Upper estimates of the approximation rate of combinations of iterates of the Bernstein operator

Borislav R. Draganov*

Abstract

We present upper estimates of the approximation rate of combinations $\mathcal{B}_{r,n}$ of iterates of the Bernstein operator B_n, defined by $I - \mathcal{B}_{r,n} = (I - B_n)^r$, $r \in \mathbb{N}$. The treatment is based on (weighted) simultaneous approximation by the Bernstein operator. We give a sufficient condition on the smoothness of the function that implies approximation rate of n^{-r}.

AMS classification: 41A10, 41A17, 41A25, 41A28, 41A35, 41A36.
Key words and phrases: Bernstein polynomials, iterates, Jackson inequality, upper error estimate, simultaneous approximation, modulus of smoothness.

1 Main results

Probably the most investigated linear approximating operator is the Bernstein polynomial, defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

It is known (see [1, Chapter 10, § 7] and [5, Chapter 9]) that there exists $n_0 \in \mathbb{N}$ such that for all $f \in C[0,1]$ and $n \geq n_0$ there holds

$$\|B_n f - f\| \leq c\, \omega_\varphi^2(f, n^{-1/2}), \tag{1.1}$$

where $\| \circ \|$ stands for the uniform norm on the interval $[0,1]$, c is an absolute constant and $\omega_\varphi^2(f,t)$ is the Ditzian-Totik modulus of smoothness of second order with step-weight $\varphi(x) = \sqrt{x(1-x)}$, defined by (see [5, Chapter 1])

$$\omega_\varphi^2(f,t) = \sup_{0<h\leq t} \|\Delta_{h\varphi}^2 f\|$$

*Supported by grant No. 133/2012 of the National Science Fund to the University of Sofia.

and

$$\Delta^2_{h\varphi(x)} f(x) = \begin{cases} f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)), & x \pm h\varphi(x) \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

For $f \in AC^1_{loc}(0,1)$ and $n \in \mathbb{N}$ we have

$$\|B_n f - f\| \leq \frac{c}{n} \|\varphi^2 f''\|. \tag{1.2}$$

Moreover, $B_n f$ cannot tend to f in $C[0,1]$ faster than n^{-1} unless f is a linear function, in which case we have $B_n f = f$ for all n (see e.g. [1, Chapter 10, § 5]).

One way to modify the Bernstein operator in order to get larger approximation rate is to form an appropriate linear combination of its iterates. Here we shall consider the bounded linear operator $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$, defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}$. Our main objective is to establish the following upper estimate of the error of $\mathcal{B}_{r,n}$.

Theorem 1.1. *For $f \in C^{2r-2}[0,1]$ and $r \geq 2$, there holds*

$$\|\mathcal{B}_{r,n} f - f\| \leq \frac{c}{n^{r-1}} \left(\omega^2_\varphi(\varphi^{2r-2} f^{(2r-2)}, n^{-1/2}) + \frac{1}{n}\|f^{(2r-2)}\| + \frac{1}{n}\|f^{(2)}\| \right).$$

The value of the constant c is independent of f and n.

The above implies a sufficient condition on the smoothness of the function, which yields an approximation order of n^{-r}.

Corollary 1.2. *Let $f \in C[0,1]$ and $n, r \in \mathbb{N}$ as $r \geq 2$. Then:*

(a) $\|\mathcal{B}_{r,n} f - f\| \leq \frac{c}{n^{r-1/2}} \left(\|\varphi^{2r-1} f^{(2r-1)}\| + \|f^{(2r-2)}\| + \|f^{(2)}\| \right),$

$f \in AC^{2r-2}_{loc}(0,1);$

(b) $\|\mathcal{B}_{r,n} f - f\| \leq \frac{c}{n^{r}} \left(\|\varphi^{2r} f^{(2r)}\| + \|f^{(2r-2)}\| + \|f^{(2)}\| \right), \quad f \in AC^{2r-1}_{loc}(0,1).$

The value of the constant c is independent of f and n.

