm BB}}), g_{\rm BB} \neq 0$. Denoting $\tilde{g}_{\rm BF} = g_{\rm BF}\alpha^2\gamma^2 m_{\rm B}, \tilde{\omega}_0 = \omega_0\gamma^2 m_{\rm B}, \tilde{\omega}_j = \omega_j\gamma^2 m_{\rm F}, \tilde{C}_j^2 = C_j^2\gamma^2/\alpha^4, \tilde{C}_0^2 = C_0^2\gamma^2/\beta^4$ we reach

$$\frac{1}{2}\frac{d^2\tilde{q}_0}{d\tilde{x}^2} - \tilde{q}_0^3 - \tilde{g}_{\rm BF}\left(\sum_{i=1}^{N_f}\tilde{q}_i^2\right)\tilde{q}_0 + \tilde{\omega}_0\tilde{q}_0 = \frac{\tilde{C}_0^2}{2\tilde{q}_0^3}, \tag{4.1.7}$$

$$\frac{1}{2}\frac{d^2\tilde{q}_j}{d\tilde{x}^2} - \tilde{g}_{\rm BF}\tilde{q}_0^2\tilde{q}_j + \tilde{\omega}_j\tilde{q}_j = \frac{\tilde{C}_j^2}{2\tilde{q}_j^3}, \quad j = 1,\dots,N_f.$$

To simplify notations we skip the tildes, write t instead of x and denote $p_j = \dot{q}_j, j = 0,\dots,N_f$, $(. = d/dt)$. Then the system (4.1.7) can be presented as a Hamiltonian system with the Hamiltonian

$$H = \frac{p_0^2}{2} + \frac{1}{2}\sum_1^{N_f} p_j^2 + \omega_0 q_0^2 + \sum_1^{N_f}\omega_j q_j^2 - g_{\rm BF}q_0^2\sum_1^{N_f} q_j^2 - \frac{q_0^4}{2} + \frac{C_0^2}{2q_0^2} + \frac{1}{2}\sum_1^{N_f}\frac{C_j^2}{q_j^2}. \tag{4.1.8}$$

For the Hamiltonian system with the Hamiltonian (4.1.8) we consider the cases:

1) $C_0 = 0$, $C_j \neq 0$, $\sum C_j \neq 0$, $\omega_j = \omega^2/2, j = 1,\dots,N_f$;

2) $C_0 \neq 0$, $C_j = 0, j = 1,\dots,N_f$, $g_{\rm BF} = n(n+1)/2, n \notin \mathbb{Z}$;

3) $C_0 \neq 0$, $C_1 \neq 0, N_f = 1$, $g_{\rm BF}$ sufficiently small.

Our result is the following:

Theorem 4.1.1. *For the cases given above, the Hamiltonian system corresponding to (4.1.8) is non-integrable in the Liouville sense unless* $g_{\mathrm{BF}} = 0$.

In other words, the Hamiltonian system under consideration is integrable only when it is separable.

The proof of the above result is based on the Differential Galois approach and the Ziglin-Morales-Ramis method. This method has been applied for the studying the integrability to a number of Hamiltonian systems, in particular systems with homogeneous potentials, see [70, 75, 76, 81]. The classification of all integrable two degrees of freedom systems with polynomial potentials of degree 3 is obtained in [62]. The above mentioned approach is used in [1] for obtaining non-integrability results for some two degrees of freedom Hamiltonians with rational potentials. Note that the system in this paper is not of that kind.

For the natural Hamiltonian systems with two degrees of freedom, similar to (4.1.8)

$$H = \frac{p_1^2 + p_2^2}{2} + U(q_1, q_2)$$

there is an integrable generalization of Garnier's system found by Wojciechowski [105], namely

$$U = Aq_1^2 + Bq_2^2 + (q_1^2 + q_2^2)^2 + \frac{C}{q_1^2} + \frac{D}{q_2^2},$$

with a rational first integral depending on A, B, C, D (see also [80]). Note that in the system under consideration, the symmetry is lost, so it is natural to expect integrability only in the separable case.

Let us turn to the case 2 above : $C_0 \neq 0, C_j = 0, j = 1, \ldots, N_f$ when $g_{\mathrm{BF}} = n(n+1)/2, n \in \mathbb{Z}$. We formulate the following

Conjecture. The Hamiltonian system corresponding to (4.1.8) when $C_0 \neq 0, C_j = 0, j = 1, \ldots, N_f$ and $g_{\mathrm{BF}} = n(n+1)/2, n \in \mathbb{Z}$ is non-integrable in the Liouville sense unless $g_{\mathrm{BF}} = 0$.

This statement is formulated in that way because we have checked it only for $n = 1$ and $n = 2$. Nevertheless, we think that the system (4.1.8) is also non-integrable for arbitrary integer $n > 2$.

The chapter is organized as follows. In the next section the proof of Theorem 4.1.1 is given. Some comments are made in Section 4.3. We also give the proofs that the corresponding system is indeed non-integrable for the cases $n = 1$ and $n = 2$ in the above Conjecture there.

4.2 Proof of the main Theorem

In what follows we assume that $t, q_0(t), q_j(t)$ are complex quantities, but we keep the parameters real. The proof goes in the following lines. For the first two parts we find particular solutions. Then we study the variational equation (VE) along these solutions. The first part is the simplest, that is why we start with it. The variational equation (VE) is reduced to a particular case of the double confluent Heun equation, which differential Galois group is more or less known.

The second part needs more steps. The identity component of the Galois group of (VE) is not commutative except for some discrete values of g_{BF}. By studying higher variational equations we find a logarithmic term in solutions of (VE_3) when $g_{\mathrm{BF}} \neq 0$, which implies non commutativity of the identity component of $Gal(L_3/K)$ and hence, non-integrability of our Hamiltonian system.

For the third part we use a perturbational technique which is still related to the Differential Galois approach. We study the Poincaré-Arnold-Melnikov integral in order to show that a complex separatrix self-intersects.

4.2.1 The case $C_0 = 0, C_j \neq 0$.

In this case the Hamiltonian (4.1.8) becomes

$$H = \frac{p_0^2}{2} + \frac{1}{2}\sum_1^{N_f} p_j^2 + \omega_0 q_0^2 + \sum_1^{N_f} \omega_j q_j^2 - g_{\rm BF} q_0^2 \sum_1^{N_f} q_j^2 - \frac{q_0^4}{2} + \frac{1}{2}\sum_1^{N_f} \frac{C_j^2}{q_j^2}. \tag{4.2.9}$$

The equations corresponding to the Hamiltonian (4.2.9) are

$$\begin{aligned} \dot{q_0} &= p_0, \qquad \dot{p_0} = -2\omega_0 q_0 + 2q_0^3 + 2g_{\rm BF} q_0 \sum_1^{N_f} {q_j}^2, \\ \dot{q_j} &= p_j, \qquad \dot{p_j} = -2\omega_j q_j + 2g_{\rm BF} q_0^2 q_j + \frac{C_j^2}{q_j^3}, \quad j = 1, \dots, N_f. \end{aligned} \tag{4.2.10}$$

Proposition 4.2.1. *The system (4.2.10) has a particular solution of the form*

$$\begin{aligned} q_0 &= p_0 = 0, \\ {q_j}^2 &= \frac{C_j}{\sqrt{2\omega_j}} \sinh(2i\sqrt{2\omega_j} t), \qquad p_j = \dot{q_j}, \quad j = 1, \dots, N_f. \end{aligned} \tag{4.2.11}$$

Proof. We put $q_0 = p_0 = 0$ in (4.2.10). The general solution of the system with respect to $(q_j, p_j), j = 1, \dots, N_f$ is

$${q_j}^2 = \frac{h_j}{2\omega_j} + \sqrt{\frac{C_j^2}{2\omega_j} - \frac{h_j^2}{4\omega_j^2}} \sinh 2i\sqrt{2\omega_j}(t - t_0), \quad p_j = \dot{q_j}, \quad j = 1, \dots, N_f, \tag{4.2.12}$$

here h_j are arbitrary constants. Then we set $h_j = 0$ and $t_0 = 0$ to obtain our particular solution. □

Denote the variations by $\xi_0 = dq_0$ and $\eta_0 = dp_0$. It is easy to see that the (NVE) are written in variables ξ_0, η_0, namely

$$\dot{\xi_0} = \eta_0, \qquad \dot{\eta_0} = \left[-2\omega_0 + 2g_{\rm BF} \sum_1^{N_f} {q_j}^2 \right] \xi_0. \tag{4.2.13}$$

We rewrite (4.2.13) as a second order equation

$$\ddot{\xi_0} + \left[2\omega_0 - 2g_{\rm BF} \sum_1^{N_f} \frac{C_j}{\sqrt{2\omega_j}} \sinh(2i\sqrt{2\omega_j} t) \right] \xi_0 = 0. \tag{4.2.14}$$

The study of the identity component of the Galois group of (4.2.14) is a difficult task. That is why we assume that all ω_j are equal. We put $\omega_j = \frac{\omega^2}{2}$, $j = 1, \dots, N_f$. Then we get a variant of the Mathieu equation

$$\ddot{\xi_0} + [A_1 + B_1 \sinh(2i\omega t)]\, \xi_0 = 0, \tag{4.2.15}$$

where

$$A_1 = 2\omega_0, \qquad B_1 = -\frac{2}{\omega} g_{\rm BF} \sum_1^{N_f} C_j. \tag{4.2.16}$$

Since C_j are constants of integration, we can always assume that $\sum C_j \neq 0$.

Next, by changing the independent variable $x = e^{2i\omega t}$ we get an algebraic version of (4.2.15)

$$\xi_0'' + \frac{1}{x}\xi_0' + \left[\frac{B}{x} + \frac{A}{x^2} - \frac{B}{x^3}\right]\xi_0 = 0, \tag{4.2.17}$$

where $' = \frac{d}{dx}$, $A = -\frac{A_1}{4\omega^2}$, $B = -\frac{B_1}{8\omega^2}$. It is obvious that when $B = 0$ this equation becomes an Euler equation which is solvable. Further, we reduce (4.2.17) to the standard form by putting $y = \sqrt{x}\xi_0$,

$$y'' = r(x)y, \qquad r(x) = -\frac{B}{x} - \frac{A+\frac{1}{4}}{x^2} + \frac{B}{x^3}. \tag{4.2.18}$$

The equation (4.2.18) is a particular case of the double confluent Heun equation. For this equation the points 0 and ∞ are irregular singular ones and one natural way to study the Galois group is the Kovacic algorithm. This is done by A. Duval and M. Loday-Richaud in [25] p.237. We just apply their result which simply says that if $B \neq 0$ the Galois group of (4.2.18) is $\mathrm{SL}(2,\mathbb{C})$. In our case

$$B = \frac{g_{\mathrm{BF}}}{4\omega^3}\sum_1^{N_f} C_j,$$

which means that under the assumption $\sum_1^{N_f} C_j \neq 0$

$$B = 0 \Leftrightarrow g_{\mathrm{BF}} = 0,$$

that is, the identity component of the Galois group is noncommutative if $g_{\mathrm{BF}} \neq 0$. Therefore, by Theorem 1.0.1 the Hamiltonian system (4.2.9) is non-integrable unless $g_{\mathrm{BF}} = 0$. This finishes the proof of this part of Theorem 4.1.1.

Remark 2. Let us note that in [1, 2, 3] a systematic procedure is presented, called Hamiltonian Algebrization, which transforms second order linear differential equations with non-rational coefficients into differential equations with rational coefficients. As an example, the Mathieu equation is considered, see for instance section 2.1 in [1]. The conclusion is the same: the Mathieu equation is not integrable for $B \neq 0$.

4.2.2 The case $C_0 \neq 0, C_j = 0$.

Let us first find a particular solution.

Proposition 4.2.2. *The Hamiltonian system generated by the Hamiltonian (4.1.8) with $C_j = 0$ has a particular solution in the form*

$$\bar{q}_0^2(t) = \frac{2}{3}\omega_0 + \wp(t; g_2, g_3), \quad \bar{p}_0(t) = \dot{\bar{q}}_0(t), \quad q_j = p_j = 0, \quad j = 1, \ldots, N_f, \tag{4.2.19}$$

where $\wp(t; g_2, g_3)$ is the Weierstrass elliptic function satisfying

$$\Gamma : \dot{v}^2 = 4v^3 - g_2 v - g_3 \tag{4.2.20}$$

with $g_2 = \frac{16}{3}\omega_0^2 - 4h$, $g_3 = 4C_0^2 - \frac{8}{3}\omega_0 h + \frac{64}{27}\omega_0^3$ and h is level of the Hamiltonian (4.1.8), chosen so that $\Delta = g_2^3 - 27g_3^2 \neq 0$.

Proof. We put $q_j = p_j = 0$, $j = 1, \dots, N_f$ (recall $C_j = 0$) in (4.1.8) to obtain

$$H = \frac{p_0^2}{2} + \omega_0 q_0^2 - \frac{q_0^4}{2} + \frac{C_0^2}{2q_0^2} = \frac{h}{2}. \tag{4.2.21}$$

We rewrite this expression in the form

$$\dot{q}_0^2 = -2\omega_0 q_0^2 + q_0^4 - \frac{C_0^2}{q_0^2} + h. \tag{4.2.22}$$

Then denoting $u = q_0^2$ and also $u = v + \frac{2}{3}\omega_0$ we obtain the general solution of (4.2.22)

$$\bar{q}_0^2(t) = \frac{2}{3}\omega_0 + \wp(t - t_0; g_2, g_3), \quad \bar{p}_0(t) = \dot{\bar{q}}_0(t). \tag{4.2.23}$$

We set $t_0 = 0$ to get the desired result.

□

Next we write the variational equations (VE) along the particular solution (4.2.19). Denote $\xi_0 = dq_0$, $\eta_0 = dp_0$, $\xi_j = dq_j$, $\eta_j = dp_j$. Then the (VE) can be written as

$$\dot{\xi}_0 = \eta_0, \qquad \dot{\eta}_0 = \left(-2\omega_0 + 6\bar{q}_0^2(t) - \frac{3C_0^2}{\bar{q}_0^4(t)}\right)\xi_0, \tag{4.2.24}$$

$$\dot{\xi}_j = \eta_j, \qquad \dot{\eta}_j = \left(-2\omega_j + 2g_{\mathrm{BF}}\bar{q}_0^2(t)\right)\xi_j, \quad j = 1, \dots, N_f. \tag{4.2.25}$$

The equation (4.2.24) forms the tangent part of (VE) and the equations (4.2.25) form the normal part of (VE), actually (NVE). It is seen from (4.2.25) that the (NVE) splits into a system of N_f independent equations (NVE_j), $j = 1, \dots, N_f$. Hence, the (NVE) is integrable if, and only if, each of the (NVE_j) is integrable. In other words, the identity component of the Galois group of the (NVE) is solvable (commutative) if, and only if, each of identity components of the Galois groups of the (NVE_j) is solvable (commutative). Therefore, it is enough to study one of them. Let us write the (NVE_j) for certain particular j as a second order equation

$$\ddot{\xi}_j + \left(2\omega_j - 2g_{\mathrm{BF}}\bar{q}_0^2(t)\right)\xi_j = 0. \tag{4.2.26}$$

Taking into account the particular solution (4.2.19) the Eq. (4.2.26) is a Lamé equation

$$\ddot{\xi}_j + \left(2\omega_j - \frac{4}{3}g_{\mathrm{BF}}\omega_0 - 2g_{\mathrm{BF}}\wp(t)\right)\xi_j = 0. \tag{4.2.27}$$

Further, we study the tangential part of the (VE) - Eq. (4.2.24). The theory gives that its Galois group is solvable. In fact, we have

Proposition 4.2.3. *The Galois group of (4.2.24) is abelian.*

Proof. It is well known that the system (4.2.24) has a particular solution $(\xi_{0,1}, \dot{\xi}_{0,1}) = (\bar{p}_0(t), \dot{\bar{p}}_0(t))$. The other solution is obtained via D'Alembert's formula

$$\xi_{0,2} = \xi_{0,1}\int_0^t \frac{d\tau}{(\xi_{0,1})^2}.$$

Denote the coefficient field of (4.2.24) by $K = \mathbb{C}\left(\wp(t), \wp'(t)\right)$. This field is isomorphic to the field of meromorphic functions $\mathcal{M}(\Gamma)$ on Γ.

It can be seen from the obtained solutions that one part of them lie in a quadratic extension of the field K and the another part is obtained with single quadrature of the elements of this extension. Therefore the Galois group of (4.2.24) acts in the following way: $\sigma \in Gal(L/K)$, $\sigma(\xi_{0,1}) = \xi_{0,1}$ and $\sigma(\xi_{0,2}) = \xi_{0,2} + \nu_0\xi_{0,1}$, $\alpha_0 \in \mathbb{C}$. Let $\Xi(t)$ is the fundamental matrix of (4.2.24)

$$\Xi = \begin{pmatrix} \xi_{0,2} & \xi_{0,1} \\ \dot{\xi}_{0,2} & \dot{\xi}_{0,1} \end{pmatrix}.$$

Then $\sigma \in Gal(L/K)$ can be represented by the matrix R_{ν_0}, $\sigma\Xi(t) = \Xi(t)R_{\nu_0}$, where $R_{\nu_0} = \begin{pmatrix} 1 & 0 \\ \nu_0 & 1 \end{pmatrix}$. It is clear that the group $\left\{\begin{pmatrix} 1 & 0 \\ \nu_0 & 1 \end{pmatrix}\right\}$ is abelian.

□

Now we pass to the analysis of the normal part of the variational equations given by (4.2.27). Usually the parameter A in the Lamé equation $\ddot{\xi} - (A\wp(t) + B)\xi = 0$ is replaced by a parameter n, where $A = n(n+1)$, and thus for (4.2.27) we have

$$g_{\mathrm{BF}} = \frac{n(n+1)}{2}, \quad n \in \mathbb{R} \tag{4.2.28}$$

and (recall that the parameters are real)

$$\ddot{\xi}_j - \left[n(n+1)\wp(t) + B_j\right]\xi_j = 0, \tag{4.2.29}$$

where $B_j = \frac{2}{3}\omega_0 n(n+1) - 2\omega_j$. The cases for which the Lamé equation (4.2.29) is solvable are well known:

(i) The Lamé and Hermite solutions. In this case $n \in \mathbb{Z}$ and g_2, g_3, B_j are arbitrary parameters;

(ii) The Brioschi-Halphen-Crowford solutions. Here $m := n + 1/2 \in \mathbb{N}$ and the parameters g_2, g_3, B_j must satisfy an algebraic equation.

(iii) The Baldassarri solutions. Now $n+1/2 \in \frac{1}{3}\mathbb{Z}\cup\frac{1}{4}\mathbb{Z}\cup\frac{1}{5}\mathbb{Z}\setminus\mathbb{Z}$ with additional algebraic relations between the other parameters.

Note that in the case (i) the identity component of the Galois group G^0 is of the form $\begin{pmatrix} 1 & 0 \\ \nu_j & 1 \end{pmatrix}$ and in the cases (ii) and (iii) $G^0 = id$ (G is finite). And these are the all cases when the Lamé equation is integrable.

Therefore, together with the result of Proposition 4.2.2 we have that the identity component of Galois group of the (VE) is represented by the block-diagonal matrices of the kind

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\nu_0 & 1 & 0 & 0 & & 0 & 0 \\
\cdots & & \cdots & & \cdots & \cdots & \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \nu_j & 1 & & 0 & 0 \\
\cdots & & \cdots & & \cdots & \cdots & \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & & \nu_{N_f} & 1
\end{pmatrix}$$

and it is clearly commutative.

The integrability of Hamiltonian systems with two degrees of freedom which (NVE) are Lamé equations is studied in [73, 70]. We summarize the facts and the result (Theorem 4.4.1), that gives necessary conditions for integrability in an Appendix. Since in our case the (NVE) splits into a system of N_f equations, the result for two degrees of freedom can be applied.

The potential $\varphi(q_0)$ is obtained from (4.2.22). Denote $\alpha(t,h) := n(n+1)\wp(t)+B_j$. We calculate the coefficients of the polynomial $P(\alpha,h)$ (compare with (A.4) and (4.4.71))

$$P(\alpha,h) = (a_1+ha_2)\alpha^3 + (b_1+hb_2)\alpha^2 + (c_1+hc_2)\alpha + (d_1+hd_2).$$

In our case these coefficients are:

$$\begin{aligned} a_1 = \tfrac{4}{n(n+1)}, \quad a_2 = 0, \quad b_1 = -\tfrac{12B_j}{n(n+1)}, \quad b_2 = 0 \\ c_1 = \tfrac{12B_j^2}{n(n+1)} - \tfrac{16}{3}\omega_0^2 n(n+1), \quad c_2 = 4n(n+1), \\ d_1 = \tfrac{16}{3}b_j\omega_0^2 n(n+1) - \tfrac{4}{n(n+1)}B_j^3 - n^2(n+1)^2\left(4C_0^2+\tfrac{64}{27}\omega_0^3\right), \quad d_2 = 8n(n+1)\omega_j. \end{aligned} \tag{4.2.30}$$

Now we are ready to apply Theorem 4.4.1 (see the Appendix). Suppose that $g_{\mathrm{BF}} = n(n+1)/2 \neq 0$.

The condition 3 is not fulfilled: $c_2 \neq 0$ and $c_2b_1 - 3a_1d_2 = -32\omega_0 n(n+1)$, which is nonzero by the assumption that ω_0, ω_j are positive numbers, made in the very beginning. In particular, there are no Baldassarri solutions.

We proceed with the cases of the condition 2. In the case 2.1, $m=1$, $b_1=0$ is equivalent to

$$B_j = 0, \quad \text{or equivalently,} \quad \omega_j = \omega_0/4, \quad j=1,\dots,N_f. \tag{4.2.31}$$

If for some j $B_j \neq 0$ then the system is not integrable for this m. We will consider the case when all $B_j = 0$ in what follows.

The case 2.2 $m=2$ does not occur here since $c_2 \neq 0$.

In the case 2.3, $m=3$ the necessary conditions

$$16a_1d_2 + 11b_1c_2 = 0, \quad 1024a_1^2d_1 + 704a_1b_1c_1 + 45b_1^3 = 0$$

yield correspondingly

$$B_j = \frac{32}{33}\omega_j \ (55\omega_0 = 28\omega_j), \quad 7^3C_0^2 = 72\omega_0^3. \tag{4.2.32}$$

If any of the above conditions is violated, then the system is non-integrable for this m. We will consider the case when relations (4.2.32) are valid for all j in what follows.

The case 2.m, $m>3$ does not occur here since $c_2 \neq 0$ and $d_2 \neq 0$.

Finally, if $n \notin \mathbb{Z}$ the condition 1 is not fulfilled.

In order to resolve the case 2.1 with (4.2.31) and the case 2.3 with (4.2.32) of the condition 2 in the Theorem 4.4.1 we need to study the Galois groups of higher variational equations and to apply Theorem 1.0.5. To compute higher variations we put

$$\begin{aligned} q_0 &= \bar{q}_0 + \varepsilon\xi_0^{(1)} + \varepsilon^2\xi_0^{(2)} + \varepsilon^3\xi_0^{(3)} + \dots, \\ p_0 &= \bar{p}_0 + \varepsilon\eta_0^{(1)} + \varepsilon^2\eta_0^{(2)} + \varepsilon^3\eta_0^{(3)} + \dots, \\ q_j &= 0 + \varepsilon\xi_j^{(1)} + \varepsilon^2\xi_j^{(2)} + \varepsilon^3\xi_j^{(3)} + \dots, \\ p_j &= 0 + \varepsilon\eta_j^{(1)} + \varepsilon^2\eta_j^{(2)} + \varepsilon^3\eta_j^{(3)} + \dots, \quad j=1,\dots,N_f. \end{aligned} \tag{4.2.33}$$

and substitute these expressions into the original Hamiltonian system. Comparing the terms with the same order in ε, we get consecutively the variational equations up to order 3.

The first variational equation is

$$\dot{\xi}_0^{(1)} = \eta_0^{(1)}, \qquad \dot{\eta}_0^{(1)} = \left(-2\omega_0 + 6\bar{q}_0^2 - \frac{3C_0^2}{\bar{q}_0^4}\right)\xi_0^{(1)}, \tag{4.2.34}$$

$$\dot{\xi}_j^{(1)} = \eta_j^{(1)}, \qquad \dot{\eta}_j^{(1)} = (-2\omega_j + 2g_{\mathrm{BF}}\bar{q}_0^2)\xi_j^{(1)}, \quad j = 1, \ldots, N_f, \tag{4.2.35}$$

but of course we know it (see (4.2.24, (4.2.25)). For the second variational equation we have

$$\dot{\xi}_0^{(2)} = \eta_0^{(2)}, \qquad \dot{\eta}_0^{(2)} = \left(-2\omega_0 + 6\bar{q}_0^2 - \frac{3C_0^2}{\bar{q}_0^4}\right)\xi_0^{(2)} + K_0^{(2)}, \tag{4.2.36}$$

$$\dot{\xi}_j^{(2)} = \eta_j^{(2)}, \qquad \dot{\eta}_j^{(2)} = (-2\omega_j + 2g_{\mathrm{BF}}\bar{q}_0^2)\xi_j^{(2)} + K_j^{(2)}, \quad j = 1, \ldots, N_f. \tag{4.2.37}$$

The third variational equation is

$$\dot{\xi}_0^{(3)} = \eta_0^{(3)}, \qquad \dot{\eta}_0^{(3)} = \left(-2\omega_0 + 6\bar{q}_0^2 - \frac{3C_0^2}{\bar{q}_0^4}\right)\xi_0^{(3)} + K_0^{(3)}, \tag{4.2.38}$$

$$\dot{\xi}_j^{(3)} = \eta_j^{(3)}, \qquad \dot{\eta}_j^{(3)} = (-2\omega_j + 2g_{\mathrm{BF}}\bar{q}_0^2)\xi_j^{(3)} + K_j^{(3)}, \quad j = 1, \ldots, N_f. \tag{4.2.39}$$

Here

$$\begin{aligned}
K_0^{(2)} &= 2g_{\mathrm{BF}}\bar{q}_0 \sum (\xi_j^{(1)})^2 + 6\bar{q}_0(\xi_0^{(1)})^2 + 6C_0^2\frac{(\xi_0^{(1)})^2}{\bar{q}_0^5}, \\
K_j^{(2)} &= 4g_{\mathrm{BF}}\bar{q}_0\xi_0^{(1)}\xi_j^{(1)}, \quad j = 1, \ldots, N_f, \\
K_0^{(3)} &= 2g_{\mathrm{BF}}\left[2\bar{q}_0 \sum \xi_j^{(1)}\xi_j^{(2)} + \xi_0^{(1)} \sum (\xi_j^{(1)})^2\right] + 2(\xi_0^{(1)})^3 + 12\bar{q}_0\xi_0^{(1)}\xi_0^{(2)} \\
&- \frac{C_0^2}{\bar{q}_0^6}\left[10(\xi_0^{(1)})^3 - 12\bar{q}_0\xi_0^{(1)}\xi_0^{(2)}\right], \\
K_j^{(3)} &= 2g_{\mathrm{BF}}\left[(\xi_0^{(1)})^2\xi_j^{(1)} + 2\bar{q}_0\left(\xi_0^{(1)}\xi_j^{(2)} + \xi_0^{(2)}\xi_j^{(1)}\right)\right], \quad j = 1, \ldots, N_f.
\end{aligned} \tag{4.2.40}$$

Then, in our notation from Section 2, we have

$$\begin{aligned}
f_2 &= \left[0, K_0^{(2)}, 0, K_1^{(2)}, \ldots, 0, K_{N_f}^{(2)}\right]^T, \\
f_3 &= \left[0, K_0^{(3)}, 0, K_1^{(3)}, \ldots, 0, K_{N_f}^{(3)}\right]^T.
\end{aligned} \tag{4.2.41}$$

First, we have to solve (VE_1). Let $\xi_{0,1}^{(1)}, \xi_{0,2}^{(1)}$ be two linearly independent solutions of (4.2.34) with Wronskian equal to unity, i.e., $\xi_{0,1}^{(1)}\dot{\xi}_{0,2}^{(1)} - \dot{\xi}_{0,1}^{(1)}\xi_{0,2}^{(1)} = 1$. Similarly, $\xi_{j,1}^{(1)}, \xi_{j,2}^{(1)}$ are linearly independent solutions of (4.2.35) with Wronskian equal to unity. Then the fundamental matrix $X(t)$ of (4.2.34), (4.2.35) and its inverse have the block-diagonal form

$$X(t) = \begin{pmatrix} \xi_{0,1}^{(1)} & \xi_{0,2}^{(1)} & 0 \ \ 0 & \dots & 0 \ \ 0 \\ \dot{\xi}_{0,1}^{(1)} & \dot{\xi}_{0,2}^{(1)} & 0 \ \ 0 & & 0 \ \ 0 \\ \dots & & \dots & \dots & \dots \\ 0 & 0 & \xi_{j,1}^{(1)} \ \ \xi_{j,2}^{(1)} & \dots & 0 \ \ 0 \\ 0 & 0 & \dot{\xi}_{j,1}^{(1)} \ \ \dot{\xi}_{j,2}^{(1)} & & 0 \ \ 0 \\ \dots & & \dots & \dots & \dots \\ 0 & 0 & 0 \ \ 0 & \dots & \xi_{N_f,1}^{(1)} \ \ \xi_{N_f,2}^{(1)} \\ 0 & 0 & 0 \ \ 0 & & \dot{\xi}_{N_f,1}^{(1)} \ \ \dot{\xi}_{N_f,2}^{(1)} \end{pmatrix}, \tag{4.2.42}$$

$$X^{-1}(t) = \begin{pmatrix} \dot{\xi}_{0,2}^{(1)} & -\xi_{0,2}^{(1)} & 0 \ \ 0 & \dots & 0 \ \ 0 \\ -\dot{\xi}_{0,1}^{(1)} & \xi_{0,1}^{(1)} & 0 \ \ 0 & & 0 \ \ 0 \\ \dots & & \dots & \dots & \dots \\ 0 & 0 & \dot{\xi}_{j,2}^{(1)} \ \ -\xi_{j,2}^{(1)} & \dots & 0 \ \ 0 \\ 0 & 0 & -\dot{\xi}_{j,1}^{(1)} \ \ \xi_{j,1}^{(1)} & & 0 \ \ 0 \\ \dots & & \dots & \dots & \dots \\ 0 & 0 & 0 \ \ 0 & \dots & \dot{\xi}_{N_f,2}^{(1)} \ \ -\xi_{N_f,2}^{(1)} \\ 0 & 0 & 0 \ \ 0 & & -\dot{\xi}_{N_f,1}^{(1)} \ \ \xi_{N_f,1}^{(1)} \end{pmatrix} \tag{4.2.43}$$

The first variational equations (VE_1) (4.2.34), (4.2.35) have a singular point at $t = 0$ (the pole of $\wp(t)$). We calculate the expansion of the solutions of variational equations around the point $t = 0$. Note that

$$\bar{q}_0(t) = \frac{1}{t} + \frac{\omega_0}{3}t + \left(\frac{g_2}{40} - \frac{\omega_0^2}{18}\right)t^3 + \dots. \tag{4.2.44}$$

Here and further dots denote the higher order terms with respect to t. In a neighborhood of $t = 0$ we have the following expansions for the solutions of the tangential part of (VE_1) Eq. (4.2.34)

$$\xi_{0,1}^{(1)} = \frac{1}{t^2} - \frac{\omega_0}{3} - \left(\frac{3g_2}{40} - \frac{\omega_0^2}{6}\right)t^2 + \dots, \quad \xi_{0,2}^{(1)} = \frac{t^3}{5} + \frac{\omega_0}{35}t^5 + \dots. \tag{4.2.45}$$

After this preparation, we are ready to proceed with the cases unresolved from the analysis of the first variational equations (VE_1).

We continue the study of the case 2.1 with (4.2.31). Here $n = \frac{1}{2}$ and $g_{\mathrm{BF}} = \frac{3}{8}$. One can find the following expansions of the solutions of (4.2.35) in this case

$$\xi_{j,1}^{(1)} = t^{-1/2}\left(1 + \frac{3}{640}g_2t^4 + \dots\right), \quad \xi_{j,2}^{(1)} = t^{3/2}\left(\frac{1}{2} + \frac{1}{2.640}g_2t^4 + \dots\right).$$

There are no logarithms in the expansions around $t = 0$ of the local solutions of the second variational equation (VE_2) due to (4.2.31). We take

$$\xi_0^{(1)} = \xi_{0,1}^{(1)}, \quad \xi_j^{(1)} = \xi_{j,2}^{(1)}.$$

With the above choice, we find that

$$\xi_0^{(2)} = \xi_{0,2}^{(2)} = \frac{1}{t^3} + O(t), \quad \xi_j^{(2)} = \xi_{j,1}^{(2)} = \frac{1}{\sqrt{t}} - \frac{3}{4}\sqrt{t} + O(t^{3/2}).$$

We will show that a logarithmic term appears in a local solution of (VE_3). For this purpose, it is enough to show that at least one component of $X^{-1}f_3$ has a nonzero residue at $t=0$, see formulae (1.0.17), (1.0.18). We calculate j-th component $j=1,\dots,N_f$ of $X^{-1}f_3$, which looks like

$$(-\xi_{j,2}^{(1)}K_j^{(3)},\xi_{j,1}^{(1)}K_j^{(3)})^T. \tag{4.2.46}$$

Then the first term in (4.2.46) has the following expansion around $t=0$

$$\mu_3=-\xi_{j,2}^{(1)}K_j^{(3)}=-\frac{3}{8}\left[\frac{2}{t^2}-\frac{2\omega_0}{3t}+\dots\right],$$

that is, μ_3 has a pole at $t=0$ with non-zero residue $\frac{\omega_0}{4}$. Therefore, the identity component of the Galois group of (VE_3) is not abelian and hence, in this case, the Hamiltonian system (4.1.8) is not integrable due to Theorem 1.0.5.

Next we consider the case 2.3 with (4.2.32). Here $n=\frac{5}{2}$ and $g_{\mathrm{BF}}=\frac{35}{8}$. One can find the following expansions of the solutions of (4.2.35) in this case

$$\xi_{j,1}^{(1)}=t^{-5/2}\left(\frac{1}{6}+O(t^2)\right),\quad \xi_{j,2}^{(1)}=t^{7/2}\left(1+O(t^2)\right).$$

There are no logarithms in the expansions around $t=0$ of the local solutions of the second variational equation (VE_2) due to (4.2.32).

We take

$$\xi_0^{(1)}=\xi_{0,1}^{(1)},\quad \xi_j^{(1)}=\xi_{j,1}^{(1)}.$$

With the above choice, we find that

$$\xi_0^{(2)}=\xi_{0,2}^{(2)}=\frac{5N_f}{144t^4}+\frac{1}{t^3}-\frac{N_f\omega_0}{224t^2}-\frac{\omega_0}{5t}+O(t^0),$$

$$\xi_j^{(2)}=\xi_{j,2}^{(2)}=t^{7/2}(1+O(t^2)+t^{-7/2}\left(\frac{5}{12}-\frac{\omega_j}{99}t^2+\dots\right).$$

Again the first term in (4.2.46) has the expansion around $t=0$

$$\mu_3=-\xi_{j,2}^{(1)}K_j^{(3)}=-\frac{175N_f}{1728t^4}-\frac{35}{3t^3}-\frac{5N_f\omega_0}{576t^2}+\frac{7\omega_0}{12t}+O(t^0),$$

that is, μ_3 has a non-zero residue $\frac{7}{12}\omega_0$ at $t=0$. Therefore, the identity component of the Galois group of (VE_3) is not abelian and hence, in this case, the Hamiltonian system (4.1.8) is not integrable due to Theorem 1.0.5.

This finishes the proof of this part of Theorem 4.1.1.

4.2.3 The case $C_0\neq 0, C_1\neq 0$.

Here we consider the Hamiltonian (4.1.8) only for two degrees of freedom (see comments in the next section)

$$H=\frac{p_0^2}{2}+\omega_0q_0^2-\frac{q_0^4}{2}+\frac{C_0^2}{2q_0^2}+\frac{p_1^2}{2}+\omega_1q_1^2+\frac{C_1^2}{2q_1^2}-g_{\mathrm{BF}}q_0^2q_1^2. \tag{4.2.47}$$

Denote $\varepsilon:=g_{\mathrm{BF}}$ and assume that ε is small enough. We can rewrite (4.2.47) as

$$H=H_0+\varepsilon H_1, \tag{4.2.48}$$

where

$$H_0 = \frac{p_0^2}{2} + \omega_0 q_0^2 - \frac{q_0^4}{2} + \frac{C_0^2}{2q_0^2} + \frac{p_1^2}{2} + \omega_1 q_1^2 + \frac{C_1^2}{2q_1^2}, \quad H_1 = -q_0^2 q_1^2. \tag{4.2.49}$$

The unperturbed system ($\varepsilon = 0$) is separable.

$$\dot{q}_0 = p_0, \quad \dot{p}_0 = -2\omega_0 q_0 + 2q_0^3 + \frac{C_0^2}{q_0^3}, \tag{4.2.50}$$

$$\dot{q}_1 = p_1, \qquad \dot{p}_1 = -2\omega_1 q_1 + \frac{C_1^2}{q_1^3} \tag{4.2.51}$$

From the proof of Proposition 4.2.2 the general solution of (4.2.50) is found in (4.2.23). From the proof of Proposition 4.2.1 the general solution of (4.2.51) is

$$q_1^2 = \frac{h_1}{2\omega_1} + \sqrt{\frac{C_1^2}{2\omega_1} - \frac{h_1^2}{4\omega_1^2}} \sinh 2i\sqrt{2\omega_1}(t - t_0), \quad p_1 = \dot{q}_1. \tag{4.2.52}$$

First, we put the Hamiltonian H in the context of the theory recalled in Chapter 1. It is assumed that at this point the variables are real. We introduce action-angle variables (I, φ), so that $H_0 = H_0(q_0, p_0, I)$. To do so, we need to find a generating function $S(I, q_1)$:

$$(p_1, q_1) \overset{S(I,q_1)}{\longrightarrow} (I, \varphi), \quad p_1 = \frac{\partial S}{\partial q_1}, \quad \varphi = \frac{\partial S}{\partial I},$$

such that

$$\frac{p_1^2}{2} + \omega_1 q_1^2 + \frac{C_1^2}{2q_1^2} = h_1 \to h_1(I) := I. \tag{4.2.53}$$

Note that the real ovals for the curve (p_1, q_1) in (4.2.53) exist for $h_1 > \frac{C_1^2\sqrt{2\omega_1}}{\sqrt{C_1^2}}$. Then the formula (4.2.52) becomes

$$q_1^2 = \frac{h_1}{2\omega_1} - \sqrt{\frac{h_1^2}{4\omega_1^2} - \frac{C_1^2}{2\omega_1}} \sin 2\sqrt{2\omega_1}(t - t_0). \tag{4.2.54}$$

The generating function S can be found explicitly, but we do not need it, we just set

$$I := \frac{p_1^2}{2} + \omega_1 q_1^2 + \frac{C_1^2}{2q_1^2}, \quad \varphi := \int \frac{dq_1}{p_1}. \tag{4.2.55}$$

Note that $dI \wedge d\varphi = dp_1 \wedge dq_1$, φ is multivalued, but $\dot{\varphi} = 1$, that is, t and φ are interchangeable.

Next, we fix I to an arbitrary constant greater than $\frac{C_1^2\sqrt{2\omega_1}}{\sqrt{C_1^2}}$ and again consider $t, q_0(t), p_0(t)$ as complex variables. Our system becomes an one-and-a-half degrees of freedom system with a Hamiltonian $H = H_0 + \varepsilon H_1$, where

$$H_0 = \frac{p_0^2}{2} + \omega_0 q_0^2 - \frac{q_0^4}{2} + \frac{C_0^2}{2q_0^2} + I, \quad H_1 = -q_0^2\left(\frac{I}{2\omega_1} - \sqrt{\frac{I^2}{4\omega_1^2} - \frac{C_1^2}{2\omega_1}} \sin 2\sqrt{2\omega_1}(t - t_0)\right). \tag{4.2.56}$$

We need to find a separatrix in the dynamics of (q_0, p_0). Denote $\tilde{h} = h - I$ and $\tilde{g}_2 = \frac{16}{3}\omega_0^2 - 4\tilde{h}$, $\tilde{g}_3 = 4C_0^2 - \frac{8}{3}\omega_0\tilde{h} + \frac{64}{27}\omega_0^3$ (compare with the corresponding formulas in the Proposition 4.2.2). Let h^* be the biggest real root of

$$\Delta(\tilde{h}) = \tilde{g}_2^3 - 27\tilde{g}_3^2 = -64\left(\tilde{h}^3 - \omega_0^2\tilde{h}^2 - 9C_0^2\omega_0\tilde{h} + 8C_0^2\omega_0^3 + \frac{27}{4}C_0^4\right) = 0. \tag{4.2.57}$$

Assume that $4\omega_0^2 - 3h^* > 0$. Further, we denote

$$a := \frac{\sqrt{4\omega_0^2 - 3h^*}}{3} > 0.$$

Then the unperturbed system (4.2.56) has a separatrix

$$\Gamma_0 : q_0^2(t) = \frac{2}{3}\omega_0 + a + \frac{3a}{\sinh^2(\sqrt{3a}t)}, \quad p_0(t) = \dot{q}_0(t). \tag{4.2.58}$$

The perturbed variational equation (PVE) of (4.2.56) along Γ_{t_0} is given by (see [32, 74])

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \eta \\ \nu \end{pmatrix} = \begin{pmatrix} H_{0,q_0p_0} & H_{0,p_0p_0} & H_{1,p_0} \\ -H_{0,q_0q_0} & -H_{0,q_0p_0} & -H_{1,q_0} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \nu \end{pmatrix}, \tag{4.2.59}$$

where all coefficients are restricted to Γ_{t_0}. In order to study the Galois group of (PVE) we fix the coefficient field K in (4.2.59). From the expressions for the separatrix (4.2.58) and the perturbation H_1 (4.2.56)

$$K := \mathbb{C}(e^{\sqrt{3a}t}, e^{2\sqrt{2\omega_1}it}).$$

Then, to obtain the fundamental matrix of (4.2.59) a quadrature is needed, namely $\delta = \delta(t) = \int \frac{H_{0,p_0p_0}}{H_{0,p_0}^2}{}_{|\Gamma_0} dt$ (see [74] for details). In our case $\delta = \int \frac{dt}{p_0^2(t)}$ equals

$$\delta = \frac{1}{(3a)^3}\Big(\frac{2\omega_0 + 3a}{12\sqrt{3a}}\sinh(\sqrt{3a}t)\cosh^3(\sqrt{3a}t) + \frac{10\omega_0 + 27a}{8\sqrt{3a}}\sinh(\sqrt{3a}t)\cosh(\sqrt{3a}t)$$
$$+ \frac{2\omega_0 + 12a}{3\sqrt{3a}}\tanh(\sqrt{3a}t) + \frac{26\omega_0 + 99a}{8}t\Big).$$

It is clear that $\delta = \delta(t)$ is uniform and $\delta \notin K$. Then, the Picard-Vessiot extension of (4.2.59) is $L_1 = K(\delta) = \mathbb{C}(e^{\sqrt{3a}t}, e^{2\sqrt{2\omega_1}it}, t)$. It remains to find $d(t_0)$. Let γ be a loop around the pole $t = 0$. Then simple calculations give that the Poincaré-Arnold-Melnikov integral is

$$d(t_0) = \int_\gamma \{H_0, H_1\}(q_0(t), p_0(t), t - t_0)dt = 12\pi i a\sqrt{2\omega_1}\sqrt{\frac{I^2}{4\omega_1^2} - \frac{C_1^2}{2\omega_1}}\sin 2\sqrt{2\omega_1}t_0. \tag{4.2.60}$$

It is seen that $d(t_0)$ has simple zeroes and by Theorem 4, the perturbed separatrix self-intersects transversally. Also since $d(t_0)$ is not identically zero, the Galois group of the perturbed variational equation is not abelian [74]. Hence, when $\varepsilon = g_{\mathrm{BF}} \neq 0$ sufficiently small, there is no additional meromorhic first integral. This finishes the proof of this part and therefore, the proof of the Theorem 4.1.1.

■

4.3 Concluding Remarks

In this paper we use variational equations to obtain a necessary and sufficient condition for integrability of a system which describes the stationary solutions in the time dependent mean field equations of Bose–Fermi mixture. Here we make some remarks.

We start with some restrictions to our methods. In subsection 4.2.1 we don't know how to study the Galois group of a second order linear equation with quasi-periodic coefficient, that is why we assume that all ω_j are equal. It is an open problem to develop a Picard-Vessiot Theory for the coefficient field $K = \mathbb{C}(e^{\alpha_1 x}, \ldots, e^{\alpha_m x})$ with $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and to relate this result with the integrability of the corresponding linear equation, see [85], p. 408.

In the second part of Theorem 4.1.1 considered in the subsection 4.2.2, it turns out that the (NVE) splits into a system of second order linear equations of Lamé type. By applying Theorem 4.4.1 and by studying the Galois groups of the variational equations up to order three, we prove that the Hamiltonian system (4.1.8) is non-integrable when $g_{\mathrm{BF}} = n(n+1)/2, n \notin \mathbb{Z}$. Unfortunately, we do not give a complete answer for the integrability of the system (4.1.8) in the Lamé and Hermite case when $g_{\mathrm{BF}} = n(n+1)/2, n \in \mathbb{Z}$. That is why we have formulated the Conjecture in the Introduction. We consider only $n = 1$ and $n = 2$ by technical reasons. As it is seen from subsection 4.2.2 and the Appendix, in the Lamé and Hermite case it is not possible to derive additional integrability conditions from the analysis of the (VE_1). Therefore, in order to study the integrability for arbitrary $n \in \mathbb{Z}$ in $g_{\mathrm{BF}} = n(n+1)/2$, one needs to know the exact coefficients in expansions of the Lamé solutions of (4.2.35) and eventually the expansions of the higher (than order three) variations. The formulas are quite involved. However, it is highly unlikely that the system is integrable for some integer $n > 2$, which is justified by the result in subsection 4.2.3.

The cases $n = 1$ and $n = 2$ from the Conjecture.

We take $n = 1$, but we keep writing g_{BF} instead 1. In the vicinity $t = 0$ we have the following expansions for the solutions $\xi_{j,1}^{(1)}, \xi_{j,2}^{(1)}$ of (4.2.35) ($n = 1$)

$$\xi_{j,1}^{(1)} = \frac{1}{t} + \frac{B_j}{2}t + \left(\frac{g_2}{40} - \frac{B_j^2}{8}\right)t^3 + \ldots, \quad \xi_{j,2}^{(1)} = \frac{t^2}{3} - \frac{a_j}{30}t^4 + \ldots, \tag{4.3.61}$$

where $B_j = 2\omega_j - 4\omega_0/3$.

There are no logarithms in the expansions around $t = 0$ of the local solutions of the second variational equation (VE_2).

