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On simultaneous approximation by iterated Boolean sums of Bernstein operators

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Abstract

The paper presents upper estimates of the error of weighted and unweighted simultaneous approximation by the Bernstein operators and their iterated Boolean sums. The estimates are stated in terms of the Ditzian-Totik modulus of smoothness or appropriate K-functionals.

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1 Main results

One of the most investigated linear approximating operators is the Bernstein polynomial, defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

It is known (see [1, Chapter 10, § 7] and [4, Chapter 9]) that there exists $n_0 \in \mathbb{N}$ such that for all $f \in C[0,1]$ and $n \geq n_0$ we have

$$\|B_n f - f\| \leq c\, \omega_\varphi^2(f, n^{-1/2}), \tag{1.1}$$

where $\| \circ \|$ stands for the uniform norm on the interval $[0,1]$, c is an absolute constant and $\omega_\varphi^2(f,t)$ is the Ditzian-Totik modulus of smoothness of second order with step-weight $\varphi(x) = \sqrt{x(1-x)}$. To recall, the Ditzian-Totik modulus of order r with step-weight φ is defined by (see [4, Chapter 1])

$$\omega_\varphi^r(f,t) = \sup_{0<h\leq t} \|\Delta_{h\varphi}^r f\|$$

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and

$$\Delta^r_{h\varphi(x)} f(x) = \begin{cases} \displaystyle\sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right), & x \pm rh\varphi(x)/2 \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

For $f \in AC^1_{loc}(0,1)$ and $n \in \mathbb{N}$ we have

$$\|B_n f - f\| \le \frac{c}{n} \|\varphi^2 f''\|. \tag{1.2}$$

Moreover, $B_n f$ cannot tend to f in $C[0,1]$ faster than n^{-1} unless f is a linear function, in which case we have $B_n f = f$ for all n (see e.g. [1, Chapter 10, § 5]).

One way to modify the Bernstein operator in order to get higher approximation rate is to form an appropriate linear combination of its iterates. Here we shall consider the bounded linear operator $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$, defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}$. These operators can be regarded as iterated Boolean sums (see [11]).

An important and nice characterization of the error of $\mathcal{B}_{r,n}$ was given by Gonska ans Zhou [11]. They established the following upper estimate

$$\|\mathcal{B}_{r,n} f - f\| \le c \left(\omega^{2r}_{\varphi}(f, n^{-1/2}) + \frac{1}{n^r} \|f\| \right). \tag{1.3}$$

A Stechkin-type converse inequality was also proved. That enabled them to deduce the trivial class of the operator and a big O equivalence characterization of the error.

Replacing in (1.3) f with $f - p_1$, where p_1 is the polynomial of degree 1 of best approximation of f, we immediately arrive at

$$\|\mathcal{B}_{r,n} f - f\| \le c \left(\omega^{2r}_{\varphi}(f, n^{-1/2}) + \frac{1}{n^r} E_1(f) \right), \tag{1.4}$$

where $E_1(f)$ denotes the best approximation of f by algebraic polynomials of degree 1 in uniform norm on $[0,1]$.

One of our main goals is to extend (1.3) to simultaneous approximation. We shall establish the following upper estimate.

Theorem 1.1. *Let $r, s, \ell \in \mathbb{N}$ and $\ell < s$. Then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and all $n \in \mathbb{N}$ there holds*

$$\|\varphi^{2\ell} (\mathcal{B}_{r,n} f - f)^{(s)}\| \le c \left(\omega^{2r}_{\varphi}(f^{(s)}, n^{-1/2})_{\varphi^{2\ell}} + \frac{1}{n^r} \|\varphi^{2\ell} f^{(s)}\| \right).$$

Above $\omega_\varphi^r(f,t)_w$ is the weighted Ditzian-Totik modulus of smoothness in uniform norm on the interval $[0,1]$, defined in [4, Appendix B]. Instead, we can use the modification of this modulus considered in [7, Chapter 3, Section 10] and [13].

Several years ago Ding and Cao [2] proved the Jackson-type estimate

$$\|\mathcal{B}_{r,n}f - f\| \le c\, K_r(f, n^{-r}), \quad n \in \mathbb{N}, \tag{1.5}$$

where the K-functional $K_r(f,t)$ is defined by

$$K_r(f,t) = \inf_{g \in C^{2r}[0,1]} \{\|f-g\| + t\|D^r g\|\},$$

and $Dg = \varphi^2 g''$. A corresponding strong converse inequality of type D (according to the terminology of [3]) was also established. Actually, C. Ding and F. Cao proved their characterization in the multivariate case.

Let us point out that though (1.5) seems more precise than (1.4) both estimates are equivalent. We shall consider this below.

We shall establish a generalization of (1.5) in the case of simultaneous approximation. In particular, we shall show that if $f \in C^{2(r+\ell-1)}[0,1] \cap AC_{loc}^{2(r+\ell)-1}(0,1)$, then for all $n \in \mathbb{N}$ there holds

$$\|D^\ell(\mathcal{B}_{r,n}f - f)\| \le \frac{c}{n^r}\, \|D^{r+\ell} f\|.$$

This extends estimate (1.2).

The contents of the paper are organized as follows. In the next section we collect a number of embedding inequalities, which we shall use. In Section 3 we establish upper estimates of the error in simultaneous approximation by the Bernstein polynomials in weighted and unweighted uniform norm. Then in Section 4 we use them to derive upper estimates about weighted and unweighted simultaneous approximation by $\mathcal{B}_{r,n}$ and, in particular, Theorem 1.1. Finally, in Section 5 we consider the equivalence of the upper estimates of Gonska-Zhou and Ding-Cao.

The results presented here improve and generalize estimates established in [6].

2 Embedding inequalities

Here we recall and extend several embedding inequalities, which we shall use frequently in the proofs. We begin with the very well-known inequality

$$\|f^{(j)}\|_J \le c\big(\|f\|_J + \|f^{(m)}\|_J\big), \quad j = 0, \dots, m, \tag{2.1}$$

where $\|\circ\|_J$ denotes the sup norm on the interval J.

Through (2.1) it can be shown that (see [5, Lemma 1])

$$\|\chi^{\alpha+j} f^{(j)}\| \le c\left(\|\chi^\alpha f\| + \|\chi^{\alpha+m} f^{(m)}\|\right), \quad j = 0, \dots, m,$$

where $\chi(x) = x$ and $\alpha \in \mathbb{R}$. It directly implies the following inequalities.

Proposition 2.1. *For $f \in AC_{loc}^{m-1}(0,1)$ and $i \in \mathbb{N}_0$, there hold*

$$\|\varphi^{2(i+j)} f^{(j)}\| \le c\left(\|\varphi^{2i} f\| + \|\varphi^{2(i+m)} f^{(m)}\|\right), \quad j = 0, \dots, m.$$

The following assertion is a modification of [4, p. 135, (a) and (b)] in the case of uniform norm.

Proposition 2.2. *Let $i, m, p, q \in \mathbb{N}_0$ as $p < q$. Let also $f \in AC_{loc}^{q-1}(0,1)$. The following inequalities hold true:*

(a) If $p \le q - m$, then

$$\|\varphi^2 f^{(j)}\| \le c\left(\|\varphi^{2i} f^{(p)}\| + \|\varphi^{2m} f^{(q)}\|\right), \quad j = p, \dots, q - m;$$

(b) If $j_0 = \max\{p, q - m + 1\}$, then

$$\|\varphi^{2(j+m-q)} f^{(j)}\| \le c\left(\|\varphi^{2i} f^{(p)}\| + \|\varphi^{2m} f^{(q)}\|\right), \quad j = j_0, \dots, q.$$

Proof. We follow the argument in [4, pp. 136-137].