In order to extend the estimates above for every continuous functions we can introduce the K-functional

$$K_r(f,t) = \inf_{g \in AC^{2r-1}_{loc}} \left\{ \|f-g\| + t\big(\|\varphi^{2r} g^{(2r)}\| + \|g^{(2r-2)}\| + \|g^{(2)}\|\big) \right\}.$$

for $f \in C[0,1]$, $t > 0$ and $r \in \mathbb{N}$ with $r \geq 2$. Standard considerations imply the following Jackson-type inequality from Corollary 1.2 (b).

Theorem 1.3. *Let $f \in C[0,1]$ and $n, r \in \mathbb{N}$ as $r \geq 2$. Then*

$$\|\mathcal{B}_{r,n} f - f\| \leq c\, K_r(f, n^{-r}).$$

The value of the constant c is independent of f and n.

Let us note that

$$K_r(f, t^{2r}) \leq c\big(\omega^{2r}(f,t) + t^{2r}\|f\|\big), \quad f \in C[0,1], \quad t > 0, \tag{1.3}$$

where $\omega^\ell(f,t)$ is the classical fixed-step modulus of smoothness of order ℓ, defined by

$$\omega^\ell(f,t) = \sup_{0<h\leq t} \|\Delta_h^\ell f\|$$

and Δ_h^ℓ is the ℓth symmetric finite difference

$$\Delta_h^\ell f(x) = \begin{cases} \displaystyle\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} f\left(x + \left(\frac{\ell}{2} - k\right) h\right), & x \pm \dfrac{\ell h}{2} \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

The inequality (1.3) follows from the embedding inequality

$$\|f^{(m)}\| \leq c\big(\|f\| + \|f^{(\ell)}\|\big), \quad m = 0, \dots, \ell, \tag{1.4}$$

and the well-known result of Johnen (see e.g. [1, Chapter 6, Theorem 2.4])

$$\inf_{g \in AC^{\ell-1}[0,1]} \left\{ \|f - g\| + t^\ell \|g^{(\ell)}\| \right\} \leq c\, \omega^\ell(f,t), \quad f \in C[0,1].$$

All estimates with the Ditzian-Totik modulus are established for $n \geq n_0$ with some absolute constant n_0. However, the assertions of Corollary 1.2 and Theorem 1.3 are valid for all n (see Remark 3.1 at the end).

We base our proof of Theorem 1.1 on upper estimates for simultaneous approximation by Bernstein polynomials. They are established in the next section. This approach lays stronger conditions on the function than necessary but provides us with a simple proof. We verify Theorem 1.1 (and its corollary) in the third and final section.

2 Simultaneous approximation by Bernstein polynomials

There is a simple method for deriving upper estimates for combinations of iterates of a linear operator by iterating the estimate for the operator (see [4, Theorem 10.2 and Corollary 10.3]). However, it is not applicable in the case of the Bernstein operator because it does not commute with the associated differential operator $Dg = \varphi^2 g''$. Another difficulty of a technical character lies with the fact that $\mathcal{B}_{r,n}$ is not generally a positive operator. In order to get round the latter, we shall establish upper estimates that are similar to (1.1) for simultaneous approximation. This will allow us to get the result about $\mathcal{B}_{r,n}$ still by a certain iteration. This approach has a shortcoming. It misses the point that $\mathcal{B}_{r,n}$ provides better approximation near the ends of the interval $[0,1]$ (it interpolates f at 0 and 1). The simultaneous approximation by B_n does not possess this property.

Our first result concerns the unweighted simultaneous approximation by B_n.

Theorem 2.1. *For $f \in C^s[0,1]$ there holds*

$$\|(B_n f - f)^{(s)}\| \le c\left(\omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega(f^{(s)}, n^{-1}) + \frac{1}{n}\|f^{(s)}\|\right).$$

The value of the constant c is independent of f and n.

Proof. The assertion is trivial for $n < s$. For $n \ge s$ it is known (see [14] or [1, Chapter 10, (2.3)], [5, p. 125]) that

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!}\sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}(x), \tag{2.1}$$

where $\overrightarrow{\Delta}_h^s f(x) = \Delta_h^s f(x + sh/2)$ are the forward differences of order s.