We take

$$\xi_0^{(1)} = \xi_{0,2}^{(1)}, \quad \xi_j^{(1)} = \xi_{j,1}^{(1)}. \tag{4.3.62}$$

With this choice we find

$$\xi_{0,1}^{(2)} = \frac{1}{t^2} + \frac{g_{\mathrm{BF}}N_f}{2}\frac{1}{t} - \frac{\omega_0}{3} + \ldots, \quad \xi_{0,2}^{(2)} = \frac{g_{\mathrm{BF}}N_f}{2}\frac{1}{t} + \ldots \tag{4.3.63}$$

and

$$\xi_{j,1}^{(2)} = \frac{1}{t} + \frac{B_j}{2}t + \ldots, \quad \xi_{j,2}^{(2)} = \frac{t^2}{3} + \ldots. \tag{4.3.64}$$

Taking the first term in (4.2.46), namely

$$\mu_3 = -\xi_{j,2}^{(1)}K_j^{(3)} = -\xi_{j,2}^{(1)}2g_{\mathrm{BF}}\left[(\xi_0^{(1)})^2\xi_j^{(1)} + 2\bar{q}_0\left(\xi_0^{(1)}\xi_j^{(2)} + \xi_0^{(2)}\xi_j^{(1)}\right)\right]$$

with the choice (4.3.62) and $\xi_0^{(2)} = \xi_{0,2}^{(2)}$ and $\xi_j^{(2)} = \xi_{j,1}^{(2)}$ we can see that μ_3 has a simple pole at $t = 0$ with residue $-2g_{\mathrm{BF}}^2N_f/3$, which is non-zero. Therefore, the identity component of the Galois group of (VE_3) is not commutative and hence, in this case, the Hamiltonian system (4.1.8) is not integrable due to Theorem 1.0.5.

Similarly, for $n = 2$ and $g_{\rm BF} = 3$ we have the following expansions for the solutions $\xi^{(1)}_{j,1}, \xi^{(1)}_{j,2}$ of (4.2.35)

$$\xi^{(1)}_{j,1} = \frac{1}{t^2} - \frac{B_j}{6} + O(t^2), \quad \xi^{(1)}_{j,2} = \frac{t^3}{5} + \frac{B_j t^5}{70} + \dots, \tag{4.3.65}$$

where $B_j = 4\omega_0 - 2\omega_j (n = 2)$. Let us study first the expansions of the local solutions of the (VE_2) around $t = 0$. The calculation of $\mu_2 = \xi^{(1)}_{j,1} K^{(2)}_j$ with $\xi^{(1)}_0 = \xi^{(1)}_{0,1}, \xi^{(1)}_j = \xi^{(1)}_{j,1}$ gives

$$\mu_2 = \frac{12}{t^7} - \frac{4B_j}{t^5} + \frac{\frac{4}{3}B_j^2 - \frac{12}{5}g_2}{t^3} + \frac{B_j(g_2 - \frac{1}{3}B_j^2)}{t} + O(t).$$

Since g_2 depends on h, which is arbitrary provided $\Delta = g_2^3 - 27g_3^2 \neq 0$, the only possibility for the residue of μ_2 to be zero is $B_j = 0$, or equivalently, $\omega_j = 2\omega_0$. If there exists at least one ω_j, such that $B_j \neq 0$, then a logarithm appears in the solutions of (VE_2) around $t = 0$.

We proceed with the case when all $B_j = 0$, or equivalently, $\omega_j = 2\omega_0, j = 1, \dots, N_f$. Choosing

$$\xi^{(1)}_0 = \xi^{(1)}_{0,1} = \frac{1}{t^2} - \frac{\omega_0}{3} + \dots, \quad \xi^{(1)}_j = \xi^{(1)}_{j,2} = \frac{t^3}{5} + \dots. \tag{4.3.66}$$

we find that

$$\xi^{(2)}_{0,1} = \frac{1}{t^3} + \frac{1}{t^2} - \frac{\omega_0}{5t} - \frac{\omega_0}{3} + \dots, \qquad \xi^{(2)}_{0,2} = \frac{1}{t^3} - \frac{\omega_0}{5t} + \dots$$

and

$$\xi^{(2)}_{j,1} = \frac{1}{t^2} + \dots, \qquad \xi^{(2)}_{j,2} = -\frac{3}{5}t^2 + \frac{t^3}{5} + \dots.$$

Taking the second term in (4.2.46) $\mu_3 = \xi^{(1)}_{j,1} K^{(3)}_j$ with the choice (4.3.66) and $\xi^{(2)}_0 = \xi^{(2)}_{0,2}, \xi^{(2)}_j = \xi^{(2)}_{j,2}$ one can see that μ_3 has a simple pole with a residue $-\omega_0 72/25$, which is nonzero since $\omega_0 \neq 0$ by assumption.

In either of the cases above, the identity component of the Galois group of (VE_2) or (VE_3) is not commutative and the Hamiltonian system (4.1.8) is not integrable due to Theorem 1.0.5.

We notice that the non-integrability result obtained in this part are also valid for the limiting case $C_0 = 0$ and $C_j = 0, j = 1, \dots, N_f$.

In the general case $C_0 \neq 0$ and $C_j \neq 0, j = 1, \dots, N_f$ (VE) does not split in nice way as in the previous cases. Because of this reason, we consider the system (4.1.8) with two degrees of freedom. Even then, studying the Galois group of (NVE) is not so simple due to the number of the parameters. That is why in subsection 4.2.3 we use a perturbational approach, which is still related to the Differential Galois approach. Furthermore, this approach gives a dynamical meaning to the algebraic obstructions to integrability. Note that, the using Poincaré-Arnold-Melnikov integrals in more degrees of freedom for real Hamiltonian systems needs certain KAM-conditions.

The above results allow us to think that the system (4.1.8) is not integrable unless $g_{\rm BF} = 0$. Moreover, the formulas (4.2.12) and (4.2.23) give the general solution to the separable system ($g_{\rm BF} = 0$).

4.4 Appendix: Necessary conditions for integrability of Hamiltonian systems which have (NVE) of Lamé type

In this appendix we recall some facts concerning the integrability of Hamiltonian systems with two degrees of freedom, an invariant plane and which (NVE) are of Lamé type. More details can be found in [73, 70]. In our case the (NVE) splits into a system of N_f equations of Lamé type, and therefore, these arguments can be applied.

Classically the Lamé equation is written in the form

$$\ddot{\xi} - [n(n+1)\wp(t) + B]\xi = 0, \tag{4.4.67}$$

where $\wp(t)$ is the Weierstrass function with invariants g_2 and g_3, satisfying $\dot{v}^2 = 4v^3 - g_2 v - g_3$ with $\Delta = g_2^3 - 27g_3^2 \neq 0$.

The known (mutually exclusive) cases of closed form solutions of (4.4.67) are:

(i) The Lamé and Hermite solutions. In this case $n \in \mathbb{Z}$ and g_2, g_3, B are arbitrary parameters;

(ii) The Brioschi-Halphen-Crowford solutions. Here $m := n + 1/2 \in \mathbb{N}$ and the parameters g_2, g_3, B must satisfy an algebraic equation.

(iii) The Baldassarri solutions. Now $n+1/2 \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$ with additional algebraic relations between the other parameters.

Note that in the case (i) the identity component of the Galois group G^0 is of the form $\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}$ and in the cases (ii) and (iii) $G^0 = id$ (G is finite). And these are the all cases when the Lamé equation is integrable.

Now consider a natural two degrees of freedom Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{4.4.68}$$

$q_j(t) \in \mathbb{C}, p_j(t) = \dot{q}_j, j = 1, 2$. We assume that there exists a family of solutions of the form

$$\Gamma_h : q_2 = p_2 = 0, \quad q_1 = q_1(t, h), \quad p_1(t, h) = \dot{q}_1(t, h)$$

and $q_1(t, h)$ is a solution of

$$\frac{1}{2}\dot{q}_1^2 + \varphi(q_1) = h, \quad h \in \mathbb{R}.$$

The (NVE) along Γ_h is

$$\ddot{\xi} - \alpha(t, h)\xi = 0, \tag{4.4.69}$$

where $\alpha(t, h) = \alpha(q_1(t, h))$ is such that (4.4.69) is of type (4.4.67).

In [73, 70] the type of the potentials V with this property are obtained as well as the necessary conditions for the integrability of the Hamiltonian systems with the Hamiltonian (4.4.68). In order to formulate the result we need certain additional quantities.

Since $\alpha(t, h)$ depends linearly on $\wp(t)$, then $\dot{\alpha}^2$ is a cubic polynomial in α, depending also in h, namely

$$\dot{\alpha}^2 := P(\alpha, h) = P_1(\alpha) + hP_2(\alpha). \tag{4.4.70}$$

The following coefficients are introduced

$$P(\alpha, h) = (a_1 + ha_2)\alpha^3 + (b_1 + hb_2)\alpha^2 + (c_1 + hc_2)\alpha + (d_1 + hd_2). \tag{4.4.71}$$

Now we are ready to give the corresponding result. Note that the following Theorem gives necessary conditions only from the analysis of the first variational equation.

Theorem 4.4.1. *(Theorem 6.2 [70]). Assume that a natural Hamiltonian system has (NVE) of Lamé type, associated to the family of solutions Γ_h, lying on the plane $q_2 = 0$ and parametrized by the energy h. Then, a necessary conditions for integrability is that the related polynomials P_1 and P_2 satisfy $a_2 = 0$, and one of the following conditions holds:*

1. $a_1 = \frac{4}{n(n+1)}$ for some $n \in \mathbb{N}$;

2. $a_1 = \frac{16}{4m^2-1}$ for some $m \in \mathbb{N}$. Then, assuming the conjecture above is true, one should have $b_2 = 0$ and we should be in one of the following cases:

2.1) $m = 1$ and $b_1 = 0$,

2.2) $m = 2$ and $c_2 = 0$, $16a_1c_1 + 3b_1^2 = 0$,

2.3) $m = 3$ and $16a_1d_2 + 11b_1c_2 = 0$, $1024a_1^2d_1 + 704a_1b_1c_1 + 45b_1^3 = 0$,

2.m) $m > 3$. Then, we should have $b_1 = 0$ and, furthermore, either $c_1 = c_2 = 0$ if m is congruent with $1, 2, 4$ or 5 modulo 6, or $d_1 = d_2 = 0$ if m is odd;

3. $a_1 = \frac{4}{n(n+1)}$ with $n + 1/2 \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$, $b_2 = 0$ and either $c_2 = 0, b_1^2 - 3a_1c_1 = 0$ or $c_2b_1 - 3a_1d_2 = 0, 2b_1^3 - 9a_1b_1c_1 + 27a_1^2d_1 = 0$.

It is clear that the condition 1. in the above Theorem gives the Lamé and Hermite solutions (i), the condition 2.– the Brioschi-Halphen-Crowford solutions (ii), and the condition 3. – the Baldassarri solutions (iii).

Chapter 5

Near Integrability in Low Dimensional Gross-Neveu Models

The low dimensional Gross-Neveu models are studied. For the systems, related to the Lie algebras $\mathrm{so}(4), \mathrm{so}(5), \mathrm{sp}(4), \mathrm{sl}(3)$ we prove that they have Birkhoff-Gustavson normal forms which are integrable and non - degenerate in KAM-theory sense. Unfortunately, this is not the case for three degrees of freedom systems, related to the Lie algebras $\mathrm{so}(6) \sim \mathrm{sl}(4), \mathrm{so}(7), \mathrm{sp}(6)$ – their Birkhoff-Gustavson normal forms are proven to be non-integrable in Liouville sense. The last result can easily be extended to higher dimensions. The results of this chapter are published in [16].

5.1 Introduction and motivation

The Gross-Neveu models are Hamiltonian systems related with the root systems of simple Lie algebras

$$H = \frac{1}{2}(y, y) + \sum_{\alpha} \exp[(\alpha, x)], \tag{5.1.1}$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ are the canonical coordinates in $\mathbb{R}^{2n}$, $(\,,\,)$ denotes the standard inner product and α is a root of a simple Lie algebra $\mathfrak{g}$. The sum is extended over the entire root system of $\mathfrak{g}$ or over its appropriate subspace, depending on the model.

Such models are considered by Shankar [90] in his research on the Gross-Neveu model [31] in the two dimensional field theory. As a matter of fact, physical Gross-Neveu model describing a set of fermionic fields with the local quartic interaction is related with the Lie algebra $\mathrm{o}(2n)$ for small n, but Shankar asked the question about integrability for all simple Lie algebras. So, the Hamiltonian systems (5.1.1) are known as Gross-Neveu models. To mention only such kind of systems with exponential interactions, defined by simple Lie algebras appeared in investigations in two-dimensional classical and quantum field theories and statistical physics.

The Hamiltonian functions (5.1.1), related with the root systems of the classical simple Lie algebras $\mathrm{so}(2n), \mathrm{so}(2n+1), \mathrm{sl}(n+1), \mathrm{sp}(2n)$ are of kind

$$H_{\mathfrak{g}} = \frac{1}{2}\sum_{i=1}^{N} y_i^2 + V_{\mathfrak{g}}(x),$$

where $N = n + 1$ for $\mathrm{sl}(n+1)$ and $N = n$ for the remaining algebras and the potential $V_{\mathfrak{g}}$ has the form

$$
\begin{aligned}
V_{\mathrm{so}(2n)} &= \sum_{i,j=1,\, i>j}^{N} \left(e^{x_i+x_j} + e^{-(x_i+x_j)}\right) + \sum_{i,j=1, i\neq j}^{N} e^{x_i-x_j}, \\
V_{\mathrm{so}(2n+1)} &= \sum_{i=1}^{N}(e^{x_i} + e^{-x_i}) + \sum_{i,j=1,\, i>j}^{N} \left(e^{x_i+x_j} + e^{-(x_i+x_j)}\right) + \sum_{i,j=1,\, i\neq j}^{N} e^{x_i-x_j}, \\
V_{\mathrm{sl}(n+1)} &= \sum_{i,j=1,\, i\neq j}^{N} e^{x_i-x_j}, \\
V_{\mathrm{sp}(2n)} &= \sum_{i=1}^{N}(e^{2x_i} + e^{-2x_i}) + \sum_{i,j=1,\, i>j}^{N} \left(e^{x_i+x_j} + e^{-(x_i+x_j)}\right) + \sum_{i,j=1,\, i\neq j}^{N} e^{x_i-x_j}.
\end{aligned}
$$

Except the Hamiltonian H, we have an obvious first integral only for the case of $\mathrm{sl}(n+1)$, namely $\sum y_i = const.$. Hence, the Gross-Neveu model for $\mathrm{sl}(2)$ is integrable. It turns out that the model for $\mathrm{so}(4)$ is also integrable. The Hamiltonian systems for the remaining cases are non-integrable, more precisely the Hamiltonian systems with two and three degrees of freedom were proven to be non-integrable by Horozov [39] with a modification of Ziglin's method [107] while the rest were proven to be non-integrable by Maciejewski et al. [60] with the differential Galois theory approach.

A motivation for this work is a series of papers of Rink [83, 84] who presented the famous Fermi-Pasta-Ulam (FPU) system as a perturbation of one integrable and KAM non-degenerate system, namely the normal form of order four in the vicinity of an equilibrium. Non-degenerate in KAM theory sense integrable system means that its frequency map is a local diffeomorphism.

Our aim is to check whether this fact is true for the Gross-Neveu models. Unfortunately, this is not the case for the Gross-Neveu models with exceptions of two degrees of freedom cases.

We already have recalled some facts about normal forms of Hamiltonian systems in Chapter 2. Our results are presented in the following theorems.

Theorem 5.1.1. *The Hamiltonian systems, corresponding to the Gross-Neveu models for algebras* $\mathrm{so}(4), \mathrm{so}(5), \mathrm{sp}(4), \mathrm{sl}(3)$ *have Birkhoff-Gustavson normal forms* $\bar{H}^{tr} = H_2 + H_4$ *integrable and non-degenerate in KAM theory sense.*

Theorem 5.1.2. *The Hamiltonian systems, corresponding to the Gross - Neveu models for* $\mathrm{so}(6) \sim \mathrm{sl}(4), \mathrm{so}(7), \mathrm{sp}(6)$ *have non integrable Birkhoff - Gustavson normal forms* $\bar{H}^{tr} = H_2 + H_4$.

One should note that the results are not surprising. The Hamiltonian systems for Gross - Neveu models enjoy $1 : 1 : \ldots : 1$ resonance, as well many symmetries. Due to these symmetries there are no third order resonant terms in the truncated up to order four Hamiltonians. In the two degrees of freedom cases, where these truncations are integrable, it is natural to expect non-degeneracy. From the other side, apparently this resonance and the symmetries are not sufficient to assure integrability in the systems with more degrees of freedom as in the case of FPU chains, which is indeed very rare.

The chapter is organized as follows. The proof of Theorem 5.1.1 is presented in section 2. The systems are naturally divided in two parts. In the first part the integrals are quadratic and in the second part integrals are quartic. Thus, we need different approaches. In section 3 we prove Theorem 5.1.2. The proof is based on Morales - Ramis theory using Differential Galois groups of the linearized system along a particular solution. In fact, we explore only the monodromy group and prove that it is non abelian. As it is contained in the differential Galois group, the result follows from Morales - Ramis theorem. We study so(6) model in details and give the main points for the other cases so(7), sp(6).

5.2 Proof of Theorem 5.1.1

In this section we consider the Gross - Neveu models, related to Lie algebras so(4), so(5), sp(4), sl(3) referred to as low dimensional Gross-Neveu models. They correspond to some two degrees of freedom Hamiltonian systems (sl(3) after reduction). These systems near the origin can be considered as perturbations of their normal forms which are integrable and KAM-non degenerate. These systems naturally fall in two subclasses.

For the cases sl(3) and so(4) the second integral is quadratic. This fact allows us to construct action-angle variables explicitly following [84]. Hence, the corresponding Hamiltonians of the normal forms are easily expressed via action variables, which makes the verification of Kolmogorov's condition straightforward.

For the cases so(5) and sp(4) the second integral is quartic. The expressions of the corresponding Hamiltonians of the normal forms via action variables are not explicit. So, we adopt Horozov's approach [37] for verification of Kolmogorov's condition in these cases. We consider sl(3) and so(5) in details and give the key points for the other cases.

5.2.1 sl(3)

The Gross-Neveu model related with sl(3) is actually a three degrees of freedom system described with the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{3} y_j^2 + (e^{x_1-x_2} + e^{-x_1+x_2}) + (e^{x_2-x_3} + e^{-x_2+x_3}) + (e^{x_1-x_3} + e^{-x_1+x_3}). \tag{5.2.2}$$

It is easy to check that the total momentum $y_1+y_2+y_3$ is conserved. Hence, the motion of the mass center $\frac{1}{3}\sum_{j=1}^{3} x_j$ is linear and therefore unbounded.

In order to follow our aim, we reduce the Hamiltonian (5.2.2) to one with two degrees of freedom with the help of the above integral. Let us perform the following canonical transformation

$$Q_1 = x_1 - x_2, \quad Q_2 = x_2 - x_3, \quad Q_3 = x_1 + x_2 + x_3,$$

$$P_1 = \frac{1}{3}(2y_1 - y_2 - y_3), \quad P_2 = \frac{1}{3}(y_1 + y_2 - 2y_3), \quad P_3 = \frac{1}{3}(y_1 + y_2 + y_3).$$

In these coordinates (5.2.2) reads

$$H = P_1^2 - P_1P_2 + P_2^2 + \frac{3}{2}P_3^2 + (e^{Q_1} + e^{-Q_1}) + (e^{Q_2} + e^{-Q_2}) + (e^{Q_1+Q_2} + e^{-(Q_1+Q_2)}).$$

Hence, Q_3 is cyclic and P_3 is an integral. We leave out P_3 and denote by H_R the reduced Hamiltonian

$$H_R = P_1^2 - P_1P_2 + P_2^2 + (e^{Q_1} + e^{-Q_1}) + (e^{Q_2} + e^{-Q_2}) + (e^{Q_1+Q_2} + e^{-(Q_1+Q_2)}). \tag{5.2.3}$$

The corresponding equations of motion are

$$\dot{Q}_1 = 2P_1 - P_2, \quad \dot{P}_1 = -\left[(e^{Q_1} - e^{-Q_1}) + (e^{Q_1+Q_2} - e^{-(Q_1+Q_2)})\right],$$

$$\dot{Q}_2 = 2P_2 - P_1, \quad \dot{P}_2 = -\left[(e^{Q_2} - e^{-Q_2}) + (e^{Q_1+Q_2} - e^{-(Q_1+Q_2)})\right].$$

The point $(Q, P) = (0, 0)$ is an equilibrium point. Linearizing about it one gets

$$\dot{\xi}_1 = 2\eta_1 - \eta_2, \quad \dot{\eta}_1 = -4\xi_1 - 2\xi_2,$$

$$\dot{\xi}_2 = 2\eta_2 - \eta_1, \quad \dot{\eta}_2 = -2\xi_1 - 4\xi_2.$$

The eigenvalues of the linearized system are $\pm i\sqrt{6}, \pm i\sqrt{6}$ that is $1:1$ resonance. Expanding H_R about $(Q, P) = (0, 0)$ and neglecting irrelevant additive constant, we obtain

$$H_R = P_1^2 - P_1P_2 + P_2^2 + 2(Q_1^2 + Q_2^2 + Q_1Q_2) + \frac{1}{12}\left[Q_1^4 + Q_2^4 + (Q_1 + Q_2)^4\right] + O(||\,Q||^6).$$

First, we bring the quadratic part of the above Hamiltonian in diagonal form via coordinate change

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

and then scale $p_j \to \sqrt[4]{6}p_j, \quad q_j \to q_j/\sqrt[4]{6}$ to obtain

$$H_R = \frac{\sqrt{6}}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{1}{16}(q_1^2 + q_2^2)^2 + O(||\,q||^6).$$

Next, we put

$$q_j = \frac{1}{2}(z_j + w_j), \quad p_j = \frac{1}{2i}(z_j - w_j), \quad j = 1, 2. \tag{5.2.4}$$

The resonant terms of order four are $z_1^2w_1^2$, $z_2^2w_2^2$, $z_1^2w_2^2$, $z_2^2w_1^2$, $z_1w_1z_2w_2$. Since we are interested in the normal form truncated up to order four, we just remove the non-resonant terms and get

$$\bar{H}_R^{tr} = \frac{\sqrt{6}}{2}(z_1w_1 + z_2w_2) + 2^{-7}\left[3(z_1w_1 + z_2w_2)^2 + (z_1w_2 - z_2w_1)^2\right]. \tag{5.2.5}$$

It was mentioned earlier that the truncated normal form has two integrals $H_2 = z_1w_1 + z_2w_2$ and $\bar{H}_R^{tr}$ or equivalently here H_2 and $BB = z_1w_2 - z_2w_1$.

The Hamiltonian $\bar{H}_R^{tr}$ (5.2.5) of truncated up to order four normal form and the quadratic integrals in cartesian coordinates (q, p) take the form

$$\bar{H}_R^{tr} = \frac{\sqrt{6}}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{1}{2^7}\left[3(p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 - 4(p_1q_2 - q_1p_2)^2\right],$$

$$a = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2), \quad b = p_1q_2 - q_1p_2.$$

In order to introduce action variables we need to find the set of regular values of the energy momentum map

$$EM : (p_1, p_2, q_1, q_2) \to (a, b).$$

This is already done in [84]. Denote by $U_r = \{(a, b) \in \mathbb{R}^2, a > 0, |\,b| < a\}$. Then for all $(a, b) \in U_r$, the level sets of $EM^{-1}(a, b)$ are diffeomorphic to two - tori.

Let $\arg : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}/2\pi\mathbb{Z}$ be the argument function $\arg(r\cos\Phi, r\sin\Phi) \to \Phi$. Define the following set of variables (a, b, Φ, Ψ), a, b as above and

$$\Phi = \frac{1}{2}\arg(p_2 - q_1, p_1 + q_2) + \frac{1}{2}\arg(-p_2 - q_1, p_1 - q_2),$$

$$\Psi = \frac{1}{2}\arg(p_2 - q_1, p_1 + q_2) - \frac{1}{2}\arg(-p_2 - q_1, p_1 - q_2).$$

These functions are well defined since $(a, b) \in U_r$. With the formula $d\arg(x, y) = \frac{xdy - ydx}{x^2+y^2}$ one can verify that the set (a, b, Φ, Ψ) are canonical coordinates, actually action-angle coordinates

$$dp_1 \wedge dq_1 + dp_2 \wedge dq_2 = da \wedge d\Phi + db \wedge d\Psi.$$

The truncated Hamiltonian $\bar{H}_R^{tr}$ is a function of actions a, b

$$\bar{H}_R^{tr} = \sqrt{6}a + 2^{-5}(3a^2 - b^2). \tag{5.2.6}$$

Now, the non-degeneracy is straightforward

$$det\left(\frac{\partial^2 \bar{H}_R^{tr}}{\partial a \partial b}\right) = det\begin{pmatrix} \frac{6}{2^5} & 0 \\ 0 & -\frac{2}{2^5} \end{pmatrix} = -3.2^{-8} < 0.$$

5.2.2 so(4)

The Gross-Neveu model related with so(4) is a two degrees of freedom system described with the Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) + e^{x_1+x_2} + e^{-x_1-x_2} + e^{x_2-x_1} + e^{x_1-x_2}, \tag{5.2.7}$$

which is integrable. The second integral is $B = (1/2)(y_1 + y_2)^2 + 2(\exp(x_1 + x_2) + \exp(-x_1 - x_2))$. Nevertheless, we concentrate our attention on the truncated normal form.

The corresponding equations of motion are

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -(e^{x_1+x_2} - e^{-x_1-x_2} - e^{-x_1+x_2} + e^{x_1-x_2}),$$

$$\dot{x}_2 = y_2, \quad \dot{y}_2 = -(e^{x_1+x_2} - e^{-x_1-x_2} + e^{-x_1+x_2} - e^{x_1-x_2}).$$

The point $(x, y) = (0, 0)$ is an equilibrium point. Linearizing about it one gets

$$\dot{\xi}_i = \eta_i, \quad \dot{\eta}_i = -4\xi_i, \quad i = 1, 2.$$

The eigenvalues of the linearized system are $\pm i2, \pm i2$ that is $1:1$ resonance. Expanding H about $(x, y) = (0, 0)$ and neglecting irrelevant additive constant, we obtain

$$H = \frac{1}{2}(y_1^2 + y_2^2) + 2(x_1^2 + x_2^2) + \frac{1}{6}(x_1^4 + x_2^4 + 6x_1^2x_2^2) + O(||\,x||^6).$$

Further, we perform a canonical change of variables $y_j = \sqrt{2}p_j, \quad x_j = q_j/\sqrt{2}, j = 1, 2$ to obtain

$$H = p_1^2 + p_2^2 + q_1^2 + q_2^2 + \frac{1}{24}(q_1^4 + q_2^4 + 6q_1^2q_2^2) + O(||\,q||^6).$$

Next, we put as usual (5.2.4). The resonant terms of order four are already known from the previous subsection. Since we are interested in the normal form truncated up to order four, we just remove the non-resonant terms and get

$$\bar{H}^{tr} = z_1w_1 + z_2w_2 + 2^{-6}\left[(z_1w_1 + z_2w_2)^2 + (z_1w_2 + z_2w_1)^2\right]. \tag{5.2.8}$$

It was mentioned earlier that the truncated normal form has two integrals $H_2 = z_1w_1 + z_2w_2$ and $\bar{H}^{tr}$ or equivalently here H_2 and $BB = z_1w_2 + z_2w_1$.

The Hamiltonian $\bar{H}^{tr}$ (5.2.8) of truncated up to order four normal form and the quadratic integrals in cartesian coordinates (q,p) take the form

$$\bar{H}^{tr} = p_1^2 + p_2^2 + q_1^2 + q_2^2 + \frac{1}{2^6}\left[(p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 + 4(p_1p_2 + q_1q_2)^2\right],$$

$$a = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2), \quad b = p_1p_2 + q_1q_2.$$

As before for $(a,b) \in U_r = \{(a,b) \in \mathbb{R}^2, a > 0, |\,b| < a\}$ the level set of integrals is torus and the following functions are well defined

$$\begin{aligned}\Phi &= -\frac{1}{2}\arg(q_1 - q_2, p_1 - p_2) - \frac{1}{2}\arg(q_1 + q_2, p_1 + p_2),\\ \Psi &= \ \ \frac{1}{2}\arg(q_1 - q_2, p_1 - p_2) - \frac{1}{2}\arg(q_1 + q_2, p_1 + p_2).\end{aligned}$$

One can verify that the set (a, b, Φ, Ψ) are canonical coordinates, actually action-angle coordinates. The truncated Hamiltonian $\bar{H}^{tr}$ is a function of actions a, b

$$\bar{H}^{tr} = 2a + 2^{-4}(a^2 + b^2). \tag{5.2.9}$$

Now, the non-degeneracy is immediate.

5.2.3 so(5)

The Gross-Neveu model related with so(5) is a two degrees of freedom system described with the Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) + e^{x_1} + e^{-x_1} + e^{x_2} + e^{-x_2} + e^{x_1-x_2} + e^{-(x_1-x_2)} + e^{x_1+x_2} + e^{-(x_1+x_2)}. \tag{5.2.10}$$

The corresponding equations of motion are

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -(e^{x_1} - e^{-x_1} + e^{x_1-x_2} - e^{-(x_1-x_2)} + e^{x_1+x_2} - e^{-(x_1+x_2)}),$$

$$\dot{x}_2 = y_2, \quad \dot{y}_2 = -(e^{x_2} - e^{-x_2} - e^{x_1-x_2} + e^{-(x_1-x_2)} + e^{x_1+x_2} - e^{-(x_1+x_2)}).$$

Recall that this system is not integrable [39, 60]. Clearly, $(0,0)$ is an equilibrium point. The eigenvalues of the linearized system are $\pm i\sqrt{6}, \pm i\sqrt{6}$, that is $1:1$ resonance. Expanded around $(0,0)$ Hamiltonian (5.2.10) up to irrelevant constant reads

$$H = \frac{1}{2}(y_1^2 + y_2^2) + 3(x_1^2 + x_2^2) + \frac{1}{4}(x_1^4 + x_2^4 + 4x_1^2x_2^2) + O(||\,x||^6).$$

Next, we scale $x_j = q_j/\sqrt[4]{6}, y_j = p_j\sqrt[4]{6}$, put as usual (5.2.4), remove the non - resonant terms and get the normal form up to order four

$$\bar{H}^{tr} = \frac{\sqrt{6}}{2}(z_1w_1+z_2w_2)+\frac{1}{3.2^6}\left[3(z_1w_1+z_2w_2)^2+\frac{3}{2}(z_1w_2+z_2w_1)^2+\frac{1}{2}(z_1w_2-w_1z_2)^2\right]. \quad (5.2.11)$$

As we know, the truncated normal form is integrable and the two integrals are H_2 and $\bar{H}^{tr}$ or equivalently H_2 and $BB = 3(z_1w_2+z_2w_1)^2+(z_1w_2-w_1z_2)^2$.

Next, we put

$$z_j = \sqrt{2a_j}e^{-i\psi_j}, \quad w_j = \sqrt{2a_j}e^{i\psi_j} \quad (5.2.12)$$

and after that we perform the following canonical change of variables

$$J_1 = \frac{a_1+a_2}{2},\ J_2 = \frac{a_1-a_2}{2},\ \chi_1 = \psi_1+\psi_2,\ \chi_2 = \psi_1-\psi_2 \quad (5.2.13)$$

to obtain

$$\bar{H}^{tr} = 2\sqrt{6}J_1 + \frac{1}{24}\left[6J_1^2+(J_1^2-J_2^2)(2\cos(2\chi_2)+1)\right]. \quad (5.2.14)$$

So, χ_1 is a cyclic variable and J_1 is a first integral. Note that in these coordinates, the symplectic form is the exact two - form $d\sigma$, where

$$\sigma = J_1d\chi_1 + J_2d\chi_2. \quad (5.2.15)$$

In order to get rid of the linear term in $\hat{H}^{tr}$ we continue with the canonical transformation

$$J_j \to J'_j,\ \chi_1 \to \chi'_1 + 2\sqrt{6}t,\ \chi_2 \to \chi'_2,\ \bar{H}^{tr} \to \bar{H'}^{tr}.$$

To simplify the notations we drop the primes and the multiplier 1/24 and reach the Hamiltonian, we will work with

$$\bar{H}^{tr} = 6J_1^2 + (J_1^2-J_2^2)(2\cos(2\chi_2)+1), \quad (5.2.16)$$

which admits the integrals $\bar{H}^{tr} = h$ and $F = J_1 = f \geq 0$.

In order to construct the action variables, we need to find the set of regular values of the energy momentum map

$$EM : (J_1, J_2, \chi_1, \chi_2) \to (\bar{H}^{tr}, F).$$

These turn out to be

$$U_r = U_{r1} \cup U_{r2}, \quad (5.2.17)$$

where $U_{r1} = \{(h,f) \in \mathbb{R}^2, f > 0, 6f^2 > h > 5f^2\}$ and $U_{r2} = \{(h,f) \in \mathbb{R}^2, f > 0, 9f^2 > h > 6f^2\}$. Moreover, for each $(h,f) \in U_r$, the level set $EM^{-1}(h,f)$ is a two - torus $T_{h,f}$.

Choose a basis γ_1, γ_2 of the homology group $H_1(T_{h,f}, \mathbb{Z})$ with the following representatives. For γ_1 we take the circle on $T_{h,f}$ defined by fixing χ_2, J_1 and J_2 and letting χ_1 run through $[0, 2\pi]$. For γ_2 we fix χ_1 and let J_2, χ_2 make one circle on the curve given by the equation

$$6f^2 + (f^2-J_2^2)(2\cos 2\chi_2+1) = h.$$

The corresponding action variables $I_j = \int_{\gamma_j}\sigma$, where σ is the one - form (5.2.15), have the following form

$$I_1 = 2\pi f, \quad I_2 = 2\int_{\chi_2^-}^{\chi_2^+}\sqrt{\frac{f^2(2\cos 2\chi_2+7)-h}{2\cos 2\chi_2+1}}\,d\chi_2, \quad (5.2.18)$$

where $\chi_2^- < \chi_2^+$ are the two roots of the equation

$$f^2 - \frac{h - 6f^2}{2\cos 2\chi_2 + 1} = 0 \quad \text{in} \quad (0, \pi).$$

Put $z = \cos 2\chi_2, |z| \leq 1, y^2 = (2z+1)(1-z^2)(f^2(2z+7)-h)$ and denote by γ an oval of the curve

$$\Gamma_{h,f} = \{(y,z) \in \mathbb{C}^2 : y^2 = (2z+1)(1-z^2)(f^2(2z+7)-h)\}.$$

Then we have

$$\psi(h,f) \overset{def}{=} I_2 = \int_\gamma \frac{y dz}{(2z+1)(1-z^2)} \tag{5.2.19}$$

Denote by $H(I_1, I_2)$ the Hamiltonian of the truncated normal form (5.2.16) expressed in action variables. Earlier in [37] it was proven that

$$(2\pi)^2(\psi_h)^4 \det\left(\frac{\partial^2 H}{\partial I_i \partial I_j}\right) = \det\begin{pmatrix}\psi_{hh} & \psi_{hf}\\ \psi_{fh} & \psi_{ff}\end{pmatrix}. \tag{5.2.20}$$

Since

$$\psi_h = -\frac{1}{2}\int_\gamma \frac{dz}{y} \neq 0 \quad \text{in} \quad U_r,$$

one can see that Kolmogorov's condition is equivalent to the condition that $D = \psi_{hh}\psi_{ff} - (\psi_{hf})^2 \neq 0$. In the following we will express D in terms of Abelian integrals. Since we can homotope the curve γ to another without changing ψ, it follows that we can take partial derivatives under the integral sign. Denoting by E the integral $E = \int_\gamma \frac{(2z+1)(2z+7)(1-z^2)}{y^3} dz$ we get successively the following expressions for the derivatives of ψ

$$\psi_{hh} = -\frac{1}{4}\int_\gamma \frac{(2z+1)(1-z^2)}{y^3} dz, \quad \psi_{hf} = \frac{f}{2}E, \quad \psi_{ff} = -hE.$$

From here D becomes

$$D = \frac{1}{4}E\int_\gamma \frac{h(2z+1)(1-z^2) - f^2(2z+1)(2z+7)(1-z^2)}{y^3} dz = \frac{1}{2}\psi_h E,$$

that is, $D \neq 0 \leftrightarrow E \neq 0$ in U_r. Note that

$$E = \frac{2}{f}\psi_{hf} = \frac{2}{f}\frac{\partial}{\partial f}\psi_h = -\frac{1}{f}\frac{\partial}{\partial f}\left(\int_\gamma \frac{dz}{y}\right). \tag{5.2.21}$$

To show that $E \neq 0$, we first consider $(h,f) \in U_{r1}$. Then $z_3 = \frac{1}{2}(\frac{h}{f^2} - 7) \in (-1, -1/2)$ and

$$\int_\gamma \frac{dz}{y} = 2\int_{-1}^{z_3} \frac{dz}{\sqrt{4f^2(z+1/2)(1-z^2)(z-z_3)}} = \frac{4}{f\sqrt{2(1-z_3)}} K\left(\sqrt{\frac{3(z_3+1)}{1-z_3}}\right), \tag{5.2.22}$$

where $K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$ is the complete elliptic integral of first kind. By putting $k = \sqrt{\frac{3(z_3+1)}{1-z_3}}$, $k \in (0,1)$, we obtain that $f = \sqrt{\frac{h}{3}}\sqrt{\frac{k^2+3}{3k^2+5}}$. Then (5.2.22) becomes

$$\int_\gamma \frac{dz}{y} = \frac{2}{\sqrt{h}}\sqrt{3k^2+5}K(k).$$

Therefore,

$$E = -\frac{2}{f\sqrt{h}}\frac{1}{f'(k)}\frac{\partial}{\partial k}\left(\sqrt{3k^2+5}\,K(k)\right) \neq 0,$$

since $K(k)$ is an increasing function in k.

Next, consider $(h, f) \in U_{r2}$. In this case $z_3 \in (-1/2, 1)$ and

$$\int_\gamma \frac{dz}{y} = 2\int_{-1}^{-1/2} \frac{dz}{\sqrt{4f^2(z+1/2)(1-z^2)(z-z_3)}} = \frac{4}{f\sqrt{6(1+z_3)}}K\left(\sqrt{\frac{1-z_3}{3(1+z_3)}}\right). \quad (5.2.23)$$

Put $k = \sqrt{\frac{1-z_3}{3(1+z_3)}}$, $k \in (0, 1)$. Then, we obtain that $f = \sqrt{\frac{h}{3}}\sqrt{\frac{3k^2+1}{5k^2+3}}$. Thus, (5.2.23) reads

$$\int_\gamma \frac{dz}{y} = \frac{2}{\sqrt{h}}\sqrt{5k^2+3}K(k).$$

From this we get

$$E = -\frac{2}{f\sqrt{h}}\frac{1}{f'(k)}\frac{\partial}{\partial k}\left(\sqrt{5k^2+3}\,K(k)\right) \neq 0$$

due to above mentioned arguments.

5.2.4 sp(4)

The Gross-Neveu model related with sp(4) is a two degrees of freedom system described with the Hamiltonian

$$H = \frac{1}{2}(y_1^2+y_2^2) + e^{2x_1} + e^{-2x_1} + e^{2x_2} + e^{-2x_2} + e^{x_1-x_2} + e^{-(x_1-x_2)} + e^{x_1+x_2} + e^{-(x_1+x_2)}. \quad (5.2.24)$$

The equations of motion can be written in the standard way and one can obtain the eigenvalues of the linearized equations about the equilibrium as $\pm i\sqrt{12}, \pm i\sqrt{12}$ that is they are in $1:1$ resonance. After similar transformations as in the previous cases, the truncated up to order four normal form is

$$\bar{H}^{tr} = \sqrt{3}(z_1w_1+z_2w_2) + \frac{1}{3.2^6}\left[9(z_1w_1+z_2w_2)^2 - 3(z_1w_2+z_2w_1)^2 + 4(z_1w_2-w_1z_2)^2\right]. \quad (5.2.25)$$

We know that the truncated normal form is integrable and the two integrals are H_2 and $\bar{H}^{tr}$ or equivalently H_2 and $BB = -3(z_1w_2+z_2w_1)^2 + 4(z_1w_2-w_1z_2)^2$.

Performing consequently the changes of variables (5.2.12), (5.2.13) and removing the linear term, we reduce $\bar{H}^{tr}$ to a Hamiltonian with a cyclic variable

$$\bar{H}^{tr} = 18J_1^2 + (J_1^2 - J_2^2)(\cos(2\chi_2) - 7), \quad (5.2.26)$$

which admits the integrals $\bar{H}^{tr} = h$ and $F = J_1 = f \geq 0$.

The regular values of the energy momentum mapping here are

$$U_r = \{(h, f) \in \mathbb{R}^2,\ f > 0,\ 10f^2 < h < 12f^2\}.$$

Then the corresponding action variables are

$$I_1 = 2\pi f, \quad I_2 = 2\int_{\chi_2^-}^{\chi_2^+} \sqrt{\frac{h - f^2(\cos 2\chi_2 + 11)}{7 - \cos 2\chi_2}}\, d\chi_2, \quad (5.2.27)$$

where $\chi_2^- < \chi_2^+$ are the two roots of the equation

$$h - f^2\left(\cos(2\chi_2) + 11\right) = 0 \quad \text{in} \quad (0, \pi).$$

Now we put $z = \cos 2\chi_2, |z| \le 1, y^2 = (7-z)(1-z^2)(h - f^2(z+11))$ and denote by γ the oval of the curve

$$\Gamma_{h,f} = \{(y,z) \in \mathbb{C}^2 : y^2 = (7-z)(1-z^2)(h - f^2(z+11))\}.$$

Then we have

$$\psi(h,f) \stackrel{def}{=} I_2 = \int_\gamma \frac{ydz}{(7-z)(1-z^2)} \tag{5.2.28}$$

Since $\psi_h = \frac{1}{2}\int_\gamma \frac{dz}{y} \neq 0$ in U_r from (5.2.20) it is seen that in order to verify Kolmogorov's condition, one needs to show that the Hessian of the function ψ - $D = \psi_{hh}\psi_{ff} - (\psi_{hf})^2$ is nonzero. We again express the entries of D via Abelian integrals. Denote this time $E = \int_\gamma \frac{(z+11)(7-z)(1-z^2)}{y^3} dz$. Then

$$\psi_{hh} = -\frac{1}{4}\int_\gamma \frac{(7-z)(1-z^2)}{y^3}dz, \quad \psi_{hf} = \frac{f}{2}E, \quad \psi_{ff} = -hE.$$

Hence

$$D = \frac{1}{4}E\int_\gamma \frac{h(7-z)(1-z^2) - f^2(z+1)(7-z)(1-z^2)}{y^3}dz = \frac{1}{4}E\int_\gamma \frac{dz}{y} = \frac{1}{2}\psi_h E$$

and $D \neq 0 \leftrightarrow E \neq 0$. As before, E can be presented in the following way

$$E = \frac{2}{f}\psi_{hf} = \frac{1}{f}\frac{\partial}{\partial f}\int_\gamma \frac{dz}{y}.$$

To show that $E \neq 0$, we consider $(h,f) \in U_r$. Then $z_2 = \frac{h}{f^2} - 11 \in (-1,1)$ and

$$\int_\gamma \frac{dz}{y} = 2\int_{-1}^{z_2} \frac{dz}{\sqrt{f^2(7-z)(1-z^2)(z_2-z)}} = \frac{2\sqrt{2}}{f\sqrt{7-z_2}}K\left(\sqrt{\frac{3(z_3+1)}{7-z_2}}\right). \tag{5.2.29}$$

By putting $k = \sqrt{\frac{3(z_2+1)}{7-z_2}}, k \in (0,1)$ we obtain that $f = \sqrt{\frac{h}{6}}\sqrt{\frac{k^2+3}{3k^2+5}}$. Then

$$E = \frac{\sqrt{2}}{f\sqrt{h}}\frac{1}{f'(k)}\frac{\partial}{\partial k}\left(\sqrt{3k^3+5}K(k)\right) \neq 0.$$

This completes the proof of Theorem 5.1.1. ∎

Remark 1. The variables

$$W_0 = \frac{1}{2}(z_1w_1 + z_2w_2), W_1 = \frac{i}{2}(z_1w_2 - z_2w_1), W_2 = \frac{1}{2}(z_1w_2 + z_2w_1), W_3 = \frac{1}{2}(z_2w_2 - z_1w_1)$$

are known as Hopf variables. They satisfy the relation $W_1^2 + W_2^2 + W_3^2 = W_0^2$. In fact, every truncated normalized Hamiltonian with two equal frequencies can be written as a function of these variables [56]. See also [23] for a nice geometrical treatment of some classical integrable systems using these variables.

Remark 2. Kummer [56] along his studies on periodic solutions of Hamiltonians with two equal frequencies, verifies Arnold-Moser's condition. Let us show how the condition (1.0.5) can be treated in these particular cases. For the cases sl(3) (reduced) and so(4) one can obtain that the determinant D_1 is not zero from the Hamiltonians (5.2.6) and (5.2.9) respectively, since a, b are the action variables. For the cases so(5) and sp(4) we will show that the map (1.0.5) $F_h, h = const$ is regular in U_r. Note that f can be taken as a coordinate on the set $h = const$ in U_r, so $F_h = F_h(f)$. One can infer from [38] that

$$F_h(f) = -\frac{1}{2\pi}\psi_f.$$

Hence, F_h is regular iff $\psi_{ff} \neq 0$ in U_r. But this is indeed the case because $\psi_{ff} = -hE \neq 0$.

5.3 Proof of Theorem 5.1.2

In this section we consider the Hamiltonian systems with three degrees of freedom, describing the Gross-Neveu models, corresponding to Lie algebras so(6) $\sim$ sl(4), sp(6) and so(7). As it was mentioned above, they are non-integrable. Here, we will show that truncated normal forms up to order four are also non-integrable in Liouville sense. The proof is based on the Morales-Ramis method using Differential Galois theory.

Having the truncated up to order 4 Hamiltonian we bring it to the truncated normal form with near-identity symplectic transform, which preserves integrability [35]. This means that the truncated Hamiltonian and the truncated normal form are simultaneously integrable or non-integrable. In this case it is more convenient to prove the non-integrability for the truncated Hamiltonians, corresponding to the above algebras. We consider the so(6) case in details and give the key points for the other cases.

5.3.1 so(6)

Let us recall the Hamiltonian for the so (6) Gross-Neveu model

$$H = \frac{1}{2}\sum_{j=1}^{3} y_j^2 + \sum_{3\geq j>k\geq 1} e^{x_j+x_k} + e^{-x_j-x_k} + \sum_{j\neq k} e^{x_j-x_k}. \tag{5.3.30}$$

The corresponding equations of motion are

$$\dot{x}_j = y_j, \quad \dot{y}_j = -\sum_{k\neq j}\left(e^{x_j+x_k} - e^{-x_j-x_k}\right) - \sum_{k\neq j}\left(e^{x_j-x_k} - e^{x_k-x_j}\right). \tag{5.3.31}$$

After linearization about the stationary point $(x, y) = (0, 0)$ we obtain

$$\dot{\xi}_j = \eta_j, \quad \dot{\eta}_j = -8\xi_j, \qquad j = 1, 2, 3.$$

The eigenvalues of this system are $\pm 2\sqrt{2}i, \pm 2\sqrt{2}i, \pm 2\sqrt{2}i$, and thus in $1:1:1$ resonance.