Let $p < j \le q$. For any $\alpha > 1$ and $x \in (0, 1/2]$ we have

$$\begin{aligned} |f^{(j-1)}(x)| &\le |f^{(j-1)}(1/2)| + \int_x^{1/2} |f^{(j)}(u)|\, du \\ &\le c\left(|f^{(j-1)}(1/2)| + x^{1-\alpha} \|\chi^\alpha f^{(j)}\|_{[0,1/2]}\right). \end{aligned}$$

By (2.1) we have

$$\begin{aligned} |f^{(j-1)}(1/2)| &\le c\left(\|f^{(p)}\|_{[1/4,3/4]} + \|f^{(q)}\|_{[1/4,3/4]}\right) \\ &\le c\left(\|\varphi^{2i} f^{(p)}\| + \|\varphi^{2m} f^{(q)}\|\right). \end{aligned}$$

Consequently,

$$\|\varphi^{2(\alpha-1)} f^{(j-1)}\|_{[0,1/2]} \le c\left(\|\varphi^{2i} f^{(p)}\| + \|\varphi^{2m} f^{(q)}\| + \|\varphi^{2\alpha} f^{(j)}\|_{[0,1/2]}\right).$$

By symmetry we get the analogue of the last inequality on the interval $[1/2, 1]$. Thus we establish

$$\|\varphi^{2(\alpha-1)} f^{(j-1)}\| \le c\left(\|\varphi^{2i} f^{(p)}\| + \|\varphi^{2m} f^{(q)}\| + \|\varphi^{2\alpha} f^{(j)}\|\right).$$

Hence (b) follows by induction.

To establish (a) we observe that

$$|f^{(j-1)}(x)| \le c|\log x|\left(|f^{(j-1)}(1/2)| + \|\varphi^2 f^{(j)}\|\right), \quad x \in (0, 1/2]; \tag{2.2}$$

hence

$$\|\varphi^2 f^{(j-1)}\|_{[0,1/2]} \le c\left(\|\varphi^{2i} f^{(p)}\| + \|\varphi^2 f^{(j)}\|\right)$$

and, by symmetry, such an inequality holds on $[1/2,1]$ too.

Then we proceed by induction to get for $j = p,\dots,q-m$

$$\|\varphi^2 f^{(j)}\| \le c\left(\|\varphi^{2i} f^{(p)}\| + \|\varphi^2 f^{(q-m+1)}\|\right).$$

Now, (a) follows from (b) with $j = j_0 = q-m+1$. □

Let us especially note the following corollary of the last result.

Corollary 2.3. *Let $i \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Let also $f \in AC^{2r-1}_{loc}(0,1)$. Then*

$$\|\varphi^{2(k+i)} f^{(2k)}\| \le c\left(\|\varphi^{2i} f\| + \|\varphi^{2(r+i)} f^{(2r)}\|\right), \quad k = 0,\dots,r.$$

Proof. We apply Proposition 2.2 with $m = r+i$, $p = 0$ and $q = 2r$. Then (a) and (b) imply respectively

$$\|\varphi^{2(k+i)} f^{(2k)}\| \le \|\varphi^2 f^{(2k)}\| \le c\left(\|\varphi^{2i} f\| + \|\varphi^{2(r+i)} f^{(2r)}\|\right),$$
$$1 \le k \le (r-i)/2, \quad r-i \ge 2,$$

and

$$\begin{aligned}\|\varphi^{2(k+i)} f^{(2k)}\| &\le \|\varphi^{2(2k-r+i)} f^{(2k)}\| \\ &\le c\left(\|\varphi^{2i} f\| + \|\varphi^{2(r+i)} f^{(2r)}\|\right), \quad \max\{0,(r-i)/2\} < k \le r,\end{aligned}$$

where we have also used that $2k-r+i \le k+i$ for $k \le r$. □

Remark 2.4. (a) As it is seen from the proof, Proposition 2.2 actually holds true with $\|f^{(p)}\|_{[1/4,3/4]}$ in the place of $\|\varphi^{2i} f^{(p)}\|$.

(b) Similarly to [4, p. 135, (a)] Proposition 2.2(a) is even valid with $\|f^{(j)}\|$ on the left provided that $j < q-m$.

The last inequalities are due to Gonska and Zhou [11, (1), (2) and (4)].

Proposition 2.5. *For $f \in C^{2r}[0,1]$ there hold:*

(a) $\|D^r f\| \le c\left(\|f\| + \|\varphi^{2r} f^{(2r)}\|\right)$;

(b) $\|\varphi^{2r} f^{(2r)}\| \le c\,\|D^r f\|$;

(c) $\|D^j f\| \le c\,\|D^r f\|, \quad j = 1,\dots,r.$

Actually, Gonska and Zhou [11] stated the assertions above only for algebraic polynomials since that was what they needed, but the same considerations verify them for all functions in $C^{2r}[0,1]$.

Remark 2.6. There is an elegant Taylor-type formula through which the embedding inequalities in Proposition 2.5 can be verified.

Let $f \in AC^1_{loc}(0,1)$ be such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0} \varphi^2(x) f'(x) = \lim_{x\to 1} \varphi^2(x) f'(x) = 0.$$

Then

$$f(x) = \int_0^1 [xu - \min\{x,u\}] f''(u)\, du, \quad x \in [0,1]. \tag{2.3}$$

This formula is verified by integration by parts.

If $f \in C[0,1]$ is such that $f(0) = f(1) = 0$, $f \in AC^1_{loc}(0,1)$ and $\varphi^2 f'' \in L_\infty[0,1]$, then (2.2) with $j = 2$ implies $\lim_{x\to 0} \varphi^2(x) f'(x) = \lim_{x\to 1} \varphi^2(x) f'(x) = 0$. Formula (2.3) is applicable and yields

$$|f(x)| \le \int_0^1 \frac{\min\{x,u\} - xu}{\varphi^2(u)}\, du\, \|Df\|, \quad x \in [0,1].$$

Hence, taking into account that,

$$\int_0^1 \frac{\min\{x,u\} - xu}{\varphi^2(u)}\, du = -x\log x - (1-x)\log(1-x) \le \log 2, \quad x \in (0,1),$$

we arrive at the inequality

$$\|f\| \le \log 2\, \|Df\|.$$

Iterating it, we get Proposition 2.5(c) for $f \in C^{2r-2}[0,1]$ such that $f^{(2r-2)} \in AC^1_{loc}(0,1)$.

Formula (2.3) can be extended. Let $r \in \mathbb{N}$ and $f \in C^{2r-2}[0,1]$ be such that $f^{(2r-2)} \in AC^1_{loc}(0,1)$ and $f(0) = f(1) = 0$. Then

$$f(x) = \int_0^1 K_r(x,u) D^r f(u)\, du, \quad x \in [0,1], \tag{2.4}$$

where the kernel is defined by the recurrence relation

$$K_1(x,u) = \frac{xu - \min\{x,u\}}{\varphi^2(u)}, \quad K_{j+1}(x,u) = \int_0^1 K_j(x,v) K_1(v,u)\, dv.$$

The kernel possesses various properties. They include the symmetries

$$K_j(x,u) = K_j(u,x), \quad K_j(x,u) = K_j(1-x, 1-u)$$

and the relation

$$\varphi^2(x)\, \frac{\partial^2 K_{j+1}}{\partial x^2}(x,u) = K_j(x,u).$$

However, its explicit form is quite complicated even for $j = 2$. So it is easier to verify Proposition 2.5 (a) and (b) by the method used by H. Gonska and X.-l. Zhou rather than by (2.4).

