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Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums

Borislav R. Draganov*

Abstract

We establish matching direct and two-term strong converse estimates of the rate of weighted simultaneous approximation by the Bernstein operator and its iterated Boolean sums for smooth functions in L_p-norm, $1 < p \leq \infty$. We consider Jacobi weights. The characterization is stated in terms of appropriate moduli of smoothness or K-functionals. Also, analogous results concerning the generalized Kantorovich operators are derived.

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1 Main results

The Bernstein operator is defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

As is known its saturation order is n^{-1} and the differential operator which describes its rate of approximation is $Df(x) = \varphi^2(x) f''(x)$ with $\varphi(x) = \sqrt{x(1-x)}$ (see e.g. [4, Chapter 10, Theorems 3.1 and 5.1]). More precisely, Voronovskaya's classic result states

$$\lim_{n\to\infty} n\,(B_n f(x) - f(x)) = \frac{1}{2} Df(x) \quad \text{uniformly on } [0,1] \tag{1.1}$$

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for $f \in C^2[0,1]$.

One way to increase the approximation rate of B_n is to form its iterated Boolean sums $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$, defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}$. In [27] it was shown that their saturation order is n^{-r}. Gonska and Zhou [18] established a neat direct estimate for $\mathcal{B}_{r,n}$ and a Stechkin-type inverse inequality. Also, they made a historical overview of the study of that kind of operators and explained why they can be regarded as iterated Boolean sums. Later on Ding and Cao [5] characterized the error of the multivariate generalization of $\mathcal{B}_{r,n}$ on the simplex and further improved the lower estimate. They used a K-functional with a differential operator that reduces in the univariate case to D^r, i.e. exactly the rth iterate of the differential operator associated with B_n as it should be expected.

Here we shall consider simultaneous approximation by $\mathcal{B}_{r,n}$. It is known that the derivatives of the Bernstein polynomial of a smooth function approximate the corresponding derivatives of the function (see [4, Chapter 10, Theorem 2.1]). López-Moreno, Martínez-Moreno and Muñoz-Delgado [24] and Floater [14] extended (1.1) showing that for $f \in C^{s+2}[0,1]$ we have

$$\lim_{n\to\infty} n\left((B_n f(x))^{(s)} - f^{(s)}(x)\right) = \frac{1}{2}(Df(x))^{(s)} \quad \text{uniformly on } [0,1]. \tag{1.2}$$

Hence the differential operator that describes the simultaneous approximation by B_n is $(d/dx)^s D$. Results about the rate of convergence in (1.2) were established in [15, 16, 17].

So, it is reasonable to expect that the differential operator related to the simultaneous approximation by $\mathcal{B}_{r,n}$ is $(d/dx)^s D^r$ and the saturation order is n^{-r}. That turns out to be indeed so. Before stating our main results let us note that, since the derivative of the Bernstein polynomial is closely related to the Kantorovich polynomial, it makes sense to consider approximation not only in the uniform norm but also in the L_p-norm. Moreover, weights of the form $\varphi^{2\ell}$ with $\ell \in \mathbb{N}$ appear naturally in the study of the approximation rate of $\mathcal{B}_{r,n}$ (see the proof of [18, Theorem 1(i)] we gave in [9, pp. 35-36]). So, it is appropriate to consider simultaneous approximation by $\mathcal{B}_{r,n}$ with Jacobi weights. We set

$$w(x) = w(\gamma_0, \gamma_1; x) = x^{\gamma_0}(1-x)^{\gamma_1}, \quad x \in (0,1), \tag{1.3}$$

where $\gamma_0, \gamma_1 > -1/p$ for $1 \le p < \infty$ or $\gamma_0, \gamma_1 \ge 0$ for $p = \infty$. To characterize the rate of the simultaneous approximation by $\mathcal{B}_{r,n}$, we shall use the K-functional

$$K_{r,s}(f,t)_{w,p} = \inf_{g \in C^{2r+s}[0,1]} \left\{ \|w(f - g^{(s)})\|_p + t\|w(D^r g)^{(s)}\|_p \right\}.$$

We denote by $\|f\|_p$ the L_p-norm of the function f on the interval $[0,1]$. When the norm is taken on a subinterval $J \subset [0,1]$, we shall write $\|f\|_{p(J)}$. As usual, $AC^k[a,b]$ stands for the set of all functions, which along with their derivatives up to order $k \in \mathbb{N}_0$ are absolutely continuous on $[a,b]$; $AC^k_{loc}(0,1)$ is the set of the functions, which are in $AC^k[a,b]$ for all $0 < a < b < 1$ (see e.g. [23] for the basic properties of these and related spaces). By c we shall denote positive constants, not necessarily the same at each occurrence, which are independent of the functions involved or the degree n of the operators.

We shall establish the following characterization of the error of simultaneous approximation by $\mathcal{B}_{r,n}$.

Theorem 1.1. *Let $r, s \in \mathbb{N}$, $1 < p \le \infty$ and $w = w(\gamma_0, \gamma_1)$ be given by (1.3) as*

$$\begin{aligned} -1/p <&\gamma_0, \gamma_1 < s - 1/p \quad \textit{if} \quad 1 < p < \infty, \\ 0 \le&\gamma_0, \gamma_1 < s \quad \textit{if} \quad p = \infty. \end{aligned}$$

Then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\|w(\mathcal{B}_{r,n} f - f)^{(s)}\|_p \le c\, K_{r,s}(f^{(s)}, n^{-r})_{w,p}.$$

Conversely, there exists $R \in \mathbb{N}$ such that for all $f \in C[0,1]$ with $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $k, n \in \mathbb{N}$ with $k \ge Rn$ there holds

$$K_{r,s}(f^{(s)}, n^{-r})_{w,p} \le c \left(\frac{k}{n}\right)^r \left(\|w(\mathcal{B}_{r,n} f - f)^{(s)}\|_p + \|w(\mathcal{B}_{r,k} f - f)^{(s)}\|_p\right).$$

In particular,

$$K_{r,s}(f^{(s)}, n^{-r})_{w,p} \le c \left(\|w(\mathcal{B}_{r,n} f - f)^{(s)}\|_p + \|w(\mathcal{B}_{r,Rn} f - f)^{(s)}\|_p\right).$$

This characterization is valid for $p = 1$ as well. However, the proof requires additional considerations, which would make the exposition even longer (see Remark 4.6 below). That is why we shall omit this case despite its importance.

The estimates in Theorem 1.1 can be simplified. The involved K-functional $K_{r,s}(f,t)_{w,p}$ can be characterized by the simpler ones given by

$$K_{m,\varphi}(f,t)_{w,p} = \inf_{g \in AC^{m-1}_{loc}} \left\{\|w(f-g)\|_p + t\|w\varphi^m g^{(m)}\|_p\right\}$$

and

$$K_m(f,t)_{w,p} = \inf_{g \in AC^{m-1}_{loc}} \left\{\|w(f-g)\|_p + t\|wg^{(m)}\|_p\right\}.$$

For the unweighted case $w = 1$ we set $K_{m,\varphi}(f,t)_\infty = K_{m,\varphi}(f,t)_{1,\infty}$ and $K_m(f,t)_\infty = K_m(f,t)_{1,\infty}$. We say that $\Phi(f,t)$ and $\Psi(f,t)$ are equivalent and write $\Phi(f,t) \sim \Psi(f,t)$ if there exists a constant c such that $c^{-1}\Phi(f,t) \le \Psi(f,t) \le c\,\Phi(f,t)$ for all f and t under consideration. There follows the characterization of $K_{r,s}(f,t)_{w,p}$ by means of $K_{m,\varphi}(f,t)_{w,p}$ and $K_m(f,t)_{w,p}$.

Theorem 1.2. *Let $1 < p \le \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *with $-1/p < \gamma_0, \gamma_1 < s - 1/p$. Then for all $wf \in L_p[0,1]$ and $0 < t \le 1$ there holds*

$$K_{r,s}(f,t)_{w,p} \sim \begin{cases} K_{2r,\varphi}(f,t)_{w,p} + K_1(f,t)_{w,p}, & s = 1, \\ K_{2r,\varphi}(f,t)_{w,p} + t\,\|wf\|_p, & s \ge 2. \end{cases}$$

The result in the case $w = 1$ and $p = \infty$ is of a different form.

Theorem 1.3. *Let $r, s \in \mathbb{N}$. Then for all $f \in C[0,1]$ and $0 < t \le 1$ there holds*

$$K_{r,s}(f,t)_{1,\infty} \sim \begin{cases} K_{2r,\varphi}(f,t)_\infty + K_r(f,t)_\infty + K_1(f,t)_\infty, & s = 1, \\ K_{2r,\varphi}(f,t)_\infty + K_r(f,t)_\infty + t\,\|f\|_\infty, & s \ge 2. \end{cases}$$

Remark 1.4. Let us note that the assertion of Theorem 1.2 in the case $p = \infty$ and $r = s = 1$ actually holds for all $0 \le \gamma_0, \gamma_1 < 1$, as it will be briefly shown in its proof.

Further, we can take into account that $K_{2r,\varphi}(f, t^{2r})_{w,p}$ is equivalent to the weighted Ditzian-Totik modulus of smoothness $\omega_\varphi^{2r}(f,t)_{w,p}$ [8, Chapter 6] provided that $\gamma_0, \gamma_1 \ge 0$, and to its modification introduced and considered in [13, Chapter 3, Section 10] and [22] if $w = \varphi^s$. Generally, for $\gamma_0, \gamma_1 > -1/p$ we can use the modulus defined in [10, 11, 12].

Theorems 1.1-1.3 and the equivalence between the K-functionals and the moduli of smoothness imply the following Jackson-type estimates.

Corollary 1.5. *Let $r, s \in \mathbb{N}$, $1 < p \le \infty$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *as*

$$\begin{aligned} 0 \le &\gamma_0, \gamma_1 < s - 1/p \quad \text{if} \quad 1 < p < \infty, \\ 0 < &\gamma_0, \gamma_1 < s \quad \text{if} \quad p = \infty. \end{aligned}$$

Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_p \le c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2})_{w,p} + \omega_1(f', n^{-r})_{w,p}, & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2})_{w,p} + \dfrac{1}{n^r}\,\|wf^{(s)}\|_p, & s \ge 2. \end{cases}$$

Corollary 1.6. *Let $r, s \in \mathbb{N}$. Then for all $f \in C^s[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\begin{aligned} &\|(\mathcal{B}_{r,n}f - f)^{(s)}\|_\infty \\ &\qquad \le c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2})_\infty + \omega_r(f', n^{-1})_\infty + \omega_1(f', n^{-r})_\infty, & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2})_\infty + \omega_r(f^{(s)}, n^{-1})_\infty + \dfrac{1}{n^r}\,\|f^{(s)}\|_\infty, & s \ge 2. \end{cases} \end{aligned}$$

Here $\omega_r(f,t)_\infty$ denotes the unweighted fixed-step modulus of smoothness of order r in the uniform norm and $\omega_r(f,t)_{w,p}$ its analogue in weighted L_p-spaces (see e.g. [8, Appendix B]). We shall give brief details about the proof of the two corollaries in the last section.

In particular, for the weighted simultaneous approximation by the Bernstein operator (i.e. $r=1$) in weighted uniform norm we have

$$\begin{aligned}&\|w(B_nf-f)^{(s)}\|_\infty\\ &\le c\begin{cases}\omega_\varphi^2(f',n^{-1/2})_{w,\infty}+\omega_1(f',n^{-1})_{w,\infty}, & s=1,\ 0\le\gamma_0,\gamma_1<1,\\[2mm] \omega_\varphi^2(f^{(s)},n^{-1/2})_\infty+\omega_1(f^{(s)},n^{-1})_\infty+\dfrac{1}{n}\,\|f^{(s)}\|_\infty, & s\ge 2,\ \gamma_0=\gamma_1=0,\\[2mm] \omega_\varphi^2(f^{(s)},n^{-1/2})_{w,\infty}+\dfrac{1}{n}\,\|wf^{(s)}\|_\infty, & s\ge 2,\ 0<\gamma_0,\gamma_1<s.\end{cases}\end{aligned}$$

Remark 1.7. The middle term on the right-hand side in the characterization in Theorem 1.3 *cannot* be omitted. Indeed, if $f^{(s)}(x)=x^r\log x$, then $f^{(s)},\varphi^{2r}f^{(2r+s)}\in L_\infty[0,1]$ and $f^{(s+1)}\in L_\infty[0,1]$ (the latter in the case $r\ge 2$), but $f^{(r+s)}\notin L_\infty[0,1]$.

Results about the simultaneous approximation by the Bernstein operator can be easily transferred to the Kantorovich operator. Let $1\le p\le\infty$. The Kantorovich polynomials are defined for $f\in L_p[0,1]$ and $x\in[0,1]$ by

$$K_nf(x)=\sum_{k=0}^n(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)}f(t)\,dt\,p_{n,k}(x),\quad p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}.$$

They are related to the Bernstein polynomials as follows

$$K_nf(x)=(B_{n+1}F(x))',\quad F(x)=\int_0^x f(t)\,dt.$$

More generally, we set for $f\in L_p[0,1]$, $1\le p\le\infty$, and $m\in\mathbb{N}$ (see [29])

$$K_n^{\langle m\rangle}f(x)=\left(B_{n+m}F_m(x)\right)^{(m)},$$

where

$$F_m(x)=\frac{1}{(m-1)!}\int_0^x(x-t)^{m-1}f(t)\,dt.$$

The operator $K_n^{\langle m\rangle}$ is referred to as the generalized Kantorovich operator of order m. That generalization of the Kantorovich polynomials or similar modifications of related operators are studied in [2, 3, 15, 16, 17, 20].

Further, we set

$$\mathcal{K}_{r,n}^{\langle m\rangle}=I-(I-K_n^{\langle m\rangle})^r.$$

Using that B_n is degree reducing w.r.t. the algebraic polynomials, it can be verified by induction on j that

$$\left(K_n^{\langle m\rangle}\right)^j f=\left(B_{n+m}^jF_m\right)^{(m)};$$

hence
$$\mathcal{K}_{r,n}^{\langle m\rangle} f = (\mathcal{B}_{r,n+m} F_m)^{(m)} .$$
This enables us to transfer all the above results about simultaneous approximation by $\mathcal{B}_{r,n}$ to $\mathcal{K}_{r,n}$.

Theorem 1.8. *Let $m, r \in \mathbb{N}$, $s \in \mathbb{N}_0$, $1 < p \leq \infty$ and $w = w(\gamma_0, \gamma_1)$ be given by (1.3) as*

$$\begin{aligned} -1/p <&\gamma_0, \gamma_1 < s + m - 1/p \quad \textit{if} \quad 1 < p < \infty, \\ 0 \leq&\gamma_0, \gamma_1 < s + m \quad \textit{if} \quad p = \infty. \end{aligned} \tag{1.4}$$

Then for all $f \in L_p[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds
$$\|w(\mathcal{K}_{r,n}^{\langle m\rangle} f - f)^{(s)}\|_p \leq c\, K_{r,s+m}(f^{(s)}, n^{-r})_{w,p}.$$
Conversely, there exists $R \in \mathbb{N}$ such that for all $f \in L_p[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $k, n \in \mathbb{N}$ with $k \geq Rn$ there holds
$$K_{r,s+m}(f^{(s)}, n^{-r})_{w,p} \leq c \left(\frac{k}{n}\right)^r \left(\|w(\mathcal{K}_{r,n}^{\langle m\rangle} f - f)^{(s)}\|_p + \|w(\mathcal{K}_{r,k}^{\langle m\rangle} f - f)^{(s)}\|_p \right).$$
In particular,
$$K_{r,s+m}(f^{(s)}, n^{-r})_{w,p} \leq c \left(\|w(\mathcal{K}_{r,n}^{\langle m\rangle} f - f)^{(s)}\|_p + \|w(\mathcal{K}_{r,Rn}^{\langle m\rangle} f - f)^{(s)}\|_p \right).$$

In the statement of the last theorem the condition $f \in AC_{loc}^{s-1}(0,1)$ is to be ignored for $s = 0$.

Remark 1.9. As it is clear from the last theorem, the higher the order of the generalized Kantorovich operator is, the broader the space of functions it approximates is. More precisely, let us denote by $\mathcal{W}_m^s$ the set of functions, for which Theorem 1.8 is established, i.e. $\mathcal{W}_m^s$ is the set of all $f \in L_p[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$ for some Jacobi weight $w = w(\gamma_0, \gamma_1)$, which satisfies (1.4). Then we have $\mathcal{W}_m^s \subset \mathcal{W}_{m+1}^s$. Or, to put it otherwise, given an $s \in \mathbb{N}_0$ and a function $f \in AC_{loc}^{s-1}(0,1)$ such that $wf^{(s)} \in L_p[0,1]$ for some Jacobi weight $w = w(\gamma_0, \gamma_1)$, then $(\mathcal{K}_{r,n}^{\langle m\rangle} f)^{(s)}$ approximates $f^{(s)}$ in L_p with a weight w, provided that we take m large enough, namely, $m > \max\{\gamma_0, \gamma_1\} - s + 1/p$.

Remark 1.10. We can enlarge the domain of $\mathcal{K}_{r,n}^{\langle m\rangle}$ if we replace F_m in its definition by
$$f_m(x) = \frac{1}{(m-1)!} \int_{1/2}^x (x-t)^{m-1} f(t)\, dt.$$
If $\tilde{w} f \in L_p[0,1]$, where $\tilde{w} = w(\tilde{\gamma}_0, \tilde{\gamma}_1)$ with $\tilde{\gamma}_0, \tilde{\gamma}_1 < m - 1/p$, then $f_m \in C[0,1]$ (see the proof of Lemma 5.1 below). Theorem 1.8 holds for that modification of $\mathcal{K}_{r,n}^{\langle m\rangle}$ as the condition $f \in L_p[0,1]$ is replaced with $\tilde{w} f \in L_p[0,1]$, $\tilde{\gamma}_0, \tilde{\gamma}_1 < m - 1/p$.

The upper estimate for the Kantorovich operator (i.e. $r = m = 1$) in the case $w = 1$ and $s = 0$ is due to Berens and Xu [1, Theorem 6]. There a weak converse inequality was established as well. The corresponding one-term strong converse inequality ($R = 1$ in Theorem 1.8) and the characterization of the K-functional by the Ditzian-Totik modulus for all $1 \le p \le \infty$ were proved by Gonska and Zhou [19]. Mache [26] established the direct estimate for the Kantorovich operator and a weak converse one in the case $w = \varphi^{2\ell}$ and $s = 2\ell$, $\ell \in \mathbb{N}$.