Next, we expand the Hamiltonian H around $(x, y) = (0, 0)$ and truncate it to order four to obtain

$$H^{tr} = \frac{1}{2}\sum_{j=1}^{3}\left(y_j^2 + 8x_j^2\right) + \frac{1}{3}(x_1^4 + x_2^4 + x_3^4) + (x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_2^2). \tag{5.3.32}$$

In what follows, we consider the complexified system with the Hamiltonian H^{tr}, that is $(x_j(t), y_j(t)) \in \mathbb{C}^6, t \in \mathbb{C}$. The corresponding equations read

$$\dot{x}_j = y_j, \quad \dot{y}_j = -8x_j - \frac{4}{3}x_j^3 - 2x_j\left(\sum_{k\neq j} x_k^2\right). \tag{5.3.33}$$

It is easy to be seen that the equations (5.3.33) have a family of phase curves

$$\Gamma(h) : y_1^2 = 2h - 8x_1^2 - \frac{2}{3}x_1^4, \quad x_2 = x_3 = y_2 = y_3 = 0. \tag{5.3.34}$$

These curves can be parameterized as follows

$$x_1 = \sqrt{\lambda_1}\mathrm{dn}(\sqrt{\frac{2}{3}\lambda_1}\, t, k),\ y_1 = \dot{x}_1, \quad x_2 = x_3 = y_2 = y_3 = 0, \tag{5.3.35}$$

where dn is the Jacobi elliptic function [103] and λ_1, λ_2 are the roots of $\frac{2}{3}\lambda^2 + 8\lambda - 2h = 0$, $|\lambda_1| > |\lambda_2|$, $k' = \sqrt{\frac{\lambda_2}{\lambda_1}}, k' = \sqrt{1-k^2}$. So, we have a particular solution (5.3.35).

The function $\mathrm{dn}(\tau, k)$ has two periods $T_1 = \frac{2K}{\sqrt{\frac{2}{3}\lambda_1}}$, $T_2 = \frac{4iK'}{\sqrt{\frac{2}{3}\lambda_1}}$ $(K'(k) = K(k'))$ and two simple poles $t_0 = \frac{iK'}{\sqrt{\frac{2}{3}\lambda_1}}$, $t_1 = \frac{3iK'}{\sqrt{\frac{2}{3}\lambda_1}}$ in the parallelogram of periods. Geometrically, the curves $\Gamma(h)$ are complex tori with two points removed.

In order to reduce the domain of the solution (5.3.35) we consider the involution

$$\mathrm{R} : (x_1, y_1, x_2, y_2, x_3, y_3) \to (-x_1, -y_1, x_2, y_2, x_3, y_3),$$

which leaves the system (5.3.33) invariant, it maps the phase curves $\Gamma(h)$ onto themselves and it interchanges the places of the two missing points. Then, $\hat{\Gamma}(h) = \Gamma(h)/\mathrm{R}$ are tori with one point removed. Let F_R be the set of the fixed points of the involution R i.e. $F_R := (0, 0, x_2, y_2, x_3, y_3)$. Then, we can factorize M/F_R by R to obtain a symplectic manifold $\hat{M}$. The Hamiltonian H^{tr} (5.3.32) is naturally mapped into a Hamiltonian function $\hat{H}^{tr}$ on $\hat{M}$. From [107] we know that if the system (5.3.33) has three independent integrals, then the system defined by $\hat{H}^{tr}$ has also three independent integrals.

Next, we need the normal variational equations (NVE) along $\hat{\Gamma}(h)$. It is straightforward that NVE have the form

$$\dot{\xi}_j = \eta_j, \quad \dot{\eta}_j = -8\xi_j - 2x_1^2(t)\xi_j, \quad j = 2, 3. \tag{5.3.36}$$

The NVE splits into two equal subsystems each of them can be written as a second order linear differential equation $\ddot{\xi}_j + (8 + 2x_1^2(t))\xi_j = 0$, $j = 2, 3$. In order to prove non-integrability we need to show that the Galois group G_j corresponding to at least one of them is non abelian. Since they are equal, we consider one of them and drop the index

$$\ddot{\xi} + f(t)\xi = 0, \tag{5.3.37}$$

where $f(t) = 8 + 2\lambda_1\mathrm{dn}^2(\sqrt{\frac{2}{3}\lambda_1}\, t, k)$.

The function $f(t)$ has periods $T_1, T_2/2$ and in the parallelogram of these periods has only one pole – t_0. The equation (5.3.37) is of Fuchsian type. It is known that is this case the monodromy group topologically generates the Galois group [70, 11]. The differential Galois group of (5.3.37) is an algebraic subgroup of $SL(2, L)$ which is connected. Here L is the field of all elliptic functions. Now, we shall study the monodromy group $\mathcal{M}$ of the equation (5.3.37).

Let α_1 be a path over $\hat{\Gamma}(h)$ which corresponds to adding of period T_1, and α_2 be a path over $\hat{\Gamma}(h)$ which corresponds to adding of period $T_2/2$. Let $g_1 := g(\alpha_1)$ and $g_2 := g(\alpha_2)$ be the monodromy transformations which correspond to the closed paths α_1 and α_2 on $\hat{\Gamma}(h)$, respectively. The commutator $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ is the transformation which corresponds to one winding around the regular singular point t_0 of the equation (5.3.37).

It is known [20] that the eigenvalues of the commutator are given by $\exp(2\pi i \rho_{1,2})$, where $\rho_{1,2}$ are the roots of indicial equation

$$\rho(\rho - 1) + f_0 = 0$$

and where f_0 is the coefficient of the $(t - t_0)^{-2}$ in the Laurent expansion of $f(t)$. Since $f(t) = 8+2\lambda_1 \left(\frac{-i}{\sqrt{\frac{2}{3}\lambda_1}(t-t_0)} \right)^2 + \ldots = -\frac{3}{(t-t_0)^2} + \ldots$ we have $f_0 = -3$. Then, the commutator has eigenvalues $\exp(\pi i(1 \pm \sqrt{13}))$ that is $[g_1, g_2] \neq id$, so $\mathcal{M}$ is not abelian and hence G is not abelian too. According to Morales-Ramis theorem the truncated to order four Hamiltonian (also the truncated normal form) for the Lie algebra so(6) is non-integrable.

5.3.2 so(7)

The Hamiltonian for the so(7) Gross-Neveu model is

$$H = \frac{1}{2}\sum_{j=1}^{3} y_j^2 + \sum_{j=1}^{3}(e^{x_j} + e^{-x_j}) + \sum_{3\geq j>k\geq 1} e^{(x_j+x_k)} + e^{-(x_j+x_k)} + \sum_{j\neq k} e^{(x_j - x_k)}. \quad (5.3.38)$$

The corresponding equations of motion are

$$\dot{x}_j = y_j, \quad \dot{y}_j = -(e^{x_j} - e^{-x_j}) - \sum_{k\neq j}\left(e^{(x_j+x_k)} - e^{-(x_j+x_k)}\right) - \sum_{k\neq j}\left(e^{(x_j-x_k)} - e^{(x_k-x_j)}\right). \quad (5.3.39)$$

After linearization about the stationary point $(x, y) = (0, 0)$ we obtain

$$\dot{\xi}_j = \eta_j, \quad \dot{\eta}_j = -10\xi_j, \qquad j = 1, 2, 3.$$

The eigenvalues of this system are $\pm\sqrt{10}i, \pm\sqrt{10}i, \pm\sqrt{10}i$, and thus in $1 : 1 : 1$ resonance.

Next, we expand the Hamiltonian H around $(x, y) = (0, 0)$ and truncate it to order four to obtain

$$H^{tr} = \frac{1}{2}\sum_{j=1}^{3}\left(y_j^2 + 5x_j^2\right) + \frac{5}{12}(x_1^4 + x_2^4 + x_3^4) + (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_2^2). \quad (5.3.40)$$

The equations of motion of the truncated Hamiltonian read

$$\dot{x}_j = y_j, \quad \dot{y}_j = -10x_j - \frac{5}{3}x_j^3 - 2x_j\left(\sum_{k\neq j} x_k^2\right). \quad (5.3.41)$$

The curves $\Gamma(h)$ are given by the equations

$$\Gamma(h) : y_1^2 = 2h - 10x_1^2 - \frac{5}{6}x_1^4, \quad x_2 = x_3 = y_2 = y_3 = 0 \tag{5.3.42}$$

and correspond to the solutions

$$x_1 = \sqrt{\lambda_1}\mathrm{dn}(\sqrt{\frac{5}{6}\lambda_1}\, t, k),\ y_1 = \dot{x}_1, \quad x_2 = x_3 = y_2 = y_3 = 0. \tag{5.3.43}$$

The corresponding Fuchsian equation is (5.3.37) with $f(t) = 10 + 2\lambda_1 \mathrm{dn}^2(\sqrt{\frac{5}{6}\lambda_1}\, t, k)$. The eigenvalues of the commutator are $\exp(\pi i(1 \pm \sqrt{53/5}))$ that is $[g_1, g_2] \neq id$, so $\mathcal{M}$ is not abelian and hence G is not abelian too. According to Morales-Ramis theorem the truncated to order four Hamiltonian (also the truncated normal form) for the Lie algebra so(7) is non - integrable.

5.3.3 sp(6)

The Hamiltonian for the sp(6) Gross-Neveu model is

$$H = \frac{1}{2}\sum_{j=1}^{3} y_j^2 + \sum_{j=1}^{3}(e^{2x_j} + e^{-2x_j}) + \sum_{3\geq j>k\geq 1} e^{(x_j+x_k)} + e^{-(x_j+x_k)} + \sum_{j\neq k} e^{(x_j - x_k)}. \tag{5.3.44}$$

The corresponding equations of motion are

$$\dot{x}_j = y_j, \quad \dot{y}_j = -2(e^{2x_j} - e^{-2x_j}) - \sum_{k\neq j}\left(e^{(x_j+x_k)} - e^{-(x_j+x_k)}\right) - \sum_{k\neq j}\left(e^{(x_j-x_k)} - e^{(x_k-x_j)}\right). \tag{5.3.45}$$

After linearization about the stationary point $(x, y) = (0, 0)$ we obtain

$$\dot{\xi}_j = \eta_j, \quad \dot{\eta}_j = -16\xi_j, \qquad j = 1, 2, 3.$$

The eigenvalues of this system are $\pm 4i, \pm 4i, \pm 4i$, and thus in $1:1:1$ resonance.

Next, we expand the Hamiltonian H around $(x, y) = (0, 0)$ and truncate it to order four to obtain

$$H^{tr} = \frac{1}{2}\sum_{j=1}^{3}\left(y_j^2 + 16x_j^2\right) + \frac{5}{3}(x_1^4 + x_2^4 + x_3^4) + (x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_2^2). \tag{5.3.46}$$

The equations of motion of the truncated Hamiltonian read

$$\dot{x}_j = y_j, \quad \dot{y}_j = -16x_j - \frac{10}{3}x_j^3 - 2x_j\left(\sum_{k\neq j} x_k^2\right). \tag{5.3.47}$$

The curves $\Gamma(h)$ are given by the equations

$$\Gamma(h) : y_1^2 = 2h - 16x_1^2 - \frac{10}{3}x_1^4, \quad x_2 = x_3 = y_2 = y_3 = 0 \tag{5.3.48}$$

and correspond to the solutions

$$x_1 = \sqrt{\lambda_1}\mathrm{dn}(\sqrt{\frac{10}{3}\lambda_1}\, t, k),\ y_1 = \dot{x}_1, \quad x_2 = x_3 = y_2 = y_3 = 0. \tag{5.3.49}$$

The corresponding Fuchsian equation is (5.3.37) with $f(t) = 16 + 2\lambda_1 \mathrm{dn}^2(\sqrt{\frac{10}{3}\lambda_1}\, t, k)$. The eigenvalues of the commutator are $\exp(\pi i(1 \pm \sqrt{17/5}))$ that is $[g_1, g_2] \neq id$, so $\mathcal{M}$ is not abelian and hence G is not abelian too. According to Morales-Ramis theorem the truncated to order four Hamiltonian (also the truncated normal form) for the Lie algebra sp(6) is non-integrable.

This finishes the proof of Theorem 5.1.2. ∎

5.4 Conclusions

In this chapter, low dimensional Gross-Neveu models are studied. The aim is to establish whether their normal forms up to order four are integrable and non-degenerate in KAM-theory sense. It turns out that this fact holds only for two degrees of freedom systems. Application of KAM-theory implies that there exist many invariant tori on which motion is quasi periodic. Moreover, the result of Theorem 5.1.1 partially explains the following situation. Maciejewski et al. in [60] carried out some numerical experiments in order to understand a dynamical meaning for proved non - integrability of the Gross-Neveu systems. They take two systems, related with the Lie algebras sl(3) and so(5). Using suitable Poincaré cross-sections, they observe for these systems very regular structures of the phase portraits. They conclude that if the chaos is present in these systems then it must be smaller than the numerical precision. The answer is that in the neighborhood of the equilibrium, the Hamiltonians corresponding to the above Lie algebras can be considered as perturbations of their KAM non-degenerate normal forms. So, the result of Theorem 5.1.1 is numerically confirmed. For the essentially three degrees of freedom Gross-Neveu models we prove that they do not have integrable truncated normal form. In fact, this means that in vicinity of equilibrium they do not possess more than two integrals. This result can be proven in higher dimensions also.

Chapter 6

Non-integrability of the Karabut system

In order to characterize the solitary wave in a fluid of finite depth, Witting introduced a specific power series (the Witting series). Karabut demonstrated that the problem of summation of the Witting series is brought to the integration of a particular system of ordinary differential equations and solved this system in the cases when the number of the equations is three or four. We give a simple proof that the Karabut system of five equations is already non-integrable in non-Hamiltonian sense using the Differential Galois approach. The results of this chapter are published in [19].

6.1 Description of the Problem

We study a plane vortex-free stationary flow of an ideal incompressible heavy fluid over a flat bottom. Detailed formulation of the problem can be found in Karabut [48, 49, 50] and we follow mainly his description.

Let (X, Y) be a Cartesian coordinate system with X-axis aligned along the bottom, h_0 be the depth of the unperturbed fluid at infinity and u_0 be the velocity of the flow at infinity (Fig. 1).

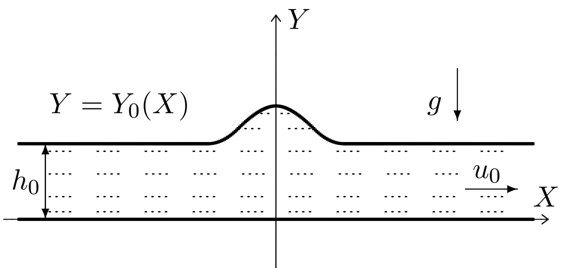


Figure 1.

The problem of constructing a solitary wave is reduced to the finding of a solution in the form $Y = Y_0(X)$, which fulfils the condition $\lim_{|X|\to\infty} Y_0(X) = h_0$. This task hinges on a single parameter. Usually the Stokes parameter θ $(0 \le \theta < \pi/2)$ is taken which in its turn is related with the Froude number $(Fr = u_0/\sqrt{gh_0} > 1)$ via $\tan\theta/\theta = Fr^2$.

Denote by Φ the velocity potential and by Ψ the streamline function. In the plane of the non-dimensional complex potential $\chi = \varphi + i\psi = \theta(\Phi + \Psi)/h_0 u_0$, the strip

$$-\infty < \varphi < \infty \quad (0 < \psi < \theta) \tag{6.1.1}$$

is in correspondence with the fluid. The solitary-wave problem will be solved if we obtain the conformal map of this strip onto the flow area. This map can be written as $Z = X + iY = \frac{h_0}{\theta}(\chi + W(\chi))$. Here the function $W(\chi)$ is a solution of the following boundary-value problem

$$\left|\frac{d(W+\chi)}{d\chi}\right|^2 = \tfrac{1}{1-2\nu \mathrm{Im}W}, \quad \nu = \cot\theta \quad (\psi = \theta,\, \varphi < \varphi_0), \tag{6.1.2}$$

$$\mathrm{Im}W = 0, \qquad (\psi = 0,\, \varphi < \varphi_0), \tag{6.1.3}$$

$$\lim_{\varphi\to-\infty} \mathrm{Im}W = 0. \tag{6.1.4}$$

Note that among the solutions of (6.1.2)–(6.1.4), the solitary-wave is subjected to the condition

$$\varphi_0 = +\infty, \quad \lim_{\varphi\to\infty} \mathrm{Im}W = 0. \tag{6.1.5}$$

In a number of earlier papers the solution of the problem of finding a solitary wave is represented by series of different types (see [104, 48, 49, 50]).

For instance, if we are looking for a solution of (6.1.2)–(6.1.4) presented as a series of the kind

$$W = \sum_{j=1}^{\infty} \theta^{2j} W^j(\chi) \tag{6.1.6}$$

this yields the shallow-water expansion. It occurs that the functions $W^j(\chi)$ can be given as polynomials of $\cosh^{-2}(\frac{\chi}{2})$. Then it is reasonably to put $\zeta = e^{\chi}$ and to rewrite (6.1.6) as

$$W = \sum_{j=1}^{\infty} E_j(\theta)\zeta^j, \quad \mathrm{Im}E_j = 0. \tag{6.1.7}$$

This type of series was suggested by Witting [104]. One can easily obtain recurrent formulas for the coefficients E_j: E_1 can be any positive number.

This series has been investigated numerically for $\theta = \pi/3$ and $\theta = \pi/4$ in [104]. Karabut [48, 49, 50] has shown that for $\theta = m\pi/n$, where m and n are integers, the problem of exact summation of the Witting series is equivalent to the solution of a special system of n ordinary differential equations. The following functions are introduced

$$P_j(\chi) = W(\zeta\omega^{2j-2}), \quad \omega = e^{i\theta}, \quad j = 1,\ldots,n \tag{6.1.8}$$

and it turns out that they satisfy the following system

$$\left(\frac{dP_{j+1}}{d\chi} + 1\right)\left(\frac{dP_j}{d\chi} + 1\right) = \frac{1}{f_j}, \quad P_{n+1} \equiv P_1, \quad j = 1,\ldots,n, \tag{6.1.9}$$

where $f_j = 1 + i\nu(P_{j+1} - P_j)$. Therefore, to deal with the boundary-value problem (6.1.2) – (6.1.4) in the form of the Witting series (6.1.7), it is enough to integrate the system (6.1.9) and to take $W = P_1$.

The system (6.1.9) has been integrated for $\theta = \pi/3$ in [48] and for $\theta = \pi/4$ in [50], that is, in these cases the Witting series are summed up exactly.

Here, we consider the case $\theta = \pi/5$ when (6.1.9) contains five equations. In this case (and more generally, for all $\theta = \pi/n$, n is an odd integer), the system (6.1.9) can be written in the standard form.

Denote $\triangle = \sqrt{f_1 f_2 f_3 f_4 f_5}$. Then the equations (6.1.9) can be written with respect to the variables f_j in the following way:

$$\frac{df_1}{d\chi} = i\nu \frac{f_3 f_5 - f_2 f_4}{\triangle}, \quad \frac{df_2}{d\chi} = i\nu \frac{f_4 f_1 - f_3 f_5}{\triangle}, \quad \frac{df_3}{d\chi} = i\nu \frac{f_5 f_2 - f_4 f_1}{\triangle},$$
$$\frac{df_4}{d\chi} = i\nu \frac{f_1 f_3 - f_5 f_2}{\triangle}, \quad \frac{df_5}{d\chi} = i\nu \frac{f_2 f_4 - f_1 f_3}{\triangle}. \tag{6.1.10}$$

We replace f_j with y_j, $j = 1, \ldots, 5$ for convenience and bring into use a new independent variable $t = i\nu \int d\chi / \triangle$. Then, the system (6.1.9) can be brought to the following form $(. = d/dt)$

$$\begin{aligned} \dot{y}_1 &= y_3 y_5 - y_2 y_4, \\ \dot{y}_2 &= y_4 y_1 - y_3 y_5, \\ \dot{y}_3 &= y_5 y_2 - y_4 y_1, \\ \dot{y}_4 &= y_1 y_3 - y_5 y_2, \\ \dot{y}_5 &= y_2 y_4 - y_1 y_3. \end{aligned} \tag{6.1.11}$$

Apart from a number of discrete symmetries, the above system has two first integrals

$$I_1 = y_1 + y_2 + y_3 + y_4 + y_5, \quad I_2 = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right). \tag{6.1.12}$$

Systems with symmetries like (6.1.11) often have a Hamiltonian (or a Poisson) structure and are even integrable, see for instance [45] and the references therein. Thus, it is tempting to test (6.1.11) for integrability. The system (6.1.11) is obviously non-Hamiltonian and it is not clear whether one can define a suitable Poisson bracket. It is also not known if there are more integrals, that is why Karabut [50] has studied (6.1.11) numerically.

The following result answers the question for the integrability of the above systems.

Theorem 6.1.1. *The system (6.1.11) is not meromorphically integrable (in non-Hamiltonian sense).*

Corollary. The Karabut system (6.1.10) (or (6.1.9) with $n = 5$) is also not integrable (in non-Hamiltonian sense).

Recall that all necessary notions and facts about the integrability in Hamiltonian sense are given in Chapter 1.

Let us have a general dynamical system (not necessarily Hamiltonian). There exists a natural notion of (non-Hamiltonian) integrability, which is studied by Bogoyavlenskij [13]. By definition, X is integrable (in non-Hamiltonian sense) if there exist integers $k \geq 1, l \geq 0$ such that $k + l = m$, k vector fields $X_1, \ldots, X_k$ on M (linearly independent almost everywhere), which commute pairwise, that is, $[X_i, X_j] = 0$ for every i, j and such that $X = X_1$, and l first integrals $f_1, \ldots, f_l$ (functionally independent almost everywhere) for these vector fields, i.e., $X_i(f_j) = 0$ for every i, j. If the vector fields X_i and the functions f_j are meromorphic, then X is meromorphically integrable.

Ayoul and Zung have extended the Morales–Ramis result 1.0.1 to the non-Hamiltonian case by observing that any dynamical system can be transformed into a Hamiltonian one by a procedure called cotangent lifting. We make use of the following result :

Theorem 6.1.2. *([7]) Assume that a dynamical system is meromorphically integrable in non-Hamiltonian sense in a connected neighborhood of a phase curve Γ. Then the identity component G^0 of the Galois group of the (NVE) is abelian.*

The proof of Theorem 6.1.1 is given in section 2. For the proof the differential Galois approach is used. We find a particular solution of the system (6.1.11). Then we study the monodromy group of the linearized system along this solution. In our case the monodromy group topologically generates the differential Galois group of that linear system and it turns out to be non-commutative. Then the Corollary follows immediately.

Another application of the Differential Galois approach for the investigation of the non-integrability of non-Hamiltonian systems can be found, for example in [61].

6.2 Proof of the main Theorem

In this section the Theorem 6.1.1 is proved. We choose a particular solution and study the Galois group of the (NVE) along this particular solution. It turns out that investigating the corresponding monodromy group is sufficient to conclude non-integrability.

Proposition 6.2.1. *The system (6.1.11) has the following particular solution*

$$\Gamma : y_1 = -y_4 = \frac{1}{\cosh t}; \quad y_2 = -y_3 = -\tanh t; \quad y_5 = 0. \tag{6.2.13}$$

The proof is straightforward. Note that the solution (6.2.13) is on the levels of the integrals: $I_1 = 0, I_2 = 1$.

□

Let $\xi_j = dy_j$, $j = 1, \dots, 5$ be the variations. Since we have $y_5 = 0, y_1 + y_4 = 0$ and $y_2 + y_3 = 0$ along the phase curve Γ, then

$$\eta_1 = \xi_1 + \xi_4, \quad \eta_2 = \xi_2 + \xi_3 \quad \text{and} \quad \xi_5$$

can be taken as coordinates in which the (NVE) is written

$$\begin{aligned} \dot{\eta}_1 &= -y_2\eta_1 + y_1\eta_2 - 2y_2\xi_5, \\ \dot{\eta}_2 &= 2y_2\xi_5, \\ \dot{\xi}_5 &= y_2\eta_1 - y_1\eta_2. \end{aligned} \tag{6.2.14}$$

The system (6.2.14) admits an integral inherited from I_1

$$\eta_1 + \eta_2 + \xi_5 = c = const.$$

By taking $c = 0$ and expressing $\xi_5 = -\eta_1 - \eta_2$ we get

$$\begin{aligned} \dot{\eta}_1 &= y_2\eta_1 + (y_1 + 2y_2)\eta_2, \\ \dot{\eta}_2 &= -2y_2\eta_1 - 2y_2\eta_2. \end{aligned} \tag{6.2.15}$$

Further, we introduce a new independent variable $z := \sinh t$ (this transformation does not change the identity component of the Differential Galois group, see for instance [200]). Then the above system becomes

$$\eta' = \left(\frac{A^{(1)}}{z - i} + \frac{A^{(2)}}{z + i} \right) \eta, \quad ' = d/dz, \tag{6.2.16}$$

where $\eta = (\eta_1, \eta_2)^T$ and

$$A^{(1)} = \begin{pmatrix} -\frac{1}{2} & -\frac{i+2}{2} \\ 1 & 1 \end{pmatrix}, \quad A^{(2)} = \overline{A^{(1)}}.$$

The system (6.2.16) has three regular singular points: $z_{1,2} = \pm i$ and $z_3 = \infty$. Linear systems with regular singularities which are simple poles are usually called Fuchsian systems. For the systems with only regular singular points the Differential Galois group coincides with the Zariski closure in $\mathrm{GL}(n, \mathbb{C})$ of the monodromy group (see e.g. [70, 96, 110]).

Thus, we study the monodromy group of (6.2.16) in order to obtain information about the Galois group. In a vicinity of $z_3 = \infty$ we put $\tau = \frac{1}{z}$ and the system (6.2.16) takes the form

$$\frac{d\eta}{d\tau} = A^{(3)}(\tau)\eta, \tag{6.2.17}$$

where

$$A^{(3)}(\tau) = \begin{pmatrix} \frac{1}{\tau(\tau^2+1)} & \frac{2-\tau}{\tau(\tau^2+1)} \\ \frac{-2}{\tau(\tau^2+1)} & \frac{-2}{\tau(\tau^2+1)} \end{pmatrix} = \frac{1}{\tau} A_0^{(3)} + \dots$$

(here dots mean higher order terms with respect to τ). The eigenvalues of the matrix

$$A_0^{(3)} = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$$

are $-(1 \pm i\sqrt{7})/2$ and they do not differ by a positive integer (alternatively, $A_0^{(3)} = -A^{(1)} - A^{(2)}$). Then by means of a holomorphic gauge transformation the system (6.2.17) can be transformed to a simpler one (see Wasow [102], ch. 2 and also [15])

$$\frac{dP}{d\tau} = \frac{1}{\tau} A_0^{(3)} P, \tag{6.2.18}$$

which is easily solved $P(\tau) = \tau^{A_0^{(3)}} C$. Hence, the local monodromy around $\tau = 0\,(z_3 = \infty)$ is $M_3 = e^{2\pi i A_0^{(3)}}$, $det M_3 = 1$.

The eigenvalues of $A^{(1)}$ are $\lambda_{1,2} = (1 \pm \sqrt{-7-8i})/4$ and they do not differ by a positive integer. By the above arguments the local monodromy around $z_1 = i$ is $M_1 = e^{2\pi i A^{(1)}}$. Similarly, one can obtain the monodromy around $z_2 = -i$, but we don't need it since we can always choose paths around the singular points in order to have $M_1 M_2 M_3 = E$. The monodromy matrix M_1 does not belong to the identity component ($\det M_1 = -1$), that is why we consider M_1^2 which corresponds to a turning twice around $z_1 = i$.

Straightforward calculations show that M_3 and M_1^2 do not commute

$$M_3 M_1^2 \neq M_1^2 M_3. \tag{6.2.19}$$

Denote for short $a = -\cosh\sqrt{7}\pi$, $b = -(\sinh\sqrt{7}\pi)/\sqrt{7}$. Then the element of $M_3 M_1^2$ at the place (1, 2) is

$$\frac{1}{\lambda_1 - \lambda_2}\Big\{4ib\Big[(\lambda_1 - 1)e^{4\pi i\lambda_2} - (\lambda_2 - 1)e^{4\pi i\lambda_1}\Big] + \frac{i+2}{2}(e^{4\pi i\lambda_2} - e^{4\pi i\lambda_1})(a + 3ib)\Big\},$$

while the same element of $M_1^2 M_3$ is

$$\frac{1}{\lambda_1 - \lambda_2}\Big\{4ib\Big[(\lambda_1 - 1)e^{4\pi i\lambda_1} - (\lambda_2 - 1)e^{4\pi i\lambda_2}\Big] + \frac{i+2}{2}(e^{4\pi i\lambda_2} - e^{4\pi i\lambda_1})(a - 3ib)\Big\}.$$

Recall that $\lambda_{1,2}$ are the eigenvalues of $A^{(1)}$. Since the monodromy operators M_3 and M_1^2 belong to the identity component G^0 of the Differential Galois group of (6.2.16) and they do not commute, we conclude that G_0 is not abelian. Therefore, the system (6.1.11) is not meromorphically integrable in non-Hamiltonian sense due to Theorem 6.1.2. This finishes the proof.

■

So far we have proved that the system (6.1.9) is not integrable (in non-Hamiltonian sense) in a neighborhood of Γ. In some neighborhood where $y_5 \neq 0$ the systems (6.1.10) and (6.1.9) are equivalent. Hence, the Corollary follows.

6.3 Concluding Remarks

In this chapter we analyze the integrability of a dynamical system which is equivalent of the exact summation of the Witting series that describes the solitary wave in a fluid of finite depth. This system of ordinary differential equations is non-Hamiltonian. The integrability of the system under consideration has been an open research problem in the last 15 years.

We apply an extension of Morales-Ruiz and Ramis approach for studying integrability of Hamiltonian systems, developed by Ayoul and Zung, based on the investigation of the differential Galois group of the normal variational equation along certain non-trivial solution. As a result we give a simple proof that the above system is meromorphically not integrable in non-Hamiltonian sense. Alternatively, one can apply the Kovacic algorithm after reducing (6.2.15) to a second order linear equation (see e.g. [61, 70, 96]).

While we have just proved that the system under consideration is non-integrable (it can not be solved explicitly), this does not prevent the existence of a particular solution, which is a solitary wave or approximate to a solitary wave.

For the Karabut systems with more than five equations the implementation of similar analysis is a difficult task. In particular, the corresponding to (6.1.10) system when $n = 7$ consists of seven equations with cubic right-hand sides, when $n = 9$ it consists of nine equations with fourth degree right-hand sides, etc. The main difficulty here is to find a suitable particular solution which makes the higher dimensions variational equations to fall apart into manageable lower dimensional linear systems.

Nevertheless, taking into account our result, we think that these higher dimensional systems are also non-integrable and some numerical simulations, carried out in [50] confirm that claim.

Part II

Analytic and Quantitative Studies on some nonlinear PDEs

Chapter 7

Action-angle variables for the Dullin-Gottwald-Holm equation

The aim of this chapter is to compute the Poisson brackets for the scattering data and to construct the action - angle variables for the Dullin-Gottwald-Holm equation. The calculations are analogues to that of other integrable equations [127, 161, 162, 224, 225, 226], see also [212]. Here we follow closely [127], where the action - angle variables for the Camassa - Holm equation are derived. The results of this chapter are published in [145] and [146].

7.1 Preliminaries

In this chapter, we consider the following nonlinear dispersive equation for a unidirectional water wave with fluid velocity $u(x,t)$, referred to as the Dullin - Gottwald - Holm equation (DGH)

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{7.1.1}$$

where the constants α^2 and $\gamma/2\omega$ are squares of length scales and $2\omega > 0$ is the linear wave speed for undisturbed water at rest at spatial infinity.

Equation (7.1.1) was derived by using asymptotic expansions directly in the Hamiltonian for the Euler's equations in the shallow water regime [155]. Recently, the alternative derivations for (7.1.1) as a model for water waves was presented in [182].

The equation (7.1.1) is connected with two separately integrable equations [112, 122, 226]. For $\alpha = 0$ and $\gamma \neq 0$, this equation becomes the Korteweg - de Vries (KdV) equation

$$u_t + 2\omega u_x + 3uu_x = -\gamma u_{xxx}.$$

When $\gamma = 0$ and $\alpha = 1$, it is the Camassa-Holm (CH) equation

$$u_t + 2\omega u_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}.$$

KdV equation describes the unidirectional propagation of waves at free surface of shallow water under the influence of the gravity. CH equation appears first in [164] as a bi-Hamiltonian equation with infinite number of conserved functionals and later is derived in [118] by approximating directly the Hamiltonian for Euler's equations in the shallow water regime, where $u(x,t)$ represents the free surface above a flat bottom. The solitary waves of the KdV equation are smooth solitons. The solitary waves of CH are smooth for $\omega > 0$ and peaked for $\omega = 0$, which are stable [131, 132]. It was shown that the KdV equation is globally well-posed for initial data $u_0 \in L^2(\mathbb{R})$ [214] and it does not accommodate the phenomenon of wave breaking [218]. CH equation is locally well-posed for initial data $u_0 \in H^s(\mathbb{R}),\quad s > \frac{3}{2}$, and has unique global conservative solutions in $H^1(\mathbb{R})$ [116]. Moreover it has global solutions and also solutions with singularities in the form of wave breaking [123, 124, 194]. Both equations can be described as the geodesic flow of some right - invariant metric on the diffeomorphism group of the circle or on a central real extension of it, the Virasoro group [121, 128, 129, 199, 200, 203].

The Cauchy problem for the equation (7.1.1) has been studied in [216, 220]. It has been shown that this equation is locally well - posed for initial data $u_0 \in H^s(\mathbb{R}),\quad s > \frac{3}{2}$. In [216] is considered the scattering problem for equation (7.1.1) by examining the associated iso-spectral problem. The equation (7.1.1) has soliton solutions which are orbitally stable [170, 216]. The stability proofs for the soliton solutions to (7.1.1) follow the approaches developed for the CH equation in [131, 132]. In [170] is considered the connection between the smooth solitary waves and the iso-spectral problem to equation (7.1.1). See also [166] for a discussion of how equation (7.1.1) relates to the theory of hereditary symmetries.

With $m = u - \alpha^2 u_{xx}$, called the momentum variable, the equation (7.1.1) can be written in the form

$$m_t + 2\omega u_x + u m_x + 2m u_x = -\gamma u_{xxx}. \tag{7.1.2}$$

The bi-Hamiltonian form of (7.1.2) is [177]

$$m_t = -\mathcal{B}_2 \frac{\delta H_1[m]}{\delta m} = -\mathcal{B}_1 \frac{\delta H_2[m]}{\delta m}, \tag{7.1.3}$$

where the Hamiltonians are

$$H_1[m] = \frac{1}{2}\int (u^2 + \alpha^2 u_x^2)\, dx = \frac{1}{2}\int m u\, dx, \tag{7.1.4}$$

$$H_2[m] = \frac{1}{2}\int (u^3 + \alpha^2 u u_x^2 + 2\omega u^2 - \gamma u_x^2)\, dx. \tag{7.1.5}$$

We consider the solutions in the Schwartz class, denoted here by $S(\mathbb{R})$, so the integration is from $-\infty$ to ∞. The two compatible Hamiltonian operators $\mathcal{B}_1, \mathcal{B}_2$ are (∂ stands for $\partial/\partial x$)

$$\mathcal{B}_2 = \partial m + m\partial + 2\omega\partial + \gamma\partial^3, \quad \mathcal{B}_1 = \partial - \alpha^2\partial^3. \tag{7.1.6}$$

Because the equation (7.1.2) is bi - Hamiltonian, it has an infinite number of conservation laws H_n, such that

$$\mathcal{B}_2 \frac{\delta H_{n-1}[m]}{\delta m} = \mathcal{B}_1 \frac{\delta H_n[m]}{\delta m} \tag{7.1.7}$$

(in addition to H_1, H_2 we list below only few of them which we need - H_{-1} and H_0).

Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are compatible any linear combination of these operators defines a Poisson bracket,

$$\{f, h\} = -\int \frac{\delta f}{\delta m}(c_1\mathcal{B}_1 + c_2\mathcal{B}_2)\frac{\delta h}{\delta m} dx$$

with arbitrary c_1 and c_2. In order to simplify calculations, we take $c_1 = \frac{\gamma}{\alpha^2}$, $c_2 = 1$ and denote

$$\Omega = \omega + \frac{\gamma}{2\alpha^2}.$$

In this way we reduce the above bracket to the form similar as in [127]

$$\{f,h\} = -\int \frac{\delta f}{\delta m}(2\Omega\partial + \partial m + m\partial)\frac{\delta h}{\delta m}dx$$

or in more obvious antisymmetric form

$$\{f,h\}_s = -\int (m+\Omega)\left(\frac{\delta f}{\delta m}\partial\frac{\delta h}{\delta m} - \frac{\delta h}{\delta m}\partial\frac{\delta f}{\delta m}\right)dx. \tag{7.1.8}$$

Then (7.1.2) can be written as

$$m_t = \{m, \tilde{H}\}_s, \tag{7.1.9}$$

where

$$\tilde{H} := H_1 - \frac{\gamma}{\alpha^2}\left(H_0 - 2\sqrt{\Omega}\,H_{-1}\right) \tag{7.1.10}$$

and

$$H_0[m] = \int m\,dx, \quad H_{-1}[m] = \int (\sqrt{m+\Omega} - \sqrt{\Omega})\,dx. \tag{7.1.11}$$

Note that H_{-1} is a Casimir function for this bracket.

The organization is as follows. In section 2, we present some facts about the DGH equation, some of them well known: Lax pair, spectral problem, scattering data. In the end of the section we also discuss the inverse scattering problem for this equation. In section 3, we express the conservation laws, we need, through the scattering data. In section 4 we compute the Poisson brackets between elements of the scattering matrix and construct the action - angle variables for the DGH equation. Finally, in section 5 the inverse scattering method, developed by Constantin, Gerdjikov and Ivanov [125] for the Camassa - Holm equation is applied to the DGH equation. The solutions corresponding to the reflectionless potentials are constructed in terms of scattering data.

7.2 Direct and Inverse Scattering

The equation (7.1.2) can be expressed as the condition of compatibility between

$$\Psi_{xx} = \frac{1}{4\alpha^2}\Psi + \eta(m(x)+\Omega)\Psi \tag{7.2.12}$$

and

$$\Psi_t = \left(\frac{1}{2\alpha^2}(\frac{1}{\eta} + 2\gamma) - u\right)\Psi_x + \frac{1}{2}u_x\Psi + \beta\Psi \tag{7.2.13}$$

that is, $\partial_t(\psi'') = (\partial_t\psi)''$, where $\eta = \frac{\lambda}{\alpha^2 - 2\gamma\lambda}$ and β is a parameter. We will use this freedom in the sequel for a proper normalization of the eigenfunctions. Denote

$$k^2 := -\frac{1}{4\alpha^2} - \eta\,\Omega. \tag{7.2.14}$$

The spectrum of the equation (7.2.12) is described in the same way as in [122]. If $\Omega > 0$ and $m+\Omega > 0$, then the continuous spectrum of (7.2.12) in the terms of k corresponds to real k. The discrete spectrum (in the upper half plane) consists of finitely many points $k_n = i\kappa_n$, $n = 1,\ldots,N$ where κ_n is real and $0 < \kappa_1 < \ldots < \kappa_N < \frac{1}{2\alpha}$ (If $m+\Omega$ changes sign there are infinitely many positive eigenvalues accumulating at infinity, and singularities can appear in finite time in the form of wave breaking [121, 122, 123]. This justifies that it is reasonable to assume that $m(x,0)+\Omega>0$. Under this condition the DGH equation has a unique solution $m \in C^1([0,\infty), S(\mathbb{R}))$ and $m(x,t)+\Omega>0$ for all $(x,t)\in\mathbb{R}\times[0,\infty)$ (see [216])).

Under the assumptions that $m\in H^1(\mathbb{R})$ and

$$\int (1+|x|)|m(x)|\,dx<\infty \tag{7.2.15}$$

we have

$$\psi''(x,k) = -k^2\psi(x,k)+\eta m\psi(x,k) \tag{7.2.16}$$

and when $x\to\pm\infty$, then $\psi''\sim -k^2\psi$.

These suggest the introduction of the complex-valued solutions $\varphi,\overline{\varphi},\psi,\overline{\psi}$ with asymptotic behavior

$$\varphi(x,k)=e^{-ikx}+o(1), \qquad x\to-\infty, \tag{7.2.17}$$

$$\bar\varphi(x,k)=e^{ikx}+o(1), \qquad x\to-\infty, \tag{7.2.18}$$

$$\psi(x,k)=e^{-ikx}+o(1), \qquad x\to\infty, \tag{7.2.19}$$

$$\bar\psi(x,k)=e^{ikx}+o(1), \qquad x\to\infty. \tag{7.2.20}$$

Since any three solutions of (7.2.16) are linearly dependent, then for every $k\in\mathbb{R}\setminus\{0\}$ we have the relations of the form

$$\varphi(x,k)=a(k)\psi(x,k)+b(k)\bar\psi(x,k), \tag{7.2.21}$$

$$\bar\varphi(x,k)=\bar b(k)\psi(x,k)+\bar a(k)\bar\psi(x,k). \tag{7.2.22}$$

The matrix

$$T(k)=\begin{pmatrix} a(k) & b(k)\\ \bar b(k) & \bar a(k)\end{pmatrix} \tag{7.2.23}$$

is called the scattering matrix. The Wronskian antisymmetric bilinear form $W(g,h)=gh'-g'h$ is independent of $x\in\mathbb{R}$ for a pair of solutions. Therefore for $k\in\mathbb{R}\setminus\{0\}$ we have

$$W(\varphi,\bar\varphi)=2ik=-W(\psi,\bar\psi). \tag{7.2.24}$$

From (7.2.21-7.2.24) it follows that

$$|\,a(k)|^2-|\,b(k)|^2=1 \tag{7.2.25}$$

and

$$a(k)=(2ik)^{-1}W(\varphi,\bar\psi), \tag{7.2.26}$$

$$b(k)=(2ik)^{-1}W(\psi,\varphi). \tag{7.2.27}$$

The quantities $\mathcal{T}(k)=a^{-1}(k)$ and $\mathcal{R}(k)=b(k)/a(k)$ represent the transmission and reflection coefficients, respectively. In fact, the asymptotic of the eigenfunction $\varphi(x,k)/a(k)$ at $+\infty$ has the form

$$\frac{\varphi(x,k)}{a(k)}=e^{-ikx}+\mathcal{R}(k)e^{ikx}+o(1), \tag{7.2.28}$$

that is, a superposition of incident (e^{-ikx}) and reflected ($\mathcal{R}(k)e^{ikx}$) waves. For $x\to-\infty$ we have a transmitted wave

$$\frac{\varphi(x,k)}{a(k)}=\mathcal{T}(k)e^{-ikx}+o(1). \tag{7.2.29}$$

It follows from (7.2.25) that the scattering matrix is unitary, i.e.

$$|\mathcal{T}(k)|^2 + |\mathcal{R}(k)|^2 = 1. \tag{7.2.30}$$

The scattering matrix $T(k)$ contains all information about the continuous spectrum. Actually, in order to obtain $T(k)$, we have to know only $\mathcal{R}(k), k > 0$. Indeed, from (7.2.17 - 7.2.22) $\bar{a}(k) = a(-k), \bar{b}(k) = b(-k)$ and hence, $\bar{\mathcal{R}}(k) = \mathcal{R}(-k)$. From (7.2.30) we have

$$|a(k)|^2 = (1 - |\mathcal{R}(k)|^2)^{-1}, \tag{7.2.31}$$

i.e. $|\mathcal{R}(k)|$ determines $|a(k)|$. In the next section, we will recall that $|a(k)|$ uniquely determines $\arg(a(k))$.

From (7.2.26) it follows that $a(k)$ can be extended to analytic function in the upper half plane - Im$k > 0$. At the points of the discrete spectrum - $k_n = i\kappa_n$, $a(k)$ has simple zeroes (the proof is given in [216], Lemma 6.2.1 and follows the ideas from [226, 122]), therefore $W(\varphi, \bar{\psi}) = 0$. Hence, ϕ and $\bar{\psi}$ are linearly dependent:

$$\varphi(x, i\kappa_n) = b_n \bar{\psi}(x, i\kappa_n), \tag{7.2.32}$$

where

$$b_n := b(i\kappa_n). \tag{7.2.33}$$

Let

$$\varphi^{(n)}(x) \equiv \varphi(x, i\kappa_n), \tag{7.2.34}$$

be the eigenfunction, corresponding to eigenvalue $i\kappa_n$ with asymptotics

$$\varphi^{(n)}(x) = e^{\kappa_n x} + o(e^{\kappa_n x}), \quad x \to -\infty, \tag{7.2.35}$$

$$\varphi^{(n)}(x) = b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \quad x \to \infty. \tag{7.2.36}$$

The sign of b_n depends on the number of the zeroes of $\varphi^{(n)}$. From the oscillation theorem for the Sturm - Liouville problem [113], $\varphi^{(n)}$ has exactly $n-1$ zeros. Therefore

$$b_n = (-1)^{n-1} |b_n|, \quad n = 1, \dots, N. \tag{7.2.37}$$

The set

$$\mathcal{S} := \{\mathcal{R}(k),\, k > 0,\, \kappa_n,\, |b_n|,\, n = 1, \dots, N\}$$

is called scattering data. In the following sections the Hamiltonians for the DGH equation will be expressed in terms of the scattering data and Poisson brackets for the scattering data will be calculated.

From (7.2.21) with $x \to \infty$ we have

$$\varphi(x, k) = a(k) e^{-ikx} + b(k) e^{ikx} + o(1). \tag{7.2.38}$$

The substitution of $\varphi(x, k)$ in (7.2.13) with $x \to \infty$ gives

$$\psi_t = (\frac{1}{2\alpha^2}(\frac{1}{\eta} + 2\gamma))\psi_x + \beta\psi. \tag{7.2.39}$$

Now, from (7.2.38) and (7.2.39) with the choice $\beta := \frac{ik}{2\alpha^2}(\frac{1}{\eta} + 2\gamma)$, we get

$$\dot{a}(k) = 0, \quad \dot{b}(k) = \frac{ik}{\alpha^2}(\frac{1}{\eta} + 2\gamma)\, b(k). \tag{7.2.40}$$

Therefore,

$$a(k,t) = a(k,0), \quad b(k,t) = b(k,0)e^{\frac{ik}{\alpha^2}(\frac{1}{\eta}+2\gamma)t}, \tag{7.2.41}$$

$$\mathcal{T}(k,t) = \mathcal{T}(k,0), \quad \mathcal{R}(k,t) = \mathcal{R}(k,0)e^{\frac{ik}{\alpha^2}(\frac{1}{\eta}+2\gamma)t}. \tag{7.2.42}$$

Thus, $a(k)$ is independent from t and will serve as a generating function of the conservation laws.