3 Simultaneous approximation by Bernstein polynomials

The weighted upper estimates for simultaneous approximation by Bernstein polynomials given below enable us to derive analogous estimates for their iterated Boolean sums $\mathcal{B}_{r,n}$ just by iteration. Before we proceed to our results on simultaneous approximation by B_n we shall mention one straightforward property of the derivatives of $B_n f$ and one simple auxiliary inequality.

Throughout c denotes constants whose value is independent of f, n and x. It is not necessarily the same at each occurrence. For convenience we set $B_0 = I$.

Proposition 3.1. *Let $\ell, s \in \mathbb{N}_0$ and $\ell < s$. Then for all $f \in AC_{loc}^{s-1}(0,1)$ and $n \in \mathbb{N}$ there holds*

$$\|\varphi^{2\ell}(B_n f)^{(s)}\| \le c\,\|\varphi^{2\ell} f^{(s)}\|.$$

Proof. The inequality is trivial for $n < s$. For $n \ge s$ it is known (see [14], or [1, Chapter 10, (2.3)], or [4, p. 125]) that

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}(x),$$

where $\overrightarrow{\Delta}_h^m f(x) = \Delta_h^m f(x + mh/2)$ is the forward difference of order m. Hence

$$\begin{aligned} \varphi^{2\ell}(x)(B_n f)^{(s)}(x) &= \frac{n!}{\tilde{n}!} \sum_{k=\ell}^{\tilde{n}-\ell} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k-\ell}{n}\right) \frac{k!\,(\tilde{n}-k)!}{(k-\ell)!\,(\tilde{n}-k-\ell)!}\, p_{\tilde{n},k}(x) \\ &= \frac{n!\,\tilde{n}^{2\ell}}{\tilde{n}!\,n^s}\, B_{\tilde{n}}(D_{\ell,s,n} f)(x), \end{aligned} \tag{3.1}$$

where we have put

$$\tilde{n} = n - s + 2\ell, \tag{3.2}$$

$$D_{\ell,s,n} f(x_{\tilde{n},k}) = \varphi_{\ell,\tilde{n}}(x_{\tilde{n},k})\, n^s \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{\tilde{n} x_{\tilde{n},k} - \ell}{n}\right), \quad x_{\tilde{n},k} = \frac{k}{\tilde{n}}, \quad k = 0, \dots, \tilde{n},$$

and

$$\varphi_{\ell,\tilde{n}}(x) = \prod_{i=0}^{\ell-1} \left(x - \frac{i}{\tilde{n}}\right)\left(1 - x - \frac{i}{\tilde{n}}\right), \tag{3.3}$$

as $D_{\ell,s,n} f(x_{\tilde{n},k})$ is defined to be 0 for $k = 0, \dots, \ell-1, \tilde{n}-\ell+1, \dots, \tilde{n}$. As usual, we assume that an empty product equals 1.

Identity (3.1) implies

$$\|\varphi^{2\ell}(B_n f)^{(s)}\| \le c \max_{k=0,\dots,\tilde{n}} |D_{\ell,s,n} f(x_{\tilde{n},k})|.$$

It remains to show that

$$|D_{\ell,s,n}f(x_{\tilde{n},k})| \le c\,\|\varphi^{2\ell}f^{(s)}\|, \quad k=0,\dots,\tilde{n}. \tag{3.4}$$

To verify it we first note that $D_{\ell,s,n}f(x_{\tilde{n},k})=0$ for $k=0,\dots,\ell-1,\tilde{n}-\ell+1,\dots,\tilde{n}$. For $k=\ell,\dots,\tilde{n}-\ell$ we use the following representation of the forward finite difference of order s and step h (see e.g. [1, p. 45])

$$\overrightarrow{\Delta}_h^s f(x) = h^s \int_0^s M_s(u) f^{(s)}(x+hu)\,du, \quad x\in[0,1-sh], \tag{3.5}$$

where M_s is the s-fold convolution of the characteristic function of $[0,1]$ with itself.

This formula implies for $x\in[\ell/\tilde{n},1-\ell/\tilde{n}]$

$$\begin{aligned}|D_{\ell,s,n}f(x)| &\le \varphi_{\ell,\tilde{n}}(x)\int_0^s M_s(u)\left|f^{(s)}\left(\frac{\tilde{n}x-\ell+u}{n}\right)\right| du\\ &\le \varphi^{2\ell}(x)\int_0^s \frac{M_s(u)}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)}\,du\,\|\varphi^{2\ell}f^{(s)}\|.\end{aligned}$$

Thus, to complete the proof of (3.4) for $k=\ell,\dots,\tilde{n}-\ell$, it remains to show that

$$\varphi^{2\ell}(x)\int_0^s \frac{M_s(u)}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)}\,du \le c, \quad x\in[\ell/\tilde{n},1-\ell/\tilde{n}], \quad n\in\mathbb{N}.$$

Since

$$0\le M_s(u)\le c\,[u(s-u)]^{s-1}, \quad 0\le u\le s,$$

it reduces to the estimate

$$\varphi^{2\ell}(x)\int_0^s \frac{[u(s-u)]^{s-1}}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)}\,du \le c, \quad x\in[\ell/\tilde{n},1-\ell/\tilde{n}], \quad n\in\mathbb{N}. \tag{3.6}$$

Relation (3.6) is trivial for $n=s$ since then $\tilde{n}=2\ell$ and hence $x=1/2$. To verify it for $n>s$, we just need to observe that for $\tilde{n}x\in[\ell,\ell+1/2]$ we have

$$\varphi^{2\ell}(x)\int_0^s \frac{[u(s-u)]^{s-1}}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)}\,du \le c(\tilde{n}x)^{\ell}\int_0^s \frac{u^{s-1}}{u^{\ell}}\,du \le c$$

and for $\tilde{n}x\in[\ell+1/2,\tilde{n}/2]$

$$\varphi^{2\ell}(x)\int_0^s \frac{[u(s-u)]^{s-1}}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)}\,du \le c\,\frac{(\tilde{n}x)^{\ell}}{(\tilde{n}x-\ell)^{\ell}}\int_0^s [u(s-u)]^{s-1}\,du \le c.$$

Thus (3.6) is established for $x\in[\ell/\tilde{n},1/2]$. On the interval $[1/2,1-\ell/\tilde{n}]$ it follows by symmetry since $\varphi(1-x)=\varphi(x)$ and $(\tilde{n}(1-x)-\ell+u)/n = 1-(\tilde{n}x-\ell+(s-u))/n$. □

Lemma 3.2. *Let $\ell, s \in \mathbb{N}$ and $\ell \geq 2$. Then for all $n \geq s$ there hold*

$$0 \leq \varphi^{2\ell}(x) - \varphi_{\ell,\tilde{n}}(x) \leq \frac{c}{n}\,\varphi^{2\ell-2}(x), \quad x \in [\ell/\tilde{n}, 1-\ell/\tilde{n}],$$

where $\tilde{n}$ and $\varphi_{\ell,\tilde{n}}$ are given in (3.2) *and* (3.3), *respectively.*

Proof. The first inequality is trivial. For the second one we first observe that

$$\left(x - \frac{i}{\tilde{n}}\right)\left(1 - x - \frac{i}{\tilde{n}}\right) = \varphi^2(x) - \varphi^2(i/\tilde{n})$$

and hence

$$\begin{aligned}\prod_{i=0}^{\ell-1}\left(x - \frac{i}{\tilde{n}}\right)\left(1 - x - \frac{i}{\tilde{n}}\right) &= \varphi^2(x)\prod_{i=1}^{\ell-1}\left(\varphi^2(x) - \varphi^2(i/\tilde{n})\right)\\ &= \varphi^{2\ell}(x) + \sum_{k=1}^{\ell-1}(-1)^{k-1}\sigma_k\left(\varphi^2(1/\tilde{n}), \dots \varphi^2((\ell-1)/\tilde{n})\right)\varphi^{2(\ell-k)}(x),\end{aligned}$$

where σ_k are the elementary symmetric polynomials.