It seems quite plausible that the technique introduced by Knoop and Zhou to prove [21, Theorem 1.2] and further developed by Gonska and Zhou [19] can lead to such one-term strong inequality for the weighted simultaneous approximation by $\mathcal{B}_{r,n}$. We have checked that this method works at least for $p = \infty$ and $0 \le \gamma_0, \gamma_1 \le s/2$ but we shall not include that result here.

Finally, let us mention one general feature of the approach we adopt. As we noted above, the differential operator associated with the simultaneous approximation by $\mathcal{B}_{r,n}$ is $(d/dx)^s D^r$. However, it is rather involved. It is much easier to establish estimates in terms of the norms of the components into which $(D^r g)^{(s)}$ expands. They are of the form $q\varphi^{2i} g^{(j)}$, where q is an algebraic polynomial, which can be ignored, and $i, j \in \mathbb{N}$. Due to the validity of certain embedding inequalities their number can be reduced to two or three and the sum of their weighted L_p-norms is equivalent to the norm of $(D^r g)^{(s)}$. That allows us not only to get round the technical difficulties of dealing with $(d/dx)^s D^r$, but also to derive almost simultaneously both characterizations of $\|w(\mathcal{B}_{r,n} f - f)^{(s)}\|_p$: the more natural one by $K_{r,s}(f,t)_{w,p}$ and the more useful one by $K_{m,\varphi}(f,t)_{w,p}$. However, this method has a disadvantage—it fails to cover the cases when the error is characterized by $K_{r,s}(f,t)_{w,p}$ but not by $K_{m,\varphi}(f,t)_{w,p}$, that is, when the above mentioned semi-norms are not equivalent. It seems that this occurs only when $p = 1$ and γ_0 or γ_1 is equal to $s - 1$ (see [19, Theorem 1.1]).

The contents of the paper are organized as follows. In Section 2 we establish several embedding inequalities, which play a crucial role in simplifying estimates and finding relations between different K-functionals. Section 3 is devoted to technical identities concerning the Bernstein polynomial and its derivatives. Then in Section 4 we establish the main inequalities which constitute the proof of the error characterization stated in Theorem 1.1. We establish the direct estimate by means of a standard argument based on the boundedness of the operator and a Jackson-type inequality. For the converse estimate we apply the method developed by Ditzian and Ivanov [7]. Finally, in Section 5 we characterize $K_{r,s}(f,t)_{w,p}$ by $K_{2r,\varphi}(f,t)_{w,p}$ as stated in Theorems 1.2 and 1.3. There we also make a couple of comments about the proof of Corollaries 1.5 and 1.6.

The results established here extend and complement those in [9]. There we included more references to similar works.

2 Embedding inequalities

We shall extensively use embedding inequalities in order to simplify estimates or show that certain integrals are well-defined. Such inequalities are typical for that setting; see e.g. [1, Lemmas 2, 3 and 4], [8, p. 135], [18, Lemma 2] and [19, pp. 127-128].

First, we recall the well-known inequality

$$\|f^{(j)}\|_{p(J)} \le c\left(\|f\|_{p(J)} + \|f^{(m)}\|_{p(J)}\right), \quad j = 0, \dots, m, \tag{2.1}$$

where J is an interval on the real line.

Next, we shall establish a generalization of [8, p. 135, (a) and (b)] and [9, Proposition 2.2] by means of an argument similar to the one used there.

Proposition 2.1. *Let $1 \le p \le \infty$ and $j, m \in \mathbb{N}_0$ as $j < m$. Let $w_\mu = w(\gamma_{\mu,0}, \gamma_{\mu,1})$ be given by (1.3) with $\gamma_{\mu,0}, \gamma_{\mu,1} > -1/p$ for $\mu = 1, 2$ and let $\gamma_{2,\nu} \le \gamma_{1,\nu} + m - j$ for $\nu = 0, 1$. Let also $g \in AC_{loc}^{m-1}(0,1)$. Then*

$$\|w_1 g^{(j)}\|_p \le c\left(\|g\|_{p[1/4,3/4]} + \|w_2 g^{(m)}\|_p\right).$$

The constant c is independent of g.

Proof. By Taylor's formula we have

$$g^{(j)}(x) = \sum_{i=0}^{m-j-1} \frac{g^{(i+j)}(1/2)}{i!}\left(x - \frac{1}{2}\right)^i + \frac{1}{(m-j-1)!}\int_{1/2}^x (x-u)^{m-j-1} g^{(m)}(u)\, du.$$

Consequently, for $x \in (0, 1/2]$ we have

$$x^{\gamma_{1,0}}|g^{(j)}(x)| \le x^{\gamma_{1,0}} \sum_{i=0}^{m-j-1} \left|g^{(i+j)}\left(\frac{1}{2}\right)\right| + \sum_{k=0}^{m-j-1} \psi_k(x), \tag{2.2}$$

where we have set

$$\psi_k(x) = x^{k+\gamma_{1,0}} \int_x^{1/2} u^{m-j-k-1}|g^{(m)}(u)|\, du, \quad k = 0, \dots, m-j-1.$$

Next, we shall show that

$$\left|g^{(j)}\left(\frac{1}{2}\right)\right| \le c\left(\|g\|_{p[1/4,1/2]} + \|g^{(m)}\|_{p[1/4,1/2]}\right), \quad j = 0, \dots, m-1. \tag{2.3}$$

For $p = \infty$ that follows immediately from (2.1). For $p < \infty$ we introduce the function $\psi(x) = 4x - 1$ and observe that

$$\begin{aligned}
\left| g^{(j)}\left(\frac{1}{2}\right) \right| &= \left| (\psi g^{(j)})\left(\frac{1}{2}\right) - (\psi g^{(j)})\left(\frac{1}{4}\right) \right| \\
&\le \int_{1/4}^{1/2} |(\psi g^{(j)})'(u)|\, du \\
&\le c\left(\|g^{(j)}\|_{1[1/4,1/2]} + \|g^{(j+1)}\|_{1[1/4,1/2]} \right) \\
&\le c\left(\|g^{(j)}\|_{p[1/4,1/2]} + \|g^{(j+1)}\|_{p[1/4,1/2]} \right) \\
&\le c\left(\|g\|_{p[1/4,1/2]} + \|g^{(m)}\|_{p[1/4,1/2]} \right),
\end{aligned}$$

where at the last two steps we have applied Hölder's inequality and (2.1), respectively.

In order to estimate above the terms of the second sum on the right side of (2.2), we shall use Hardy's inequality (with the appropriate modification for $p = \infty$)

$$\left(\int_0^{1/2} \left| x^\gamma \int_x^{1/2} F(u)\, du \right|^p dx \right)^{1/p} \le c \left(\int_0^{1/2} |x^{\gamma+1} F(x)|^p\, dx \right)^{1/p} \tag{2.4}$$

provided that $\gamma > -1/p$.

We set $\chi(x) = x$. By (2.4) we get

$$\begin{aligned}
\|\psi_k\|_{p[0,1/2]} &\le c\, \|\chi^{m+\gamma_{1,0}-j} g^{(m)}\|_{p[0,1/2]} \\
&\le c\, \|\chi^{\gamma_{2,0}} g^{(m)}\|_{p[0,1/2]}, \quad k = 0, \dots, m-j-1,
\end{aligned} \tag{2.5}$$

as for the second estimate above we have used that $\gamma_{2,0} \le \gamma_{1,0} + m - j$.

Now, (2.2), (2.3) and (2.5) imply the inequality

$$\|\chi^{\gamma_{1,0}} g^{(j)}\|_{p[0,1/2]} \le c\left(\|g\|_{p[1/4,1/2]} + \|\chi^{\gamma_{2,0}} g^{(m)}\|_{p[0,1/2]} \right). \tag{2.6}$$

By symmetry, we get

$$\|(1-\chi)^{\gamma_{1,1}} g^{(j)}\|_{p[1/2,1]} \le c\left(\|g\|_{p[1/2,3/4]} + \|(1-\chi)^{\gamma_{2,1}} g^{(m)}\|_{p[1/2,1]} \right). \tag{2.7}$$

The last two estimates yield the assertion of the proposition. □

Remark 2.2. As a corollary of (2.1) and Proposition 2.1 we get the following embedding inequalities:

$$\|w\varphi^{2j} g^{(j)}\|_p \le c\left(\|wg\|_p + \|w\varphi^{2m} g^{(m)}\|_p \right), \quad j = 0, \dots, m, \tag{2.8}$$

$$\|w\varphi^{2j} g^{(2j)}\|_p \le c\left(\|wg\|_p + \|w\varphi^{2m} g^{(2m)}\|_p \right), \quad j = 0, \dots, m, \tag{2.9}$$

and

$$\|wg^{(j)}\|_p \le c\left(\|wg\|_p + \|wg^{(m)}\|_p\right), \quad j = 0, \dots, m, \tag{2.10}$$

where $w = w(\gamma_0, \gamma_1)$ is given by (1.3) with $\gamma_0, \gamma_1 > -1/p$ if $1 \le p < \infty$ or $\gamma_0, \gamma_1 \ge 0$ if $p = \infty$.

We proceed to several embedding inequalities, which will enable us to transfer estimates in terms of the semi-norms $\|w\varphi^{2i}g^{(j)}\|_p$ to such in terms of the more complicated one $\|w(D^r g)^{(s)}\|_p$. Their proof is based on the following Taylor-type formulas.

Lemma 2.3. *Let $s \in \mathbb{N}$ and $g \in AC^{s+1}[0,1]$.*

(a) If $s \ge 2$, then

$$g^{(s)}(x) = \int_0^1 \mathcal{K}_s(x,u)\,(Dg)^{(s)}(u)\,du, \quad x \in [0,1],$$

where

$$\mathcal{K}_s(x,u) = -\frac{1}{s-1}\begin{cases} \left(\dfrac{u}{x}\right)^{s-1}, & u \le x, \\ \left(\dfrac{1-u}{1-x}\right)^{s-1}, & x \le u. \end{cases}$$

(b) If $s \ge 1$, then

$$g^{(s+1)}(x) = \int_0^1 \mathcal{L}_s(x,u)\,(Dg)^{(s)}(u)\,du, \quad x \in [0,1],$$

where

$$\mathcal{L}_s(x,u) = \begin{cases} \dfrac{u^{s-1}}{x^s}, & u \le x, \\ -\dfrac{(1-u)^{s-1}}{(1-x)^s}, & x < u. \end{cases}$$

Proof. Assertion (a) is verified by integration by parts. More precisely, we expand $(Dg)^{(s)}(u)$ to get

$$(Dg)^{(s)}(u) = -s(s-1)g^{(s)}(u) + s(1-2u)g^{(s+1)}(u) + u(1-u)g^{(s+2)}(u). \tag{2.11}$$

Next, we evaluate the integral

$$\int_0^1 \mathcal{K}_s(x,u)\,\left[s(1-2u)g^{(s+1)}(u) + u(1-u)g^{(s+2)}(u)\right]\,du.$$

We get by integration by parts

$$\begin{aligned} &\int_0^x u^{s-1}\left[s(1-2u)g^{(s+1)}(u) + u(1-u)g^{(s+2)}(u)\right]\,du \\ &\quad = x^s(1-x)g^{(s+1)}(x) - (s-1)\int_0^x u^s g^{(s+1)}(u)\,du \\ &\quad = x^s(1-x)g^{(s+1)}(x) - (s-1)x^s g^{(s)}(x) + s(s-1)\int_0^x u^{s-1}g^{(s)}(u)\,du \end{aligned}$$

and

$$\int_x^1 (1-u)^{s-1} \left[s(1-2u) g^{(s+1)}(u) + u(1-u) g^{(s+2)}(u) \right] du$$
$$= -x(1-x)^s g^{(s+1)}(x) + (s-1) \int_x^1 (1-u)^s g^{(s+1)}(u)\, du$$
$$= -x(1-x)^s g^{(s+1)}(x) - (s-1)(1-x)^s g^{(s)}(x)$$
$$+ s(s-1) \int_x^1 (1-u)^{s-1} g^{(s)}(u)\, du.$$

Consequently,

$$\int_0^1 \mathcal{K}_s(x,u) \left[s(1-2u) g^{(s+1)}(u) + u(1-u) g^{(s+2)}(u) \right] du$$
$$= g^{(s)}(x) + s(s-1) \int_0^1 \mathcal{K}_s(x,u)\, g^{(s)}(u)\, du,$$

which, in view of (2.11), completes the proof of (a).

Assertion (b) for $s \geq 2$ is directly verified by differentiating the formula in (a). If $s = 1$, we just have

$$\frac{1}{x} \int_0^x (Dg)'(u)\, du = \frac{Dg(x)}{x} = (1-x) g''(x)$$

and

$$-\frac{1}{1-x} \int_x^1 (Dg)'(u)\, du = \frac{Dg(x)}{1-x} = x g''(x).$$

Hence (b) for $s = 1$ follows. □

Proposition 2.4. *Let $1 \leq p \leq \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by (1.3) as*

$$\begin{aligned} -1/p <&\gamma_0, \gamma_1 < s - 1/p \quad \text{if} \quad 1 \leq p < \infty, \\ 0 \leq&\gamma_0, \gamma_1 < s \quad \text{if} \quad p = \infty. \end{aligned}$$

Set $j_s = 1$ if $s = 1$, and $j_s = 0$ otherwise. Then for all $g \in AC^{2r+s-1}[0,1]$ there hold

$$\|w g^{(j+s)}\|_p \leq c\, \|w (D^r g)^{(s)}\|_p, \quad j = j_s, \ldots, r, \tag{2.12}$$

and

$$\|w \varphi^{2r} g^{(2r+s)}\|_p \leq c\, \|w (D^r g)^{(s)}\|_p. \tag{2.13}$$

The constant c is independent of g.

Proof. We shall establish the assertions by induction on r. In order to verify them for $r=1$ we apply Lemma 2.3 and Hardy's inequality (with the appropriate modification for $p=\infty$)

$$\left(\int_0^{1/2}\left|x^\gamma\int_0^x F(u)\,du\right|^p\,dx\right)^{1/p}\le c\left(\int_0^{1/2}|x^{\gamma+1}F(x)|^p\,dx\right)^{1/p} \tag{2.14}$$

provided that $\gamma<-1/p$.

We shall estimate the integrals in the formulas in Lemma 2.3. We set $\chi(x)=x$ and

$$\Psi_1(x)=x^{-s+1}\int_0^x u^{s-1}(Dg)^{(s)}(u)\,du,$$
$$\Psi_2(x)=x^{-s}\int_0^x u^{s-1}(Dg)^{(s)}(u)\,du.$$

Clearly,

$$\|w\Psi_1\|_p\le\|w\Psi_2\|_p. \tag{2.15}$$

We shall estimate the L_p-norm of $w\Psi_2$ separately on the intervals $[0,1/2]$ and $[1/2,1]$.

For the estimate on $[0,1/2]$ we use that $\gamma_0-s<-1/p$ and apply (2.14), which yields

$$\|\chi^{\gamma_0}\Psi_2\|_{p[0,1/2]}\le c\|\chi^{\gamma_0}(Dg)^{(s)}\|_{p[0,1/2]}\le c\|w(Dg)^{(s)}\|_p. \tag{2.16}$$

For the estimate on $[1/2,1]$ we observe that for $0\le u\le x\le 1$ we have

$$(1-u)^{\gamma_1}\ge\begin{cases}(1-x)^{\gamma_1} & \text{if } \gamma_1\ge 0,\\ 1 & \text{if } \gamma_1<0.\end{cases}$$

Then, using Hölder's inequality, we get

$$\begin{aligned}(1-x)^{\gamma_1}|\Psi_2(x)|&\le c\,\max\{1,(1-x)^{\gamma_1}\}\|\chi^{s-\gamma_0-1}\|_q\,\|w(Dg)^{(s)}\|_p\\ &\le c\,\max\{1,(1-x)^{\gamma_1}\}\,\|w(Dg)^{(s)}\|_p,\end{aligned}$$

where q is the conjugate exponent to p. At the last estimate above we have again taken into account that $s-\gamma_0-1>-1/q=1/p-1$.

Since $\gamma_1>-1/p$ if $1\le p<\infty$, and $\gamma_1\ge 0$ if $p=\infty$, then the L_p-norm on $[1/2,1]$ of $\max\{1,(1-x)^{\gamma_1}\}$ is finite and we deduce that

$$\|(1-\chi)^{\gamma_1}\Psi_2\|_{p[1/2,1]}\le c\,\|w(Dg)^{(s)}\|_p. \tag{2.17}$$

Inequalities (2.15)-(2.17) imply

$$\|w\Psi_1\|_p\le\|w\Psi_2\|_p\le c\,\|w(Dg)^{(s)}\|_p.$$

By symmetry, we get the analogue of the last estimates for the terms

$$(1-x)^{-s+i}\int_x^1 (1-u)^{s-1}(Dg)^{(s)}(u)\,du, \quad i=0,1.$$

Thus, we establish that

$$\left\| w\int_0^1 \mathcal{K}_s(\circ,u)\,(Dg)^{(s)}(u)\,du\right\|_p \le c\,\|w(Dg)^{(s)}\|_p, \quad s\ge 2,$$

and

$$\left\| w\int_0^1 \mathcal{L}_s(\circ,u)\,(Dg)^{(s)}(u)\,du\right\|_p \le c\,\|w(Dg)^{(s)}\|_p, \quad s\ge 1.$$

Now, we complete the proof of inequalities (2.12) for $r=1$ by Lemma 2.3. Then (2.13) follows from (2.11). The proposition is established for $r=1$.

We proceed by induction on r, so let us assume that (2.12)-(2.13) are valid for some r. Then applying (2.12) with Dg in place of g, we arrive at

$$\|w(Dg)^{(j+s)}\|_p \le c\,\|w(D^{r+1}g)^{(s)}\|_p, \quad j=j_s,\dots,r. \tag{2.18}$$

On the other hand, by what we have already shown in the case $r=1$, we have

$$\|wg^{(j'+j+s)}\|_p \le c\,\|w(Dg)^{(j+s)}\|_p, \quad j'=0,1. \tag{2.19}$$

Let us note that $j_{j+s}=0$ because $j+s\ge 2$ for $j\ge j_s$.

