Ann. Funct. Anal. (2022) 13:36, https://doi.org/10.1007/s43034-022-00185-6

A strong converse inequality for generalized sampling operators

Tuncer Acar Borislav R. Draganov

Abstract

We establish a two-term strong converse inequality for the rate of approximation of generalized sampling operators by means of the classical moduli of smoothness. It is of the same order as an already known direct estimate. We combine both estimates to derive the saturation property and class of this approximation operator. The particular cases of kernel functions—the central B-splines, linear combinations of translates of B-splines, and the Bochner-Riesz kernel, are demonstrated.

AMS classification: 41A17, 41A25, 41A27, 41A35, 41A40, 94A20.

Key words and phrases: Sampling operator, sampling series, direct inequality, Jackson-type inequality, strong converse inequality, converse estimate, Voronovskaja-type inequality, Bernstein-type inequality, saturation, modulus of smoothness, K-functional.

1 Introduction

In approximation theory, the Bernstein operator bears a significant role to prove the well-known Weierstrass approximation theorem via a simple constructive method for the case $C(I)$, $I := [0,1]$, which is the space of all continuous functions defined on I. In fact, the interval I can be translated to any compact interval of $\mathbb{R}$. Hence, roughly speaking, the Bernstein operator of order n is a mapping from $C(I)$ to $P_n(I) := Span\left\{1, x, x^2, \ldots, x^n\right\}$.

An extension of the compact interval I in an approximation process to an unbounded interval, but only the non-negative real semi axis $I := [0,\infty)$, was given by Baskakov [12], Favard [27], Mirakjan [38] and Szász [45] via transforms which are based on expansion of exponential functions.

For the counterpart of Bernstein polynomials in case of continuous functions on the whole real axis, the generalized sampling operator has been considered. The sampling operator aims to reconstruct a function f by its sample values at some discrete points and the pioneer of such a construction is the Whittaker-Kotelnikov-Shannon sampling theorem [28, 29]. A generalized version of Whittaker-Kotelnikov-Shannon sampling theorem was developed at

RWTH Aachen by P. L. Butzer and his school in the late 1970s [17, 40, 43]. More precisely, the generalized sampling operator is defined by

$$(1.1) \qquad (G_w^\chi f)(x) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wx-k), \quad x\in\mathbb{R},\ w>0,$$

where $f:\mathbb{R}\to\mathbb{R}$ is any function such that the above series is convergent for every $x\in\mathbb{R}$, and the function $\chi:\mathbb{R}\to\mathbb{R}$ (called the *kernel* of the operator) denotes a continuous, discrete approximate identity which satisfies suitable assumptions. The generalized sampling operator and its various forms are important not only from a theoretical point of view, but they also have fundamental applications in signal and image processing [4, 5, 13, 39].

One of the most important tasks in approximation theory is to establish strong converse inequalities. Such a result for the Bernstein operators in terms of a strong converse inequality of type A (classification of Ditzian and Ivanov [24]) was independently proved by Knoop and Zhou [33] and Totik [47]. As an immediate consequence of a strong converse inequality for a given approximation process, it is possible to determine the saturation order and class of the process. An earlier description of the saturation class of the Bernstein operators was established in [35] (or see [36, p. 102]). We also refer the readers to [34] about similar partial results.

On the other hand, the studies on inverse inequalities and the saturation order and class of the generalized sampling type operators are very limited in literature. They are mostly available under certain specific assumptions on the kernel [3, 19], or, particularly, for the generalized sampling Kantorovich operator [11, 21, 22].

In the present paper, we consider the above mentioned problem for a more general class of kernel functions. To do this, first, in the next section, we recall some required facts of the sampling theory and necessary tools. Next, a direct estimate for G_w^χ by means of the modulus of smoothness is stated in Theorem 3.1. Further, we give a strong converse estimate of the rate of approximation by G_w^χ in Theorem 3.3. As a result of these theorems, we obtain Corollary 3.4, in which the saturation class of G_w^χ and its invariant elements are described. Section 4 is devoted to basic estimates—a Voronovskaja-type and two Bernstein-type inequalities, which we need to prove our main results. Section 5 contains proofs of the main results. In Section 6 we illustrate the general theorems by considering several examples of kernel functions, namely, B-splines, linear combinations of their translates, and the Bochner-Riesz kernel.

2 Preliminaries and basic assumptions

Let $C(\mathbb{R})$ denote the space of the continuous (not necessarily bounded) functions on $\mathbb{R}$, $CB(\mathbb{R})$ the space of the continuous bounded functions on $\mathbb{R}$, and $UCB(\mathbb{R})$ the space of the uniformly continuous and bounded functions on $\mathbb{R}$. Further, let $\|\circ\|$ stand for the uniform norm in $CB(\mathbb{R})$. Let $C^m(\mathbb{R})$ and $CB^m(\mathbb{R})$ be the spaces of the functions that are m-times differentiable on $\mathbb{R}$ whose derivatives are continuous, respectively, continuous and bounded on $\mathbb{R}$. Also, as usual, $L(\mathbb{R})$ denotes the space of the Lebesgue summable functions on $\mathbb{R}$. Throughout the paper c stands for positive constants, not necessarily the same at each occurrence, which are independent of the considered function and the operator order.

For any function $\chi : \mathbb{R} \to \mathbb{R}$ and $s \in \mathbb{N}_0$, the function m_s, defined by

$$m_s^\chi(u) := \sum_{k\in\mathbb{Z}} \chi(u-k)\,(k-u)^s,\ u \in \mathbb{R},$$

is called the discrete moment of the function χ of order s; and the number M_σ, $\sigma \geq 0$, defined by

$$M_\sigma^\chi := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)|\,|k-u|^\sigma,$$

is called the discrete absolute moment of the function χ of order σ.

Definition 2.1. A function $\chi : \mathbb{R} \to \mathbb{R}$ is called a kernel if it satisfies the following assumptions:

(χ1) χ is continuous on $\mathbb{R}$,

(χ2) $m_0^\chi(u) = 1$ for every $u \in \mathbb{R}$,

(χ3) there exists $\sigma \geq 0$, such that the discrete absolute moments of order σ, M_σ^χ is finite.

It follows from [20, Lemma 2.1. (i)] that $M_\gamma^\chi < +\infty$ for every $0 \leq \gamma \leq \sigma$ if χ satisfies the assumptions (χ1) and (χ3).

