A New Characterization of Weighted Peetre K-Functionals

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Abstract

The purpose of this paper is to present a characterization of certain types of generalized weighted Peetre K-functionals by means of a modulus of smoothness. This new modulus is based on the classical one taken on a certain linear transform of the function. A new modulus of smoothness which describes the best algebraic approximation is introduced.

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1 Introduction

In a number of approximation processes the error is estimated by means of an appropriate K-functional. Generally it is of the form

$$K(f,t) = K(f,t;X,Y,\mathcal{D}) = \inf\big\{\|f-g\|_X + t\,\|\mathcal{D}g\|_X \ : \ g\in Y\big\}, \tag{1.1}$$

where X is a Banach space, $\mathcal{D}$ is a differential operator of the form

$$\mathcal{D}g(x) = \sum_{k=0}^{r}\varphi_k(x)g^{(k)}(x), \quad \varphi_k\in X,\ k=0,\dots,r, \quad \varphi_r>0 \quad a.\ e. \tag{1.2}$$

with a given $r\in\mathbb{N}$ and $Y\subseteq\mathcal{D}^{-1}(X)=\{g\in X : \mathcal{D}g\in X\}$ (note that $\mathcal{D}^{-1}(X)\subset X$) is a dense subspace of X. Given X, Y and $\mathcal{D}$ the quantity (1.1) is considered for every $f\in X$ and $t>0$.

We shall also use the notation (1.1) when $Y\backslash\mathcal{D}^{-1}(X)\neq\emptyset$, assuming $\|f-g\|_X + t\,\|\mathcal{D}g\|_X = +\infty$ for $g\in Y\backslash\mathcal{D}^{-1}(X)$. In such cases the infimum in the definition of the K-functional is actually taken on $Y\cap\mathcal{D}^{-1}(X)$.

As the class of functions f for which we can estimate the infimum in (1.1) for any $t\in(0,1]$ is quite narrow (because Y is large), it is useful to have an easier to calculate modulus of smoothness $\Omega(f,t)$ equivalent to the K-functional above, namely that there exists a constant $C>0$ independent of f and t such that

$$C^{-1}\Omega(f,t) \le K(f,t) \le C\Omega(f,t),$$

which we denote in short by $K(f,t)\sim\Omega(f,t)$. (We shall denote by C constants of this kind which may differ at each occurrence.) Our goal is to define such a modulus of smoothness for a number of X, Y and $\mathcal{D}$, where X is a space of functions defined on a fixed finite or infinite interval $[a,b]$ on the real line.

In the unweighted case, i.e. weights equal to 1, it is well known (see [14] and the references cited there) that for $\mathcal{D}g = D^r g := g^{(r)}$ and $X = L_p = L_p[a,b]$ with the usual L_p-norm denoted by $\|\cdot\|_p$ for $1\le p<\infty$ or $X = C = C[a,b]$ with the uniform norm denoted by $\|\cdot\|_\infty$ for $p=\infty$, we have

$$K(f,t^r;L_p,W_p^r,D^r)\sim\omega_r(f,t)_p,$$

where $\omega_r(f,t)_p$ are the classical moduli of smoothness defined by

$$\omega_r(f,t)_p = \sup_{0<h\le t}\|\Delta_h^r f(\cdot)\|_p\,, \tag{1.3}$$

and the finite difference with a fixed step h is given by

$$\Delta_h^r f(x) = \begin{cases} \sum_{k=0}^r (-1)^{r-k}\binom{r}{k} f(x+kh), & \text{if } x, x+rh\in[a,b],\\ 0, & \text{otherwise.}\end{cases} \tag{1.4}$$

Two important solutions of the considered problem are presented so far for weighted Peetre K-functionals ($\mathcal{D}=\varphi^r D^r$ with a proper weight φ, which inside (a,b) is equivalent to 1)

$$K(f,t^r;L_p,AC_{loc}^{r-1},\varphi^r D^r) = \inf\left\{\|f-g\|_p + t^r\|\varphi^r g^{(r)}\|_p \;:\; g\in AC_{loc}^{r-1}\right\}, \tag{1.5}$$

where $AC_{loc}^k = AC_{loc}^k(a,b) = \left\{g \;:\; g, g', \dots, g^{(k)} \in AC[c,d] \quad \forall a<c<d<b\right\}$ and $AC[c,d]$ is the set of the absolutely continuous functions on $[c,d]$.

On one hand Ditzian and Totik introduced in [5] the varying step moduli

$$\omega_\varphi^r(f,t)_p = \sup_{0<h\le t}\|\bar\Delta_{h\varphi(\cdot)}^r f(\cdot)\|_p, \tag{1.6}$$

where the centered finite difference with (varying) step θ is given by $\bar\Delta_\theta^r f(x) = \Delta_\theta^r f(x - r\theta/2)$. They generalized in [5] some earlier results of theirs (see [4] and [20]) and proved for certain power and logarithmic-type weights φ (see [5, Ch. 1, Sec. 1.2 and Ch. 5] about the conditions imposed on φ) the equivalence

$$K(f,t^r;L_p,AC_{loc}^{r-1},\varphi^r D^r)\sim\omega_\varphi^r(f,t)_p. \tag{1.7}$$

Finally, let us note that Ditzian-Totik moduli are useful in estimating the rate of approximation by algebraic polynomials and some well known linear operators.

On the other hand, the second author introduced the following moduli of smoothness

$$\tau_r(f;\psi(t))_{q,p} = \|\omega_r(f,\cdot;\psi(t,\cdot))_q\|_p\,, \tag{1.8}$$

where the local moduli are given by

$$\omega_r(f,x;\psi(t,x))_q = \left((2\psi(t,x))^{-1}\int_{-\psi(t,x)}^{\psi(t,x)} |\Delta_h^r f(x)|^q \, dh\right)^{1/q}, \ 1 \le q < \infty,$$
$$\omega_r(f,x;\psi(t,x))_\infty = \sup\{|\Delta_h^r f(x)| : |h| \le \psi(t,x)\}$$

and ψ is a continuous function connected with φ in a certain way ([11], [12]). Under the conditions to which the weight φ is subdued given in [12], Ivanov proved

$$K(f,t^r;L_p,AC_{loc}^{r-1},\varphi^r D^r) \sim \tau_r(f;\psi(t))_{p,p}.$$

The power and logarithmic-type weights φ are covered. These moduli were introduced in 1980 (see e.g. [11]) for characterizing the rate of convergence of the best approximations by algebraic polynomials.

In the present paper we utilize an approach that differs from the above mentioned one for finding equivalent moduli to the weighted Peetre K-functionals

(1.9)
$$K(f,t^r;L_p(w),AC_{loc}^{r-1},\varphi^r D^r) = \inf\bigl\{\|w(f-g)\|_p + t^r\|w\varphi^r g^{(r)}\|_p \ : \ g \in AC_{loc}^{r-1}\bigr\},$$

where the weighted L_p spaces are given by $L_p(w) = L_p(w)[a,b] = \bigl\{f : wf \in L_p[a,b]\bigr\}$, $L_p(1) = L_p$. Note that (1.9) reduces to (1.5) when $w \equiv 1$ and that K-functionals of the form (1.9) with some weights w are proved to be equivalent to proper modifications of (1.6) in [5, Ch. 6] (see also [6]). K-functionals of the type (1.9) with $\varphi \equiv 1$ and monotonicity requirements on the bounded weight w near the end-points are also characterized in [16]. The present approach covers more cases than those in [5, Ch. 6] and [6] (see Remark 5.2).

The idea consists of two steps. The *first step* is to study conditions on the triples (X_1,Y_1,D_1) and (X_2,Y_2,D_2) under which one can find a linear operator $\mathcal{A} : X_1 \to X_2$ such that

$$K(f,t;X_1,Y_1,D_1) \sim K(\mathcal{A}f,t;X_2,Y_2,D_2). \tag{1.10}$$

The *second step* is to choose (X_2,Y_2,D_2) in (1.10) in such way that the K-functional has a known equivalent modulus $\Omega(F,t)$, i.e.

$$K(F,t;X_2,Y_2,D_2) \sim \Omega(F,t). \tag{1.11}$$

As a consequence of (1.10) and (1.11), one gets

$$K(f,t;X_1,Y_1,D_1) \sim \Omega(\mathcal{A}f,t). \tag{1.12}$$

In order to make (1.12) effective for computations one has to require some additional properties of $\mathcal{A}$ as explicitness, simple form, easy to calculate for a given f, etc. In our opinion the operators constructed in this article possess these properties.

While in the second step one simply considers the known cases of equivalence between K-functionals and moduli, the first step needs some considerations.

Let X_1 and X_2 be Banach functional spaces and let D_1 and D_2 be differential operators. (In general, we do not require the functions in X_1 to be defined on the same interval as the functions in X_2.)

Definition 1.1. We say that the linear operator $\mathcal{A}$ maps continuously (X_1, Y_1, D_1) onto (X_2, Y_2, D_2) and write $\mathcal{A} : (X_1, Y_1, D_1) \mapsto (X_2, Y_2, D_2)$ if and only if $\mathcal{A} : X_1 \to X_2$ is invertible and together with its inverse $\mathcal{A}^{-1} : X_2 \to X_1$ satisfy the conditions

(a) $\|\mathcal{A}f\|_{X_2} \le C \|f\|_{X_1}$ for any $f \in X_1$;

(b) $\|D_2\mathcal{A}f\|_{X_2} \le C \|D_1 f\|_{X_1}$ for any $f \in Y_1 \cap D_1^{-1}(X_1)$;

(c) $\|\mathcal{A}^{-1}F\|_{X_1} \le C \|F\|_{X_2}$ for any $F \in X_2$;

(d) $\|D_1\mathcal{A}^{-1}F\|_{X_1} \le C \|D_2 F\|_{X_2}$ for any $F \in Y_2 \cap D_2^{-1}(X_2)$;

(e) $\mathcal{A}(Y_1 \cap D_1^{-1}(X_1)) = Y_2 \cap D_2^{-1}(X_2)$.

In Section 2 we show that $\mathcal{A} : (X_1, Y_1, D_1) \mapsto (X_2, Y_2, D_2)$ is a sufficient condition for (1.10). Note also that the dimensions of the null spaces of the K-functionals in (1.10) have to be equal and that $\mathcal{A}$ is an one-to-one correspondence between the null spaces.

There are several reasons for using triples $(X, Y, \mathcal{D})$ instead of the usual for interpolation theory pairs (X, Y). Among them are:

- in several problems in approximation theory it is natural to vary Y, keeping fixed the semi-norm $\|\mathcal{D}(\cdot)\|_X$ (see e.g. [13]);
- the semi-norm in Y is determined only by X and $\mathcal{D}$ and hence we introduce less definitions and notations for Y's;
- as demonstrated in Section 5 the weights in the norm of X and in the differential operator $\mathcal{D}$ play different role in establishing relations like (1.10).

Having in mind Definition 1.1 and Proposition 2.1 below, we introduce new moduli by

Definition 1.2. For given $(X, Y, \mathcal{D})$ we set for every $f \in X$ and $t > 0$

$$\Omega(f, t) = \Omega(f, t; X, Y, \mathcal{D}) := \omega_r(\mathcal{A}f, t)_p, \tag{1.13}$$

where $\mathcal{A}$ is an operator such that $\mathcal{A} : (X, Y, \mathcal{D}) \mapsto (L_p, W_p^r, D^r)$.

Note that Ω in (1.13) also depends on the choice of $\mathcal{A}$ but the dependence is not essential – varying $\mathcal{A}$ we get equivalent moduli.

Obviously $\Omega(f, t)$ inherits all properties of $\omega_r(F, t)_p$ as $(f, g \in X)$:

- $\Omega(f + g, t) \le \Omega(f, t) + \Omega(g, t)$;
- $\Omega(\lambda f, t) = |\lambda|\Omega(f, t), \quad \lambda \in \mathbb{R}$;

- $\Omega(f,t) \le C\|f\|_X$;
- $\Omega(f,\lambda t) = C\lambda^r\Omega(f,t), \quad \lambda > 0$;
- Marchaud inequality, etc.

The idea of using operators like $\mathcal{A}$ is not new. Several examples of its implementation are given in [3, Ch. 6], but it can be traced back even before the invention of the K-functional. When comparing the best approximations by trigonometric polynomials and the best approximations by algebraic polynomials, several mathematicians used the mapping $(\mathcal{A}f)(y) = f(\cos y)$ in order to establish the so-called "effect of the end-points". It is well known that this mapping solves the following problem in the case $r = 1$ and $p = \infty$.

Problem 1.1. *Given $r \in \mathbb{N}$ and $1 \le p \le \infty$. Find an operator $\mathcal{A} : L_p[-1,1] \to L_p[0,\pi]$ such that for every $t > 0$ and $f \in L_p[-1,1]$ we have*

$$\inf\{\|f-g\|_p + t^r\|\varphi^r g^{(r)}\|_p : g \in AC_{loc}^{r-1}\} \sim \inf\{\|\mathcal{A}f - G\|_p + t^r\|G^{(r)}\|_p : G \in W_p^r\},$$

where $\varphi(x) = (1-x^2)^{1/2}$.

In the terms of Definition 1.1, we have $\mathcal{A} : (C, AC_{loc}, \varphi D) \mapsto (C, W^1_\infty, D)$ because of $(\mathcal{A}f)'(\arccos x) = -(1-x^2)^{1/2}f'(x)$. But this approach has also known difficulties when $p < \infty$ or $r \ge 2$:

i) For $p < \infty$ we have an additional weight:

$$\left\{\int_0^\pi |(\mathcal{A}f)(y)|^p\,dy\right\}^{\frac{1}{p}} = \left\{\int_{-1}^1 |(1-x^2)^{-\frac{1}{2p}} f(x)|^p\,dx\right\}^{\frac{1}{p}};$$

ii) For $r \ge 2$ the r-th derivative of $\mathcal{A}f$ contains more than one terms. For example, for $r = 2$ we have

$$(\mathcal{A}f)''(y) = \sin^2 y f''(\cos y) - \cos y f'(\cos y),$$
$$(\mathcal{A}f)''(\arccos x) = (1-x^2)f''(x) - xf'(x).$$

Maybe these difficulties caused the abandonment of the idea and the invention of the moduli of Ivanov and of Ditzian and Totik. In Sections 3 and 4 we show that one can overcome both listed difficulties with proper definitions of the operators $\mathcal{A}$. In fact, we introduce in these sections *commutative groups of operators* depending on a real parameter, which have several additional properties.

In Section 5 we show that the consecutive application of several of the operators from Sections 3 and 4 leads to the construction of operators $\mathcal{A}$ such that (1.10) holds when $(X_j, Y_j, D_j) = (L_p(w_j), AC_{loc}^{r-1}, \varphi_j^r D^r)$, $j = 1,2$, for a variety of weights w_j and φ_j (see e.g. Theorem 5.3) and, in particular, $K(\mathcal{A}f, t^r; L_p, W_p^r, D^r)$ are equivalent to K-functionals (1.9) (see e.g. Corollary 5.2). In particular, a solution of Problem 1.1 with more general φ is given in Corollary 5.3. Several possible generalizations are given in Section 6. Examples of operators $\mathcal{A}$ are shown in Section 7, while Section 8 contains applications to some areas of the approximation theory as best polynomial approximations, Bernstein, Kantorovich, Durrmeyer and Szász-Mirakjan operators.

Finally, let us mention that this investigation was motivated by the results of the first author in [7].

2 Preliminaries

The next statement is a standard relation connecting linear operators and K-functionals (see e.g. [3, Ch. 6, (1.14)]).