The time evolution of the data on the discrete spectrum is found as follows - $i\kappa_n$ are the zeroes of $a(k)$, which does not depend on t, hence $\dot{\kappa}_n = 0$. From (7.2.33) and (7.2.40) we have

$$\dot{b}_n = 4\kappa_n \left(\frac{\Omega}{1 - 4\alpha^2\kappa_n^2} - \frac{\gamma}{2\alpha^2} \right) b_n. \tag{7.2.43}$$

The inverse scattering problem for the CH equation is solved in [112, 125, 126, 130, 192]. Note that in [125] a more direct approach is used not based on the Liouville transform. Now, we will discuss the inverse scattering problem for the DGH equation. Recall that $m(x,t) + \Omega > 0$. Next, we can perform the Liouville transform

$$\Phi(y) = (m(x,t) + \Omega)^{\frac{1}{4}}\Psi(x,t),$$

where $y = \sqrt{\Omega}x + \int_{-\infty}^{x}(\sqrt{m(\xi) + \Omega} - \sqrt{\Omega})d\xi$ to convert (7.2.12) into the classical Sturm - Liouville problem

$$-\Phi'' + Q\Phi = \mu\Phi. \tag{7.2.44}$$

Here $Q(y) = \frac{1}{4\alpha^2 q(y)} + \frac{q_{yy}(y)}{4q(y)} - \frac{3q_y^2}{16q^2(y)} + \frac{1}{4\alpha^2\Omega}$, $q(y) = m(x,t) + \Omega$ and $\mu = -\frac{1}{2\alpha^2\Omega} - \eta$. Note that if $\Psi_n(x)$ is an eigenfunction for (7.2.12) corresponding to the eigenvalue $\eta_n \in \left(-\frac{1}{4\alpha^2\Omega}, 0\right)$, then $\Phi_n(y) = (m(x) + \Omega)^{\frac{1}{4}}\Psi_n(x)$ is an eigenfunction for (7.2.44) corresponding to the eigenvalue $\mu_n = -\frac{1}{4\alpha^2\Omega} - \eta_n$.

Since $H_{-1}[m] = \int(\sqrt{m+\Omega} - \sqrt{\Omega})dx$ is an integral of the DGH equation (see Sections 1, 3) we deduce from (7.2.28) and (7.2.29) that the eigenfunction $\Phi(y,t)$ corresponding to $\mu = \nu^2$ in the continuous spectrum satisfies

$$\Phi(y,t) \approx \begin{cases} e^{-\sqrt{\mu}(y - H_{-1}[m])} + \mathcal{R}(\nu,t)e^{i\sqrt{\mu}(y - H_{-1}[m])}, & y \to +\infty \\ \mathcal{T}(\nu)e^{-\sqrt{\mu}y}, & y \to -\infty \end{cases}. \tag{7.2.45}$$

In order to perform the inverse scattering transform in the presence of finitely many bound states, we also need to know the time evolutions of the normalization constants $c_n(t)$ defined by

$$c_n(t) = \left[\int \Phi_n^2(y,t)dy \right]^{-1}.$$

Let us write the equation (7.1.2) in the form

$$m_t = \frac{\gamma}{\alpha^2}m_x - 2\Omega u_x - um_x - 2mu_x. \tag{7.2.46}$$

We see that, dropping the subscript n for convenience

$$\frac{d}{dt}(c^{-1}(t)) = \frac{d}{dt}\int \Phi^2(y,t)dy = \frac{d}{dt}\int (m(x,t) + \Omega)\Psi^2(x,t)dx.$$

Using that $m(x,t)$ evolves according to equation (7.2.46) and integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}(c^{-1}(t)) &= \frac{\gamma}{\alpha^2}\int m_x\Psi^2dx - 2\Omega\int u_x\Psi^2dx - \int um_x\Psi^2dx - 2\int u_xm\Psi^2dx \\ &+\frac{1}{\alpha^2}\left(\frac{1}{\eta}+2\gamma\right)\int(m+\Omega)\Psi\Psi_xdx - 2\int(m+\Omega)u\Psi\Psi_xdx \\ &+\int(m+\Omega)u_x\Psi^2dx + 2\beta\int(m+\Omega)\Psi^2dx \\ &= \frac{1}{\eta\alpha^2}\int m\Psi\Psi_xdx + 2\beta\int(m+\Omega)\Psi^2dx.\end{aligned}$$

Multiplying the equation (7.2.12) with Ψ_x and integrating we obtain that $\int(m+\Omega)\Psi\Psi_xdx = 0$. Moreover, in the case of bound states we have that $\beta = \frac{\kappa_n}{2\alpha^2}\left(\frac{1}{\eta_n}+2\gamma\right)$. Finally, we find that

$$\frac{d}{dt}\left(c_n^{-1}(t)\right) = \frac{\kappa_n}{\alpha^2}\left(\frac{1}{\eta_n}+2\gamma\right)c_n^{-1}(t),$$

and hence

$$c_n(t) = c_n(0)e^{-\frac{\kappa_n}{\alpha^2}\left(\frac{1}{\eta_n}+2\gamma\right)t}.$$

Thus, the evolution of the scattering data under the DGH flow has been explicitly determined. Note that $Q(y,0)\in S(\mathbb{R})$, so that using the Marchenko method we can construct $Q(y,t)$. Therefore, it remains only to recover $m(x,t)$ from $Q(y,t)$.

7.3 Conservation Laws and Scattering Data

The solution of (7.2.12) can be written in the form

$$\varphi(x,k) = \exp(-ikx)\exp(\int_{-\infty}^{x}\chi(y,k)\,dy). \tag{7.3.47}$$

For $\operatorname{Im} k > 0$ and $x\to\infty$, $\varphi(x,k)e^{ikx} = a(k)$, hence

$$\ln a(k) = \int_{-\infty}^{\infty}\chi(y,k)\,dy, \quad \operatorname{Im} k > 0. \tag{7.3.48}$$

Since $a(k)$ does not depend on t, the expression $\int_{-\infty}^{\infty}\chi(y,k)\,dy$ represents integrals of motion for all k. We obtain the equation for χ substituting (7.3.47) in (7.2.12)

$$\chi_x + \chi^2 - 2ik\chi = k^2 + \frac{1}{4\alpha^2} + \eta(m+\Omega) = \eta\, m, \tag{7.3.49}$$

where from (7.2.14) we have

$$\eta = -\frac{4\alpha^2k^2+1}{4\alpha^2\Omega}. \tag{7.3.50}$$

The equation (7.3.49) admits a solution with the asymptotic expansion

$$\chi(x,k) = p_1k + p_0 + \sum_{n=1}^{\infty}\frac{p_{-n}}{k^n}. \tag{7.3.51}$$

The substitution of (7.3.51) in (7.3.49) gives the following quadratic equation for p_1, namely $p_1^2 - 2ip_1 = -\frac{m}{\Omega}$ with solutions

$$p_1 = i(1 \pm \sqrt{1 + \frac{m}{\Omega}}). \qquad (7.3.52)$$

Since $\int_{-\infty}^{+\infty} p_1(x)dx$ is an integral of the DGH equation, presumably finite, we take minus sign in (7.3.52). For k real and $\chi = \chi_R + i\chi_I$ from (7.3.49) one can obtain that p_0 and p_{-2n} are total derivatives, so we have the expansion

$$\ln a(k) = -i\sigma k + \sum_{n=1}^{\infty} \frac{I_{-n}}{k^n}, \qquad (7.3.53)$$

where σ is a positive constant (corresponding to H_{-1})

$$\sigma = \int_{-\infty}^{\infty} \left(\sqrt{\frac{m}{\Omega} + 1} - 1 \right) dx \qquad (7.3.54)$$

and $I_{-n} = \int_{-\infty}^{\infty} p_{-n}(x)\, dx$ are the other integrals, whose densities p_{-n} are obtained recurrently from (7.3.49) and (7.3.51). For example, we have

$$p_0 = -\frac{q_x}{4q}, \qquad q(x) := m(x) + \Omega, \qquad (7.3.55)$$

$$p_{-1} = \frac{p_1}{8\alpha^2} + \frac{i\sqrt{\Omega}}{8} \left[\frac{1}{\alpha^2\sqrt{q}} - \frac{1}{\alpha^2\sqrt{\Omega}} + \frac{q_x^2}{4(q)^{5/2}} + \left(\frac{q_x}{q^{3/2}} \right)_x \right], \qquad (7.3.56)$$

etc., that is,

$$I_{-1} = \frac{-i\sigma}{8\alpha^2} + \frac{i\sqrt{\Omega}}{8} \int_{-\infty}^{\infty} \left(\frac{1}{\alpha^2\sqrt{q}} - \frac{1}{\alpha^2\sqrt{\Omega}} + \frac{q_x^2}{4q^{5/2}} \right) dx, \ldots \qquad (7.3.57)$$

The asymptotic of $a(k)$ for $\operatorname{Im} k > 0$ and $|k| \to \infty$ from (7.3.53) is $a(k) \to e^{-i\sigma k}$, or

$$a(k)e^{i\sigma k} \to 1, \quad \operatorname{Im} k > 0, \quad |k| \to \infty. \qquad (7.3.58)$$

Now, let us consider the function

$$a_1(k) = e^{i\sigma k} a(k) \prod_{n=1}^{N} \frac{k + i\kappa_n}{k - i\kappa_n}. \qquad (7.3.59)$$

This function is analytic for $\operatorname{Im} k > 0$, but does not have any zeroes there. Hence, $\ln a_1(k)$ is analytic in the upper half plane and from (7.3.58) $\ln a_1(k) \to 0$ for $|k| \to \infty$. Moreover, for real k $|a_1(k)| = |a(k)|$ and the Kramers - Kronig dispersion relation [181] for the function

$$\ln a_1(k) = \ln |a(k)| + i \arg a_1(k)$$

gives

$$\arg a_1(k) = -\frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\ln |a(k')|}{k' - k} dk',$$

where P denotes the principal value. Combining the last two formulas, we obtain for real k

$$\ln a_1(k) = \ln|a(k)| - \frac{i}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\ln|a(k')|}{k'-k}dk'.$$

With the help of Sohotski - Plemelj formula [181, 226] we have

$$\ln a_1(k) = \frac{1}{i\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{\ln|a(k')|}{k'-k-i0}dk', \tag{7.3.60}$$

which together with (7.3.59) gives

$$\ln a(k) = -i\sigma k + \sum_{n=1}^{N} \ln\frac{k-i\kappa_n}{k+i\kappa_n} + \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\ln|a(k')|}{k'-k-i0}\,dk'. \tag{7.3.61}$$

Next, we will show the validity of (7.3.60), (7.3.61) in $\operatorname{Im} k \geq 0$. We use a contour Γ consisting of real line and the infinite semicircle in the upper half plane, where $\ln a_1(k) = 0$. Then, the Cauchy theorem for the function $\ln a_1(k)$ and $\operatorname{Im} k > 0$ yields

$$\ln a_1(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln|a(k')|}{k'-k}\,dk'. \tag{7.3.62}$$

The substitution of (7.3.60) into (7.3.62) gives

$$\ln a_1(k) = \frac{1}{2(\pi i)^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{dk'}{(k'-k)(k''-k'-i0)}\right)\ln|a(k'')|\,dk''. \tag{7.3.63}$$

Now, using the residue theorem to compute the integral in the bracket, the contour Γ is the same as before and noting that the pole at $k' = k'' - i0$ is outside the contour since k'' is real, we obtain

$$\ln a_1(k) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln|a(k'')|}{k''-k}\,dk'', \quad \operatorname{Im} k > 0. \tag{7.3.64}$$

Therefore, from (7.3.59) and (7.3.64) for $\operatorname{Im} k > 0$ we have

$$\ln a(k) = -i\sigma k + \sum_{n=1}^{N} \ln\frac{k-i\kappa_n}{k+i\kappa_n} + \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\ln|a(k')|}{k'-k}\,dk'. \tag{7.3.65}$$

The equation (7.3.49) can also be written in the form

$$\chi_x + (\chi - ik)^2 = \frac{1}{4\alpha^2} + \eta(m+\Omega) \tag{7.3.66}$$

and admits a solution with the asymptotic expansion

$$\chi(x,k) = ik + \frac{1}{2\alpha} + \sum_{n=1}^{\infty} p_n \eta(k)^n. \tag{7.3.67}$$

Since $\eta(\frac{i}{2\alpha}) = 0$, then $\chi(x, \frac{i}{2\alpha}) = 0$. Then from (7.3.48) $\ln(a(\frac{i}{2\alpha})) = 0$. Now (7.3.65) for $k = \frac{i}{2\alpha}$ gives the integral σ (7.3.54) in terms of the scattering data

$$\sigma = 2\alpha\sum_{n=1}^{N} \ln\frac{1+2\kappa_n\alpha}{1-2\kappa_n\alpha} - \frac{8\alpha^2}{\pi}\int_{0}^{\infty} \frac{\ln|a(k')|}{(4\alpha^2 k'^2+1)}dk'. \tag{7.3.68}$$

We know from (7.2.31) that $|\mathcal{R}(k)|$ determines $|a(k)|$. But $|a(k)|$ determines uniquely $a(k)$ for real k due to (7.3.65) and (7.3.68). Since $b(k)=a(k)\mathcal{R}(k)$, the entire information about $T(k)$ is provided by $\mathcal{R}(k)$, $k>0$.

From (7.3.66) and (7.3.67) we get

$$\frac{p_1}{\alpha}+(p_1)_x=m+\Omega \tag{7.3.69}$$

with a solution

$$p_1=\alpha u-\alpha^2u_x+\alpha\Omega. \tag{7.3.70}$$

Taking integral of (7.3.69) we obtain

$$\int_{-\infty}^{\infty}p_1dx=\alpha H_0+\alpha\int_{-\infty}^{\infty}\Omega dx. \tag{7.3.71}$$

The last term in (7.3.71) has infinite contribution - all such contributions in formulas here and below cancel eventually the infinite term $\int_{-\infty}^{\infty}(\frac{1}{2\alpha}+ik)dx$ also from (7.3.67) when it is substituted in (7.3.48).

The next equation from (7.3.66) and (7.3.67) is

$$\frac{p_2}{\alpha}+(p_2)_x+p_1^2=0. \tag{7.3.72}$$

Then, taking into consideration (7.1.4) and (7.1.11) we obtain

$$\int_{-\infty}^{\infty}p_2dx=-\int_{-\infty}^{\infty}p_1^2dx=-\alpha^3\left(2H_1+2\Omega H_0+\int_{-\infty}^{\infty}\Omega^2dx\right). \tag{7.3.73}$$

The equation for p_3 from (7.3.66) and (7.3.67) is

$$\frac{p_3}{\alpha}+(p_3)_x+2p_1p_2=0 \tag{7.3.74}$$

and the next integral $\int_{-\infty}^{\infty}p_3dx=-2\alpha\int_{-\infty}^{\infty}p_1p_2dx$ is obtained after long, but straightforward computations to be

$$\int_{-\infty}^{\infty}p_3dx=4\alpha^5H_2+4\alpha^3\left(\gamma+\alpha^2\Omega\right)H_1+6\alpha^5\Omega^2H_0+2\alpha^5\int_{-\infty}^{\infty}\Omega^3dx. \tag{7.3.75}$$

Now, let us expand $\ln a(k)$ about the point $k=\frac{i}{2\alpha}$. Define a new parameter l by

$$k=\frac{i}{2\alpha}\sqrt{1+4\alpha^2l},\quad \eta=\frac{l}{\Omega} \tag{7.3.76}$$

and therefore reduce the expansion about $l=0$. Using (7.3.76) and (7.3.67) in (7.3.48), and the expressions (7.3.71), (7.3.73) and (7.3.75) we reach

$$\ln a(k(l))=\frac{l}{\Omega}\alpha H_0-2\frac{l^2\alpha^3}{\Omega^2}\left(H_1+\Omega H_0\right)$$
$$+\frac{l^3\alpha^5}{\Omega^3}\left[4H_2+4\left(\frac{\gamma}{\alpha^2}+\Omega\right)H_1+6\Omega^2H_0\right]+o(l^3). \tag{7.3.77}$$

On the other hand, we substitute (7.3.76) into (7.3.65) and again expand about $l = 0$

$$\begin{aligned}\ln a(k(l)) = &\frac{\sigma}{2\alpha}\left[1 + 2\alpha^2 l - 2\alpha^4 l^2 + 4\alpha^6 l^3\right] + \sum_{n=1}^{N}\Bigg[\ln\frac{1-2\alpha\kappa_n}{1+2\alpha\kappa_n}\\ &+ l\frac{8\alpha^3\kappa_n}{4\alpha^2\kappa_n^2 - 1} + l^2\frac{8\alpha^5\kappa_n(4\alpha^2\kappa_n^2-3)}{(4\alpha^2\kappa_n^2-1)^2} - l^3\frac{16\alpha^7\kappa_n(48\alpha^4\kappa_n^4 - 40\alpha^2\kappa_n^2+15)}{3(4\alpha^2\kappa_n^2-1)^3}\Bigg]\\ &+\frac{4\alpha}{\pi}\Bigg[\int_0^\infty \frac{\ln|a(k')|}{4\alpha^2k'^2+1}dk' + l\int_0^\infty\frac{2\alpha^2(4\alpha^2k'^2-1)\ln|a(k')|}{(4\alpha^2k'^2+1)^2}dk'\\ &- l^2\int_0^\infty\frac{2\alpha^4(16\alpha^4k'^4+24\alpha^2k'^2-3)\ln|a(k')|}{(4\alpha^2k'^2+1)^3}dk'\\ &+ l^3\int_0^\infty\frac{4\alpha^6(64\alpha^6k'^6+80\alpha^4k'^4+60\alpha^2k'^2-5)\ln|a(k')|}{(4\alpha^2k'^2+1)^4}dk'\Bigg] + o(l^3).\end{aligned} \tag{7.3.78}$$

Finally, equating the coefficients of equal powers of l in (7.3.77) and (7.3.78) and using (7.3.68) we obtain the Hamiltonians in terms of the scattering data:

$$H_0 = 2\alpha\Omega\left[\sum_{n=1}^{N}\ln\frac{1+2\kappa_n\alpha}{1-2\kappa_n\alpha} + \frac{4\alpha\kappa_n}{1-4\alpha^2\kappa_n^2} - \frac{8\alpha}{\pi}\int_0^\infty\frac{\ln|a(k')|}{(4\alpha^2k'^2+1)^2}dk'\right], \tag{7.3.79}$$

$$\begin{aligned}H_1 = \alpha\Omega^2\Bigg[&\sum_{n=1}^{N}\ln\frac{1-2\kappa_n\alpha}{1+2\kappa_n\alpha} + 4\alpha\frac{\kappa_n(4\alpha^2\kappa_n^2+1)}{(1-4\alpha^2\kappa_n^2)^2}\\ &+\frac{128\alpha^3}{\pi}\int_0^\infty\frac{k'^2\ln|a(k')|}{(4\alpha^2k'^2+1)^3}dk'\Bigg].\end{aligned} \tag{7.3.80}$$

$$\begin{aligned}H_2 = \alpha\Omega^3\Bigg[&\sum_{n=1}^{N}\ln\frac{1-2\kappa_n\alpha}{1+2\kappa_n\alpha} + \frac{4\alpha}{3}\frac{\kappa_n(-48\alpha^4\kappa_n^4+32\alpha^2\kappa_n^2+3)}{(1-4\alpha^2\kappa_n^2)^3}\\ &+\frac{2^8\alpha^3}{\pi}\int_0^\infty\frac{k'^2\ln|a(k')|}{(4\alpha^2k'^2+1)^4}dk'\Bigg] - \frac{\gamma}{\alpha^2}H_1.\end{aligned} \tag{7.3.81}$$

The higher conservation laws can be expressed through the scattering data in the same way. From H_1, H_0 and $H_{-1}\,(\sigma)$ the expression for $\tilde{H}$ is straightforward.

7.4 Poisson Brackets for the Scattering Data

In this part we are going to compute the Poisson brackets between the elements of the scattering matrix (7.2.23). Let us consider, for example, $\{a(k_1), b(k_2)\}_s$. Recall that $q(x) = m(x) + \Omega$.

$$\{a(k_1), b(k_2)\}_s = -\int_{-\infty}^{\infty} q(x)\left(\frac{\delta a(k_1)}{\delta m(x)}\partial\frac{\delta b(k_2)}{\delta m(x)} - \frac{\delta b(k_2)}{\delta m(x)}\partial\frac{\delta a(k_1)}{\delta m(x)}\right)dx. \tag{7.4.82}$$

For the computation of $\delta a(k_1)/\delta m(x)$ and $\delta a(k_1)/\delta m(x)$ we use (7.2.26) and (7.2.27).

$$\begin{aligned}\frac{\delta a(k)}{\delta m(x)} &= (2ik)^{-1}\times \\ &\times\Big[\frac{\delta\varphi(y,k)}{\delta m(x)}\frac{\partial}{\partial y}\bar{\psi}(y,k) - \frac{\delta\bar{\psi}(y,k)}{\delta m(x)}\frac{\partial}{\partial y}\varphi(y,k) \\ &+ \varphi(y,k)\frac{\partial}{\partial y}\frac{\delta\bar{\psi}(y,k)}{\delta m(x)} - \bar{\psi}(y,k)\frac{\partial}{\partial y}\frac{\delta\varphi(y,k)}{\delta m(x)}\Big].\end{aligned} \tag{7.4.83}$$

The function $G(x,y,k) := \delta\varphi(y,k)/\delta m(x)$ satisfies the equation, obtained as a variational derivative of (7.2.12)

$$\left(\partial_y^2 - \eta m(y) + k^2\right) G(x,y,k) = \eta\delta(x-y)\varphi(y,k). \tag{7.4.84}$$

Due to the delta function, the right-hand side is zero for $y < x$. On the other hand, since $\varphi(y,k)$ does not depend on $m(x)$, then the solution of (7.4.84) has to satisfy

$$G(x,y,k) = 0, \quad y < x. \tag{7.4.85}$$

As a solution of (7.4.84) $G(x,y,k)$ is a continuous function of y, however due to the delta function in the right-hand side $\partial G(x,y,k)/\partial y$ has a finite jump at $x = y$. Integrating both sides of (7.4.84) from $x-\varepsilon$ to $x+\varepsilon$ and taking $\varepsilon \to +0$, we found

$$\frac{\partial G}{\partial y}(x,y,k)_{|y=x+0} = \eta\varphi(x,k). \tag{7.4.86}$$

Since left-hand side of (7.4.83) does not depend on y, we put $y = x+\varepsilon$ and then take $\varepsilon \to 0$. When $y \to \infty$ $\psi(y,k)$ is defined via its asymptotic (7.2.19) and does not depend on $m(x)$, hence by analogous arguments, we infer that $\delta\psi(y,k)/\delta m(x) = 0$ for $y > x$. Then, from (7.4.83) it follows

$$\frac{\delta a(k)}{\delta m(x)} = -\frac{\eta(k)}{2ik}\bar{\psi}(x,k)\varphi(x,k). \tag{7.4.87}$$

Similarly

$$\frac{\delta b(k)}{\delta m(x)} = \frac{\eta(k)}{2ik}\psi(x,k)\varphi(x,k). \tag{7.4.88}$$

Putting (7.4.87) and (7.4.88) into (7.4.82), we reach

$$\begin{aligned}\{a(k_1), b(k_2)\}_s &= \frac{\eta(k_1)\eta(k_2)}{(2i)^2k_1k_2}\times \\ &\times\int_{-\infty}^{\infty} q(x)\Big[\bar{\psi}(x,k_1)\varphi(x,k_1)(\psi(x,k_2)\varphi(x,k_2))_x - \\ &\quad - \psi(x,k_2)\varphi(x,k_2)(\bar{\psi}(x,k_1)\varphi(x,k_1))_x\Big]dx.\end{aligned} \tag{7.4.89}$$

The expression under the integral in (7.4.89) is a total derivative. Indeed, let f_1, g_1 and f_2, g_2 be two pairs of solutions of (7.2.12) with spectral parameters k_1 and k_2

$$\begin{aligned}(f_{1,2})_{xx} &= \left(\frac{1}{4\alpha^2} + \eta(k_{1,2})q(x)\right) f_{1,2}, \\ (g_{1,2})_{xx} &= \left(\frac{1}{4\alpha^2} + \eta(k_{1,2})q(x)\right) g_{1,2}.\end{aligned} \tag{7.4.90}$$

It is straightforward from (7.4.90) to obtain the following identity

$$q(x)\left(f_1g_1(f_2g_2)_x - f_2g_2(f_1g_1)_x\right) = \frac{1}{\eta(k_2)-\eta(k_1)}\left[(g_1(g_2)_x - g_2(g_1)_x)(f_1(f_2)_x - f_2(f_1)_x)\right]_x . \quad (7.4.91)$$

Now we take $f_1 = \bar{\psi}(x,k_1), f_2 = \psi(x,k_2), g_1 = \varphi(x,k_1), g_2 = \varphi(x,k_2)$ and substitute in (7.4.89). Then, together with (7.4.91) and with the asymptotic representations

$$\psi(x,k) \to e^{-ikx}, \quad \varphi(x,k) \to a(k)e^{-ikx} + b(k)e^{ikx}, \quad x \to \infty \quad (7.4.92)$$

$$\varphi(x,k) \to e^{-ikx}, \quad \psi(x,k) \to \bar{a}(k)e^{-ikx} - b(k)e^{ikx}, \quad x \to -\infty \quad (7.4.93)$$

we obtain

$$\begin{aligned}\{a(k_1), b(k_2)\}_s = {} & \frac{\Omega\eta(k_1)\eta(k_2)}{(2i)^2k_1k_2(k_1^2-k_2^2)}\times \\ & \times\Big[\lim_{x\to\infty}\Big(a(k_1)a(k_2)(k_1^2-k_2^2)e^{-2ik_2x} + (k_1+k_2)^2a(k_1)b(k_2) \\ & - (k_1+k_2)^2a(k_2)b(k_1)e^{2i(k_1-k_2)x} - (k_1^2-k_2^2)b(k_1)b(k_2)e^{2ik_1x}\Big) \\ & - \lim_{x\to-\infty}\Big((k_1^2-k_2^2)a(k_1)\bar{a}(k_2)e^{2ik_2x} - (k_1-k_2)^2a(k_1)b(k_2) \\ & + (k_2-k_1)^2\bar{b}(k_1)\bar{a}(k_2)e^{-2i(k_1+k_2)x} + (k_2^2-k_1^2)\bar{b}(k_1)b(k_2)e^{-2ik_1x}\Big)\Big].\end{aligned} \quad (7.4.94)$$

The expression on the right-hand side in (7.4.94) is defined only as a distribution. Taking into account that $\lim_{x\to\infty}\mathrm{P}\frac{e^{ikx}}{k} = \pi i\delta(k)$ and assuming $k_{1,2} > 0$, we get

$$\{\ln a(k_1), \ln b(k_2)\}_s = \Omega\eta(k_1)\eta(k_2)\left[\frac{-k_1^2-k_2^2}{2k_1k_2(k_1^2-k_2^2)} + \frac{\pi i}{2k_1}\delta(k_1-k_2)\right]. \quad (7.4.95)$$

In the same way, the rest of the Poisson brackets between the scattering data can be computed. Define the quantities

$$\rho(k) := -\frac{2k}{\pi\Omega\eta(k)^2}\ln|\,a(k)|, \quad \phi(k) := \arg b(k), \quad k > 0. \quad (7.4.96)$$

Their Poisson brackets have the canonical form

$$\{\phi(k_1), \rho(k_2)\}_s = \delta(k_1-k_2), \quad \{\phi(k_1), \phi(k_2)\}_s = \{\rho(k_1), \rho(k_2)\}_s = 0 \quad (7.4.97)$$

and hence (7.4.97) are the action - angle variables for the DGH equation, related to the continuous spectrum. Note that from (7.4.97), (7.1.10), (7.3.68), (7.3.79) and (7.3.80), we have

$$\dot{\phi} = \{\phi, \tilde{H}\}_s = \frac{k}{\alpha^2}\left(\frac{1}{\eta} + 2\gamma\right), \quad (7.4.98)$$

which agrees with (7.2.40).

Let us proceed with the discrete spectrum. Denote $\eta_n := \eta(i\kappa_n)$. We will need variational derivatives $\delta\eta_n/\delta m(x)$ and $\delta b_n/\delta m(x)$. Due to (7.2.33), we use the expression (7.4.88), taking the limit $k \to i\kappa_n$ to obtain

$$\frac{\delta b_n}{\delta m(x)} = -\frac{\eta_n}{2\kappa_n}\psi(x, i\kappa_n)\varphi(x, i\kappa_n). \tag{7.4.99}$$

In order to find $\delta\eta_n/\delta m(x)$, we proceed in the following way. Differentiating the equation

$$\varphi^{(n)}_{xx} = \left[\frac{1}{4\alpha^2} + \eta(m+\Omega)\right]\varphi^{(n)}, \tag{7.4.100}$$

we obtain ($\delta q = \delta m$)

$$\delta\varphi^{(n)}_{xx} = \left(\frac{1}{4\alpha^2} + \eta q\right)\delta\varphi^{(n)} + \delta\eta_n\varphi^{(n)} + \eta_n(\delta m)\varphi^{(n)}. \tag{7.4.101}$$

From (7.4.100) and (7.4.101), it follows

$$(\varphi^{(n)}\delta\varphi^{(n)}_x - \varphi^{(n)}_x\delta\varphi^{(n)})_x = (\delta\eta_n)q(\varphi^{(n)})^2 + \eta_n(\delta m)(\varphi^{(n)})^2. \tag{7.4.102}$$

After integration of (7.4.102), we reach

$$(\delta\eta_n)\int_{-\infty}^{\infty} q(x)(\varphi^{(n)}(x))^2 dx = -\eta_n\int_{-\infty}^{\infty}(\delta m(x))(\varphi^{(n)}(x))^2 dx \tag{7.4.103}$$

or

$$\frac{\delta \ln\eta_n}{\delta m(x)} = -\frac{\varphi^{(n)}(x))^2}{\int_{-\infty}^{\infty} q(y)(\varphi^{(n)}(y))^2 dy}. \tag{7.4.104}$$

Since φ is a solution of (7.2.12) it is straightforward to obtain

$$(\varphi\varphi_{x\eta} - \varphi_x\varphi_\eta)_x = q\varphi^2. \tag{7.4.105}$$

We integrate (7.4.105) and then take the limit $k \to i\kappa_n$ i.e. $\eta \to \eta_n$ using the asymptotics

$$\varphi(x, k) \to b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \quad x \to \infty, \tag{7.4.106}$$

$$\varphi_\eta(x, k) \to \frac{a'(i\kappa_n)}{\eta'(i\kappa_n)}e^{\kappa_n x} + o(e^{-\kappa_n x}), \quad x \to \infty. \tag{7.4.107}$$

This yields

$$i\Omega b_n a'(i\kappa_n) = \int_{-\infty}^{\infty} q(y)(\varphi^{(n)}(y))^2 dy \tag{7.4.108}$$

and finally

$$\frac{\delta \ln\eta_n}{\delta m(x)} = \frac{i\varphi^{(n)}(x))^2}{\Omega b_n a'(i\kappa_n)}. \tag{7.4.109}$$

It remains to compute the expression

$$\begin{aligned}\{\ln\eta_n, b_l\}_s &= -\int_{-\infty}^{\infty} q(x)\left(\frac{\delta\ln\eta_n}{\delta m(x)}\partial\frac{b_l}{\delta m(x)} - \frac{b_l}{\delta m(x)}\partial\frac{\delta\ln\eta_n}{\delta m(x)}\right)dx \\ &= \frac{i\eta_l}{2\kappa_l\Omega b_n a'(i\kappa_n)}\int_{-\infty}^{\infty} q(x)\,(\lim_{k_j\to i\kappa_j}[\varphi^2(x,k_n)(\psi(x,k_l)\varphi(x,k_l))_x - \\ &\quad - \psi(x,k_l)\varphi(x,k_l)(\varphi^2(x,k_n))_x])\,dx.\end{aligned} \tag{7.4.110}$$

Taking $f_1 = g_1 = \varphi(x, k_n), f_2 = \psi(x, k_l), g_2 = \varphi(x, k_l)$ in (7.4.91) and using the asymptotic expressions (7.4.92) and (7.4.93) for $x \to \pm\infty$, we get

$$\{\ln \eta_n, b_l\}_s = \frac{i\eta_l}{2\kappa_l b_n a'(i\kappa_n)} \times \\ \times \lim_{x\to\infty} \lim_{k_j \to i\kappa_j} \frac{(k_n + k_l)\left(a(k_n)b(k_n)b(k_l) - a(k_l)b^2(k_n)e^{2i(k_n - k_l)x}\right)}{k_n - k_l}. \quad (7.4.111)$$

Evidently, the right-hand side of vanishes if $\kappa_n \neq \kappa_l$, since $a(i\kappa_n) = 0$. After applying the l'Hospital's rule to handle the limit $\kappa_l \to \kappa_n$, we obtain

$$\{\ln \eta_n, b_l\}_s = -\eta_n b_l \delta_{nl}. \quad (7.4.112)$$

Define the quantities

$$\rho_n = \eta_n^{-1}, \quad \phi_n = -\ln|\, b_n|, \quad n = 1, \dots, N. \quad (7.4.113)$$

Then, their Poisson brackets have the canonical form

$$\{\phi_n, \rho_l\}_s = \delta_{nl}, \quad \{\phi_n, \phi_l\}_s = \{\rho_n, \rho_l\}_s = 0 \quad (7.4.114)$$

and hence (7.4.114) are the action - angle variables for the DGH equation, related to the discrete spectrum. They also commute with the variables on the continuous spectrum (7.4.97). Note that from (7.4.114), (7.1.10), (7.3.68), (7.3.79) and (7.3.80), we have

$$\dot{\phi}_n = \{\phi_n, \tilde{H}\}_s = \{\phi_n, \kappa_n\}_s \frac{\partial \tilde{H}}{\partial \kappa_n} = 4\left(\frac{\gamma}{2\alpha^2} - \frac{\Omega}{1 - 4\alpha^2\kappa_n^2}\right)\kappa_n, \quad (7.4.115)$$

which agrees with (7.2.43).

7.5 Inverse Scattering Transform for the DGH equation

Let us now consider the asymptotic of the Jost solutions, starting for instance with $\psi(x, k)$ (7.2.19). One can check that the asymptotic for $|k| \to \infty$ has the form

$$\psi(x, k) = e^{-ikx + kG(x)}\eta(x, k), \quad \eta(x, k) = X_0(x) + \frac{X_1(x)}{k} + \frac{X_2(x)}{k^2} + \dots, \quad (7.5.116)$$

where from (7.2.19) we have $G(x) \to 0$ and $\eta(x, k) \to 1$ for $x \to \infty$. After substitution of (7.5.116) into (7.2.12) one gets explicit expressions for $G(x), X_0(x), X_1(x), \dots$.

$$\psi(x, k) = e^{-ik\left(x + \int_\infty^x (\sqrt{\frac{m(y)+\Omega}{\Omega}} - 1)dy\right)} \left[\left(\frac{\Omega}{m(x) + \Omega}\right)^{1/4} + \frac{X_1(x)}{k} + \dots\right]. \quad (7.5.117)$$

Define the function

$$\xi(x) := \exp\left[x + \int_\infty^x \left(\sqrt{\frac{m(y) + \Omega}{\Omega}} - 1\right) dy\right]. \quad (7.5.118)$$

Then (7.5.117) can be written as

$$\psi(x,k) = [\xi(x)]^{-ik}\left[\left(\frac{\xi(x)}{\xi'(x)}\right)^{1/2} + \frac{X_1(x)}{k} + \dots\right]. \tag{7.5.119}$$

Next, the function $\underline{\chi}(x,k) := \psi(x,k)e^{ikx}$ is analytic for Im $k < 0$. This follows from the representation

$$\underline{\chi}(x,k) = 1 - \frac{\eta}{k}\int_x^{\infty}\frac{e^{2ik(x-y)}-1}{2i}\, m(y)\,\underline{\chi}(y,k)\,dy. \tag{7.5.120}$$

Notice that $\int_\infty^x \left(\sqrt{\frac{m(y)+\Omega}{\Omega}}-1\right)dy$ is bounded for all x - trivial estimates give that

$$\left|\int_\infty^x \left(\sqrt{\frac{m(y)+\Omega}{\Omega}}-1\right)dy\right| = \left|\int_\infty^x \frac{m(y)dy}{\Omega(1+\sqrt{\frac{m(y)+\Omega}{\Omega}})}\right| \le \int_{-\infty}^{\infty}\frac{|m(y)|}{\Omega}dy < \infty$$

since $m(x) \in S(\mathbb{R})$. Therefore the function

$$\underline{\psi}(x,k) = \psi(x,k)\,[\xi(x)]^{ik} \tag{7.5.121}$$

is also analytic for Im $k < 0$.

Similarly

$$\begin{aligned}\underline{\varphi}(x,k) &= \varphi(x,k)\exp\left\{ik\left[x + \int_{-\infty}^x\left(\sqrt{\frac{m(y)+\Omega}{\Omega}}-1\right)dy\right]\right\}\\ &= \left(\frac{\xi(x)}{\xi'(x)}\right)^{1/2} + \frac{\tilde{X}_1(x)}{k} + \frac{\tilde{X}_2(x)}{k^2} + \dots\end{aligned} \tag{7.5.122}$$

is analytic for Im $k > 0$.

Multiplying (7.2.21) by $[\xi(x)]^{ik}/a(k)$ and using (7.3.54), (7.5.121) and (7.5.122) we obtain

$$\frac{\underline{\varphi}(x,k)}{e^{ik\sigma}a(k)} = \underline{\psi}(x,k) + \mathcal{R}(k)\bar{\underline{\psi}}(x,k)\,[\xi(x)]^{2ik}. \tag{7.5.123}$$

The function $\underline{\varphi}(x,k)/(e^{ik\sigma}a(k))$ is analytic for Im $k > 0$, $\underline{\psi}(x,k)$ is analytic for Im $k < 0$. Therefore, (7.5.123) represents an additive Riemann - Hilbert Problem with jump on the real line, given by $\mathcal{R}(k)\bar{\underline{\psi}}(x,k)\,[\xi(x)]^{2ik}$.

Suppose k is an arbitrary number from lower half plane (Im $k < 0$). Then applying the Residue Theorem and using (7.3.54) and (7.2.32) the following integral can be evaluated as

$$\frac{1}{2\pi i}\oint_{C_+}\frac{\underline{\varphi}(x,k')}{e^{ik'\sigma}a(k')}\frac{dk'}{k'-k} = \sum_{n=1}^{N}\frac{b_n\underline{\psi}(x,-i\kappa_n)(\xi(x))^{-2\kappa_n}}{a'(i\kappa_n)(i\kappa_n-k)}, \tag{7.5.124}$$

where C_+ is the closed contour in the upper half plane (Fig. 1). On the other hand, taking advantage of (7.5.123) this integral can be computed directly as

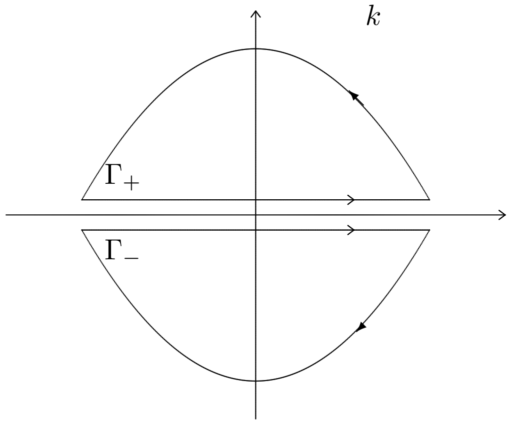


Fig. 1. The contours $\Gamma_\pm$

$$\frac{1}{2\pi i}\oint_{C_+}\frac{\underline{\varphi}(x,k')}{e^{ik'\sigma}a(k')}\frac{dk'}{k'-k}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\underline{\psi}(x,k')+\mathcal{R}(k')\bar{\underline{\psi}}(x,k')(\xi(x))^{2ik'})\frac{dk'}{k'-k} \quad (7.5.125)$$
$$+\frac{1}{2\pi i}\int_{\Gamma_+}\frac{\underline{\varphi}(x,k')}{e^{ik'\sigma}a(k')}\frac{dk'}{k'-k},$$

where Γ_+ is the infinite semi circle in the upper half plane (Fig. 1). Using the expressions (7.5.122) and (7.3.58) we compute that the integral over Γ_+ is amount to $(1/2)(\xi(x)/\xi'(x))^{1/2}$.

Next, due to Cauchy Theorem we have

$$-\underline{\psi}(x,k)=\frac{1}{2\pi i}\int_{C_-}\underline{\psi}(x,k')\frac{dk'}{k'-k}= \quad (7.5.126)$$
$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\underline{\psi}(x,k')\frac{dk'}{k'-k}+\frac{1}{2\pi i}\int_{\Gamma_-}\underline{\psi}(x,k')\frac{dk'}{k'-k},$$

where C_- is the closed contour in the lower half plane and Γ_- is the infinite semi circle (Fig. 1). Using the expressions (52) and (54) the integral over Γ_- is $-(1/2)(\xi(x)/\xi'(x))^{1/2}$.

Now, from (7.5.125) - (7.5.126) it follows that for Im $k<0$

$$\underline{\psi}(x,k)=\left(\frac{\xi(x)}{\xi'(x)}\right)^{1/2}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathcal{R}(k')\bar{\underline{\psi}}(x,k')(\xi(x))^{2ik'}\frac{dk'}{k'-k} \quad (7.5.127)$$
$$+\sum_{n=1}^{N}\frac{b_n}{a'(i\kappa_n)}\frac{\underline{\psi}(x,-i\kappa_n)}{k-i\kappa_n}(\xi(x))^{-2\kappa_n}.$$

The above expression (7.5.127) taken at $k=-i\kappa_p$, $p=1,\ldots,N$ gives

$$\underline{\psi}(x,-i\kappa_p)=\left(\frac{\xi(x)}{\xi'(x)}\right)^{1/2}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathcal{R}(k')\bar{\underline{\psi}}(x,k')(\xi(x))^{2ik'}\frac{dk'}{k'+i\kappa_p} \quad (7.5.128)$$
$$-\sum_{n=1}^{N}\frac{b_n}{ia'(i\kappa_n)}\frac{\underline{\psi}(x,-i\kappa_n)}{\kappa_n+\kappa_p}(\xi(x))^{-2\kappa_n}.$$

The equations (7.5.127) - (7.5.128) afford us to express $\underline{\psi}(x,k)$ and $\underline{\psi}(x,-i\kappa_p)$ through ξ.

Recall from (7.2.14) that $\eta(-i/2\alpha) = 0$. Since $\psi(x,k)$ does not depend on m for $\eta = 0$ (7.2.12) and since $\psi(x,k)$ is defined by its asymptotics at $-\infty$, it follows that $\psi(x,-i/2\alpha) = e^{-\frac{x}{2\alpha}}$. Hence, the substitution of $k = -i/2\alpha$ in (7.5.127) gives

$$e^{-\frac{x}{2\alpha}}(\xi(x))^{\frac{1}{2\alpha}} = \left(\frac{\xi(x)}{\xi'(x)}\right)^{1/2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathcal{R}(k')\underline{\bar{\psi}}(x,k')(\xi(x))^{2ik'}\frac{dk'}{k'+\frac{1}{2\alpha}} - \sum_{n=1}^{N}\frac{b_n}{ia'(i\kappa_n)}\frac{\underline{\psi}(x,-i\kappa_n)}{\kappa_n+\frac{1}{2\alpha}}(\xi(x))^{-2\kappa_n}. \tag{7.5.129}$$

Since $\underline{\psi}(x,k)$ and $\underline{\psi}(x,-i\kappa_p)$ are known from (7.5.127) - (7.5.128), the equation (7.5.129) is a first order differential equation for ξ. Thus, (7.5.127) - (7.5.129) represent a system of singular integral equations for $\underline{\psi}(x,k), \underline{\psi}(x,-i\kappa_p)$ and $\xi(x)$.

The time evolution of the scattering data is known from (7.2.43), the dependence on t - $\xi(x,t)$ is also expressed by the scattering data. Hence, the set $\mathcal{S}$ uniquely determines $m(x,t)$ from (7.5.118)

$$m(x,t) = \Omega\left[\left(\frac{\xi_x(x,t)}{\xi(x,t)}\right)^2 - 1\right]. \tag{7.5.130}$$

The inverse scattering is simplified in the case of so - called reflectionless potentials which correspond to the case $\mathcal{R}(k) = 0$ for all real k. This class potentials are usually connected with the N - soliton solutions. In this case $b(k) = 0$ and $|a(k)| = 1$. From (7.3.65) and (7.3.68) one can find that $ia'(i\kappa_p)$ is real:

$$ia'(i\kappa_p) = \frac{1}{2\kappa_p}e^{\sigma\kappa_p}\prod_{n\neq p}\frac{\kappa_p-\kappa_n}{\kappa_p+\kappa_n}, \tag{7.5.131}$$

where

$$\sigma = 2\alpha\sum_{n=1}^{N}\ln\frac{1+2\alpha\kappa_n}{1-2\alpha\kappa_n}. \tag{7.5.132}$$

It is proven [226] that $ia'(i\kappa_n)$ has the same sign as b_n (7.2.37), so

$$\beta_n := \frac{b_n}{ia'(i\kappa_n)} > 0. \tag{7.5.133}$$

The time evolution for β_n is obtained from (7.2.43)

$$\beta_n(t) = \beta_n(0)\exp\left[4\kappa_n\left(\frac{\Omega}{1-4\alpha^2\kappa_n^2} - \frac{\gamma}{2\alpha^2}\right)t\right]. \tag{7.5.134}$$

When $\mathcal{R}(k) = 0$ (7.5.128) becomes a linear system with respect to $\underline{\psi}(x,-i\kappa_p)$

$$\underline{\psi}(x,-i\kappa_p) = \left(\frac{\xi(x)}{\xi'(x)}\right)^{1/2} - \sum_{n=1}^{N}\frac{b_n}{ia'(i\kappa_n)}\frac{\underline{\psi}(x,-i\kappa_p)}{\kappa_n+\kappa_p}(\xi(x))^{-2\kappa_n}, \quad p = 1,\dots,N \tag{7.5.135}$$

that is

$$\underline{\psi}(x,-i\kappa_n) = \left(\frac{\xi(x)}{\xi'(x)}\right)^{1/2}\left(A^{-1}B\right)_n, \tag{7.5.136}$$

where

$$A_{p,n} := \delta_{np} + \frac{\beta_n(t)}{\kappa_n + \kappa_p}\xi^{-2\kappa_n}, \qquad B := (1, 1, \ldots, 1)^t, \tag{7.5.137}$$

and finally

$$\underline{\psi}(x, -i\kappa_n, t) = \left(\frac{\xi(x,t)}{\xi'_x(x,t)}\right)^{1/2} \sum_{n=1}^{N} A^{-1}_{n,p}(\xi, t), \quad n = 1, \ldots, N. \tag{7.5.138}$$

Similarly, when $\mathcal{R}(k) = 0$ (7.5.129) gives ($\xi(-\infty, t) = 0$)

$$x := X(\xi, t) = \alpha \ln \frac{1}{\alpha} \int_0^{\xi} \underline{\xi}^{\frac{1-\alpha}{\alpha}} \left[1 - \sum_{n,p=1}^{N} \frac{\beta_n(t)}{\kappa_n + 1/2\alpha} \underline{\xi}^{-2\kappa_n} A^{-1}_{n,p}(\underline{\xi}, t)\right]^{-2} d\underline{\xi}. \tag{7.5.139}$$

This implicit relation represents ξ as a function of x and t. Thus, in this case scattering data uniquely determine $\xi = \xi(x, t)$ and due to (7.5.130) $m(x, t)$.