It is known (see [40, Lemma 1 and Corollary 1]) that if χ satisfies (χ1) and the series $\sum_{k\in\mathbb{Z}} |\chi(u-k)|$ is uniformly convergent on $[0,1]$, then M_0^χ is finite and $G_w^\chi f \in CB(\mathbb{R})$ for all $f \in CB(\mathbb{R})$ and $w > 0$, as

$$\|G_w^\chi f\| \leq M_0^\chi \|f\|. \tag{2.1}$$

Thus $\{G_w^\chi\}_w$ is a family of uniformly bounded linear operators, mapping $CB(\mathbb{R})$ into itself. On the other hand, in view of assumption (χ2), $G_w^\chi f = f$ for every $f \equiv$ const and all $w > 0$.

The modulus of smoothness of order $s \in \mathbb{N}_+$ of $f \in CB(\mathbb{R})$ is defined for $t > 0$ by

$$\omega_s(f,t) := \sup_{0\leq h\leq t} \|\Delta_h^s f\|,$$

where $\Delta_h f(x) := f(x+h) - f(x)$, $x, h \in \mathbb{R}$, and $\Delta_h^s := \Delta_h(\Delta_h^{s-1})$. The expanded form of $\Delta_h^s f$ is

$$\Delta_h^s f(x) = \sum_{\ell=0}^{s} (-1)^\ell \binom{s}{\ell} f(x + (s-\ell)h), \quad x \in \mathbb{R}.$$

Let $\mathrm{Lip}(1, UCB)$ denote the Lipschitz space, consisting of the functions $f \in UCB(\mathbb{R})$ with $\omega_1(f,t) = O(t)$; or, equivalently, the functions $f \in UCB(\mathbb{R})$, for which there exists $C_f \in \mathbb{R}$ such that $|f(x+h) - f(x)| \leq C_f h$ for all $x \in \mathbb{R}$ and $h > 0$. We will also need the space $W^s_\infty(\mathbb{R})$, which consists of the functions $f : \mathbb{R} \to \mathbb{R}$ with locally absolutely continuous derivatives up to order $s-1$ such that $f^{(s)}$ is essentially bounded on $\mathbb{R}$. As is known (see e.g. [23, Chapter 6, Theorem 3.1 and Chapter 2, Theorem 9.3]), $f \in W^s_\infty(\mathbb{R})$ iff $\omega_s(f,t) = O(t^s)$.

The K-functional of order $s \in \mathbb{N}_+$ is defined for $f \in CB(\mathbb{R})$ and $t > 0$ by

$$K_s(f,t) := \inf_{g \in CB^s(\mathbb{R})} \left\{ \|f - g\| + t\|g^{(s)}\| \right\}.$$

It is known (see e.g. [23, Chapter 6, Theorem 2.4]) that there exists a positive constant c such that for all $f \in CB(\mathbb{R})$ and all $t > 0$ there holds

$$c^{-1}\omega_s(f,t) \leq K_s(f,t^s) \leq c\,\omega_s(f,t). \tag{2.2}$$

3 Main results

This section consists of the main results of the paper. We first state a direct estimate for the generalized sampling operators G^χ_w for functions belonging to $CB(\mathbb{R})$. Secondly, a converse estimate of the rate of approximation by G^χ_w is presented. Combining these two estimates, we obtain a saturation result as a corollary.

Ries and Stens [40, Theorem 3] showed that if χ is a kernel according to Definition 2.1 with $\sigma = s \in \mathbb{N}_+$, the series $\sum_{k\in\mathbb{Z}} |\chi(u-k)|$ is uniformly convergent on $[0,1]$, and $m^\chi_j(u) \equiv 0$, $j = 1, \ldots, s-1$, then

$$\|G^\chi_w f - f\| \leq \frac{M^\chi_s}{s!\,w^s}\,\|f^{(s)}\|, \quad f \in CB^s(\mathbb{R}), \quad w > 0. \tag{3.1}$$

This estimate was stated in [10, Remark 2.3, b)], and previously established under certain more restrictive assumptions on the kernel in [19, Theorem 2].

Estimate (3.1) can be generalized, by means of a short standard argument, which also uses (2.1), to the following direct inequality for all $f \in CB(\mathbb{R})$.

Theorem 3.1. *Let χ be a kernel according to Definition 2.1 with $\sigma = s \in \mathbb{N}_+$, and the series $\sum_{k\in\mathbb{Z}} |\chi(u-k)|$ is uniformly convergent on $[0,1]$. Further, let $m^\chi_j(u) \equiv 0$, $j = 1, \ldots, s-1$. Then for all $f \in CB(\mathbb{R})$ and all $w > 0$ there holds*

$$\|G^\chi_w f - f\| \leq c\,\omega_s(f, 1/w). \tag{3.2}$$

For the sake of completeness, we give the short proof of this theorem in Section 5.

Let us explicitly mention that if the kernel χ satisfies the assumptions of Theorem 3.1, then this theorem and a well-known property of the modulus of smoothness (see e.g. [23, Chapter 2, (7.12)]) imply (cf. (3.1))

$$\|G^\chi_w f - f\| = O(w^{-s}), \quad f \in W^s_\infty(\mathbb{R}). \tag{3.3}$$

Remark 3.2. As is known, for $f \in CB(\mathbb{R})$, we have $\omega_s(f,t) \to 0$ as $t \to 0^+$ iff f is uniformly continuous on $\mathbb{R}$. Thus, though (3.2) is valid for all $f \in CB(\mathbb{R})$, it implies the convergence of $G_w^\chi f$ to f in the uniform norm on $\mathbb{R}$ only if f is uniformly continuous on $\mathbb{R}$. On the other hand, (3.6) below implies that if f is *not* uniformly continuous on $\mathbb{R}$, then $\|G_w^\chi f - f\| \not\to 0$ as $w \to \infty$.

The estimate in Theorem 3.1 in the case $s = 1$ was previously established in [40, Theorem 2] (see also [16, Theorem 3(b)]). Another form is presented with the absolute constant on the right given explicitly in [10, Theorems 2.1 and 2.2, and Remark 2.3, a)]. General s were considered in [40, Corollary 2] and [16, Theorem 3(c)], where, however, the result was stated for functions, which are continuously differentiable up to order s, and by means of the modulus of continuity of their s-th derivative. Also, estimate (3.2) was established for certain types of kernels in [30, 31, 32, 46]. Its analogue using the averaged modulus of smoothness [42], extended for functions on $\mathbb{R}$ in [6], was given in [7, Theorem 4.3].