Proposition 2.1. *Let the linear operator $\mathcal{A}$ map continuously (X_1, Y_1, D_1) onto (X_2, Y_2, D_2). Then for every $f \in X_1$ and $t > 0$ we have*

$$K(f, t; X_1, Y_1, D_1) \sim K(\mathcal{A}f, t; X_2, Y_2, D_2).$$

Proof. For $g \in Y_1$ we set $G = \mathcal{A}g$. Then $G \in Y_2$ in view of Definition 1.1 (e). Using (a) and (b) of the same definition we get

$$\begin{aligned} K(\mathcal{A}f, t; X_2, Y_2, D_2) &= \inf\bigl\{\|\mathcal{A}f - G\|_{X_2} + t\|D_2 G\|_{X_2} \ : \ G \in Y_2\bigr\} \\ &= \inf\bigl\{\|\mathcal{A}(f-g)\|_{X_2} + t\,\|D_2\mathcal{A}g\|_{X_2} \ : \ g \in Y_1\bigr\} \\ &\le C\inf\bigl\{\|f-g\|_{X_1} + t\,\|D_1 g\|_{X_1} \ : \ g \in Y_1\bigr\} \\ &= CK(f, t; X_1, Y_1, D_1). \end{aligned}$$

The inequality

$$K(f, t; X_1, Y_1, D_1) \le CK(\mathcal{A}f, t; X_2, Y_2, D_2)$$

is verified in the same way using the properties of $\mathcal{A}^{-1}$. □

We shall use the following generalization of Hardy's inequalities (see [10, p. 245]) given in [18].

Proposition 2.2. *Suppose U, V are non-negative measurable functions on $(0,\infty)$, $1 \le p \le \infty$ and p' is the conjugate exponent of p, i.e. $1/p + 1/p' = 1$ with the usual modification for either $p = 1$ or $p = \infty$. Then for every measurable function f on $(0,\infty)$ we have*

$$\Biggl(\int_0^\infty \Bigl|U(x)\int_0^x f(y)\,dy\Bigr|^p dx\Biggr)^{\frac{1}{p}} \le C\Biggl(\int_0^\infty |V(x)f(x)|^p\,dx\Biggr)^{\frac{1}{p}} \tag{2.1}$$

if and only if

$$\sup_{\xi>0}\Biggl(\int_\xi^\infty U(x)^p\,dx\Biggr)^{\frac{1}{p}}\Biggl(\int_0^\xi V(x)^{-p'}\,dx\Biggr)^{\frac{1}{p'}} < \infty. \tag{2.2}$$

Also

$$(2.3)\qquad \left(\int_0^\infty \left|U(x)\int_x^\infty f(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \le C\left(\int_0^\infty |V(x)f(x)|^p\,dx\right)^{\frac{1}{p}}$$

if and only if

$$(2.4)\qquad \sup_{\xi>0}\left(\int_0^\xi U(x)^p\,dx\right)^{\frac{1}{p}}\left(\int_\xi^\infty V(x)^{-p'}\,dx\right)^{\frac{1}{p'}} < \infty.$$

From Proposition 2.2 we get

Corollary 2.1. *Let $\zeta < \eta$ and let F be a measurable function on $[\zeta, \eta]$.*
a) If $1 \le p \le \infty$, $\beta > 0$, $\gamma \le \beta$ or $p = 1$, $\beta = 0$, $\gamma < 0$ then

$$\left(\int_\zeta^\eta \left|(x-\zeta)^{-\gamma-\frac{1}{p}}\int_\zeta^x F(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \le C\left(\int_\zeta^\eta |(x-\zeta)^{-\beta+1-\frac{1}{p}}F(x)|^p\,dx\right)^{\frac{1}{p}}.$$

b) If $1 \le p \le \infty$, $\beta \le \gamma$, $\gamma > 0$ or $p = \infty$, $\beta < 0$, $\gamma = 0$ then

$$\left(\int_\zeta^\eta \left|(x-\zeta)^{\gamma-\frac{1}{p}}\int_x^\eta F(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \le C\left(\int_\zeta^\eta |(x-\zeta)^{\beta+1-\frac{1}{p}}F(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Throughout the paper we shall use the following notations. For $c \in \mathbb{R}$ set $\chi_c(x) = |x-c|$. For $n \in \mathbb{N}\cup\{0\}$ set Π_n to be the set of all algebraic polynomials of degree at most n. $D = \frac{d}{dx}$ means first derivative and D^r means r-th derivative. We often do not indicate the dependence of the objects on the fixed $r \in \mathbb{N}$, which always stays for the power of the leading term of the differential operator.

All constants denoted by C can be explicitly evaluated using algebraic expressions and the constants in the Hardy-type inequalities (which are known).

3 Operators that change the weight in both terms of the K-functional

Let $r \in \mathbb{N}$ be fixed. For $\rho \in \mathbb{R}$ we define the operator $A(\rho) : L_{1,loc}(0,\infty) \to L_{1,loc}(0,\infty)$ by

$$(3.1)\qquad (A(\rho)f)(x) = x^\rho f(x) + \sum_{k=1}^r \alpha_{r,k}(\rho)\psi_k(x),$$

where

$$(3.2)\qquad \psi_k(x) = x^{k-1}\int_1^x y^{-k+\rho}f(y)dy, \quad k = 1, 2, \ldots$$

and

$$(3.3)\qquad \alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!}\binom{r-1}{k-1}\prod_{\nu=0}^{r-1}(\rho + r - k - \nu), \quad k = 1, 2, \ldots, r.$$

Obviously, operators of type (3.1), (3.2) preserve the smoothness properties of the functions on intervals $[a,b] \subset (0,\infty)$.

We shall use the combinatorial identity for $M,N,P \in \mathbb{N} \cup \{0\}, M \le N$

$$\sum_{K=0}^{\min\{P,N-M\}} (-1)^K \binom{N-K}{M}\binom{P}{K} = \binom{N-P}{M-P}. \tag{3.4}$$

Lemma 3.1. *For $s = 1,2,\ldots,r$ we have*

$$\sum_{k=s}^{r} \frac{(k-1)!}{(k-s)!}\alpha_{r,k}(\rho) = (-1)^s \binom{r}{s} \prod_{\nu=0}^{s-1}(\rho-\nu). \tag{3.5}$$

Proof. Both sides of (3.5) are polynomials in ρ of degree r. Thus, (3.5) will be proved if we show that both sides take one and the same values for $\rho = i$, $i = 0,1,\ldots,r$. We have $\prod_{\nu=0}^{r-1}(i+r-k-\nu) = 0$ for $0 \le i < k$ and $\prod_{\nu=0}^{r-1}(i+r-k-\nu) = \frac{(i+r-k)!}{(i-k)!}$ for $k \le i \le r$. Thus, both sides of (3.5) are 0 for $0 \le i < s$. For $s \le i \le r$ using (3.3) and (3.4) with $K = k-s, P = r-s, M = r, N = r+i-s$ we get

$$\begin{aligned}
\sum_{k=s}^{r} \frac{(k-1)!}{(k-s)!}\alpha_{r,k}(i) &= \sum_{k=s}^{i} \frac{(-1)^k (i+r-k)!}{(k-s)!(r-k)!(i-k)!} \\
&= \frac{r!}{(r-s)!} \sum_{k=s}^{i} (-1)^k \binom{r-s}{k-s}\binom{r+i-k}{r} \\
&= (-1)^s \frac{r!}{(r-s)!}\binom{i}{s} = (-1)^s \binom{r}{s}\prod_{\nu=0}^{s-1}(i-\nu).
\end{aligned}$$

This proves the lemma. □

Lemma 3.2. *Let $f \in AC_{loc}^{r-1}(0,\infty)$. Then for $s = 1,2,\ldots,r$ and $x \in (0,\infty)$ we have*

$$\begin{aligned}
(A(\rho)f)^{(s-1)}(x) &= \sum_{j=0}^{s-1} (-1)^j \binom{r+j-s}{j} \prod_{\nu=0}^{j-1}(\rho-\nu) x^{\rho-j} f^{(s-j-1)}(x) \\
&+ \sum_{k=s}^{r} \frac{(k-1)!}{(k-s)!}\alpha_{r,k}(\rho) x^{-s+1}\psi_k(x).
\end{aligned} \tag{3.6}$$

Proof. We proceed by induction. For $s = 1$ (3.6) reduces to (3.1). Let (3.6) be true for some $s < r$. Using that $(x^{-s+1}\psi_k(x))' = x^{\rho-s}f(x) + (k-s)x^{-s}\psi_k(x)$

and Lemma 3.1 we get from (3.6)

$$
\begin{aligned}
&(A(\rho)f)^{(s)}(x)\\
&=\sum_{j=0}^{s-1}(-1)^j\binom{r+j-s}{j}\prod_{\nu=0}^{j-1}(\rho-\nu)\left\{(\rho-j)x^{\rho-j-1}f^{(s-j-1)}(x)+x^{\rho-j}f^{(s-j)}(x)\right\}\\
&+\sum_{k=s}^{r}\frac{(k-1)!}{(k-s)!}\alpha_{r,k}(\rho)\left\{x^{\rho-s}f(x)+(k-s)x^{-s}\psi_k(x)\right\}\\
&=\sum_{j=0}^{s}(-1)^j\left\{\binom{r+j-s}{j}-\binom{r+j-1-s}{j-1}\right\}\prod_{\nu=0}^{j-1}(\rho-\nu)x^{\rho-j}f^{(s-j)}(x)\\
&+\sum_{k=s+1}^{r}\frac{(k-1)!}{(k-s-1)!}\alpha_{r,k}(\rho)x^{-s}\psi_k(x)\\
&=\sum_{j=0}^{s}(-1)^j\binom{r+j-s-1}{j}\prod_{\nu=0}^{j-1}(\rho-\nu)x^{\rho-j}f^{(s-j)}(x)\\
&+\sum_{k=s+1}^{r}\frac{(k-1)!}{(k-s-1)!}\alpha_{r,k}(\rho)x^{-s}\psi_k(x).
\end{aligned}
$$

This proves the lemma. $\square$

Theorem 3.1. *Let $r\in\mathbb{N}$ and $\rho\in\mathbb{R}$. Then for every $f\in AC_{loc}^{r-1}(0,\infty)$ we have*

$$(A(\rho)f)^{(r)}(x)=x^{\rho}f^{(r)}(x)\qquad a.e. \tag{3.7}$$

and

(3.8)

$$(A(\rho)f)^{(s-1)}(1)=\sum_{i=1}^{s}(-1)^{s-i}\binom{r-i}{s-i}\prod_{\nu=0}^{s-i-1}(\rho-\nu)f^{(i-1)}(1),\qquad s=1,2,\ldots,r.$$

Proof. Lemma 3.2 with $s=r$ gives

$$
\begin{aligned}
(A(\rho)f)^{(r-1)}(x) &= \sum_{j=0}^{r-1}(-1)^j\prod_{\nu=0}^{j-1}(\rho-\nu)x^{\rho-j}f^{(r-j-1)}(x)\\
&+ (-1)^r\prod_{\nu=0}^{r-1}(\rho-\nu)\int_1^x y^{-r+\rho}f(y)dy.
\end{aligned}
$$

Differentiating this equality we get (3.7). The relation (3.8) follows from Lemma 3.2 with $i=s-j$ and $x=1$ because of $\psi_k(1)=0$. This proves the theorem. $\square$

Lemma 3.3. *For $0\le m\le n$ we have*

$$\sum_{j=0}^{m}\binom{n-j}{m-j}\binom{n}{j}\prod_{\nu=0}^{m-j-1}(x-\nu)\prod_{\nu=0}^{j-1}(y-\nu)=\binom{n}{m}\prod_{\nu=0}^{m-1}(x+y-\nu). \tag{3.9}$$

Proof. Both sides of (3.9) are polynomials in x and y of total degree m. Thus, (3.9) will be proved if we show that both sides take one and the same values for $x=\ell$, $\ell=0,1,\ldots$, $y=k$, $k=0,1,\ldots$, $k+\ell\le m$. Both sides of (3.9) are 0 for $0\le k+\ell<m$. For $k+\ell=m$ only the term for $j=k$ in the left-hand side is not 0. Hence

$$\sum_{j=0}^{m}\binom{n-j}{m-j}\binom{n}{j}\prod_{\nu=0}^{m-j-1}(\ell-\nu)\prod_{\nu=0}^{j-1}(k-\nu)=\binom{n-k}{m-k}\binom{n}{k}(m-k)!k!=\binom{n}{m}m!.$$

This proves the lemma. $\square$

Theorem 3.2. *Let $r\in\mathbb{N}$, $\rho,\sigma\in\mathbb{R}$. Then $A(\rho)A(\sigma)=A(\rho+\sigma)$.*

Proof. Using Theorem 3.1 for every $f\in AC_{loc}^{r-1}(0,\infty)$ we have

$$(A(\rho)A(\sigma)f)^{(r)}(x)=x^{\rho}(A(\sigma)f)^{(r)}(x)=x^{\rho+\sigma}f^{(r)}(x)=(A(\rho+\sigma)f)^{(r)}(x)$$

and for $s=1,2,\ldots,r$

$$\begin{aligned}
(A(\rho)A(\sigma)f)^{(s-1)}(1)&=\sum_{i=1}^{s}(-1)^{s-i}\binom{r-i}{s-i}\prod_{\nu=0}^{s-i-1}(\rho-\nu)(A(\sigma)f)^{(i-1)}(1)\\
=&\sum_{i=1}^{s}(-1)^{s-i}\binom{r-i}{s-i}\prod_{\nu=0}^{s-i-1}(\rho-\nu)\sum_{k=1}^{i}(-1)^{i-k}\binom{r-k}{i-k}\prod_{\nu=0}^{i-k-1}(\sigma-\nu)f^{(k-1)}(1)\\
=&\sum_{k=1}^{s}(-1)^{s-k}\left[\sum_{i=k}^{s}\binom{r-i}{s-i}\binom{r-k}{i-k}\prod_{\nu=0}^{s-i-1}(\rho-\nu)\prod_{\nu=0}^{i-k-1}(\sigma-\nu)\right]f^{(k-1)}(1)\\
=&\sum_{k=1}^{s}(-1)^{s-k}\binom{r-k}{s-k}\prod_{\nu=0}^{s-k-1}(\rho+\sigma-\nu)f^{(k-1)}(1)=(A(\rho+\sigma)f)^{(s-1)}(1),
\end{aligned}$$

where we have used Lemma 3.3 with $x=\rho$, $y=\sigma$, $m=s-k$, $n=r-k$ and $j=i-k$. Now Taylor formula gives $A(\rho)A(\sigma)f=A(\rho+\sigma)f$ for any $f\in AC_{loc}^{r-1}(0,\infty)$. We complete the proof using the boundedness of the linear operators $A(\rho)A(\sigma)$ and $A(\rho+\sigma)$ and the density of $W_1^r[a,b]$ in $L_1[a,b]$ for every $0<a<b<\infty$. $\square$

Corollary 3.1. *$\{A(\rho)\}_{\rho\in\mathbb{R}}$ is a commutative group of operators with $A(0)$ as the identity element. In particular, $A(\rho)^{-1}=A(-\rho)$.*

From Theorem 3.1 and Corollary 3.1 we get

Corollary 3.2. *Let $r\in\mathbb{N}$ and $\rho\in\mathbb{R}$. Then $A(\rho)(\Pi_{r-1})=\Pi_{r-1}$ and $A(\rho)(AC_{loc}^{k-1})=AC_{loc}^{k-1}$ for any $k\in\mathbb{N}$.*

In the next statement we collect some additional combinatorial properties of the coefficients $\alpha_{r,k}(\rho)$ which will be used later.

Proposition 3.1. *Let $r \in \mathbb{N}$ and $\sigma, \rho \in \mathbb{R}$. For $\alpha_{r,k}(\rho)$ given in* (3.3) *we have*

$$\sum_{k=1}^{r} \frac{\alpha_{r,k}(\rho)}{\rho - k + i} = -1, \quad i = 1, 2, \ldots, r, \tag{3.10}$$

$$\alpha_{r,k}(\rho) - \sum_{i=1}^{r} \frac{\alpha_{r,i}(\sigma)\alpha_{r,k}(\rho)}{\rho - k + i} = \alpha_{r,k}(\rho + \sigma), \quad k = 1, 2, \ldots, r, \tag{3.11}$$

where we define by continuity

$$\left. \frac{\alpha_{r,k}(\rho)}{\rho - k + i} \right|_{\rho = k - i} = (-1)^{k+i-1} \frac{(r-i)!(i-1)!}{(r-k)!(k-1)!}, \quad i, k = 1, 2, \ldots, r. \tag{3.12}$$

Proof. To prove the first identity we just notice that for $f_i(x) = x^{i-1}$, $i = 1, 2, \ldots, r$, and $\rho - k + i \neq 0$, (3.1), (3.2) and (3.7) imply

$$(A(\rho)f_i)(x) = x^{\rho+i-1} + \sum_{k=1}^{r} \frac{\alpha_{r,k}(\rho)}{\rho - k + i}(x^{\rho+i-1} - x^{k-1}) \in \Pi_{r-1}.$$

Hence we get (3.10) for an irrational ρ. But as the left-hand side of (3.10) is an algebraic polynomial in ρ, we get it for any ρ by continuity.