Now, since

$$0 < \sigma_k\left(\varphi^2(1/\tilde{n}), \dots \varphi^2((\ell-1)/\tilde{n})\right) \leq \frac{1}{\tilde{n}^k} \leq \frac{c}{n}\,\varphi^{2k-2}(x), \quad x \in [1/\tilde{n}, 1-1/\tilde{n}],$$

we get the second inequality in the lemma. $\square$

Now, we are ready to establish an estimate concerning weighted simultaneous approximation by the Bernstein polynomials. For the sake of completeness we include also an estimate in the unweighted case.

Theorem 3.3. *Let $\ell, s \in \mathbb{N}$ and $\ell \leq s$. Then the following assertions hold true for all $f \in C[0,1]$ such that $f \in AC^{s+1}_{loc}(0,1)$ and all $n \in \mathbb{N}$:*

(a) $\|(B_n f - f)^{(s)}\| \leq \dfrac{c}{n}\left(\|f^{(s)}\| + \|f^{(s+1)}\| + \|\varphi^2 f^{(s+2)}\|\right);$

(b) $\|\varphi^{2\ell}(B_n f - f)^{(s)}\| \leq \dfrac{c}{n}\left(\|\varphi^{2\ell} f^{(s)}\| + \|\varphi^{2\ell+2} f^{(s+2)}\|\right).$

Proof. The assertions are trivial for $n < s$. Let $n \geq s$. We get by means of (1.2) and (3.1) that

$$\begin{aligned}&\|\varphi^{2\ell}(B_n f - f)^{(s)}\|\\ &\quad\leq \|B_{\tilde{n}}(\varphi^{2\ell} f^{(s)}) - \varphi^{2\ell} f^{(s)}\| + \left(1 - \frac{n!\,\tilde{n}^{2\ell}}{\tilde{n}!\,n^s}\right)\|B_{\tilde{n}}(\varphi^{2\ell} f^{(s)})\|\\ &\qquad + \left\|\varphi^{2\ell}(B_n f)^{(s)} - \frac{n!\,\tilde{n}^{2\ell}}{\tilde{n}!\,n^s}\,B_{\tilde{n}}(\varphi^{2\ell} f^{(s)})\right\|\\ &\quad\leq c\Big(\frac{1}{n}\,\|\varphi^2(\varphi^{2\ell} f^{(s)})''\| + \frac{1}{n}\,\|\varphi^{2\ell} f^{(s)}\|\\ &\qquad + \max_{k=0,\dots,\tilde{n}}|D_{\ell,s,n} f(x_{\tilde{n},k}) - \varphi^{2\ell}(x_{\tilde{n},k}) f^{(s)}(x_{\tilde{n},k})|\Big).\end{aligned} \tag{3.7}$$

For the first term above with $\ell = 1$ we have

$$\begin{aligned}\|\varphi^2(\varphi^2 f^{(s)})''\| &\le c\left(\|\varphi^2 f^{(s)}\| + \|\varphi^2 f^{(s+1)}\| + \|\varphi^4 f^{(s+2)}\|\right)\\ &\le c\left(\|\varphi^2 f^{(s)}\| + \|\varphi^4 f^{(s+2)}\|\right),\end{aligned}$$

where for the second inequality we have applied Proposition 2.2(b) with $i = 1$, $m = 2$, $p = s$, $q = s + 2$ and $j = s + 1$ to get

$$\|\varphi^2 f^{(s+1)}\| \le c\left(\|\varphi^2 f^{(s)}\| + \|\varphi^4 f^{(s+2)}\|\right). \tag{3.8}$$

For $\ell \ge 2$ we have

$$\begin{aligned}\|\varphi^2(\varphi^{2\ell} f^{(s)})''\| &\le c\left(\|\varphi^{2\ell-2} f^{(s)}\| + \|\varphi^{2\ell} f^{(s+1)}\| + \|\varphi^{2\ell+2} f^{(s+2)}\|\right)\\ &\le c\left(\|\varphi^{2\ell-2} f^{(s)}\| + \|\varphi^{2\ell+2} f^{(s+2)}\|\right),\end{aligned}$$

where for the second inequality we have applied Proposition 2.1 with $f^{(s)}$ in the place of f, $i = \ell - 1$, $m = 2$ and $j = 1$ to get

$$\|\varphi^{2\ell} f^{(s+1)}\| \le c\left(\|\varphi^{2\ell-2} f^{(s)}\| + \|\varphi^{2\ell+2} f^{(s+2)}\|\right). \tag{3.9}$$

Next, Proposition 2.2(b) with $i = \ell$, $m = \ell + 1$, $j = p = s$ and $q = s + 2$ yields

$$\|\varphi^{2\ell-2} f^{(s)}\| \le c\left(\|\varphi^{2\ell} f^{(s)}\| + \|\varphi^{2\ell+2} f^{(s+2)}\|\right). \tag{3.10}$$

So we have

$$\|\varphi^2(\varphi^{2\ell} f^{(s)})''\| \le c\left(\|\varphi^{2\ell} f^{(s)}\| + \|\varphi^{2\ell+2} f^{(s+2)}\|\right), \quad \ell \in \mathbb{N}. \tag{3.11}$$

Further, we shall show that

$$|D_{0,s,n} f(x_{\tilde{n},k}) - f^{(s)}(x_{\tilde{n},k})| \le \frac{c}{n}\,\|f^{(s+1)}\| \tag{3.12}$$

and

$$\begin{aligned}&|D_{\ell,s,n} f(x_{\tilde{n},k}) - \varphi^{2\ell}(x_{\tilde{n},k}) f^{(s)}(x_{\tilde{n},k})|\\ &\qquad\le \frac{c}{n}\left(\|\varphi^{2\ell} f^{(s)}\| + \|\varphi^{2\ell+2} f^{(s+2)}\|\right), \quad \ell \in \mathbb{N},\end{aligned} \tag{3.13}$$

for $k = 0, 1 \dots, \tilde{n}$.

Then assertion (a) follows from (3.7) and (3.12), and assertion (b) from (3.7), (3.11) and (3.13).