Our main goal is to establish the following converse estimate of the rate of approximation by G_w^χ.

Theorem 3.3. *Let $s \in \mathbb{N}_+$. Let $\chi \in C^{s+1}(\mathbb{R})$ be such that $m_0^\chi(u) \equiv 1$, $m_j^\chi(u) \equiv 0$, $j = 1, \ldots, s-1$, $m_s^\chi(u) \equiv \mathrm{const} \neq 0$, and $M_{s+1}^\chi, M_s^{\chi^{(s+1)}} < +\infty$. Also, let the series*

$$\sum_{k\in\mathbb{Z}} |\chi^{(j)}(u-k)|,\ j = 1, \ldots, s, \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |\chi^{(s+1)}(u-k)||u-k|^{s-1} \tag{3.4}$$

be uniformly convergent on $[0,1]$. Then there exists $r > 0$ such that for all $f \in CB(\mathbb{R})$ and all $w, v > 0$ with $v \geq rw$ there holds

$$\omega_s(f, 1/w) \leq c \left(\frac{v}{w}\right)^s \left(\|G_w^\chi f - f\| + \|G_v^\chi f - f\|\right). \tag{3.5}$$

In particular,

$$\omega_s(f, 1/w) \leq c \left(\|G_w^\chi f - f\| + \|G_{rw}^\chi f - f\|\right). \tag{3.6}$$

Earlier, a strong converse inequality, involving the supremum of $\|G_v^\chi f - f\|$ for $v \geq w$ (that is a strong converse inequality of type D in the terminology of [24]), was proved in [19, Lemma 2] for spline kernels. An equivalence relation between the norm of the approximation error of G_w^χ and a modulus of smoothness was proved in [3, Theorem 6] in the multivariate case under certain assumptions on the kernel. Theorem 3.3 includes kernels not covered there and vice versa (in the one dimensional case).

In combination with the direct inequality (3.2) (or (3.3)), Theorem 3.3 implies the following saturation result.

Corollary 3.4. *Let $s \in \mathbb{N}_+$. Let $\chi \in C^{s+1}(\mathbb{R})$ be such that $m_0^\chi(u) \equiv 1$, $m_j^\chi(u) \equiv 0$, $j = 0, \ldots, s-1$, $m_s^\chi(u) \equiv \mathrm{const} \neq 0$, and $M_{s+1}^\chi, M_s^{\chi^{(s+1)}} < +\infty$. Also, let the series in (3.4) be uniformly convergent on $[0,1]$. Then the approximation process $\{G_w^\chi\}_w$ is saturated in $UCB(\mathbb{R})$ with order $O(w^{-s})$, its saturation class is $W_\infty^s(\mathbb{R})$ and its invariant elements are the constant functions.*

In particular, in the case $s = 1$ we have that $\{G^\chi_w\}_w$ is saturated in $UCB(\mathbb{R})$ with order $O(1/w)$ and its saturation class is $\mathrm{Lip}(1, UCB)$.

Previously, the optimal rate of approximation of $\{G^\chi_w\}_w$ was established for certain types of kernels (see e.g. [19, Theorem 4]), or can be derived from the converse estimates cited after Theorem 3.3, but under certain specific assumptions on the kernel. Dryanov [25, 26] settled it for the Fejér kernel. Also, the characterization of the rate of approximation by generalized sampling operators can be reduced to estimating convolution operators for certain classes of kernels by means of the equivalence results in [14, p. 27] (or [18, Theorem 3.5(3)]) and [18, Theorem 4.3(5)].

4 Basic estimates

To establish the strong converse inequality (3.5), we will apply [24, Theorem 3.2]. To this end, besides (2.1), we need a Voronovskaja-type inequality and two Bernstein-type inequalities.

First, we establish a quantitative Voronovskaja-type inequality.

Proposition 4.1. *Let χ be a kernel according to Definition 2.1 with $\sigma = s + 1$, where $s \in \mathbb{N}_+$. Let also $m^\chi_j(u) \equiv 0$, $j = 1, \dots, s-1$, and $m^\chi_s(u) \equiv \mathrm{const} \neq 0$. Then for all $f \in CB^{s+1}(\mathbb{R})$ and all $w > 0$ there holds*

$$\left\| G^\chi_w f - f - \frac{m^\chi_s}{s!\, w^s} f^{(s)} \right\| \le \frac{M^\chi_{s+1}}{(s+1)!\, w^{s+1}} \, \|f^{(s+1)}\|.$$

A pointwise Voronovskaja inequality in terms of the least concave majorant of the modulus of continuity of $f^{(s)}$ was established in [10, Theorem 2.8]. A Voronovskaja convergence formula was given in [21, Theorem 4.1]. The case $s = 2$ was considered in [8] (see also [9]).

Proof of Proposition 4.1. This estimate is easily established by means of Taylor's formula. We have

$$f(t) = \sum_{j=0}^{s} \frac{f^{(j)}(x)}{j!} (t-x)^j + R_{s,x} f(t), \quad x, t \in \mathbb{R}, \tag{4.1}$$

where we have set

$$R_{s,x} f(t) := \frac{1}{s!} \int_x^t (t-u)^s f^{(s+1)}(u) \, du.$$

We apply the linear operator G^χ_w to both sides of (4.1), regarding t as the variable and x fixed. Thus, after taking into consideration (χ2) and $m^\chi_j(u) \equiv 0$, $j = 1, \dots, s-1$, we arrive at

$$G^\chi_w f(x) - f(x) - \frac{m^\chi_s}{s!\, w^s} f^{(s)}(x) = G^\chi_w (R_{s,x} f)(x), \quad x \in \mathbb{R}. \tag{4.2}$$

Let us estimate the right-hand side. We have for any $x \in \mathbb{R}$ that

$$
\begin{aligned}
|G_w^\chi(R_{s,x}f)(x)| &\leq \frac{1}{s!} \sum_{k\in\mathbb{Z}} \left| \int_x^{k/w} \left| \frac{k}{w} - u \right|^s |f^{(s+1)}(u)|\, du \right| |\chi(wx-k)| \\
&\leq \frac{\|f^{(s+1)}\|}{s!} \sum_{k\in\mathbb{Z}} \left| \int_x^{k/w} \left(\frac{k}{w} - u \right)^s du \right| |\chi(wx-k)| \\
&= \frac{\|f^{(s+1)}\|}{(s+1)!\, w^{s+1}} \sum_{k\in\mathbb{Z}} |k-wx|^{s+1} |\chi(wx-k)| \\
&\leq \frac{M_{s+1}^\chi}{(s+1)!\, w^{s+1}} \|f^{(s+1)}\|.
\end{aligned}
$$

Now, the assertion of the proposition follows from (4.2). ∎

Next, we will establish two Bernstein-type inequalities.