Using Theorem 3.2 and applying twice (3.1), (3.2) and (3.3) we get

$$\begin{aligned}
&(A(\rho + \sigma)f)(x) = (A(\rho)A(\sigma)f)(x) \\
&= x^{\rho+\sigma} f(x) + \sum_{i=1}^{r} \alpha_{r,i}(\sigma)\Big[1 + \sum_{k=1, k-i \neq \rho}^{r} \frac{\alpha_{r,k}(\rho)}{\rho - k + i}\Big] x^{\rho+i-1} \int_1^x y^{-i+\sigma} f(y)\, dy \\
&\quad + \sum_{k=1}^{r} \Big[\alpha_{r,k}(\rho) - \sum_{i=1, k-i \neq \rho}^{r} \frac{\alpha_{r,i}(\sigma)\alpha_{r,k}(\rho)}{\rho - k + i}\Big] x^{k-1} \int_1^x y^{-k+\rho+\sigma} f(y)\, dy \\
&= x^{\rho+\sigma} f(x) + \sum_{i=1}^{r} \alpha_{r,i}(\sigma)\Big[1 + \sum_{k=1}^{r} \frac{\alpha_{r,k}(\rho)}{\rho - k + i}\Big] x^{\rho+i-1} \int_1^x y^{-i+\sigma} f(y)\, dy \\
&\quad + \sum_{k=1}^{r} \Big[\alpha_{r,k}(\rho) - \sum_{i=1}^{r} \frac{\alpha_{r,i}(\sigma)\alpha_{r,k}(\rho)}{\rho - k + i}\Big] x^{k-1} \int_1^x y^{-k+\rho+\sigma} f(y)\, dy \\
&= x^{\rho+\sigma} f(x) + \sum_{k=1}^{r} \Big[\alpha_{r,k}(\rho) - \sum_{i=1}^{r} \frac{\alpha_{r,i}(\sigma)\alpha_{r,k}(\rho)}{\rho - k + i}\Big] x^{k-1} \int_1^x y^{-k+\rho+\sigma} f(y)\, dy,
\end{aligned}$$

where we have applied (3.10) in the last equality. If we let $f(x) = 1$ in the last relation and in (3.1) for $A(\rho + \sigma)$ we get (3.11) for an irrational $\rho + \sigma$ and then by continuity for any $\rho, \sigma \in \mathbb{R}$. □

Remark 3.1. In the proof of Proposition 3.1 we have actually shown that (3.10), (3.11) are equivalent to Theorem 3.2.

Now, we give some boundedness properties of $A(\rho)$.

Proposition 3.2. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\gamma > -1/p$, $\rho \in \mathbb{R}$. Then for every $f \in L_p(\chi_0^{\rho+\gamma})[0,b]$ we have*

$$\|\chi_0^{\gamma} A(\rho) f\|_{p,[0,b]} \le C \|\chi_0^{\rho+\gamma} f\|_{p,[0,b]}.$$

Also for every $\tau \in \mathbb{R}$, measurable and non-negative φ and $g \in AC_{loc}^{r-1}$ we have

$$\|\chi_0^{\tau} \varphi^r (A(\rho) g)^{(r)}\|_{p,[0,b]} = \|\chi_0^{\tau+\rho} \varphi^r g^{(r)}\|_{p,[0,b]}.$$

Proof. The second statement follows immediately from (3.7). From Corollary 2.1 we get $\|\chi_0^{\gamma} \psi_k\|_p \le C \|\chi_0^{\rho+\gamma} f\|_p$ for $k = 1, 2, \dots, r$. Now the first statement follows from (3.1). $\square$

From Proposition 3.2 applied for $A(\rho)$ and $A(-\rho)$, Corollary 3.2 and Proposition 2.1 we get

Proposition 3.3. *Let φ be a non-negative weight on $[0,b]$, $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $\gamma > -1/p$ and $\gamma + \rho > -1/p$. Then we have*

$$A(\rho) : (L_p(\chi_0^{\rho+\gamma}), AC_{loc}^{r-1}, \varphi^r D^r) \mapsto (L_p(\chi_0^{\gamma}), AC_{loc}^{r-1}, \varphi^r D^r).$$

4 Operators that change the weight in the second term of the K-functional

Let $r \in \mathbb{N}$ be fixed. For $\sigma \neq 0$ we define the operator $B(\sigma) : L_{1,loc}(0,\infty) \to L_{1,loc}(0,\infty)$ by

$$(B(\sigma) f)(x) = f(x^{\sigma}) + \sum_{k=1}^{r} \beta_{r,k}(\sigma) \psi_k(x), \tag{4.1}$$

where

$$\psi_k(x) = x^{k-1} \int_1^x y^{-k} f(y^{\sigma})\, dy, \quad k = 1, 2, \dots \tag{4.2}$$

and

$$\beta_{r,1}(\sigma) \equiv 0, \quad \beta_{r,k}(\sigma) = \frac{(-1)^{r-k}}{(r-2)!} \binom{r-2}{k-2} \prod_{i=1}^{r-1} (k-1-i\sigma), \quad k = 2, 3, \dots, r. \tag{4.3}$$

Obviously, operators of type (4.1), (4.2) preserve the smoothness properties of the functions on intervals $[a,b] \subset (0,\infty)$.

In this section we apply an alternative method for studying the properties of the operator than the one in the previous section.

Theorem 4.1. *Let $r \in \mathbb{N}$ and $\sigma \in \mathbb{R}\backslash\{0\}$. Then for every $f \in AC_{loc}^{r-1}(0,\infty)$ we have*

$$(B(\sigma)f)^{(r)}(x) = \sigma^r x^{r(\sigma-1)} f^{(r)}(x^\sigma) \qquad a.e. \tag{4.4}$$

In order to prove Theorem 4.1 we need two auxiliary statements.

Lemma 4.1. *For $\sigma \neq 0$ there are unique numbers $\beta_{r,k} = \beta_{r,k}(\sigma)$, $k = 1, \ldots, r$, such that $B(\sigma)$ defined in (4.1), (4.2) satisfies (4.4) for any $f \in AC_{loc}^{r-1}(0,\infty)$. Moreover, $\beta_{r,k}(\sigma)$ are continuous functions of $\sigma \in \mathbb{R}\backslash\{0\}$ and $\beta_{r,1} \equiv 0$.*

Proof. It is enough to show that there exist unique numbers $\beta'_{r,i} = \beta'_{r,i}(\sigma)$, $i = 1, \ldots, r$, which are continuous functions of $\sigma \neq 0$, such that

$$(B(\sigma)f)(x) = f(x^\sigma) + \sum_{i=1}^{r} \beta'_{r,i} \Psi_i(x) \tag{4.5}$$

satisfies (4.4), where

$$\Psi_i(x) = \frac{1}{(i-1)!} \int_1^x (x-y)^{i-1} y^{-i} f(y^\sigma)\, dy = \sum_{k=1}^{i} \frac{(-1)^{i-k}}{(i-1)!} \binom{i-1}{k-1} \psi_k(x)$$

for $i = 1, 2, \ldots$. Then

$$\beta_{r,k} = \sum_{i=k}^{r} \frac{(-1)^{i-k}}{(i-1)!} \binom{i-1}{k-1} \beta'_{r,i}.$$

We use the following formula for the k-th derivative ($k \geq 1$) of a composition of functions

$$(f \circ \theta)^{(k)} = \sum_{j=1}^{k} \frac{f^{(j)}(\theta)}{j!} \sum_{l=0}^{j-1} (-1)^l \binom{j}{l} (\theta^{j-l})^{(k)} \theta^l. \tag{4.6}$$

Using the above formula we have

$$\Psi_i^{(r)}(x) = \left(\Psi_i^{(i)}(x)\right)^{(r-i)} = (x^{-i} f(x^\sigma))^{(r-i)} = \sum_{k=0}^{r-i} \binom{r-i}{k} (x^{-i})^{(r-i-k)} (f(x^\sigma))^{(k)}$$

$$= (-1)^{r-i} \frac{(r-1)!}{(i-1)!} x^{-r} f(x^\sigma)$$

$$+ \sum_{k=1}^{r-i} \binom{r-i}{k} (-1)^{r-i-k} \frac{(r-k-1)!}{(i-1)!} x^{k-r} \sum_{j=1}^{k} \frac{f^{(j)}(x^\sigma)}{j!} \sum_{l=0}^{j-1} (-1)^l \binom{j}{l} (x^{\sigma(j-l)})^{(k)} x^{\sigma l}$$

$$= \sum_{k=1}^{r-i} \sum_{j=1}^{k} \sum_{l=0}^{j-1} (-1)^{r-i-k+l} \binom{r-i}{k} \binom{j}{l} \frac{(r-k-1)!}{j!\,(i-1)!} \prod_{n=0}^{k-1} [\sigma(j-l)-n] x^{\sigma j - r} f^{(j)}(x^\sigma)$$

$$+ (-1)^{r-i} \frac{(r-1)!}{(i-1)!} x^{-r} f(x^\sigma) = \sum_{j=0}^{r-i} a_{r,i,j}(\sigma) x^{\sigma j - r} f^{(j)}(x^\sigma),$$

where we have put $a_{r,i,0}(\sigma) = (-1)^{r-i}\frac{(r-1)!}{(i-1)!}$ and

$$a_{r,i,j}(\sigma) = \sum_{k=j}^{r-i}\sum_{l=0}^{j-1}(-1)^{r-i-k+l}\binom{r-i}{k}\binom{j}{l}\frac{(r-k-1)!}{j!\,(i-1)!}\prod_{n=0}^{k-1}[\sigma(j-l)-n]$$

for $j = 1, \ldots, r-i$. Hence we get from (4.5) and (4.6)

$$\begin{aligned}
(B(\sigma)f)^{(r)}(x) &= (f(x^\sigma))^{(r)} + \sum_{i=1}^{r}\beta'_{r,i}\Psi_i^{(r)}(x)\\
&= \sum_{j=1}^{r}\frac{f^{(j)}(x^\sigma)}{j!}\sum_{l=0}^{j-1}(-1)^l\binom{j}{l}\prod_{n=0}^{r-1}[\sigma(j-l)-n]x^{\sigma j-r}\\
&\quad+\sum_{i=1}^{r}\beta'_{r,i}\sum_{j=0}^{r-i}a_{r,i,j}(\sigma)x^{\sigma j-r}f^{(j)}(x^\sigma)\\
&= b_{r,r}(\sigma)x^{r(\sigma-1)}f^{(r)}(x^\sigma)\\
&\quad+\sum_{j=1}^{r-1}\left(\sum_{i=1}^{r-j}a_{r,i,j}(\sigma)\beta'_{r,i}+b_{r,j}(\sigma)\right)x^{\sigma j-r}f^{(j)}(x^\sigma),
\end{aligned}$$

where we have used the notation

$$b_{m,j}(\sigma) = \frac{1}{j!}\sum_{l=0}^{j-1}(-1)^l\binom{j}{l}\prod_{n=0}^{m-1}[\sigma(j-l)-n].$$

Next we observe that $b_{r,r}(\sigma) = r!\sigma^r/r! = \sigma^r$ as the sum above with $m = j = r$ turns into the r-th finite difference with step 1 of the r-th degree algebraic polynomial $\sigma x(\sigma x-1)\cdots(\sigma x-r+1)$ at the point 0. Further, we have $a_{r,r-j,j}(\sigma) = b_{j,j}(\sigma) = \sigma^j \neq 0$ for $j = 0, \ldots, r-1$ and then $\beta'_{r,i}$, $i = 1, \ldots, r$, are the unique solution of the triangular linear system

$$\sum_{i=1}^{r-j}a_{r,i,j}(\sigma)\beta'_{r,i} = -b_{r,j}(\sigma),\ j = 0, \ldots, r-1.$$

Moreover, Crammer's formulae imply that the solution of the linear system above consists of rational functions of σ, as $a_{r,i,j}(\sigma)$, $i = 1, \ldots, r-j$, $j = 0, \ldots, r-1$ and $b_{r,j}(\sigma)$, $j = 0, \ldots, r-1$ are polynomials of σ and the determinant of the system is $\sigma^{r(r-1)/2}$. Thus, $\beta'_{r,i}$ and hence $\beta_{r,i}$ exist, they are unique and they are continuous functions of $\sigma \neq 0$.

Finally, in order to prove $\beta_{r,1} \equiv 0$ we let in (4.1) $f \equiv 1$. Then, in view of (4.4) and (4.2), (4.1) implies $\beta_{r,1}\psi_1$ is an algebraic polynomial of degree $r-1$. But $\psi_1(x) = \log x$. Hence $\beta_{r,1}(\sigma) \equiv 0$. This completes the proof of the lemma. □

Lemma 4.2. *Let $a_i \neq a_k$ and $b_i \neq b_k$ for $i \neq k$, $i, k = 1, \dots, n$. If $a_i + b_j \neq 0$ for $i, j = 1, \dots, n$ then the linear system of equations*

$$\sum_{j=1}^{n} \frac{x_j}{a_i + b_j} = 1, \quad i = 1, \dots, n,$$

has a unique solution given by

$$x_j = \prod_{i=1}^{n} (a_i + b_j) \prod_{i=1, i \neq j}^{n} (b_j - b_i)^{-1}, \quad j = 1, \dots, n.$$

Proof. Set

$$\Delta(a_1, \dots, a_n; b_1, \dots, b_n) = \det \left(\frac{1}{a_i + b_j} \right)_{i,j=1}^{n}.$$

The following relation, known as Cauchy theorem, holds (see for example [19, p. 327])

$$\Delta(a_1, \dots, a_n; b_1, \dots, b_n) = \prod_{1 \leq i < k \leq n} [(a_k - a_i)(b_k - b_i)] \prod_{i,k=1}^{n} (a_i + b_k)^{-1}. \tag{4.7}$$

(4.7) shows that the system has a unique solution. The proof of (4.7) is by induction. In order to obtain a recursive relation for the determinants one subtracts the last row from the previous ones, takes the common multipliers out of the determinant and repeat the same with the last column. Crammer's formulae (with similar computations for the numerators) yield that the solution x_j, $j = 1, \dots, n$ of the system is given by

$$x_j = \prod_{i=1}^{n-1} (a_n - a_i) \prod_{k=1, k \neq j}^{n} (a_n + b_k)^{-1} (-1)^{n+j} \times \frac{\Delta(a_1, \dots, a_{n-1}; b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n)}{\Delta(a_1, \dots, a_n; b_1, \dots, b_n)},$$

which proves the lemma in view of (4.7). □

Proof of Theorem 4.1. Let $\beta_{r,k}$ be the coefficients from Lemma 4.1. Set $f_i(x) = x^{i-1}$ for $i = 2, 3, \dots, r$. Then (4.1), (4.2) for $(i-1)\sigma - k + 1 \neq 0$ and (4.4) implies

$$(B(\sigma) f_i)(x) = x^{\sigma(i-1)} + \sum_{k=2}^{r} \beta_{r,k} \frac{x^{\sigma(i-1)} - x^{k-1}}{(i-1)\sigma - k + 1} \in \Pi_{r-1}.$$

Hence for irrational σ we have

$$\sum_{k=2}^{r} \frac{\beta_{r,k}}{k - 1 - (i-1)\sigma} = 1, \quad i = 2, 3, \dots, r. \tag{4.8}$$

From Lemma 4.2 with $j = k-1,\ n = r-1,\ a_i = -(i-1)\sigma,\ b_j = j$ we get that $\beta_{r,k}(\sigma)$, $k = 2,\dots,r$, from (4.3) are the solution of (4.8). This proves the theorem for irrational σ in view of the uniqueness in Lemma 4.1 and Lemma 4.2. For rational $\sigma \neq 0$ the statement follows from the continuous dependance of $\beta_{r,k}(\sigma)$ on σ in (4.3) and Lemma 4.1. $\square$