Now, (2.18)-(2.19) yield

$$\|wg^{(j+s)}\|_p \le c\,\|w(D^{r+1}g)^{(s)}\|_p, \quad j=j_s,\dots,r+1.$$

Thus (2.12) is verified for $r+1$ in place of r.

To complete the proof of (2.13), we need to show that

$$\|w\varphi^{2r+2}g^{(2r+s+2)}\|_p \le c\,\|w(D^{r+1}g)^{(s)}\|_p.$$

In view of (2.11) with $2r+s$ in place of s, that will follow from the inequalities

$$\|w\varphi^{2r}(Dg)^{(2r+s)}\|_p \le c\,\|w(D^{r+1}g)^{(s)}\|_p \tag{2.20}$$

and

$$\|w\varphi^{2r}g^{(j+2r+s)}\|_p \le c\,\|w(D^{r+1}g)^{(s)}\|_p, \quad j=0,1. \tag{2.21}$$

Inequality (2.20) follows from (2.13) with Dg in place of g. To establish (2.21) we first apply (2.12) with $r=1$, $w\varphi^{2r}$ in place of w, and $2r+s$ in place of s and thus get

$$\|w\varphi^{2r}g^{(j+2r+s)}\|_p \le c\,\|w\varphi^{2r}(Dg)^{(2r+s)}\|_p, \quad j=0,1. \tag{2.22}$$

Inequalities (2.20) and (2.22) imply (2.21). $\square$

3 Auxiliary identities concerning the Bernstein operator

In this section we shall present some of the basic properties of the Bernstein operator, which we shall use.

Direct computation yields the following formulas for the derivatives of the polynomials $p_{n,k}$, $k = 0, \dots, n$ (see e.g. [4, Chapter 10, (2.1)]):

$$p'_{n,k}(x) = n[p_{n-1,k-1}(x) - p_{n-1,k}(x)] \tag{3.1}$$

and

$$p'_{n,k}(x) = \varphi^{-2}(x)(k - nx)p_{n,k}(x), \tag{3.2}$$

where we have set for convenience $p_{n,k} = 0$ if $k < 0$ or $k > n$.

For a sequence $\{a_k\}_{k\in\mathbb{Z}}$ we set $\Delta a_k = a_k - a_{k-1}$. Now, if we put $p_k(n, x) = p_{n,k}(x)$, then iterating (3.1) we get

$$p^{(s)}_{n,k}(x) = (-1)^s \frac{n!}{(n-s)!}\,\Delta^s p_k(n - s, x). \tag{3.3}$$

Similarly, using (3.2), it is verified by induction that (cf. [8, (9.4.8)])

$$p^{(s)}_{n,k}(x) = \varphi^{-2s}(x)\,p_{n,k}(x)\sum_{j=0}^{s}(k - nx)^j \sum_{0\le i\le (s-j)/2} q_{s,j,i}(x)\left(n\varphi^2(x)\right)^i,$$

where $q_{s,j,i}(x)$ are polynomials, whose coefficients are independent of n. Rearranging the summands, we get

$$p^{(s)}_{n,k}(x) = \varphi^{-2s}(x)\,p_{n,k}(x)\sum_{0\le i\le s/2}\left(n\varphi^2(x)\right)^i \sum_{j=0}^{s-2i} q_{s,j,i}(x)(k - nx)^j. \tag{3.4}$$

We shall often use the quantities

$$T_{n,\ell}(x) = \sum_{k=0}^{n}(k - nx)^{\ell} p_{n,k}(x).$$

It is known (see [4, Chapter 10, Theorem 1.1]) that

$$T_{n,\ell}(x) = \sum_{1\le\rho\le\ell/2} t_{\ell,\rho}(x)\left(n\varphi^2(x)\right)^{\rho}, \quad \ell\in\mathbb{N}, \tag{3.5}$$

where $t_{\ell,\rho}(x)$ are polynomials, whose coefficients are independent of n.

In particular (see e.g. [4, p. 304] and [25, p. 14]),

$$\begin{aligned} &T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0, \quad T_{n,2}(x) = n\varphi^2(x),\\ &T_{n,3}(x) = (1 - 2x)n\varphi^2(x), \quad T_{n,4}(x) = 3n^2\varphi^4(x) + n\varphi^2(x)(1 - 6\varphi^2(x)). \end{aligned} \tag{3.6}$$

Identity (3.5) implies (see also [8, Lemma 9.4.4]) that

$$0 \le T_{n,2m}(x) \le c \begin{cases} n\varphi^2(x), & n\varphi^2(x) \le 1, \\ \left(n\varphi^2(x)\right)^m, & n\varphi^2(x) \ge 1. \end{cases} \tag{3.7}$$

Let $\alpha > 0$. We fix $m \in \mathbb{N}$ such that $2m/\alpha > 1$. Then Hölder's inequality, (3.7) and the identity $\sum_{k=0}^n p_{n,k}(x) \equiv 1$ imply

$$0 \le \sum_{k=0}^{n} |k - nx|^\alpha p_{n,k}(x) \le T_{n,2m}^{\alpha/(2m)}(x) \le c \begin{cases} 1, & n\varphi^2(x) \le 1, \\ \left(n\varphi^2(x)\right)^{\alpha/2}, & n\varphi^2(x) \ge 1. \end{cases} \tag{3.8}$$

We shall need the analogue of $T_{n,\ell}$ associated with the differentiated Bernstein polynomial. We set

$$T_{s,n,\ell}(x) = \sum_{k=0}^{n} (k - nx)^\ell p_{n,k}^{(s)}(x).$$

The following formula, similar to (3.5), holds.

Lemma 3.1. *Let $\ell, n, s \in \mathbb{N}$. Then*

$$T_{s,n,\ell}(x) = \sum_{\rho=1}^{s} \tilde{t}_{s,\ell,\rho}(x)\, n^\rho + n^s \sum_{1 \le \rho \le (\ell - s)/2} t_{s,\ell,\rho}(x) \left(n\varphi^2(x)\right)^\rho,$$

where $t_{s,\ell,\rho}(x)$ and $\tilde{t}_{s,\ell,\rho}(x)$ are polynomials, whose coefficients are independent of n.

Above we follow the usual convention that an empty sum is considered to be equal to 0.

Proof of Lemma 3.1. Let $\ell \ge 2$. We apply (3.4). Then we sum on k, use (3.5) and finally reorder the summands to get

$$\begin{aligned} T_{s,n,\ell}(x) &= n^s \sum_{0 \le i \le s/2} \left(n\varphi^2(x)\right)^{i-s} \sum_{j=0}^{s-2i} q_{s,j,i}(x)\, T_{n,j+\ell}(x) \\ &= n^s \sum_{0 \le i \le s/2} \ \sum_{1 \le \rho \le (s+\ell-2i)/2} t_{s,i,\ell,\rho}(x) \left(n\varphi^2(x)\right)^{i+\rho-s}, \end{aligned}$$

where we have set

$$t_{s,i,\ell,\rho}(x) = \sum_{j=\max\{0,2\rho-\ell\}}^{s-2i} q_{s,j,i}(x)\, t_{j+\ell,\rho}(x).$$

Let us note that $t_{s,i,\ell,\rho}(x)$ are polynomials, whose coefficients are independent of n.

Consequently,

$$T_{s,n,\ell}(x) = n^s \sum_{1-s\le\rho\le(\ell-s)/2} t_{s,\ell,\rho}(x)\left(n\varphi^2(x)\right)^\rho$$

with some polynomials $t_{s,\ell,\rho}(x)$, whose coefficients are independent of n. To get the assertion of the lemma for $\ell \ge 2$, we need only take into account that the left-hand side of the last identity is a polynomial in x; hence so is $\varphi^{2\rho}(x)t_{s,\ell,\rho}(x)$ for each negative ρ. Here we also use that $t_{s,\ell,\rho}(x)$ are independent of n.

Minor changes in the above argument establish the lemma for $\ell = 1$ too. $\square$

We proceed to several identities concerning the derivatives of the error of the Bernstein operators. We shall use them to establish Jackson- and Voronovskaya-type estimates. We denote the set of the algebraic polynomials of degree at most j by π_j.

Lemma 3.2. *Let $s \in \mathbb{N}$, $f \in C[0,1]$, $f \in AC_{loc}^{s+1}(0,1)$ and $\varphi^{2s+2}f^{(s+2)} \in L[0,1]$. Then*

$$(3.9)\quad (B_nf(x) - f(x))^{(s)} = \frac{1}{n}\,A_{s,n}f^{(s)}(x) + \frac{1}{n}\,B_{s,n}(x)f^{(s+1)}(x) + \frac{1}{(s+1)!}\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\int_x^{k/n}\left(\frac{k}{n} - u\right)^{s+1} f^{(s+2)}(u)\,du, \quad x \in (0,1),$$

where

$$A_{s,n} = \sum_{\nu=0}^{s-2} a_{s,\nu}\,n^{-\nu}, \quad B_{s,n}(x) = \sum_{\nu=0}^{s-1} b_{s,\nu}(x)\,n^{-\nu},$$

and $a_{s,\nu}$ and $b_{s,\nu}(x)$ are respectively constants and linear functions, which are independent of n.

Above we again use the usual convention that an empty sum is zero. Note that the order of the derivatives on the right of (3.9) is at least $\max\{2, s\}$.

Proof of Lemma 3.2. Let us make two observations that will justify our usage of Taylor's expansions, integration by parts and induction on s below.

First, if $f \in AC_{loc}^{\sigma+1}(0,1)$ and $\varphi^{2\sigma+2}f^{(\sigma+2)} \in L[0,1]$ for some $\sigma \in \mathbb{N}$, then

$$(3.10)\qquad \varphi^{2\sigma}f^{(\sigma+1)} \in L[0,1].$$

That follows from Proposition 2.1 with $p = 1$, $g = f$, $j = \sigma + 1$, $m = \sigma + 2$, $w_1 = \varphi^{2\sigma}$ and $w_2 = \varphi^{2\sigma+2}$.

Further, using the representation

$$u^{\sigma+1}f^{(\sigma+1)}(u) = \frac{1}{2^{\sigma+1}}\,f^{(\sigma+1)}\left(\frac{1}{2}\right) - (\sigma+1)\int_u^{1/2} v^\sigma f^{(\sigma+1)}(v)\,dv - \int_u^{1/2} v^{\sigma+1}f^{(\sigma+2)}(v)\,dv, \quad u \in (0,1),$$

we deduce that $\lim_{u\to 0+0} u^{\sigma+1} f^{(\sigma+1)}(u)$ exists as a finite limit. Moreover, if we assume that it is not 0, then we shall get that $u^{\sigma}|f^{(\sigma+1)}(u)| \ge C/u$ for $u \in (0,\delta)$ with some positive constants C and δ, which contradicts $\varphi^{2\sigma} f^{(\sigma+1)} \in L[0,1]$. Consequently,

$$\lim_{u\to 0+0} u^{\sigma+1} f^{(\sigma+1)}(u) = 0. \tag{3.11}$$

By symmetry, we get

$$\lim_{u\to 1-0} (1-u)^{\sigma+1} f^{(\sigma+1)}(u) = 0. \tag{3.12}$$

Let us proceed to the proof of the lemma. We shall establish it by means of induction on s. To check it for $s = 1$ we note that by (3.10) with $\sigma = 1$ we have $\varphi^2 f'' \in L[0,1]$ and we can expand $f(t)$ at $x \in (0,1)$ by Taylor's formula to get

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u) f''(u)\,du, \quad t \in [0,1].$$

Then we apply the operator B_n to both sides of the above identity, take into account that it preserves the linear functions and arrive at

$$B_n f(x) - f(x) = \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\,du. \tag{3.13}$$

We differentiate (3.13), integrate by parts as we take into account (3.11)-(3.12) with $\sigma = 1$ and use (3.2) and (3.6) to derive

$$\begin{aligned}
(B_n f(x) - f(x))' &= -\frac{1}{n} T_{n,1}(x) f''(x) + \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right) f''(u)\,du \\
&= \frac{\varphi^{-2}(x)}{2n^2} T_{n,3}(x) f''(x) \\
&\quad + \frac{1}{2} \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^2 f'''(u)\,du \\
&= \frac{1-2x}{2n} f''(x) + \frac{1}{2} \sum_{k=0}^{n} p'_{n,k}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^2 f'''(u)\,du.
\end{aligned}$$

Thus the lemma is verified for $s = 1$.

Next, let us assume that the assertion of the lemma is true for some s and let $f \in C[0,1]$, $f \in AC_{loc}^{s+2}(0,1)$ and $\varphi^{2s+4} f^{(s+3)} \in L[0,1]$. Then by (3.10) with $\sigma = s+1$ we have $\varphi^{2s+2} f^{(s+2)} \in L[0,1]$. Therefore, by the induction hypothesis, formula (3.9) is valid for that s. We differentiate it and integrate by parts using

(3.11)-(3.12) with $\sigma = s+1$. Thus we arrive at

$$
\begin{aligned}
(B_n f(x) - f(x))^{(s+1)} &= \frac{1}{n}\left(A_{s,n} + B'_{s,n}(x)\right) f^{(s+1)}(x) \\
&+ \frac{1}{n}\, B_{s,n}(x) f^{(s+2)}(x) \\
&- \frac{1}{(s+1)!}\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\left(\frac{k}{n} - x\right)^{s+1} f^{(s+2)}(x) \\
&+ \frac{1}{(s+2)!}\sum_{k=0}^{n} p_{n,k}^{(s+1)}(x)\left(\frac{k}{n} - x\right)^{s+2} f^{(s+2)}(x) \\
&+ \frac{1}{(s+2)!}\sum_{k=0}^{n} p_{n,k}^{(s+1)}(x)\int_x^{k/n}\left(\frac{k}{n} - u\right)^{s+2} f^{(s+3)}(u)\, du.
\end{aligned}
\tag{3.14}
$$

According to the induction hypothesis the expression $A_{s+1,n} = A_{s,n} + B'_{s,n}(x)$ is of the form $\sum_{\nu=0}^{s-1} a_{s+1,\nu}\, n^{-\nu}$ with some constants $a_{s+1,\nu}$, which are independent of n.

Let us denote by $B_{s+1,n}(x)$ the factor of $f^{(s+2)}(x)/n$ in the expansion (3.14). From the induction hypothesis and Lemma 3.1 with $\ell = s+1$ it follows that it is of the form

$$
B_{s+1,n}(x) = \sum_{\nu=0}^{s} b_{s+1,\nu}(x)\, n^{-\nu}, \tag{3.15}
$$

where $b_{s+1,\nu}(x)$ are polynomials, whose coefficients are independent of n. To show that they are of degree 1, we set in (3.14) $f(x) = x^{s+2}$. We get

$$
(B_n f(x) - f(x))^{(s+1)} = \frac{A_{s+1,n}\,(s+2)!}{n}\, x + \frac{(s+2)!}{n}\, B_{s+1,n}(x).
$$

Since $B_n f \in \pi_{s+2}$, we deduce that $B_{s+1,n} \in \pi_1$; hence $b_{s+1,\nu} \in \pi_1$ because their coefficients are independent of n.

This completes the proof of the lemma. □

Lemma 3.3. *Let $s \in \mathbb{N}$, $f \in C[0,1]$, $f \in AC_{loc}^{s+2}(0,1)$ and $\varphi^{2s+4} f^{(s+3)} \in L[0,1]$. Then*

$$
\begin{aligned}
\left(B_n f(x) - f(x) - \frac{1}{2n}\, Df(x)\right)^{(s)} &= \frac{1}{n^2}\, \widetilde{A}_{s,n} f^{(s)}(x) + \frac{1}{n^2}\, \widetilde{B}_{s,n}(x) f^{(s+1)}(x) \\
+ \frac{1}{n^2}\, \widetilde{C}_{s,n}(x) f^{(s+2)}(x) &+ \frac{1}{(s+2)!}\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\int_x^{k/n}\left(\frac{k}{n} - u\right)^{s+2} f^{(s+3)}(u)\, du, \\
&\qquad x \in (0,1),
\end{aligned}
$$

where

$$
\widetilde{A}_{s,n} = \sum_{\nu=0}^{s-3} \tilde{a}_{s,\nu}\, n^{-\nu}, \quad \widetilde{B}_{s,n}(x) = \sum_{\nu=0}^{s-2} \tilde{b}_{s,\nu}(x)\, n^{-\nu}, \quad \widetilde{C}_{s,n}(x) = \sum_{\nu=0}^{s-1} \tilde{c}_{s,\nu}(x)\, n^{-\nu}
$$

and $\tilde{a}_{s,\nu}$*,* $\tilde{b}_{s,\nu}(x)$ *and* $\tilde{c}_{s,\nu}(x)$ *are polynomials of degree respectively* 0*,* 1 *and* 2*, whose coefficients are independent of* n*.*

Let us note that the order of the derivatives on the right of the formula in the lemma is at least $\max\{3, s\}$.

Proof of Lemma 3.3. We verify the lemma just similarly to the previous one.
To check it for $s = 1$ we apply (3.10) with $\sigma = 2$ and get $\varphi^4 f''' \in L[0,1]$. Then

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}\,(t-x)^2 f''(x) + \frac{1}{2}\int_x^t (t-u)^2 f'''(u)\,du, \quad t \in [0,1].$$

We apply the operator B_n to both sides of the above identity, take into account that it preserves the linear functions and also that $T_{n,2}(x) = n\varphi^2(x)$ (see (3.6)) and arrive at

$$B_n f(x) - f(x) - \frac{1}{2n}\,Df(x) = \frac{1}{2}\sum_{k=0}^{n} p_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n} - u\right)^2 f^{(3)}(u)\,du. \tag{3.16}$$

We set

$$V_n f(x) = B_n f(x) - f(x) - \frac{1}{2n}\,Df(x). \tag{3.17}$$

We differentiate (3.16), integrate by parts, taking into account (3.11)-(3.12) with $\sigma = 2$, and apply (3.2). Thus we arrive at

$$\begin{aligned}(V_n f)'(x) &= -\frac{1}{2n^2}\,T_{n,2}(x) f^{(3)}(x) + \frac{1}{2}\sum_{k=0}^{n} p'_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n} - u\right)^2 f^{(3)}(u)\,du\\ &= \frac{1}{6n^2}\left(\frac{\varphi^{-2}(x)}{n}\,T_{n,4}(x) - 3T_{n,2}(x)\right) f^{(3)}(x)\\ &\quad + \frac{1}{3!}\sum_{k=0}^{n} p'_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n} - u\right)^3 f^{(4)}(u)\,du.\end{aligned}$$

To complete the proof for $s = 1$ we apply (3.6), which yields

$$\frac{\varphi^{-2}(x)}{n}\,T_{n,4}(x) - 3T_{n,2}(x) = 1 - 6\varphi^2(x).$$

Next, let us assume that the lemma is valid for some s. Let $f \in C[0,1]$, $f \in AC_{loc}^{s+3}(0,1)$ and $\varphi^{2s+6} f^{(s+4)} \in L[0,1]$. Then by (3.10) with $\sigma = s+2$ we have $\varphi^{2s+4} f^{(s+3)} \in L[0,1]$; hence the formula of the lemma is true for that s. We differentiate it and integrate by parts as we use (3.11)-(3.12) with $\sigma = s+2$.