Proposition 4.2. *Let $s \in \mathbb{N}_+$. Let $\chi \in C^s(\mathbb{R})$ be such that the series*

$$\sum_{k\in\mathbb{Z}} |\chi^{(j)}(u-k)|, \quad j = 1, \dots, s,$$

are uniformly convergent on $[0,1]$, and $M_0^\chi < +\infty$. Then $G_w^\chi f \in CB^s(\mathbb{R})$ for all $f \in CB(\mathbb{R})$ and all $w > 0$, as, moreover,

$$\|(G_w^\chi f)^{(s)}\| \leq M_0^{\chi^{(s)}}\, w^s \|f\|. \tag{4.3}$$

Proof. Since the series $\sum_{k\in\mathbb{Z}} |\chi^{(j)}(u-k)|$, $j = 1, \dots, s$, are uniformly convergent on $[0,1]$, then their sums are 1-periodic and the series are uniformly convergent on the compact intervals of $\mathbb{R}$. Therefore, using also $\chi \in C^s(\mathbb{R})$, $M_0^\chi < +\infty$ and that f is bounded on $\mathbb{R}$, we get $G_w^\chi f \in C^s(\mathbb{R})$, the series defining $G_w^\chi f$ can be differentiated term-by-term s times on $\mathbb{R}$ and

$$(G_w^\chi f)^{(s)}(x) = w^s \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \chi^{(s)}(wx-k), \quad x \in \mathbb{R};$$

hence (4.3) readily follows. Let us note that $M_0^{\chi^{(s)}} < +\infty$ because the sum of $\sum_{k\in\mathbb{Z}} |\chi^{(s)}(u-k)|$ is a continuous 1-periodic function, as it follows from $\chi^{(s)} \in C(\mathbb{R})$ and the uniform convergence of the series.

To complete the proof of $G_w^\chi f \in CB^s(\mathbb{R})$, it remains to show that the functions $(G_w^\chi f)^{(j)}(x)$, $j = 1, \dots, s-1$, are bounded on $\mathbb{R}$. This follows from $G_w^\chi f, (G_w^\chi f)^{(s)} \in CB(\mathbb{R})$ and the well-known inequality for the intermediate derivatives (see e.g. [23, Chapter 2, Theorem 5.6])

$$|g^{(j)}(x)| \leq c\big(\|g\| + \|g^{(s)}\|\big), \quad x \in \mathbb{R}, \quad j = 1, \dots, s-1,$$

where c is a constant and $g \in C^s(\mathbb{R})$ is such that $g, g^{(s)} \in CB(\mathbb{R})$. ∎

Proposition 4.3. *Let $s \in \mathbb{N}_+$. Let $\chi \in C^{s+1}(\mathbb{R})$ be such that the series*

$$\sum_{k \in \mathbb{Z}} |\chi^{(j)}(u-k)|, \quad j = 1, \ldots, s+1,$$

are uniformly convergent on $[0,1]$, $m_j^{\chi^{(s+1)}}(u) \equiv 0$ for $j = 0, \ldots, s-1$, and $M_0^\chi, M_s^{\chi^{(s+1)}} < +\infty$. Then $G_w^\chi f \in CB^{s+1}(\mathbb{R})$ for all $f \in CB^s(\mathbb{R})$ and all $w > 0$, as, moreover,

$$\|(G_w^\chi f)^{(s+1)}\| \le \frac{M_s^{\chi^{(s+1)}}}{s!}\, w\, \|f^{(s)}\|. \tag{4.4}$$

Proof. Just as in the proof of the previous proposition we verify that $G_w^\chi f \in C^{s+1}(\mathbb{R})$ and

$$(G_w^\chi f)^{(s+1)}(x) = w^{s+1} \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \chi^{(s+1)}(wx-k), \quad x \in \mathbb{R}. \tag{4.5}$$

We expand $f(k/w)$ by Taylor's formula at the point x to get

$$f\left(\frac{k}{w}\right) = \sum_{j=0}^{s-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{w} - x\right)^j + \frac{1}{(s-1)!} \int_x^{k/w} \left(\frac{k}{w} - u\right)^{s-1} f^{(s)}(u)\, du.$$

Then, taking into account that $m_j^{\chi^{(s+1)}}(u) \equiv 0$ for $j = 0, \ldots, s-1$, we get from (4.5) the relation

$$\begin{aligned}
&(G_w^\chi f)^{(s+1)}(x) \\
&\quad = \frac{w^{s+1}}{(s-1)!} \sum_{k \in \mathbb{Z}} \int_x^{k/w} \left(\frac{k}{w} - u\right)^{s-1} f^{(s)}(u)\, du\, \chi^{(s+1)}(wx-k), \quad x \in \mathbb{R}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(G_w^\chi f)^{(s+1)}(x)| &\le \frac{w^{s+1}\|f^{(s)}\|}{(s-1)!} \sum_{k \in \mathbb{Z}} \left| \int_x^{k/w} \left(\frac{k}{w} - u\right)^{s-1} du \right| |\chi^{(s+1)}(wx-k)| \\
&\le \frac{M_s^{\chi^{(s+1)}}}{s!}\, w\, \|f^{(s)}\|, \quad x \in \mathbb{R}.
\end{aligned}$$

Finally, we show that $(G_w^\chi f)^{(j)}(x)$, $j = 1, \ldots, s$, are also bounded on $\mathbb{R}$ just as in the proof of Proposition 4.2. ∎

5 Proof of the main results

For the sake of completeness, we give the proof of the direct estimate of the rate of approximation of G_w^χ stated in Theorem 3.1.

Proof of Theorem 3.1. The argument, which verifies the direct inequality (3.2) is quite standard—it is based on the Jackson-type inequality in (3.1) and the equivalence of the moduli of smoothness to the K-functionals given in (2.2).