Remark 4.1. In the proof we have established that $\beta_{r,k}(\sigma)$, $k = 2,\dots,r$, from (4.3) satisfy (4.8) for every $\sigma \neq 0$ if we understand that the ratio in (4.8) is defined by continuity when $\sigma = (k-1)/(i-1)$ for some $i, k = 2, 3, \dots, r$, i.e.

$$\left.\frac{\beta_{r,k}(\sigma)}{k-1-(i-1)\sigma}\right|_{\sigma=(k-1)/(i-1)} = (-1)^{k+i}\frac{(r-i)!(i-2)!(k-1)^{r-2}}{(r-k)!(k-2)!(i-1)^{r-2}}. \tag{4.9}$$

Theorem 4.2. *Let $r \in \mathbb{N}$, $\rho, \sigma \in \mathbb{R}\backslash\{0\}$. Then $B(\sigma)B(\rho) = B(\sigma\rho)$.*

Proof. Applying twice (4.1) we have for every $f \in L_{1,loc}(0,\infty)$

$$\begin{aligned}
&(B(\sigma)B(\rho)f)(x)\\
&= (B(\rho)f)(x^\sigma) + \sum_{k=2}^{r}\beta_{r,k}(\sigma)x^{k-1}\int_1^x y^{-k}(B(\rho)f)(y^\sigma)\,dy\\
&= f(x^{\sigma\rho}) + \sum_{l=2}^{r}\beta_{r,l}(\rho)x^{\sigma(l-1)}\int_1^{x^\sigma} y^{-l}f(y^\rho)\,dy\\
&+ \sum_{k=2}^{r}\beta_{r,k}(\sigma)x^{k-1}\int_1^x y^{-k}\left(f(y^{\sigma\rho}) + \sum_{l=2}^{r}\beta_{r,l}(\rho)y^{\sigma(l-1)}\int_1^{y^\sigma} u^{-l}f(u^\rho)\,du\right)dy\\
&= f(x^{\sigma\rho}) + \sum_{l=2}^{r}\sigma\beta_{r,l}(\rho)x^{\sigma(l-1)}\int_1^{x} u^{-\sigma(l-1)-1}f(u^{\sigma\rho})\,du\\
&+ \sum_{k=2}^{r}\beta_{r,k}(\sigma)x^{k-1}\int_1^x y^{-k}f(y^{\sigma\rho})\,dy\\
&+ \sum_{k=2}^{r}\sum_{l=2}^{r}\sigma\beta_{r,k}(\sigma)\beta_{r,l}(\rho)x^{k-1}\int_1^x y^{\sigma(l-1)-k}\left(\int_1^{y} u^{-\sigma(l-1)-1}f(u^{\sigma\rho})\,du\right)dy.
\end{aligned}$$

If $\sigma(l-1) = k-1$ then $\beta_{r,k}(\sigma) = 0$ and if $\sigma(l-1) \neq k-1$ then

$$\begin{aligned}
x^{k-1}\int_1^x y^{\sigma(l-1)-k}\left(\int_1^y u^{-\sigma(l-1)-1}f(u^{\sigma\rho})\,du\right)dy &= \frac{1}{\sigma(l-1)-(k-1)}\\
\times\left(x^{\sigma(l-1)}\int_1^x u^{-\sigma(l-1)-1}f(u^{\sigma\rho})\,du - x^{k-1}\int_1^x u^{-k}f(u^{\sigma\rho})\,du\right)&.
\end{aligned}$$

Hence

$$
\begin{aligned}
(B(\sigma)B(\rho)f)(x) &= f(x^{\sigma\rho})\\
&\quad+\sum_{l=2}^{r}\Bigg(\sigma\beta_{r,l}(\rho)+\sigma\sum_{\substack{k=2\\ k-1\neq\sigma(l-1)}}^{r}\frac{\beta_{r,l}(\rho)\beta_{r,k}(\sigma)}{\sigma(l-1)-(k-1)}\Bigg)\\
&\qquad\qquad x^{\sigma(l-1)}\int_1^x u^{-\sigma(l-1)-1}f(u^{\sigma\rho})\,du\\
&\quad+\sum_{k=2}^{r}\Bigg(\beta_{r,k}(\sigma)-\sigma\sum_{\substack{l=2\\ \sigma(l-1)\neq k-1}}^{r}\frac{\beta_{r,k}(\sigma)\beta_{r,l}(\rho)}{\sigma(l-1)-(k-1)}\Bigg)x^{k-1}\int_1^x u^{-k}f(u^{\sigma\rho})\,du.
\end{aligned}
$$

Adding and subtracting in the above the terms for $k=\sigma(l-1)+1$ with the convention (4.9) and applying (4.8) afterwards, we get for every $f\in L_{1,loc}(0,\infty)$

(4.10)

$$
\begin{aligned}
&(B(\sigma)B(\rho)f)(x)\\
&=f(x^{\sigma\rho})+\sum_{k=2}^{r}\Bigg(\beta_{r,k}(\sigma)-\sigma\sum_{l=2}^{r}\frac{\beta_{r,k}(\sigma)\beta_{r,l}(\rho)}{\sigma(l-1)-(k-1)}\Bigg)x^{k-1}\int_1^x u^{-k}f(u^{\sigma\rho})\,du.
\end{aligned}
$$

On the other hand for any $f\in AC_{loc}^{r-1}(0,\infty)$ (4.4) implies

$$
\begin{aligned}
(B(\sigma)B(\rho)f)^{(r)}(x) &= \sigma^r x^{r(\sigma-1)}(B(\rho)f)^{(r)}(x^\sigma)=\sigma^r x^{r(\sigma-1)}\rho^r x^{\sigma r(\rho-1)}f^{(r)}(x^{\sigma\rho})\\
&=(\sigma\rho)^r x^{r(\sigma\rho-1)}f^{(r)}(x^{\sigma\rho})=(B(\sigma\rho)f)^{(r)}(x).
\end{aligned}
$$

Hence $B(\sigma)B(\rho)f-B(\sigma\rho)f\in\Pi_{r-1}$. Thus, (4.10) and (4.1) imply

$$
\sum_{k=2}^{r}\Bigg(\beta_{r,k}(\sigma)-\sigma\sum_{l=2}^{r}\frac{\beta_{r,k}(\sigma)\beta_{r,l}(\rho)}{\sigma(l-1)-(k-1)}-\beta_{r,k}(\sigma\rho)\Bigg)x^{k-1}\int_1^x u^{-k}f(u^{\sigma\rho})\,du
$$

is an algebraic polynomial of degree at most $r-1$ for any $f\in AC_{loc}^{r-1}(0,\infty)$. If we put $f(x)=x^{r/\sigma\rho}\exp(x^{1/\sigma\rho})$, $f\in AC_{loc}^{r-1}(0,\infty)$, in the above relation and use that the system of functions $\{x^k\}_{k=0}^{r-1}\cup\{x^{k-1}\int_1^x u^{r-k}e^u\,du\}_{k=2}^{r}$ is linearly independent we get

$$
\beta_{r,k}(\sigma)-\sigma\sum_{l=2}^{r}\frac{\beta_{r,k}(\sigma)\beta_{r,l}(\rho)}{\sigma(l-1)-(k-1)}=\beta_{r,k}(\sigma\rho),\quad k=2,\dots,r. \tag{4.11}
$$

Finally, using (4.11) in (4.10) and comparing with (4.1), we prove the theorem. □

As an immediate corollary of the last theorem, we get the following important property of the linear operators $B(\sigma)$.

Corollary 4.1. $\{B(\sigma)\}_{\sigma\in\mathbb{R}\setminus\{0\}}$ *and* $\{B(\sigma)\}_{\sigma\in(0,\infty)}$ *are commutative groups of operators with* $B(1)$ *as the identity element. In particular* $B(\sigma)^{-1} = B(\sigma^{-1})$.

From Theorem 4.1 and Corollary 4.1 we get

Corollary 4.2. *Let* $r \in \mathbb{N}$ *and* $\sigma \neq 0$. *Then* $B(\sigma)(\Pi_{r-1}) = \Pi_{r-1}$ *and* $B(\sigma)(AC_{loc}^{k-1}) = AC_{loc}^{k-1}$ *for any* $k \in \mathbb{N}$.

We can change the order of applying the operators A and B as follows.

Proposition 4.1. *Let* $r \in \mathbb{N}$, $\rho \in \mathbb{R}$ *and* $\sigma \neq 0$. *Then* $B(\sigma)A(\rho) = A(\rho\sigma)B(\sigma)$.

Proof. As in the proof of Theorem 4.2 using (3.10), we get for any $f \in L_{1,loc}(0,\infty)$

$$(4.12)\quad \begin{aligned}(A(\rho\sigma)B(\sigma)f)(x) &= x^{\rho\sigma} f(x^\sigma)\\ &+ \sum_{k=1}^{r}\Big(\alpha_{r,k}(\rho\sigma) - \sum_{i=2}^{r} \frac{\alpha_{r,k}(\rho\sigma)\beta_{r,i}(\sigma)}{\rho\sigma - k + i}\Big) x^{k-1} \int_1^x y^{\rho\sigma-k} f(y^\sigma)\,dy.\end{aligned}$$

Similarly, using (4.8), we get

$$(4.13)\quad \begin{aligned}(B(\sigma)A(\rho)f)(x) &= x^{\rho\sigma} f(x^\sigma) + \sigma\alpha_{r,1}(\rho)\Big(1 - \sum_{i=2}^{r} \frac{\beta_{r,i}(\sigma)}{i-1}\Big) \int_1^x y^{\rho\sigma-1} f(y^\sigma)\,dy\\ &+ \sum_{k=2}^{r}\Big(\beta_{r,k}(\sigma) + \sigma \sum_{i=1}^{r} \frac{\alpha_{r,i}(\rho)\beta_{r,k}(\sigma)}{k-1-\sigma(i-1)}\Big) x^{k-1} \int_1^x y^{\rho\sigma-k} f(y^\sigma)\,dy.\end{aligned}$$

For any $f \in AC_{loc}^{r-1}(0,\infty)$ (3.7) and (4.4) imply

$$\begin{aligned}(A(\rho\sigma)B(\sigma)f)^{(r)}(x) &= x^{\rho\sigma}(B(\sigma)f)^{(r)}(x) = \sigma^r x^{\rho\sigma + r(\sigma-1)} f^{(r)}(x^\sigma)\\ &= \sigma^r x^{r(\sigma-1)}(A(\rho)f)^{(r)}(x^\sigma) = (B(\sigma)A(\rho)f)^{(r)}(x).\end{aligned}$$

Hence $A(\rho\sigma)B(\sigma)f - B(\sigma)A(\rho)f \in \Pi_{r-1}$. Thus, as in the proof of Theorem 4.2, if we put $f(x) = x^{-\rho + r/\sigma}\exp(x^{1/\sigma})$ then (4.12) and (4.13) imply

$$(4.14)\quad \alpha_{r,1}(\rho\sigma) - \sum_{i=2}^{r} \frac{\alpha_{r,1}(\rho\sigma)\beta_{r,i}(\sigma)}{\rho\sigma - 1 + i} = \sigma\alpha_{r,1}(\rho)\Big(1 - \sum_{i=2}^{r} \frac{\beta_{r,i}(\sigma)}{i-1}\Big)$$

and for $k = 2, 3, \ldots, r$

$$(4.15)\quad \alpha_{r,k}(\rho\sigma) - \sum_{i=2}^{r} \frac{\alpha_{r,k}(\rho\sigma)\beta_{r,i}(\sigma)}{\rho\sigma - k + i} = \beta_{r,k}(\sigma) + \sigma \sum_{i=1}^{r} \frac{\alpha_{r,i}(\rho)\beta_{r,k}(\sigma)}{k-1-\sigma(i-1)}.$$

Now, replacing (4.14) and (4.15) in (4.12) and (4.13) we complete the proof. $\square$

Now, we give some boundedness properties of $B(\sigma)$.

Proposition 4.2. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\gamma > -1 - 1/p$, $\sigma > 0$. Then with $\lambda = 1 - 1/\sigma$ for every $f \in L_p(\chi_0^{\gamma-\lambda(\gamma+1/p)})[0, b^\sigma]$ we have*

$$\|\chi_0^\gamma B(\sigma) f\|_{p,[0,b]} \le C \|\chi_0^{\gamma-\lambda(\gamma+1/p)} f\|_{p,[0,b^\sigma]}.$$

Also for every $\tau \in \mathbb{R}$ and $g \in AC_{loc}^{r-1}$ we have

$$\|\chi_0^\tau (B(\sigma) g)^{(r)}\|_{p,[0,b]} = \sigma^{r-1/p} \|\chi_0^{\tau-\lambda(\tau+1/p)} \chi_0^{\lambda r} g^{(r)}\|_{p,[0,b^\sigma]}.$$

Proof. The second statement follows immediately from (4.4). From Corollary 2.1 we get $\|\chi_0^\gamma \psi_k\|_p \le C \|\chi_0^{\gamma-\lambda(\gamma+1/p)} f\|_p$ for $\gamma + k - 1 > -1/p$, $k = 2, \ldots, r$. Now the first statement follows from (4.1) and (4.3). $\square$

From Proposition 4.2 applied for $B(\sigma)$, $\tau = \gamma$ and for $B(1/\sigma)$, $\tau = \gamma - (\gamma + 1/p)(1 - 1/\sigma)$, Corollary 4.2 and Proposition 2.1 we get

Proposition 4.3. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $\gamma > -1 - 1/p$ and $\gamma > -\sigma - 1/p$. Then with $\lambda = 1 - 1/\sigma$ we have*

$$B(\sigma) : (L_p(\chi_0^{\gamma-\lambda(\gamma+1/p)}), AC_{loc}^{r-1}, \chi_0^{\lambda r} D^r) \mapsto (L_p(\chi_0^\gamma), AC_{loc}^{r-1}, D^r).$$

5 Compositions of operators

In this section we combine the results from the previous two sections and construct operators acting on spaces of functions defined on a given finite interval $[a, b]$.

First, we observe that one can replace the point 1 in the definitions of ψ_k in (3.2) and (4.2) by any other point ξ inside $(0, \infty)$ without affecting the properties of operators A and B. Indeed, such a change adds $r-1$-st degree polynomials to $A(\rho) f$ and $B(\sigma) f$ and hence Theorem 3.1 and Theorem 4.1 remain true. On the other hand the group properties are also valid because they depend only on combinatorial identities (one only has to replace (3.8) with (3.6) for $x = \xi$).

Second, Proposition 4.2 shows that it is convenient (but not necessary) to work with changes of the variable that keep unchanged the domain of the functions.

Definition 5.1. Let $r \in \mathbb{N}$. For a given finite interval $[a, b]$ let s be one of the points a or b, let e be the other point and let $\xi \in [a, b]$, $\xi \ne s$ be fixed. For every function f which is integrable on any $[c, d] \subset [a, b]$, $c \ne s \ne d$, and every $x \in [a, b]$ we set

$$(5.1) \qquad (A(\rho; s, e; \xi) f)(x) = \left(\frac{x-s}{e-s}\right)^\rho f(x)$$
$$+ \frac{1}{e-s} \sum_{k=1}^{r} \alpha_{r,k}(\rho) \left(\frac{x-s}{e-s}\right)^{k-1} \int_\xi^x \left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y) dy,$$

where $\alpha_{r,k}(\rho)$ are given in (3.3), $\rho \in \mathbb{R}$, and

$$(B(\sigma; s, e; \xi)f)(x) = f\left(s + (e-s)\left(\frac{x-s}{e-s}\right)^{\sigma}\right) \tag{5.2}$$

$$+\frac{1}{e-s}\sum_{k=2}^{r}\beta_{r,k}(\sigma)\left(\frac{x-s}{e-s}\right)^{k-1}\int_{\xi}^{x}\left(\frac{y-s}{e-s}\right)^{-k} f\left(s + (e-s)\left(\frac{y-s}{e-s}\right)^{\sigma}\right) dy,$$

where $\beta_{r,k}(\sigma)$ are given in (4.3), $\sigma > 0$.