Thus we arrive at

$$
\begin{aligned}
(V_n f)^{(s+1)} &= \frac{1}{n^2}\left(\widetilde{A}_{s,n} + \widetilde{B}'_{s,n}(x)\right) f^{(s+1)}(x) \\
&+ \frac{1}{n^2}\left(\widetilde{B}_{s,n}(x) + \widetilde{C}'_{s,n}(x)\right) f^{(s+2)}(x) \\
&+ \frac{1}{n^2}\left(\widetilde{C}_{s,n}(x) + \widetilde{D}_{s,n}(x)\right) f^{(s+3)}(x) \\
&+ \frac{1}{(s+3)!}\sum_{k=0}^{n} p_{n,k}^{(s+1)}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^{s+3} f^{(s+4)}(u)\,du,
\end{aligned}
\tag{3.18}
$$

where we have set

$$
\widetilde{D}_{s,n}(x) = \frac{n^2}{(s+3)!}\left(\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\left(\frac{k}{n}-x\right)^{s+3}\right)'.
$$

The induction hypothesis implies that the factors of $f^{(s+1)}(x)$ and $f^{(s+2)}(x)$ are of the stated form. To establish that for the factor of $f^{(s+3)}(x)$, we use Lemma 3.1 with $\ell = s+3$ to deduce that

$$
\widetilde{D}_{s,n}(x) = \sum_{\nu=0}^{s} \tilde{d}_{s,\nu}(x)\, n^{-\nu}
$$

with some polynomials $\tilde{d}_{s,\nu}$, whose coefficients do not depend on n. Consequently, if we set

$$
\widetilde{C}_{s+1,n}(x) = \widetilde{C}_{s,n}(x) + \widetilde{D}_{s,n}(x),
$$

then

$$
\widetilde{C}_{s+1,n}(x) = \sum_{\nu=0}^{s} \tilde{c}_{s+1,\nu}(x)\, n^{-\nu}
$$

with some polynomials $\tilde{c}_{s+1,\nu}$, whose coefficients do not depend on n. To prove that they are of degree 2, we set $f(x) = x^{s+3}$ in (3.18) and argue as in the proof of Lemma 3.2. $\square$

4 Basic estimates for the simultaneous approximation by B_n and $\mathcal{B}_{r,n}$

In this section we shall establish the basic inequalities which imply the characterization of the error of the simultaneous approximation by means of $\mathcal{B}_{r,n}$. We use techniques, which have already become standard for this set of problems (see [8, Chapters 9 and 10]). To establish the converse estimate we apply the general method given in [7, Theorem 3.2]. At the end we shall be able to prove Theorem 1.1.

We begin with the following basic estimates concerning the boundedness of the weighted L_p-norms of $(B_n f)^{(s)}$ and $(\mathcal{B}_{r,n} f)^{(s)}$.

Proposition 4.1. *Let $1 \le p \le \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *as*

$$\begin{aligned} -1 < \gamma_0, \gamma_1 \le s-1 \quad &\text{if} \quad p = 1, \\ -1/p < \gamma_0, \gamma_1 < s - 1/p \quad &\text{if} \quad 1 < p < \infty, \\ 0 \le \gamma_0, \gamma_1 < s \quad &\text{if} \quad p = \infty. \end{aligned}$$

Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there hold:

(a) $\|w(B_n f)^{(s)}\|_p \le c\,\|wf^{(s)}\|_p$;

(b) $\|w(\mathcal{B}_{r,n} f)^{(s)}\|_p \le c\,\|wf^{(s)}\|_p$.

Proof. Let us establish assertion (a). The inequality is trivial for $n < s$. For $n \ge s$ it is known (see [28], or [4, Chapter 10, (2.3)], or [8, p. 125]) that

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}(x), \tag{4.1}$$

where $\overrightarrow{\Delta}_h f(x) = f(x+h) - f(x)$, $x \in [0, 1-h]$, and $\overrightarrow{\Delta}_h^s = \overrightarrow{\Delta}_h(\overrightarrow{\Delta}_h^{s-1})$; hence

$$\overrightarrow{\Delta}_h^s f(x) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} f(x + (s-i)h), \quad x \in [0, 1-sh].$$

Formula (4.1) can be established by means of (3.3) and Abel's transform.

It is also known that (see e.g. [4, p. 45])

$$\overrightarrow{\Delta}_h^s f(x) = h^s \int_0^s M_s(u) f^{(s)}(x+hu)\,du, \quad x \in [0, 1-sh],$$

where M_s is the s-fold convolution of the characteristic function of $[0,1]$ with itself. Hence, by Hölder's inequality, we arrive at

$$\left| \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) \right| \le \frac{w_{p,n,k}}{n^s}\, \|wf^{(s)}\|_{p[k/n,(k+s)/n]}, \quad k = 0, \dots, n-s, \tag{4.2}$$

where

$$w_{p,n,k} = n \left\| \frac{M_s(n \circ - k)}{w} \right\|_{q[k/n,(k+s)/n]} \tag{4.3}$$

and q is the conjugate exponent to p.

Relations (4.1)-(4.2) yield

$$|w(x)(B_n f)^{(s)}(x)| \le c\, w(x) \sum_{k=0}^{n-s} w_{p,n,k}\, p_{n-s,k}(x)\, \|wf^{(s)}\|_{p[k/n,(k+s)/n]}. \tag{4.4}$$

We shall show that the L_p-norm of the right-hand side of (4.4) is bounded above by $c\,\|wf^{(s)}\|_p$. In view of the Riesz-Thorin interpolation theorem, we need only do that for $p=1$ and $p=\infty$. Also, due to symmetry it is sufficient to consider only the summands for $k=0,\dots,[(n-s)/2]$ on the right-hand side of (4.4). Indeed, for $\bar{w}(x)=w(1-x)$, $\bar{f}(x)=f(1-x)$, and $\bar{w}_{p,n,k}$, defined by (4.3) with $\bar{w}$ in place of w, we have

$$\begin{aligned} &w_{p,n,n-s-k}=\bar{w}_{p,n,k}, \quad \|wf^{(s)}\|_p=\|\bar{w}\bar{f}^{(s)}\|_p,\\ &\|wf^{(s)}\|_{p[(n-s-k)/n,(n-k)/n]}=\|\bar{w}\bar{f}^{(s)}\|_{p[k/n,(k+s)/n]}, \end{aligned} \tag{4.5}$$

as for the first relation above we have taken into account that $M_s(s-u)=M_s(u)$. Consequently, with $y=1-x$ we have

$$\begin{aligned} &\sum_{(n-s)/2\le k\le n-s} w_{p,n,k}\,p_{n-s,k}(x)\,\|wf^{(s)}\|_{p[k/n,(k+s)/n]}\\ &\qquad\qquad=\sum_{0\le k\le (n-s)/2} \bar{w}_{p,n,k}\,p_{n-s,k}(y)\,\|\bar{w}\bar{f}^{(s)}\|_{p[k/n,(k+s)/n]}. \end{aligned} \tag{4.6}$$

Thus it is sufficient to consider only the summands for $k=0,\dots,[(n-s)/2]$ on the right-hand side of (4.4).

It is known that

$$0\le M_s(u)\le c[u(s-u)]^{s-1},\quad 0\le u\le s.$$

Hence (a) follows for $n=s$. Let $n>s$. We have

$$\frac{M_s(nu)}{w(u)}\le c\,n^{\gamma_0}\,(nu)^{s-\gamma_0-1},\quad u\in(0,s/n],$$

and

$$\frac{M_s(nu-k)}{w(u)}\le c\,n^{\gamma_0}k^{-\gamma_0},\quad u\in[k/n,(k+s)/n],\quad 1\le k\le (n-s)/2;$$

hence, under the assumptions on γ_0, we get

$$w_{p,n,k}\le c\,n^{1/p}\left(\frac{n}{k+1}\right)^{\gamma_0},\quad 0\le k\le (n-s)/2. \tag{4.7}$$

Let $p=\infty$. Inequality (4.7) and Hölder's inequality imply

$$\begin{aligned} \sum_{k=0}^{[(n-s)/2]} w_{\infty,n,k}\,p_{n-s,k}(x)&\le c\sum_{k=0}^{n-s}\left(\frac{n}{k+1}\right)^{\gamma_0}p_{n-s,k}(x)\\ &\le c\left(\sum_{k=0}^{n-s}\left(\frac{n}{k+1}\right)^{s}p_{n-s,k}(x)\right)^{\gamma_0/s}. \end{aligned} \tag{4.8}$$

There holds (see [8, (10.2.4)])

$$\sum_{k=0}^{n}\left(\frac{n}{k+1}\right)^{s} p_{n,k}(x) \le c\,x^{-s}, \quad x\in(0,1]. \tag{4.9}$$

Consequently,

$$w(x)\sum_{k=0}^{[(n-s)/2]} w_{\infty,n,k}\, p_{n-s,k}(x) \le c, \quad x\in[0,1]. \tag{4.10}$$

Now, (4.4), (4.6) and (4.10) imply (a) for $p=\infty$ and $n>s$.

For $p=1$ we use instead the estimate (see [8, (10.2.6)])

$$\int_0^1 x^{\gamma_0} p_{n,k}(x)\,dx \le \frac{c}{n}\left(\frac{k+1}{n}\right)^{\gamma_0}, \quad k=0,\dots,n,$$

which implies

$$\int_0^1 w(x)p_{n-s,k}(x)\,dx \le \frac{c}{n}\left(\frac{k+1}{n}\right)^{\gamma_0}, \quad 0\le k\le (n-s)/2. \tag{4.11}$$

Now, (a) for $p=1$ and $n>s$ follows from (4.4), (4.6) and (4.7).

Assertion (b) follows from (a) by iteration. $\square$

Now, we shall establish Jackson-type estimates for the operators $(B_n f)^{(s)}$ and $(\mathcal{B}_{r,n}f)^{(s)}$. We shall use the following technical result.

Lemma 4.2. *Let $\alpha,\beta,\delta\in\mathbb{R}$ be such that $0\le\alpha,\beta\le\delta$. Set $\gamma=\min\{\alpha,\beta\}$. Then for $x,t\in(0,1)$ and u between x and t there holds*

$$\frac{|t-u|^{\delta}}{u^{\alpha}(1-u)^{\beta}} \le 2^{|\gamma-1|}\,\frac{|t-x|^{\delta}}{x^{\alpha}(1-x)^{\beta}}.$$

Proof. For u between t and x such that $x,t\in(0,1)$ we have the inequalities:

$$\frac{|t-u|}{u}\le\frac{|t-x|}{x}, \quad \frac{|t-u|}{1-u}\le\frac{|t-x|}{1-x}. \tag{4.12}$$

The first one is checked directly and the second one follows from it by symmetry.

Next, we shall show that under the same conditions on x, t and u we have

$$\frac{|t-u|^{\mu}}{[u(1-u)]^{\mu}} \le 2^{|\mu-1|}\,\frac{|t-x|^{\mu}}{[x(1-x)]^{\mu}}, \quad \mu\ge 0. \tag{4.13}$$

To establish that we raise each of the inequalities in (4.12) to the power of μ and sum them up. Thus we arrive at

$$|t-u|^{\mu}\,\frac{u^{\mu}+(1-u)^{\mu}}{[u(1-u)]^{\mu}} \le |t-x|^{\mu}\,\frac{x^{\mu}+(1-x)^{\mu}}{[x(1-x)]^{\mu}}.$$

To get (4.13), it remains to observe that $\min\{1, 2^{1-\mu}\} \le x^\mu + (1-x)^\mu \le \max\{1, 2^{1-\mu}\}$ for $x \in [0,1]$.

Further, we set $\hat{\gamma} = \max\{\alpha, \beta\}$ and

$$\phi(x) = \begin{cases} x, & \alpha \ge \beta, \\ 1-x, & \beta > \alpha. \end{cases}$$

Now, to prove the lemma we need only multiply the inequalities:

$$\frac{|t-u|^\gamma}{[u(1-u)]^\gamma} \le 2^{|\gamma-1|} \frac{|t-x|^\gamma}{[x(1-x)]^\gamma}, \tag{4.14}$$

$$\left(\frac{|t-u|}{\phi(u)}\right)^{\hat{\gamma}-\gamma} \le \left(\frac{|t-x|}{\phi(x)}\right)^{\hat{\gamma}-\gamma} \tag{4.15}$$

and

$$|t-u|^{\delta-\hat{\gamma}} \le |t-x|^{\delta-\hat{\gamma}}. \tag{4.16}$$

Inequality (4.14) is (4.13) with $\mu = \gamma \ge 0$, (4.15) follows from (4.12) and $\gamma \le \hat{\gamma}$, and (4.16) from $\hat{\gamma} \le \delta$.

□

Proposition 4.3. *Let $1 < p \le \infty$, $s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by (1.3). Set $s' = \max\{2, s\}$. If $-1/p < \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+1}(0,1)$ and $wf^{(s')}, w\varphi^2 f^{(s+2)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds*

$$\|w(B_n f - f)^{(s)}\|_p \le \frac{c}{n}\left(\|wf^{(s')}\|_p + \|w\varphi^2 f^{(s+2)}\|_p\right). \tag{4.17}$$

For $p = \infty$ we may allow $\gamma_0\gamma_1 = 0$, while still assuming $0 \le \gamma_0, \gamma_1 < s$, and have

$$\|w(B_n f - f)^{(s)}\|_\infty \le \frac{c}{n}\left(\|wf^{(s')}\|_\infty + \|wf^{(s+1)}\|_\infty + \|w\varphi^2 f^{(s+2)}\|_\infty\right), \tag{4.18}$$

provided that $wf^{(s+1)} \in L_\infty[0,1]$ too.

Remark 4.4. Let us note that (4.17) is, in general, not true in the case $s \ge 2$, $\gamma_0\gamma_1 = 0$ and $p = \infty$. To avoid certain technical details we shall show that for $\gamma_0 = \gamma_1 = 0$. Let $f^{(s)}(x) = x \log x$. Then $f^{(s)}, \varphi^2 f^{(s+2)} \in L_\infty[0,1]$ but $f^{(s+1)} \notin L_\infty[0,1]$. If (4.17) was true for $s \ge 2$, $\gamma_0 = \gamma_1 = 0$ and $p = \infty$, then the last assertion of Theorem 1.1 and Theorem 1.3, both with $r = 1$, would imply $K_1(f^{(s)}, n^{-1})_\infty = O(n^{-1})$; hence $f^{(s+1)} \in L_\infty[0,1]$ (see [4, Chapter 2, Theorem 9.3 and Chapter 6, Theorem 2.4]), which is a contradiction. Let us explicitly note that the fact that (4.17) is not generally valid in the case $p = \infty$ and $\gamma_0\gamma_1 = 0$ is not used in the proofs of Theorems 1.1 and 1.3 (see also Remark 1.7).

Proof of Proposition 4.3. The proof is based on Lemma 3.2. We use Hölder's inequality and $w\varphi^2 f^{(s+2)} \in L_p[0,1]$ to derive $\varphi^{2s+2} f^{(s+2)} \in L[0,1]$; and hence the lemma is applicable.

We shall prove that if $1 < p < \infty$ and $-1/p < \gamma_0, \gamma_1 \leq s$, or $p = \infty$ and $0 \leq \gamma_0, \gamma_1 \leq s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+1}(0,1)$ and $wf^{(s')}, wf^{(s+1)}, w\varphi^2 f^{(s+2)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\|w(B_n f - f)^{(s)}\|_p \leq \frac{c}{n}\left(\|wf^{(s')}\|_p + \|wf^{(s+1)}\|_p + \|w\varphi^2 f^{(s+2)}\|_p\right). \tag{4.19}$$

That contains, in particular, (4.18), and estimate (4.17) follows from (4.19) and the inequality

$$\|wf^{(s+1)}\|_p \leq c\left(\|wf^{(s')}\|_p + \|w\varphi^2 f^{(s+2)}\|_p\right),$$

which is established by means of Proposition 2.1 with $g = f^{(s')}$, $j = s - s' + 1$, $m = s - s' + 2$, $w_1 = w$ and $w_2 = w\varphi^2$.

Let us set

$$R_{s,n} f(x) = \frac{1}{(s+1)!} \sum_{k=0}^{n} p_{n,k}^{(s)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{s+1} f^{(s+2)}(u)\, du.$$

We shall show that

$$\|wR_{s,n} f\|_p \leq \frac{c}{n}\left(\|wf^{(s+1)}\|_p + \|w\varphi^2 f^{(s+2)}\|_p\right), \tag{4.20}$$

which verifies (4.19) in view of Lemma 3.2.

Let $F(u) = |w(u)\varphi^2(u) f^{(s+2)}(u)|$ and let $M_F(x)$ be its Hardy-Littlewood maximal function defined by

$$M_F(x) = \sup_{t\in[0,1]} \left|\frac{1}{t-x}\int_x^t F(u)\, du\right|.$$

As is known, if $1 < p \leq \infty$, then

$$\|M_F\|_p \leq c\,\|F\|_p. \tag{4.21}$$

In order to simplify our argument, we shall consider two cases for the domain of x.

Case 1. Let $n\varphi^2(x) \geq 1$. We make use of (3.4) and Lemma 4.2 with $\delta = s+1$, $\alpha = \gamma_0 + 1$ and $\beta = \gamma_1 + 1$ to get

$$\begin{aligned} &|w(x)\, R_{s,n} f(x)| \\ &\quad \leq \frac{c}{n} \sum_{0\leq i\leq s/2} \left(n\varphi^2(x)\right)^{i-s-1} \sum_{j=0}^{s-2i} \sum_{k=0}^{n} p_{n,k}(x)|k - nx|^{s+j+2} M_F(x). \end{aligned} \tag{4.22}$$

Further, we apply estimate (3.8) and get

$$\sum_{0\le i\le s/2} \big(n\varphi^2(x)\big)^{i-s-1} \sum_{j=0}^{s-2i}\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^{s+j+2} \tag{4.23}$$

$$\le c \sum_{0\le i\le s/2}\sum_{j=0}^{s-2i} \big(n\varphi^2(x)\big)^{(2i+j-s)/2} \le c,$$

as at the last inequality we have taken into account that $n\varphi^2(x)\ge 1$ and $2i+j-s\le 0$.