In order to obtain N - soliton solutions let (x_0, t_0) be arbitrary but fixed coordinates. From (7.5.139) x is monotonically increasing function of ξ and $\xi_x = X_\xi^{-1}(\xi, t)$. Hence, there exists a unique $\xi_0 > 0$ (considered as a parameter from now on), such that $x_0 = X(\xi_0, t_0)$ and from (7.5.130) we have

$$m(x_0, t_0) = m(X(\xi_0, t_0), t_0) = \Omega\left[(\xi_0 X_\xi)^{-2} - 1\right]. \tag{7.5.140}$$

Next

$$u(x_0, t_0) = u(X(\xi_0, t_0), t_0) = \frac{1}{2\alpha}\int_0^{\infty} e^{-\frac{1}{\alpha}|X(\xi_0,t_0) - X(\underline{\xi},t_0)|} m(X(\underline{\xi}, t_0), t_0) X_\xi(\underline{\xi}, t_0) d\underline{\xi} =$$

$$\frac{\Omega}{2\alpha}\int_0^{\infty} e^{-\frac{1}{\alpha}|X(\xi_0,t_0) - X(\underline{\xi},t_0)|} \underline{\xi}^{-2} X_\xi^{-1} d\underline{\xi} - \Omega.$$

Finally, the N - soliton solution is

$$u(x, t) = \frac{\Omega}{2\alpha}\int_0^{\infty} e^{-\frac{1}{\alpha}|x - X(\xi,t)|} \xi^{-2} X_\xi^{-1}(\xi, t) d\xi - \Omega. \tag{7.5.141}$$

Note that $X(\xi, t)$ is an explicitly defined function (7.5.139) in terms of scattering data. Hence, the solution (7.5.141) does not depend on any additional parameter.

For example, for the one-soliton solution the function $X(\xi, t)$ is

$$X(\xi, t) = \alpha \ln \frac{1}{\alpha}\int_0^{\xi} \underline{\xi}^{\frac{1-\alpha}{\alpha}} \left[\frac{1 + \frac{\beta_1(t)}{2\kappa_1}\underline{\xi}^{-2\kappa_1}}{1 + \beta_1(t)\left(\frac{1}{2\kappa_1} - \frac{1}{\kappa_1 + \frac{1}{2\alpha}}\right)\underline{\xi}^{-2\kappa_1}}\right]^2 d\underline{\xi}. \tag{7.5.142}$$

Due to $0 < \kappa_1 < 1/2\alpha$ and $\beta_1(t) > 0$ both the nominator and the denominator in the last formula are positive and singularities do not appear.

7.6 Concluding remarks

In this chapter, the action - angle variables for the DGH equation are computed. For the solutions in the Schwartz class, they are expressed in terms of scattering data for this equation.

In order to simplify the calculations, a special linear combination of the Hamiltonian operators is used to define a Poisson bracket. The Casimir function H_{-1} is added in the corrected Hamiltonian $\tilde{H}$ only to match the evolution of the scattering data. Again, like in the CH equation, the question about the behavior of the scattering data at $k = 0$ needs further consideration (see [161] for the KdV case). Also, the situation when the condition $m(x,t) + \Omega > 0$ does not hold, requires additional analysis.

One may argue that the things are the same as in the CH equation, because γ can be removed. Recall from [155] that upon shifting the velocity variable by $u_0 = const$ and moving into a Galilean frame $\xi = x - ct$ with velocity c so that $u(x,t) = \tilde{u}(\xi,t) + c + u_0$, the equation (7.1.2) transforms to

$$\tilde{m}_t + c_0\tilde{u}_\xi + \tilde{u}\tilde{m}_\xi + 2\tilde{m}\tilde{u}_\xi = -\tilde{\gamma}\tilde{u}_{\xi\xi\xi},$$

where $c_0 = (2\omega + 2c + 3u_0)$ and $\tilde{\gamma} = (\gamma - u_0\alpha^2)$. Obvious choice of u_0 removes $\tilde{\gamma}$, but the boundary conditions at spatial infinity become nonzero ones, which makes the spectral problem more complicated.

Chapter 8

Non-uniform continuity of the periodic Holm-Staley b-family of equations

We consider a family of non-evolutionary partial differential equations known as Holm - Staley b - family which includes the integrable Camassa-Holm and Degasperis-Procesi equations. We show that the solution map is not uniformly continuous. The proof relies on a construction of smooth periodic traveling waves with small amplitude. The results of this chapter are published in [149] and [147].

8.1 Preliminaries

In [175, 176] D. Holm and M. Staley studied an one-dimensional version of an active fluid transport that is described by the following nonlinear equation

$$m_t + um_x + bu_x m = 0, \tag{8.1.1}$$

with $m = u - u_{xx}$, $u(x,t)$ representing the fluid velocity, while the constant b is a balance or a bifurcation parameter for the solution behavior. It has been shown in [153] that equation (8.1.1) is integrable only for $b = 2$ and $b = 3$.

In this chapter we study the periodic Cauchy problem for the b - family of equations (8.1.1), namely

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad u(0) = u_0, \quad t \geq 0,\, x \in \mathbb{S}. \tag{8.1.2}$$

If $b = 2$, then (8.1.2) becomes the Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{8.1.3}$$

Our aim here is to enlarge the result of Himonas and Misiolek [173] (originally proved for the CH equation) for all real $b \neq 0$.

Equation (8.1.3) was first derived by Fokas and Fuchssteiner [164] as a bi-Hamiltonian system, and then by Camassa and Holm [118] as a model for shallow water waves. There are also alternative derivations for the CH equation and the Degasperis-Procesi equation (see below) in Ivanov [180] and Constantin and Lannes [142]. For the CH equation an inverse scattering/inverse spectral analysis is performed in the periodic case in [133, 140], on the line in [125], and complete integrability in sense of an infinite-dimensional Hamiltonian system is proven.

The Cauchy problem for the CH equation in both periodic and non periodic case was studied extensively. It has been shown that the Camassa-Holm equation is locally well-posed in H^s, $s > \frac{3}{2}$ with solutions depending continuously on initial data [121, 138, 124, 194, 208]. The Camassa-Holm equation has global solutions but also solutions which blow-up in finite time (see [121, 134, 135, 123, 138, 124, 229]). The blow-up occurs as wave breaking, that is, the solution remains bounded but its slope becomes infinite in finite time. In [116, 117] the question about the behavior of a solution after wave breaking is resolved for the CH equation. Such a solution can be extended uniquely either as a conservative global solution or as a dissipative global solution.

When $b = 3$ in (8.1.2), we recover Degasperis-Procesi (DP) equation which is a model for nonlinear shallow water dynamics,

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{8.1.4}$$

The DP equation is also completely integrable, see for example [141]. The Cauchy problem for the DP equation in both periodic and non periodic case is studied in [159, 195, 221, 222, 230]. The wave breaking phenomenon takes place for the DP equation, too.

The b-family of equations (8.1.1) admits peaked traveling wave solutions, named peakons. These peakons replicate a feature that is characteristic for the waves of great height - waves of large amplitude which are exact solutions of the governing equations for water waves, see for example Constantin [137] and Constantin and Escher [139].

Sometimes, it is more appropriate to consider other version of well-posedness problem, for example if one strengthens the notion of well-posedness, requiring that the mapping data-solution is uniformly continuous. The ill-posedness of some classical nonlinear dispersive equations (for instance Korteweg-de Vries equation, modified Korteweg-de Vries equation, cubic Schrödinger equation, and Benjamin-Ono equation) in both periodic and non-periodic cases are studied in [111, 114, 115]. The approach in these papers is based on the existence and suitable properties of the traveling wave solutions associated to the equations. In particular, a good behavior of their Fourier transforms is required. In [173] Himonas and Misiolek showed that for $s \geq 2$ the solution map $u_0 \to u$ for the CH equation is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S})$ into $C([0,T], \mathrm{H}^s(\mathbb{S}))$. A key step in the proof of that result is a construction of a sequence of smooth traveling waves. Himonas, Kenig and Misiolek [174] extend the result to the range $\frac{3}{2} < s < 2$. Their proof is based on the approximation of solutions by terms containing high and low frequencies and exploring the conservation of the H^1 norm. Note that H^1 norm is a conservation law for equation (8.1.2) only for $b = 2$.

Recently Gui, Liu, and Tian [217] considered the equation (8.1.1) on the real line. They proved that the equation is locally well-posed in the Sobolev space $\mathrm{H}^s(\mathbb{R})$ for $s > \frac{3}{2}$. Moreover, they gave the precise blow-up scenario of strong solution of the equation with certain initial data. In [231] Zhou established blow-up results for this family of equations under various classes of initial data. He also proved that the solutions with compact support initial data do not have compact support. In the periodic case, sufficient conditions on the initial data are obtained in [147] to guarantee the finite time blow-up and global existence. Using the method from [173], it is also established there the non-uniform continuity of DP equation.

The first result in this chapter is the following.

Theorem 8.1.1. *For any $s \geq 3$, the solution map $u_0 \to u$ for the equation (8.1.2) with $b \neq 0$, is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S})$ into $\mathrm{C}([0,t_0], \mathrm{H}^s(\mathbb{S}))$. More precisely, for each $s \geq 3$ there exist constants $c_{1,2} > 0$ and two sequences of smooth solutions u_n, v_n of the equation*

(8.1.2) such that for any $t \in [0,1]$

$$\sup_n ||\, u_n(t)||_{\mathrm{H}^s} + \sup_n ||v_n(t)||_{\mathrm{H}^s} \le c_1,$$
$$\lim_{n\to\infty} ||\, u_n(0) - v_n(0)||_{\mathrm{H}^s} = 0,$$
$$\liminf_n ||\, u_n(t) - v_n(t)||_{\mathrm{H}^s} \ge c_2 \sin\left(\frac{t}{2}\right).$$

The idea of the proof is borrowed from [173]. Two sequences of exact periodic smooth solutions are constructed taking advantage of a scaling property of the b - family. While their initial states converge in H^s - norms, the solutions remain apart at certain time. We use two different parameters but equivalent to those in [173] in order to define appropriate families of solutions. The careful choice of these two parameters is crucial in deriving the H^s estimates. Due to the transcendent dependence on b, here we do not give the sharp estimates for these parameters and merely say that they are sufficiently small.

For similar results, we refer to Olson [201] where the following equation is studied

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{8.1.5}$$

where $\gamma \neq 0$. This equation is known as the hyperelastic rod equation and it was first derived by Dai [152]. When $\gamma = 1$ equation (8.1.5) becomes the CH equation. It is shown in [201] that solutions of the periodic Cauchy problem for (8.1.5) do not depend uniformly continuously on initial data in H^s when $s = 1$ or $s \ge 2$.

The equation (8.1.5) is studied by several authors to mention only Constantin and Strauss [143], where the stability of a class of solitary waves is established.

The chapter is organized as follows. In section 2 the periodic traveling waves of the b-family of equations are studied. Although the corresponding conservative system describing the traveling waves is somehow transcendent and depends on several parameters, the things are arranged so that we study an equivalent Hamiltonian quadratic system for which the conditions for the existence of periodic solutions are more or less known.

The main difficulty here is to establish estimates for the period. This is done in section 3 by calculating the first two terms in the expansion of the period function for periodic traveling waves with small amplitude.

In section 4 we obtain upper estimates for these solutions in H^s - norm and carry on the proof of the main result for $b \neq 0, \pm 1$.

We summarize the corresponding results for the case $b = \pm 1$ in subsection 5. This approach is not applicable to the case $b = 0$ due to lack of periodic solutions. Finally, in Section 6 we consider the same type of result for the Degasperis-Procesi equation ($b = 3$), but in the range $s \ge 2$. Note that the Degasperis-Procesi equation is integrable.

8.2 Periodic traveling waves

In this section we investigate the periodic traveling wave solutions of the b - family equation

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad b \in \mathbb{R}. \tag{8.2.6}$$

Note that if $u(x,t)$ is a classical solution of (8.2.6), then such is the function

$$u_c(x,t) = cu(x,ct), \quad \text{for any constant } c.$$

First, take velocity $c = 1$ and look for a traveling-wave solution of (8.2.6) of the form $u(x,t) = \varphi(x-t)$. The function φ must satisfy equation

$$-\varphi' + \varphi''' + (b+1)\varphi\varphi' = b\varphi'\varphi'' + \varphi\varphi'''. \tag{8.2.7}$$

By integrating once, one obtains

$$(1-\varphi)\varphi'' - \frac{b-1}{2}\varphi'^2 + \frac{b+1}{2}\varphi^2 - \varphi = C_1.$$

We then multiply both sides by $2\varphi'(1-\varphi)|1-\varphi|^{b-3}$ and integrate once again which yields, if $b \neq 1$,

$$|1-\varphi|^{b-1}\left[\varphi'^2 - \varphi^2 + \frac{2C_1}{b-1}\right] = 2C_2, \tag{8.2.8}$$

and, if $b = 1$,

$$\varphi'^2 - \varphi^2 + 2C_1 \ln|1-\varphi| = 2C_2, \tag{8.2.9}$$

where C_1, C_2 are constants of integration. Both (8.2.8) and (8.2.9) can be considered as first integrals of one-degree of freedom Hamiltonian systems. For the case $b = 1$, the equation (8.2.9) is easily studied for periodic solutions in Section 5.

The general case $b \neq 1$ is not so easy due to transcendent dependency on b. So, in order to study the periodic solutions of (8.2.8) we take a different way. In the (X,Y)-plane with $X = \varphi$, $Y = \varphi'$ consider the autonomous quadratic system

$$\begin{aligned} \dot{X} &= H_Y/M = 2Y(1-X), \\ \dot{Y} &= -H_X/M = 2X(1-X) + (b-1)(Y^2 - X^2 + d), \end{aligned} \tag{8.2.10}$$

having a first integral H and an integrating factor M, respectively, as follows:

$$H(X,Y) = |1-X|^{b-1}(Y^2 - X^2 + d), \quad M(X) = (1-X)|1-X|^{b-3}, \tag{8.2.11}$$

where it is taken for short $d = 2C_1(b-1)^{-1}$. We see that equation $H(\varphi,\varphi') = 2C_2$ coincides with (8.2.8). Therefore, to study the periodic solutions of (8.2.8), one can use the quadratic system (8.2.10) having the same first integral H and hence, the same periodic trajectories as (8.2.8).

On the other hand, it is well known fact that in a quadratic system with a first integral such as H, any periodic trajectory surrounds a unique critical point which is a center. Hence, system (8.2.10) has a periodic solution if and only if it has a center. The coordinates (X,Y) of a center of (8.2.10) must satisfy

$$(1+b)X^2 - 2X + (1-b)d = 0, \quad Y = 0; \quad [1-X][1-(b+1)X] < 0. \tag{8.2.12}$$

Using this condition, one can easily verify the following statement.

Proposition 8.2.1. *Let $b \neq 0, \pm 1$. System (8.2.10) has a center if and only if one of the following conditions holds:*

(i) $|b| > 1, \quad \dfrac{1}{1-b^2} < d < 1.$

(ii) $|b| < 1, \quad 1 < d < \dfrac{1}{1-b^2}.$

(iii) $b < -1, \quad d \geq 1.$

We observe that $\Delta = 1 + d(b^2 - 1) > 0$ for all cases. See the corresponding phase portraits of the systems with a center on Figure 1. Note that, by (8.2.12), there are no periodic orbits in (8.2.10) if $b = 0$. Besides, cases $b = \pm 1$ will be considered separately in Section 5. Therefore, we will assume below that $b \neq 0, \pm 1$.

The following facts are not essential for our study in rest part of the paper. We give them for completeness and because they are very useful in verifying the bifurcation diagram on Figure 1, as well as for the proof of Proposition 8.2.3 below.

The types of quadratic centers are well known (see e.g. [232]). Writing a quadratic system with a center at the origin in the (x, y)-plane as a complex equation with respect to $z = x + iy$, after rescaling we obtain the following types [179]:

$$\begin{array}{lr}
\dot{z} = -iz - z^2 + 2|z|^2 + (A + iB)\bar{z}^2; & \text{Hamiltonian,} \\
\dot{z} = -iz + Az^2 + 2|z|^2 + B\bar{z}^2; & \text{Reversible,} \\
\dot{z} = -iz + 4z^2 + 2|z|^2 + (A + iB)\bar{z}^2, \quad A^2 + B^2 = 4; & \text{Codimension 4,} \\
\dot{z} = -iz + z^2 + (A + iB)\bar{z}^2; & \text{Generalized Lotka-Volterra,} \\
\dot{z} = -iz + \bar{z}^2; & \text{Hamiltonian triangle.}
\end{array}$$

In the equations above, A and B are real parameters.

By passing to the respective normal form, one can prove the following structure result concerning the types of centers in (8.2.10).

Proposition 8.2.2. *Up to an affine transformation of the variables, the center of (8.2.10) belongs to the following type:*

(i) Hamiltonian triangle, if $b = 2$, $d = 0$;

(ii) Lotka-Volterra, if $|b| > 1$, $b \neq 2$, $d = 0$, *with* $(A, B) = \left(\frac{b}{b-2}, 0\right)$;

(iii) Reversible, if $d \neq 0$, $(b - \sqrt{\Delta})(b + 1) > 0$, *with*

$$(A, B) = \left(\frac{b\sqrt{\Delta} - 4\sqrt{\Delta} + b}{b(\sqrt{\Delta} - 1)}, \frac{\sqrt{\Delta} + 1}{\sqrt{\Delta} - 1}\right);$$

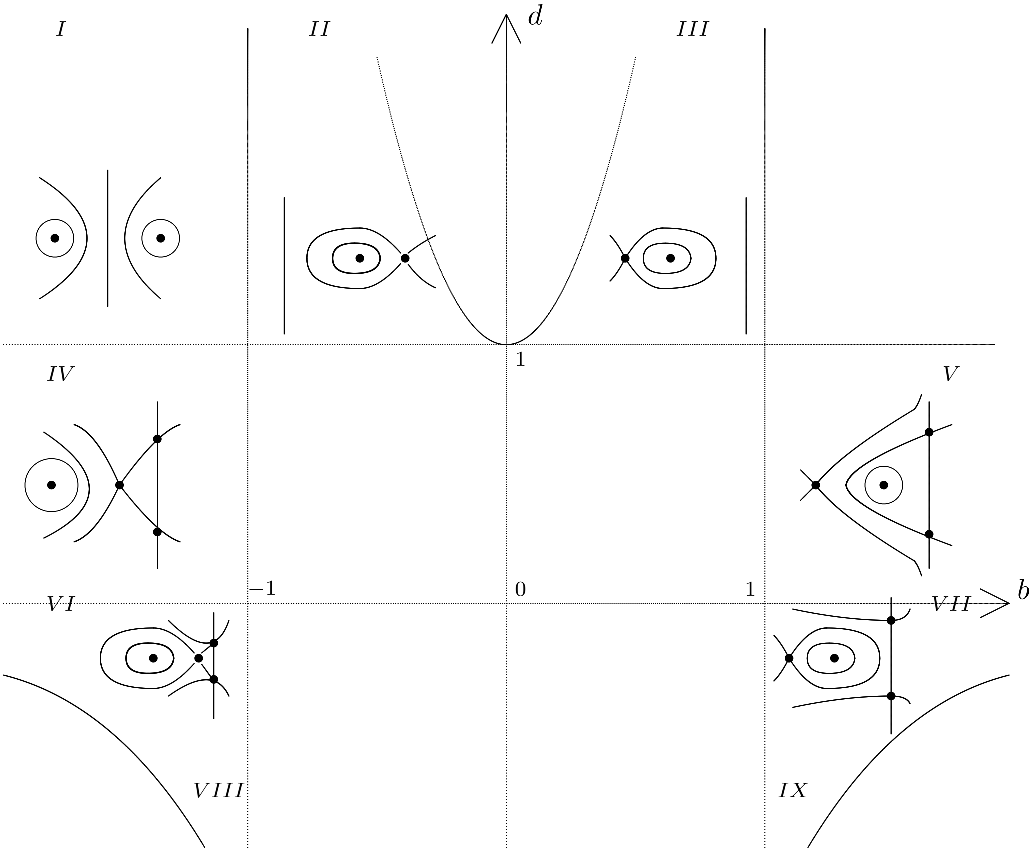
(iv) Reversible, if $d \neq 0$, $(b + \sqrt{\Delta})(b + 1) < 0$, *with*

$$(A, B) = \left(\frac{b\sqrt{\Delta} - 4\sqrt{\Delta} - b}{b(\sqrt{\Delta} + 1)}, \frac{\sqrt{\Delta} - 1}{\sqrt{\Delta} + 1}\right).$$

We next proceed to determine the interval Σ where the periodic orbits exist. This piece of information is essential for our study in Section 3. Namely, given b and d as in Proposition 8.2.1, to find the maximal open interval $\Sigma = \Sigma(b, d)$ such that for any $e \in \Sigma$ the level curve

$$H(X, Y) = e \tag{8.2.13}$$

contains an oval (a simple closed curve without critical points). Clearly, one of the endpoints of Σ is the level e_c corresponding to the center and the other is the level e_s corresponding to the contour at which the period annulus around the center terminates.



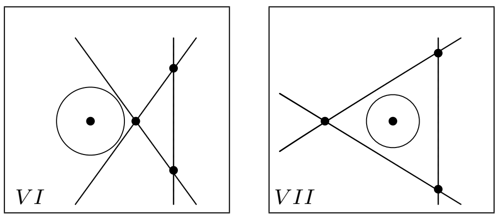


Figure 8.1: Bifurcation diagram of system (8.2.10) in the (b, d)-plane. Phase portraits of the systems having periodic solutions are shown only. The vertical invariant line in the (X, Y)-space is always $X = 1$.

Proposition 8.2.3. *The system (8.2.10) has a periodic solution for energy levels* $e \in \Sigma = (e_c, e_s)$*, where:*

$$\Sigma = \left(2\frac{|b-\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)-\sqrt{\Delta}-1), 0\right)$$
$$\text{for} \quad b>1,\ 0\le d<1, \quad \text{and for} \quad b<-1,\ d\ge 0,$$

$$\Sigma = \left(2\frac{|b-\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)-\sqrt{\Delta}-1), 2\frac{|b+\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)+\sqrt{\Delta}-1)\right)$$
$$\text{for} \quad |b|>1,\ \frac{1}{1-b^2}<d<0 \quad \text{and for} \quad 0<b<1,\ 1<d<\frac{1}{1-b^2},$$

$$\Sigma = \left(2\frac{|b+\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)+\sqrt{\Delta}-1), 2\frac{|b-\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)-\sqrt{\Delta}-1)\right)$$
$$\text{for} \quad -1<b<0,\ 1<d<\frac{1}{1-b^2},$$

$$\Sigma = \left(2\frac{|b+\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)+\sqrt{\Delta}-1), 0\right)$$
$$\text{for} \quad b<-1,\ d>1.$$

And, finally, if $T = T(e)$, $e \in \Sigma$ is the (minimal) period of the orbit contained in (8.2.13), one can find the limits $T_c(b,d) = \lim_{e\to e_c} T(e)$ and $T_s(b,d) = \lim_{e\to e_s} T(e)$ (T_s might be infinity). Then, for any T from the open interval with endpoints T_c and T_s, there will be (at least one) periodic orbit of (8.2.10) having T as a period. The following Proposition is immediate.

Proposition 8.2.4. *Let* x_c *be the abscissa of a center of system (8.2.10). Then*

$$x_c = \frac{1\pm\sqrt{\Delta}}{1+b}, \qquad T_c = 2\pi\sqrt{\frac{1-x_c}{(b+1)x_c-1}}, \qquad T_s = \infty. \tag{8.2.14}$$

8.3 The period function for small-amplitude traveling - wave solutions

Below we calculate the first two terms in the expansion of the period function in the case when the periodic wave φ we study has a small amplitude. That is, $x_1 - x_0$ is close to zero where $x_1 = \max\varphi$, $x_0 = \min\varphi$.

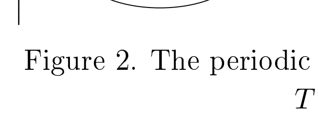


Figure 2. The periodic solution

Therefore the periodic trajectory of (8.2.10) corresponding to φ is entirely contained in a small neighborhood of a center $(x_c, 0) \in \mathbb{R}^2$, see Proposition 8.2.4. Let us recall that the period function has an expansion

$$T(\xi) = T_c + T_{2k}\varepsilon^{2k} + T_{2k+1}\varepsilon^{2k+1} + T_{2k+2}\varepsilon^{2k+2} + \dots$$

with respect to ε, the distance between the center at $(x_c, 0)$ and the intersection point of the orbit with the x-axis $(x_0, 0)$ (see Figure 2, where the case $x_c < 1$ is depicted). The series begins always with an even-degree coefficient T_{2k} for some $k = 1, 2, \ldots$ which is called the k th isochronous constant. Namely, $T_{2k} = 0$ implies also $T_{2k+1} = 0$. If all isochronous constants vanish, all orbits around the center have the same period T_c and the center is isochronous. We will need for our purposes however another "weighted" expansion with respect to $\eta = \varepsilon/(1 - x_c)$ which we are going to handle below.

Proposition 8.3.1. *The explicit expression of the first isochronous constant is determined from formulas* (8.3.24) *and* (8.3.25) *below.*

Proof. Take a small positive ε and let $x_0 = x_c - \varepsilon$. Then using (8.2.11) and (8.2.13) one obtains by direct calculations

$$\begin{aligned} e = H(x_0, 0) &= |1 - x_c + \varepsilon|^{b-1}[d - (x_c - \varepsilon)^2] \\ &= |1 - x_c|^{b-1}\left(1 + \frac{\varepsilon}{1 - x_c}\right)^{b-1}[d - x_c^2 + 2\varepsilon x_c - \varepsilon^2]. \end{aligned}$$

Then using the identity (equivalent to 8.2.12)

$$d - x_c^2 = \frac{2x_c(x_c - 1)}{b - 1} \tag{8.3.15}$$

and denoting $\eta = \varepsilon/(1 - x_c)$, we derive the formula

$$\begin{aligned} e &= (d - x_c^2)|1 - x_c|^{b-1}(1 + \eta)^{b-1}\left[1 - (b-1)\eta + \frac{(b-1)(1 - x_c)}{2x_c}\eta^2\right] \\ &= (d - x_c^2)|1 - x_c|^{b-1}e_1(\eta). \end{aligned} \tag{8.3.16}$$

An expansion in series with respect to η yields immediately that

$$e_1(\eta) = 1 + p_2\eta^2 + p_3\eta^3 + p_4\eta^4 + \ldots, \tag{8.3.17}$$

$$p_j = \binom{b-1}{j} - (b-1)\binom{b-1}{j-1} + \frac{(b-1)(1-x_c)}{2x_c}\binom{b-1}{j-2}, \quad j \geq 2.$$

As x_c is far from zero, we see that all coefficients p_j are uniformly bounded for b fixed for all centers in all regions (I)–(IX), see Fig. 1. Moreover,

$$p_2 = \frac{(b-1)(1-(b+1)x_c)}{2x_c} \neq 0 \text{ and } p_j = \binom{b-1}{j-2}\left[p_2 + \tfrac{j-2}{2j}b(b+1)\right], \quad j \geq 3.$$

Similarly, from $e = H(x_0, 0) = H(x_1, 0)$, one can calculate the function ψ in $x_1 = x_c + \psi(\varepsilon)$. Even more conveniently, taking $\psi(\varepsilon) = (1 - x_c)\Phi(\eta)$, one obtains as above the following equation for Φ: $e_1(\eta) = e_1(-\Phi)$. Expanding both sides, we derive by calculations the following expansion formula,

$$\Phi(\eta) = \eta + q_2\eta^2 + q_3\eta^3 + q_4\eta^4 + O(\eta^5) \tag{8.3.18}$$

with

$$q_2 = \frac{p_3}{p_2}, \quad q_3 = \frac{p_3^2}{p_2^2}, \quad q_4 = \frac{2p_3^3 - 2p_2p_3p_4 + p_2^2p_5}{p_2^3}.$$

Let us denote

$$U(x) = e|1 - x|^{1-b} + x^2 - d.$$

Then (8.2.13) becomes $Y^2 = U(X)$ and the period function is determined by

$$T = 2\int_{x_0}^{x_1} \frac{dX}{\sqrt{U(X)}}. \tag{8.3.19}$$

We perform a change of the variables

$$X = \frac{x_1 - x_0}{2} z + \frac{x_1 + x_0}{2}$$

in (8.3.19) and obtain

$$T = \int_{-1}^{1} \frac{(x_1 - x_0)dz}{\sqrt{U(x_c + D(z,\eta))}}, \tag{8.3.20}$$

where

$$D(z,\eta) = (1 - x_c)\left(\frac{\Phi(\eta)+\eta}{2} z + \frac{\Phi(\eta)-\eta}{2}\right) := (1 - x_c)M(z,\eta).$$

Next, making use of (8.3.16), we get

$$\begin{aligned} U(x_c + D) &= e|1 - x_c - D|^{1-b} + x_c^2 - d + 2x_c D + D^2 \\ &= (d - x_c^2)\left[e_1(\eta)(1-M)^{1-b} - 1 - (b-1)M - \frac{(b-1)(1-x_c)}{2x_c}M^2\right]. \end{aligned}$$

Conditions $U(x_0) = U(x_1) = 0$ imply that U vanishes for both $M = -\eta$ and $M = \Phi(\eta)$. Hence, using analyticity with respect to M, one can rewrite $U(x_c + D)$ as

$$\begin{aligned} U &= (d - x_c^2)(M + \eta)(\Phi(\eta) - M)(A_0 + A_1 M + A_2 M^2 + \ldots) \\ &= \tfrac{1}{4}(d - x_c^2)(\eta + \Phi(\eta))^2(1 - z^2)(A_0 + A_1 M + A_2 M^2 + \ldots). \end{aligned} \tag{8.3.21}$$

Comparing the coefficients at the corresponding degrees M^j, $j = 0, 1, 2$, we obtain the following equations for A_j:

$$\begin{aligned} &\eta\Phi A_0 = e_1(\eta) - 1, \\ &\eta\Phi A_1 + (\Phi - \eta)A_0 = (b-1)(e_1(\eta) - 1), \\ &\eta\Phi A_2 + (\Phi - \eta)A_1 - A_0 = \frac{b(b-1)}{2} e_1(\eta) - \frac{(b-1)(1-x_c)}{2x_c}, \\ &\eta\Phi A_j + (\Phi - \eta)A_{j-1} - A_{j-2} = (-1)^j \binom{1-b}{j} e_1(\eta), \quad j \geq 3. \end{aligned}$$

By using the expansions (8.3.17), (8.3.18) and equality $e_1(\eta) = e_1(-\Phi)$, we calculate

$$\begin{aligned} A_0 &= p_2 + \frac{p_2 p_4 - p_3^2}{p_2}\eta^2 + \frac{p_2 p_3 p_4 - p_3^3}{p_2^2}\eta^3 + O(\eta^4), \\ A_1 &= (b-1)p_2 - p_3 + \frac{(b-1)(p_2 p_4 - p_3^2) + p_3 p_4 - p_2 p_5}{p_2}\eta^2 + O(\eta^3), \\ A_2 &= \frac{b(b-1)}{2}p_2 - (b-1)p_3 + p_4 + O(\eta^2), \\ A_3 &= \frac{b(b^2-1)}{6}p_2 - \frac{b(b-1)}{2}p_3 + (b-1)p_4 - p_5 + O(\eta). \end{aligned}$$

On the other hand, from $M=\frac{1}{2}(\Phi+\eta)z+\frac{1}{2}(\Phi-\eta)$ one obtains

$$\begin{aligned}
&M=\eta(1+\tfrac{1}{2}q_2\eta+\tfrac{1}{2}q_3\eta^2)z+\eta^2(\tfrac{1}{2}q_2+\tfrac{1}{2}q_3\eta)+O(\eta^4),\\
&M^2=\eta^2(1+q_2\eta)z^2+\eta^3q_2z+O(\eta^4),\\
&M^3=\eta^3z^3+O(\eta^4).
\end{aligned}$$

Therefore, by direct calculations, we can derive the expression

$$\begin{aligned}
&A_0+A_1M+A_2M^2+A_3M^3+O(M^4)\\
&=p_2[1+a_1z\eta+(b_0+b_1z+b_2z^2)\eta^2+(c_0+c_1z+c_2z^2+c_3z^3)\eta^3+O(\eta^4)]
\end{aligned}$$

where

$$a_1=\frac{(b-1)p_2-p_3}{p_2},\quad b_0=\frac{(b-1)p_2p_3+2p_2p_4-3p_3^2}{2p_2^2},\quad b_1=\tfrac{1}{2}q_2a_1,$$

$$b_2=\frac{b(b-1)p_2-2(b-1)p_3+2p_4}{2p_2},\quad c_0=q_2b_0,\quad c_2=q_2b_2.$$

Next, modulo odd-degree terms with respect to z, one obtains

$$[1+a_1z\eta+\ldots]^{-1/2}=1+(\tfrac{3}{8}a_1^2z^2-\tfrac{1}{2}b_0-\tfrac{1}{2}b_2z^2)(\eta^2+q_2\eta^3)+O(\eta^4). \tag{8.3.22}$$

Finally, using (8.3.15) and the definition of Φ we get by direct calculations

$$\frac{1}{4}(d-x_c^2)(\eta+\Phi(\eta))^2p_2=\frac{(b+1)x_c-1}{4(1-x_c)}(x_1-x_0)^2. \tag{8.3.23}$$

Therefore, by (8.3.22), (8.3.23) and (8.3.21), one obtains (modulo odd-degree terms)

$$\frac{x_1-x_0}{\sqrt{U(x_c+D)}}=2\sqrt{\frac{1-x_c}{(b+1)x_c-1}}.\frac{1+(\frac{3}{8}a_1^2z^2-\frac{1}{2}b_0-\frac{1}{2}b_2z^2)(\eta^2+q_2\eta^3)+O(\eta^4)}{\sqrt{1-z^2}}$$

and therefore by (8.3.20)

$$T=2\pi\sqrt{\frac{1-x_c}{(b+1)x_c-1}}\left[1+K(\eta^2+q_2\eta^3)+O(\eta^4)\right] \tag{8.3.24}$$

with

$$\begin{aligned}
K&=\tfrac{3}{16}a_1^2-\tfrac{1}{2}b_0-\tfrac{1}{4}b_2=\frac{(b^2-4b+3)p_2^2-6(b-1)p_2p_3-12p_2p_4+15p_3^2}{16p_2^2}\\
&=\frac{b[2(b-3)(b+1)^2x_c^2-9(b-2)(b+1)x_c+12(b-1)]}{48[(b+1)x_c-1]^2}.
\end{aligned} \tag{8.3.25}$$

□

Let us recall that our aim is to obtain sequences of $2\pi/n$-periodic solutions φ_n satisfying appropriate bounds in Sobolev H^s norms. For that purpose, we need the following relations

$$T=\frac{2\pi}{n},\quad |1-x_c|=\varepsilon^{2/s}\equiv(x_c-x_0)^{2/s},\text{where}\quad s\geq 3. \tag{8.3.26}$$

We first establish the existence of solutions φ of (8.2.7) satisfying (8.3.26). Fix $b\neq 0,\pm 1$.

Proposition 8.3.2. *Given $b \neq 0, \pm 1$ and $s \geq 3$, then there is $N_0 = N_0(b,s)$ sufficiently large, so that for any $n \geq N_0$ there exists a periodic solution $\varphi = \varphi_n$ of (8.2.7) satisfying (8.3.26).*

Proof. Clearly, the period T could be small only provided that T_c is small, see (8.2.14) and (8.3.24). Therefore, $|1 - x_c| = \varepsilon^{2/s}$ is small and such is $|\eta| = \varepsilon^{1-\frac{2}{s}}$. To calculate K in (8.3.25) at first-order approximation, we take $x_c = 1$ to obtain $K = \frac{1}{48}(2b^2 - 11b + 11)$. This implies that $T = T_c = 2\pi/n$, at first-order approximation, which by (8.2.14) yields

$$x_c = 1 - \frac{b}{n^2} + o(n^{-2}). \tag{8.3.27}$$

Therefore,

$$\varepsilon = \frac{|b|^{s/2}}{n^s} + o(n^{-s}), \quad \eta = \frac{|b|^{s/2}}{bn^{s-2}} + o(n^{2-s}).$$

Next, by (8.3.15),

$$d = 1 + \frac{2b^2}{(1-b)n^2} + o(n^{-2}), \quad \Delta = b^2 \left[1 - \frac{2(b+1)}{n^2} + o(n^{-2})\right]. \tag{8.3.28}$$

We replace this value of d in conditions (i)-(iii) of Proposition 8.2.1 (neglecting the remainder $o(n^{-2})$) to verify that all they hold, provided that $n^2 > 2(1+b)$. Therefore, Proposition 8.2.1 holds as long as $n \geq N_0 = N_0(b,s)$ and N_0 is sufficiently large. To verify Proposition 8.2.3 we need to calculate Σ in any of the cases and check that $e \in \Sigma$. Unfortunately, first-order approximations do not suffice to verify Proposition 8.2.3. For that reason, we can proceed as follows. Using the above asymptotical values, we conclude that solutions of small period φ_n can exist only for parameters b, d in domains I (right period annulus), II, III and V, see the bifurcation diagram on Figure 1. Therefore, in domains I and V, it suffices to check that $\sqrt{d} < x_0 = x_c - \varepsilon$ because $(\sqrt{d}, 0)$ is the intersection point of the right branch of the invariant hyperbola $y^2 - x^2 + d = 0$ with the abscissa. At first-order approximation, this inequality is equivalent to

$$\sqrt{d} < x_c \quad \Leftrightarrow \quad 1 + \frac{b^2}{(1-b)n^2} < 1 - \frac{b}{n^2}$$

which clearly holds if $|b| > 1$. It remains to consider domains II and III where $|b| < 1$, $d > 1$. The function $H(x,0)$ then has just two critical points x_c and x_s (a minimum at x_c and a maximum at x_s) corresponding to the center and the saddle. Moreover,

$$1 < x_c < x_s \quad \text{in II}, \qquad x_s < x_c < 1 \quad \text{in III}.$$

In both cases, $H(x,0)$ goes to infinity as $x \to 1$ and to minus infinity as $|x| \to \infty$. This information implies that it suffices to prove only

$$e = H(x_0, 0) < H(x_s, 0) = e_s \quad \text{in II, III}, \quad x_0 > 1 \text{ in II}, \quad x_0 > x_s \text{ in III}. \tag{8.3.29}$$

By (8.2.12), one obtains

$$x_s = \frac{1-b}{1+b} + \frac{b}{n^2} + o(n^{-2}),$$

by (8.3.16), (8.3.17) we have

$$e = \frac{2b}{(1-b)n^2} \left|\frac{b}{n^2}\right|^{b-1} [1 + o(1)]$$

and by (8.2.11)

$$e_s = \frac{4b|\,2b|^{b-1}}{(1+b)^{1+b}}\left[1+O(n^{-2})\right].$$

As $x_0 = x_c$ at first order approximation, all conditions in (8.3.29) are obviously satisfied. Thus, Proposition 8.3.2 is proved.

□

So, the solutions $\varphi = \varphi_n(\tau)$ we just constructed have high frequency since $|1-x_c|$ is close to zero. This fact will be used in what follows.

Our next goal is to obtain simple estimates in terms of x_c for the period of the periodic solutions φ having sufficiently small amplitude and high frequency. For $b \neq 0, \pm 1$ an arbitrary but fixed number, such solutions exist in domains I, II, III and V, provided that both d and x_c are close enough to 1 (as shown above). By (8.3.27), one obtains immediately

$$\frac{|b|}{4n^2} \leq |1-x_c| \leq \frac{4|b|}{n^2}, \quad n \geq N_0(b,s)$$

as long as N_0 is large enough. This is obviously equivalent to

$$\pi\frac{|1-x_c|^{1/2}}{|b|^{1/2}} \leq T \leq 4\pi\frac{|1-x_c|^{1/2}}{|b|^{1/2}}, \quad n \geq N_0(b,s). \tag{8.3.30}$$

Below, we write $T \simeq |1-x_c|^{1/2}$ for the sake of (8.3.30).

Finally, let us rewrite equation (8.2.13) in the form

$$\varphi'^2 = \varphi^2 - d + e(1-\varphi)^{1-b} = U(\varphi), \quad ' = d/d\tau. \tag{8.3.31}$$

We shall need also the derivatives in the next section

$$\varphi'' = \varphi + \frac{e(b-1)}{2(1-\varphi)^b} = \frac{1}{2}U'(\varphi), \qquad \varphi''' = \left[1+\frac{eb(b-1)}{2(1-\varphi)^{b+1}}\right]\varphi' = \frac{1}{2}U''(\varphi)\varphi'. \tag{8.3.32}$$

Up to now we have seen that equation (8.3.31) admits a nonconstant even T - periodic solution (in the corresponding domains of (b,d)) which solves the initial value problem

$$\varphi'' = \varphi + \frac{e(b-1)}{2(1-\varphi)^b}, \quad \varphi(0) = x_0 = x_c - \varepsilon, \quad \varphi'(0) = 0.$$

We conclude this section with an estimate for the incomplete period, proceeding in the same way as above. Take $\alpha \in \left(0, \frac{x_1-x_0}{x_c-x_0}\right)$ and denote

$$\tau(x_0+\alpha\varepsilon) = \int_{x_0}^{x_0+\alpha\varepsilon} \frac{dX}{\sqrt{U(X)}}.$$

Applying the same change of variables, we obtain with

$$\zeta = \frac{2\alpha\varepsilon}{x_1-x_0} - 1 \in (-1,1)$$

the formula (instead of (8.3.20))

$$\tau(x_0+\alpha\varepsilon)=\frac{1}{2}\int_{-1}^{\zeta}\frac{(x_1-x_0)dz}{\sqrt{U(x_c+D(z,\eta))}}.$$

Then, including in the calculation of (8.3.22) all terms up to $O(\eta^2)$, we obtain

$$[1+a_1z\eta+\ldots]^{-1/2}=1-a_1z(\tfrac{1}{2}\eta+\tfrac{1}{4}q_2\eta^2)+(\tfrac{3}{8}a_1^2z^2-\tfrac{1}{2}b_0-\tfrac{1}{2}b_2z^2)\eta^2+O(\eta^3),$$

instead. Calculating the elementary integral, we get

$$\begin{aligned}\tau(x_0+\alpha\varepsilon)\ &=\sqrt{\frac{1-x_c}{(b+1)x_c-1}}\Big\{\left(1+G_1\eta^2\right)\left(\tfrac{1}{2}\pi+\arcsin\zeta\right)\\ &+\left[a_1(\tfrac{1}{2}\eta+\tfrac{1}{4}q_2\eta^2)-\zeta G_2\eta^2\right]\sqrt{1-\zeta^2}+O(\eta^3)\Big\},\end{aligned}\tag{8.3.33}$$

where $G_1=\frac{3}{16}a_1^2-\frac{1}{2}b_0-\frac{1}{4}b_2, G_2=\frac{3}{16}a_1^2-\frac{1}{4}b_2$. Recall that

$$x_1-x_0=(1-x_c)(2\eta+q_2\eta^2+q_3\eta^3+\ldots).$$

So, we have

$$\zeta=-1+\frac{\alpha}{1+\frac{q_2}{2}\eta+\frac{q_3}{2}\eta^2+\ldots}=\zeta_0+\zeta_1\eta+O(\eta^2),$$

where $\zeta_0=\alpha-1,\zeta_1=-\alpha q_2/2$. Substituting this expression into (8.3.33) gives

$$\tau(x_0+\alpha\varepsilon)=\frac{|\,1-x_c|^{1/2}}{\Delta^{1/4}}\left\{\frac{\pi}{2}+\arcsin\zeta_0+\eta\left(\frac{\zeta_1}{\sqrt{1-\zeta_0^2}}+\frac{a_1\sqrt{1-\zeta_0^2}}{2}\right)+O(\eta^2)\right\}.\tag{8.3.34}$$

Again, by analyticity argument we can take η or ε small enough that the expression in the brackets in (8.3.34) can be estimated as follows

$$\frac{|1-x_c|^{1/2}}{2\Delta^{1/4}}\left(\frac{\pi}{2}+\arcsin(\alpha-1)\right)\le\tau(x_0+\alpha\varepsilon)\le 2\frac{|1-x_c|^{1/2}}{\Delta^{1/4}}\left(\frac{\pi}{2}+\arcsin(\alpha-1)\right).\tag{8.3.35}$$

We shall use this estimate later in section 4.

8.4 Non-uniform continuity

In this section we establish appropriate estimates in Sobolev norms of the periodic solutions φ derived in the previous section and then prove our main theorem. The proof of Theorem 8.1.1 proceeds in the line of [173].

Below, we will use the notation introduced in the previous sections. In the proof of our main theorem, we are going to exploit the properties of small-amplitude high-frequency periodic solutions φ.

First, let us choose the parameter $b\neq 0,\pm 1$ and freeze it. Next, we choose x_c so that $|\,1-x_c|$ is sufficiently small. And, finally, we choose a periodic orbit sufficiently close to the center $(x_c,0)$.

That is, we choose the parameter e in (8.2.13) be so close to e_c in order to ensure that the amplitude $x_1 - x_0$ of the corresponding periodic solution φ will satisfy $x_1 - x_0 << 1 - x_c$. Therefore,

$$\varepsilon << 1 - x_c, \quad |\varphi - x_c| \leq |x_1 - x_0|; \text{ and } \frac{|\varphi - x_c|}{|1 - x_c|} << 1. \tag{8.4.36}$$

In the sequel we need U and several its derivatives evaluated at x_c. Trivial calculations give

$$U(x_c) = P\varepsilon^2 + Q\varepsilon^3 + R\varepsilon^4 + O(\varepsilon^5), \tag{8.4.37}$$
$$U'(x_c) = \frac{b-1}{1-x_c}U(x_c), \quad U''(x_c) = -2P + \frac{b(b-1)}{(1-x_c)^2}U(x_c), \quad \text{etc.}\ldots,$$

where

$$P = \frac{(b+1)x_c - 1}{1 - x_c}, \quad Q = \frac{(b+1)(2b-3)x_c - 3(b-1)}{3(1-x_c)^2},$$
$$R = \frac{(b-2)[(b+1)(b-2)x_c - 2(b-1)]}{4(1-x_c)^3}.$$

We begin with L^∞ - estimates of the derivatives.