Thus, the operators in (5.1) and (5.2) are designed to treat singularities of the weights at the point s (which is not necessarily the left end of the interval) starting the integration from any point $\xi \neq s$. For $s = 0, e \geq 1, \xi = 1$ operators $A(\rho; s, e; \xi)$ and $B(\sigma; s, e; \xi)$ reduce to $A(\rho)$ and $B(\sigma)$ from Sections 3 and 4. If, for example, we would like to treat a singularity at 1 for a function defined in $[0, 1]$ then for a fixed $\xi \in (0, 1)$ (5.1) and (5.2) become

$$(A(\rho; 1, 0; \xi)f)(x) = (1-x)^{\rho} f(x) - \sum_{k=1}^{r} \alpha_{r,k}(\rho)(1-x)^{k-1}\int_{\xi}^{x}(1-y)^{-k+\rho} f(y)dy,$$

$$(B(\sigma; 1, 0; \xi)f)(x)$$

$$= f(1-(1-x)^{\sigma}) - \sum_{k=2}^{r}\beta_{r,k}(\sigma)(1-x)^{k-1}\int_{\xi}^{x}(1-y)^{-k} f(1-(1-y)^{\sigma})\, dy.$$

The main algebraic properties of A and B are given by

Theorem 5.1. *Let $r \in \mathbb{N}$, $\rho, \sigma \in \mathbb{R}$. Then:*
a) for every $f \in AC_{loc}^{r-1}(a, b)$ we have

$$(A(\rho; s, e; \xi)f)^{(r)}(x) = \left(\frac{x-s}{e-s}\right)^{\rho} f^{(r)}(x) \qquad a.e.$$

b) $A(\rho; s, e; \xi)A(\sigma; s, e; \xi) = A(\rho + \sigma; s, e; \xi)$.

Theorem 5.2. *Let $r \in \mathbb{N}$, $\sigma, \rho > 0$. Then:*
a) for every $f \in AC_{loc}^{r-1}(a, b)$ we have

$$(B(\sigma; s, e; \xi)f)^{(r)}(x) = \sigma^{r}\left(\frac{x-s}{e-s}\right)^{r(\sigma-1)} f^{(r)}\left(s + (e-s)\left(\frac{x-s}{e-s}\right)^{\sigma}\right) \qquad a.e.$$

b) $B(\rho; s, e; \xi)B(\sigma; s, e; \xi) = B(\rho\sigma; s, e; \xi)$.

From Theorem 5.1 and Theorem 5.2 we get

Corollary 5.1. *Let $r \in \mathbb{N}$, $\rho \in \mathbb{R}$ and $\sigma > 0$. Then*

$$A(\rho; s, e; \xi)(\Pi_{r-1}) = \Pi_{r-1}; \qquad A(\rho; s, e; \xi)(AC_{loc}^{k-1}) = AC_{loc}^{k-1}, \quad \forall k \in \mathbb{N};$$

$$B(\sigma; s, e; \xi)(\Pi_{r-1}) = \Pi_{r-1}; \qquad B(\sigma; s, e; \xi)(AC_{loc}^{k-1}) = AC_{loc}^{k-1}, \quad \forall k \in \mathbb{N}.$$

In addition to the group properties from Sections 3 and 4 we observe that operators (5.1) commute in the following way.

Proposition 5.1. *Let* $r \in \mathbb{N}$, $\rho, \sigma \in \mathbb{R}$ *and* $\xi \in (a,b)$. *Then*

$$A(\rho; a, b; \xi)A(\sigma; b, a; \xi) = A(\sigma; b, a; \xi)A(\rho; a, b; \xi).$$

Proof. For every $f \in AC_{loc}^{r-1}(a,b)$ applying twice Theorem 5.1 a) we get

$$(A(\rho; a, b; \xi)A(\sigma; b, a; \xi)f)^{(r)}(x) = \left(\frac{x-a}{b-a}\right)^{\rho}\left(\frac{b-x}{b-a}\right)^{\sigma} f^{(r)}(x).$$

Hence

$$(A(\rho; a, b; \xi)A(\sigma; b, a; \xi)f)^{(r)}(x) = (A(\sigma; b, a; \xi)A(\rho; a, b; \xi)f)^{(r)}(x). \tag{5.3}$$

Set $v[F,\xi] = (F(\xi), F'(\xi), \dots, F^{(r-1)}(\xi))^T$. Then Lemma 3.2 and (5.1) give

$$v[A(\rho; a, b; \xi)f, \xi] = M(\rho; a, b; \xi)v[f, \xi],$$

where $M(\rho; a, b; \xi) = (\mu_{k,l}(\rho; a, b; \xi))_{k,l=1}^r$, $\mu_{k,l}(\rho; a, b; \xi) = 0$ for $k < l$ and

$$\mu_{k,l}(\rho; a, b; \xi) = (-1)^{k-l}\binom{r-l}{k-l}\prod_{\nu=0}^{k-l-1}(\rho-\nu)\left(\frac{\xi-a}{b-a}\right)^{\rho-k+l}(b-a)^{l-k}, \quad l \le k.$$

Hence

$$v[A(\rho; a, b; \xi)A(\sigma; b, a; \xi)f, \xi] = M(\rho; a, b; \xi)M(\sigma; b, a; \xi)v[f, \xi].$$

From the definition of $\mu_{k,l}$ we observe that:

i) $\mu_{i,n}(\rho; a, b; \xi)\mu_{n,j}(\sigma; b, a; \xi) = 0$ if $n < j$ or $i < n$;

ii) $\mu_{i,n}(\rho; a, b; \xi)\mu_{n,j}(\sigma; b, a; \xi) = \mu_{i,i+j-n}(\sigma; b, a; \xi)\mu_{i+j-n,j}(\rho; a, b; \xi)$ if $j \le n \le i$.

Hence

$$\sum_{n=1}^{r}\mu_{i,n}(\rho; a, b; \xi)\mu_{n,j}(\sigma; b, a; \xi) = \sum_{n=1}^{r}\mu_{i,n}(\sigma; b, a; \xi)\mu_{n,j}(\rho; a, b; \xi)$$

and

$$M(\rho; a, b; \xi)M(\sigma; b, a; \xi) = M(\sigma; b, a; \xi)M(\rho; a, b; \xi).$$

Therefore

$$v[A(\rho; a, b; \xi)A(\sigma; b, a; \xi)f, \xi] = v[A(\sigma; b, a; \xi)A(\rho; a, b; \xi)f, \xi].$$

Thus, for every $f \in AC_{loc}^{r-1}(a,b)$ applying (5.3) and the above equality in the Taylor formula we get $A(\rho; a, b; \xi)A(\sigma; b, a; \xi)f = A(\sigma; b, a; \xi)A(\rho; a, b; \xi)f$. Now we complete the proof using the boundedness of the linear operators $A(\rho; a, b; \xi)A(\sigma; b, a; \xi)$ and $A(\sigma; b, a; \xi)A(\rho; a, b; \xi)$ and the density of $W_1^r[c,d]$ in $L_1[c,d]$ for any $a < c < d < b$. $\square$

Remark 5.1. Analogues of Proposition 5.1 with operators B cannot be true because we have for smooth f

$$(B(\rho;a,b;\xi)B(\sigma;b,a;\xi)f)^{(r)}(x) \neq (B(\sigma;b,a;\xi)B(\rho;a,b;\xi)f)^{(r)}(x)$$

and

$$(A(\rho;a,b;\xi)B(\sigma;b,a;\xi)f)^{(r)}(x) \neq (B(\sigma;b,a;\xi)A(\rho;a,b;\xi)f)^{(r)}(x).$$

In order to combine several operators of type (5.1) and (5.2) we have to prove their boundedness in weighted L_p spaces with more general weights than those in Proposition 3.2 and Proposition 4.2. In order to describe some conditions on the weight powers we give

Definition 5.2. $\Gamma_+(p) = (-1/p, +\infty)$ for $1 \leq p < \infty$ and $\Gamma_+(\infty) = [0, +\infty)$. $\Gamma_-(p) = (-\infty, 1-1/p)$ for $1 < p \leq \infty$ and $\Gamma_-(1) = (-\infty, 0]$.

Note that $\Gamma_-(p) \cap \Gamma_+(p)$ is a (semi-)open interval of length 1 for fixed p and $0 \in \Gamma_\pm(p)$ for every p.

Proposition 5.2. *Let $r \in \mathbb{N}$, $1 \leq p \leq \infty$, $w(x) = \chi_a^{\gamma_a}(x)\chi_b^{\gamma_b}(x) = (x-a)^{\gamma_a}(b-x)^{\gamma_b}$ with $\gamma_a > -1/p$, $\gamma_b \in \mathbb{R}$ and $\rho \in \mathbb{R}$. Then for every $f \in L_p(\chi_a^\rho w)[a,b]$ we have*

$$\|w(A(\rho;a,b;\xi)f)\|_{p,[a,b]} \leq C\|w\chi_a^\rho f\|_{p,[a,b]}, \tag{5.4}$$

where $\xi \in (a,b)$ if $\gamma_b \in \Gamma_+(p)$ and/or $\xi = b$ if $\gamma_b \in \Gamma_-(p)$. Also for every non-negative measurable ϕ in (a,b) and every $g \in AC_{loc}^{r-1}[a,b]$ we have

$$\|w\phi(A(\rho;a,b;\xi)g)^{(r)}\|_{p,[a,b]} = (b-a)^{-\rho}\|w\chi_a^\rho\phi g^{(r)}\|_{p,[a,b]}.$$

Proof. The second statement follows immediately from Theorem 5.1 a).

In order to prove the first statement we multiply (5.1) by w, take L_p norm and apply Minkowski's inequality according to the terms on the right-hand side of (5.1). The first norm is the norm on the right-hand side of (5.4). Every of the other norms $\|w\psi_k\|_{p,[a,b]}$, $k = 1, 2, \ldots, r$, where

$$\psi_k(x) = \left(\frac{x-a}{b-a}\right)^{k-1} \int_\xi^x \left(\frac{y-a}{b-a}\right)^{-k+\rho} f(y)dy,$$

is estimated by applying twice Corollary 2.1 – for the subintervals $[a,c]$ and $[c,b]$, where $c = \xi$ if $\gamma_b \in \Gamma_+(p)$ or $c = \frac{a+b}{2}$ if $\gamma_b \in \Gamma_-(p)$.

In $[a,c]$ we have $w(x) \sim (x-a)^{\gamma_a}$ and Corollary 2.1 b) with $\beta = \gamma = \gamma_a + k - 1 + 1/p > 0$ estimates $\|w\psi_k\|_{p,[a,c]}$ with the norm on the right-hand side of (5.4).

In $[c,b]$ we have $w(x) \sim (b-x)^{\gamma_b}$. If $\gamma_b \in \Gamma_+(p)$ then Corollary 2.1 b) with $\gamma = \gamma_b + 1/p > 0$, $\beta = \gamma - 1$ gives $\|w\psi_k\|_{p,[c,b]} \leq C\|wf\|_{p,[c,b]} \leq C\|w\chi_a^\rho f\|_{p,[a,b]}$. If $\gamma_b \in \Gamma_-(p)$ then Corollary 2.1 a) with $\beta = -\gamma_b + 1 - 1/p > 0$, $\gamma = \beta - 1$ gives $\|w\psi_k\|_{p,[c,b]} \leq C\|wf\|_{p,[c,b]} \leq C\|w\chi_a^\rho f\|_{p,[a,b]}$. This completes the proof. □

Proposition 5.3. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $w(x) = \chi_a^{\gamma_a}(x)\chi_b^{\gamma_b}(x) = (x-a)^{\gamma_a}(b-x)^{\gamma_b}$ with $\gamma_a > -1-1/p$, $\gamma_b \in \mathbb{R}$ and $\sigma > 0$. Then with $\lambda = 1 - 1/\sigma$ for every $f \in L_p(\chi_a^{-(\gamma_a+1/p)\lambda} w)[a,b]$ we have*

$$\|w(B(\sigma;a,b;\xi)f)\|_{p,[a,b]} \le C\|w\chi_a^{-(\gamma_a+1/p)\lambda} f\|_{p,[a,b]}, \tag{5.5}$$

where $\xi \in (a,b)$ if $\gamma_b \in \Gamma_+(p)$ and/or $\xi = b$ if $\gamma_b \in \Gamma_-(p)$. Also for every $\tau_a, \tau_b \in \mathbb{R}$, $\phi(x) = \chi_a^{\tau_a}(x)\chi_b^{\tau_b}(x) = (x-a)^{\tau_a}(b-x)^{\tau_b}$ and $g \in AC_{loc}^{r-1}[a,b]$ we have

$$\|w\phi(B(\sigma;a,b;\xi)g)^{(r)}\|_{p,[a,b]} \sim \|w\chi_a^{-(\gamma_a+1/p)\lambda}\phi\chi_a^{(r-\tau_a)\lambda} g^{(r)}\|_{p,[a,b]}.$$

Proof. The second statement follows from Theorem 5.2 a) by applying the change of the variable $((x-a)/(b-a))^\sigma = (y-a)/(b-a)$ and taking into account that $(b-a)^{1/\sigma} - (y-a)^{1/\sigma} \sim b - y$ for $y \in [a,b]$.

In order to prove the first statement we multiply (5.2) by w, take L_p norm and apply Minkowski's inequality according to the terms on the right-hand side of (5.2). The first norm is directly evaluated by the above mentioned change of the variable with the norm on the right-hand side of (5.5). Every of the other norms $\|w\psi_k\|_{p,[a,b]}$, $k = 2, 3, \dots, r$, where

$$\psi_k(x) = \left(\frac{x-a}{b-a}\right)^{k-1} \int_\xi^x \left(\frac{y-a}{b-a}\right)^{-k} f\left(a + (b-a)\left(\frac{y-a}{b-a}\right)^\sigma\right) dy,$$

is estimated by applying twice Corollary 2.1 – for the subintervals $[a,c]$ and $[b,c]$, where $c = \xi$ if $\gamma_b \in \Gamma_+(p)$ or $c = \frac{a+b}{2}$ if $\gamma_b \in \Gamma_-(p)$.

In $[a,c]$ we have $w(x) \sim (x-a)^{\gamma_a}$ and Corollary 2.1 b) with $\beta = \gamma = \gamma_a + k - 1 + 1/p > 0$ estimates $\|w\psi_k\|_{p,[a,c]}$ with the norm on the right-hand side of (5.5).

In $[c,b]$ we have $w(x) \sim (b-x)^{\gamma_b}$. If $\gamma_b \in \Gamma_+(p)$ then Corollary 2.1 b) with $\gamma = \gamma_b + 1/p > 0$, $\beta = \gamma - 1$ gives $\|w\psi_k\|_{p,[c,b]} \le C\|wf\|_{p,[\bar{c},b]} \le C\|w\chi_a^{-(\gamma_a+1/p)\lambda} f\|_{p,[a,b]}$ where $\bar{c} = a + (b-a)^{1-\sigma}(c-a)^\sigma$. If $\gamma_b \in \Gamma_-(p)$ then Corollary 2.1 a) with $\beta = -\gamma_b + 1 - 1/p > 0$, $\gamma = \beta - 1$ gives $\|w\psi_k\|_{p,[c,b]} \le C\|wf\|_{p,[\bar{c},b]} \le C\|w\chi_a^{-(\gamma_a+1/p)\lambda} f\|_{p,[a,b]}$. This completes the proof. $\square$

Proposition 5.2 shows the operator $A(\rho)$ clears the multiplier χ_a^ρ from the weights in both terms of the K-functional, where the weight in the first term is restricted by Hardy's inequality and there are practically no restrictions on the second term weight.

Proposition 5.3 shows the operator $B(\sigma)$ clears the multiplier $\chi_a^{(r-\tau_a)(1-1/\sigma)}$ from the second term of the K-functional, but also clears $\chi_a^{-(\gamma_a+1/p)(1-1/\sigma)}$ as an additional weight in both terms. Once more the weight in the first term is restricted by Hardy's inequality and there are practically no restrictions on the second term weight.