Thus to complete the proof of the theorem it remains to establish (3.12) and (3.13). For $k = 0$ and $k = \tilde{n}$, we have $D_{\ell,s,n} f(x_{\tilde{n},k}) = \varphi^{2\ell}(x_{\tilde{n},k}) = 0$ and for $k = 1, \dots, \ell - 1, \tilde{n} - \ell + 1, \dots, \tilde{n} - 1$, $\ell \ge 2$, we directly get

$$\begin{aligned}|D_{\ell,s,n} f(x_{\tilde{n},k}) - \varphi^{2\ell}(x_{\tilde{n},k}) f^{(s)}(x_{\tilde{n},k})| &= \varphi^{2\ell}(x_{\tilde{n},k})|f^{(s)}(x_{\tilde{n},k})|\\ &\le \frac{c}{n}\|\varphi^{2\ell-2} f^{(s)}\|.\end{aligned} \tag{3.14}$$

For $k = \ell, \dots, \tilde{n} - \ell$ we again use (3.5) and

$$\int_0^s M_s(u)\, du = 1$$

to get for $x \in [\ell/\tilde{n}, 1 - \ell/\tilde{n}]$

$$\begin{aligned}
|D_{\ell,s,n} f(x) - \varphi^{2\ell}(x) f^{(s)}(x)| &\le \left(\varphi^{2\ell}(x) - \varphi_{\ell,\tilde{n}}(x)\right) |f^{(s)}(x)| \\
&\quad + \varphi^{2\ell}(x) \int_0^s M_s(u) \left| f^{(s)}\left(\frac{\tilde{n}x - \ell + u}{n}\right) - f^{(s)}(x) \right| du.
\end{aligned} \tag{3.15}$$

Next, we shall show that

$$\begin{aligned}
\varphi^{2\ell}(x) \int_0^s M_s(u) \left| f^{(s)}\left(\frac{\tilde{n}x - \ell + u}{n}\right) - f^{(s)}(x) \right| du& \\
\le \frac{c}{n} \|\varphi^{2\ell} f^{(s+1)}\|, \quad x \in [\ell/\tilde{n}, 1 - \ell/\tilde{n}].&
\end{aligned} \tag{3.16}$$

We have with $x'_{s,\ell} = (s - 2\ell)x + \ell$

$$\begin{aligned}
&\int_0^s M_s(u) \left| f^{(s)}\left(\frac{\tilde{n}x - \ell + u}{n}\right) - f^{(s)}(x) \right| du \\
&\quad \le \frac{c}{n} \int_0^s [u(s-u)]^{s-1} \left| \int_{x'_{s,\ell}}^u \left| f^{(s+1)}\left(\frac{\tilde{n}x - \ell + v}{n}\right) \right| dv \right| du \\
&\quad = \frac{c}{n} \int_0^{x'_{s,\ell}} [u(s-u)]^{s-1} \left(\int_u^{x'_{s,\ell}} \left| f^{(s+1)}\left(\frac{\tilde{n}x - \ell + v}{n}\right) \right| dv \right) du \\
&\qquad + \frac{c}{n} \int_{x'_{s,\ell}}^s [u(s-u)]^{s-1} \left(\int_{x'_{s,\ell}}^u \left| f^{(s+1)}\left(\frac{\tilde{n}x - \ell + v}{n}\right) \right| dv \right) du.
\end{aligned} \tag{3.17}$$

Let us note that $x'_{s,\ell}$ is between ℓ and $s - \ell$; hence it is bounded away from 0 and s for all $x \in [0, 1]$ unless $\ell = 0$ or $\ell = s$. Estimate (3.16) for $\ell = 0$ follows directly from (3.17). For $0 < \ell < s$ we interchange the order of integration in each of the iterated integrals above. Thus we arrive at the estimates

$$\begin{aligned}
&\int_0^{x'_{s,\ell}} [u(s-u)]^{s-1} \left(\int_u^{x'_{s,\ell}} \left| f^{(s+1)}\left(\frac{\tilde{n}x - \ell + v}{n}\right) \right| dv \right) du \\
&\quad = \int_0^{x'_{s,\ell}} \left| f^{(s+1)}\left(\frac{\tilde{n}x - \ell + v}{n}\right) \right| \left(\int_0^v [u(s-u)]^{s-1}\, du \right) dv \\
&\quad \le c \int_0^{x'_{s,\ell}} \frac{[u(s-u)]^s}{\varphi^{2\ell}\left(\frac{\tilde{n}x - \ell + u}{n}\right)}\, du\, \|\varphi^{2\ell} f^{(s+1)}\| \\
&\quad \le c \int_0^s \frac{[u(s-u)]^s}{\varphi^{2\ell}\left(\frac{\tilde{n}x - \ell + u}{n}\right)}\, du\, \|\varphi^{2\ell} f^{(s+1)}\|
\end{aligned} \tag{3.18}$$

and

$$
(3.19)\quad \begin{aligned}
&\int_{x'_{s,\ell}}^{s} [u(s-u)]^{s-1} \left(\int_{x'_{s,\ell}}^{u} \left| f^{(s+1)} \left(\frac{\tilde{n}x-\ell+v}{n} \right) \right| dv \right) du \\
&\qquad = \int_{x'_{s,\ell}}^{s} \left| f^{(s+1)} \left(\frac{\tilde{n}x-\ell+v}{n} \right) \right| \left(\int_{v}^{s} [u(s-u)]^{s-1} \, du \right) dv \\
&\qquad \le c \int_{x'_{s,\ell}}^{s} \frac{[u(s-u)]^{s}}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)} \, du \, \|\varphi^{2\ell} f^{(s+1)}\| \\
&\qquad \le c \int_{0}^{s} \frac{[u(s-u)]^{s}}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)} \, du \, \|\varphi^{2\ell} f^{(s+1)}\|.
\end{aligned}
$$

Just similarly to (3.6) we verify that

$$
(3.20)\quad \varphi^{2\ell}(x) \int_{0}^{s} \frac{[u(s-u)]^{s}}{\varphi^{2\ell}\left(\frac{\tilde{n}x-\ell+u}{n}\right)} \, du \le c, \quad x \in [\ell/\tilde{n}, 1-\ell/\tilde{n}], \quad n \ge s, \quad \ell \le s.
$$

Inequalities (3.17)-(3.20) yield (3.16) for $\ell = 1, \dots, s-1$ and $s \ge 2$.

For $\ell = s$ we get just as above

$$
\begin{aligned}
&\int_{0}^{x'_{s,s}} [u(s-u)]^{s-1} \left(\int_{u}^{x'_{s,s}} \left| f^{(s+1)} \left(\frac{\tilde{n}x-s+v}{n} \right) \right| dv \right) du \\
&\qquad\qquad \le c \int_{0}^{s(1-x)} \frac{u^{s}}{\varphi^{2s}\left(\frac{\tilde{n}x-s+u}{n}\right)} \, du \, \|\varphi^{2s} f^{(s+1)}\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{x'_{s,s}}^{s} [u(s-u)]^{s-1} \left(\int_{x'_{s,s}}^{u} \left| f^{(s+1)} \left(\frac{\tilde{n}x-s+v}{n} \right) \right| dv \right) du \\
&\qquad\qquad \le c \int_{s(1-x)}^{s} \frac{(s-u)^{s}}{\varphi^{2s}\left(\frac{\tilde{n}x-s+u}{n}\right)} \, du \, \|\varphi^{2s} f^{(s+1)}\|.
\end{aligned}
$$