Now, for $n = s$ the above formula immediately implies the assertion of the theorem. Let $n > s$. We set

$$\widetilde{D}_{s,n} f(x) = n^s \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{n-s}{n}\,x\right), \quad x \in [0,1].$$

Then by (2.1)

$$(B_n f)^{(s)}(x) = \frac{n!}{n^s(n-s)!}\, B_{n-s}(\widetilde{D}_{s,n} f)(x), \quad x \in [0,1]. \tag{2.2}$$

Hence

$$\left\|\frac{n^s(n-s)!}{n!}\,(B_n f)^{(s)} - B_{n-s}(f^{(s)})\right\| \le \|\widetilde{D}_{s,n} f - f^{(s)}\|.$$

Consequently,

$$\begin{aligned}\|(B_n f - f)^{(s)}\| \le \left(\frac{n^s(n-s)!}{n!} - 1\right)\|(B_n f)^{(s)}\| \\ + \|\widetilde{D}_{s,n} f - f^{(s)}\| + \|B_{n-s}(f^{(s)}) - f^{(s)}\|.\end{aligned} \tag{2.3}$$

We shall estimate the three quantities on the right above separately.

First, due to (2.2), we have

$$\begin{aligned}\left(\frac{n^s(n-s)!}{n!} - 1\right)\|(B_n f)^{(s)}\| = \left(1 - \frac{n!}{n^s(n-s)!}\right)\|B_{n-s}(\widetilde{D}_{s,n} f)\| \\ \le \frac{c}{n}\,\|\widetilde{D}_{s,n} f\| \le \frac{c}{n}\,\|f^{(s)}\|.\end{aligned} \tag{2.4}$$

The finite forward difference of order s of $F \in AC^{s-1}[a,b]$ can be represented in the integral form

$$\overrightarrow{\Delta}_h^s F(x) = h^{s-1} \int_0^{sh} M_s(u/h) F^{(s)}(x+u)\, du, \quad x \in [a, b-sh], \tag{2.5}$$

where M_s is the s-fold convolution of the characteristic function of $[0,1]$ with itself (see e.g. [1, p. 45]). Consequently,

$$\widetilde{D}_{s,n} f(x) = n \int_0^{s/n} M_s(nu) f^{(s)} \left(\frac{n-s}{n} x + u \right) du, \quad x \in [0,1],$$

and

(2.6)

$$\begin{aligned} |\widetilde{D}_{s,n} f(x) - f^{(s)}(x)| &\le n \int_0^{s/n} M_s(nu) \left| f^{(s)} \left(\frac{n-s}{n} x + u \right) - f^{(s)}(x) \right| du \\ &\le c\, \omega(f^{(s)}, n^{-1}), \quad x \in [0,1]. \end{aligned}$$

Above we have used that

$$\int_0^s M_s(u)\, du = 1.$$

Finally, by (1.1) and [5, Theorem 4.1.2] we get that there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$\|B_{n-s}(f^{(s)}) - f^{(s)}\| \le c\, \omega_\varphi^2(f^{(s)}, (n-s)^{-1/2}) \le c\, \omega_\varphi^2(f^{(s)}, n^{-1/2}). \tag{2.7}$$

Now, (2.3), (2.4), (2.6) and (2.7) imply the assertion of the theorem. $\square$

Remark 2.2. Based on Ditzian [3], Jiang and Xie [11] (or see [12, (16)]) gave a pointwise generalization of

$$\left\| \frac{n^s (n-s)!}{n!} (B_n f)^{(s)} - f^{(s)} \right\| \le c \left(\omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega(f^{(s)}, n^{-1}) \right).$$

Theorem 2.1, the property of the moduli (see [5, Theorem 2.1.1] or [1, Chapter 6, Theorem 6.1])

$$\omega_\varphi^2(f,t) \le c\, t^2 \|\varphi^2 f''\|, \quad f \in AC_{loc}^1(0,1),$$

and (1.4) imply the following estimate for the simultaneous approximation by the Bernstein polynomials (cf. [7]).