By virtue of (2.1) and (3.1), we have for any $g \in CB^s(\mathbb{R})$ and $w > 0$ that

$$\begin{aligned}\|G_w^\chi f - f\| &\le \|G_w^\chi (f-g)\| + \|G_w^\chi g - g\| + \|g - f\| \\ &\le (M_0^\chi + 1)\|f - g\| + \frac{M_s^\chi}{s!\, w^s}\|g^{(s)}\| \\ &\le \left(1 + M_0^\chi + \frac{M_s^\chi}{s!}\right)\left(\|f-g\| + \frac{1}{w^s}\|g^{(s)}\|\right).\end{aligned}$$

Taking the infimum on $g \in CB^s(\mathbb{R})$, we arrive at

$$\|G_w^\chi f - f\| \le \left(1 + M_0^\chi + \frac{M_s^\chi}{s!}\right) K_s(f, w^{-s}), \quad w > 0. \tag{5.1}$$

Now, the assertion of the theorem follows from the right inequality in (2.2). ∎

We proceed to the proof of our main results.

Proof of Theorem 3.3. First, we observe that $M_{s+1}^\chi < +\infty$ implies $M_0^\chi, M_{s-1}^\chi < +\infty$, and $M_s^{\chi^{(s+1)}} < +\infty$ implies $M_{s-1}^{\chi^{(s+1)}} < +\infty$.

We set $g_1(u) := u^{s-1}\chi(u)$ and $g_2(u) := u^{s-1}\chi^{(s+1)}(u)$. Since $M_0^{g_1} = M_{s-1}^\chi < +\infty$ and $M_0^{g_2} = M_{s-1}^{\chi^{(s+1)}} < +\infty$, then $g_1, g_2 \in L(\mathbb{R})$ (see e.g. [40, Lemma 1, (2.3)]).

Also, since the series $\sum_{k\in\mathbb{Z}} |\chi^{(s+1)}(u-k)||u-k|^{s-1}$ is uniformly convergent on $[0,1]$, then so are the series $\sum_{k\in\mathbb{Z}} |\chi^{(s+1)}(u-k)||u-k|^{j}$ for $j = 0, \dots, s-2$; hence $m_j^{\chi^{(s+1)}} \in C(\mathbb{R})$, $j = 0, \dots, s-1$.

Now, $m_j^\chi(u) \equiv \mathrm{const}$, $j = 0, \dots, s-1$, imply $m_j^{\chi^{(s+1)}}(u) \equiv 0$ for $j = 0, \dots, s-1$ (see e.g. [1, Lemma 3] and its proof).

Thus, estimate (2.1) holds and the hypotheses of Propositions 4.1, 4.2 and 4.3 are satisfied.

We apply [24, Theorem 3.2] to the operators G_w^χ, as $n = w$, $k = v$, $X = CB(\mathbb{R})$ is equipped with the uniform norm on $\mathbb{R}$, $Y = CB^s(\mathbb{R})$ and $Z = CB^{s+1}(\mathbb{R})$.

Relation (2.1) shows that [24, (3.3)] is satisfied.

By virtue of Proposition 4.1, we have that [24, (3.4)] holds with $\lambda(w) = |m_s^\chi|/(s!\, w^s)$, $Df = -[\operatorname{sgn} m_s^\chi] f^{(s)}$, $\lambda_1(w) = M_{s+1}^\chi/((s+1)!\, w^{s+1})$ and $\Phi(f) = \|f^{(s+1)}\|$.

Next, Proposition 4.3 with $G_w^\chi f$ in place of f yields [24, (3.5)] with $m = 2$, $\ell = 1$ and $A = M_{s+1}^\chi M_s^{\chi^{(s+1)}}/((s+1)!\, |m_s^\chi|)$.

Finally, Proposition 4.2 implies [24, (3.6)].

We determine $r > 0$ so that we have

$$A\frac{\lambda(w)}{\lambda(v)}\,\frac{\lambda_1(v)}{\lambda_1(w)} = \frac{M_{s+1}^\chi M_s^{\chi^{(s+1)}}}{(s+1)!\,|m_s^\chi|}\,\frac{w}{v} \le \frac{1}{2} \quad \text{for} \quad v \ge rw.$$

Now, [24, Theorem 3.2] yields (3.5). ∎

Proof of Corollary 3.4. If $\|G_w^\chi f - f\| = o(w^{-s})$, then, by virtue of (3.6), we have $\omega_s(f,t) = o(t^s)$. This implies that f is an algebraic polynomial of degree at most $s-1$ (see e.g. [23, Chapter 2, Proposition 7.1]), but since it is bounded on $\mathbb{R}$, it is constant and hence $G_w^\chi f = f$. Thus f is an invariant element of $\{G_w^\chi\}_w$.

To complete the proof that $\{G_w^\chi\}_w$ possesses the saturation property and its optimal approximation order is $O(w^{-s})$, it remains to observe that, clearly, there exists a noninvariant element $f \in UCB(\mathbb{R})$ such that $\|G_w^\chi f - f\| = O(w^{-s})$—by Theorem 3.1 (or see (3.3)) any function f in $W_\infty^s(\mathbb{R})$, which is not identically constant, is such. Let us note that this theorem is applicable since $M_{s+1}^\chi < +\infty$ implies $M_s^\chi < +\infty$.

Let us show that the saturation class of $\{G_w^\chi\}_w$ is $W_\infty^s(\mathbb{R})$. Indeed, Theorem 3.1 implies (3.3) and if $\|G_w^\chi f - f\| = O(w^{-s})$, then, by virtue of Theorem 3.3 (in particular, (3.6)), we have $\omega_s(f,t) = O(t^s)$; hence $f \in W_\infty^s(\mathbb{R})$ (see e.g. [23, Chapter 6, Theorem 3.1 and Chapter 2, Theorem 9.3]). ∎

6 Examples

We will illustrate the general results we established by several concrete well-known kernels. In order to evaluate the discrete moments of the kernel, we will use the Poisson summation formula (see e.g. [15, pp. 201–202]), which implies that if the function $u^j\chi(u)$ is in $L(\mathbb{R})$, then the Fourier series of $m_j^\chi(u)$ is

$$(-i)^j \sum_{k\in\mathbb{Z}} \widehat{\chi}^{(j)}(2\pi k) e^{i2\pi k u}.$$

The Fourier transform of $\chi \in L(\mathbb{R})$ is defined by

$$\widehat{\chi}(v) := \int_{\mathbb{R}} \chi(u)\, e^{-ivu} du, \quad v \in \mathbb{R}.$$

Let us note that, if χ satisfies $(\chi 1)$ and M_0^χ is finite, then $\chi \in L(\mathbb{R})$ (see e.g. [40, Lemma 1]).