For $x \in [s/\tilde{n}, 1-s/\tilde{n}]$ inequality (3.20) with $\ell = s$ yields

$$
\varphi^{2s}(x) \int_{0}^{\min\{s/2, s(1-x)\}} \frac{u^{s}}{\varphi^{2s}\left(\frac{\tilde{n}x-s+u}{n}\right)} \, du \le c \, \varphi^{2s}(x) \int_{0}^{s} \frac{[u(s-u)]^{s}}{\varphi^{2s}\left(\frac{\tilde{n}x-s+u}{n}\right)} \, du \le c.
$$

Also, for $x \in [s/\tilde{n}, 1/2]$ and $u \in [s/2, s(1-x)]$ we have $(\tilde{n}x - s + u)/n \le 1/2$; consequently,

$$
\varphi^{2s}(x) \int_{s/2}^{s(1-x)} \frac{u^{s}}{\varphi^{2s}\left(\frac{\tilde{n}x-s+u}{n}\right)} \, du \le c \, \frac{(\tilde{n}x)^{s}}{(\tilde{n}x - s/2)^{s}} \le c.
$$

These considerations show that

$$
\varphi^{2s}(x) \int_{0}^{s(1-x)} \frac{u^{s}}{\varphi^{2s}\left(\frac{\tilde{n}x-s+u}{n}\right)} \, du \le c, \quad x \in [s/\tilde{n}, 1-s/\tilde{n}], \quad n \ge s.
$$

For the other integral we need only observe that

$$\int_{s(1-x)}^{s} \frac{(s-u)^s}{\varphi^{2s}\left(\frac{\tilde{n}x-s+u}{n}\right)}\,du = \int_0^{s(1-(1-x))} \frac{u^s}{\varphi^{2s}\left(\frac{\tilde{n}(1-x)-s+u}{n}\right)}\,du.$$

This completes the proof of (3.16) for $\ell = s$.

Now, we are ready to complete the proof. Inequalities (3.15) and (3.16) with $\ell = 0$ imply (3.12). Inequalities (3.15) and (3.16) with $\ell = 1$ yield

$$|D_{1,s,n}f(x_{\tilde{n},k}) - \varphi^2(x_{\tilde{n},k})f^{(s)}(x_{\tilde{n},k})| \le \frac{c}{n}\,\|\varphi^2 f^{(s+1)}\|, \quad k = 0,\dots,\tilde{n},$$

Taking into account (3.8), we establish (3.13) for $\ell = 1$.

Finally, for $\ell \ge 2$, inequalities (3.14)-(3.16) and Lemma 3.2 imply

$$\begin{aligned}|D_{\ell,s,n}f(x_{\tilde{n},k}) &- \varphi^{2\ell}(x_{\tilde{n},k})f^{(s)}(x_{\tilde{n},k})|\\ &\le \frac{c}{n}\left(\|\varphi^{2\ell-2}f^{(s)}\| + \|\varphi^{2\ell}f^{(s+1)}\|\right), \quad k = 0,\dots,\tilde{n},\end{aligned}$$

which, along with (3.9) and (3.10), gives (3.13) for $\ell \ge 2$. □

Remark 3.4. For functions of lower smoothness, similarly we can verify that

$$\|\varphi^{2\ell}(B_n f - f)^{(s)}\| \le \frac{c}{\sqrt{n}}\left(\|\varphi^{2\ell}f^{(s)}\| + \|\varphi^{2\ell}f^{(s+1)}\|\right),$$

where $\ell \in \mathbb{N}_0$ and $\ell \le s$.

To see that we only need to use instead of (1.2) the estimate

$$\|B_n f - f\| \le \frac{c}{\sqrt{n}}\,\|\varphi f'\|$$

valid for $f \in AC_{loc}(0,1)$ and all n. For large n it follows from (1.1) and (see [4, Theorems 2.1.1 and 4.1.3])

$$\omega_\varphi^2(f,t) \le c\,\omega_\varphi^1(f,t) \le c\,t\,\|\varphi f'\|, \quad f \in AC_{loc}(0,1), \quad 0 < t \le t_0,$$

with some $t_0 > 0$. Whereas for small n it is verified directly by means of Taylor's formula.

Remark 3.5. In view of (2.1), Theorem 3.3(a) implies (cf. [8])

$$\|(B_n f - f)^{(s)}\| \le \frac{c}{n}\left(\|f^{(s)}\| + \|f^{(s+2)}\|\right).$$

Inequalities like the one in Theorem 3.3 but in unweighted norm and in terms of the classical moduli of smoothness were earlier established in [9] and [12]. Also, Gonska, Heilmann and Raşa [10] established a quantitative Voronovskaya-type theorem about simultaneous approximation by the Bernstein operator.

A very neat though generally less practical upper estimate of the error of simultaneous approximation by the Bernstein operator can be stated in terms of the differential operator $D = \varphi^2(d/dx)^2$.

Theorem 3.6. *Let $\ell \in \mathbb{N}_0$. Then for all $f \in C^{2\ell+2}[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\|D^\ell(B_n f - f)\| \leq \frac{c}{n}\,\|D^{\ell+1} f\|.$$

Proof. For $\ell = 0$ the estimate reduces to (1.2). Otherwise, it is a direct corollary of Theorem 3.3(b) and Proposition 2.5. Indeed, applying consecutively Proposition 2.5(a), Theorem 3.3(b) with $s = 2\ell$, (1.2), and Proposition 2.5 (b) and (c), we get

$$\begin{aligned}
\|D^\ell(B_n f - f)\| &\leq c\left(\|B_n f - f\| + \|\varphi^{2\ell}(B_n f - f)^{(2\ell)}\|\right)\\
&\leq \frac{c}{n}\left(\|Df\| + \|\varphi^{2\ell} f^{(2\ell)}\| + \|\varphi^{2\ell+2} f^{(2\ell+2)}\|\right)\\
&\leq \frac{c}{n}\,\|D^{\ell+1} f\|.
\end{aligned}$$

Thus the assertion of the theorem is verified. □

4 Simultaneous approximation by $\mathcal{B}_{r,n}$

Here we shall formulate and prove our main results about simultaneous approximation by iterated Boolean sums of Bernstein polynomials.

First, we shall show how the result of Gonska and Zhou (1.3) can be derived from the upper estimates on simultaneous approximation by B_n, presented in the previous section. In my opinion, such an approach is more elementary and more straightforward (though not shorter) than the one used by H. Gonska and X.-l. Zhou. It is more elementary because essentially it uses only Taylor's formula and simple integral estimates, whereas highly non-trivial results on best approximation by algebraic polynomials are applied in [11]. Besides that it is more straightforward because it is independent on the close relation between best algebraic approximation and approximation by the Bernstein polynomials, as in both cases the weight $\varphi(x) = \sqrt{x(1-x)}$ plays an important role. However, it should be noted, the method used by H. Gonska and X.-l. Zhou enabled them to prove also an important converse inequality. The approach of Ding and Cao [2] was similar to that of H. Gonska and X-l. Zhou.