Now, (4.21)-(4.23) imply

$$\|wR_{s,n}f\|_{p(I_n)} \le \frac{c}{n}\,\|w\varphi^2 f^{(s+2)}\|_p, \tag{4.24}$$

where $I_n=\{x\in[0,1] : n\varphi^2(x)\ge 1\}$.

Case 2. Let $n\varphi^2(x)\le 1$. Due to symmetry, we may also assume that $x\le 1/2$. Therefore, $x\le 2/n$. By means of (3.3) and Abel's transform we derive for $n\ge s$ the relation (cf. (4.1))

$$R_{s,n}f(x)=\frac{1}{(s+1)!}\,\frac{n!}{(n-s)!}\sum_{k=0}^{n-s}\overrightarrow{\Delta}^{s}_{1/n} r_{s,x}\left(\frac{k}{n}\right) p_{n-s,k}(x),$$

where we have set

$$r_{s,x}(t)=\int_x^t (t-u)^{s+1}\, f^{(s+2)}(u)\,du.$$

Consequently,

$$|w(x)R_{s,n}f(x)|\le c\,n^s \max_{i=0,\dots,s}\sum_{k=0}^{n-s}\left|w(x)\,r_{s,x}\left(\frac{k+i}{n}\right)\right| p_{n-s,k}(x). \tag{4.25}$$

Just as in Case 1 we estimate $r_{s,x}(t)$ by means of Lemma 4.2 and get

$$\left|w(x)\,r_{s,x}\left(\frac{k+i}{n}\right)\right|\le c\,\varphi^{-2}(x)\left|\frac{k+i}{n}-x\right|^{s+2} M_F(x). \tag{4.26}$$

Next, we observe that for $k\ge 1$ and $i=0,\dots,s$ we have $k+i+1\ge 2\ge nx$. Therefore for $n>s$ there holds

$$\sum_{k=1}^{n-s}\left|\frac{k+i}{n}-x\right|^{s+2} p_{n-s,k}(x)\le \frac{x}{n^{s+1}}\sum_{k=0}^{n-s-1}|k+i+1-nx|^{s+2}p_{n-s-1,k}(x)$$

$$\le \frac{c\,x}{n^{s+1}}\left(1+\sum_{k=1}^{n-s-1}(k+i+1-nx)^{s+2}p_{n-s-1,k}(x)\right)$$

$$\le \frac{c\,x}{n^{s+1}}\left(1+\sum_{k=1}^{n-s-1}(k+s+1-nx)^{s+2}p_{n-s-1,k}(x)\right).$$

Further, we use the binomial formula to represent $(k+s+1-nx)^{s+2}$ in the form

$$\begin{aligned}(k+s+1-nx)^{s+2} &= \big([k-(n-s-1)x]+[(s+1)(1-x)]\big)^{s+2}\\ &= \sum_{j=0}^{s+2}\binom{s+2}{j}[k-(n-s-1)x]^j[(s+1)(1-x)]^{s-j+2}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{k=1}^{n-s}\left|\frac{k+i}{n}-x\right|^{s+2} p_{n-s,k}(x)\\ &\quad\le \frac{c\,x}{n^{s+1}}\left(1+\sum_{j=0}^{s+2}\sum_{k=1}^{n-s-1}|k-(n-s-1)x|^j p_{n-s-1,k}(x)\right)\\ &\quad\le \frac{c\,x}{n^{s+1}},\end{aligned}$$

where at the last estimate, we applied (3.8). Consequently, by (4.26) we get

$$\sum_{k=1}^{n-s}\left|w(x)\,r_{s,x}\left(\frac{k+i}{n}\right)\right| p_{n-s,k}(x)\le \frac{c}{n^{s+1}}\,M_F(x),\quad i=0,\ldots,s;$$

hence by (4.21) we arrive at

$$\left\|\sum_{k=1}^{n-s} w\,r_{s,\circ}\left(\frac{k+i}{n}\right)p_{n-s,k}\right\|_{p(I_n')}\le \frac{c}{n^{s+1}}\,\|F\|_p,\quad i=0,\ldots,s, \tag{4.27}$$

where $I_n'=\{x\in[0,1/2]: n\varphi^2(x)\le 1\}$.

It remains to estimate the terms for $k=0$ in (4.25). First, we observe that by (4.26) with $k=i=0$ we have

$$|w(x)r_{s,x}(0)|\le c\,x^{s+1}M_F(x)\le \frac{c}{n^{s+1}}\,M_F(x);$$

hence

$$\|w\,r_{s,\circ}(0)\|_{p(I_n')}\le \frac{c}{n^{s+1}}\,\|F\|_p. \tag{4.28}$$

To estimate $w(x)r_{s,x}(i/n)$ for $i=1,\ldots,s$, we expand $(i/n-u)^{s+1}$ by the binomial formula and get

$$\left|w(x)r_{s,x}\left(\frac{i}{n}\right)\right|\le \frac{c\,x^{\gamma_0}}{n^{s+1}}\sum_{j=0}^{s+1}\left|\int_x^{i/n}(nu)^j f^{(s+2)}(u)\,du\right|. \tag{4.29}$$

Further, taking into account that in the case under consideration we have $nx\le 2$, we get for $i=2,\ldots,s$ and $n\ge s$ but not $i=n=s$ the inequality

$$\left|w(x)r_{s,x}\left(\frac{i}{n}\right)\right|\le \frac{c}{n^{s+1}}\,x^{\gamma_0}\int_x^{s/(s+1)}|f^{(s+2)}(u)|\,du.$$

In order to estimate the L_p-norm of $w(x)r_{s,x}(i/n)$ for $\gamma_0 > -1/p$, we apply Hardy's inequality (2.4) with $s/(s+1)$ in place of $1/2$ as the upper bound in the integrals. Thus we arrive at

$$\left\| w\, r_{s,\circ}\left(\frac{i}{n}\right)\right\|_{p(I'_n)} \le \frac{c}{n^{s+1}}\,\|F\|_p \tag{4.30}$$

for $\gamma_0 > -1/p$, $1 < p \le \infty$, $i = 2, \dots, s$ and $n \ge s$ but not $i = n = s$.

For $\gamma_0 > -1/p$, $i = 1$, $n \ge s$ but not $n = s = 1$ we split the interval I'_n into two intervals. On $[0, 1/n]$ (note that $n \ge 2$), the same considerations as above yield

$$\left\| w\, r_{s,\circ}\left(\frac{1}{n}\right)\right\|_{p[0,1/n]} \le \frac{c}{n^{s+1}}\,\|F\|_p. \tag{4.31}$$

Let us denote the right end of the interval I'_n by x_n. We have $x_n \le 2/n$. Then for $x \in [1/n, x_n]$ there hold

$$\int_{1/n}^{x} |f^{(s+2)}(u)|\,du \le c\,n^{\gamma_0+1}\int_{1/n}^{x} F(u)\,du \le c\,x^{-\gamma_0}M_F(x).$$

Consequently (with the appropriate modification for $p = \infty$),

$$\left(\int_{1/n}^{x_n}\left(x^{\gamma_0}\int_{1/n}^{x}|f^{(s+2)}(u)|\,du\right)^p dx\right)^{1/p} \le c\,\|M_F\|_p \le c\,\|F\|_p,$$

as at the second step we have applied (4.21). Thus, in view of (4.29), we have established

$$\left\| w\, r_{s,\circ}\left(\frac{1}{n}\right)\right\|_{p[1/n,x_n]} \le \frac{c}{n^{s+1}}\,\|F\|_p. \tag{4.32}$$

Combining (4.31) and (4.32), we get

$$\left\| w\, r_{s,\circ}\left(\frac{1}{n}\right)\right\|_{p(I'_n)} \le \frac{c}{n^{s+1}}\,\|F\|_p \tag{4.33}$$

for $\gamma_0 > -1/p$, $1 < p \le \infty$.

For $p = \infty$, $\gamma_0 = 0$, $i = 1, \dots, s$ and $n \ge s$ but not $i = n = s$ we apply (4.29) to derive

(4.34)

$$\begin{aligned}\left| w(x) r_{s,x}\left(\frac{i}{n}\right)\right| &\le \frac{c}{n^{s+1}}\left|\int_x^{i/n} f^{(s+2)}(u)\,du\right| + \frac{c}{n^{s+1}}\sum_{j=1}^{s+1}\left|\int_x^{i/n}(nu)^j|f^{(s+2)}(u)|\,du\right| \\ &\le \frac{c}{n^{s+1}}\left|\int_x^{i/n} f^{(s+2)}(u)\,du\right| + \frac{c}{n^{s}}\left|\int_x^{i/n} u|f^{(s+2)}(u)|\,du\right|.\end{aligned}$$

For the first term on the right above we have

$$\left|\int_x^{i/n} f^{(s+2)}(u)\,du\right| \le |f^{(s+1)}(x)| + \left|f^{(s+1)}\left(\frac{i}{n}\right)\right| \le 2\,\|f^{(s+1)}\|_{\infty[0,s/(s+1)]} \le c\,\|wf^{(s+1)}\|_{\infty}. \tag{4.35}$$

We estimate the second term on the right of (4.34) in the following way:

$$\left|\int_x^{i/n} u|f^{(s+2)}(u)|\,du\right| \le \frac{c}{n}\,\|\chi f^{(s+2)}\|_{\infty[0,s/(s+1)]} \le \frac{c}{n}\,\|w\varphi^2 f^{(s+2)}\|_{\infty}. \tag{4.36}$$

Combining (4.34)-(4.36) we deduce that

$$\left\|w\,r_{s,\circ}\left(\frac{i}{n}\right)\right\|_{\infty(I_n')} \le \frac{c}{n^{s+1}}\left(\|wf^{(s+1)}\|_{\infty} + \|w\varphi^2 f^{(s+2)}\|_{\infty}\right) \tag{4.37}$$

for $\gamma_0 = 0$, $i = 1, \ldots, s$ and $n \ge s$ except $i = n = s$.

It remains to estimate the L_p-norm of $w(x)r_{s,x}(i/n)$ on I_n' for $i = n = s$. It is enough to do so for the function $x^{\gamma_0} r_{s,x}(1)$ on $[0, 1/2]$. To this end we split the integral in $r_{s,x}(1)$ by means of the intermediate point $1/2$ and consider the two quantities separately. To the first one we apply Hardy's inequality (2.4) and get (with the appropriate modification for $p = \infty$)

$$\begin{aligned}&\left(\int_0^{1/2}\left|x^{\gamma_0}\int_x^{1/2}(1-u)^{s+1}f^{(s+2)}(u)\,du\right|^p dx\right)^{1/p}\\ &\le c\left(\int_0^{1/2}\left|x^{\gamma_0+1}(1-x)^{\gamma_1+1}(1-x)^{s-\gamma_1}f^{(s+2)}(x)\right|^p dx\right)^{1/p}\\ &\le c\|F\|_p.\end{aligned} \tag{4.38}$$

For the other one we simply have (with the appropriate modification for $p = \infty$)

$$\begin{aligned}&\left(\int_0^{1/2}\left|x^{\gamma_0}\int_{1/2}^{1}(1-u)^{s+1}f^{(s+2)}(u)\,du\right|^p dx\right)^{1/p}\\ &\le c\left(\int_0^{1/2}\left|x^{\gamma_0}\int_{1/2}^{1}F(u)\,du\right|^p dx\right)^{1/p}\\ &\le c\|F\|_1 \le c\|F\|_p,\end{aligned} \tag{4.39}$$

where at the last step we have applied Hölder's inequality.

Relations (4.38)-(4.39) show that

$$\|w\,r_{s,\circ}(1)\|_{p[0,1/2]} \le c\,\|F\|_p. \tag{4.40}$$

To summarize, (4.28), (4.30), (4.33), (4.37) and (4.40) yield

$$\left\| w\, r_{s,\circ}\left(\frac{i}{n}\right)\right\|_{p(I'_n)} \le \frac{c}{n^{s+1}}\left(\|wf^{(s+1)}\|_p + \|w\varphi^2 f^{(s+2)}\|_p\right) \tag{4.41}$$

for $i = 0, \ldots, s$, $n \ge s$ and a weight w satisfying the assumptions in assertion (4.19). Let us explicitly note that (4.37) is used only if $\gamma_0 = 0$ and $p = \infty$. So the term $\|wf^{(s+1)}\|_p$ in (4.41) is redundant except when $\gamma_0 = 0$ and $p = \infty$.

Now, (4.25), (4.27) and (4.41) imply

$$\|wR_{s,n}f\|_{p(I'_n)} \le \frac{c}{n}\left(\|wf^{(s+1)}\|_p + \|w\varphi^2 f^{(s+2)}\|_p\right). \tag{4.42}$$

Finally, estimates (4.24) and (4.42) yield (4.20). Thus (4.19) is verified. $\square$

Remark 4.5. It seems that the assertions of the proposition continue to hold for $\gamma_0, \gamma_1 < s + 1 - 1/p$ (and unchanged lower bounds). However, the proof is much more technical. What is explicitly established above is quite enough in view of the restrictions on w in the other propositions (especially Proposition 4.1).

Remark 4.6. The assertion of Proposition 4.3 remains valid for $p = 1$ too. However, the proof is much more complicated. To get an idea of how it can be carried out, the interested reader can refer to e.g. [8, pp. 145-147].

Corollary 4.7. *Let $1 < p \le \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by (1.3). Set $s' = \max\{2, s\}$. If $-1/p < \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s-1}(0,1)$ and $wf^{(s')}, w\varphi^{2r} f^{(2r+s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds*

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_p \le \frac{c}{n^r}\left(\|wf^{(s')}\|_p + \|w\varphi^{2r} f^{(2r+s)}\|_p\right).$$

For $p = \infty$ we may allow $\gamma_0\gamma_1 = 0$, while still assuming $0 \le \gamma_0, \gamma_1 \le s$, and have

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_\infty \le \frac{c}{n^r}\left(\|wf^{(s')}\|_\infty + \|wf^{(r+s)}\|_\infty + \|w\varphi^{2r} f^{(2r+s)}\|_\infty\right)$$

provided that $wf^{(r+s)} \in L_\infty[0,1]$ too.

Proof. We shall prove that if $1 < p < \infty$ and $-1/p < \gamma_0, \gamma_1 \le s$, or $p = \infty$ and $0 \le \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s-1}(0,1)$ and $wf^{(s')}, wf^{(r+s)}, w\varphi^{2r} f^{(2r+s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_p \le \frac{c}{n^r}\left(\|wf^{(s')}\|_p + \|wf^{(r+s)}\|_p + \|w\varphi^{2r} f^{(2r+s)}\|_p\right). \tag{4.43}$$

That already contains the second assertion of the corollary; to get the first one we apply

$$\|wf^{(r+s)}\|_p \le c\left(\|wf^{(s')}\|_p + \|w\varphi^{2r} f^{(2r+s)}\|_p\right), \tag{4.44}$$

which follows from Proposition 2.1 with $g = f^{(s')}$, $j = r+s-s'$, $m = 2r+s-s'$, $w_1 = w$ and $w_2 = w\varphi^{2r}$.

To establish (4.43) for $s \geq 2$ we use Proposition 4.3 to derive by induction on r the estimate

$$\|w[(B_n - I)^r f]^{(s)}\|_p \leq \frac{c}{n^r} \sum_{i=0}^{r} \sum_{j=2i}^{i+r} \|w\varphi^{2i} f^{(j+s)}\|_p. \tag{4.45}$$

In order to estimate above the terms with $i = 0$ on the right side of the last relation, we apply (2.10) with $g = f^{(s)}$ and $m = r$ to get for $j = 0, \ldots, r$

$$\|wf^{(j+s)}\|_p \leq c\left(\|wf^{(s)}\|_p + \|wf^{(r+s)}\|_p\right), \tag{4.46}$$

whereas to estimate above the terms with $i > 0$, we apply Proposition 2.1 with $g = f^{(s)}$, $m = 2r$, $w_1 = w\varphi^{2i}$ and $w_2 = w\varphi^{2r}$ to get for $j = 2i, \ldots i + r$

$$\|w\varphi^{2i} f^{(j+s)}\|_p \leq c\left(\|wf^{(s)}\|_p + \|w\varphi^{2r} f^{(2r+s)}\|_p\right). \tag{4.47}$$

Now, (4.43) for $s \geq 2$ follows from (4.45)-(4.47).

To prove (4.43) for $s = 1$ we first observe that Proposition 4.3 and what we have already established yield

$$\begin{aligned}
\|w(\mathcal{B}_{r,n}f - f)'\|_p &\leq \frac{c}{n}\left(\|w(\mathcal{B}_{r-1,n}f - f)''\|_p + \|w\varphi^2(\mathcal{B}_{r-1,n}f - f)'''\|_p\right) \\
&\leq \frac{c}{n^r}\Big(\|wf''\|_p + \|wf^{(r+1)}\|_p + \|w\varphi^{2r-2} f^{(2r)}\|_p \\
&\qquad + \|w\varphi^2 f'''\|_p + \|w\varphi^2 f^{(r+2)}\|_p + \|w\varphi^{2r} f^{(2r+1)}\|_p\Big).
\end{aligned}$$

Next, to complete the proof in this case, we use that

$$\|w\varphi^{2j} f^{(r+j+1)}\|_p \leq c\left(\|wf^{(r+1)}\|_p + \|w\varphi^{2r} f^{(2r+1)}\|_p\right), \quad j = 1, r-1,$$

and

$$\|w\varphi^2 f'''\|_p \leq c\left(\|wf''\|_p + \|w\varphi^{2r} f^{(2r+1)}\|_p\right),$$

which follow from Proposition 2.1 respectively with $g = f^{(r+1)}$, $m = r$, $w_1 = w\varphi^{2j}$, $w_2 = w\varphi^{2r}$ (or see (2.8)) and $g = f''$, $j = 1$, $m = 2r-1$, $w_1 = w\varphi^2$, $w_2 = w\varphi^{2r}$. ☐

The upper estimate can be stated in a more concise form in terms of the differential operator $(d/dx)^s D^r$. This result follows directly from Proposition 2.4 and Corollary 4.7.