This approach to evaluating the discrete moments is quite standard (see e.g. [19, Lemmas 2 and 3] and [16, Theorem 3(b)]).

6.1 B-spline kernels

One of the well-known examples of duration limited kernels is given by the central B-splines of order $n \in \mathbb{N}_+$, defined by

$$(6.1) \qquad B_n(x) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{n}{2} + x - j\right)_+^{n-1}, \quad x \in \mathbb{R},$$

where $(x)_+^k := \max\{x^k, 0\}$, $x \in \mathbb{R}$; see [41, p. 176] or [2, 48]. Clearly, $B_n(x)$ has finite support—the interval $\left[-\frac{n}{2}, \frac{n}{2}\right]$, and it is bounded for all n; also $B_n \in C(\mathbb{R})$ for $n \geq 2$.

The Fourier transform of B_n is (see e.g. [19, (5.2.10)])

$$\widehat{B}_n(v) = \left(\frac{\sin(v/2)}{v/2}\right)^n, \quad v \in \mathbb{R}.$$

Therefore, if $n \geq 3$, then

$$\widehat{B}_n^{(j)}(2\pi k) = \begin{cases} 1, & k = j = 0, \\ 0, & k \in \mathbb{Z},\ k \neq 0,\ j = 0 \\ 0, & k \in \mathbb{Z},\ j = 1, \\ -\dfrac{n}{12}, & k = 0,\ j = 2, \\ 0, & k \in \mathbb{Z},\ k \neq 0,\ j = 2. \end{cases} \tag{6.2}$$

Now, by means of the Poisson summation formula we arrive at

$$m_0^{B_n}(u) = 1, \quad m_1^{B_n}(u) = 0, \quad m_2^{B_n}(u) = \frac{n}{12}, \quad u \in \mathbb{R}.$$

Hence the conditions concerning the discrete moments of the kernel in Theorem 3.3 and Corollary 3.4 hold for B_n with $s = 2$. Since B_n is bounded and of finite support, then $M_3^{B_n} < +\infty$.

Finally, we have $B_n \in C^3(\mathbb{R})$ if $n \geq 5$, as $B_n^{(j)}$, $j = 1, 2, 3$, are of finite support. Consequently, we trivially have that $M_2^{B_n'''} < +\infty$ and the series in (3.4) with $\chi = B_n$ are uniformly convergent on $[0, 1]$.

Thus, Theorem 3.3 and Corollary 3.4 are applicable with $s = 2$ and they yield the following result.

Corollary 6.1. *Let B_n be the centeral B-spline given in* (6.1) *with $n \geq 5$. Then there exists $r > 0$ such that for all $f \in CB(\mathbb{R})$ and all $w, v > 0$ with $v \geq rw$, there holds*

$$\omega_2(f, 1/w) \leq c\left(\frac{v}{w}\right)^2 \left(\|G_w^{B_n} f - f\| + \|G_v^{B_n} f - f\|\right).$$

In particular,

$$\omega_2(f, 1/w) \leq c\left(\|G_w^{B_n} f - f\| + \|G_{rw}^{B_n} f - f\|\right).$$

Furthermore, the approximation process $\{G_w^{B_n}\}_w$ is saturated in $UCB(\mathbb{R})$ with order $O(w^{-2})$, its saturation class is $W_\infty^2(\mathbb{R})$ and its invariant elements are the constant functions.

6.2 Kernels of linear combinations of translates of B-splines

To obtain a kernel with higher order zero moments, we can apply [19, Theorem 3]. In particular, Bardaro et. al [10, Section 4, II] considered the combination

of translations of B-splines
(6.3)
$$\varphi_n(u) := \left(3-\frac{n}{24}\right) B_n(x-1) + \left(\frac{n}{12}-3\right) B_n(x-2) + \left(1-\frac{n}{24}\right) B_n(x-3),$$

where $n \geq 3$. The Fourier transform of φ_n is

$$\widehat{\varphi}_n(v) = \widehat{B}_n(v)\left(\left(3-\frac{n}{24}\right)e^{-iv} + \left(\frac{n}{12}-3\right)e^{-2iv} + \left(1-\frac{n}{24}\right)e^{-3iv}\right). \tag{6.4}$$

As in the previous subsection, it can be obtained for $n \geq 4$ that (see [10, Section 4, II])

$$m_1^{\varphi_n}(u) = m_2^{\varphi_n}(u) = 0,\ m_3^{\varphi_n}(u) = -6 + \frac{n}{2}, \quad u \in \mathbb{R}.$$

On the other hand, φ_n is bounded and has compact support—the interval $\left[1-\frac{n}{2}, 3+\frac{n}{2}\right]$, which immediately yields that $M_4^{\varphi_n} < +\infty$.

Consequently, the conditions on the moments of the kernel in Theorem 3.3 and Corollary 3.4 hold for φ_n, $n \geq 4$, $n \neq 12$, with $s = 3$.

Just as in the previous subsection we see that if $n \geq 6$, then $\varphi_n \in C^4(\mathbb{R})$, $M_3^{\varphi_n^{(4)}} < +\infty$ and the series in (3.4) with $\chi = \varphi_n$ are uniformly convergent on $[0,1]$.

Now, Theorem 3.3 and Corollary 3.4 with $s = 3$ imply

Corollary 6.2. *Let φ_n be defined by* (6.3) *with $n \geq 6$ and $n \neq 12$. Then there exists $r > 0$ such that for all $f \in CB(\mathbb{R})$ and all $w, v > 0$ with $v \geq rw$, there holds*

$$\omega_3(f, 1/w) \leq c\left(\frac{v}{w}\right)^3 \left(\|G_w^{\varphi_n} f - f\| + \|G_v^{\varphi_n} f - f\|\right).$$

In particular,

$$\omega_3(f, 1/w) \leq c\left(\|G_w^{\varphi_n} f - f\| + \|G_{rw}^{\varphi_n} f - f\|\right).$$

Furthermore, the approximation process $\{G_w^{\varphi_n}\}_w$ is saturated in $UCB(\mathbb{R})$ with order $O(w^{-3})$, its saturation class is $W_\infty^3(\mathbb{R})$ and its invariant elements are the constant functions.

Remark 6.3. If $n = 12$ in the previous corollary, since $m_3^{\varphi_{12}}(u)$ is also equal to 0 for all u, and $m_4^{\varphi_{12}}(u) \equiv \text{const} \neq 0$ (see [10, Section 4, II]), then, similarly, it is established that $\{G_w^{\varphi_{12}}\}_w$ is saturated in $UCB(\mathbb{R})$ with order $O(w^{-4})$ and its saturation class is $W_\infty^4(\mathbb{R})$.