Corollary 2.3. *For $f \in C^{s+2}[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\|(B_n f - f)^{(s)}\| \le \frac{c}{n} \left(\|f^{(s+2)}\| + \|f^{(s)}\| \right).$$

The value of the constant c is independent of f and n.

Let us mention that Gonska, Heilmann and Raşa [9] established a quantitative Voronovskaya-type theorem about simultaneous approximation by B_n. They also gave an account of other similar results.

Inequalities like the one in Theorem 2.1 but in terms of the classical moduli of smoothness were earlier established in [8] and [13].

A somewhat neater upper estimate holds in terms of the differential operator $\varphi^{2s}(d/dx)^{2s}$.

Theorem 2.4. *For $f \in C^{2s}[0,1]$ there holds*

$$\|\varphi^{2s}(B_n f - f)^{(2s)}\| \le c\left(\omega_\varphi^2(\varphi^{2s} f^{(2s)}, n^{-1/2}) + \frac{1}{n}\|f^{(2s)}\|\right).$$

The value of the constant c is independent of f and n.

Proof. The assertion is trivial for $n < 2s$. Let $n \ge 2s$. Using (2.1) we get

$$\begin{aligned}\varphi^{2s}(x)(B_n f)^{(2s)}(x) &= \sum_{k=s}^{n-s} \Delta_{1/n}^{2s} f\left(\frac{k}{n}\right) \frac{k!\,(n-k)!}{(k-s)!\,(n-k-s)!}\, p_{n,k}(x) \\ &= B_n(D_{s,n} f)(x),\end{aligned} \tag{2.8}$$

where we have set

$$D_{s,n} f(x_{n,k}) = \varphi_{s,n}(x_{n,k})\, n^{2s} \Delta_{1/n}^{2s} f(x_{n,k}), \quad x_{n,k} = \frac{k}{n}, \quad k = 0, 1, \ldots, n,$$

and

$$\varphi_{s,n}(x) = \prod_{i=0}^{s-1}\left(x - \frac{i}{n}\right)\left(1 - x - \frac{i}{n}\right),$$

as $D_{s,n} f(x_{n,k})$ is defined to be 0 for $k = 0, \ldots, s-1, n-s+1, \ldots, n$.

Next, we get by means of (1.1) and (2.8) that for $n \ge n_0$ with some $n_0 \in \mathbb{N}$

$$\begin{aligned}&\|\varphi^{2s}(B_n f - f)^{(2s)}\| \\ &\quad \le \|B_n(\varphi^{2s} f^{(2s)}) - \varphi^{2s} f^{(2s)}\| + \|\varphi^{2s}(B_n f)^{(2s)} - B_n(\varphi^{2s} f^{(2s)})\| \\ &\quad \le c\left(\omega_\varphi^2(\varphi^{2s} f^{(2s)}, n^{-1/2}) + \max_{k=0,\ldots,n} |D_{s,n} f(x_{n,k}) - \varphi^{2s}(x_{n,k}) f^{(2s)}(x_{n,k})|\right).\end{aligned}$$

For $k = 0$ and $k = n$, we have $D_{s,n} f(x_{n,k}) = \varphi^{2s}(x_{n,k}) = 0$. For $k = 1, \ldots, s-1, n-s+1, \ldots, n-1$, $s \ge 2$, we directly get

$$\begin{aligned}|D_{s,n} f(x_{n,k}) - \varphi^{2s}(x_{n,k}) f^{(2s)}(x_{n,k})| &= \varphi^{2s}(x_{n,k}) |f^{(2s)}(x_{n,k})| \\ &\le \frac{c}{n^s}\|f^{(2s)}\|.\end{aligned}$$