Theorem 4.1. *Let $r \in \mathbb{N}$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r-1}(0,1)$ and all $n \in \mathbb{N}$ there holds*

$$\|\mathcal{B}_{r,n} f - f\| \leq \frac{c}{n^r}\left(\|f\| + \|\varphi^{2r} f^{(2r)}\|\right).$$

Proof. Let us set $F_\rho = (B_n f - f)^\rho$. We first apply (1.2) to get

$$\|F_r\| \leq \frac{c}{n}\|\varphi^2 F''_{r-1}\|.$$

Next, if $r \geq 2$, we estimate the norm on the right above by Theorem 3.3(b) with $\ell = 1$ and $s = 2$ and thus arrive at

$$\|F_r\| \leq \frac{c}{n^2}\left(\|\varphi^2 F^{(2)}_{r-2}\| + \|\varphi^4 F^{(4)}_{r-2}\|\right).$$

If $r \geq 3$, we proceed in a similar fashion, i.e. we estimate above each of the two terms on the right by means of Theorem 3.3(b). Note that at each such step:

(i) The power of n increases by one,

(ii) The number of iterates of $B_n - I$ decreases by one,

(iii) The range of the index ℓ of the terms $\|\varphi^{2\ell} F_\rho^{(2\ell)}\|$ increases by one.

The inequality between the power of φ^2 and the order of the derivative in Theorem 3.3 is always satisfied.

Thus we arrive at the upper estimate

$$\|\mathcal{B}_{r,n} f - f\| \leq \frac{c}{n^r} \sum_{k=1}^{r} \|\varphi^{2k} f^{(2k)}\|. \tag{4.1}$$

To complete the proof, we need only apply Corollary 2.3 with $i = 0$. □

By a standard argument we derive from Theorem 4.1 the estimate

$$\|\mathcal{B}_{r,n} f - f\| \leq c \left(K_{2r,\varphi}(f, n^{-r}) + \frac{1}{n^r} \|f\| \right). \tag{4.2}$$

where $K_{m,\varphi}(f,t)$ is the K-functional defined for $f \in C[0,1]$ and $t > 0$ by

$$K_{m,\varphi}(f,t) = \inf_{g \in AC_{loc}^{m-1}(0,1)} \left\{ \|f - g\| + t \|\varphi^m g^{(m)}\| \right\}.$$

As Ditzian and Totik showed in [4, Theorem 2.1.1],

$$K_{m,\varphi}(f, t^m) \sim \omega_\varphi^m(f,t), \quad 0 < t \leq t_0, \tag{4.3}$$

with some t_0. Here the relation $\psi_1(f,t) \sim \psi_2(f,t)$ means that there exists a constant c such that for all f and t under consideration

$$c^{-1} \psi_1(f,t) \leq \psi_2(f,t) \leq c\, \psi_1(f,t).$$

Relations (4.2) and (4.3) imply (1.3) for $n \geq n_0$ with some fixed $n_0 \in \mathbb{N}$. For $n \leq n_0$ it is trivial.

Remark 4.2. Let us set for $f \in C[0,1]$ and $t > 0$

$$\widetilde{K}_r(f,t) = \inf_{g \in AC_{loc}^{2r-1}(0,1)} \left\{ \|f - g\| + t \left(E_1(g) + \|\varphi^{2r} g^{(2r)}\| \right) \right\}.$$

By means of this K-functional estimate (1.4) can be stated in the equivalent form

$$\|\mathcal{B}_{r,n} f - f\| \leq c\, \widetilde{K}_r(f, n^{-r}), \quad n \in \mathbb{N}.$$

It may seem that the seminorm on the right hand-side of (4.1) is essentially smaller than the seminorm $\|\varphi^{2r}g^{(2r)}\|+E_1(g)$ in the definition of $\widetilde{K}_r(f,t)$. Recall that by Corollary 2.3 we have, for all $g \in AC_{loc}^{2r-1}(0,1)$, the estimate

$$\sum_{k=1}^{r} \|\varphi^{2k}g^{(2k)}\| \le c\left(\|g\| + \|\varphi^{2r}g^{(2r)}\|\right);$$

hence

$$\sum_{k=1}^{r} \|\varphi^{2k}g^{(2k)}\| \le c\left(E_1(g) + \|\varphi^{2r}g^{(2r)}\|\right).$$

But the converse inequality is also valid since by (1.2) we have

$$E_1(g) \le \|B_1 g - g\| \le c\,\|\varphi^2 g''\|, \quad g \in AC_{loc}^1(0,1). \tag{4.4}$$

Now, let us proceed to simultaneous approximation by $\mathcal{B}_{r,n}$. We shall prove the following two estimates. The first one concerns the unweighted case, the other the weighted.

Theorem 4.3. *Let $r, s, \ell \in \mathbb{N}$ and $\ell \le s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s-1}(0,1)$ and all $n \in \mathbb{N}$ the following assertions hold true:*

(a) $\|(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \dfrac{c}{n^r}\left(\|f^{(s)}\| + \|f^{(s+r)}\| + \|\varphi^{2r}f^{(2r+s)}\|\right)$;

(b) $\|\varphi^{2\ell}(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \dfrac{c}{n^r}\left(\|\varphi^{2\ell}f^{(s)}\| + \|\varphi^{2r+2\ell}f^{(2r+s)}\|\right)$.

Proof. Iterating the estimates of Theorem 3.3 we get

$$\|(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \frac{c}{n^r}\left(\sum_{k=0}^{r}\|f^{(k+s)}\| + \sum_{k=1}^{r}\|\varphi^{2k}f^{(k+s+r)}\|\right).$$

The embedding inequality (2.1) implies

$$\|f^{(k+s)}\| \le c\left(\|f^{(s)}\| + \|f^{(s+r)}\|\right), \quad k = 1, \ldots, r-1,$$

whereas Proposition 2.1 with $f^{(s+r)}$ in the place of f, $i = 0$ and $m = r$ yields

$$\|\varphi^{2k}f^{(k+s+r)}\| \le c\left(\|f^{(s+r)}\| + \|\varphi^{2r}f^{(2r+s)}\|\right), \quad k = 1, \ldots, r-1.$$

Thus assertion (a) is established.

To verify (b) we proceed just in the same way as in the proof of Theorem 4.1 but we skip the initial application of (1.2). Thus we arrive at

$$\|\varphi^{2\ell}(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \frac{c}{n^r}\sum_{k=0}^{r}\|\varphi^{2(k+\ell)}f^{(2k+s)}\|.$$

Then we complete the proof by means of Corollary 2.3 with $i = \ell$ and $f^{(s)}$ in the place of f. $\square$

Remark 4.4. In view of Remark 2.4, even the following estimate holds true

$$\|\varphi^{2\ell}(\mathcal{B}_{r,n}f-f)^{(s)}\| \le \frac{c}{n^r}\left(\|f^{(s)}\|_{[1/4,3/4]}+\|\varphi^{2r+2\ell}f^{(2r+s)}\|\right),$$

where $\ell \in \mathbb{N}$.

Now, we are ready to establish our main result. Let us recall that by [4, Theorem 6.1.1] there exists t_0 such that

$$K_{m,\varphi}(f,t^m)_{\varphi^{2\ell}} \sim \omega_\varphi^m(f,t)_{\varphi^{2\ell}}, \quad 0<t\le t_0, \tag{4.5}$$

where the K-functional $K_{m,\varphi}(f,t)_w$ with a weight w is defined for locally continuous functions f such that $wf \in C[0,1]$ by

$$K_{m,\varphi}(f,t)_w = \inf_{g\in AC_{loc}^{m-1}(0,1)}\{\|w(f-g)\|+t\|w\varphi^m g^{(m)}\|\}.$$

Proof of Theorem 1.1. We follow the same standard argument we used to derive (4.2) from Theorem 4.1.