6.3 The Bochner-Riesz kernels

The Bochner-Riesz kernel is defined by

$$b^\gamma(x) := \frac{2^\gamma}{\sqrt{2\pi}}\Gamma(\gamma+1)\,|x|^{-\frac{1}{2}-\gamma} J_{\frac{1}{2}+\gamma}(|x|) \text{ for } \gamma > 0, \tag{6.5}$$

where J_λ is the Bessel function of order λ (see [15, p. 467], or [44, pp. 170–171]). It is well known that the Fourier transform of b^γ is (see e.g. [15, p. 468])

$$\widehat{b^\gamma}(v) = \begin{cases} \left(1-v^2\right)^\gamma, & |v| \leq 1, \\ 0, & |v| > 1. \end{cases}$$

Since $J_\lambda(x) = O(|x|^{-1/2})$, as $x \to \pm\infty$ (see e.g. [37, p. 204]), then $b^\gamma(x) = O(|x|^{-\gamma-1})$, as $x \to \pm\infty$. Thus, $u^2 b^\gamma(u)$ belongs to $L(\mathbb{R})$ for $\gamma > 2$ and, in the light of the Poisson summation formula, we have

$$m_0^{b^\gamma}(u) = 1, \quad m_1^{b^\gamma}(u) = 0, \quad m_2^{b^\gamma}(u) = 2\gamma, \quad u \in \mathbb{R}.$$

Thus, the conditions concerning the discrete moments of the kernel in Theorem 3.3 and Corollary 3.4 hold with $s = 2$, provided $\gamma > 2$.

Furthermore, since $b^\gamma(u) = O(|u|^{-\gamma-1})$, as $u \to \pm\infty$, then $M_3^{b^\gamma} < +\infty$ for $\gamma > 3$ (see [8]).

It remains to observe that $b^\gamma \in C^\infty(\mathbb{R})$ and, since $J'_\lambda(x) = (J_{\lambda-1}(x) - J_{\lambda+1}(x))/2$ (see e.g. [37, p. 48]), then $(b^\gamma)^{(j)}(x) = O(|x|^{-\gamma-1})$, as $x \to \pm\infty$; hence the series in (3.4) with $\chi = b^\gamma$ and $s = 2$ are uniformly convergent on $[0, 1]$ for any $\gamma > 1$, and $M_2^{(b^\gamma)'''} < +\infty$ for any $\gamma > 2$.

Now, Theorem 3.3 and Corollary 3.4 with $s = 2$ imply the following result.

Corollary 6.4. *Let b^γ be the Bochner-Riesz kernel, defined in* (6.5) *with* $\gamma > 3$. *Then there exists $r > 0$ such that for all $f \in CB(\mathbb{R})$ and all $w, v > 0$ with $v \geq rw$, there holds*

$$\omega_2(f, 1/w) \leq c \left(\frac{v}{w}\right)^2 \left(\|G_w^{b^\gamma} f - f\| + \|G_v^{b^\gamma} f - f\|\right).$$

In particular,

$$\omega_2(f, 1/w) \leq c \left(\|G_w^{b^\gamma} f - f\| + \|G_{rw}^{b^\gamma} f - f\|\right).$$

Furthermore, the approximation process $\{G_w^{b^\gamma}\}_w$ is saturated in $UCB(\mathbb{R})$ with order $O(w^{-2})$, its saturation class is $W^2_\infty(\mathbb{R})$ and its invariant elements are the constant functions.

References

[1] T. Acar, D. Costarelli and G. Vinti, Linear prediction and simultaneous approximation by m-th order Kantorovich type sampling series, Banach J. Math. Anal. 14 (2020), 1481–1508.

[2] G. Allasia, R. Cavoretto, A. De Rossi, A class of spline functions for landmark-based image registration, Math. Methods Appl. Sci. 35 (2012), 923–934.

[3] S. Artamonov, K. V. Runovski, H. J. Schmeisser, Approximation by families of generalized sampling series, realizations of generalized K-functionals and generalized moduli of smoothness, J. Math. Anal. Appl. 489 (2020), 124138.

[4] F. Asdrubali, G. Baldinelli, F. Bianchi, D. Costarelli, L. Evangelisti, A. Rotili, M. Seracini, G. Vinti, A model for the improvement of thermal bridges quantitative assessment by infrared thermography, Appl. Energy 211 (2018), 854–864.

[5] F. Asdrubali, G. Baldinelli, F. Bianchi, D. Costarelli, A. Rotili, M. Seracini, G. Vinti, Detection of thermal bridges from thermographic images by means of image processing approximation algorithms, Appl. Math. Comput. 317 (2018), 160–171.

[6] C. Bardaro, P. L. Butzer, R. L. Stens, G. Vinti, Approximation error of the Whittaker cardinal series in terms of an averaged modulus of smoothness covering discontinuous signals, J. Math. Anal. Appl. 316(1) (2006), 269–306.

[7] C. Bardaro, P. L. Butzer, R. L. Stens, G. Vinti, Prediction by samples from the past with error estimates covering discontinuous signals, IEEE Trans. Inf. Theory 56(1) (2010), 614–633.

[8] C. Bardaro, I. Mantellini, A quantitative Voronovskaja formula for generalized sampling operators, East J. Approx. 15 (2009), 429–441.

[9] C. Bardaro, I. Mantellini, A Voronovskaya-type theorem for a general class of discrete operators, Rocky Mountain J. Math. 39 (2009), 1411–1442.

[10] C. Bardaro, I. Mantellini, Asymptotic formulae for linear combinations of generalized sampling operators, Z. Anal. Anwend. 32 (2013), 279–298.

[11] B. Bartoccini, D. Costarelli, G. Vinti, Extension of saturation theorems for the sampling Kantorovich operators, Complex Anal. Oper. Theory 13 (2019), 1161–1175.

[12] V. A. Baskakov, An example of a sequence of linear positive operators in space of continuous functions, Dokl. Akad. Nauk. SSSR 113 (1957), 249–251 (in Russian).

[13] T. Blu, M. Unser, Approximation error for quasi-interpolators and (multi-) wavelet expansions, Appl. Comput. Harmon. Anal. 6 (1999), 219–251.

[14] Z. Burinska, K. Runovski, H.-J. Sehmeisser, On the approximation by generalized sampling series in L_p-metrics, Sampl. Theory Signal Image Process. 5 (2006), 59–87.