Corollary 4.8. *Let* $1 < p \leq \infty$, $r, s \in \mathbb{N}$ *and* $w = w(\gamma_0, \gamma_1)$ *be given by* (1.3) *as*

$$\begin{aligned}
-1/p <&\gamma_0, \gamma_1 < s - 1/p \quad \textit{if} \quad 1 < p < \infty, \\
0 \leq&\gamma_0, \gamma_1 < s \quad \textit{if} \quad p = \infty.
\end{aligned}$$

Then for all $f \in AC^{2r+s-1}[0,1]$ *such that* $w\varphi^{2r} f^{(2r+s)} \in L_p[0,1]$, *and all* $n \in \mathbb{N}$ *there holds*

$$\|w(\mathcal{B}_{r,n} f - f)^{(s)}\|_p \le \frac{c}{n^r} \|w(D^r f)^{(s)}\|_p.$$

We proceed to Voronovskaya-type estimates.

Proposition 4.9. *Let* $1 < p \le \infty$, $s \in \mathbb{N}$ *and* $w = w(\gamma_0, \gamma_1)$ *be given by* (1.3). *Set* $s'' = \max\{3, s\}$. *If* $-1/p < \gamma_0, \gamma_1 \le s+1$, *then for all* $f \in C[0,1]$ *such that* $f \in AC_{loc}^{s+3}(0,1)$ *and* $wf^{(s'')}, w\varphi^4 f^{(s+4)} \in L_p[0,1]$, *and all* $n \in \mathbb{N}$ *there holds*

$$\left\| w\left(B_n f - f - \frac{1}{2n}\, Df\right)^{(s)} \right\|_p \le \frac{c}{n^2} \left(\|wf^{(s'')}\|_p + \|w\varphi^4 f^{(s+4)}\|_p \right).$$

For $p = \infty$ *we may allow* $\gamma_0\gamma_1 = 0$, *while still assuming* $0 \le \gamma_0, \gamma_1 \le s+1$, *and have*

$$\begin{aligned} &\left\| w\left(B_n f - f - \frac{1}{2n}\, Df\right)^{(s)} \right\|_\infty \\ &\qquad \le \frac{c}{n^2} \left(\|wf^{(s'')}\|_\infty + \|wf^{(s+2)}\|_\infty + \|w\varphi^4 f^{(s+4)}\|_\infty \right) \end{aligned}$$

provided that $wf^{(s+2)} \in L_\infty[0,1]$ *too.*

Proof. The proof is based on Lemma 3.3 and is similar to that of the previous proposition.

Using $\|w\varphi^4 f^{(s+4)}\|_p < \infty$, we get by Proposition 2.1 with $g = f$, $j = s+3$, $m = s+4$, $w_1 = \varphi^{2s+4}$ and $w_2 = w\varphi^4$ that $\varphi^{2s+4} f^{(s+3)} \in L[0,1]$ and we can apply Lemma 3.3.

We shall prove that if $1 < p < \infty$ and $-1/p < \gamma_0, \gamma_1 \le s+1$, or $p = \infty$ and $0 \le \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+3}(0,1)$ and $wf^{(s'')}, wf^{(s+2)}, w\varphi^4 f^{(s+4)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\begin{aligned} &\left\| w\left(B_n f - f - \frac{1}{2n}\, Df\right)^{(s)} \right\|_p \\ &\qquad \le \frac{c}{n^2} \left(\|wf^{(s'')}\|_p + \|wf^{(s+2)}\|_p + \|w\varphi^4 f^{(s+4)}\|_p \right). \end{aligned} \tag{4.48}$$

That establishes the second assertion of the proposition; the first one follows from (4.48) and

$$\|wf^{(s+2)}\|_p \le c\left(\|wf^{(s'')}\|_p + \|w\varphi^4 f^{(s+4)}\|_p \right),$$

which is established by Proposition 2.1 with $g = f^{(s'')}$, $j = s - s'' + 2$, $m = s - s'' + 4$, $w_1 = w$ and $w_2 = w\varphi^4$.

Let us set

$$\widetilde{R}_{s,n}f(x) = \frac{1}{(s+2)!}\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^{s+2} f^{(s+3)}(u)\,du.$$

We shall show that

$$\|w\widetilde{R}_{s,n}f\|_p \le \frac{c}{n^2}\left(\|wf^{(s+2)}\|_p + \|w\varphi^2 f^{(s+3)}\|_p + \|w\varphi^4 f^{(s+4)}\|_p\right). \tag{4.49}$$

Then Lemma 3.3 implies
(4.50)

$$\|w(V_nf)^{(s)}\|_p \le \frac{c}{n^2}\left(\sum_{k=s''}^{s+2}\|wf^{(k)}\|_p + \|w\varphi^2 f^{(s+3)}\|_p + \|w\varphi^4 f^{(s+4)}\|_p\right),$$

where $V_nf(x)$ is defined in (3.17). By (2.10) with $g = f^{(s)}$, $j = 1$ and $m = 2$ we have for $s \ge 3$

$$\|wf^{(s+1)}\|_p \le c\left(\|wf^{(s'')}\|_p + \|wf^{(s+2)}\|_p\right), \tag{4.51}$$

and by (2.8) with $g = f^{(s+2)}$, $j = 1$ and $m = 2$ we have

$$\|w\varphi^2 f^{(s+3)}\|_p \le c\left(\|wf^{(s+2)}\|_p + \|w\varphi^4 f^{(s+4)}\|_p\right). \tag{4.52}$$

Now, estimate (4.48) follows from (4.50)-(4.52).

It remains to prove (4.49). We consider two cases for the domain of x.

Case 1. Let $n\varphi^2(x) \ge 1$. Hölder's inequality implies that since $w\varphi^4 f^{(s+4)} \in L_p[0,1]$, then $\varphi^{2s+6}f^{(s+4)} \in L[0,1]$; hence (3.11)-(3.12) are valid for $\sigma = s+2$. Using them we integrate by parts in $\widetilde{R}_{s,n}f$ and represent it in the form

$$\widetilde{R}_{s,n}f(x) = \widetilde{S}_{s,n}f(x) + \widetilde{R}'_{s,n}f(x),$$

where

$$\widetilde{S}_{s,n}f(x) = \frac{1}{(s+3)!}\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\left(\frac{k}{n}-x\right)^{s+3} f^{(s+3)}(x)$$

and

$$\widetilde{R}'_{s,n}f(x) = \frac{1}{(s+3)!}\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\int_x^{k/n}\left(\frac{k}{n}-u\right)^{s+3} f^{(s+4)}(u)\,du.$$

We shall show that

$$\left|\sum_{k=0}^{n} p_{n,k}^{(s)}(x)\left(\frac{k}{n}-x\right)^{s+3}\right| \le \frac{c}{n^2}\,\varphi^2(x), \quad x \in I_n, \tag{4.53}$$

and

$$\|w\widetilde{R}'_{s,n}f\|_{p(I_n)} \le \frac{c}{n^2}\,\|w\varphi^4 f^{(s+4)}\|_p, \tag{4.54}$$

where $I_n = \{x \in [0,1] : n\varphi^2(x) \ge 1\}$. Then it will follow that

$$\|w\widetilde{R}_{s,n}f\|_{p(I_n)} \le \frac{c}{n^2}\left(\|w\varphi^2 f^{(s+3)}\|_p + \|w\varphi^4 f^{(s+4)}\|_p\right). \tag{4.55}$$

Estimate (4.53) follows directly from Lemma 3.1 with $\ell = s+3$ and from $n^{-1} \le \varphi^2(x)$.

To establish (4.54) we set $F(u) = |w(u)\varphi^4(u)f^{(s+4)}(u)|$ and denote by $M_F(x)$ its Hardy-Littlewood maximal function.

We make use of (3.4) and Lemma 4.2 with $\delta = s+3$, $\alpha = \gamma_0 + 2$ and $\beta = \gamma_1 + 2$ to get

$$\begin{aligned} &|w(x)\,\widetilde{R}'_{s,n}f(x)| \\ &\qquad \le \frac{c}{n^2}\sum_{0\le i\le s/2}\big(n\varphi^2(x)\big)^{i-s-2}\sum_{j=0}^{s-2i}\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^{s+j+4}M_F(x). \end{aligned} \tag{4.56}$$

Further, we apply estimate (3.8) and get

$$\begin{aligned} &\sum_{0\le i\le s/2}\big(n\varphi^2(x)\big)^{i-s-2}\sum_{j=0}^{s-2i}\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^{s+j+4} \\ &\qquad\qquad \le c\sum_{0\le i\le s/2}\sum_{j=0}^{s-2i}\big(n\varphi^2(x)\big)^{(2i+j-s)/2} \le c. \end{aligned} \tag{4.57}$$

Now, (4.21), (4.56) and (4.57) imply (4.54).

Case 2. Let $n\varphi^2(x) \le 1$ and, because of the symmetry, we may also assume that $x \le 1/2$. Just as in the proof of Proposition 4.3, case 2, we represent $\widetilde{R}_{s,n}f$ in the form

$$\widetilde{R}_{s,n}f(x) = \frac{1}{(s+2)!}\,\frac{n!}{(n-s)!}\sum_{k=0}^{n-s}\overrightarrow{\Delta}^s_{1/n}r_{s+1,x}\left(\frac{k}{n}\right)p_{n-s,k}(x)$$

and derive (cf. (4.25))

$$|w(x)\widetilde{R}_{s,n}f(x)| \le c\,n^s \max_{i=0,\dots,s}\sum_{k=0}^{n-s}\left|w(x)\,r_{s+1,x}\left(\frac{k+i}{n}\right)\right|\,p_{n-s,k}(x). \tag{4.58}$$

Just similarly to (4.27) and (4.41) we establish the following estimates

$$\left\|\sum_{k=1}^{n-s} w\,r_{s+1,\circ}\left(\frac{k+i}{n}\right)p_{n-s,k}\right\|_{p(I'_n)} \le \frac{c}{n^{s+2}}\,\|w\varphi^2 f^{(s+3)}\|_p$$

and

$$\left\| w\, r_{s+1,\circ}\left(\frac{i}{n}\right)\right\|_{p(I'_n)} \le \frac{c}{n^{s+2}}\left(\|wf^{(s+2)}\|_p + \|w\varphi^2 f^{(s+3)}\|_p\right)$$

for $i = 0, \dots, s$ and $n \ge s$. Actually the second estimate follows directly from (4.41).

Consequently,

$$\|w\widetilde{R}_{s,n}f\|_{p(I'_n)} \le \frac{c}{n^2}\left(\|wf^{(s+2)}\|_p + \|w\varphi^2 f^{(s+3)}\|_p\right). \tag{4.59}$$

Estimates (4.55) and (4.59) yield (4.49). Thus (4.48) is verified. □

Remark 4.10. In Proposition 4.9 we have assumed higher degree of smoothness than usual — $w\varphi^4 f^{(s+4)} \in L_p[0,1]$ rather than the weaker $w\varphi^3 f^{(s+3)} \in L_p[0,1]$. However, the latter assumption yields an order of $n^{-3/2}$ on the right in the corresponding Voronovskaya-type estimate. It still can be used to prove the converse inequality about simultaneous approximation by B_n, but the order of n^{-2} as in Proposition 4.9 seems more natural in this setting and is easier to work with (see [19, Lemma 2.1]).

Corollary 4.11. *Let $1 < p \le \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by (1.3). Set $s'' = \max\{3, s\}$. If $-1/p < \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s+1}(0,1)$ and $wf^{(s'')}, w\varphi^{2r+2} f^{(2r+s+2)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds*

$$\left\| w\left(\mathcal{B}_{r,n}f - f - \frac{(-1)^{r-1}}{(2n)^r}\, D^r f\right)^{(s)}\right\|_p \le \frac{c}{n^{r+1}}\left(\|wf^{(s'')}\|_p + \|w\varphi^{2r+2} f^{(2r+s+2)}\|_p\right).$$

For $p = \infty$ we may allow $\gamma_0\gamma_1 = 0$, while still assuming $0 \le \gamma_0, \gamma_1 \le s+1$, and have

$$\left\| w\left(\mathcal{B}_{r,n}f - f - \frac{(-1)^{r-1}}{(2n)^r}\, D^r f\right)^{(s)}\right\|_\infty \le \frac{c}{n^{r+1}}\left(\|wf^{(s'')}\|_\infty + \|wf^{(r+s+1)}\|_\infty + \|w\varphi^{2r+2} f^{(2r+s+2)}\|_\infty\right).$$

provided that $wf^{(r+s+1)} \in L_\infty[0,1]$ too.

Proof. We shall establish that if $1 < p < \infty$ and $-1/p < \gamma_0, \gamma_1 \le s+1$, or $p = \infty$ and $0 \le \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{2r+s+1}(0,1)$ and $wf^{(s'')}, wf^{(r+s+1)}, w\varphi^{2r+2} f^{(2r+s+2)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\left\| w\left(\mathcal{B}_{r,n}f - f - \frac{(-1)^{r-1}}{(2n)^r}\, D^r f\right)^{(s)}\right\|_p \le \frac{c}{n^{r+1}}\left(\|wf^{(s'')}\|_p + \|wf^{(r+s+1)}\|_p + \|w\varphi^{2r+2} f^{(2r+s+2)}\|_p\right). \tag{4.60}$$

That verifies the second estimate in the corollary; to get the first one we also use the inequality

$$\|wf^{(r+s+1)}\|_p \le c\left(\|wf^{(s'')}\|_p + \|w\varphi^{2r+2} f^{(2r+s+2)}\|_p\right),$$

which follows from Proposition 2.1 with $g = f^{(s'')}$, $j = r + s - s'' + 1$, $m = 2r + s - s'' + 2$, $w_1 = w$ and $w_2 = w\varphi^{2r+2}$.

So, let us proceed to the proof of (4.60). We set

$$V_{r,n}f = \mathcal{B}_{r,n}f - f - \frac{(-1)^{r-1}}{(2n)^r}\, D^r f.$$

For $s \ge 3$ we establish by induction on r that

$$\|w(V_{r,n}f)^{(s)}\|_p \le \frac{c}{n^{r+1}} \sum_{i=0}^{r+1} \sum_{j=2i}^{i+r+1} \|w\varphi^{2i} f^{(j+s)}\|_p. \tag{4.61}$$

To this end, we use the relation

$$\|w(V_{r+1,n}f)^{(s)}\|_p \le \|w(V_{1,n}F_{r,n})^{(s)}\|_p + \frac{1}{n}\, \|w(DV_{r,n}f)^{(s)}\|_p, \tag{4.62}$$

where $F_{r,n} = (B_n - I)^r f$, as we estimate $\|w(V_{1,n}F_{r,n})^{(s)}\|_p$ by means of Proposition 4.9 and (4.45), and the term $\|w(DV_{r,n}f)^{(s)}\|_p$ by (see (2.11))

$$\begin{aligned} &\|w(DV_{r,n}f)^{(s)}\|_p \\ &\quad \le c\left(\|w(V_{r,n}f)^{(s)}\|_p + \|w(V_{r,n}f)^{(s+1)}\|_p + \|w\varphi^2 (V_{r,n}f)^{(s+2)}\|_p\right) \end{aligned} \tag{4.63}$$

and the induction hypothesis.

Next, we estimate above the terms of (4.61) with $i = 0$ by means of (2.10) with $g = f^{(s)}$ and $m = r + 1$ to get for $j = 0, \dots, r + 1$

$$\|wf^{(j+s)}\|_p \le c\left(\|wf^{(s)}\|_p + \|wf^{(r+s+1)}\|_p\right). \tag{4.64}$$

For the terms with $i > 0$, we apply Proposition 2.1 with $g = f^{(s)}$, $m = 2r + 2$, $w_1 = w\varphi^{2i}$ and $w_2 = w\varphi^{2r+2}$ to get for $j = 2i, \dots, i + r + 1$

$$\|w\varphi^{2i} f^{(j+s)}\|_p \le\ c\left(\|wf^{(s)}\|_p + \|w\varphi^{2r+2} f^{(2r+s+2)}\|_p\right). \tag{4.65}$$

Now, estimate (4.60) for $s \ge 3$ follows from (4.61)-(4.65).

The proof in the case $s = 2$ is similar. We verify by induction on r that

$$\|w(V_{r,n}f)''\|_p \le \frac{c}{n^{r+1}} \sum_{i=0}^{r+1} \sum_{j=\max\{1,2i\}}^{i+r+1} \|w\varphi^{2i} f^{(j+2)}\|_p,$$

as besides (4.62), (4.63), Proposition 4.9 and (4.45) we also use (4.61). Then we complete the proof by means of Proposition 2.1 just similarly as in the case $s \geq 3$.

Finally, in the case $s = 1$ we deduce from (4.62) with $s = 1$ and $r-1$ in place of r, Proposition 4.9, Corollary 4.7, the estimate

$$\|w(DV_{r-1,n}f)'\|_p \leq c\left(\|w(V_{r-1,n}f)''\|_p + \|w\varphi^2(V_{r-1,n}f)'''\|_p\right)$$

and what we have already established that

$$\begin{aligned}\|w(V_{r,n}f)'\|_p \leq \frac{c}{n^{r+1}}\Big(&\|wf^{(3)}\|_p + \|wf^{(r+2)}\|_p + \|w\varphi^4 f^{(5)}\|_p + \|w\varphi^2 f^{(r+3)}\|_p\\ &+ \|w\varphi^{2r-2}f^{(2r+1)}\|_p + \|w\varphi^{2r}f^{(2r+2)}\|_p + \|w\varphi^{2r+2}f^{(2r+3)}\|_p\Big).\end{aligned}$$

To complete the proof of (4.60) for $s = 1$ we need only take into account the inequalities

$$\|w\varphi^{2j}f^{(r+j+2)}\|_p \leq c\left(\|wf^{(r+2)}\|_p + \|w\varphi^{2r+2}f^{(2r+3)}\|_p\right), \quad j = 1, r-1, r,$$

and

$$\|w\varphi^4 f^{(5)}\|_p \leq c\left(\|wf^{(3)}\|_p + \|w\varphi^{2r+2}f^{(2r+3)}\|_p\right),$$

which follow from Proposition 2.1 respectively with $g = f^{(r+2)}$, $m = r+1$, $w_1 = w\varphi^{2j}$, $w_2 = w\varphi^{2r+2}$ and $g = f^{(3)}$, $j = 2$, $m = 2r$, $w_1 = w\varphi^4$, $w_2 = w\varphi^{2r+2}$. $\square$

Similarly to Corollary 4.8 we get by Proposition 2.4 and Corollary 4.11 the following Voronovskaya-type estimate.