Lemma 8.4.1. *There exist constants $C_k(b)$, $k \in \mathbb{N}$, so that the following estimates hold*

$$|\varphi^{(k)}| \leq C_k(b)\frac{\varepsilon}{|1 - x_c|^{k/2}}. \tag{8.4.38}$$

Proof. Recall the equation $\varphi'^2 = U(\varphi)$ and its derivatives (8.3.32). Expanding U around x_c and using the values of U and its derivatives at x_c we calculated earlier in (8.4.37), we obtain

$$\begin{aligned}
|U(\varphi)| &\leq |U(x_c)| + |\varphi - x_c||U'(x_c)| + \frac{1}{2}|\varphi - x_c|^2|U''(x_c)| + O(|\varphi - x_c|^3)\\
&\leq \left[1 + |\varphi - x_c|\frac{|b-1|}{|1-x_c|} + |\varphi - x_c|^2\frac{|b||b-1|}{2|1-x_c|^2}\right]\left[P\varepsilon^2 + |Q|\varepsilon^3 + O(\varepsilon^4)\right]\\
&+ P|\varphi - x_c|^2 + O(|\varphi - x_c|^3)\\
&\leq P\varepsilon^2\left(5 + |b-1| + \frac{|b||b-1|}{2}\right) + O(\varepsilon^3) \leq C(b)P\varepsilon^2
\end{aligned}$$

because of (8.4.36). Since $P < |b|/|1 - x_c|$, we obtain the estimate

$$|\varphi'| \leq C_1(b)\frac{\varepsilon}{|1 - x_c|^{1/2}}.$$

In a similar way, developing $U'(\varphi)$, we verify the estimate

$$|\varphi''| \leq C_2(b)\frac{\varepsilon}{|1 - x_c|}.$$

Next, we are going to proceed by induction. Taking kth-order derivative of the both sides of (8.2.7), $k = 0, 1, 2, \ldots$, we obtain the equation

$$(1 - \varphi)\varphi^{(k+3)} = \varphi^{(k+1)} + \sum_{i=0}^{k}\left[c_i\varphi^{(i+1)}\varphi^{(k-i+2)} + d_i\varphi^{(i)}\varphi^{(k-i+1)}\right], \tag{8.4.39}$$

where c_i, d_i are certain constants depending on k and b. Applying the induction hypothesis and the first bound from (8.4.36), we conclude that

$$|1-\varphi||\varphi^{(k+3)}| \leq C_k(b)\frac{\varepsilon}{|1-x_c|^{(k+1)/2}}.$$

As $|1-\varphi| > |1-x_c-O(\varepsilon)| > \frac{1}{2}|1-x_c|$, the claim follows.

□

Next we turn to L^2 - estimates.

Lemma 8.4.2. *There exist constants $D_k(b)$, $k \in \mathbb{N}$, so that the following estimates hold*

$$||\varphi'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} \leq D_1(b)\frac{\varepsilon^2}{|1-x_c|^{1/2}}. \tag{8.4.40}$$

and for any $k = 2, 3, \dots$

$$||\varphi^{(k)}||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} \leq D_k(b)\frac{||\varphi'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]}}{|1-x_c|^{(k-1)}}.$$

Proof. For the first derivative, we have

$$||\varphi'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi'^2 d\tau \leq C_1^2(b)\frac{\varepsilon^2}{|1-x_c|}T \leq D_1(b)\frac{\varepsilon^2}{|1-x_c|^{1/2}}.$$

Next, we get by (8.3.32), (8.3.16) and $|1-\varphi| > \frac{1}{2}|1-x_c|$

$$||\varphi''||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi''^2 d\tau = -\int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi'''\varphi' d\tau = -\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} U''(\varphi)\varphi'^2 d\tau =$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\frac{b(1-x_c)^b}{(1-\varphi)^{b+1}}e_1(\eta) - 1\right)\varphi'^2 d\tau \leq D_2(b)\frac{||\varphi'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]}}{|1-x_c|}.$$

Finally, we again proceed by induction. Lemma 8.4.2 holds for $k = 2$. By using (8.4.39), (8.4.38) and the inductive hypothesis, one easily obtains

$$\begin{aligned}
&||(1-\varphi)\varphi^{(k+3)}||_{L^2[-\frac{T}{2},\frac{T}{2}]} \leq \\
&||\varphi^{(k+1)}||_{L^2[-\frac{T}{2},\frac{T}{2}]} + \sum_{i=0}^{k}(c_i||\varphi^{(i+1)}\varphi^{(k-i+2)}||_{L^2[-\frac{T}{2},\frac{T}{2}]} + d_i||\varphi^{(i)}\varphi^{(k-i+1)}||_{L^2[-\frac{T}{2},\frac{T}{2}]}) \leq \\
&\left[\frac{D_{k+1}}{|1-x_c|^{k/2}} + \sum_{i=0}^{k}\left(\frac{c_i D_{i+1}}{|1-x_c|^{i/2}}\cdot\frac{C_{k-i+2}\varepsilon}{|1-x_c|^{(k-i+2)/2}} + \frac{d_i C_i\varepsilon}{|1-x_c|^{i/2}}\cdot\frac{D_{k-i+1}}{|1-x_c|^{(k-i)/2}}\right)\right]||\varphi'|| \\
&\leq \frac{D_{k+3}}{|1-x_c|^{k/2}}||\varphi'||_{L^2[-\frac{T}{2},\frac{T}{2}]}.
\end{aligned} \tag{8.4.41}$$

As $||(1-\varphi)\varphi^{(k+3)}||_{L^2[-\frac{T}{2},\frac{T}{2}]} \geq \frac{1}{2}|1-x_c|||\varphi^{(k+3)}||_{L^2[-\frac{T}{2},\frac{T}{2}]}$, the statement follows by induction.

□

Recall the Sobolev norm

$$||f||^2_{\mathrm{H}^s} = \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^s|\hat{f}(\xi)|^2,$$

where $\hat{f}(\xi)$ is the Fourier transform of f.

Lemma 8.4.3. *Let $\varphi = \varphi_n$ be the $T = \frac{2\pi}{n}$-periodic solution constructed in the end of the previous section. For any $s \geq 3$, there is a positive constant $c_{s,b}$ depending only on s and b, such that*

$$||\varphi||^2_{\mathrm{H}^s(-\pi,\pi)} \leq c_{s,b}\left(\frac{1}{|1-x_c|^{s-1}}||\varphi'||^2_{L^2(-\pi,\pi)} + x_1^2\right).$$

Proof. Let $s = k$, where $k = 3, 4, \ldots$. Using the facts that

$$||\varphi^{(k)}||^2_{L^2(-\pi,\pi)} = n||\varphi^{(k)}||^2_{L^2(\frac{-\pi}{n},\frac{\pi}{n})} \quad \text{and} \quad x_0 \leq \varphi \leq x_1$$

these estimates follow from Lemma 8.4.2.

Let now $s = k + \sigma$, where $k \geq 3$ is a positive integer and $0 < \sigma < 1$. We follow Proposition 3.3 in [173]

$$||\varphi||^2_{\mathrm{H}^s(-\pi,\pi)} \lesssim ||\varphi^{(k)}||^2_{\mathrm{H}^\sigma(-\pi,\pi)} + ||\varphi'||^2_{L^2(-\pi,\pi)} + ||\varphi||^2_{L^2(-\pi,\pi)}.$$

We have $||\varphi||^2_{L^2(-\pi,\pi)} = n||\varphi||^2_{L^2(-\pi/n,\pi/n)} \backsimeq 2\pi x_1^2$. It remains to estimate the H^σ - norm of $\varphi^{(k)}$. It is proven in [173] that for any smooth f the following inequality holds

$$||f||_{\mathrm{H}^\sigma(-\pi,\pi)} \lesssim ||f||^{1-\sigma}_{L^2(-\pi,\pi)}||f||^\sigma_{\mathrm{H}^1(-\pi,\pi)}.$$

Applying this to $\varphi^{(k)}$ yields

$$||\varphi^{(k)}||_{\mathrm{H}^\sigma(-\pi,\pi)} \lesssim ||\varphi^{(k)}||^{1-\sigma}_{L^2(-\pi,\pi)}||\varphi^{(k)}||^\sigma_{\mathrm{H}^1(-\pi,\pi)}.$$

Since $|1-x_c| < 1$, using the estimates from Lemma 8.4.2 we obtain

$$||\varphi^{(k)}||_{\mathrm{H}^1(-\pi,\pi)} \backsimeq ||\varphi^{(k)}||_{L^2(-\pi,\pi)} + ||\varphi^{(k+1)}||_{L^2(-\pi,\pi)} \lesssim$$

$$\left(\frac{1}{|1-x_c|^{(k-1)/2}} + \frac{1}{|1-x_c|^{k/2}}\right)||\varphi'||_{L^2(-\pi,\pi)} \lesssim \frac{||\varphi'||_{L^2(-\pi,\pi)}}{|1-x_c|^{k/2}}.$$

Combining these inequalities, we get

$$||\varphi^{(k)}||_{\mathrm{H}^\sigma(-\pi,\pi)} \lesssim \frac{||\varphi'||^{1-\sigma}_{L^2(-\pi,\pi)}}{|1-x_c|^{(k-1)(1-\sigma)/2}}\frac{||\varphi'||^\sigma_{L^2(-\pi,\pi)}}{|1-x_c|^{k\sigma/2}} \lesssim \frac{||\varphi'||_{L^2(-\pi,\pi)}}{|1-x_c|^{(k+\sigma-1)/2}}$$

from where the lemma follows.

□

Proof of Theorem 8.1.1. Let $s \geq 3$ and let φ_n be the $2\pi/n$ - periodic smooth solution, constructed above. Recall from section 3 that

$$n \backsimeq \frac{1}{|1-x_c|^{1/2}}. \tag{8.4.42}$$

Consider the following two sequences of traveling wave solutions

$$u_n(x,t) = \varphi_n(x-t), \qquad v_n(x,t) = c_n\varphi_n(x-c_nt) \tag{8.4.43}$$

and take

$$c_n = 1 + \frac{1}{n}.$$

As in [173] we show that these sequences are bounded, their difference goes to zero at time $t=0$ and stays apart from zero at $t>0$.

The boundedness and the limit at the time $t=0$ are almost straightforward. Taking into account (8.4.40), (8.4.42) it is obtained

$$||\,\varphi_n'||^2_{L^2(-\pi,\pi)} = n||\,\varphi_n'||^2_{L^2(\frac{-\pi}{n},\frac{\pi}{n})} \lesssim \frac{1}{|1-x_c|^{1/2}}\frac{D_1(b)\varepsilon^2}{|1-x_c|^{1/2}}.$$

Also, we have from (8.4.43) and Lemma 8.4.3 that

$$||v_n(t)||^2_{\mathrm{H}^s(-\pi,\pi)} = c_n^2||\,\varphi_n||^2_{\mathrm{H}^s(-\pi,\pi)} \lesssim c_n^2 c_{b,s}\frac{\varepsilon^2}{|1-x_c|^s} + x_1^2,$$

where $s\geq 3$. The choice of parameters $|1-x_c|^s = \varepsilon^2$ assures that the both sequences u_n and v_n are bounded in H^s - norms.

Further,

$$||\,v_n(0)-u_n(0)||^2_{\mathrm{H}^s(\mathbb{S})} = ||\,c_n\varphi_n - \varphi_n||^2_{\mathrm{H}^s(\mathbb{S})} = (c_n-1)^2||\,\varphi_n||^2_{\mathrm{H}^s(\mathbb{S})} \cong \frac{1}{n^2},$$

which goes to 0 when $n\to\infty$.

Finally, the behavior at time $t>0$ can be established in the following way.

$$||\,v_n(t)-u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} = \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^s|\,c_n\widehat{\varphi_n}(\cdot - c_nt)(\xi) - \widehat{\varphi_n}(\cdot - t)(\xi)|^2,$$

where $\widehat{\varphi_n}(\cdot - c_nt)(\xi)$ is the Fourier transform of the function $\varphi_n(x-c_nt)$ with respect to x, that is after changing the variables $\widehat{\varphi_n}(\cdot - c_nt)(\xi) = e^{-itc_n\xi}\widehat{\varphi_n}(\xi)$. Hence,

$$||\,v_n(t)-u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} = \sum_{\xi\in\mathbb{Z}}(1+\xi^2)^s\left|(e^{\frac{-it\xi}{n}}-1)+\frac{1}{n}e^{\frac{-it\xi}{n}}\right|^2|\,\widehat{\varphi}(\xi)|^2,$$

and

$$||\,v_n(t)-u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \geq (1+n^2)^s|(e^{-it}-1)+\frac{1}{n}e^{-it}|^2|\,\hat{\varphi}_n(n)|^2.$$

Since φ_n is a $2\pi/n$ - periodic, even function and after integrating by parts, we get

$$\hat{\varphi}_n(n) = \frac{n}{\sqrt{2\pi}}\int_{-\pi/n}^{\pi/n} e^{-in\tau}\varphi_n(\tau)d\tau = -\frac{2}{\sqrt{2\pi}}\int_0^{\pi/n}\sin(n\tau)\varphi'(\tau)d\tau.$$

Therefore

$$||\,v_n(t)-u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \geq \frac{2}{\pi}(1+n^2)^s|(e^{-it}-1)+\frac{1}{n}e^{-it}|^2|\,B_n|^2, \tag{8.4.44}$$

where we denote

$$B_n = \int_0^{\pi/n}\sin(n\tau)\varphi_n'(\tau)d\tau.$$

The integral for B_n can be estimated from below in the same line as Lemma 4.1 in [173].

Lemma 8.4.4. *There exists a constant $c_0 > 0$ independent of n such that*

$$B_n \geq c_0 \varepsilon.$$

Proof. We have

$$B_n = \int_0^{\pi/n} \sin(n\tau)\varphi_n'(\tau)d\tau = \int_{x_0}^{x_1} \sin(n\tau(\varphi))d\varphi.$$

For any $\alpha \in (0, \frac{x_1 - x_0}{x_c - x_0})$,

$$B_n \geq \int_{x_0+\alpha\varepsilon/2}^{x_0+\alpha\varepsilon} \sin(n\tau(\varphi))d\varphi. \tag{8.4.45}$$

We take α to satisfy the condition

$$n\tau(x_0 + \alpha\varepsilon) \leq \frac{\pi}{2}. \tag{8.4.46}$$

To do this, let us first recall the estimate (8.3.30) on the period. Next, by (8.3.28), we have $\Delta = b^2[1 + O(n^{-2})]$, therefore one can rewrite (8.3.30) as

$$\pi \frac{|1 - x_c|^{1/2}}{\Delta^{1/4}} \leq T = \frac{2\pi}{n} \leq 4\pi \frac{|1 - x_c|^{1/2}}{\Delta^{1/4}}.$$

Further, taking advantage from the estimate on the incomplete period (8.3.35), we get

$$\frac{1}{4}\left(\frac{\pi}{2} + \arcsin(\alpha - 1)\right) \leq n\tau(x_0 + \alpha\varepsilon) \leq 4\left(\frac{\pi}{2} + \arcsin(\alpha - 1)\right). \tag{8.4.47}$$

Thus, to satisfy the condition (8.4.46) we take α so that

$$4\left(\frac{\pi}{2} + \arcsin(\alpha - 1)\right) = \frac{\pi}{2},$$

or $\arcsin(\alpha - 1) = -\frac{3\pi}{8}$. With this choice of α inequality (8.4.45) gives

$$\begin{aligned} B_n &\geq \int_{x_0+\alpha\varepsilon/2}^{x_0+\alpha\varepsilon} \sin(n\tau(x_0 + \frac{\alpha}{2}\varepsilon))d\varphi \\ &= \sin\left(n\tau(x_0 + \frac{\alpha}{2}\varepsilon)\right)\frac{\alpha}{2}\varepsilon \geq \left[\frac{\alpha}{2}\sin\left(\frac{1}{4}\left(\frac{\pi}{2} + \arcsin(\frac{\alpha}{2} - 1)\right)\right)\right]\varepsilon, \end{aligned}$$

where the last inequality follows from the lower bound in (8.4.47) and α is replaced by $\alpha/2$. This proves the lemma.

□

Returning to (8.4.44) one gets

$$|| v_n(t) - u_n(t)||^2_{\mathrm{H}^s(S)} \gtrsim n^{2s}\varepsilon^2 \left|(e^{-it} - 1) + \frac{1}{n}e^{-it}\right|^2.$$

Thus, the desired estimate is obtained as in [173] using (8.4.42) and $|1 - x_c|^s = \varepsilon^2$.

■

8.5 The cases b = ±1.

Here we study the cases $b = \pm 1$ in the Holm - Staley equation (8.1.1). Since most of the computations and estimates are similar to those in sections 2, 3 and 4, we give only the key results and differences.

The case b = −1.

Equation (8.1.2) with $b = -1$ has no hydrodynamical relevance, but we consider it here due to its simplicity. One should start with it, because all things are transparent. By (8.2.8), we obtain the conic curve

$$(\varphi - 1)^{-2}(\varphi'^2 - \varphi^2 + d) = e. \qquad (8.5.48)$$

There are periodic solutions φ for $d > 1$ and $e \in \left(\frac{d}{1-d}, -1\right)$. They surround the center at $(d, 0)$. One can rewrite (8.5.48) as

$$\varphi'^2 + \frac{1}{e}(\varphi - 1)^2 - 2(\varphi - 1) + d = 0,$$

with new d and e (equal to $d - 1$ and $-(e + 1)^{-1}$, respectively). Hence, periodic solutions exist for $e > d > 0$. Denote $A = \sqrt{e(e - d)}$. Then they are given explicitly by

$$\varphi(\tau) = 1 + e - A \cos \frac{\tau}{\sqrt{e}}$$

with period $T = 2\pi\sqrt{e}$. Assuming A and e small, one can find integer n such that $n \simeq \frac{1}{\sqrt{e}}$ and $T = \frac{2\pi}{n}$.

As before, let us take the following two sequences of solutions

$$u_n(x, t) = \varphi_n(x - t), \quad v_n(x, t) = c_n\varphi_n(x - c_nt), \quad c_n = 1 + \frac{1}{n}.$$

It is sufficient to estimate v_n. A direct computation gives

$$||\, v_n(t)||^2_{\mathrm{H}^s(-\pi,\pi)} = c_n^2 \left[(1 + e)^2 + \frac{1}{4}(1 + n^2)^s A^2\right] \le 4 \left[(1 + e)^2 + 2^{s-2}n^{2s}A^2\right].$$

Boundedness is achieved upon the condition $A^2 = e^s, s \ge 3$. The limit at $t = 0$ is the same as above. It remains to consider the estimate

$$||\, v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \ge (1 + n^2)^s |(e^{-it} - 1) + \frac{1}{n}e^{-it}|^2 |\, \hat{\varphi}_n(n)|^2.$$

Trivial computations yield that $\hat{\varphi}_n(n) = -A/2$, so

$$||\, v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \ge \frac{1}{4}A^2n^{2s}|(e^{-it} - 1) + \frac{1}{n}e^{-it}|^2 = \frac{1}{4}|(e^{-it} - 1) + \frac{1}{n}e^{-it}|^2$$

in view of relation $A^2 = e^s, s \ge 3$. Hence, the result follows as in [173].

Case b = 1.

By (8.2.9), traveling-wave solutions of (8.1.1) of the form $u = y(x - t)$, $y < 1$ will satisfy

$$H(y, y') \equiv y'^2 - y^2 - 2d \ln(1 - y) = e. \qquad (8.5.49)$$

Hence, we have a conservative system with a Newtonian first integral H. The critical points of the potential $H(y, 0)$ are

$$y_c = \frac{1 + \sqrt{1 - 4d}}{2}, \quad y_s = \frac{1 - \sqrt{1 - 4d}}{2}, \quad d < 1/4,$$

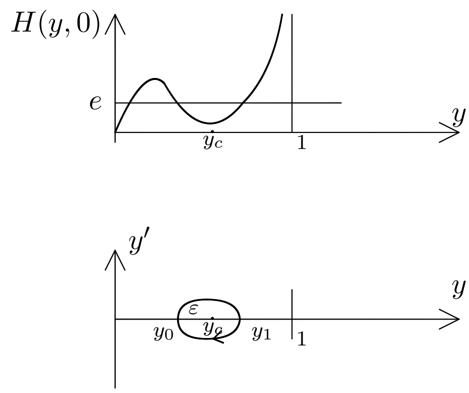


Figure 8.3: The periodic solution of (8.5.49)

where c stands for the center and s for the saddle. It is straightforward to verify that for $d \in (0, 1/4)$ and $H(y_c, 0) < e < H(y_s, 0)$ there are periodic solutions (see also Figure 3). Let $y_1 = \max y$ and $y_0 = \min y$. We assume that $y_1 - y_0$ is small. So, there is a periodic solution to (8.5.49) which satisfies the following initial value problem

$$y'' = y - \frac{d}{1-y}, \quad y(0) = y_0, \quad y'(0) = 0.$$

The period function has an expansion

$$T(\varepsilon) = T_c + \varepsilon^2 T_2 + \dots,$$

where ε is defined as

$$y_0 = y_c - \varepsilon \quad \text{and} \quad T_c = 2\pi\sqrt{\frac{1-y_c}{2y_c-1}}.$$

Note that, when $d < 1/4$, then $y_c > 1/2$. Similar computations as in Section 2 give

$$e = H(y_0, 0) = -y_c^2 - 2d\ln(1-y_c) + P\varepsilon^2 + Q\varepsilon^3 + R\varepsilon^4 + \dots,$$

where

$$P = \frac{2y_c - 1}{1-y_c}, \quad Q = -\frac{2y_c}{3(1-y_c)^2}, \quad R = \frac{y_c}{2(1-y_c)^3}.$$

In terms of η the energy e becomes

$$e = H(y_0, 0) = -y_c^2 - 2d\ln(1-y_c) + p_2\eta^2 + p_3\eta^3 + p_4\eta^4 + \dots,$$

where

$$p_2 = (2y_c - 1)(1-y_c), \quad p_3 = -\frac{2}{3}y_c(1-y_c), \quad p_4 = \frac{y_c}{2}(1-y_c).$$

For these calculations we have used the identity $y_c - d/(1-y_c) = 0$. From $H(y_0, 0) = H(y_1, 0)$ one obtains that

$$y_1 = y_c + (1-y_c)\Phi(\eta), \quad \Phi(\eta) = \eta + r\eta^2 + r^2\eta^3 + \dots$$

with $r = p_3/p_2$. Denote $U(y) = y^2 + 2d\ln(1-y) + e$. Then (8.5.49) becomes $y'^2 = U(y)$ and the period function is

$$T = 2\int_{y_0}^{y_1} \frac{dy}{\sqrt{U(y)}}.$$

As before we put

$$y = \frac{y_1 - y_0}{2}z + \frac{y_1 + y_0}{2},$$

thus

$$T = \int_{-1}^{1} \frac{(y_1 - y_0)\,dz}{\sqrt{U(y_c + D(z,\eta))}}.$$

In the same line of computations we obtain the formula

$$T = 2\pi\sqrt{\frac{1-y_c}{2y_c - 1}}\left(1 + \frac{y_c(9 - 8y_c)}{24(2y_c - 1)^2}\eta^2 + \dots\right). \qquad (8.5.50)$$

As above one can take η so small, that the expression in the brackets in (8.5.50) will take values in $[\frac{1}{2}, 2]$. This gives

$$\pi\sqrt{\frac{1-y_c}{2y_c-1}} \le T \le 4\pi\sqrt{\frac{1-y_c}{2y_c-1}}.$$

We write $T \simeq \sqrt{1-y_c}$ and for any sufficiently large integer n one can find y_c so that and $1 - y_c$ is sufficiently small in order to achieve

$$T = \frac{2\pi}{n} \quad \text{and} \quad n \simeq \frac{1}{\sqrt{1-y_c}}.$$

Hence, we have constructed high-frequency solution $y = y_n(t)$ with period $T = 2\pi/n$.

Next, in order to estimate the incomplete integral $\tau(y_0 + \alpha\varepsilon)$, by long but straightforward computations similar to those in section 3 we obtain

$$\frac{1}{2}\sqrt{\frac{1-y_c}{2y_c-1}}\left(\frac{\pi}{2} + \arcsin(\alpha - 1)\right) \le \tau(y_0 + \alpha\varepsilon) \le 2\sqrt{\frac{1-y_c}{2y_c-1}}\left(\frac{\pi}{2} + \arcsin(\alpha - 1)\right).$$

Finally, we need some estimates in order to obtain upper bounds for these solutions. We need writing (8.5.49) in the form $y'^2 = U(y)$ and then calculate the derivatives

$$y'' = \frac{1}{2}U'(y), \qquad y''' = \frac{1}{2}U''(y)y'. \qquad (8.5.51)$$

Assume that y_c is close enough to 1 and $y_1 - y_0 << 1 - y_c$, and also

$$\varepsilon << 1 - y_c, \quad |y - y_c| \le |y_1 - y_0| \text{ and } \frac{|y - y_c|}{1 - y_c} << 1. \qquad (8.5.52)$$

Expanding U around y_c, using (8.5.52) and that $P \le 1/(1 - y_c)$ we obtain the estimate

$$|y'| \le \frac{\sqrt{10}\,\varepsilon}{(1 - y_c)^{1/2}}.$$

In a similar way, developing $U'(y)$ we get $|y''| \leq 4\varepsilon/(1-y_c)$. Again, induction arguments give the estimates

$$|y^{(k)}| \leq C_k \frac{\varepsilon}{(1-y_c)^{k/2}}.$$

From the above expressions we obtain L^2-estimate for the first derivative

$$||y'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} = \int_{-T/2}^{T/2} y'^2 d\tau \leq C_1 \frac{\varepsilon^2}{1-y_c} T \leq D_1 \frac{\varepsilon^2}{(1-y_c)^{1/2}}.$$

Finally, from (8.5.51) we obtain an estimate for the second derivative

$$||y''||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} = \int_{-\frac{T}{2}}^{\frac{T}{2}} y''^2 d\tau = -\int_{-\frac{T}{2}}^{\frac{T}{2}} y''' y' d\tau = -\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} U''(y) y'^2 d\tau \leq D_2 \frac{||y'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]}}{1-y_c}.$$

Now, we can proceed by induction to obtain similar estimates for the higher-order derivatives as in Lemma 8.4.2. The proof of Theorem 8.1.1 is then finished in the same way as in the general case.

8.6 Ill - posedness for the DP equation

In this section we establish the second result of this chapter concerning non uniform well - posedness for the periodic Cauchy problem for the Degasperis - Procesi equation (DP)

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad u(0) = u_0,\ t \geq 0,\ x \in \mathbb{S}. \tag{8.6.53}$$

As it is mentioned above this equation appears in the family (8.1.2) for $b = 3$ and it is formally integrable in the sense that it admits a Lax pair, which should lead to integrability if an inverse spectral theory is developed, as it was done for $b = 2$ in Constantin and McKean [140]. Here we follow closely Himonas and Misiolek [173] where such kind a result is originally proven for the Camassa - Holm equation ($b = 2$).

Recall that we say that a Cauchy problem is uniformly well - posed in $\mathrm{H}^s(\mathbb{S})$ if there exist $T > 0$ and a unique solution u in $\mathrm{C}([0,T],\mathrm{H}^s(\mathbb{S}))$ such that the map $u_0 \to u$ is uniformly continuous from any bounded set of $\mathrm{H}^s(\mathbb{S}))$ into $\mathrm{C}([0,T],\mathrm{H}^s(\mathbb{S}))$.

Theorem 8.6.1. *For any $s \geq 2$, the solution map $u_0 \to u$ for the Degasperis - Procesi equation (8.6.53) is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S}))$ into $\mathrm{C}([0,T],\mathrm{H}^s(\mathbb{S})))$. More precisely, for each $s \geq 2$ there exist constants $c_{1,2} > 0$ and two sequences of smooth solutions u_n, v_n of the equation (8.6.53) such that for any $t \in [0,1]$*

$$\begin{gathered}\sup_n ||u_n(t)||_{\mathrm{H}^s} + \sup_n ||v_n(t)||_{\mathrm{H}^s} \leq c_1, \\ \lim_{n\to\infty} ||u_n(0) - v_n(0)||_{\mathrm{H}^s} = 0, \\ \liminf_n ||u_n(t) - v_n(t)||_{\mathrm{H}^s} \geq c_2 \sin\left(\frac{t}{2}\right).\end{gathered} \tag{8.6.54}$$

In order to prove this Theorem we proceed in the following way. As in [173] first we construct a family of high - frequency smooth traveling wave solutions of (8.6.53).

Note that if $u(x,t)$ is a classical solution of (8.6.53), then the function

$$u_c(x,t) = cu(x,ct) \tag{8.6.55}$$

is a solution also for arbitrary constant c.

Let us consider the traveling wave solutions of the DP $u(x,t) = f(x-t)$, so the function f must satisfy the ordinary differential equation

$$-f' + f''' + 4ff' - 3f'f'' - ff''' = 0. \tag{8.6.56}$$

Integrating once we get $(1-f)f'' - f'^2 + 2f^2 - f = d_1$. Next we put

$$y = 1 - f \tag{8.6.57}$$

and integrate once again to obtain

$$y^2 y'^2 = y^4 - 2y^3 - dy^2 + e. \tag{8.6.58}$$

Let m and M are the minimum and the maximum of the function f, correspondingly i.e. $m \le f \le M$. We assume that $0 << m < M < 1$ or for sufficiently small $\varepsilon, \delta > 0$ we let

$$M = 1 - \delta, \quad m = 1 - \delta - \varepsilon. \tag{8.6.59}$$

It follows from (8.6.57) that

$$m_1 := \delta \le y \le \delta + \varepsilon := M_1. \tag{8.6.60}$$

Expressing d, e through M_1 and m_1 we find

$$d = \frac{(M_1 + m_1)(M_1^2 + m_1^2) - 2(M_1^2 + m_1^2 + m_1 M_1)}{M_1 + m_1}, \quad e = M_1^2 m_1^2 \frac{M_1 + m_1 - 2}{M_1 + m_1}. \tag{8.6.61}$$

Then (8.6.58) becomes

$$y'^2 = \frac{(M_1 - y)(y - m_1)(-y^2 + py + q)}{y^2} := V(y), \tag{8.6.62}$$

where $p = 2 - M_1 - m_1$, $q = M_1 m_1 \frac{2 - M_1 - m_1}{M_1 + m_1}$. It can be seen that (8.6.62) admits a nonconstant solution with period $2l$ for certain $l > 0$, which satisfies the following initial value problem (see also Fig. 1)

$$y'' = y - 1 - \frac{e}{y^3}, \quad y(0) = \delta, \; y'(0) = 0. \tag{8.6.63}$$

The above discussion can be summarized as follows

Proposition 8.6.1. *For any $\varepsilon, \delta < 1/6$, there exists a positive number $l = l(\varepsilon, \delta)$ and even $2l$ periodic smooth function $y = y(\xi)$ which solves (8.6.63) and (8.6.62). The function $y = y(\xi)$ satisfies*

$$m_1 := \delta \le y \le \delta + \varepsilon := M_1$$

and $u(x,t) = f(x-t)$, where $f(\xi) = 1 - y(\xi)$ is a traveling wave solution of the DP equation.

Next we estimate the period $2l$ in terms of ε, δ.

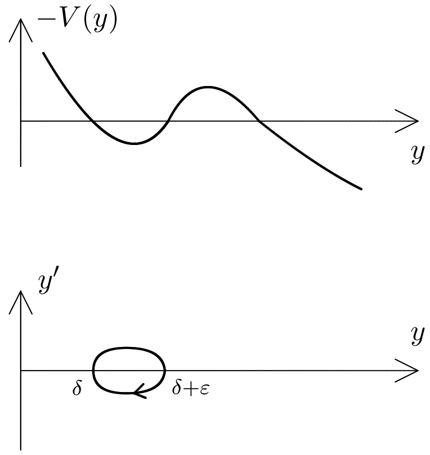


Figure 8.4: The periodic solution of (8.6.62)

Proposition 8.6.2. *The period $2l$ of the function y depends continuously on parameters ε, δ and satisfies*

$$\frac{\pi}{4}\sqrt{\varepsilon+\delta} \leq l \leq \sqrt{2}\pi\sqrt{\varepsilon+\delta}. \tag{8.6.64}$$

Proof. The half period can be expressed as

$$l = \int_{m_1}^{M_1} \frac{dy}{\sqrt{V(y)}} = \int_{m_1}^{M_1} \frac{ydy}{\sqrt{(M_1-y)(y-m_1)k(y)}}, \tag{8.6.65}$$

where $k(y) = -y^2 + py + q$. It is clear that $k_{min} = k(m_1)$ and $k_{max} = k(M_1)$. Then

$$l \leq \frac{1}{\sqrt{k(m_1)}} \int_{m_1}^{M_1} \frac{ydy}{\sqrt{(M_1-y)(y-m_1)}} = \frac{1}{\sqrt{k(m_1)}} \frac{\pi}{2}(M_1+m_1) \tag{8.6.66}$$

$$l \geq \frac{1}{\sqrt{k(M_1)}} \int_{m_1}^{M_1} \frac{ydy}{\sqrt{(M_1-y)(y-m_1)}} = \frac{1}{\sqrt{k(M_1)}} \frac{\pi}{2}(M_1+m_1). \tag{8.6.67}$$

The estimate for $k(M_1)$ is immediate - $k(M_1) \leq 4M_1$. Then from (8.6.67) it follows the lower bound

$$l \geq \frac{1}{\sqrt{k(M_1)}} \frac{\pi}{2}(M_1+m_1) \geq \frac{1}{\sqrt{M_1}} \frac{\pi}{4}(M_1+m_1) \geq \frac{\pi}{4}\sqrt{M_1} = \frac{\pi}{4}\sqrt{\varepsilon+\delta}.$$

Estimating $k(m_1)$ we have $k(m_1) \geq 2m_1(2-M_1-m_1)$. Choose M_1 and m_1 or equivalently $\delta+\varepsilon, \delta$ in such a way to achieve

$$k(m_1) \geq 2m_1(2-M_1-m_1) \geq M_1/2. \tag{8.6.68}$$

Then with the help of (8.6.66) it follows

$$l \leq \frac{\sqrt{2}}{\sqrt{M_1}} \frac{\pi}{2}(M_1+m_1) \leq \sqrt{2}\pi\sqrt{M_1} = \sqrt{2}\pi\sqrt{\varepsilon+\delta}.$$

□

We write $l \simeq \sqrt{\varepsilon+\delta}$ for the sake of (8.6.64). Since $l(\varepsilon, \delta)$ is continuous for n sufficiently large, we can find ε and $\delta = \varepsilon^{2/s}, s \geq 2$ such that

$$l = \frac{\pi}{n}, \quad (T = 2l = \frac{2\pi}{n}) \tag{8.6.69}$$

and

$$n \simeq \frac{1}{\sqrt{\varepsilon+\delta}}. \tag{8.6.70}$$

Hence, we have constructed high - frequency solution $y = y_n(\xi)$ with the period $T = 2\pi/n$.

Next we need some estimates in order to obtain upper bounds for these solutions. We start with L^∞ estimates of the derivatives.

Proposition 8.6.3. *Suppose $\delta \geq \varepsilon$. Then for any $k = 2, 3, \dots$ there exists a constant $c_k > 0$ such that*

$$| y^{(k)}| \leq \frac{c_k}{(\sqrt{\delta})^{k-2}}. \tag{8.6.71}$$

For $k = 1$ we have

$$| y'| \leq 2\sqrt{2}\frac{\varepsilon}{\sqrt{\delta}}.$$

Proof. With the help of (8.6.62), the first derivative is estimated as follows

$$| y'| = \frac{\sqrt{M_1 - y}\sqrt{y - m_1}\sqrt{k(y)}}{y} \leq \frac{(M_1 - m_1)\sqrt{k_{max}}}{m_1} \leq$$

$$\frac{2(M_1 - m_1)\sqrt{M_1}}{m_1} \leq \frac{2\varepsilon\sqrt{2\delta}}{\delta} = 2\sqrt{2}\frac{\varepsilon}{\sqrt{\delta}}.$$

For $k = 2$ using (8.6.63) we have

$$| y''| \leq 1 + | y| + \frac{| e|}{| y|^3} \leq 1 + \varepsilon + \delta + \frac{| e|}{\delta^3} = 1 + \varepsilon + \delta + \frac{\delta^2(\delta+\varepsilon)^2(2 - 2\delta - \varepsilon)}{\delta^3(2\delta+\varepsilon)} \leq 7.$$

Next, we proceed by induction and assume that (8.6.71) is true for all positive integers up to $k+2$. To estimate $(k+3)$ - order derivative we have from (8.6.63)

$$y^3 y'' = y^4 - y^3 - e. \tag{8.6.72}$$

Differentiate both sides of (8.6.72) and divide by y^2 to obtain

$$yy''' = -3y'y'' + 4yy' - 3y'. \tag{8.6.73}$$

Taking k derivatives and using Leibniz rule we have

$$\sum_{j=0}^{k} \binom{k}{j} y^{(k-j)} y^{(j+3)} = -3\sum_{j=0}^{k} \binom{k}{j} y^{(k+1-j)} y^{(j+2)} + 4\sum_{j=0}^{k} \binom{k}{j} y^{(k-j)} y^{(j+1)} - 3y^{(k+1)}.$$

Now, $y^{(k+3)}$ can be expressed as follows

$$y^{(k+3)} = \frac{1}{y}\left\{-\sum_{j=0}^{k-1}\binom{k}{j}y^{(k-j)}y^{(j+3)} - 3\sum_{j=0}^{k}\binom{k}{j}y^{(k+1-j)}y^{(j+2)} + 4\sum_{j=0}^{k}\binom{k}{j}y^{(k-j)}y^{(j+1)} - 3y^{(k+1)}\right\}.$$

The last equation together with the induction hypothesis gives

$$|y^{(k+3)}| \leq \frac{1}{\delta}\sum_{j=0}^{k-1}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\frac{c_{j+3}}{(\sqrt{\delta})^{j+3-2}} + \frac{3}{\delta}\sum_{j=0}^{k}\binom{k}{j}\frac{c_{k+1-j}}{(\sqrt{\delta})^{k+1-j-2}}\frac{c_{j+2}}{(\sqrt{\delta})^{j+2-2}} +$$

$$\frac{4}{\delta}\sum_{j=0}^{k}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\frac{c_{j+1}}{(\sqrt{\delta})^{j+1-2}} + \frac{3}{\delta}\frac{c_{k+1}}{(\sqrt{\delta})^{k+1-2}} \leq$$

$$\frac{c'_{k+3}}{\delta}\left[\sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(\sqrt{\delta})^{k-1}} + \sum_{j=0}^{k}\binom{k}{j}\frac{1}{(\sqrt{\delta})^{k-3}} + \sum_{j=0}^{k}\binom{k}{j}\frac{1}{(\sqrt{\delta})^{k-1}} + \frac{1}{(\sqrt{\delta})^{k-1}}\right] \leq \frac{c_{k+3}}{(\sqrt{\delta})^{(k+3)-2}},$$

which completes the proof.

□

Further, we proceed with L^2 estimates.

Proposition 8.6.4. *For any $k = 2, 3, \ldots$ there exists a constant $c_k > 0$ such that*

$$||\, y^{(k)}||^2_{L^2(-l,l)} \leq \frac{c_k}{(\delta)^{k-1}}||\, y'||^2_{L^2(-l,l)}. \tag{8.6.74}$$

For $k = 1$ we have

$$\int_{-l}^{l}(y')^2\, d\xi \leq \frac{\sqrt{2}\pi}{2}\frac{\varepsilon^2}{\sqrt{\delta}}.$$

Proof. For the first derivative we get

$$\int_{-l}^{l}(y')^2\, d\xi = 2\int_{0}^{l}(y')^2\, d\xi = 2\int_{\delta}^{\delta+\varepsilon}\frac{\sqrt{\delta+\varepsilon-y}\sqrt{y-\delta}\sqrt{-y^2+py+q}}{y}\, dy \leq$$

$$\frac{2}{\delta}\sqrt{k_{max}}\int_{\delta}^{\delta+\varepsilon}\sqrt{\delta+\varepsilon-y}\sqrt{y-\delta}dy \leq \frac{2\sqrt{4M_1}}{\delta}\frac{\varepsilon^2\pi}{8} \leq \frac{\sqrt{2}\pi}{2}\frac{\varepsilon^2}{\sqrt{\delta}}.$$

For the second derivative using symmetry, periodicity and integrating by parts, we have

$$\int_{0}^{l}(y'')^2 d\xi = -\int_{0}^{l}y'''y'd\xi = -\int_{\delta}^{\delta+\varepsilon}y'''dy = -\int_{\delta}^{\delta+\varepsilon}\left(1+\frac{3e}{y^4}\right)y'dy \leq$$

$$\left(\frac{3|\, e|}{|\, y|^4}-1\right)\int_{\delta}^{\delta+\varepsilon}y'dy \leq \left(\frac{3(\delta+\varepsilon)^2(2-2\delta-\varepsilon}{\delta^2(2\delta+\varepsilon)}-1\right)||\, y'||^2_{L^2(0,l)} \leq \frac{12}{\delta}||\, y'||^2_{L^2(0,l)}.$$

Proceeding by induction and using the expression for $y^{(k+3)}$ in Proposition 8.6.3, we obtain

$$||y^{(k+3)}||_{L^2(-l,l)} \leq \frac{1}{\delta}\sum_{j=0}^{k-1}\binom{k}{j}||y^{(k-j)}y^{(j+3)}||_{L^2(-l,l)} + \frac{3}{\delta}\sum_{j=0}^{k}\binom{k}{j}||y^{(k+1-j)}y^{(j+2)}||_{L^2(-l,l)}$$

$$+\frac{4}{\delta}\sum_{j=0}^{k}\binom{k}{j}||y^{(k-j)}y^{(j+1)}||_{L^2(-l,l)}+\frac{3}{\delta}||y^{(k+1)}||_{L^2(-l,l)}\leq$$

$$\frac{1}{\delta}\sum_{j=0}^{k-1}\binom{k}{j}||y^{(k-j)}||_{L^\infty}||y^{(j+3)}||_{L^2(-l,l)}+\frac{3}{\delta}\sum_{j=0}^{k}\binom{k}{j}||y^{(k+1-j)}||_{L^\infty}||y^{(j+2)}||_{L^2(-l,l)}$$

$$+\frac{4}{\delta}\sum_{j=0}^{k}\binom{k}{j}||y^{(k-j)}||_{L^\infty}||y^{(j+1)}||_{L^2(-l,l)}+\frac{3}{\delta}||y^{(k+1)}||_{L^2(-l,l)}.$$

Now, applying estimates of Proposition 8.6.3 and using the induction hypothesis we get

$$||y^{(k+3)}||_{L^2(-l,l)}\leq\frac{1}{\delta}\sum_{j=0}^{k-1}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\frac{c_{j+3}}{(\sqrt{\delta})^{j+2}}||\,y'||_{L^2(-l,l)}+$$

$$\frac{3}{\delta}\sum_{j=0}^{k}\binom{k}{j}\frac{c_{k+1-j}}{(\sqrt{\delta})^{k-j-1}}\frac{c_{j+2}}{(\sqrt{\delta})^{j+1}}||\,y'||_{L^2(-l,l)}+\frac{4}{\delta}\sum_{j=0}^{k}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\frac{c_{j+1}}{(\sqrt{\delta})^{j}}||\,y'||_{L^2(-l,l)}$$

$$+\frac{3}{\delta}\frac{c_{k+1}}{(\sqrt{\delta})^{k}}\leq\frac{c_{k+3}}{(\sqrt{\delta})^{(k+3)-1}}||\,y'||_{L^2(-l,l)},$$

which finishes the proof.

□

So far we have obtained practically the same necessary estimates as in [173] and can proceed with the proof exactly in their way. We only give the key steps for the reader's convenience.

Proposition 8.6.5. *(Proposition 3.3 [173]) Let $y = y_n$ be the $2\pi/n$ periodic smooth solution constructed in Proposition 8.6.1. for any $s \geq 2$, there is a positive constant c_s such that*

$$||y||^2_{H^s(-\pi,\pi)}\leq\left(\frac{1}{\delta^{s-1}}||y'||^2_{L^2(-\pi,\pi)}+(\delta+\varepsilon)^2\right).$$

Next,the following two sequences of traveling wave solutions are defined

$$u_n(x,t)=f_n(x-t),\qquad v_n(x,t)=c_nf_n(x-c_nt) \tag{8.6.75}$$

and take

$$c_n=1+\frac{1}{n}.$$

The boundedness and the limit at the time $t=0$ are almost straightforward. Estimating v_n, taking into account Propositions 8.6.4, 8.6.5 it is obtained

$$||v_n(t)||_{H^s(-\pi,\pi)}\leq\pi^2+c_s\frac{\varepsilon^2}{\delta^s}, \tag{8.6.76}$$

where $s\geq 2$. Note that it is already chosen $\delta^s=\varepsilon^2$. Hence, the both sequences of smooth solutions are bounded. Further,

$$||v_n(0)-u_n(0)||^2_{H^s(\mathbb{S})}=||c_nf_n-f_n||^2_{H^s(\mathbb{S})}=(c_n-1)^2||f_n||^2_{H^s(\mathbb{S})}\cong\frac{1}{n^2}\to 0. \tag{8.6.77}$$

Finally, the behaviour at time $t > 0$ can be established as follows. The norm $||v_n(t) - u_n(t)||^2_{H^s(\mathbb{S})}$ can be estimated as follows

$$||v_n(t) - u_n(t)||^2_{H^s(\mathbb{S})} \geq \frac{2}{\pi}(1+n^2)^s|(e^{-it}-1) + \frac{1}{n}e^{-it}|^2|B_n|^2, \tag{8.6.78}$$

where

$$B_n = \int_0^{\pi/n} \sin(n\xi)y'(\xi)d\xi.$$

Then, in the same line as Lemma 4.1 in [173] the integral for B_n can be estimated

$$B_n \geq c_0\varepsilon.$$

Returning to (8.6.78) one gets

$$||v_n(t) - u_n(t)||^2_{H^s(S)} \geq n^{2s}\varepsilon^2|(e^{-it}-1) + \frac{1}{n}e^{-it}|^2.$$

Thus,the desired estimate is obtained as in [173] using (8.6.70) and $\delta^s = \varepsilon^2$.

■

8.7 Concluding Remarks

In this chapter we study the Cauchy problem for the periodic Holm - Staley b - family of equations. The results by Himonas and Misiolek [173] (proved for the CH equation $b = 2$ only) and the one for the DP equation $b = 3$ [147], are extended for the general case of b-family $b \neq 0$ (Theorem 8.1.1). We show that the solution map is not uniformly continuous in $\mathrm{H}^s, s \geq 3$. The proof is based on the construction of suitable smooth periodic solutions of small amplitude. To our knowledge, this idea comes from Kato [184].

Our result for the whole b-family is weaker than the results for particular values of b in the above mentioned papers [173] and [147] where $s \geq 2$. This is because we assume that the small parameters ε and $|1-x_c|$ are related as $\varepsilon << |1-x_c|$. We need this assumption in order to estimate the period, which is the main difficulty here. Then the relation $|1-x_c|^s = \varepsilon^2$ is valid for $s > 2$.