First, let us observe that Proposition 3.1 implies

$$\|\varphi^{2\ell}(B_n^j f)^{(s)}\| \le c\|\varphi^{2\ell}f^{(s)}\|, \quad n\in\mathbb{N};$$

hence

$$\|\varphi^{2\ell}(\mathcal{B}_{r,n}f)^{(s)}\| \le c\|\varphi^{2\ell}f^{(s)}\|, \quad n\in\mathbb{N}. \tag{4.6}$$

Let $g\in AC_{loc}^{2r-1}(0,1)$ and $\tilde{g}$ be any of its s-fold integrals. Then $\tilde{g}\in AC_{loc}^{2r+s-1}(0,1)$ and we have by (4.6) and Theorem 4.3(b)

$$\begin{aligned}\|\varphi^{2\ell}(\mathcal{B}_{r,n}f-f)^{(s)}\| &\le c\left(\|\varphi^{2\ell}(f-\tilde{g})^{(s)}\|+\|\varphi^{2\ell}(\mathcal{B}_{r,n}\tilde{g}-\tilde{g})^{(s)}\|\right)\\ &\le c\left(\|\varphi^{2\ell}(f-\tilde{g})^{(s)}\|+\frac{1}{n^r}\|\varphi^{2\ell}\tilde{g}^{(s)}\|+\frac{1}{n^r}\|\varphi^{2r+2\ell}\tilde{g}^{(2r+s)}\|\right)\\ &\le c\left(\|\varphi^{2\ell}(f^{(s)}-g)\|+\frac{1}{n^r}\|\varphi^{2r+2\ell}g^{(2r)}\|\right)+\frac{c}{n^r}\|\varphi^{2\ell}f^{(s)}\|.\end{aligned}$$

Taking an infimum over g we arrive at

$$\|\varphi^{2\ell}(\mathcal{B}_{r,n}f-f)^{(s)}\| \le c\left(K_{2r,\varphi}(f^{(s)},n^{-r})_{\varphi^{2\ell}}+\frac{1}{n^r}\|\varphi^{2\ell}f^{(s)}\|\right).$$

Now, for large n the assertion of the theorem follows from (4.5), and for small n from (4.6). $\square$

Similar estimates can be stated in terms of the differential operator D. They are given in the next theorem.

Theorem 4.5. *Let $\ell, r \in \mathbb{N}$. Then:*

(a) For all $f \in C^{2(r+\ell)}[0,1]$ and $n \in \mathbb{N}$ there holds

$$\|D^\ell(\mathcal{B}_{r,n}f - f)\| \le \frac{c}{n^r}\,\|D^{r+\ell}f\|.$$

(b) For all $f \in C^{2\ell}[0,1]$ and $n \in \mathbb{N}$ there holds

$$\|D^\ell(\mathcal{B}_{r,n}f - f)\| \le c\,K_{r,\ell}(D^\ell f, n^{-r}),$$

where

$$K_{r,\ell}(F,t) = \inf_{g \in C^{2(r+\ell)}[0,1]} \{\|F - D^\ell g\| + t\|D^{r+\ell}g\|\}.$$

Proof. Assertion (a) follows directly from Theorem 3.6.

To establish (b) we proceed as in the proof of Theorem 1.1. We need to show that

$$\|D^\ell \mathcal{B}_{r,n}f\| \le c\,\|D^\ell f\|, \quad n \in \mathbb{N}. \tag{4.7}$$

To this end, we apply consecutively Proposition 2.5(a), (4.6) with $s = 2\ell$ and Proposition 2.5(b) to derive

$$\begin{aligned}\|D^\ell \mathcal{B}_{r,n}f\| &\le c\,(\|\mathcal{B}_{r,n}f\| + \|\varphi^{2\ell}(\mathcal{B}_{r,n}f)^{(2\ell)}\|)\\ &\le c\,(\|f\| + \|\varphi^{2\ell}f^{(2\ell)}\|)\\ &\le c\,(\|f\| + \|D^\ell f\|);\end{aligned}$$

hence

$$\|D^\ell \mathcal{B}_{r,n}f\| \le c\,(\|E_1(f)\| + \|D^\ell f\|).$$

To complete the proof of (4.7), we need only apply (4.4) and Proposition 2.5(c). □

Remark 4.6. Let us note that, in view of Remark 2.6, assertion (a) above holds under the weaker restriction $f \in C^{2(r+\ell-1)}[0,1] \cap AC^{2(r+\ell)-1}_{loc}(0,1)$. Moreover, the latter cannot be really relaxed. For example, if $\ell = 0$ and $r = 2$, then for $f(x) = x\log x$ we have $D^2 f = 0$, whereas $\mathcal{B}_{2,n}f \ne f$.

5 Relations between K-functionals associated with $\mathcal{B}_{r,n}$

In this section we shall show that the upper estimates (1.4) and (1.5) are equivalent. More precisely, we shall establish that the quantities on the right hand-side of (1.5) and (4.2) with $\|f\|$ replaced by $E_1(f)$ are equivalent.

Theorem 5.1. *For all $f \in C[0,1]$ and $0 < t \le 1$ we have*

$$K_r(f,t) \sim K_{2r,\varphi}(f,t) + t\,E_1(f).$$

Proof. The assertion of the theorem follows from the inequalities:

$$K_{2r,\varphi}(f,t) \le c\, K_r(f,t),$$
$$t\, E_1(f) \le c\, K_r(f,t)$$

and

$$K_r(f,t) \le c\,(K_{2r,\varphi}(f,t) + t\, E_1(f))\,.$$

The first one follows directly from Proposition 2.5(b). The second one follows from the estimate

$$\begin{aligned} t\, E_1(f) &\le E_1(f-g) + t\, E_1(g) \le c\,(\|f-g\| + t\,\|Dg\|) \\ &\le c\,(\|f-g\| + t\,\|D^r g\|)\,, \quad g \in C^{2r}[0,1], \quad 0 < t \le 1, \end{aligned}$$

where at the second step we have applied (4.4) and at the third Proposition 2.5(c).

In order to verify the third inequality, we apply Proposition 2.5(a) to get for any $g \in C^{2r}[0,1]$ and $t \le 1$ that

$$\begin{aligned} t\,\|D^r g\| &\le c\, t\left(\|\varphi^{2r} g^{(2r)}\| + E_1(g)\right) \\ &\le c\left(\|f-g\| + t\,\|\varphi^{2r} g^{(2r)}\| + t\, E_1(f)\right). \end{aligned}$$

Consequently,

$$K_r(f,t) \le c\left(\inf_{g\in C^{2r}[0,1]} \left\{\|f-g\| + t\|\varphi^{2r} g^{(2r)}\|\right\} + t\, E_1(f)\right).$$

It remains to observe that

$$\inf_{g\in C^{2r}[0,1]} \left\{\|f-g\| + t\|\varphi^{2r} g^{(2r)}\|\right\} \le c\, K_{2r,\varphi}(f,t). \tag{5.1}$$

To justify the latter, we recall that the Steklov-type function used in [4, Chapter 2] to establish the inequality

$$K_{2r,\varphi}(f,t) \le c\,\omega_\varphi^{2r}(f,t)$$

belongs to $C^{2r}[0,1]$ (or see [1, Chapter 6, § 6], where a spline was used). Therefore the K-functional on the left hand-side of (5.1) is estimated above by $\omega_\varphi^{2r}(f,t)$ (at least for $t \le t_0$); and hence, in view of (4.3), by $K_{2r,\varphi}(f,t)$. □

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