From Proposition 5.2 applied for $A(\rho)$ and $A(-\rho)$ (with weight $w(x) = (x-a)^{\gamma_a+\rho}(b-x)^{\gamma_b}$), Corollary 5.1 and Proposition 2.1 we get

Proposition 5.4. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\rho \in \mathbb{R}$, $w(x) = (x-a)^{\gamma_a}(b-x)^{\gamma_b}$ with $\gamma_a > -1/p$, $\gamma_a + \rho > -1/p$, $\gamma_b \in \mathbb{R}$ and let ϕ be measurable non-negative on $[a,b]$. Then we have*

$$A(\rho; a, b; \xi) : (L_p(\chi_a^\rho w), AC_{loc}^{r-1}, \phi D^r) \mapsto (L_p(w), AC_{loc}^{r-1}, \phi D^r),$$

where $\xi \in (a,b)$ if $\gamma_b \in \Gamma_+(p)$ and/or $\xi = b$ if $\gamma_b \in \Gamma_-(p)$.

From Proposition 5.3 applied for $B(\sigma)$ and $B(1/\sigma)$ (with $\gamma_a/\sigma - (1-1/\sigma)/p$ in the place of γ_a), Corollary 5.1 and Proposition 2.1 we get

Proposition 5.5. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$, $w(x) = (x-a)^{\gamma_a}(b-x)^{\gamma_b}$ with $\gamma_a > -1 - 1/p$, $\gamma_a > -\sigma - 1/p$, $\gamma_b \in \mathbb{R}$ and $\phi(x) = (x-a)^{\tau_a}(b-x)^{\tau_b}$ with $\tau_a, \tau_b \in \mathbb{R}$. Then we have with $\lambda = 1 - 1/\sigma$*

$$B(\sigma; a, b; \xi) : (L_p(\chi_a^{-(\gamma_a+1/p)\lambda} w), AC_{loc}^{r-1}, \phi\chi_a^{(r-\tau_a)\lambda} D^r) \mapsto (L_p(w), AC_{loc}^{r-1}, \phi D^r),$$

where $\xi \in (a,b)$ if $\gamma_b \in \Gamma_+(p)$ and/or $\xi = b$ if $\gamma_b \in \Gamma_-(p)$.

The previous statements show that one can treat separately the weight singularities at both ends of the domain. Of course, the propositions remain true if we interchange the places of a and b.

Remark 5.2. Comparing Proposition 5.5 (and Proposition 2.1) from one side with Theorem 6.5.1 in [5] and the results in [6] from another side one can find several advantages of the first statement:

- no additional terms in the equivalence relation;
- the use of K-functionals (or equivalent moduli) instead of main-part moduli (in Theorem 6.5.1);
- taking the restrictions on the parameters to natural boundaries. In the notations of Proposition 5.5 (with $\tau_a = \tau_b = 0$), the main restrictions in Theorem 6.5.1 and Theorem 1 in [6] are $\gamma_a > r - 1 - 1/p$, $r = 2, 3, \ldots$, provided $\sigma \ge 1$. It should be compared with the restriction $\gamma_a > -1 - 1/p$ in Proposition 5.5.

Now we are ready to combine operators (5.1) and (5.2) and to prove our main result.

Theorem 5.3. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\kappa_a, \kappa_b, \lambda_a, \lambda_b, \mu_a, \mu_b, \nu_a, \nu_b \in \mathbb{R}$, $\varphi(x) = (x-a)^{\lambda_a}(b-x)^{\lambda_b}$, $\bar{\varphi}(x) = (x-a)^{\nu_a}(b-x)^{\nu_b}$ with $(1-\lambda_a)(1-\nu_a) > 0$ and $(1-\lambda_b)(1-\nu_b) > 0$, $w(x) = (x-a)^{\kappa_a}(b-x)^{\kappa_b}$ with $-1/p < \kappa_a, \kappa_b$, $\bar{w}(x) = (x-a)^{\mu_a}(b-x)^{\mu_b}$ with $-1/p < \mu_a, \mu_b$. Set*

$$\mathcal{A} = A(\rho_b; b, a; \xi) B(\sigma_b; b, a; \xi) A(\rho_a; a, b; \xi) B(\sigma_a; a, b; \xi),$$

where $\xi \in (a,b)$ and

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_b = \frac{1-\nu_b}{1-\lambda_b}, \quad \rho_a = (\kappa_a + \frac{1}{p})\sigma_a - \frac{1}{p} - \mu_a, \quad \rho_b = (\kappa_b + \frac{1}{p})\sigma_b - \frac{1}{p} - \mu_b.$$

Then

$$\mathcal{A} : (L_p(w)[a,b], AC_{loc}^{r-1}, \varphi^r D^r) \mapsto (L_p(\bar{w})[a,b], AC_{loc}^{r-1}, \bar{\varphi}^r D^r).$$

Hence for $t > 0$ *and* $f \in L_p(w)[a,b]$ *we have*

$$K(f, t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{A}f, t^r; L_p(\bar{w}), AC_{loc}^{r-1}, \bar{\varphi}^r D^r).$$

Proof. First, note that $\sigma_a, \sigma_b > 0$. From Proposition 5.5 with $\tau_a = r\nu_a, \tau_b = r\lambda_b$, $\gamma_a = \rho_a + \mu_a, \gamma_b = \kappa_b$, $\sigma = \sigma_a$ we get

$$\begin{aligned} B(\sigma_a; a, b; \xi) : (L_p(\chi_a^{\kappa_a - \rho_a - \mu_a} \chi_a^{\rho_a + \mu_a} \chi_b^{\kappa_b}), AC_{loc}^{r-1}, \chi_a^{r\lambda_a} \chi_b^{r\lambda_b} D^r) \\ \mapsto (L_p(\chi_a^{\rho_a + \mu_a} \chi_b^{\kappa_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a} \chi_b^{r\lambda_b} D^r), \end{aligned}$$

because $\kappa_a - \rho_a - \mu_a = -(\rho_a + \mu_a + 1/p)(1 - \sigma_a^{-1})$ and $\rho_a + \mu_a + 1/p = (\kappa_a + 1/p)\sigma_a > 0$.

From Proposition 5.4 with $\phi = \chi_a^{r\nu_a} \chi_b^{r\lambda_b}$, $\gamma_a = \mu_a, \gamma_b = \kappa_b$, $\rho = \rho_a$ we get

$$\begin{aligned} A(\rho_a; a, b; \xi) : (L_p(\chi_a^{\rho_a + \mu_a} \chi_b^{\kappa_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a} \chi_b^{r\lambda_b} D^r) \\ \mapsto (L_p(\chi_a^{\mu_a} \chi_b^{\kappa_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a} \chi_b^{r\lambda_b} D^r), \end{aligned}$$

because $\rho_a + \mu_a > -1/p$.

From Proposition 5.5 with $\tau_a = r\nu_a, \tau_b = r\nu_b$, $\gamma_a = \mu_a, \gamma_b = \rho_b + \mu_b$, $\sigma = \sigma_b$ we get

$$\begin{aligned} B(\sigma_b; b, a; \xi) : (L_p(\chi_a^{\mu_a} \chi_b^{\kappa_b - \rho_b - \mu_b} \chi_b^{\rho_b + \mu_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a} \chi_b^{r\lambda_b} D^r) \\ \mapsto (L_p(\chi_a^{\mu_a} \chi_b^{\rho_b + \mu_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a} \chi_b^{r\nu_b} D^r), \end{aligned}$$

because $\kappa_b - \rho_b - \mu_b = -(\rho_b + \mu_b + 1/p)(1 - \sigma_b^{-1})$ and $\rho_b + \mu_b + 1/p = (\kappa_b + 1/p)\sigma_b > 0$.

From Proposition 5.4 with $\phi = \chi_a^{r\nu_a} \chi_b^{r\nu_b}$, $\gamma_a = \mu_a, \gamma_b = \mu_b$, $\rho = \rho_b$ we get

$$\begin{aligned} A(\rho_b; b, a; \xi) : (L_p(\chi_a^{\mu_a} \chi_b^{\rho_b + \mu_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a} \chi_b^{r\nu_b} D^r) \\ \mapsto (L_p(\chi_a^{\mu_a} \chi_b^{\mu_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a} \chi_b^{r\nu_b} D^r), \end{aligned}$$

because $\rho_b + \mu_b > -1/p$.

Combining the four mappings we prove the theorem. $\square$

If we decide to use only operators of type B we get

Theorem 5.4. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\kappa_a, \kappa_b, \lambda_a, \lambda_b, \nu_a, \nu_b \in \mathbb{R}$, $\varphi(x) = (x-a)^{\lambda_a}(b-x)^{\lambda_b}$, $\bar{\varphi}(x) = (x-a)^{\nu_a}(b-x)^{\nu_b}$ *with* $(1-\lambda_a)(1-\nu_a) > 0$ *and* $(1-\lambda_b)(1-\nu_b) > 0$, $w(x) = (x-a)^{\kappa_a}(b-x)^{\kappa_b}$ *with* $\kappa_a, \kappa_b \in \Gamma_+(p)$. *Set*

$$\mathcal{B} = B(\sigma_b; b, a; \xi) B(\sigma_a; a, b; \xi)$$

and $\bar{w}(x) = (x-a)^{\mu_a}(b-x)^{\mu_b}$, *where* $\xi \in (a,b)$,

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_b = \frac{1-\nu_b}{1-\lambda_b}, \quad \mu_a = (\kappa_a + \frac{1}{p})\sigma_a - \frac{1}{p}, \quad \mu_b = (\kappa_b + \frac{1}{p})\sigma_b - \frac{1}{p}.$$

Then

$$\mathcal{B} : (L_p(w)[a,b], AC_{loc}^{r-1}, \varphi^r D^r) \mapsto (L_p(\bar{w})[a,b], AC_{loc}^{r-1}, \bar{\varphi}^r D^r).$$

Hence for $t > 0$ *and* $f \in L_p(w)[a,b]$ *we have*

$$K(f, t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{B}f, t^r; L_p(\bar{w}), AC_{loc}^{r-1}, \bar{\varphi}^r D^r).$$

Proof. First, note that $\sigma_a, \sigma_b > 0$. From Proposition 5.5 with $\tau_a = r\nu_a, \tau_b = r\lambda_b$, $\gamma_a = \mu_a, \gamma_b = \kappa_b$, $\sigma = \sigma_a$ we get

$$\begin{aligned} B(\sigma_a; a, b; \xi) : (L_p(\chi_a^{\kappa_a - \mu_a}\chi_a^{\mu_a}\chi_b^{\kappa_b}), AC_{loc}^{r-1}, \chi_a^{r\lambda_a}\chi_b^{r\lambda_b} D^r) \\ \mapsto (L_p(\chi_a^{\mu_a}\chi_b^{\kappa_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\lambda_b} D^r), \end{aligned}$$

because $\kappa_a - \mu_a = -(\mu_a + 1/p)(1 - \sigma_a^{-1})$ and $\mu_a + 1/p = (\kappa_a + 1/p)\sigma_a$, i.e. $\mu_a \in \Gamma_+(p)$ iff $\kappa_a \in \Gamma_+(p)$.

From Proposition 5.5 with $\tau_a = r\nu_a, \tau_b = r\nu_b$, $\gamma_a = \mu_a, \gamma_b = \mu_b$, $\sigma = \sigma_b$ we get

$$\begin{aligned} B(\sigma_b; b, a; \xi) : (L_p(\chi_a^{\mu_a}\chi_b^{\kappa_b - \mu_b}\chi_b^{\mu_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\lambda_b} D^r) \\ \mapsto (L_p(\chi_a^{\mu_a}\chi_b^{\mu_b}), AC_{loc}^{r-1}, \chi_a^{r\nu_a}\chi_b^{r\nu_b} D^r), \end{aligned}$$

because $\kappa_b - \mu_b = -(\mu_b + 1/p)(1 - \sigma_b^{-1})$ and $\mu_b + 1/p = (\kappa_b + 1/p)\sigma_b$, i.e. $\mu_b \in \Gamma_+(p)$ iff $\kappa_b \in \Gamma_+(p)$.

Combining the two mappings we prove the theorem. □

The Jacobean weights $w = \chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\kappa_a, \kappa_b > -1/p$ are covered by Theorem 5.3 and Theorem 5.4. The restriction on the κ's cannot be weaken because in general one cannot expect to get equivalence of the K-functionals of functions in L_p and $L_p(w)$ provided $\kappa_a < -1/p$ or $\kappa_b < -1/p$. But Proposition 5.5 still lives room for varying the weight in the differential operator when $L_p(w)$ is compared with $L_p(\bar{w})$ under the restriction $\kappa_a, \kappa_b, \mu_a, \mu_b < -1/p$. Then the proof of Theorem 5.4 gives

Theorem 5.5. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\kappa_a, \kappa_b, \lambda_a, \lambda_b, \nu_a, \nu_b \in \mathbb{R}$, $\varphi(x) = (x-a)^{\lambda_a}(b-x)^{\lambda_b}$, $\bar{\varphi}(x) = (x-a)^{\nu_a}(b-x)^{\nu_b}$ *with* $(1-\lambda_a)(1-\nu_a) > 0$ *and* $(1-\lambda_b)(1-\nu_b) > 0$, $w(x) = (x-a)^{\kappa_a}(b-x)^{\kappa_b}$ *with* $|\kappa_a + 1/p| < \min\{1, (1-\lambda_a)/(1-\nu_a)\}$, $|\kappa_b + 1/p| < \min\{1, (1-\lambda_b)/(1-\nu_b)\}$. *Set*

$$\mathcal{B} = B(\sigma_b; b, a; a)B(\sigma_a; a, b; b)$$

and $\bar{w}(x) = (x-a)^{\mu_a}(b-x)^{\mu_b}$, *where*

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_b = \frac{1-\nu_b}{1-\lambda_b}, \quad \mu_a = (\kappa_a + \frac{1}{p})\sigma_a - \frac{1}{p}, \quad \mu_b = (\kappa_b + \frac{1}{p})\sigma_b - \frac{1}{p}.$$

Then

$$\mathcal{B} : (L_p(w)[a,b], AC_{loc}^{r-1}, \varphi^r D^r) \mapsto (L_p(\bar{w})[a,b], AC_{loc}^{r-1}, \bar{\varphi}^r D^r).$$

Hence for $t > 0$ *and* $f \in L_p(w)[a,b]$ *we have*

$$K(f, t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{B}f, t^r; L_p(\bar{w}), AC_{loc}^{r-1}, \bar{\varphi}^r D^r).$$

Note that in Theorem 5.5 we have $\operatorname{sgn}(\kappa_a + 1/p) = \operatorname{sgn}(\mu_a + 1/p)$, $\operatorname{sgn}(\kappa_b + 1/p) = \operatorname{sgn}(\mu_b + 1/p)$ and not all restrictions on κ_a, κ_b are sharp.

Considering the case $\bar{w} \equiv 1$, $\bar{\varphi} \equiv 1$, from Theorem 5.3 and Theorem 5.4 we get

Corollary 5.2. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\varphi(x) = (x-a)^{\lambda_a}(b-x)^{\lambda_b}$ *with* $\lambda_a, \lambda_b < 1$, $w(x) = (x-a)^{\kappa_a}(b-x)^{\kappa_b}$ *with* $-1/p < \kappa_a, \kappa_b$ *if* $p < \infty$ *and* $\kappa_a = \kappa_b = 0$ *if* $p = \infty$. *Set*

$$\mathcal{A} = A(\rho_b; b, a; \xi) B(\sigma_b; b, a; \xi) A(\rho_a; a, b; \xi) B(\sigma_a; a, b; \xi),$$

where $\xi \in (a,b)$ *and*

$$\rho_a = \frac{\kappa_a + 1/p}{1 - \lambda_a} - \frac{1}{p}, \quad \sigma_a = \frac{1}{1 - \lambda_a}, \quad \rho_b = \frac{\kappa_b + 1/p}{1 - \lambda_b} - \frac{1}{p}, \quad \sigma_b = \frac{1}{1 - \lambda_b}.$$

Then

$$\mathcal{A} : (L_p(w)[a,b], AC_{loc}^{r-1}, \varphi^r D^r) \mapsto (L_p[a,b], AC_{loc}^{r-1}, D^r).$$

Hence for $t > 0$ *and* $f \in L_p(w)[a,b]$ *we have*

$$\begin{aligned} K(f, t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r) &\sim K(\mathcal{A}f, t^r; L_p, AC_{loc}^{r-1}, D^r) \\ &\sim \omega_r(\mathcal{A}f, t)_p = \Omega(f, t; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r). \end{aligned}$$

Proof. The case $p < \infty$ follows from Theorem 5.3 with $\mu_a = \mu_b = \nu_a = \nu_b = 0$. In the case $p = \infty$ we have $\kappa_a = \kappa_b = \mu_a = \mu_b = 0$ and hence Theorem 5.4 is applicable. □

Remark 5.3. The case $p = \infty$, $\kappa_a > 0$ and/or $\kappa_b > 0$ is not covered by Corollary 5.2. In fact, $\mathcal{A}$ is not a continuous mapping under such assumptions. In such cases one can apply Theorem 5.3 with some $\mu_a, \mu_b > 0$, $\nu_a = \nu_b = 0$ and get

$$K(f, t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{A}f, t^r; L_p(\bar{w}), AC_{loc}^{r-1}, D^r).$$

The last K-functional is equivalent to proper moduli defined in [5, Ch. 6] or [16].