Corollary 4.12. *Let $1 < p \leq \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *as*

$$\begin{aligned}-1/p <&\gamma_0, \gamma_1 < s - 1/p \quad \text{if} \quad 1 < p < \infty,\\ 0 \leq&\gamma_0, \gamma_1 < s \quad \text{if} \quad p = \infty.\end{aligned}$$

Then for all $f \in AC^{2r+s+1}[0,1]$ such that $w\varphi^{2r+2}f^{(2r+s+2)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\left\|w\left(\mathcal{B}_{r,n}f - f - \frac{(-1)^{r-1}}{(2n)^r}\,D^r f\right)^{(s)}\right\|_p \leq \frac{c}{n^{r+1}}\,\|w(D^{r+1}f)^{(s)}\|_p.$$

The last several estimates, we shall need, are traditionally regarded to as Bernstein-type inequalities.

Proposition 4.13. *Let $1 \leq p \leq \infty$, $\ell, r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *as*

$$\begin{aligned}-1/p <&\gamma_0, \gamma_1 < s - 1/p \quad \text{if} \quad 1 \leq p < \infty,\\ 0 \leq&\gamma_0, \gamma_1 < s \quad \text{if} \quad p = \infty.\end{aligned}$$

Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there hold:

(a) $\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\|_p \le c\, n^\ell \|wf^{(s)}\|_p$;

(b) $\|w\varphi^{2\ell}(\mathcal{B}_{r,n} f)^{(2\ell+s)}\|_p \le c\, n^\ell \|wf^{(s)}\|_p$;

(c) $\|w\varphi^{2\ell}(\mathcal{B}_{r,n} f)^{(2\ell+s)}\|_p \le c\, n^\ell K_{2\ell,\varphi}(f^{(s)}, n^{-\ell})_{w,p}$.

Proof. Again we shall consider two cases for the domain of x.

Case 1. Let $(n-s)\varphi^2(x) \ge 1$. Differentiating (4.1) we get

$$(B_n f)^{(2\ell+s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}^{(2\ell)}(x). \tag{4.66}$$

Next, we express $p_{n-s,k}^{(2\ell)}(x)$ by means of (3.4) and estimate $\left|\overrightarrow{\Delta}_{1/n}^{s} f(k/n)\right|$ by (4.2). Thus we arrive at

$$\begin{aligned} \left|w(x)\varphi^{2\ell}(x)(B_n f)^{(2\ell+s)}(x)\right| &\le c\, n^\ell \sum_{i=0}^{\ell} \sum_{j=0}^{2(\ell-i)} \left(n\varphi^2(x)\right)^{i-\ell} \\ &\quad \times w(x) \sum_{k=0}^{n-s} p_{n-s,k}(x)\, w_{p,n,k}\, \|wf^{(s)}\|_{p[k/n,(k+s)/n]} |k-(n-s)x|^j \\ &\le c\, n^\ell \sum_{j=0}^{2\ell} \left(n\varphi^2(x)\right)^{-j/2} \\ &\quad \times w(x) \sum_{k=0}^{n-s} p_{n-s,k}(x)\, w_{p,n,k}\, \|wf^{(s)}\|_{p[k/n,(k+s)/n]} |k-(n-s)x|^j, \end{aligned} \tag{4.67}$$

where at the last step we have used that $n\varphi^2(x) \ge 1$ and $i - \ell \le -j/2$.

We have to estimate the weighted L_p-norm of the right-hand side of the last inequality. Moreover, due to the Riesz-Thorin interpolation theorem and symmetry, it is sufficient to do that only for $p = 1$ and $p = \infty$, and restrict the range of summation on k to $\{0, \dots, [(n-s)/2]\}$ (see (4.5)-(4.6) and note that $|k-(n-s)x| = |n-s-k-(n-s)y|$ with $y = 1-x$).

For $p = \infty$ we apply Cauchy's inequality to derive

$$\begin{aligned} w(x) &\sum_{k=0}^{[(n-s)/2]} p_{n-s,k}(x)\, w_{\infty,n,k}\, \|wf^{(s)}\|_{\infty[k/n,(k+s)/n]} |k-(n-s)x|^j \\ &\le \left(w^2(x) \sum_{k=0}^{[(n-s)/2]} w_{\infty,n,k}^2\, p_{n-s,k}(x)\right)^{1/2} (T_{n-s,2j}(x))^{1/2}\, \|wf^{(s)}\|_\infty. \end{aligned} \tag{4.68}$$

Further, just as in (4.10) we see that

$$w^2(x) \sum_{k=0}^{[(n-s)/2]} w_{\infty,n,k}^2 \, p_{n-s,k}(x) \le c, \quad x \in [0,1]. \tag{4.69}$$

Also, (3.7) yields

$$\left(n\varphi^2(x)\right)^{-j/2} (T_{n-s,2j}(x))^{1/2} \le c, \quad (n-s)\varphi^2(x) \ge 1. \tag{4.70}$$

Relations (4.67)-(4.70) imply

$$\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\|_{\infty(I_{n-s})} \le c\, n^{\ell} \|wf^{(s)}\|_{\infty}, \tag{4.71}$$

where, to recall, $I_n = \{x \in [0,1] : n\varphi^2(x) \ge 1\}$.

For $p = 1$ we fix $m \in \mathbb{N}$ such that $\alpha\gamma_i > -1$ for $i = 0, 1$, where $\alpha = 2m/(2m-1)$, and apply Hölder's inequality to get

$$\begin{aligned} & w_{1,n,k} \int_{I_{n-s}} w(x)\, p_{n-s,k}(x) \, \left(n\varphi^2(x)\right)^{-j/2} |k-(n-s)x|^j \, dx \\ & \le \left(w_{1,n,k}^{\alpha} \int_0^1 w^{\alpha}(x) p_{n-s,k}(x)\, dx \right)^{1/\alpha} \\ & \quad \times \left(\int_{I_{n-s}} \left(n\varphi^2(x)\right)^{-mj} (k-(n-s)x)^{2mj} p_{n-s,k}(x)\, dx \right)^{1/(2m)}. \end{aligned} \tag{4.72}$$

Estimate (4.11) with w^{α} in place of w yields

$$\int_0^1 w^{\alpha}(x) p_{n-s,k}(x)\, dx \le \frac{c}{n} \left(\frac{k+1}{n} \right)^{\alpha\gamma_0}, \quad 0 \le k \le (n-s)/2;$$

hence by (4.7) with $p = 1$ we get

$$w_{1,n,k}^{\alpha} \int_0^1 w^{\alpha}(x) p_{n-s,k}(x)\, dx \le c\, n^{\alpha-1}, \quad 0 \le k \le (n-s)/2. \tag{4.73}$$

Also, [8, Lemma 9.4.5] implies

$$\int_{I_n} \left(n\varphi^2(x)\right)^{-\mu} (k-nx)^{2\mu} p_{n,k}(x)\, dx \le \frac{c}{n}, \quad \mu \in \mathbb{N}_0.$$

Therefore

$$\int_{I_{n-s}} \left(n\varphi^2(x)\right)^{-mj} (k-(n-s)x)^{2mj} p_{n-s,k}(x)\, dx \le \frac{c}{n}. \tag{4.74}$$

Inequalities (4.72)-(4.74) imply for $j = 0, \ldots, 2\ell$ and $k = 0, \ldots, [(n-s)/2]$ the estimates

$$w_{1,n,k} \int_{I_{n-s}} w(x)\, p_{n-s,k}(x) \, \left(n\varphi^2(x)\right)^{-j/2} |k-(n-s)x|^j \, dx \le c. \tag{4.75}$$

Relations (4.67) and (4.75) yield

$$\|w\varphi^{2\ell}(B_nf)^{(2\ell+s)}\|_{1(I_{n-s})} \le c\,n^{\ell}\|wf^{(s)}\|_1. \tag{4.76}$$

Case 2. Let $(n-s)\varphi^2(x) \le 1$ and $n \ge 2\ell+s$. Differentiating ℓ times (4.1) with $\ell+s$ in place of s, we get

$$(B_nf)^{(2\ell+s)}(x) = \frac{n!}{(n-\ell-s)!}\sum_{k=0}^{n-\ell-s}\overrightarrow{\Delta}_{1/n}^{\ell+s}f\left(\frac{k}{n}\right)p_{n-\ell-s,k}^{(\ell)}(x).$$

Consequently,

$$\begin{aligned}|(B_nf)^{(2\ell+s)}(x)|\\ &\le c\,n^{\ell}\max_{\nu=0,\dots,\ell}\frac{n!}{(n-s)!}\sum_{k=0}^{n-\ell-s}\left|\overrightarrow{\Delta}_{1/n}^{s}f\left(\frac{k+\nu}{n}\right)\right||p_{n-\ell-s,k}^{(\ell)}(x)|.\end{aligned}\tag{4.77}$$

Just as in Case 1 we estimate $\left|\overrightarrow{\Delta}_{1/n}^{s}f((k+\nu)/n)\right|$ by means of (4.2) and express $p_{n-\ell-s,k}^{(\ell)}(x)$ by means of (3.4). Thus for each $\nu=0,\dots,\ell$ we have

$$\begin{aligned}&\frac{n!}{(n-s)!}w(x)\varphi^{2\ell}(x)\sum_{k=0}^{n-\ell-s}\left|\overrightarrow{\Delta}_{1/n}^{s}f\left(\frac{k+\nu}{n}\right)\right||p_{n-\ell-s,k}^{(\ell)}(x)|\\
&\quad\le c\sum_{0\le i\le\ell/2}\sum_{j=0}^{\ell-2i}\left(n\varphi^2(x)\right)^i w(x)\sum_{k=0}^{n-\ell-s}p_{n-\ell-s,k}(x)\\
&\qquad\times w_{p,n,k+\nu}\,\|wf^{(s)}\|_{p[(k+\nu)/n,(k+\nu+s)/n]}|k-(n-\ell-s)x|^j\\
&\quad\le c\sum_{j=0}^{\ell}w(x)\sum_{k=0}^{n-\ell-s}p_{n-\ell-s,k}(x)\\
&\qquad\times w_{p,n,k+\nu}\,\|wf^{(s)}\|_{p[(k+\nu)/n,(k+\nu+s)/n]}|k-(n-\ell-s)x|^j,\end{aligned}\tag{4.78}$$

where at the last estimate we have taken into account that $n\varphi^2(x)\le c$.

We proceed as in Case 1. Again due to symmetry it is sufficient to restrict the range of summation on k to $\{0,\dots,[(n-\ell-s)/2]\}$, as now we have with $\bar{k}=n-\ell-s-k$ and $\bar{\nu}=\ell-\nu$ (cf. (4.5)-(4.6)) the relations

$$\begin{aligned}&w_{p,n,\bar{k}+\nu}=\bar{w}_{p,n,k+\bar{\nu}},\\ &\|wf^{(s)}\|_{p[(\bar{k}+\nu)/n,(\bar{k}+\nu+s)/n]}=\|\bar{w}\bar{f}^{(s)}\|_{p[(k+\bar{\nu})/n,(k+\bar{\nu}+s)/n]}.\end{aligned}\tag{4.79}$$

Let us note that we still have

$$w_{p,n,k+\nu}\le c\,n^{1/p}\left(\frac{n}{k+1}\right)^{\gamma_0},\quad 0\le k\le(n-\ell-s)/2,\tag{4.80}$$

for $\nu = 0, \ldots, \ell$. Consequently, for $p = \infty$ there holds

$$w^2(x) \sum_{k=0}^{[(n-\ell-s)/2]} w_{\infty,n,k+\nu}^2 \, p_{n-\ell-s,k}(x) \le c, \quad x \in [0,1].$$

Also, (3.7) implies

$$T_{n-\ell-s,2j}(x) \le c, \quad (n-s)\varphi^2(x) \le 1.$$

Now, just similarly to Case 1, $p = \infty$ we derive from (4.77), (4.78), the symmetry on k, and the last two estimates above the inequality

$$\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\|_{\infty(I''_{n-s})} \le c\, n^{\ell} \|w f^{(s)}\|_{\infty}, \tag{4.81}$$

where $I''_n = \{x \in [0,1] : n\varphi^2(x) \le 1\}$.

For $p = 1$ we use (cf. (4.73))

$$w_{1,n,k+\nu}^{\alpha} \int_0^1 w^{\alpha}(x) p_{n-\ell-s,k}(x)\, dx \le c\, n^{\alpha-1}, \quad 0 \le k \le (n-\ell-s)/2,$$

for $\nu = 0, \ldots, \ell$, which follows from (4.80) just as in Case 1.

Also, (3.7) implies

$$\int_{I''_n} p_{n,k}(x)(k-nx)^{2\mu}\, dx \le \frac{c}{n}, \quad \mu \in \mathbb{N}_0.$$

Just similarly to Case 1, $p = 1$ we derive from (4.77), (4.78), the symmetry on k, and the last two estimates above the inequality

$$\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\|_{1(I''_{n-s})} \le c\, n^{\ell} \|w f^{(s)}\|_{1}. \tag{4.82}$$

Estimates (4.71), (4.76), (4.81) and (4.82) yield

$$\|w\varphi^{2\ell}(B_n f)^{(2\ell+s)}\|_p \le c\, n^{\ell} \|w f^{(s)}\|_p$$

for $p = 1$ and $p = \infty$ and assertion (a) follows from the Riesz-Thorin interpolation theorem.

Assertion (b) follows from (a) and Proposition 4.1 since $\mathcal{B}_{r,n}$ is a linear combination of iterates of B_n.

Finally, to prove (c) we apply (b) and Proposition 4.1 to derive for any $g \in AC_{loc}^{2\ell+s-1}(0,1)$ the estimate

$$\|w\varphi^{2\ell}(\mathcal{B}_{r,n} f)^{(2\ell+s)}\|_p \le c\, n^{\ell} \left(\|w(f^{(s)} - g^{(s)})\|_p + n^{-\ell} \|w\varphi^{2\ell} g^{(2\ell+s)}\|_p \right).$$

Taking an infimum on g we get (c). $\square$

We shall also need the following almost trivial analogue of the last proposition in the case $p = \infty$.

Proposition 4.14. *Let $\ell, r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}$ there hold:*

(a) $\|w(B_n f)^{(\ell+s)}\|_\infty \le c\, n^\ell \|wf^{(s)}\|_\infty$;

(b) $\|w(\mathcal{B}_{r,n} f)^{(\ell+s)}\|_\infty \le c\, n^\ell \|wf^{(s)}\|_\infty$.

Proof. To establish (a) we apply (4.1) with $\ell + s$ in place of s and (4.2). Thus we get

$$\begin{aligned}
|(B_n f)^{(\ell+s)}(x)| &\le \frac{n!}{(n-\ell-s)!} \sum_{k=0}^{n-\ell-s} \left| \overrightarrow{\Delta}_{1/n}^{\ell+s} f\left(\frac{k}{n}\right)\right| p_{n-\ell-s,k}(x) \\
&\le c\, n^{\ell+s} \max_{\nu=0,\dots,\ell} \sum_{k=0}^{n-\ell-s} \left| \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k+\nu}{n}\right)\right| p_{n-\ell-s,k}(x) \\
&\le c\, n^{\ell} \max_{\nu=0,\dots,\ell} \sum_{k=0}^{n-\ell-s} w_{\infty,n,k+\nu}\, p_{n-\ell-s,k}(x)\, \|wf^{(s)}\|_\infty.
\end{aligned}$$

To complete the proof we need only recall (4.80) with $p = \infty$, (4.8)-(4.9) and use the symmetry on k, see (4.79).

Assertion (b) follows from (a) just as in the previous proposition. □

Further, we shall state two analogues of the above Bernstein-type inequalities in terms of the differential operator $(D^r g)^s$. They directly follow from them and the embedding inequalities in Section 2.

Corollary 4.15. *Let $1 \le p \le \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *as*

$$\begin{aligned}
-1/p <& \gamma_0, \gamma_1 < s - 1/p \quad \text{if} \quad 1 \le p < \infty, \\
0 \le& \gamma_0, \gamma_1 < s \quad \text{if} \quad p = \infty.
\end{aligned}$$

Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\|w(D^r \mathcal{B}_{r,n} f)^{(s)}\|_p \le c\, n^r \|wf^{(s)}\|_p.$$

Proof. It can be established by induction on r that (cf. [18, p. 24])

$$D^r g = \varphi^2 \sum_{i=2}^{r+1} q_{r,i-2}\, g^{(i)} + \sum_{i=2}^{r} \varphi^{2i}\, \tilde{q}_{r,r-i}\, g^{(i+r)},$$

where $q_{r,j}, \tilde{q}_{r,j} \in \pi_j$. Hence we derive that

$$(D^r g)^{(s)} = \sum_{i=s'}^{r+s} \hat{q}_{r,s,i}\, g^{(i)} + \sum_{i=1}^{r} \varphi^{2i}\, \hat{q}_{r,s,r+s+i}\, g^{(i+r+s)}, \tag{4.83}$$

where $s' = \max\{2, s\}$ and $\hat{q}_{r,s,j}$ are polynomials. Hence

$$\|w(D^r g)^{(s)}\|_p \le c \sum_{i=s'}^{r+s} \|w g^{(i)}\|_p + \sum_{i=1}^{r} \|w\varphi^{2i} g^{(i+r+s)}\|_p.$$

The embedding inequality (2.10) yields for $i = s', \dots, r+s$

$$\begin{aligned}\|wg^{(i)}\|_p &\le c \left(\|wg^{(s')}\|_p + \|wg^{(r+s)}\|_p\right)\\ &\le c \left(\|wg^{(s)}\|_p + \|wg^{(r+s)}\|_p\right).\end{aligned}$$

Similarly, by means of (2.8) we get for $i = 1, \dots, r$

$$\|w\varphi^{2i} g^{(i+r+s)}\|_p \le c \left(\|wg^{(r+s)}\|_p + \|w\varphi^{2r} g^{(2r+s)}\|_p\right).$$

Consequently,

$$\begin{aligned}\|w(D^r g)^{(s)}\|_p &\le c \left(\|wg^{(s')}\|_p + \|wg^{(r+s)}\|_p + \|w\varphi^{2r} g^{(2r+s)}\|_p\right) && (4.84)\\ &\le c \left(\|wg^{(s)}\|_p + \|wg^{(r+s)}\|_p + \|w\varphi^{2r} g^{(2r+s)}\|_p\right) && (4.85)\end{aligned}$$

and the middle term can be omitted except when $p = \infty$ and $\gamma_0\gamma_1 = 0$ (see (4.44)).