[15] P. L. Butzer, R. J. Nessel, Fourier Analysis and Approximation, Birkhäser Verlag, Basel, 1971.

[16] P. L. Butzer, S. Ries, R. L. Stens, Approximation of continuous and discontinuous functions by generalized sampling series, J. Approx. Theory 50 (1987), 25–39.

[17] P.L. Butzer, W. Splettstösser, A sampling theorem for duration limited functions with error estimates, Inform. Contr. 34 (1977), 55–65.

[18] P.L. Butzer, W. Splettstösser, R. L. Stens, The sampling theorem and linear prediction in signal analysis, Jahresber. Dtsch. Math.-Ver. 90 (1988), 1–70.

[19] P. L. Butzer, R. L. Stens, Linear prediction by samples from the past, In: R. J. Marks II (eds), Advanced Topics in Shannon Sampling and Interpolation Theory, pp. 157–183, Springer Texts in Electrical Engineering, Springer, New York, 1993.

[20] D. Costarelli, G. Vinti, Rate of approximation for multivariate sampling Kantorovich operators on some functions spaces, J. Integr. Equ. Appl., 26(4) (2014), 455–481.

[21] D. Costarelli, G. Vinti, Inverse results of approximation and the saturation order for the sampling Kantorovich series, J. Approx. Theory 242 (2019) 64–82.

[22] D. Costarelli, G. Vinti, Saturation by the Fourier transform method for the sampling Kantorovich series based on bandlimited kernels, Anal. Math. Phys. 9 (2019), 2263–2280.

[23] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.

[24] Z. Ditzian, K. G. Ivanov, Strong converse inequalities, J. Anal. Math. 61 (1993), 61–111.

[25] D. P. Dryanov, On the convergence and saturation problem of a class of discrete linear operators of entire exponential type in $L_p(-\infty, \infty)$ spaces, In: B. Sendov, P. Petrushev, R. Maleev, S. Tashev (eds.), Proc. Internat. Conf. "Constructive Theory of Functions", Varna, Bulgaria, June 1984, pp. 312–318, Bulgarian Acad. Sci., Sofia, 1984.

[26] D. P. Dryanov, On the convergence and saturation problem of a sequence of discrete linear operators of exponential type in $L_p(-\infty, \infty)$ spaces, Acta Math. Hungar. 49 (1987), 103–127.

[27] J. Favard, Sur les multiplicateurs d'interpolation, J. Math. Pures Appl. 23(9) (1944), 219–247.

[28] H. Feichtinger, K. Gröchenig, Theory and practice of irregular sampling. In: J. Benedetto, M. Frazier (eds.), Wavelets: Mathematics and Applications, pp. 305–363, CRC Press Inc., London, 1994.

[29] K. Gröchenig, Reconstruction algorithms in irregular sampling. Math. Comput. 59 (1992), 181–194.

[30] A. Kivinukk, Approximation by typical sampling series, in: Proc. 1999 Intern. Workshop on Sampling Theory and Applications, Loen, Norway, pp. 161–166, Norwegian Univ. Sci. and Technology, 1999.

[31] A. Kivinukk, G. Tamberg, Interpolating generalized Shannon sampling operators, their norms and approximation properties, Samp. Theory Signal Image Process. 8 (2009), 77–95.

[32] A. Kivinukk, G. Tamberg, On approximation properties of sampling operators by dilated kernels, In: 8th International in Conference on Sampling Theory and Applications, SampTA 2009, May 2009, Marseille, France, pp. 18–22, 2009.

[33] H.-B. Knoop, X.-L. Zhou, The lower estimate for linear positive operators (II), Results Math. 25 (1994), 315–330.

[34] K. d. Leeuw, On the degree of approximation by Bernstein polynomials, J. Anal. Math. 7 (1959), 89–104.

[35] G. G. Lorentz, Inequalities and the saturation classes of Bernstein polynomials, In: P.L. Butzer, J. Korevaar (eds.), On Approximation Theory, ISNM International Series of Numerical Mathematics, vol. 5, pp. 200–207, Springer, Basel, 1964.

[36] G. G. Lorentz, Approximation of functions, Holt, Rinehart and Winston, Inc., New York, 1966.

[37] Y. L. Luke, The Special Functions and Their Approximations, vol. 1, Academic Press, Inc., San Diego, 1969.

[38] G. M. Mirakjan, Approximation of continuous functions with the aid of polynomials, Dokl. Acad. Nauk SSSR 31 (1941), 201–205.

[39] R. J. Ravier, R. S. Stichartz, Sampling theory with average values on the Sierpinski gasket, Constr. Approx. 44(2) (2016), 159–194.

[40] S. Ries, R. L. Stens, Approximation by generalized sampling series, In: B. Sendov, P. Petrushev, R. Maleev, S. Tashev (eds.), Proc. Internat. Conf. "Constructive Theory of Functions", Varna, Bulgaria, June 1984, pp. 746–756, Bulgarian Acad. Sci., Sofia, 1984.

[41] I. J. Schoenberg, Cardinal interpolation and spline functions, J. Approx. Theory 2 (1969), 167–206.

[42] B. Sendov, V. A. Popov, The Averaged Moduli of Smoothness, Chichester, U.K., Wiley, 1988.

[43] W. Splettstosser, On generalized sampling sums based on convolution integrals, Arch. Elek. Ubertr. 32 (1978), 267–275.

[44] E. M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Space, Princeton Univ. Press, Princeton,1971.

[45] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Nat. Bur. Standards, Sect. B 45 (1950), 239–245.

[46] G. Tamberg, Approximation by generalized Shannon sampling operators generated by band-limited kernels, Proc. Appl. Math. Mech. 8(1) (2008), 10937–10940.

[47] V. Totik, Approximation by Bernstein polynomials, Amer. J. Math. 116 (1994), 995–1018.

[48] M. Unser, Ten good reasons for using spline wavelets, Wavelets Appl. Signal Image Process. 3169(5) (1997), 422–431.

Tuncer Acar
Department of Mathematics
Faculty of Science
Selcuk University
Selcuklu
42003 Konya
Turkey
tunceracar@ymail.com

Borislav R. Draganov
Department of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5 James Bourchier Blvd.
1164 Sofia
Bulgaria
and
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
Bulgaria
bdraganov@fmi.uni-sofia.bg

This version of the article has been accepted for publication, after peer review but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online