From Corollary 5.2 with $\kappa_a = \kappa_b = 0$ we get

Corollary 5.3. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\varphi(x) = (x-a)^{\lambda_a}(b-x)^{\lambda_b}$ *with* $\lambda_a, \lambda_b < 1$. *Set*

$$\mathcal{A} = A(\rho_b; b, a; \xi) B(\sigma_b; b, a; \xi) A(\rho_a; a, b; \xi) B(\sigma_a; a, b; \xi),$$

where $\xi \in (a,b)$ *and*

$$\rho_a = \frac{\lambda_a}{1-\lambda_a}\frac{1}{p}, \quad \sigma_a = \frac{1}{1-\lambda_a}, \quad \rho_b = \frac{\lambda_b}{1-\lambda_b}\frac{1}{p}, \quad \sigma_b = \frac{1}{1-\lambda_b}.$$

Then

$$\mathcal{A} : (L_p[a,b], AC_{loc}^{r-1}, \varphi^r D^r) \mapsto (L_p[a,b], AC_{loc}^{r-1}, D^r).$$

Hence for $t > 0$ *and* $f \in L_p[a,b]$ *we have*

$$K(f,t^r; L_p, AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{A}f, t^r; L_p, AC_{loc}^{r-1}, D^r) \\ \sim \omega_r(\mathcal{A}f,t)_p = \Omega(f,t; L_p, AC_{loc}^{r-1}, \varphi^r D^r).$$

Remark 5.4. The family of operators $\mathcal{A}$ in Corollary 5.3 is uniformly bounded from L_p to L_p, $1 \le p \le \infty$ regardless of the two parts (for $p < \infty$ and for $p = \infty$) of the proof of Corollary 5.2. The reason is that we have $\|\chi_a^{\rho_a}\psi_1\|_p \sim p\|f\|_p$ in (3.1) and $\alpha_{r,1}(\rho_a) \sim \rho_a \sim 1/p$ in (3.3) when $p \to \infty$.

Remark 5.5. In principal, there are 24 permutations of the 4 operators used in Theorem 5.3. But Propositions 4.1 and 5.1 tell us that some of these permutations give one and the same operator $\mathcal{A}$. In fact, the permutations give no more than 8 slightly different operators (see Remark 5.1): $\mathcal{A}_1 = \mathcal{A}$ (the operator from Theorem 5.3);

$$\begin{aligned}
\mathcal{A}_2 &= A(\rho_a;a,b;\xi)A(\rho_b;b,a;\xi)B(\sigma_b;b,a;\xi)B(\sigma_a;a,b;\xi);\\
\mathcal{A}_3 &= A(\rho_a;a,b;\xi)B(\sigma_b;b,a;\xi)B(\sigma_a;a,b;\xi)A(\rho_b/\sigma_b;b,a;\xi);\\
\mathcal{A}_4 &= B(\sigma_b;b,a;\xi)A(\rho_a;a,b;\xi)B(\sigma_a;a,b;\xi)A(\rho_b/\sigma_b;b,a;\xi)
\end{aligned}$$

and those obtained by changing the places of a and b in $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$. Note that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ differ in form but they do one and the same job.

In order to demonstrate how the parameters change if we commute operators A and B for one and the same singularity we apply Proposition 4.1 in Corollary 5.2 and get

Corollary 5.4. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$, $\varphi(x) = (x-a)^{\lambda_a}(b-x)^{\lambda_b}$ *with* $\lambda_a, \lambda_b < 1$, $w(x) = (x-a)^{\kappa_a}(b-x)^{\kappa_b}$ *with* $-1/p < \kappa_a, \kappa_b$ *if* $p < \infty$ *and* $\kappa_a = \kappa_b = 0$ *if* $p = \infty$. *Set*

$$\mathcal{A} = B(\sigma_b;b,a;\xi)A(\rho_b';b,a;\xi)B(\sigma_a;a,b;\xi)A(\rho_a';a,b;\xi)$$

where $\xi \in (a,b)$ *and*

$$\rho_a' = \kappa_a + \frac{\lambda_a}{p}, \quad \sigma_a = \frac{1}{1-\lambda_a}, \quad \rho_b' = \kappa_b + \frac{\lambda_b}{p}, \quad \sigma_b = \frac{1}{1-\lambda_b}.$$

Then

$$\mathcal{A} : (L_p(w)[a,b], AC_{loc}^{r-1}, \varphi^r D^r) \mapsto (L_p[a,b], AC_{loc}^{r-1}, D^r).$$

Hence for $t > 0$ *and* $f \in L_p(w)[a,b]$ *we have*

$$K(f,t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{A}f, t^r; L_p, AC_{loc}^{r-1}, D^r) \\ \sim \omega_r(\mathcal{A}f,t)_p = \Omega(f,t; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r).$$

Note that the operators $\mathcal{A}$ in Corollary 5.2 and Corollary 5.4 coincide.

6 Generalizations

First, the operators from Sections 3 and 4 can be applied to spaces of functions with a domain different than the finite interval $[a, b]$. Observe that operators (3.1), (4.1) map $L_{1,loc}[1,\infty)$ to itself and treat singularities at $+\infty$. Also, $B(\sigma)$ with $\sigma < 0$ maps $L_{1,loc}[1,\infty)$ to $L_{1,loc}(0,1]$ and vice versa. Using these operators we can treat both other domains and other powers in the weights than those in Theorem 5.3 (see e.g. Theorem 8.1 below).

Second, operators (5.1), (5.2) are design for transforming the first argument of K-functionals with power-type weights. But the same ideas work with more general classes of weights.

The generalization of $A(\rho)$ when x^ρ is replaced by a smooth $\phi(x)$ is

$$(6.1)\qquad (Af)(x) = \phi(x)f(x) + \sum_{i=1}^{r}(-1)^i\binom{r}{i}\int_\xi^x \frac{(x-y)^{i-1}}{(i-1)!}\phi^{(i)}(y)f(y)\,dy.$$

Using (3.4) one gets $(Af)^{(r)} = \phi f^{(r)}$.

The generalization of $B(\sigma)$ is more complicated. Here we only sketch it for $r = 2$ and $B(\sigma; 0, 1; 1/2)$. Let φ be a non-negative measurable function on $[0, 1]$ and $1/\varphi \in L_1[0, 1]$. Set $\Phi(x) = c\int_0^x \varphi(y)^{-1}\,dy$ where the constant c is determined by the condition $\Phi(1) = 1$. Choose the smooth function θ so that

$$\theta'(x) \sim \varphi(\theta(x)),\quad x \in [0,1]$$

(one possible choice is $\theta = \Phi^{-1}$). With the notation $\eta(x) = \theta''(x)/\theta'(x)$ we set

$$(Bf)(x) = f(\theta(x)) - \int_{1/2}^x [\eta(y) + y\eta'(y)]f(\theta(y))\,dy + x\int_{1/2}^x \eta'(y)f(\theta(y))\,dy.$$

Then

$$(Bf)''(x) = (\theta'(x))^2 f''(\theta(x)) \sim \varphi^2(\theta(x))f''(\theta(x)).$$

Applying the generalization of Hardy's inequality in Proposition 2.2 we get boundedness of B under suitable restrictions on θ.

For power-type weights φ (i.e. $\varphi(x) = x^\gamma$ and hence $\theta(x) = x^\sigma$ with $\sigma = 1/(1-\gamma)$) we have $\eta(x) = const\cdot x^{-1}$. Thus, $\eta(x) + x\eta'(x) \equiv 0$ and hence the first integral in the definition of B vanishes. For more general weights this integral does not vanish but still B may be a bounded operator.

Consider two examples:

i) $\varphi(x) = x^{1/2}(\log 1/x)^{1/2}$. We can take $\theta(x) = x^2(1 + 2\log 1/x)$. Then $\eta(x) = x^{-1} - x^{-1}(\log 1/x)^{-1}$, $\eta(y) + y\eta'(y) = -y^{-1}(\log 1/y)^{-2}$ and

$$\int_{1/2}^x [\eta(y) + y\eta'(y)]f(\theta(y))\,dy = -\int_{1/2}^x y^{-1}(\log 1/y)^{-2}f(\theta(y))\,dy,$$

which, for example, is a bounded operator from $C[0, 1/2]$ to $C[0, 1/2]$.

ii) $\varphi(x) = x^\gamma(\log 1/x)^{2-2\gamma}$, $\gamma < 1$. With $\sigma = 1/(1-\gamma)$ we can take $\theta(x) = x^\sigma(\frac{1}{2}\sigma^2(\log 1/x)^2 + \sigma\log 1/x + 1)$. Then $\eta(x) = (\sigma-1)x^{-1} - 2x^{-1}(\log 1/x)^{-1}$ and $\eta(y) + y\eta'(y) = -2y^{-1}(\log 1/y)^{-2}$.

Such generalizations will appear in a forthcoming paper.

Finally, let us consider the more general form (1.2) of the differential operator $\mathcal{D}$. One trivially gets equivalent K-functionals for the triples (X, Y, D_1) and (X, Y, D_2) when $\|D_1 f\|_X \sim \|D_2 f\|_X,\ \forall f \in Y$. In many non-trivial cases the following operator

$$
\begin{aligned}
(6.2)\quad (Af)(x) = \Phi_r(x)f(x) &+ \sum_{i=1}^{r}(-1)^i \binom{r}{i} \int_\xi^x \frac{(x-y)^{i-1}}{(i-1)!}\Phi_r^{(i)}(y)f(y)\,dy \\
&+ \sum_{k=0}^{r-1}\sum_{i=0}^{k}(-1)^i \binom{k}{i} \int_\xi^x \frac{(x-y)^{r-k+i-1}}{(r-k+i-1)!}\Phi_k^{(i)}(y)f(y)\,dy
\end{aligned}
$$

can help reducing the general case (1.2) to $\mathcal{D} = \phi D^r$ because of $D^r(Af) = \sum_{k=0}^{r}\Phi_k D^k f$. Note that (6.1) is a partial case of (6.2) when $\Phi_r = \phi;\ \Phi_k = 0, k < r$. Of course, one has to ensure boundedness of A and A^{-1} which is not likely to be true provided Φ_r has a singularity. But, the setting $\Phi_r = 1;\ \Phi_k = \varphi_k/\varphi_r, k < r$ in (6.2) gives in several cases a linear operator A which is bounded together with its inverse and possesses the property $\varphi_r D^r(Af) = \sum_{k=0}^{r}\varphi_k D^k f$. It is shown in [13] that such approach can work (see also Subsection 8.4).

7 Examples

First we give an explicit form of the operators from Sections 3 and 4 for $r = 1, 2, 3$.

For $r = 1$ we have

$$
(A(\rho)f)(x) = x^\rho f(x) - \rho\int_1^x y^{\rho-1}f(y)\,dy; \quad (B(\sigma)f)(x) = f(x^\sigma).
$$

For $r = 2$ we have

$$
\begin{aligned}
&(A(\rho)f)(x) = x^\rho f(x) - (\rho+1)\rho\int_1^x y^{\rho-1}f(y)\,dy + \rho(\rho-1)x\int_1^x y^{\rho-2}f(y)\,dy; \\
&(B(\sigma)f)(x) = f(x^\sigma) + (1-\sigma)x\int_1^x y^{-2}f(y^\sigma)\,dy.
\end{aligned}
$$

For $r = 3$ we have

$$
\begin{aligned}
&(A(\rho)f)(x) = x^\rho f(x) \\
&\quad + \int_1^x \left(-\frac{(\rho+2)(\rho+1)\rho}{2y^{1-\rho}} + \frac{(\rho+1)\rho(\rho-1)x}{y^{2-\rho}} - \frac{\rho(\rho-1)(\rho-2)x^2}{2y^{3-\rho}}\right)f(y)\,dy; \\
&(B(\sigma)f)(x) = f(x^\sigma) + \int_1^x \left(-\frac{(1-\sigma)(1-2\sigma)x}{y^2} + \frac{(2-\sigma)(2-2\sigma)x^2}{y^3}\right)f(y^\sigma)\,dy.
\end{aligned}
$$

Example 7.1. Let $r = 1$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{x}$ and $w(x) = x^{1/p}$. Then for every $f \in L_p[0,1]$ and every $t > 0$ we have

$$\begin{aligned} K(f,t;L_p,AC_{loc},\varphi D) &\sim K(B(2)f,t;L_p(w),AC_{loc},D) \\ &\sim K(A(1/p)B(2)f,t;L_p,AC_{loc},D) \\ &\sim \omega_1(A(1/p)B(2)f,t)_p \\ &= \Omega(f,t;L_p,AC_{loc},\varphi D), \end{aligned}$$

where

$$(B(2)f)(x) = f(x^2), \quad (A(1/p)B(2)f)(x) = x^{1/p} f(x^2) - \frac{1}{p}\int_1^x y^{\frac{1}{p}-1} f(y^2)\,dy$$

The inverse of $B(2)$ is

$$(B(2)^{-1}F)(x) = (B(1/2)F)(x) = F(\sqrt{x}).$$

Example 7.2. Let $r = 2$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{x}$ and $w(x) = x^{1/p}$. Then for every $f \in L_p[0,1]$ and every $t > 0$ we have

$$\begin{aligned} K(f,t^2;L_p,AC^1_{loc},\varphi^2 D^2) &\sim K(B(2)f,t^2;L_p(w),AC^1_{loc},D^2) \\ &\sim K(A(1/p)B(2)f,t^2;L_p,AC^1_{loc},D^2) \\ &\sim \omega_2(A(1/p)B(2)f,t)_p \\ &= \Omega(f,t;L_p,AC^1_{loc},\varphi^2 D^2), \end{aligned}$$

where

$$(B(2)f)(x) = f(x^2) - x\int_1^x y^{-2} f(y^2)\,dy$$

and

$$(A(1/p)B(2)f)(x) = x^{\frac{1}{p}} f(x^2) - \int_1^x \Big[\frac{p^2-1}{p^2}\, y^{\frac{1}{p}-2}x + \frac{2p+1}{p^2}\, y^{\frac{1}{p}-1}\Big] f(y^2)\,dy.$$

The inverse of $B(2)$ is

$$(B(2)^{-1}F)(x) = (B(1/2)F)(x) = F(\sqrt{x}) + \frac{x}{2}\int_1^x y^{-2} F(\sqrt{y})\,dy.$$

Example 7.3. Let $r = 3$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{x}$ and $w(x) = x^{1/p}$. Then for every $f \in L_p[0,1]$ and every $t > 0$ we have

$$\begin{aligned} K(f,t^3;L_p,AC^2_{loc},\varphi^3 D^3) &\sim K(B(2)f,t^3;L_p(w),AC^2_{loc},D^3) \\ &\sim K(A(1/p)B(2)f,t^3;L_p,AC^2_{loc},D^3) \\ &\sim \omega_3(A(1/p)B(2)f,t)_p \\ &= \Omega(f,t;L_p,AC^2_{loc},\varphi^3 D^3), \end{aligned}$$

where

$$(B(2)f)(x) = f(x^2) - 3x \int_1^x y^{-2} f(y^2)\, dy$$

and

$$(A(1/p)B(2)f)(x) = x^{\frac{1}{p}} f(x^2) + \int_1^x \mathcal{K}(x,y) f(y^2)\, dy,$$

$$\mathcal{K}(x,y) = -\frac{(2p+1)(4p+1)}{2p^3} y^{\frac{1}{p}-1} - \frac{(p^2-1)(3p+1)}{p^3}\, xy^{\frac{1}{p}-2} + \frac{(4p^2-1)}{2\,p^3}\, x^2 y^{\frac{1}{p}-3}.$$