Further, for $k = s, \ldots, n-s$ we use the representation (see (2.5))

$$\begin{aligned}\Delta_h^{2s} f(x) &= h^{2s-1}\int_{-sh}^{sh} M_{2s}(u/h + s) f^{(2s)}(x+u)\, du \\ &= h^{2s-1}\int_0^{sh} M_{2s}(u/h + s)[f^{(2s)}(x+u) + f^{(2s)}(x-u)]\, du, \quad x \in [sh, 1-sh],\end{aligned}$$

to get for $x \in [s/n, 1-s/n]$

$$\begin{aligned}|D_{s,n}f(x) - \varphi^{2s}(x)f^{(2s)}(x)| &\le n\int_0^{s/n} M_{2s}(nu+s)|\Delta_u^2(\varphi^{2s}f^{(2s)})(x)|\,du\\ &+ n\int_{-s/n}^{s/n} M_{2s}(nu+s)|\varphi_{s,n}(x) - \varphi^{2s}(x+u)|\,|f^{(2s)}(x+u)|\,du\\ &\le c\left(\omega^2(\varphi^{2s}f^{(2s)}, n^{-1}) + \frac{1}{n}\|f^{(2s)}\|\right).\end{aligned}$$

Above we have also taken into account the trivial estimate

$$\begin{aligned}|\varphi_{s,n}(x) - \varphi^{2s}(x+u)| &\le |\varphi_{s,n}(x) - \varphi^{2s}(x)| + |\varphi^{2s}(x) - \varphi^{2s}(x+u)|\\ &\le \frac{c}{n} + c\,|u| \le \frac{c}{n}, \quad x \in [0,1],\ u \in \left[-\frac{s}{n}, \frac{s}{n}\right].\end{aligned}$$

To complete the proof of the theorem, we apply [5, Theorem 3.1.1], which gives that there exists t_0 such that

$$\omega^2(F, t^2) \le c\,\omega_\varphi^2(F, t), \quad 0 < t \le t_0,$$

for every $F \in C[0,1]$. □

Just as in the unweighted case, but using the embedding inequality (see [6, Lemma 1])

$$\|\chi^{\alpha+m}f^{(m)}\| \le c\left(\|\chi^\alpha f\| + \|\chi^{\alpha+\ell}f^{(\ell)}\|\right), \quad m = 0, \dots, \ell,$$

where $\chi(x) = x$ and $\alpha \in \mathbb{R}$, we derive the following estimate.

Corollary 2.5. *For $f \in C[0,1]$ such that $f \in AC_{loc}^{2s+1}(0,1)$ and $n \in \mathbb{N}$ there holds*

$$\|\varphi^{2s}(B_n f - f)^{(2s)}\| \le \frac{c}{n}\left(\|\varphi^{2s+2}f^{(2s+2)}\| + \|f^{(2s)}\|\right).$$

The value of the constant c is independent of f and n.

3 Proof of Theorem 1.1

The estimates of the error of $\mathcal{B}_{r,n}$ can now be quite straightforwardly established by means of the results on simultaneous approximation of the previous section.

Proof of Theorem 1.1. First, the estimate (1.2) implies

$$\|\mathcal{B}_{r,n}f - f\| = \|(B_n - I)^r f\| \le \frac{c}{n}\|\varphi^2[(B_n - I)^{r-1}f]''\|.$$

For $r = 2$ we estimate above the right side of this inequality by means of Theorem 2.4 and get the assertion in this case. For $r \ge 3$ we apply instead Corollary 2.5 and arrive at

$$\|\mathcal{B}_{r,n}f - f\| \le \frac{c}{n^2}\left(\|\varphi^4[(B_n - I)^{r-2}f]^{(4)}\| + \|[(B_n - I)^{r-2}f]^{(2)}\|\right).$$

Further, we estimate the first term on the right above by Corollary 2.5 and the second by Corollary 2.3 and continue in this way, applying also (1.4), until we get

$$\|\mathcal{B}_{r,n}f - f\| \le \frac{c}{n^{r-1}}\Big(\|\varphi^{2r-2}[(B_n - I)f]^{(2r-2)}\| \\ + \|[(B_n - I)f]^{(2r-4)}\| + \|[(B_n - I)f]^{(2)}\|\Big).$$