From the other hand, an interpolation argument is used to obtain the estimates for non-integer Sobolev indexes. That is the reason why the range $2 < s < 3$ is not covered. Perhaps, one should consider the case $s = 2$ separately, but this makes the estimates of the period for arbitrary b more difficult.

Chapter 9

Nonlocal symmetries of the μCamassa-Holm equation

The μ-Camassa-Holm (μCH) equation is a nonlinear integrable partial differential equation closely related to the Camassa-Holm and the Hunter-Saxton equations. This equation admits quadratic pseudo-potentials which allow us to compute some first order nonlocal symmetries. The found symmetries preserve the mean of solutions. Finally, we discuss also the associated μCH equation. The results of this chapter are published in [150].

9.1 Preliminaries

In this chapter we study the μCH equation, which was derived recently in [185, 193] as

$$\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, \tag{9.1.1}$$

where $u(x,t)$ is a spatially periodic real-valued function of time variable t and space variable $x \in S^1 = [0,1)$, $\mu(u) = \int_0^1 u dx$ denotes its mean. The μCH equation describes the propagation of weakly nonlinear orientation waves in a massive liquid crystal with external magnetic field and self-interaction. In this form the μCH equation appears as the geodesic equation on the diffeomorphism group of the circle corresponding to a natural right invariant Sobolev metric.

By introducing $m = \mathcal{A}u = \mu(u) - u_{xx}$, where $\mathcal{A} := \mu - \partial^2$ is the inertia operator (∂ stands for $\frac{\partial}{\partial x}$), the equation (9.1.1) becomes

$$m_t = -um_x - 2mu_x, \quad m = \mu(u) - u_{xx}. \tag{9.1.2}$$

The μCH equation is closely related to the Camassa-Holm (CH) equation [118, 164] with $\mathcal{A} = 1 - \partial^2$

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{9.1.3}$$

and to the Hunter-Saxton (HS) equation [178] with $\mathcal{A} = -\partial^2$

$$-u_{txx} = 2u_x u_{xx} + uu_{xxx}. \tag{9.1.4}$$

Similar to its relatives (9.1.3), (9.1.4) the μCH equation is a model for wave breaking, that is, it admits smooth solutions which break in finite time in such a way that the wave remains bounded while its slope becomes unbounded [185]. The μCH equation also admits peaked solutions (peakons): for any $c \in \mathbb{R}$, the peakon $u(x,t) = c\varphi(x - ct)$, where

$$\varphi(x) = \frac{1}{26}(12x^2 + 23), \quad \text{for } x \in [-1/2, 1/2] \tag{9.1.5}$$

and φ is extended periodically to the real line, is a solution to (9.1.1). It is proven in [119] that the periodic peakons of the μCH equation are orbitally stable in $H^1(S^1)$.

The μCH equation is also well-posed (see [185]). This equation enjoys other geometric descriptions [165], for example, it is geometrically integrable. Moreover, its Kuperschmidt deformation is also geometrically integrable [148] (see the next chapter).

The μCH equation is formally integrable (see section 2) and bi-Hamiltonian. Let us define the Hamiltonians

$$H_1 = \frac{1}{2}\int umdx, \quad H_2 = \int \left(\mu(u)u^2 + \frac{1}{2}uu_x^2\right) dx. \tag{9.1.6}$$

Then the equation (9.1.2) can be presented as

$$m_t = -\mathcal{B}^1 \frac{\delta H_2}{\delta m} = -\mathcal{B}^2 \frac{\delta H_1}{\delta m}, \tag{9.1.7}$$

where $\mathcal{B}^1 = \partial \mathcal{A} = -\partial^3, \mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators.

In fact, there exists an infinite sequence of conservation laws $H_n[m], n = 0, \pm 1, \pm 2, \ldots$, such that

$$\mathcal{B}^1 \frac{\delta H_n}{\delta m} = \mathcal{B}^2 \frac{\delta H_{n-1}}{\delta m},$$

the first few of them in the hierarchy are H_2, H_1 given above and

$$H_0 = \int mdx, \quad H_{-1} = \int \sqrt{m}dx, \quad H_{-2} = -\frac{1}{16}\int \frac{m_x^2}{m^{5/2}}dx. \tag{9.1.8}$$

Note that

$$H_0 = \int mdx = \int (\mu(u) - u_{xx})dx = \mu(u). \tag{9.1.9}$$

Then $\mu(u_t) = 0$ on solutions to the μCH equation – this fact can be seen also if we integrate both sides of the equation (9.1.1) over the circle and use the periodicity. This implies that the mean of any solution u is a constant in time and hence is completely determined by the initial condition [185]. This fact is crucial for the calculations in sections 3 and 4. Then the equation (9.1.1) can be written in the form

$$-u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx} \tag{9.1.10}$$

just as it is introduced in [185] under the name μHS equation.

The μCH equation can be written as

$$m_t = -\{m, H_2\}_1 = -\{m, H_1\}_2, \tag{9.1.11}$$

where the two compatible Poisson brackets are

$$\{f,g\}_1 = \int_0^1 \frac{\delta f}{\delta m}\mathcal{B}^1 \frac{\delta g}{\delta m}dx, \quad \{f,g\}_2 = \int_0^1 \frac{\delta f}{\delta m}\mathcal{B}^2 \frac{\delta g}{\delta m}dx. \tag{9.1.12}$$

Note that $H_0 = \int m dx = \int \mu(u) dx = \mu(u)$ is a Casimir for the first bracket and $H_{-1} = \int \sqrt{m} dx$ is a Casimir for the second bracket.

There is a lot of activity in extending the CH, HS and μCH equations to multi-component ones recently. In [213] the authors consider the geometric integrability of two-component CH and HS systems. They also obtain a class of nonlocal symmetries for these systems. In [233] the author proposes a two-component generalization of the μCH equation. Then he shows that this generalization is a bi-hamiltonian Euler equation and can be viewed as a bi-variational equation. In [187] the author studies the periodic $\mu - b$ equation which contains the μCH equation for $b = 2$ and μ Degasperis-Procesi equation for $b = 3$, respectively. Then the author shows that the $\mu - b$ equation can be realized as a metric Euler equation on $\mathrm{Diff}^\infty(S^1)$ if and only if $b = 2$ i.e. for the μCH equation.

Let us mention also the paper [219] where the authors introduce the multi-component Hunter-Saxton and μ-Camassa-Holm systems. They show that these multi-component systems are geometrically integrable. For the three-component CH and HS systems they find nonlocal symmetries depending on the pseudo-potentials.

The organization is as follows. Section 2 contains some facts around the notion of a scalar partial differential equation describing pseudo-spherical surface. Then the pseudo-spherical character of the μCH equation is recalled. A quadratic pseudo-differential is presented. In Section 3 we first review the theory of nonlocal symmetries of partial differential equations. We follow mainly [172]. The more detailed description can be found in Krasil'shchik and Vinogradov [188, 189] (see also [207]). Another point of view can be seen in [160] where higher degree potential symmetries are introduced which lead to nonlocal conservation laws and nonlocal transformations for the equations. Then we give nonlocal symmetries for the μCH equation. We consider only symmetries which preserve the mean of solutions, because they are found in a simple way. In the general case, one has to solve an integro-differential equation for the characteristic of the symmetry. The general approach for finding symmetries of nonlocal equations is given by Zawistowski [228]. In his approach there is no need in introducing nonlocal variables, so it is different form the approach taken in this paper. The Zawistowski's approach naturally leads to the solving of a system of integro-differential equations for the coefficient determining the generator of the symmetry. We conclude the section by constructing a Darboux-like transformation for the considered equation.

In Section 4 we first discuss the existence of other symmetries, different from those found in Section 3. Then the associated μCH (AμCH) equation is introduced by analogy. The Lie algebra of nonlocal symmetries for AμCH is presented and an one-parameter family of solutions is given.

9.2 The μCH equation and pseudo-spherical surfaces

In this section we recall some definitions and facts about the equations of pseudo-spherical type. They are introduced by Chern and Tenenblat [120]. One can consult, for example [205, 206] for more details.

Definition 1. A scalar differential equation $\Xi(x, t, u, u_x, \ldots, u_{x^n t^m}) = 0$ in two independent variables x, t, where $u_{x^n t^m} = \partial^{n+m} u / (\partial x^n \partial t^n)$, is of pseudo-spherical type (or, it describes pseudo-spherical surfaces) if there exist one-forms $\omega^\alpha \neq 0$

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \ldots, u_{x^r t^p})\, dx + f_{\alpha 2}(x, t, u, \ldots, u_{x^s t^q})\, dt,\ \alpha = 1, 2, 3, \tag{9.2.13}$$

whose coefficients $f_{\alpha\beta}$ are smooth functions which depend on x, t and finite number of derivatives of u, such that the 1-forms $\bar{\omega}^\alpha = \omega^\alpha(u(x,t))$ satisfy the structure equations

$$d\bar{\omega}^1 = \bar{\omega}^3 \wedge \bar{\omega}^2,\ d\bar{\omega}^2 = \bar{\omega}^1 \wedge \bar{\omega}^3,\ d\bar{\omega}^3 = \bar{\omega}^1 \wedge \bar{\omega}^2, \tag{9.2.14}$$

whenever $u = u(x,t)$ is a solution of $\Xi = 0$.

Equations (9.2.14) can be interpreted as follows. The graphs of local solutions of equations of pseudo-spherical type can be equipped with structure of pseudo-spherical surface (see [120, 205, 206]) : if $\bar{\omega}^1 \wedge \bar{\omega}^2 \neq 0$ the tensor $\bar{\omega}^1 \otimes \bar{\omega}^1 + \bar{\omega}^2 \otimes \bar{\omega}^2$ defines a Riemannian metric of constant Gaussian curvature -1 on the graph of solution $u(x,t)$ and $\bar{\omega}^3$ is the corresponding metric connection one-form.

An equation of pseudo-spherical type is the integrability condition for a sl$(2,\mathbb{R})$-valued problem

$$d\psi = \Omega\psi,$$

where Ω is the matrix-valued one-form

$$\Omega = Xdx + Tdt = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}. \tag{9.2.15}$$

Definition 2. An equation $\Xi = 0$ is geometrically integrable if it describes a non-trivial one-parameter family of pseudo-spherical surfaces.

Here, by a non-trivial one-parameter family of pseudo-spherical surfaces we mean that it is not a constant and further, the parameter cannot be removed via transformations which preserve the Riemannian structure of the pseudo-spherical surface (see [168] for a discussion).

Hence, if $\Xi = 0$ is geometrically integrable, it is the integrability condition of one-parameter family of linear problems $\psi_x = X\psi$, $\psi_t = T\psi$. In fact, this is equivalent to the zero curvature equation

$$X_t - T_x + [X,T] = 0, \tag{9.2.16}$$

which is an essential ingredient of integrable equations.

Another important property of equations of pseudo-spherical type is that they admit quadratic pseudo-potentials. Pseudo-potentials are a generalization of conservation laws.

Proposition 9.2.1. *[205] Let $\Xi = 0$ be a differential equation describing pseudo-spherical surfaces with associated one-forms ω^α. The following two Pfaffian systems are completely integrable whenever $u(x,t)$ is a solution of $\Xi = 0$:*

$$-2d\Gamma = \omega^3 + \omega^2 - 2\Gamma\omega^1 + \Gamma^2(\omega^3 - \omega^2) \tag{9.2.17}$$

and

$$2d\gamma = \omega^3 - \omega^2 - 2\gamma\omega^1 + \gamma^2(\omega^3 + \omega^2). \tag{9.2.18}$$

Moreover, the one-forms

$$\Theta = \omega^1 - \Gamma(\omega^3 - \omega^2) \quad and \quad \hat{\Theta} = -\omega^1 + \gamma(\omega^3 + \omega^2) \tag{9.2.19}$$

are closed whenever $u(x,t)$ is a solution of $\Xi = 0$ and Γ (resp. γ) is a solution of (9.2.17) (resp. (9.2.18)).

□

Geometrically, the Pfaffian systems (9.2.17) and (9.2.18) determine geodesic coordinates on the pseudo-spherical surfaces associated with the equation $\Xi = 0$ [120, 205].

Now consider the μCH equation (9.1.2).

Proposition 9.2.2. *The μCH equation (9.1.2) describes pseudo-spherical surfaces, and hence, is geometrically integrable.*

For validation of the Proposition 9.2.2 we give the associated with (9.1.2) 1-forms (see for example [165, 148]). Note that $\mu(u_x) = \mu(u_t) = 0$ is used since the structure equations are valid on the solutions to the μCH equation

$$\begin{aligned}
\omega^1 &= \frac{1}{2}\left(\eta m - \frac{\eta^2}{2} + 2\right) dx + \frac{1}{2}\left[\frac{\eta^2}{2}u - \eta\left(u_x + um + \frac{1}{2}\right) + \mu(u) - 2u + \frac{2}{\eta}\right] dt, \\
\omega^2 &= \eta dx + (1 - \eta u + u_x)\, dt, \qquad (9.2.20) \\
\omega^3 &= \frac{1}{2}\left(\eta m - \frac{\eta^2}{2} - 2\right) dx + \frac{1}{2}\left[\frac{\eta^2}{2}u - \eta\left(u_x + um + \frac{1}{2}\right) + \mu(u) + 2u - \frac{2}{\eta}\right] dt.
\end{aligned}$$

□

For the matrices X and T in (9.2.16) we get

$$X = \frac{1}{2}\begin{pmatrix} \eta & 2 \\ \eta m - \frac{\eta^2}{2} & -\eta \end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix}, \qquad (9.2.21)$$

where

$$T_{11} = 1 - \eta u + u_x, \quad T_{12} = 2\left(-u + \frac{1}{\eta}\right), \quad T_{21} = \frac{\eta^2}{2}u - \eta\left(u_x + um + \frac{1}{2}\right) + \mu(u). \qquad (9.2.22)$$

Hence, we have a zero curvature representation $X_t - T_x + [X, T] = 0$ for the system (9.1.2). From (9.2.21) it is straightforward to obtain the corresponding scalar linear problem

$$\begin{aligned}
\psi_{xx} &= \left(\frac{\eta}{2}m\right)\psi, \qquad (9.2.23) \\
\psi_t &= \left(-u - \eta w + \frac{1}{\eta}\right)\psi_x + \frac{u_x + \eta w_x}{2}\,\psi,
\end{aligned}$$

which coincides with those in [185] upon setting $\lambda = \eta/2$.

In order to find pseudo-potentials for the the μCH equation we denote

$$\omega^1_{new} = \omega^2, \quad \omega^2_{new} = -\omega^1, \quad \omega^3_{new} = \omega^3.$$

With these forms the Pfaffian system (9.2.18) becomes

$$2\gamma_x = -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \qquad (9.2.24)$$

$$2\gamma_t = -\frac{2\gamma^2}{\eta} + 2\gamma^2 u - 2\gamma(1 - \eta u + u_x) + \left[\frac{\eta^2}{2}u - \eta\left(u_x + m + \frac{1}{2}\right) + \mu(u)\right]. \qquad (9.2.25)$$

After some manipulations the above system obtains the form

$$\begin{aligned} 2\gamma_x &= -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \\ 2\gamma_t &= -\frac{2}{\eta}\gamma^2 - [(2\gamma+\eta)u]_x + \mu(u) - 2\gamma - \frac{\eta}{2}. \end{aligned}$$

Applying the transform $\gamma \mapsto \gamma - \eta/2$ we get

$$\gamma_x = -\gamma^2 + \frac{\eta}{2}m, \tag{9.2.26}$$

$$\gamma_t = -\frac{\gamma^2}{\eta} - (\gamma u)_x + \frac{\mu(u)}{2}. \tag{9.2.27}$$

Multiplying the first equation (9.2.26) by $-1/\eta$ and then adding the result to the second equation (9.2.27) we get the following result denoting $\lambda = \eta/2$.

Proposition 9.2.3. *[148] The μCH equation (9.1.2) admits a quadratic pseudo-potential γ, defined by the equations*

$$m = \frac{\gamma^2}{\lambda} + \frac{\gamma_x}{\lambda}, \tag{9.2.28}$$

$$\gamma_t = -\frac{2\gamma^2}{\lambda} - (\gamma u)_x + \frac{\mu(u)}{2}, \tag{9.2.29}$$

where $\lambda \neq 0, m = \mu(u) - u_{xx}$. Moreover, the equation (9.1.2) possesses the parameter dependent conservation law

$$\gamma_t = \frac{1}{2\lambda}\left(\gamma + \lambda u_x - 2\lambda u\gamma\right)_x. \tag{9.2.30}$$

□

As an application we use pseudo-potential γ to obtain some conserved densities for the μCH equation. One possible expansion of γ is

$$\gamma = \lambda^{1/2}\gamma_1 + \gamma_0 + \sum_{n=1}^{\infty} \lambda^{-n/2}\gamma_{-n}. \tag{9.2.31}$$

Substituting this into (9.2.26) gives

$$\gamma_1 = \sqrt{m}, \quad \gamma_0 = -\frac{m_x}{4m}, \quad \gamma_{-1} = \frac{1}{32}\frac{m_x^2}{m^{5/2}} + \frac{1}{8}\left(\frac{m_x}{m^{3/2}}\right)_x$$

and the other γ_{-n} are obtained recurrently by

$$\gamma_{-(n+1)} = -\frac{1}{2\gamma_1}\left[(\gamma_{-n})_x + \sum_{j=0}^{n} \gamma_{-j}\gamma_{j-n}\right], \quad n \geq 2.$$

In this way, we can obtain local functionals H_{-1}, H_{-2} (9.1.8) and so forth, see also [185].

9.3 Nonlocal symmetries for the μCH equation

Nonlocal symmetries have been studied rigorously by Krasil'schik and Vinogradov [188, 189]. Here we give a brief description of the accompanying notions and facts. Note that there is a substantial geometry which we do not even present here. We follow mainly [171, 172]. Before starting we recall some usual conventions. The independent variables are denoted by $x^i, i = 1, \dots, n$ and dependent variables by $u^\alpha, \alpha = 1, \dots, m$. Partial derivatives with respect to x^i are indicated with sub-indices. The total derivative with respect x^i is denoted by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{n} \sum_{\#J \geq 0} u^\alpha_{Ji} \frac{\partial}{\partial u^\alpha_J}, \tag{9.3.32}$$

where the unordered k-tuple $J = (j_1, \dots, j_k), 0 \leq j_1, j_2, \dots, j_k \leq n$ indicates a multi-index of order $\#J = k, u^\alpha_{Ji} = \frac{\partial u^\alpha_J}{\partial x^i}$ and $D_J = D_{j_1} D_{j_2} \cdots D_{j_k}$.

Definition 3. Let N be a nonzero integer or $N = \infty$. An N-dimensional covering π of a (system of) partial differential equation(s) $\Xi_a = 0, a = 1, \dots, k$, is a triplet

$$\Big(\{\gamma^b, b = 1, \dots, N\}; \{X_{ib}, b = 1, \dots, N, i = 1, \dots, n\}; \{\tilde{D}_i, i = 1, \dots, n\}\Big) \tag{9.3.33}$$

of variables γ^b – "nonlocal variables", smooth functions X_{ib} depending on x^i, u^α, γ^b and finite number of partial derivatives of u^α, and linear operators

$$\tilde{D}_i = D_i + \sum_{b=1}^{N} X_{ib} \frac{\partial}{\partial \gamma^b} \tag{9.3.34}$$

such that the the equations

$$\tilde{D}_i(X_{jb}) = \tilde{D}_j(X_{ib}), \quad i, j = 1, \dots, n, b = 1, \dots, N \tag{9.3.35}$$

hold whenever $u^\alpha(x^i)$ is a solution to $\Xi_a = 0$.

The operators $\tilde{D}_i$ satisfy $\tilde{D}_i(\gamma^b) = X_{ib}$ and these equations are compatible due to (9.3.35). Since we expect that on solutions to the system $\Xi_a = 0$ the total derivatives $\tilde{D}_i$ become usual partial derivatives, the equations

$$\frac{\partial \gamma^b}{\partial x^i} = X_{ib} \tag{9.3.36}$$

have to be satisfied on the solutions $u^\alpha(x^i)$ of $\Xi_a = 0$. These compatible equations give the relations between u^α and new dependent variables γ^b. Conversely, a set of equations of the form (9.3.36) which are compatible on solutions to the system $\Xi_a = 0$, determines a covering $\pi = (\gamma^b, X_{ib}, \tilde{D}_i)$ where the differential operators are defined as in (9.3.34).

We define the nonlocal symmetries as follows

Definition 4. Let $\Xi_a = 0, a = 1, \dots, k$ be a system of partial differential equations, $\pi = (\gamma^b, X_{ib}, \tilde{D}_i)$ be a covering of $\Xi_a = 0$. A nonlocal π-symmetry of $\Xi_a = 0$ is a generalized symmetry

$$X = \sum_i \xi^i \frac{\partial}{\partial x^i} + \sum_\alpha \phi^\alpha \frac{\partial}{\partial u^\alpha} + \sum_b \varphi^b \frac{\partial}{\partial \gamma^b}$$

of the augmented system

$$\Xi_a = 0, \quad \frac{\partial \gamma^b}{\partial x^i} = X_{ib}. \tag{9.3.37}$$

Hence, in order to find nonlocal symmetries, we can proceed as in the local case considered, for example, in Olver [202] We need to check the conditions [189, 202]

$$prX(\Xi_a) = 0, \quad \text{and} \quad prX\left(\frac{\partial \gamma^b}{\partial x^i} - X_{ib}\right) = 0 \tag{9.3.38}$$

in which

$$prX = X + \sum_{\alpha,J} \phi_J^\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{b,J} \varphi_J^b \frac{\partial}{\partial \gamma_J^b}$$

and

$$\phi_J^\alpha = D_J\left(\phi^\alpha - \sum_i \xi^i u_i^\alpha\right) + \sum_i \xi^i u_{Ji}^\alpha, \quad \varphi_J^b = D_J\left(\varphi^b - \sum_i \xi^i \gamma_i^b\right) + \sum_i \xi^i \gamma_{Ji}^b.$$

As elaborated in [202] ch. 5, it is enough to consider 'evolutionary' vector fields X of the form

$$X = \sum_{\alpha=1}^{m} G^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{b=1}^{N} H^b \frac{\partial}{\partial \gamma^b}. \tag{9.3.39}$$

Using this form of the vector field X the symmetry conditions (9.3.38) can be transformed further (see [171, 172]). Note that the nonlocal symmetries send solutions to the system $\Xi_a = 0$ into the solutions of the same system.

We now examine the nonlocal symmetries of the μCH equation (9.1.10) considered as a system of equations for the variables m and u, namely (9.1.2). We search the nonlocal symmetries that preserve the mean of solutions, that is, the integral $\mu(u) = \int u dx$ remains the same constant after the action of any symmetry on a solution.

We have already found a pseudo-potential γ in (9.2.28) and (9.2.29), given by

$$\gamma_x = \lambda m - \gamma^2, \quad \gamma_t = \left(\frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma\right)_x. \tag{9.3.40}$$

Let δ be the potential defined via compatible system of equations

$$\delta_x = \gamma, \quad \delta_t = \frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma. \tag{9.3.41}$$

Proposition 9.3.1. *The evolutionary vector field*

$$V = G^1 \frac{\partial}{\partial u} + G^2 \frac{\partial}{\partial m} = \gamma e^{2\delta} \frac{\partial}{\partial u} - \lambda(m_x + 4m\gamma)e^{2\delta} \frac{\partial}{\partial m}. \tag{9.3.42}$$

is a nonlocal symmetry for the μCH equation (9.1.2).

For the proof of this Proposition we need to explore only the first part in the symmetry conditions (9.3.38). Then a long, but straightforward computations give the result.

Following [205, 171, 172], we note that Proposition 9.3.1 simply says that the infinitesimal variations of u and m along the flow of the vector field V are given by

$$u_\tau = \gamma e^{2\delta}, \quad m_\tau = -\lambda(m_x + 4m\gamma)e^{2\delta}, \tag{9.3.43}$$

where τ is the parameter along the flow and for each solution $u(x,t), m(x,t)$ of the μCH equation (9.1.2), the deformed $u(x,t) + \tau u_\tau(x,t)$ and $m(x,t) + \tau m_\tau(x,t)$ satisfy (9.1.2) to first order in τ. Note that as u and m move along the flow of V, so do γ and δ. Hence, in order to find the flow of V we need to find their variations with respect to V.

Let us consider the potential β determined by the compatible system of equations

$$\beta_x = me^{2\delta}, \quad \beta_t = \left(\frac{\gamma^2}{2\lambda^2} - um\right)e^{2\delta}. \tag{9.3.44}$$

The system of equations (9.3.40), (9.3.41) and (9.3.44) allow us to define a three dimensional covering π of the μCH equation (9.1.2) with the nonlocal variables γ, δ and β.

Theorem 9.3.1. *The following vector fields are the first-order generalized symmetries for the augmented μCH system (9.1.2), (9.3.40), (9.3.41) and (9.3.44), which preserve the mean of the solutions to the μCH equation (9.1.2)*

$$\begin{aligned} W_1 = {} & -u_t\frac{\partial}{\partial u} + (m_x u + 2mu_x)\frac{\partial}{\partial m} - \left[\frac{\mu(u)}{2} + \gamma^2\left(u - \frac{1}{2\lambda}\right) - \gamma u_x - \lambda um\right]\frac{\partial}{\partial \gamma} \\ & - \left(\frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma\right)\frac{\partial}{\partial \delta} - \left(\frac{\gamma^2}{2\lambda^2} - um\right)e^{2\delta}\frac{\partial}{\partial \beta}, \end{aligned} \tag{9.3.45}$$

$$W_2 = u_x\frac{\partial}{\partial u} + m_x\frac{\partial}{\partial m} + (\lambda m - \gamma^2)\frac{\partial}{\partial \gamma} + \gamma\frac{\partial}{\partial \delta} + me^{2\delta}\frac{\partial}{\partial \beta}, \tag{9.3.46}$$

$$W_3 = \frac{\partial}{\partial \delta} + 2\beta\frac{\partial}{\partial \beta}, \tag{9.3.47}$$

$$W_4 = \frac{\partial}{\partial \beta}, \tag{9.3.48}$$

$$W_5 = \gamma e^{2\delta}\frac{\partial}{\partial u} - \lambda(m_x + 4m\gamma)e^{2\delta}\frac{\partial}{\partial m} - \lambda^2 me^{2\delta}\frac{\partial}{\partial \gamma} - \lambda^2\beta\frac{\partial}{\partial \delta} - (\lambda me^{4\delta} + \lambda^2\beta^2)\frac{\partial}{\partial \beta}. \tag{9.3.49}$$

Consequently, these vector fields are nonlocal symmetries of the μCH equation (9.1.2).

Again, the proof of Theorem 9.3.1 is a straightforward computation (see a comment on the availability of other symmetries at the beginning of the next section).

Corollary 2.

The five nonlocal symmetries (9.3.45)-(9.3.49) generate a Lie algebra $\mathcal{L}$ and their commutators are presented in the Table 1.

Table 9.1: The commutation table of μCH nonlocal symmetry algebra.

	W_1	W_2	W_3	W_4	W_5
W_1	0	0	0	0	0
W_2	0	0	0	0	0
W_3	0	0	0	$-2W_4$	$2W_5$
W_4	0	0	$2W_4$	0	$-\lambda^2 W_3$
W_5	0	0	$-2W_5$	$\lambda^2 W_3$	0

Remark 1. If we introduce the vector fields $h := -W_3, \quad e := \frac{1}{\lambda}W_4, \quad f := -\frac{1}{\lambda}W_5$, we find that the commutators $[h,e] = 2e, [h,f] = -2f, [e,f] = h$, i.e. e, f, h generate a copy of $sl(2,\mathbb{R})$.

Therefore, $\mathcal{L}$ is isomorphic to the direct sum of $sl(2,\mathbb{R})$ and the Abelian Lie algebra, generated by W_1 and W_2.

Remark 2. Note that W_1 and W_2 are merely the generators of the shifts with respect to the independent variables – they are $\frac{\partial}{\partial t}$ and $-\frac{\partial}{\partial x}$, respectively.

Next we study the flow of the vector field (9.3.49). We take it because the others are simpler – W_1 and W_2 correspond to translations with respect to t and x, respectively, and W_3 and W_4 do not involve the main variables u and m. Let again τ be a parameter along the flow, so the governing equations are

$$\frac{\partial u}{\partial \tau} = \gamma e^{2\delta}, \tag{9.3.50}$$

$$\frac{\partial m}{\partial \tau} = -\lambda \left(\frac{\partial m}{\partial x} + 4m\gamma \right) e^{2\delta}, \tag{9.3.51}$$

$$\frac{\partial \gamma}{\partial \tau} = -\lambda^2 m e^{2\delta}, \tag{9.3.52}$$

$$\frac{\partial \delta}{\partial \tau} = -\lambda^2 \beta, \tag{9.3.53}$$

$$\frac{\partial \beta}{\partial \tau} = -\lambda m e^{4\delta} - \lambda^2 \beta^2, \tag{9.3.54}$$

with initial conditions

$$u(x,t,0) = u_0,\ m(x,t,0) = m_0,\ \gamma(x,t,0) = \gamma_0,\ \delta(x,t,0) = \delta_0,\ \beta(x,t,0) = \beta_0 \tag{9.3.55}$$

in which $u_0(x,t)$, $m_0(x,t)$, $\gamma_0(x,t)$, $\delta_0(x,t)$ and $\beta_0(x,t)$ are particular solutions to (9.1.2), (9.3.40), (9.3.41) and (9.3.44).

We can obtain from the general theorems on existence, uniqueness, and regularity of solutions to symmetric hyperbolic quasi-linear systems such as (9.3.50)-(9.3.54) (see Taylor [215] ch. 16) that if we start with initial data

$$u_0(x,t),\ m_0(x,t),\ \gamma_0(x,t),\ \delta_0(x,t),\ \beta_0(x,t) \tag{9.3.56}$$

belonging to the Sobolev space $H^k(S^1)$, with $k > 3/2$, then the system (9.3.50)-(9.3.54) with the initial conditions (9.3.55) has solutions $u(x,t,\tau), m(x,t,\tau), \gamma(x,t,\tau), \delta(x,t,\tau)$ and $\beta(x,t,\tau)$ on an interval I, $\tau = 0 \in I$, belonging to $L^\infty(I, H^k(S^1)) \bigcap Lip(I, H^{k-1}(S^1))$. The local solutions $u(x,t,\tau), m(x,t,\tau)$, $\gamma(x,t,\tau), \delta(x,t,\tau)$ and $\beta(x,t,\tau)$ are smooth provided that the initial data are smooth. Moreover, if we start with smooth initial conditions (9.3.56), globally defined for $x \in S^1$, we can find (at least for small values of τ) families of solutions to the μCH equation also globally defined for $x \in S^1$.

Our aim is to obtain explicit formulas for the functions $u(x,t,\tau)$, $m(x,t,\tau)$, $\gamma(x,t,\tau)$, $\delta(x,t,\tau)$ and $\beta(x,t,\tau)$. Similarly to the case of the CH equation [205, 171, 172], we have

Proposition 9.3.2. *If the variables m and u are related by (9.1.2), the functions γ, δ, β are defined by the equations (9.3.40), (9.3.41), (9.3.44) and m, γ, δ, β satisfy equations (9.3.51)-(9.3.54), then u satisfies (9.3.50).*

Proof. We compute the derivative $\beta_{t,\tau}$ using (9.3.44), $\beta_{\tau,t}$ using (9.3.54) and simplify the obtained expressions using (9.3.40), (9.3.41), (9.3.44) and (9.3.51)-(9.3.54). The result is intuitively clear since the operator $\mathcal{A} = \mu - \partial^2$ is invertible [185]. □

Therefore, we can restrict ourselves to the projection of (9.3.49) on the space of m, γ, δ and β.

$$W_{pr} = -\lambda(m_x + 4m\gamma)e^{2\delta}\frac{\partial}{\partial m} - \lambda^2 m e^{2\delta}\frac{\partial}{\partial \gamma} - \lambda^2\beta\frac{\partial}{\partial \delta} - (\lambda m e^{4\delta} + \lambda^2\beta^2)\frac{\partial}{\partial \beta} \tag{9.3.57}$$

or we study the equations (9.3.51)-(9.3.54) with initial conditions

$$m(x,t,0) = m_0,\ \gamma(x,t,0) = \gamma_0,\ \delta(x,t,0) = \delta_0,\ \beta(x,t,0) = \beta_0. \tag{9.3.58}$$

As in [172] we change the independent variables τ, x with the variables $\xi = \tau, \eta = \eta(\tau, x)$, subjected to the conditions

$$\eta(\tau = 0, x) = x, \quad \eta_\tau = -\lambda\eta_x e^{2\delta}. \tag{9.3.59}$$

Then after simplifying the resulting equations with the expressions for $\gamma_x, \delta_x, \beta_x$ from (9.3.40) (9.3.41) and (9.3.44), we get

$$\frac{\partial m}{\partial \tau} = -4\lambda m(\tau,\eta)\gamma(\tau,\eta)e^{2\delta(\tau,\eta)}, \tag{9.3.60}$$

$$\frac{\partial \gamma}{\partial \tau} = -\lambda\gamma(\tau,\eta)^2 e^{2\delta(\tau,\eta)}, \tag{9.3.61}$$

$$\frac{\partial \delta}{\partial \tau} = \lambda\gamma(\tau,\eta)e^{2\delta(\tau,\eta)} - \lambda^2\beta(\tau,\eta), \tag{9.3.62}$$

$$\frac{\partial \beta}{\partial \tau} = -\lambda^2\beta(\tau,\eta)^2, \tag{9.3.63}$$

together with (9.3.59), which is equivalent to

$$\frac{\partial x}{\partial \tau} = \lambda e^{2\delta(\tau,\eta)}. \tag{9.3.64}$$

The following Proposition provides the explicit solution of the above system (note that η appears here as a parameter).

Proposition 9.3.3. *The initial value problem (9.3.60)-(9.3.64), with initial conditions $m_0 = m(0,\eta), \gamma_0 = \gamma(0,\eta), \delta_0 = \delta(0,\eta), \beta_0 = \beta(0,\eta)$ and $x_0 = x(0,\eta) = \eta$, has the solution*

$$m = \left[\frac{1 + \tau(\lambda^2\beta_0 - \lambda\gamma_0 e^{2\delta_0})}{1 + \lambda^2\beta_0\tau}\right]^4 m_0, \tag{9.3.65}$$

$$\gamma = \gamma_0 - \lambda\gamma_0^2 e^{2\delta_0}\frac{\tau}{1 + \lambda^2\beta_0\tau}, \tag{9.3.66}$$

$$\delta = \delta_0 - \ln(1 + \tau\lambda^2\beta_0 - \tau\lambda\gamma_0 e^{2\delta_0}), \tag{9.3.67}$$

$$\beta = \frac{\beta_0}{1 + \lambda^2\beta_0\tau}, \tag{9.3.68}$$

$$x = \eta + \frac{\tau\lambda e^{2\delta_0}}{1 + \tau\lambda^2\beta_0 - \tau\lambda\gamma_0 e^{2\delta_0}}. \tag{9.3.69}$$

□

Corollary 3. Let $u_0(x,t)$ be a solution of the μCH equation. Then the solution $u(x,t,\tau)$ to the initial problem

$$\frac{\partial u}{\partial \tau} = \gamma(x,\tau)e^{2\delta(x,\tau)}, \quad u(x,t,0) = u_0(x,t), \tag{9.3.70}$$

in which $\gamma(x,t,\tau)$ and $\delta(x,t,\tau)$ are determined by (9.3.66), (9.3.67) and (9.3.69), is a one-parameter family of solutions to the μCH equation (9.1.2) .

Remark 3. The formulas of the above type are used in [172] and [207] for computation of explicit particular solutions to the CH equation and the Kaup-Kupershmidt equation, respectively. Let us recall that the solutions of the μCH equation are periodic by its definition. The trivial solution and the constant solution do not give much. The next in order of complexity smooth periodic solutions are the traveling waves [185]. They are expressed via elliptic functions. This makes the calculations in applying the above formulas and Corollary 9.3 very difficult. Anyway, the derivation of the formulas (9.3.64)-(9.3.68) is not in vain. We will use them in what follows.

We turn to the construction of Darboux-like transformation for the μCH equation. We follow [172], but it seems that this method was presented firstly in [211] (see also [207]). We start with the equation (9.3.65)

$$m = \left[1 - \frac{\lambda\gamma_0 e^{2\delta_0}}{B}\tau\right]^4 m_0,$$

where $B = 1 + \lambda^2\beta_0\tau$. Since at level '0' $x = \eta$ using the first equation of (9.3.40) we get

$$m = \left[1 - \frac{\lambda(e^{2\delta_0})_\eta}{2B}\tau\right]^4 m_0.$$

Now we replace (9.3.44) into the last expression and obtain

$$m = \left[1 - \frac{\lambda\tau}{2B}\left(\frac{(\beta_0)_\eta}{m_0}\right)_\eta\right]^4 m_0,$$

or using B instead β_0

$$m(\tau,\eta) = \left[1 - \frac{1}{2\lambda B}\left(\frac{B_\eta}{m_0}\right)_\eta\right]^4 m_0. \tag{9.3.71}$$

This is the first component of the Darboux transform. Next we consider (9.3.69). Proceeding as above we obtain

$$x(\tau,\eta) = \eta + \frac{\frac{B_\eta}{\lambda B m_0}}{1 - \frac{1}{2\lambda B}\left(\frac{B_\eta}{m_0}.\right)_\eta} \tag{9.3.72}$$

Then omitting the subscript "0" and not writing explicitly the dependance on τ, we obtain the following

Proposition 9.3.4. *The μCH equation (9.1.2), understood as an equation for m, is invariant under the transformation $\eta \mapsto x$ and $m(\eta,t) \mapsto \bar{m}(x,t)$, in which*

$$x(\tau,\eta) = \eta + \frac{\frac{B_\eta}{\lambda B m}}{1 - \frac{1}{2\lambda B}\left(\frac{B_\eta}{m}\right)_\eta} \tag{9.3.73}$$

and $\bar{m}(x,t)$ is obtained by inverting (9.3.73) and replacing into

$$\bar{m}(x,t) = \frac{1}{\lambda^4[x(\eta,t)-\eta]^4}\frac{B_\eta}{B}\frac{1}{m^3}. \tag{9.3.74}$$

In these equations $B = 1 + \lambda^2\beta(\eta, t)\tau$ and the functions $m(\eta, t), \beta(\eta, t)$ are related by

$$m = \frac{1}{4\lambda}\frac{\partial^2}{\partial\eta^2}\ln\frac{\beta_\eta}{m} + \frac{1}{4\lambda}\left(\frac{\partial}{\partial\eta}\ln\frac{\beta_\eta}{m}\right)^2 \tag{9.3.75}$$

and $\beta(\eta, t)$ is a solution to the equation obtained from replacing (9.3.75) into

$$\beta_t = \frac{\beta_\eta}{m}\left[\frac{1}{8\lambda^2}\left(\frac{\partial}{\partial\eta}\ln\frac{\beta_\eta}{m}\right)^2 - um\right]. \tag{9.3.76}$$

Proof. We already have the equation (9.3.73). The equation (9.3.74) is obtained by replacing (9.3.73) into (9.3.71) To get the equation (9.3.75) we write $\delta = \frac{1}{2}\ln\frac{\beta_\eta}{m}$ then use $\delta_\eta = \gamma$ to obtain $\gamma = \frac{1}{2}\left(\ln\frac{\beta_\eta}{m}\right)_\eta$ and finally substitute this expression into the first equation of (9.3.40). In order to obtain the equation (9.3.76) we write δ and δ_η in terms of β and put them into the second equation of (9.3.44).

□

9.4 Discussion

In this chapter we use the approach from [172] to compute some first order nonlocal symmetries for the μCH equation. Only symmetries that preserve the mean of solutions are considered. It is needed to point out that we do not find all nonlocal symmetries, since they depend essentially on the possibility to construct nonlocal variables and the corresponding equations. There exist other symmetries for certain. Here is an example of a symmetry which do not preserve $\mu(u)$

$$x \mapsto x, \quad t \mapsto \frac{t}{\tau}, \quad u \mapsto \tau u,$$

where τ is a parameter. Then the μCH equation is invariant. However, this symmetry can not be extended to a symmetry for the augmented μCH system (9.1.2), (9.3.40), (9.3.41) and (9.3.44).

It may be interesting to see another object connected with the μCH equation. Recall that the so called associated Camassa-Holm (ACH) equation is introduced by Schiff [210]. Let us give by analogy the associated μCH equation. Define

$$p = \sqrt{m}, \quad dy = pdx - pudt, \quad dT = dt \tag{9.4.77}$$

and replace in equation (9.1.2). Note that this change of variables is justified since if $m(0)$ is positive, then $m(x) > 0$ as long as $u(x, t)$ exists (see [185] for the proof). One finds

$$p_T = -p^2 u_y, \quad -p\left(\frac{p_T}{p}\right)_y + \frac{p^2}{2} = \mu(u). \tag{9.4.78}$$

This is the analogue of the ACH equation – the associated μ Camassa-Holm (AμCH) equation. It is not clear yet whether this equation is of use. Nevertheless, our aim is to study nonlocal symmetries of the AμCH equation. First of all, we transform the equations for γ, δ and β (9.3.40), (9.3.41) and (9.3.44) using (9.4.77).

Proposition 9.4.1. *The AμCH equation (9.4.78) admits a pseudo-potential γ and potentials δ, β determined by the compatible equations, respectively*

$$\gamma_y = \frac{\lambda p}{2} - \frac{\gamma^2}{p}, \quad \gamma_T = \frac{\mu(u)}{2} - \frac{\gamma^2}{2\lambda} - p\gamma u_y, \tag{9.4.79}$$

$$\delta_y = \frac{\gamma}{p}, \quad \delta_T = \frac{pu_y}{2} + \frac{\gamma}{2\lambda}, \tag{9.4.80}$$

$$\beta_y = \frac{p}{2}e^{2\delta}, \quad \beta_T = \frac{\gamma^2}{2\lambda^2}e^{2\delta}. \tag{9.4.81}$$

□

As in the previous section, we consider symmetry vector fields which preserve $\mu(u)$.

$$W = G^1\frac{\partial}{\partial u} + G^2\frac{\partial}{\partial p} + H^1\frac{\partial}{\partial \gamma} + H^2\frac{\partial}{\partial \delta} + H^3\frac{\partial}{\partial \beta}, \tag{9.4.82}$$

where G^a, H^b are functions of the variables $y, T, u, p, \gamma, \delta, \beta$ and p_y, u_y, u_T.

Theorem 9.4.1. *The following vector fields are first order generalized symmetries for the augmented AμCH system (9.4.78)-(9.4.81)*

$$W_1 = u_T\frac{\partial}{\partial u} - p^2u_y\frac{\partial}{\partial p} + \left(\frac{\mu(u)}{2} - \frac{\gamma^2}{2\lambda} - p\gamma u_y\right)\frac{\partial}{\partial \gamma} + \left(\frac{pu_y}{2} + \frac{\gamma}{2\lambda}\right)\frac{\partial}{\partial \delta} + \frac{\gamma^2}{2\lambda^2}e^{2\delta}\frac{\partial}{\partial \beta}, \tag{9.4.83}$$

$$W_2 = u_y\frac{\partial}{\partial u} + p_y\frac{\partial}{\partial p} + \left(\frac{\lambda p}{2} - \frac{\gamma^2}{p}\right)\frac{\partial}{\partial \gamma} + \frac{\gamma}{p}\frac{\partial}{\partial \delta} + \frac{p}{2}e^{2\delta}\frac{\partial}{\partial \beta}, \tag{9.4.84}$$

$$W_3 = \frac{1}{2}\frac{\partial}{\partial \delta} + \beta\frac{\partial}{\partial \beta}, \tag{9.4.85}$$

$$W_4 = \frac{\partial}{\partial \beta}, \tag{9.4.86}$$

$$W_5 = -(\gamma + \lambda pu_y)e^{2\delta}\frac{\partial}{\partial u} + 2\lambda p\gamma e^{2\delta}\frac{\partial}{\partial p} + \lambda\gamma^2e^{2\delta}\frac{\partial}{\partial \gamma} + (\lambda^2\beta - \lambda\gamma e^{2\delta})\frac{\partial}{\partial \delta} + \lambda^2\beta^2\frac{\partial}{\partial \beta}. \tag{9.4.87}$$

Therefore, these vector fields are nonlocal symmetries for the AμCH equation (9.4.78).

□

Corollary 4. The five nonlocal symmetries (9.4.83)-(9.4.87) generate a Lie algebra $\mathcal{L}$ and their commutators are presented in the Table 2.

Table 9.2: The commutation table of AμCH nonlocal symmetry algebra.

	W_1	W_2	W_3	W_4	W_5
W_1	0	0	0	0	0
W_2	0	0	0	0	0
W_3	0	0	0	$-W_4$	W_5
W_4	0	0	W_4	0	$-2\lambda^2W_3$
W_5	0	0	$-W_5$	$2\lambda^2W_3$	0

Note that $\mathcal{L}$ is again isomorphic to a direct sum of $sl(2,\mathbb{R})$ and Abelian algebra generated by W_1 and W_2, which are equivalent to $-\frac{\partial}{\partial T}, -\frac{\partial}{\partial y}$, respectively.

One can find a Darboux transform for the AμCH equation just in a way described in [210] and [172]). Instead taking this direction, we conclude the section by obtaining solutions to the AμCH equation using the nonlocal symmetries. We consider solutions generated by the flow of the vector field (9.4.87). The flow of (9.4.87) is governed by the system of equations

$$\frac{\partial u}{\partial \tau} = -(\gamma + \lambda p u_y)e^{2\delta}, \tag{9.4.88}$$

$$\frac{\partial p}{\partial \tau} = 2\lambda p\gamma e^{2\delta}, \tag{9.4.89}$$

$$\frac{\partial \gamma}{\partial \tau} = \lambda\gamma^2 e^{2\delta}, \tag{9.4.90}$$

$$\frac{\partial \delta}{\partial \tau} = -\lambda\gamma e^{2\delta} + \lambda^2\beta, \tag{9.4.91}$$

$$\frac{\partial \beta}{\partial \tau} = \lambda^2\beta^2, \tag{9.4.92}$$

with initial conditions $u(y,T,0) = u_0$, $p(y,T,0) = p_0$, $\gamma(y,T,0) = \gamma_0$, $\delta(y,T,0) = \delta_0$, $\beta(y,T,0) = \beta_0$. Easy calculations produce

$$\gamma(\tau) = \gamma_0\left(1 + \frac{\tau\lambda\gamma_0 e^{2\delta_0}}{1-\tau\lambda^2\beta_0}\right), \quad \delta(\tau) = \delta_0 - \ln(1 - \tau\lambda^2\beta_0 + \tau\lambda\gamma_0 e^{2\delta_0}), \tag{9.4.93}$$

$$p(\tau) = p_0\left(1 + \frac{\tau\lambda\gamma_0 e^{2\delta_0}}{1-\tau\lambda^2\beta_0}\right)^2, \quad \beta(\tau) = \frac{\beta_0}{1-\tau\lambda^2\beta_0}. \tag{9.4.94}$$

It remains to obtain $u(\tau)$ from (9.4.88). Note that in contrast to the ACH equation, here it is not possible to get $u(\tau)$ directly from (9.4.78). To find $u(\tau)$ we need to solve the initial value problem

$$\frac{\partial u}{\partial \tau} = -\left(\gamma(y,T,\tau) + \lambda p(y,T,\tau)\frac{\partial u}{\partial y}\right)e^{2\delta(y,T,\tau)}, \quad u(y,T,0) = u_0(y,T), \tag{9.4.95}$$

where $u_0(y,T)$ is a particular solution to the AμCH equation. Solving this initial value problem, we will find 1-parameter family of solutions to the AμCH equation.