The inverse of $B(2)$ is

$$(B(2)^{-1}F)(x) = (B(1/2)F)(x) = F(\sqrt{x}) + \frac{3x^2}{2} \int_1^x y^{-3} F(\sqrt{y})\, dy.$$

Example 7.4. Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{1-x^2}$ and $w(x) = (1-x^2)^{1/p}$. Then for every $f \in L_p[-1,1]$ and every $t > 0$ we have

$$K(f, t^r; L_p, AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{B}f, t^r; L_p(w), AC_{loc}^{r-1}, D^r)$$

where $\mathcal{B} = B(2;1,-1;0)B(2;-1,1;0)$ and

$$\begin{aligned}(B(2;1,-1;0)f)(x) &= f(x + \frac{1-x^2}{2}) \\ &\quad - \sum_{k=1}^{[r/2]} \beta_{r,2k}(2)(1-x)^{2k-1} \int_0^x \frac{f(y + \frac{1-y^2}{2})}{(1-y)^{2k}}\, dy, \\ (B(2;-1,1;0)f)(x) &= f(x - \frac{1-x^2}{2}) \\ &\quad + \sum_{k=1}^{[r/2]} \beta_{r,2k}(2)(1+x)^{2k-1} \int_0^x \frac{f(y - \frac{1-y^2}{2})}{(1+y)^{2k}}\, dy,\end{aligned}$$

where $$\beta_{r,2k}(2) = \frac{(-1)^k 2^{1-r}(2r-2k)!}{(k-1)!(r-k)!(r-2k)!}, \quad 1 \le k \le [r/2].$$

The operators in Theorem 5.3 are based on the change of the variable of the type $\theta(x) = x^\sigma$ but, as mentioned in Section 6 our method allows a broader class of "equivalent" in a certain sense changes of the variable. In the next example we show that four different operators, based on the four changes of the variable $\theta_1(x) = (3x - x^3)/2 + (1-x^2)^2/8$, $\theta_2(x) = (3x-x^3)/2 - (1-x^2)^2/8$, $\theta_3(x) = (3x - x^3)/2$ and $\theta_4(x) = \cos x$, produce equivalent K-functionals.

Example 7.5. Let $r = 2$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{1-x^2}$, $w(x) = (1-x^2)^{1/p}$ and $\bar{w}(x) = (\sin x)^{1/p}$. Then for every $f \in L_p[-1,1]$ and every $t > 0$ we have

$$\begin{gathered} K(f, t^2; L_p, AC_{loc}^1, \varphi^2 D^2) \\ \sim K(\mathcal{B}_1 f, t^2; L_p(w)[-1,1], AC_{loc}^1, D^2) \sim K(\mathcal{B}_2 f, t^2; L_p(w)[-1,1], AC_{loc}^1, D^2) \\ \sim K(\mathcal{B}_3 f, t^2; L_p(w)[-1,1], AC_{loc}^1, D^2) \sim K(\mathcal{B}_4 f, t^2; L_p(\bar{w})[0,\pi], AC_{loc}^1, D^2), \end{gathered}$$

where

$$
\begin{aligned}
\mathcal{B}_1 f(x) &= (B(2;1,-1;0)B(2;-1,1;1/2)f)(x)\\
&= f\Big(\frac{3x-x^3}{2}+\frac{(1-x^2)^2}{8}\Big)+\int_0^x \mathcal{K}_1(x,y) f\Big(\frac{3y-y^3}{2}+\frac{(1-y^2)^2}{8}\Big)\,dy;\\
\mathcal{B}_2 f(x) &= (B(2;-1,1;0)B(2;1,-1;-1/2)f)(x)\\
&= f\Big(\frac{3x-x^3}{2}-\frac{(1-x^2)^2}{8}\Big)+\int_0^x \mathcal{K}_2(x,y) f\Big(\frac{3y-y^3}{2}-\frac{(1-y^2)^2}{8}\Big)\,dy;\\
\mathcal{B}_3 f(x) &= f\Big(\frac{3x-x^3}{2}\Big)+\int_0^x \Big[\frac{1-x}{(1-y)^2}-\frac{1+x}{(1+y)^2}\Big] f\Big(\frac{3y-y^3}{2}\Big)\,dy;\\
\mathcal{B}_4 f(x) &= f(\cos x)-\int_{\pi/2}^x \frac{x-y+\sin y\cos y}{\sin^2 y} f(\cos y)\,dy,\ x\in[0,\pi]
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{K}_1(x,y) = -2\frac{(1+x)(3-x)(1-y)}{(1+y)^2(3-y)^2}+\frac{1-x}{(1-y)^2}&\\
+2\frac{(1-x)(y^2-y+4)-(x^2-x+4)(1-y)}{(1+y)^2(3-y)^2}&,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_2(x,y) = 2\frac{(1-x)(3+x)(1+y)}{(1-y)^2(3+y)^2}-\frac{1+x}{(1+y)^2}&\\
-2\frac{(1+x)(y^2+y+4)-(x^2+x+4)(1+y)}{(1-y)^2(3+y)^2}&.
\end{aligned}
$$

The inverse operators are given by

$$
\begin{aligned}
\mathcal{B}_1^{-1}F(x) &= (B(1/2;-1,1;1/2)B(1/2;1,-1;0)F)(x);\\
\mathcal{B}_2^{-1}F(x) &= (B(1/2;1,-1;-1/2)B(1/2;-1,1;0)F)(x);\\
\mathcal{B}_3^{-1}F(x) &= F(\eta(x))\\
&\quad+\frac{3}{2}\int_0^x [3(x-y)(1+3\eta^2(y))(\eta'(y))^2-2\eta(y)](\eta'(y))^2 F(\eta(y))\,dy;\\
\mathcal{B}_4^{-1}F(x) &= F(\arccos x)+\frac{1}{2}\int_0^x \Big(\frac{1+x}{(1+t)^2}-\frac{1-x}{(1-t)^2}\Big)F(\arccos y)\,dy,
\end{aligned}
$$

where $\eta(x)$ is the inverse of $\theta_3(x)=(3x-x^3)/2$ in the interval $[-1,1]$ and hence $\eta'(y)=\frac{2}{3}(1-\eta^2(y))^{-1}$.

Note that $\mathcal{B}_3$ has probably the simplest form, but we cannot write down explicitly its inverse (without using inverse functions or cubic roots). $\mathcal{B}_4$ and its inverse have also simple forms but a change of the variable, mapping $[0,\pi]$ onto $[-1,1]$, is used. Additional difficulties occur when we study the analogues of $\mathcal{B}_4$ for $r>2$. The form of $\mathcal{B}_1$ and $\mathcal{B}_2$ is the most complicated. Moreover, they are defined by non-symmetric changes of the variable (with respect to the domain). Also we took $\xi \neq 0$ in one of the operators B forming $\mathcal{B}_1$ and $\mathcal{B}_2$ in order to get one and the same bounds in the integrals. But these operators and their inverse are easier to work with and they allow separate treatment of the singularities.

Of course, one can write many other operators by choosing other changes of the variable or varying ξ, e.g. $\mathcal{B}f = B(2;1,-1;0)B(2;-1,1;0)f$ which equals $\mathcal{B}_1 f$ modulus a linear function.

Example 7.6. The operators from Example 7.5 transformed for the interval $[0,1]$ ($r=2$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$) are

$$\begin{aligned}
\mathcal{B}_1 f(x) &= (B(2;1,0;1/2)B(2;0,1;3/4)f)(x);\\
\mathcal{B}_2 f(x) &= (B(2;0,1;1/2)B(2;1,0;1/4)f)(x);\\
\mathcal{B}_3 f(x) &= f(3x^2-2x^3) - \int_{1/2}^{x} \Big(\frac{x}{y^2} - \frac{1-x}{(1-y)^2}\Big) f(3y^2-2y^3)\, dy;\\
\mathcal{B}_4 f(x) &= f\Big(\frac{1+\cos\pi x}{2}\Big) - \pi \int_{1/2}^{x} \frac{\pi x - \pi y + \sin\pi y\cos\pi y}{\sin^2\pi y} f\Big(\frac{1+\cos\pi y}{2}\Big)\, dy.
\end{aligned}$$

8 Applications

In this section we discuss some examples of approximation processes whose rate of approximation is estimated by a K-functional of the form (1.1).

8.1 Best algebraic polynomial approximation

The best approximation of a function $f \in L_p[-1,1]$, $1 \le p \le \infty$ by algebraic polynomials of degree n is given by $E_n(f)_p = \inf\{\|f-Q\|_p \;:\; Q \in \Pi_n\}$.

In 1980 the second author proved a characterization (strong direct and weak inverse theorems, see e.g. [11]) for any $r \in \mathbb{N}$ and $1 \le p \le \infty$ of the best algebraic approximations in terms of moduli (1.8) with $\psi(t,x) = t\sqrt{1-x^2} + t^2$. Later it was proved in [12] that these moduli and the K-functionals (1.5) with $\varphi(x) = \sqrt{1-x^2}$ are equivalent. Meanwhile, Ditzian and Totik introduced the weighted moduli (1.6) and proved both equivalence with the K-functionals (1.5) and characterization of the best algebraic approximations.

Now Corollary 5.3 with $\lambda_{-1} = \lambda_1 = 1/2$, $\varphi = \sqrt{1-x^2}$ gives a new characteristic of the best algebraic approximations:

$$E_n(f)_p \le C_r \omega_r(\mathcal{A}f, n^{-1})_p = C_r \Omega(f, n^{-1}; L_p[-1,1], AC_{loc}^{r-1}, \varphi^r D^r), \quad n > r,$$

$$\Omega(f,t;L_p[-1,1], AC_{loc}^{r-1}, \varphi^r D^r) = \omega_r(\mathcal{A}f,t)_p \le C_r t^r \sum_{0 \le k \le 1/t} (k+1)^{r-1} E_k(f)_p,$$

where $\mathcal{A}$ is defined in Corollary 5.3.

Let us note that $\omega_r(\mathcal{A}f,t)_\infty$ turns out to be a solution (in a certain sense because $\mathcal{A}f$ is not periodic) of the following problem posed by S. Gal in [8] (which is a variant of Problem 1.1):

Find a 2π-periodic continuous function F, depending on $f \in C[-1,1]$ and r, such that

$$\omega_\varphi^r(f,t)_\infty \sim \omega_{r_1}(F|_{[0,2\pi]};t)_\infty,\ 0 \le t \le t_0,$$

as not necessarily $r = r_1$.

G. Mastroianni and P. Vértesi also gave a solution of the above mentioned problem for $r = 1$. For $f \in C[-1,1]$ they define $g_f(\theta) = f(\cos\theta)$ and show in [17]

$$\omega_\varphi^1(f,t)_\infty \sim \omega_1(g_f,t)_\infty,\ 0 \le t \le t_0. \tag{8.1}$$

J. Bustamante noticed in [1] that a relation like (8.1) is not valid for $r > 1$ though a weaker one holds.

8.2 Bernstein polynomials

The Bernstein polynomials $B_n f$ are probably the most studied approximation operators.

Totik in [21] and H.-B. Knoop and X.-l. Zhou in [15] proved

$$\|f - B_n f\|_\infty \sim K(f,n^{-1};C[0,1],AC^1_{loc},\varphi^2D^2) \sim \omega_\varphi^2(f,n^{-1/2})_\infty,$$

where $\varphi(x) = \sqrt{x(1-x)}$. Corollary 5.3 yields another characterization

$$\begin{aligned}\|f - B_n f\|_\infty \sim \Omega(f,n^{-1/2};C[0,1],AC^1_{loc},\varphi^2D^2) &= \omega_2(\mathcal{B}f,n^{-1/2})_\infty \\ &\sim K(\mathcal{B}f,n^{-1};C[0,1],AC^1_{loc},D^2),\end{aligned}$$

where $\mathcal{B}$ is any of the operators in Example 7.6.

8.3 Szász-Mirakjan operators

For the Szász-Mirakjan operators $S_n f$ defined for $f \in C[0,\infty)$ V. Totik proved in [21]

$$\|f - S_n f\|_\infty \sim \omega_\varphi^2(f,n^{-1/2})_\infty \sim K(f,n^{-1};C[0,\infty),AC^1_{loc},\varphi^2D^2), \tag{8.2}$$

where $\varphi(x) = \sqrt{x}$. Using the method demonstrated in Section 4 we get

Theorem 8.1. *Let $f \in C[0,\infty)$ and $n \in \mathbb{N}$. Then*

$$\|f - S_n f\|_\infty \sim \omega_2(B(2)f,n^{-1/2})_\infty = \Omega(f,n^{-1/2};C[0,\infty),AC^1_{loc},\varphi^2D^2), \tag{8.3}$$

where $B(2) : C[0,\infty) \to C[0,\infty)$ is defined as (see Example 7.2)

$$(B(2)f)(x) = f(x^2) + x\int_x^\infty y^{-2} f(y^2)\,dy.$$

Although we have not discussed in the present article unbounded domains, we have all the ingredients to give a simple proof of (8.3).

Proof. The inverse of $B(2)$ is

$$(B(1/2)f)(x) = f(\sqrt{x}) - \frac{x}{2}\int_x^\infty y^{-2} f(\sqrt{y})\, dy.$$

From the definitions of $B(2)$ and $B(1/2)$ we get $\|B(2)f\|_\infty \leq 2\|f\|_\infty$ and $\|B(1/2)f\|_\infty \leq \frac{3}{2}\|f\|_\infty$, which, together with Theorem 4.1 and Corollary 4.2, imply

$$B(2) : (C[0,\infty), AC^1_{loc}, \varphi^2 D^2) \mapsto (C[0,\infty), AC^1_{loc}, D^2).$$

In view of Proposition 2.1, this continuous mapping together with (8.2) proves (8.3). $\square$

8.4 Kantorovich and Durrmeyer operators

$K(f, t; L_p[0,1], C^2, D\phi D)$ with $\phi(x) = x(1-x)$ is the K-functional that is equivalent to the approximation errors of Kantorovich $P_n f$ and Durrmeyer $M_n f$ operators. Note that the differential operator $D\phi D$ differs from the usual ϕD^2. The set on which the infimum is taken is $Y = C^2$, which gives here a different K-functional than the usual $Y = AC^1_{loc}$.

Chen, Ditzian and Ivanov in [2] and Gonska and Zhou in [9] proved for every $f \in L_p[0,1]$, $1 \leq p \leq \infty$ and $n \in \mathbb{N}$

$$\|f - M_n f\|_p \sim K(f, n^{-1}; L_p[0,1], C^2, D\phi D) \sim \|f - P_n f\|_p.$$

Gonska and Zhou also proved in [9]

$$K(f, t^2; L_p[0,1], C^2, D\phi D) \sim \omega^2_{\sqrt{\phi}}(f,t)_p + \omega_1(f,t^2)_p, \quad 1 < p \leq \infty.$$

The above equivalence is not true for $p = 1$. Using the idea of [7] the second author proved in [13] that

$$K(f, t^2; L_1[0,1], C^2, D\phi D) \sim K(\mathcal{A}f, t^2; L_1[0,1], C^2, \phi D^2) + \omega_1(f, t^2)_1,$$

where

$$(\mathcal{A}f)(x) = f(x) + \int_{1/2}^x \left(\frac{x}{y^2} - \frac{1-x}{(1-y)^2}\right) f(y)dy.$$

This equivalence together with the equivalence relations for the second K-functional gives

$$\|f - M_n f\|_1 \sim \|f - P_n f\|_1 \sim \omega_2(\mathcal{B}\mathcal{A}f, n^{-1/2})_1 + \omega_1(f, n^{-1})_1,$$

where $\mathcal{B}$ is the operator from Corollary 5.3 with $r = 2$, $p = 1$, $\lambda_0 = \lambda_1 = 1/2$.

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