Now, the assertion of the theorem follows from Theorem 2.4, Corollary 2.3 and (1.4). □

Proof of Corollary 1.2. Assertion (a) follows from Theorem 1.1 and the property (see [5, Theorems 2.1.1 and 4.1.3] or [1, Chapter 6, Theorem 6.1])

$$\omega_\varphi^2(f,t) \le c\,t\,\|\varphi f'\|, \quad f \in AC_{loc}(0,1), \quad 0 < t \le t_0.$$

Assertion (b) follows from Theorem 1.1 just as Corollary 2.5 from Theorem 2.4. □

Remark 3.1. Let us note that in all estimates with the Ditzian-Totik modulus we had to assume that $n \ge n_0$ with some absolute constant n_0 since (1.1) was proved under this restriction and some of the properties of the modulus we used are known only for t small enough. However, (1.2) as well as its analogue with $n^{-1/2}\|\varphi f'\|$ on the right are valid for all $n \in \mathbb{N}$ and hence all the corollaries as well as Theorem 1.3 are valid for all n.

Note added in proof. After submission I learned of the papers of H. Gonska and X.-l. Zhou [10], and of C. Ding and F. Cao [2], where results that are similar to and somewhat stronger than Theorem 1.3 were established. The techniques used there are different. Also, I learned of a paper by Sevy [15] who established upper estimates for the unweighted simultaneous approximation by such combinations of iterates of an operator, following just the same idea like the one used in the proof of Theorem 1.1. I am thankful to Prof. G. Tachev (University of Architecture, Civil Engineering and Geodesy, Sofia) for helping me find out those papers. In a subsequent publication I am going to show how the results proved in the present paper can be improved to include those in the above-mentioned works (in the univariate case).

References

[1] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.

[2] C. Ding, F. Cao, K-functionals and multivariate Bernstein polynomials, J. Approx. Theory 155(2008) 125–135.

[3] Z. Ditzian, Direct Estimates for Bernstein Polynomials, J. Approx. Theory 79(1994), 165–166.

[4] Z. Ditzian, K. G. Ivanov, Strong converse inequalities, J. D'Analyse Math. 61(1993), 61–111.

[5] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer-Verlag, New York, 1987.

[6] Z. Ditzian, V. Totik, K-functionals and weighted moduli of smoothness, J. Approx. Theory 63(1990), 3–29.

[7] M. S. Floater, On the convergence of derivatives of Bernstein approximation, J. Approx. Theory 134(2005), 130–135.

[8] H. Gonska, Quantitative Korovkin-type theorems on simultaneous approximation, Math. Z. 186(1984), 419–433.

[9] H. Gonska, M. Heilmann, I. Raşa, Asymptotic behaviour of differentiated Bernstein polynomials revisited, Gen. Math. 18(2010), 45–53.

[10] H. Gonska, X.-l. Zhou, Approximation theorems for the iterated Boolean sums of Bernstein operators, J. Comput. Appl. Math. 53(1994), 21–31.

[11] H. B. Jiang, L. S. Xie, Simultaneous approximation by Bernstein operators, Pure Appl. Math. (Xi'an) 4:22(2006), 471–476.

[12] X. J. Jiang, L. S. Xie, Simultaneous approximation by Bernstein-Sikkema operators, Anal. Theory Appl. 24:3(2008), 237–246.

[13] H. Knoop, P. Pottinger, Ein satz vom Korovkin-typ für C^k-raüme, Math. Z. 148(1976), 23–32.

[14] R. Martini, On the approximation of functions together with their derivatives by certain linear positive operators, Intag. Math. 31(1969), 473–481.

[15] J. C. Sevy, Convergence of iterated boolean sums of simultaneous approximation, Calcolo 30:1(1993), 41–68.

Borislav R. Draganov
Dept. of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
Bulgaria
bdraganov@fmi.uni-sofia.bg

Inst. of Mathematics and Informatics
Bulgarian Academy of Science
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
Bulgaria