Example. As we mentioned above $\mu(u)$ is a constant on solutions. Let us suppose that $\mu(u) = \lambda > 0$. It is not difficult to be verified that the system (9.4.78)-(9.4.81) has a particular solution

$$\beta_0 = \frac{z}{\lambda^2(2-z)}, \quad \gamma_0 = \lambda\left(\frac{2+z}{2-z}\right), \quad \delta_0 = \frac{1}{2}(\sqrt{2\lambda}y + T) - \ln\frac{2+z}{2}, \tag{9.4.96}$$

$$p_0 = \sqrt{2\lambda}\left(\frac{2+z}{2-z}\right)^2, \quad u_0 = -\frac{4}{\lambda}\frac{z}{(2+z)^2}, \tag{9.4.97}$$

where $z := \lambda^2\exp(\sqrt{2\lambda}y + T)$. Then using (9.4.93) and (9.4.94) we find

$$\beta(y,T,\tau) = \frac{z}{\lambda^2[2 - z(\tau+1)]}, \tag{9.4.98}$$

$$\gamma(y,T,\tau) = \lambda\left(\frac{2+z(\tau+1)}{2-z(\tau+1)}\right), \tag{9.4.99}$$

$$\delta(y,T,\tau) = \frac{1}{2}\ln\frac{4z}{\lambda^2[2+z(\tau+1)]}, \tag{9.4.100}$$

$$p(y,T,\tau) = \sqrt{2\lambda}\left(\frac{2+z(\tau+1)}{2-z(\tau+1)}\right)^2. \tag{9.4.101}$$

Now with these $\gamma(y,T,\tau), \delta(y,T,\tau), p(y,T,\tau)$ we solve the initial value problem (9.4.95), where $u_0(y,T) = u_0 = -\frac{4}{\lambda}\frac{z}{(2+z)^2}$. Some computations produce

$$u(y,T,\tau) = -\frac{4}{\lambda}\frac{z(\tau+1)}{[2+z(\tau+1)]^2}. \tag{9.4.102}$$

Therefore, the functions $p(y,T,\tau), u(y,T,\tau)$ from (9.4.101) and (9.4.102) provide one parametric family of solution to the AμCH equation (9.4.78).

Chapter 10

Geometric Integrability of Some Generalizations of the CH-equation

We show that Kupershmidt deformations of the Camassa - Holm (CH) equation and the μCH equation describe pseudo-spherical surfaces and hence are geometrically integrable. We derive the corresponding quadratic pseudo-potentials which turn out to be very useful in obtaining conservation laws and symmetries. We also construct canonically conjugated variables for the μCH equation. At the end we make some speculations about "modified" μCH equation. The results of this chapter are published in [148] and [151].

10.1 Preliminaries

The modern theory of integrable nonlinear partial differential equations arose as a result of the inverse scattering method (ISM) discovered by Gardner, Green, Kruskal, Miura [167] for Korteweg de Vries (KdV) equation. Soon after, it was realized that this method can be applied to several important nonlinear equations like nonlinear Shrödinger equation, sine - Gordon etc.

Sasaki [209] gave a natural geometric interpretation for ISM in terms of pseudo-spherical surfaces. Motivated by Sasaki, Chern and Tenenblat [120] introduce the notion of a scalar equation of pseudo-spherical type and study systematically the evolution equations that describe pseudo-spherical surfaces. It appears that almost all important equations and systems in mathematical physics enjoy this property [120, 154, 168, 205, 206]. The advantage of this geometric treatment is that most of the ingredients connected with the integrable equations such as Lax pair, zero curvature representation, conservation laws, symmetries come naturally. We have already introduced it in Section 9.2.

Recently in [183] a new 6th - order wave equation, named KdV6, was derived. After some rescaling this equation can be presented as the following system

$$\begin{aligned} u_t &= 6uu_x + u_{xxx} - w_x, \\ w_{xxx} &+ 4uw_x + 2u_x w = 0. \end{aligned} \tag{10.1.1}$$

This system gives a perturbation to the KdV equation ($w = 0$) and since the constrain on w is differential, this is a nonholonomic deformation.

Kupershmidt [191] suggested a general construction applicable to any bi - Hamiltonian system providing a nonholonomic perturbation on it. This perturbation is conjectured to preserve integrability. In the case of KdV6, the system (10.1.1) can be converted into

$$\begin{aligned} u_t &= \mathcal{B}^1 \frac{\delta H_{n+1}}{\delta u} - \mathcal{B}^1(w) = \mathcal{B}^2 \frac{\delta H_n}{\delta u} - \mathcal{B}^1(w), \\ \mathcal{B}^2(w) &= 0, \end{aligned} \tag{10.1.2}$$

where $\mathcal{B}^1 = \partial, \mathcal{B}^2 = \partial^3 + 2(m\partial + \partial m)$ are the two standard Hamiltonian operators of the KdV hierarchy and

$$H_1 = \int u dx, \quad H_2 = \frac{1}{2}\int u^2 dx, \dots. \tag{10.1.3}$$

In the same article Kupershmidt verifies the integrability of KdV6, as well as the integrability of the such nonholonomic deformations for some representative cases: the classical long - wave equation, the Toda lattice (both continuous and discrete), and the Euler top.

In fact, Kersten et al. [186] prove that the Kupershmidt deformation of every bi - Hamiltonian equation is again bi-Hamiltonian system and every hierarchy of conservation laws of the original bi-Hamiltonian system gives rise to a hierarchy of conservation laws of the Kuperschmidt deformation.

Yao and Zeng [223] propose a generalized Kupreshmidt deformation and verify that this generalized deformation also preserve integrability in few representative cases: KdV equation, Boussinesq equation, Jaulent - Miodek equation and CH equation.

Kundu et all [190] consider slightly generalized form of deformation for the KdV equation and extend this approach to mKdV equation and to AKNS system.

Guha [169] uses Kirillov's theory of coadjoint representation of Virasoro algebra to obtain a large class of KdV6 - type equations, equivalent to the original one. Also, applying the Adler - Konstant - Symes scheme, he constructs a new nonholonomic deformation of the coupled KdV equation.

It is natural to think that maybe there exists a general link in a sense:

Conjecture. The Kuperschmidt deformation of geometrically integrable system is again geometrically integrable.

We haven't succeeded in establishing such a link up to now, that is why we restrict ourselves with some relevant examples. However, we believe that this conjecture is true at least for the systems with local Hamiltonian pair of operators.

The first aim of this chapter is to show that the CH and the μCH equations have geometrically integrable Kupershmidt deformations. We show that the KdV6 equation and two - component CH system [144] are also geometrically integrable.

10.2 Some Examples of Geometrically Integrable Kupershmidt deformations

In this section we consider the nonholonomic deformation of CH equation and μCH equation. We show that they are geometrically integrable and consider their quadratic pseudo-potentials. The non-local symmetries will be studied elsewhere.

The CH equation.

Recall the CH equation and its bi-Hamiltonian form with the corresponding Hamiltonian operators $\mathcal{B}^1$ and $\mathcal{B}^2$.

$$m_t + um_x + 2mu_x, \quad m = u_{xx} - u. \tag{10.2.4}$$

The bi - Hamiltonian form of (10.2.4) is [118, 164]

$$m_t = -\mathcal{B}^1 \frac{\delta H_2[m]}{\delta m} = -\mathcal{B}^2 \frac{\delta H_1[m]}{\delta m}, \tag{10.2.5}$$

where $\mathcal{B}^1 = \partial - \partial^3, \mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators.

Following Kupershmidt's construction we introduce the nonholonomic deformation of the CH equation.

$$\begin{aligned} m_t &= -um_x - 2mu_x - \mathcal{B}^1(w), \\ \mathcal{B}^2(w) &= 0. \end{aligned} \tag{10.2.6}$$

Proposition 10.2.1. *The system (10.2.6) describes pseudo - spherical surface and hence is geometrically integrable.*

Let us give the corresponding 1 - forms:

$$\begin{aligned}
\omega^1 &= (m+1)dx + \left[-um + \frac{\eta+1}{\eta}(u_x - u) + \frac{1}{\eta} - w_{xx} + (\eta+1)w_x - \eta w(m+1)\right] dt, \\
\omega^2 &= (\eta+1)dx + \left[\frac{\eta+1}{\eta} - (\eta+1)u + u_x - \eta(\eta+1)w + \eta w_x\right] dt, \\
\omega^3 &= (m+1+\eta)dx + \\
&\left[-um + \frac{\eta-(\eta+1)^2}{\eta}u + \frac{\eta+1}{\eta}u_x + \frac{\eta+1}{\eta} - w_{xx} + (\eta+1)w_x - \eta w(m+1+\eta)\right] dt.
\end{aligned} \tag{10.2.7}$$

For the proof of Proposition 10.2.1 we need only to verify the structure equations (9.2.14) and to check that the parameter η is intrinsic.

For the matrices X and T in we get

$$X = \frac{1}{2}\begin{pmatrix} \eta+1 & -\eta \\ 2(m+1)+\eta & -(\eta+1) \end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix} \tag{10.2.8}$$

where

$$\begin{aligned}
T_{11} &= \frac{\eta+1}{\eta} - (\eta+1)u + u_x - \eta(\eta+1)w + \eta w_x, \\
T_{12} &= \eta u + \eta^2 w - 1 \\
T_{21} &= -2um - \frac{(\eta+1)^2+1}{\eta}u + \frac{2(\eta+1)}{\eta}u_x \\
&+ \frac{\eta+2}{\eta} - 2w_{xx} + 2(\eta+1)w_x - \eta(\eta+2)w - 2\eta wm.
\end{aligned} \tag{10.2.9}$$

Hence, we have a zero curvature representation $X_t - T_x + [X, T] = 0$ for the system (10.2.6). From (10.2.8) it is straightforward to obtain the corresponding scalar linear problem

$$\begin{aligned}
\psi_{xx} &= \left(\frac{1}{4} - \frac{\eta}{2}m\right)\psi, \\
\psi_t &= \left(-u - \eta w + \frac{1}{\eta}\right)\psi_x + \frac{u_x + \eta w_x}{2}\psi.
\end{aligned} \tag{10.2.10}$$

In order to apply the Proposition 9.2.1 to nonholonomic deformation of the CH equation, we consider new 1 - forms ω^α_{new}

$$\omega^1_{new} = \omega^2, \quad \omega^2_{new} = -\omega^1, \quad \omega^3_{new} = \omega^3. \tag{10.2.11}$$

With these forms the Pfaffian system (9.2.18) becomes

$$2\gamma_x = \eta\gamma^2 - 2(\eta+1)\gamma + 2(m+1) + \eta, \tag{10.2.12}$$

$$\begin{aligned} 2\gamma_t &= \gamma^2 - 2w_{xx} - 2\left[\eta\gamma - (\eta+1)\right] w_x - 2w\eta\gamma_x \\ &- 2\left[\gamma - \frac{\eta+1}{\eta}\right] u_x - 2u(\gamma_x + \frac{1}{\eta}) + \frac{\eta+2}{\eta} - 2\gamma\frac{\eta+1}{\eta}. \end{aligned} \tag{10.2.13}$$

Applying the transform

$$\gamma \mapsto \gamma + \frac{\eta+1}{\eta},$$

after some algebraic manipulations and setting $\lambda = -1/\eta$, we obtain the following result

Proposition 10.2.2. *The nonholonomic deformation of the CH equation (10.2.6) admits a quadratic pseudo - potential γ, defined by the equations*

$$m = \frac{\gamma^2}{2\lambda} + \gamma_x - \frac{\lambda}{2}, \tag{10.2.14}$$

$$\begin{aligned} \gamma_t &= \frac{\gamma^2}{2}\left[1 + \frac{1}{\lambda}\left(u - \frac{w}{\lambda}\right)\right] - \gamma\left(u - \frac{w}{\lambda}\right)_x \\ &- \left(u - \frac{w}{\lambda}\right)\left(m + \frac{\lambda}{2}\right) + \lambda u - \frac{\lambda^2}{2}, \end{aligned} \tag{10.2.15}$$

where $\lambda \neq 0, m = u_{xx} - u$. Moreover, (10.2.6) possesses the parameter dependent conservation law

$$\gamma_t = \lambda\left[(u+w)_x - \gamma - \frac{1}{\lambda}\left(u - \frac{w}{\lambda}\right)\gamma\right]_x. \tag{10.2.16}$$

Conservation densities can be obtained by expanding (10.2.14) and (10.2.16) in powers of λ. Note that the left hand side of (10.2.16) and (10.2.14) do not depend on w as it should be. The corresponding expansions are performed in [205].

The μCH equation

Consider now the μCH equation, its bi - Hamiltonian form with the corresponding Hamiltonian operators $\mathcal{B}^1, \mathcal{B}^2$.

$$m_t + um_x + 2mu_x = 0, \quad m = \mu(u) - u_{xx}. \tag{10.2.17}$$

The bi - Hamiltonian form of (10.2.17) is

$$m_t = -\mathcal{B}^1\frac{\delta H_2[m]}{\delta m} = -\mathcal{B}^2\frac{\delta H_1[m]}{\delta m}, \tag{10.2.18}$$

where $\mathcal{B}^1 = \partial A = -\partial^3, \mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators.

Applying Kupershmidt's procedure to (10.2.17) we obtain the nonholonomic deformation of the μCH equation

$$\begin{aligned} m_t &= -um_x - 2mu_x - \mathcal{B}^1(w), \\ \mathcal{B}^2(w) &= 0. \end{aligned} \tag{10.2.19}$$

or

$$\begin{aligned} m_t &= -um_x - 2mu_x + w_{xxx}, \\ 2mw_x &+ wm_x = 0, \quad m = \mu(u) - u_{xx}. \end{aligned} \tag{10.2.20}$$

Proposition 10.2.3. *The nonholonomic deformation of the μCH equation (10.2.20) describes pseudo-spherical surfaces and, hence, is geometrically integrable.*

For validation of the Proposition 10.2.3 we give the associated with (10.2.20) 1 - forms

$$\begin{aligned} \omega^1 &= \frac{1}{2}\left(\eta m - \frac{\eta^2}{2} + 2\right) dx + \\ &\frac{1}{2}\left[\frac{\eta^2}{2}u - \eta(u_x + um + \frac{1}{2}) + \mu(u) - 2u + \frac{2}{\eta} + (\frac{\eta^3}{2} - 2\eta)w - \eta^2 w_x + \eta w_{xx} - \eta^2 mw\right] dt, \\ \omega^2 &= \eta dx + \left(1 - \eta u + u_x - \eta^2 w + \eta w_x\right) dt, \\ \omega^3 &= \frac{1}{2}\left(\eta m - \frac{\eta^2}{2} - 2\right) dx + \\ &\frac{1}{2}\left[\frac{\eta^2}{2}u - \eta(u_x + um + \frac{1}{2}) + \mu(u) + 2u - \frac{2}{\eta} + (\frac{\eta^3}{2} + 2\eta)w - \eta^2 w_x + \eta w_{xx} - \eta^2 mw\right] dt. \end{aligned} \tag{10.2.21}$$

For the matrices X and T in we get

$$X = \frac{1}{2}\begin{pmatrix} \eta & 2 \\ \eta m - \frac{\eta^2}{2} & -\eta \end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix} \tag{10.2.22}$$

where

$$\begin{aligned} T_{11} &= 1 - \eta u + u_x - \eta^2 w + \eta w_x, \\ T_{12} &= 2(-u + \frac{1}{\eta} - \eta w) \\ T_{21} &= \frac{\eta^2}{2}u - \eta(u_x + um + \frac{1}{2}) + \mu(u) - \eta^2 w_x + \eta w_{xx} - \eta^2 mw + \frac{\eta^3}{2}w. \end{aligned} \tag{10.2.23}$$

Hence, we have a zero curvature representation $X_t - T_x + [X, T] = 0$ for the system (10.2.20). From (10.2.8) it is straightforward to obtain the corresponding scalar linear problem

$$\begin{aligned} \psi_{xx} &= \left(\frac{\eta}{2}m\right)\psi, \\ \psi_t &= \left(-u - \eta w + \frac{1}{\eta}\right)\psi_x + \frac{u_x + \eta w_x}{2}\psi, \end{aligned} \tag{10.2.24}$$

which coincides with those in [185] upon setting $w = 0$ and $\lambda = \eta/2$.

In order to find pseudo-potentials for the nonholonomic deformation of the μCH equation we proceed as before denoting

$$\omega^1_{new} = \omega^2, \quad \omega^2_{new} = -\omega^1, \quad \omega^3_{new} = \omega^3.$$

With these forms the Pfaffian system (9.2.18) becomes

$$2\gamma_x = -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \tag{10.2.25}$$

$$\begin{aligned} 2\gamma_t &= -\frac{2\gamma^2}{\eta} + 2\gamma^2(u + \eta w) - 2\gamma(1 - \eta u + u_x - \eta^2 w + \eta w_x) \\ &+ \left[\frac{\eta^2}{2}u - \eta(u_x + m + \frac{1}{2}) + \mu(u) + \frac{\eta^3}{2}w - \eta^2 w_x + \eta w_{xx} - \eta^2 m w\right]. \end{aligned} \tag{10.2.26}$$

After some manipulations the above system obtains the form

$$\begin{aligned} 2\gamma_x &= -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \\ 2\gamma_t &= -\frac{2}{\eta}\gamma^2 + \eta w_{xx} - [(2\gamma + \eta)(u + \eta w)]_x + \mu(u) - 2\gamma - \frac{\eta}{2}. \end{aligned}$$

Applying the transform $\gamma \mapsto \gamma - \eta/2$ we get

$$\gamma_x = -\gamma^2 + \frac{\eta}{2}m, \tag{10.2.27}$$

$$\gamma_t = -\frac{\gamma^2}{\eta} + \frac{\eta}{2}w_{xx} - [\gamma(u + \eta w)]_x + \frac{\mu(u)}{2}. \tag{10.2.28}$$

Multiplying the first equation (10.2.27) by $-1/\eta$ and then adding the result to the second equation (10.2.27) we get the following result denoting $\lambda = \eta/2$

Proposition 10.2.4. *The nonholonomic deformation of the μCH equation (10.2.20) admits a quadratic pseudo - potential γ, defined by the equations*

$$m = \frac{\gamma^2}{\lambda} + \frac{\gamma_x}{\lambda}, \tag{10.2.29}$$

$$\gamma_t = -\frac{2\gamma^2}{\lambda}\lambda w_{xx} - [\gamma(u + 2\lambda w)]_x + \frac{\mu(u)}{2}, \tag{10.2.30}$$

where $\lambda \neq 0, m = \mu(u) - u_{xx}$. Moreover, (10.2.20) possesses the parameter dependent conservation law

$$\gamma_t = \frac{1}{2\lambda}\left[\gamma + \lambda(u + 2\lambda w)_x - 2\lambda(u + 2\lambda w)\gamma\right]_x. \tag{10.2.31}$$

As the conserved densities for the nonholonomic deformation are the same as for the original bi-Hamiltonian system, we make use of the pseudo-potentials to obtain them for the μCH equation. One possible expansion of γ is

$$\gamma = \lambda^{1/2}\gamma_1 + \gamma_0 + \sum_{j=1}^{\infty} \lambda^{-j/2}\gamma_{-j}. \tag{10.2.32}$$

Substituting this into (10.2.29) yields

$$\gamma_1 = \sqrt{m}, \quad \gamma_0 = -\frac{m_x}{4m}, \quad \gamma_{-1} = \frac{1}{32}\frac{m_x^2}{m^{5/2}} + \frac{1}{8}\left(\frac{m_x}{m^{3/2}}\right)_x, \ldots \text{etc.}$$

In this way, we can obtain local functionals, see [185].

We finish this section with the geometric integrability of one of the most popular two-component generalization of CH equation and of KdV6 equation.

Another generalization of the Camassa - Holm equation is the following integrable two-component CH system [144]

$$\begin{aligned} u_t &- u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} + \sigma\rho\rho_x, \\ \rho_t &+ (u\rho)_x = 0, \end{aligned} \tag{10.2.33}$$

where $\sigma = \pm 1$. Introducing a new variable $v = \rho^2/2$, the above system becomes

$$\begin{aligned} u_t &- u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} + \sigma v_x, \\ v_t &+ 2vu_x + uv_x = 0. \end{aligned} \tag{10.2.34}$$

The system (10.2.34) is geometrically integrable. The corresponding 1-forms, satisfying the structure equations (9.2.14) are the following

$$\begin{aligned} \omega^1 &= (u_{xx} - u - \sigma\eta v + 1)dx \\ &+ \left[u^2 - uu_{xx} + \frac{\eta+1}{\eta}(u_x - u) + \frac{1}{\eta} - \sigma v + \eta\sigma uv\right] dt, \\ \omega^2 &= (\eta+1)dx + \left(\frac{\eta+1}{\eta} - (\eta+1)u + u_x\right) dt, \\ \omega^3 &= (u_{xx} - u - \sigma\eta v + \eta + 1)dx \\ &+ \left[u^2 - uu_{xx} + \frac{\eta - (\eta+1)^2}{\eta}u + \frac{\eta+1}{\eta}(u_x + 1) - \sigma v + \eta\sigma uv\right] dt. \end{aligned}$$

We could easily include two-component Hunter-Saxon system [127] into this picture (see also [206]).

Finally, we note that nonholonomic perturbation of KdV equation, known as KdV6 equation, is also of pseudo-spherical type, that is, KdV6 equation is geometrically integrable. We just give the corresponding 1-forms

$$\begin{aligned} \omega^1 &= (1-u)dx \\ &+ \left[-u_{xx} + \eta u_x - 2u^2 + \frac{1}{\eta}w_x - \frac{1}{\eta^2}(w_{xx} + 2uw) + (2-\eta^2)u + \frac{2}{\eta^2}w + \eta^2\right] dt, \\ \omega^2 &= \eta dx + \left(\eta^3 + 2\eta u + \frac{2}{\eta}w - 2u_x - \frac{2}{\eta^2}w_x\right) dt, \\ \omega^3 &= -(1+u)dx \\ &+ \left[-u_{xx} + \eta u_x - 2u^2 + \frac{1}{\eta}w_x - \frac{1}{\eta^2}(w_{xx} + 2uw) - (2+\eta^2)u - \frac{2}{\eta^2}w - \eta^2\right] dt, \end{aligned}$$

which coincide with those for KdV equation [120] when $w \to 0$.

10.3 Canonically conjugate variables for the μCH equation

In this section we deal again with the μCH equation introduced in the previous chapter as

$$-u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}. \tag{10.3.35}$$

We construct canonically conjugated variables with respect to the both brackets (9.1.12). We follow Penskoi [204], where the conjugated variables are obtained for the periodic CH equation, although many essential facts can be found in Flaschka, McLaughlin [163], where the conjugated variables for the periodic KdV equation are constructed.

Recall also that the equation (10.3.35) can be expressed as a condition of compatibility between

$$\psi_{xx} = -\lambda m\psi \tag{10.3.36}$$

and

$$\psi_t = -(\frac{1}{2\lambda} + u)\psi_x + \frac{1}{2}u_x\psi, \tag{10.3.37}$$

that is, $(\psi_{xx})_t = (\psi_t)_{xx}$, where λ is a spectral parameter.

In what follows we assume that $m(0) > 0$. It is shown in [185] that then $m(x) > 0$ as long as $u(x,t)$ exists.

The disposition of the spectra is similar to the cases of the Korteweg de Vries equation and the CH equation. If $m \in C^2[0,1]$ with the above assumption the Liouville transformation $\psi \to m^{-1/4}\Phi(\bar{x}), \bar{x} = \int_0^x \sqrt{m}d\tau$ brings (10.3.36) to a Hill's equation

$$-\frac{d^2\Phi}{d\bar{x}^2} + \left(\frac{m_{xx}}{4m^2} - \frac{5m_x^2}{16m^3}\right)\Phi = \lambda\Phi.$$

However, we shall proceed as in [196, 158] or [136].

Consider the spectral problem (10.3.36). Recall that $u(x+1) = u(x)$ and $m(x+1) = m(x)$.

Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be a fundamental system of solutions of (10.3.36) subjected to the normalization

$$y_1(0,\lambda) = 1, \quad y_1'(0,\lambda) = 0,$$
$$y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1.$$

Every solution ψ of (10.3.36) can be expressed as a linear combination of $y_{1,2}$:

$$\psi(x,\lambda) = \psi(0,\lambda)y_1(x,\lambda) + \psi'(0,\lambda)y_2(x,\lambda). \tag{10.3.38}$$

Then we have the formula

$$\begin{pmatrix}\psi(x,\lambda)\\ \psi'(x,\lambda)\end{pmatrix} = \begin{pmatrix}y_1(x,\lambda) & y_2(x,\lambda)\\ y_1'(x,\lambda) & y_2'(x,\lambda)\end{pmatrix}\begin{pmatrix}\psi(0,\lambda)\\ \psi'(0,\lambda)\end{pmatrix} \tag{10.3.39}$$

and denote the matrix in the last formula by $U(x,\lambda)$. From the definition of $y_{1,2}$ we have that $\det U(x,\lambda) = Wr(y_1,y_2) = Wr(0) = 1$. Let us define also the discriminant

$$\Delta(\lambda) = \frac{1}{2}\mathrm{tr}\, U(1,\lambda) = \frac{1}{2}\left(y_1(1,\lambda) + y_2'(1,\lambda)\right). \tag{10.3.40}$$

We first consider (10.3.36) conditioned by the periodic boundary conditions

$$\psi(0) = \psi(1), \quad \psi'(0) = \psi'(1).$$

There exists an infinite sequence of eigenvalues

$$\lambda_0^+ < \lambda_1^+ \leq \lambda_2^+ < \lambda_3^+ \ldots, \quad \lambda_n^+ \to \infty \text{ as } n \to \infty.$$

Next we consider the antiperiodic eigenvalue problem, that is, the boundary conditions for (10.3.36) are of the form

$$\psi(1) = -\psi(0), \quad \psi'(1) = -\psi'(0).$$

The corresponding sequence of eigenvalues is

$$\lambda_1^- \leq \lambda_2^- < \lambda_3^- \leq \lambda_4^- \ldots, \quad \lambda_n^- \to \infty \text{ as } n \to \infty.$$

The quantities $\lambda_m^\pm$ are the roots of $\Delta(\lambda) = \pm 1$, λ_0^+ is always simple. It is known that

$$\lambda_0^+ < \lambda_1^- \leq \lambda_2^- < \lambda_1^+ \leq \lambda_2^+ < \lambda_3^- \leq \lambda_4^- < \lambda_3^+ \ldots.$$

The intervals

$$(\lambda_0^+, \lambda_1^-), (\lambda_2^-, \lambda_1^+), (\lambda_2^+, \lambda_3^-), \ldots$$

are called intervals of stability. Similarly we can name the other intervals - the intervals of instability or gaps. Some of intervals of instability may disappear - $(-\infty, \lambda_0)$ always is present. Trivial argument show that in our case $\lambda_0^+ = 0$ and for $\lambda \in (-\infty, 0)$ the solutions of (10.3.36) are unbounded.

Recall that a solution of (10.3.36) is said to be a Floquet solution if there exists a number ρ called a Floquet multiplier satisfying

$$\psi(x+1, \lambda) = \rho\psi(x, \lambda).$$

It is straightforward from (10.3.39) that a Floquet solution and the corresponding ρ are an eigenvector and an eigenvalue of $U(x, \lambda)$.

Hence, ρ is obtained from

$$\rho^2 - 2\Delta(\lambda)\rho + 1 = 0. \tag{10.3.41}$$

Now let us consider the auxiliary eigenvalues μ_j defined as solutions of the equation $y_2(1, \mu_j) = 0$. Since $m(x)$ is periodic, $y_2(x+1, \mu_j)$ is a solution of (10.3.36) for $\lambda = \mu_j$. Due to (10.3.38) we have

$$y_2(x+1, \mu_j) = y_2'(1, \mu_j) y_2(x, \mu_j),$$

that is, $y_2(x, \mu_j)$ is a Floquet solution with $\rho_j = y_2'(1, \mu_j)$. So, we have a root of (10.3.41) for $\lambda = \mu_j$ namely ρ_j. The other root is $\tilde{\rho}_j = \frac{1}{\rho_j}$. Denote by $y(x, \mu_j)$ the corresponding to $\tilde{\rho}_j$ Floquet solution

$$y(x+1, \mu_j) = \tilde{\rho}_j y(x, \mu_j). \tag{10.3.42}$$

Since, y and y_2 are linearly independent, we normalize y by $y(0, \mu_j) = 1$.

The points of the "auxiliary spectrum" μ_j must lie in the gaps (see Fig. 1). Indeed, at $Wr(y_1, y_2) = y_1 y_2' - y_1' y_2 = 1$. Then at $x = 1$ $Wr(y_1, y_2)(\mu_j) = y_1 y_2' = 1$ so,

$$|\Delta(\mu_j)| = \frac{1}{2}|(y_1(1\mu_j) + y_2'(1, \mu_j))| = \frac{1}{2}\Big|\left(y_1(1, \mu_j) + \frac{1}{y_1(1, \mu_j)}\right)\Big| \geq 1.$$

Remark. If m changes the sign, there are infinite sequences of positive and negative eigenvalues for both periodic and antiperiodic spectra. This result goes back to Lyapunov.

The following lemma is more or less known.

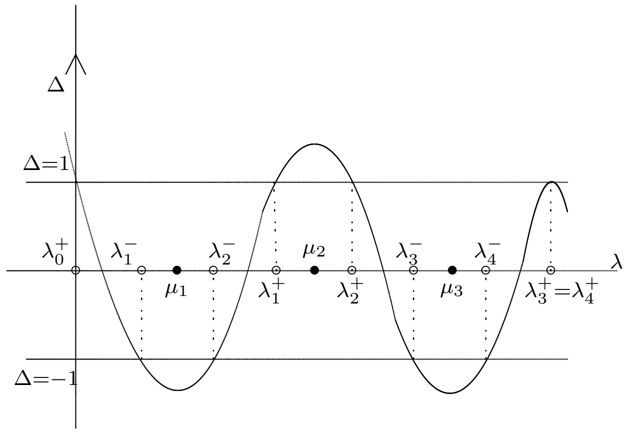


Figure 10.1: Disposition of Spectra

Lemma 10.3.1. *Let ψ, φ be solutions (not necessarily different) of the spectral problem (10.3.36) for the same λ. Then the following identity holds*

$$\lambda \mathcal{B}^2 \psi\varphi = \mathcal{B}^1 \psi\varphi. \tag{10.3.43}$$

The proof is straightforward.

Let us also give an additional formula which will be used in the next section. Suppose the functions p and q are such that p, p', q, q' are zero at $0, 1$. Then we have

$$\int_0^1 p\mathcal{B}^s q dx = -\int_0^1 q\mathcal{B}^s p dx, \quad s = 1, 2. \tag{10.3.44}$$

Since $\mu_j \neq 0$ we can define the following variables $f_j = -\frac{\ln|\rho_j|}{\mu_j^2}$ and $g_j = -\frac{\ln|\rho_j|}{\mu_j^3}$. Our main result is the following

Theorem 10.3.1. *The variables*

a) μ_i and $f_j = -\frac{\ln|\rho_j|}{\mu_j^2}$ are conjugate with respect to the bracket $\{,\}_2$;

b) μ_i and $g_j = -\frac{\ln|\rho_j|}{\mu_j^3}$ are conjugate with respect to the bracket $\{,\}_1$.

Proof. We will prove only part a) of the Theorem 10.3.1. The part b) goes in a similar fashion. Since we follow Penskoi, only the key points will be given.

We need to show that

$$\{\mu_i, \mu_j\}_2 = \int_0^1 \frac{\delta\mu_i}{\delta m} \mathcal{B}^2 \frac{\delta\mu_j}{\delta m} dx = 0, \tag{10.3.45}$$

$$\{\mu_i, f_j\}_2 = \int_0^1 \frac{\delta\mu_i}{\delta m} \mathcal{B}^2 \frac{\delta f_j}{\delta m} dx = \delta_{ij}, \tag{10.3.46}$$

$$\{f_i, f_j\}_2 = \int_0^1 \frac{\delta f_i}{\delta m} \mathcal{B}^2 \frac{\delta f_j}{\delta m} dx = 0. \tag{10.3.47}$$

Let us first calculate $\frac{\delta\mu_i}{\delta m}$. We have

$$y_2''(x,\mu_i) = -\mu_i m y_2(x,\mu_i), \tag{10.3.48}$$

which we write $y_2'' = -\mu m y_2$ for short. The variation of (10.3.48) reads

$$\delta y_2'' = -\delta\mu m y_2 - \mu\delta m y_2 - \mu m \delta y_2. \tag{10.3.49}$$

We multiply this identity by y_2 and integrate. Then the l.h.s. is transformed by integrating by parts to obtain

$$0 = -\delta\mu \int_0^1 m y_2^2 dx - \int_0^1 \mu\delta m y_2^2 dx. \tag{10.3.50}$$

Since, $\int_0^1 m y_2^2 dx \neq 0$, we get

$$\frac{\delta\mu_i}{\delta m} = -A_i\mu_i y_2^2(x,\mu_i), \tag{10.3.51}$$

where $A_i = [\int_0^1 m y_2^2(x,\mu_i)dx]^{-1}$.

To calculate $\frac{\delta\rho_i}{\delta m}$ we first multiply (10.3.49) by $y(x,\mu_i)$, defined in (10.3.42). Next, we multiply $y'' = -\mu m y$ by δy_2, subtract so obtained identities and, finally integrate to obtain

$$\int_0^1 (y\delta y_2'' - y''\delta y_2)dx = -\int_0^1 \delta\mu m y y_2 dx - \int_0^1 \mu\delta m y y_2 dx.$$

The l.h.s. gives

$$\int_0^1 (y\delta y_2'' - y''\delta y_2)dx = \int_0^1 (y\delta y_2' - y'\delta y_2)' dx =$$

$$(y\delta y_2' - y'\delta y_2)|_0^1 = \delta y_2'(1,\mu_i)y(1,\mu_i) = \frac{\delta\rho_i}{\rho_i} = \delta\ln|\rho_i|.$$

Then

$$\delta\ln|\rho_i| = -\delta\mu_i B_i - \mu_i \int_0^1 \delta m y_2 y dx,$$

where $B_i = \int_0^1 m y_2(x,\mu_i)y(x,\mu_i)dx$. Using (10.3.51) we get

$$\frac{\delta\rho_i}{\delta m} = A_i B_i \mu_i y_2^2(x,\mu_i) - \mu_i y_2(x,\mu_i)y(x,\mu_i). \tag{10.3.52}$$

Now we are ready to calculate the brackets (10.3.45) - (10.3.47).

$$\{\mu_i,\mu_j\}_2 = \int_0^1 \frac{\delta\mu_i}{\delta m}\mathcal{B}^2\frac{\delta\mu_j}{\delta m}dx = A_i A_j \mu_i\mu_j \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^2 y_2^2(x,\mu_j)dx.$$

The last integral is zero. This can be seen from

$$\mu_i\mu_j \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^2 y_2^2(x,\mu_j)dx \overset{L1}{=} \mu_i \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^1 y_2^2(x,\mu_j)dx \overset{(10.3.44)}{=}$$

$$-\mu_i \int_0^1 y_2^2(x,\mu_j)\mathcal{B}^1 y_2^2(x,\mu_i)dx \overset{L1}{=} -\mu_i^2 \int_0^1 y_2^2(x,\mu_j)\mathcal{B}^2 y_2^2(x,\mu_i)dx = \mu_i^2 \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^2 y_2^2(x,\mu_j)dx.$$

Hence, $\{\mu_i, \mu_j\}_2 = 0$.

Next, we will show that $\{\mu_i, \ln|\rho_j|\}_2 = \mu_i^2\delta_{ij}$ from where (10.3.46) follows. The case $i \neq j$ is treated similarly as above. Let us consider the case $i = j$.

$$\{\mu_i, \ln|\rho_i|\}_2 = \int_0^1 \frac{\delta\mu_i}{\delta m}\mathcal{B}^2\frac{\delta \ln|\rho_i|}{\delta m}dx = -\int_0^1 A_i\mu_i y_2^2(\mu_i)\mathcal{B}^2\left(A_iB_i\mu_i y_2^2(\mu_i) - \mu_i y_2\mu_i)y(\mu_i)\right)dx =$$

$$-A_i^2B_i\mu_i^2\int_0^1 y_2^2(\mu_i)\mathcal{B}^2 y_2^2(\mu_i)dx + A_i\mu_i^2\int_0^1 y_2^2(\mu_i)\mathcal{B}^2 y_2(\mu_i)y(\mu_i)dx =$$

$$A_i\mu_i^2\int_0^1 y_2^2(\mu_i)(m\partial + \partial m)y_2(\mu_i)y(\mu_i)dx = A_i\mu_i^2\int_0^1 my_2^2(\mu_i)\left(y'(\mu_i)y_2(\mu_i) - y_2'(\mu_i)y(\mu_i)\right)dx.$$

The expression $y'(\mu_i)y_2(\mu_i) - y_2'(\mu_i)y(\mu_i)$ is the Wronskian $Wr(y, y_2)$ which is a constant. Then

$$Wr(y, y_2) = y'(1, \mu_i)y_2(1, \mu_i) - y_2'(1, \mu_i)y(1, \mu_i) = -\frac{y(0, \mu_i)}{\rho_i}y_2'(1, \mu_i) = -1.$$

So, $\{\mu_i, \ln|\rho_i|\}_2 = -A_i\mu_i^2\int_0^1 my_2^2(\mu_i)dx = -\mu_i^2$ and hence

$$\{\mu_i, \ln|\rho_j|\}_2 = -\mu_i^2\delta_{ij} \quad \text{and} \quad \{\mu_i, f_j\}_2 = \delta_{ij}.$$

It remains to verify that $\{\ln|\rho_i|, \ln|\rho_j|\}_2 = 0$. These calculations are similar to those for the bracket $\{\mu_i, \mu_j\}_2 = 0$. Therefore, $\{f_i, f_j\}_2 = 0$. This finishes the proof of the part a) of the Theorem 1. The part b) follows in an analogous way.

∎

It is natural to express the Hamiltonians H_n via the variables μ_i, f_j, for example. It turns out that this is a difficult task. That is why we shall study the motion of the auxiliary spectrum. To do this we assume first that y_1, y_2 are the Floquet solutions of (10.3.36)

$$y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0,$$
$$y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1,$$

in particular

$$y_1(x+1, \mu_n) = y_1(1, \mu_n)y_1(x, \mu_n), \quad y_2(x+1, \mu_n) = y_2'(1, \mu_n)y_2(x, \mu_n)$$

and according to the Wronskian relation

$$y_1(1, \mu_n)y_2'(1, \mu_n) = 1. \tag{10.3.53}$$

Moreover, we also assume that

$$y_1(1, \mu_n) = \Delta - \sqrt{\Delta^2 - 1}, \quad y_2'(1, \mu_n) = \Delta + \sqrt{\Delta^2 - 1}. \tag{10.3.54}$$

If we denote $y_2^\bullet$ to be the derivative with respect to λ, an easy calculation gives that

$$\int_0^1 my_2^2(x, \mu_n)dx = y_2^\bullet y_2'(1, \mu_n).$$

We may write the formula (10.3.51) as

$$\frac{\delta \mu_n}{\delta m} = -\mu_n \frac{y_2^2(x,\mu_n)}{y_2^\bullet y_2'(1,\mu_n)}. \tag{10.3.55}$$

Next, we compute (see [140])

$$\frac{\delta \Delta}{\delta m} = -\frac{\lambda}{2}\left(y_2(x+1,\lambda)y_1(x,\lambda) - y_2(x,\lambda)y_1(x+1,\lambda)\right) = -\frac{\lambda}{2} y_2^{+x}(1,\lambda), \tag{10.3.56}$$

where the superscript $+x$ means that $y_2(1,\lambda)$ is computed for m translated in amount $0 \le x < 1$. Since $\frac{\delta\Delta}{\delta m}$ is a linear combination of products of solutions of (10.3.36), it satisfies Lemma 1.

$$\lambda \mathcal{B}^2 \frac{\delta \Delta}{\delta m} = \mathcal{B}^1 \frac{\delta \Delta}{\delta m}. \tag{10.3.57}$$

Now, with $h = \frac{\delta \mu_n}{\delta m}$ we have

$$\lambda\{\mu_n, \Delta(\lambda)\}_2 = \int_0^1 h\lambda \mathcal{B}^2 \frac{\delta\Delta}{\delta m} dx = \int_0^1 h\mathcal{B}^1 \frac{\delta\Delta}{\delta m} dx =$$

$$-\frac{1}{2}\left[h\left(\frac{\delta\Delta}{\delta m}\right)'' - h'\left(\frac{\delta\Delta}{\delta m}\right)' + h''\frac{\delta\Delta}{\delta m}\right]_0^1 - \int_0^1 \frac{\delta\Delta}{\delta m}\mathcal{B}^1 h dx.$$

Note that $h(0) = h(1) = h'(0) = h'(1) = 0$, so

$$\lambda\{\mu_n, \Delta(\lambda)\}_2 = -\frac{1}{2}\left[h''\frac{\delta\Delta}{\delta m}\right]_0^1 - \mu_n \int_0^1 \frac{\delta\Delta}{\delta m}\mathcal{B}^2 h dx$$

or

$$\lambda\{\mu_n, \Delta(\lambda)\}_2 = -\frac{1}{2}\left[h''\frac{\delta\Delta}{\delta m}\right]_0^1 + \mu_n \int_0^1 h\mathcal{B}^2 \frac{\delta\Delta}{\delta m} dx.$$

Now, it is easy to obtain from here

$$\{\mu_n, \Delta(\lambda)\}_2 = -\mu_n \frac{\lambda}{2} \frac{(y_2'(1,\mu_n))^2 - 1}{y_2^\bullet y_2'(1,\mu_n)} \frac{y_2(1,\lambda)}{\lambda - \mu_n}$$

and with the help of (10.3.53) and (10.3.54)

$$\{\mu_n, \Delta(\lambda)\}_2 = -\mu_n \lambda \frac{\sqrt{\Delta^2 - 1}}{y_2^\bullet(1,\mu_n)} \frac{y_2(1,\lambda)}{\lambda - \mu_n}. \tag{10.3.58}$$

It is known that the Hamiltonians $H_n, n = 1, 2, \ldots$ are coefficients in an expansion of $\Delta(\lambda)$: $\Delta(\lambda) = 1 - \sum_{n=1}^{\infty} H_n \lambda^n$. Since $\Delta^\bullet(0) = -H_1$, from (10.3.58) we can obtain the motion of the auxiliary spectrum under the flow of the μCH equation

$$\dot{\mu}_n = \{\mu_n, H_1\}_2 = \mu_n \frac{\sqrt{\Delta^2 - 1}}{y_2^\bullet(1,\mu_n)} \frac{y_2(1,0)}{-\mu_n} = -\frac{\sqrt{\Delta^2 - 1}}{y_2^\bullet(1,\mu_n)}, \quad n \ge 1. \tag{10.3.59}$$

Similarly we can obtain the motion μ_n under the flows of the higher Hamiltonians from (10.3.58).

It is seen that (10.3.59) is a system of infinitely many nonlinear differential equations in infinitely many variables. Only in the case of so called finite-gap potentials (10.3.59) becomes a finite system whose solutions are usually expressed via theta functions.

10.4 Concluding Remarks

In this chapter, we study the CH equation and some of its generalizations from the geometric point of view. We show that Kupershmidt deformations for CH and μCH equations preserve integrability and derive some important objects like quadratic pseudo-potentials which turn out to be useful for obtaining conservation laws and nonlocal symmetries. It is also shown that the KdV6 equation and two-component CH system are also geometrically integrable.

Having at hand these examples of geometrically integrable Kuperschmidt deformations it is natural to think that maybe there exists a general link in this sense: a Kuperschmidt deformation of geometrically integrable system is again geometrically integrable. We haven't succeeded in establishing such a link up to now, but we believe that this is true at least for the systems with local Hamiltonian pair of operators as in the above examples.

Let us return, however, to the μCH equation ($w = 0$). It is obvious that pseudo-potentials for the μCH equation (10.2.17) and parameter dependent conservation law are obtained from

$$m = \frac{\gamma^2}{\lambda} + \frac{\gamma_x}{\lambda}, \tag{10.4.60}$$

$$\gamma_t = \frac{1}{2\lambda}\left(\gamma + \lambda u_x - 2\lambda u\gamma\right)_x = \frac{\gamma_x}{2\lambda} + \frac{u_{xx}}{2} - \partial(\gamma u). \tag{10.4.61}$$

The equation (10.4.60) is an analogue of the Miura transformation of KdV theory. We can repeat, purely formally, the procedure for obtaining the "modified" CH (mCH) equation [168] in this case. However, it is clear that since μCH contains nonlocal term, one can expect that the "modified" equation also will have nonlocal terms.

Denote by A the operator $A = \mu - \partial^2, A(u) = m = \mu(u) - u_{xx}$. The operators A^{-1} and ∂ commute and $\mu(u) = \mu(Au)$.

We have

$$u = A^{-1}m, \quad u_x = A^{-1}m_x, \quad u_{xx} = A^{-1}m_{xx}, \tag{10.4.62}$$

in which m is determined by (10.4.60). Then, the second equation (10.4.61) takes the form

$$\gamma_t = \frac{\gamma_x}{2\lambda} + \frac{A^{-1}m_{xx}}{2} - \gamma A^{-1}m_x - \gamma_x A^{-1}m$$

or

$$A\gamma_t = \frac{A\gamma_x}{2\lambda} + \frac{m_{xx}}{2} - A\partial(\gamma A^{-1}m).$$

Formally, this equation can be named as a "modified" μCH equation. One can simplify further this equation using (10.4.60) or even to present it as a system as in [168] - it remains nonlocal and, hence, it is of no immediate advantage.

It turns out that the conjugate variables obtained here for the μCH equation are practically the same as for the periodic CH equation. Perhaps, the reason is that these equations, although different, have similar bi-hamiltonian structures.

Let us return to the μCH equation (10.3.35). Formally we may think of μCH in a following way. We take the HS equation and add a nonlocal term

$$-2\mu(u)u_x = -2H_0[m]u_x,$$

where H_0 was defined in the Introduction.

What if we consider an equation obtained in this way, but the other conserved quantity is taken instead H_0? For example, we may take H_1

$$-u_{txx} = -2H_1[m]u_x + 2u_x u_{xx} + uu_{xxx}.$$

Of course, the physical interpretation is missing, but the question is: Whether this equation, obtained in this formal way is integrable?

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