Now, the assertion of the corollary follows from (4.85) with $g = \mathcal{B}_{r,n} f$ and Propositions 4.1, 4.13(b) and 4.14(b) with $\ell = r$. $\square$

Corollary 4.16. *Let $1 \le p \le \infty$, $r, s \in \mathbb{N}$ and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *as*

$$\begin{aligned}-1/p < &\gamma_0, \gamma_1 < s - 1/p \quad \textit{if} \quad 1 \le p < \infty,\\ 0 \le &\gamma_0, \gamma_1 < s \quad \textit{if} \quad p = \infty.\end{aligned}$$

Then for all $f \in C[0,1]$ such that $f \in AC^{2r+s-1}[0,1]$ and $w\varphi^{2r} f^{(2r+s)} \in L_p[0,1]$, and all $n \in \mathbb{N}$ there holds

$$\|w(D^{r+1}\mathcal{B}_{r,n} f)^{(s)}\|_p \le c\, n \, \|w(D^r f)^{(s)}\|_p.$$

Proof. Just as in the previous proof, we apply (4.84) with $r+1$ in place of r and $g = \mathcal{B}_{r,n} f$, Proposition 4.1, Proposition 4.13(b) with $\ell = 1$, $w\varphi^{2r}$ in place of w, and $2r+s$ in place of s, and Proposition 4.14(b) with $\ell = 1$ and $r+s$ in place of s to derive the estimate

$$\|w(D^{r+1}\mathcal{B}_{r,n} f)^{(s)}\|_p \le c \left(\|wf^{(s')}\|_p + n\,\|wf^{(r+s)}\|_p + n\,\|w\varphi^{2r} f^{(2r+s)}\|_p\right).$$

Note that the term $\|w(\mathcal{B}_{r,n} f)^{(r+s+1)}\|_p$ appears only in the case $p = \infty$ and $\gamma_0\gamma_1 = 0$.

Now, the assertion of the corollary follows from Proposition 2.4. $\square$

The estimates verified in this section enable us to prove Theorem 1.1.

Proof of Theorem 1.1. The direct estimate follows from Proposition 4.1(b) and Corollary 4.8 via a standard argument (see e.g. [7, Theorem 3.4]). Namely, for any $g \in C^{2r+s}[0,1]$ we have

$$\begin{aligned}\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_p &\le \|w(f^{(s)} - g^{(s)})\|_p + \|w(\mathcal{B}_{r,n}g - g)^{(s)}\|_p \\ &\quad + \|w(\mathcal{B}_{r,n}(f-g))^{(s)}\|_p \\ &\le c\left(\|w(f^{(s)} - g^{(s)})\|_p + \frac{1}{n^r}\|w(D^r g)^{(s)}\|_p\right).\end{aligned}$$

Taking an infimum on $g \in C^{2r+s}[0,1]$, we arrive at

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_p \le c\, K_{r,s}(f^{(s)}, n^{-r})_{w,p}.$$

To establish the converse estimate we apply [7, Theorem 3.2] with the operator $Q_n = \mathcal{B}_{r,n}$ on the space

$$X = \{f \in C[0,1] : f \in AC^{s-1}_{loc}(0,1), wf^{(s)} \in L_p[0,1]\}$$

with a semi-norm $\|f\|_X = \|wf^{(s)}\|_p$. Let us note that [7, Theorem 3.2] continues to hold for a semi-norm $\|\circ\|_X$ since in its proof the property that distinguishes a norm from a semi-norm is not used. Let also $Y = C^{2r+s}[0,1]$ and $Z = C^{2r+s+2}[0,1]$.

Proposition 4.1(b) implies that Q_n is a bounded operator on X, so that [7, (3.3)] holds.

By virtue of Corollary 4.12, we have for $\Phi(f) = \|w(D^{r+1}f)^{(s)}\|_p$ and $f \in Z$

$$\left\| w\left(Q_n f - f - \frac{(-1)^{r-1}}{(2n)^r}\, D^r f\right)^{(s)} \right\|_p \le \frac{c}{n^{r+1}}\,\Phi(f),$$

which shows that [7, (3.4)] is valid with $(-1)^{r-1}D^r$ in place of D, $\lambda(n) = (2n)^{-r}$ and $\lambda_1(n) = c\,n^{-r-1}$, where the constant c is the one from Corollary 4.12.

Further, we set $g = \mathcal{B}_{r,n}f$ for $f \in X$ and apply Corollary 4.16 to obtain

$$\Phi(Q_n^2 f) = \Phi(\mathcal{B}_{r,n}g) \le c\,n\,\|w(D^r g)^{(s)}\|_p = c\,n\,\|w(D^r\mathcal{B}_{r,n}f)^{(s)}\|_p.$$

Hence [7, (3.5)] is established with $m = 2$ and $\ell = 1$.

Finally, Corollary 4.15 yields for $f \in X$

$$\|w(D^r Q_n f)^{(s)}\|_p \le c\,n^r\|wf^{(s)}\|_p,$$

which is [7, (3.6)].

Now, [7, Theorem 3.2] implies the converse estimate in Theorem 1.1. □

5 Relations between K-functionals

In this section we shall verify the assertions of Theorems 1.2 and 1.3 as well as of Remark 1.4. First, we shall present a couple of auxiliary inequalities between K-functionals.

It is known that in the case $w = 1$ in the definition of $K_{m,\varphi}(f,t)_p$ the infimum can be equivalently taken on $C^m[0,1]$. That is evident from the proof of [8, Theorem 2.1.1] (see also [6, p. 110]). That equivalence probably holds for any Jacobi weight w. For our purposes weaker relations will suffice. They are given in the lemma below. Using them one can derive the above mentioned equivalence under the conditions of the lemma, but we shall not establish that here.

Lemma 5.1. *Let $1 < p \le \infty$, $r, s \in \mathbb{N}$, and $w = w(\gamma_0, \gamma_1)$ be given by* (1.3) *with $-1/p < \gamma_0, \gamma_1 < s - 1/p$. Then for all $wf \in L_p[0,1]$ and $0 < t \le 1$ there holds*

$$\begin{aligned}(5.1)\qquad \inf_{g\in C^{2r+s}[0,1]} &\left\{\|w(f-g^{(s)})\|_p + t\,\|w\varphi^{2r}g^{(2r+s)}\|_p\right\}\\ &\le c\,(K_{2r,\varphi}(f,t)_{w,p} + t\,\|wf\|_p)\,,\quad s\ge 2,\end{aligned}$$

and

$$\begin{aligned}(5.2)\qquad \inf_{g\in C^{2r+1}[0,1]} &\left\{\|w(f-g')\|_p + t\,\|w\varphi^{2r}g^{(2r+1)}\|_p\right\}\\ &\le c\,(K_{2r,\varphi}(f,t)_{w,p} + K_1(f,t)_{w,p})\,.\end{aligned}$$

Proof. For a given function f such that $wf \in L_p[0,1]$ with $\gamma_0, \gamma_1 < s - 1/p$, $s \in \mathbb{N}$, we set

$$f_s(x) = \frac{1}{(s-1)!}\int_{1/2}^{x}(x-u)^{s-1}f(u)\,du,\quad x\in[0,1].$$

Hölder's inequality implies that $\varphi^{2s-2}f \in L[0,1]$. Hence $f_s(x)$ is well-defined and finite at $x = 0$ and $x = 1$; moreover, $f_s \in C[0,1]$. The continuity of $f_s(x)$ at every interior point for any s as well as at $x = 0, 1$ for $s = 1$ is clear. To see that $f_s(x)$ is continuous at $x = 0, 1$ for $s \ge 2$ we can apply Lebesgue's dominated convergence theorem.

Now, we are ready to verify the inequalities in the lemma. We set $n = [t^{-1/r}] + 1$ and $g_t = \mathcal{B}_{r,n} f_s$. In view of the above remarks, g_t is well defined and clearly $g_t \in C^{2r+s}[0,1]$. To verify (5.1) and (5.2) it is enough to show that

$$(5.3)\qquad \|w(f-g_t^{(s)})\|_p \le c\,(K_{2r,\varphi}(f,t)_{w,p} + t\,\|wf\|_p)\,,\quad s\ge 2,$$

$$(5.4)\qquad \|w(f-g_t')\|_p \le c\,(K_{2r,\varphi}(f,t)_{w,p} + K_1(f,t)_{w,p})$$

and

$$(5.5)\qquad t\,\|w\varphi^{2r}g_t^{(2r+s)}\|_p \le c\,K_{2r,\varphi}(f,t)_{w,p},\quad s\ge 1.$$

Let $G \in AC_{loc}^{2r-1}(0,1)$ with $wG, w\varphi^{2r}G^{(2r)} \in L_p[0,1]$ be arbitrarily fixed. Then $G_s \in C[0,1]$. Let $s \geq 2$. Applying Proposition 4.1(b), Corollary 4.7 and the trivial estimate

$$\|wG\|_p \leq \|w(f-G)\|_p + \|wf\|_p, \tag{5.6}$$

we get

$$\begin{aligned}
\|w(f-g_t^{(s)})\|_p &\leq \|w(f-G)\|_p + \|w(\mathcal{B}_{r,n}G_s - G_s)^{(s)}\|_p \\
&\quad + \|w(\mathcal{B}_{r,n}(f_s - G_s))^{(s)}\|_p \\
&\leq c\,\|w(f-G)\|_p + \frac{c}{n^r}\left(\|wG_s^{(s)}\|_p + \|w\varphi^{2r}G_s^{(2r+s)}\|_p\right) \\
&\leq c\left(\|w(f-G)\|_p + t\,\|w\varphi^{2r}G^{(2r)}\|_p + t\,\|wf\|_p\right).
\end{aligned}$$

We take an infimum on G and arrive at (5.3).

For $s=1$ by means of a similar argument we arrive at

$$\|w(f-g_t')\|_p \leq c\left(\|w(f-G)\|_p + t\,\|w\varphi^{2r}G^{(2r)}\|_p + t\,\|wG'\|_p\right). \tag{5.7}$$

Next, we estimate the last term on the right above by means of Proposition 2.1 with $j=1$, $m=2r$, $w_1 = w$ and $w_2 = w\varphi^{2r}$. Thus we get

$$\|wG'\|_p \leq c\left(\|wG\|_p + \|w\varphi^{2r}G^{(2r)}\|_p\right).$$

Consequently, for an arbitrary real α we have

$$\|wG'\|_p \leq c\left(\|w(G-\alpha)\|_p + \|w\varphi^{2r}G^{(2r)}\|_p\right).$$

Setting $E_0(f)_{w,p} = \inf_{\alpha\in\mathbb{R}} \|w(f-\alpha)\|_p$, we arrive at the estimate

$$t\,\|wG'\|_p \leq c\left(\|w(f-G)\|_p + t\,\|w\varphi^{2r}G^{(2r)}\|_p\right) + ct\,E_0(f)_{w,p}. \tag{5.8}$$

For $wf \in L_p[0,1]$, $\gamma_0, \gamma_1 > -1/p$, $1 \leq p < \infty$ or $\gamma_0, \gamma_1 \geq 0$, $p = \infty$ and $0 < t \leq 1$ we have

$$t\,E_0(f)_{w,p} \leq c\,K_1(f,t)_{w,p}. \tag{5.9}$$

That easily follows from the estimate

$$E_0(g)_{w,p} \leq \|w(g - g(1/2))\|_p = \left\|w\int_{1/2}^{\circ} g'(t)\,dt\right\|_p \leq c\,\|wg'\|_p,$$

where $g \in AC_{loc}(0,1)$.

Combining (5.7)-(5.9) we arrive at (5.4).

Finally, to verify (5.5) we apply Proposition 4.13(c) with $\ell = r$ to get

$$t\,\|w\varphi^{2r}g_t^{(2r+s)}\|_p \leq c\,tn^r K_{2r,\varphi}(f_s^{(s)}, n^{-r})_{w,p} \leq c\,K_{2r,\varphi}(f,t)_{w,p}.$$

$\square$

Let us proceed now to the proof of Theorems 1.2 and 1.3.

Proof of Theorems 1.2 and 1.3. Let $-1/p < \gamma_0, \gamma_1 < s - 1/p$, $1 < p < \infty$ or $0 \le \gamma_0, \gamma_1 < s$, $p = \infty$. Let $g \in C^{2r+s}[0,1]$. Proposition 2.4 yields

$$\|wf\|_p \le \|w(f - g^{(s)})\|_p + c\,\|w(D^r g)^{(s)}\|_p, \quad s \ge 2, \tag{5.10}$$

$$\|wg^{(j+s)}\|_p \le c\,\|w(D^r g)^{(s)}\|_p, \quad j = 1, r, \quad s \ge 1, \tag{5.11}$$

and

$$\|w\varphi^{2r} g^{(2r+s)}\|_p \le c\,\|w(D^r g)^{(s)}\|_p, \quad s \ge 1. \tag{5.12}$$

Taking an infimum on $g \in C^{2r+s}[0,1]$ in (5.10) we get for $0 < t \le 1$

$$t\,\|wf\|_p \le c\,K_{r,s}(f,t)_{w,p}, \quad s \ge 2.$$

Next, since $g^{(s)} \in AC_{loc}^{j-1}(0,1)$ for $j = 1, r$, we derive from (5.11) that

$$K_j(f,t)_{w,p} \le c\left(\|w(f - g^{(s)})\|_p + t\,\|w(D^r g)^{(s)}\|_p\right), \quad s \ge 1.$$

Taking an infimum on $g \in C^{2r+s}[0,1]$ we arrive at

$$K_j(f,t)_{w,p} \le c\,K_{r,s}(f,t)_{w,p}, \quad j = 1, r, \quad s \ge 1.$$

Just similarly, using that $g^{(s)} \in AC_{loc}^{2r-1}(0,1)$ and (5.12), we establish that

$$K_{2r,\varphi}(f,t)_{w,p} \le c\,K_{r,s}(f,t)_{w,p}, \quad s \ge 1.$$

Thus we have shown that $K_{r,s}(f,t)_{w,p}$ estimates above the quantities on the right-hand side of the relations in Theorems 1.2 and 1.3.

Let us proceed to the reverse inequalities. Let $-1/p < \gamma_0, \gamma_1 < s - 1/p$, $1 < p \le \infty$. Let $g \in C^{2r+s}[0,1]$. By (4.84) (see also (4.44)), we have

$$\|w(D^r g)^{(s)}\|_p \le c\left(\|wg^{(s')}\|_p + \|w\varphi^{2r} g^{(2r+s)}\|_p\right), \tag{5.13}$$

where $s' = \max\{2, s\}$. Hence, using (5.6) with $g^{(s)}$ in place of G, we get for $s \ge 2$ the estimate

$$\|w(D^r g)^{(s)}\|_p \le c\left(\|w(f - g^{(s)})\|_p + \|w\varphi^{2r} g^{(2r+s)}\|_p + \|wf\|_p\right).$$

Consequently, for $s \ge 2$ we have

$$\begin{aligned} K_{r,s}(f,t)_{w,p} &\le c\left(\inf_{g \in C^{2r+s}[0,1]} \left\{\|w(f - g^{(s)})\|_p + t\,\|w\varphi^{2r} g^{(2r+s)}\|_p\right\} + t\,\|wf\|_p\right) \\ &\le c\,(K_{2r,\varphi}(f,t)_{w,p} + t\,\|wf\|_p). \end{aligned}$$

Here we have taken into account (5.1).

Similarly, relations (5.13) with $s = 1$, (5.8) with g' in place of G, (5.9) and (5.2) yield

$$K_{r,1}(f,t)_{w,p} \le c\,(K_{2r,\varphi}(f,t)_{w,p} + K_1(f,t)_{w,p})\,.$$

This completes the proof of Theorem 1.2.

To establish the upper estimate of $K_{r,s}(f,t)_{1,\infty}$ in Theorem 1.3 we use the quasi-interpolant $Q(f) = Q_T(f)$ constructed in the proof of [4, Chapter 6, Theorem 6.2] but with $2r$ in place of r and for the interval $[0,1]$ instead of $[-1,1]$. It has the properties (see [4, p. 191]):

$$\|f - Q(f)\|_\infty \le c\,\omega_\varphi^{2r}(f,t)_\infty$$

and

$$t^{2r}\|\varphi^{2r}Q(f)^{(2r)}\|_\infty \le c\,\omega_\varphi^{2r}(f,t)_\infty,$$

where $t = 1/m$, $m \in \mathbb{N}$ and $m \ge m_0$ with some fixed $m_0 \in \mathbb{N}$. Likewise, by means of [4, Chapter 5, Proposition 4.6 and Chapter 6, Theorem 4.2] we get

$$t^{2r}\|Q(f)^{(j)}\|_\infty \le c\,t^{2(r-j)}\omega_j(f,t^2)_\infty, \quad j = 1, r,$$

for $t = 1/m$, $m \in \mathbb{N}$ and $m \ge m_0$. Hence, taking into account the inequalities $\omega_j(f,t)_\infty \le c\,K_j(f,t^j)_\infty$ and $\omega_\varphi^{2r}(f,t)_\infty \le c\,K_{2r,\varphi}(f,t^{2r})_\infty$, we arrive at

$$\begin{aligned}
&\|f - Q(f)\|_\infty \le c\,K_{2r,\varphi}(f,t^{2r})_\infty,\\
&t^{2r}\|\varphi^{2r}Q(f)^{(2r)}\|_\infty \le c\,K_{2r,\varphi}(f,t^{2r})_\infty,\\
&t^{2r}\|Q(f)^{(j)}\|_\infty \le c\,K_j(f,t^{2r})_\infty, \quad j = 1, r,
\end{aligned} \tag{5.14}$$

for $t = 1/m$, $m \in \mathbb{N}$ and $m \ge m_0$.

Now, the upper estimate of $K_{r,s}(f,t)_{1,\infty}$ for all $t \in (0,1]$ follows from (4.84) with $p = \infty$, $w = 1$ and $g^{(s)} = Q(f)$, (5.6) with $Q(f)$ in place of G, or [4, Chapter 5, Theorem 4.4] (if $s \ge 2$), and the basic property of the K-functionals

$$K(f,t_1) \le \max\left\{1, \frac{t_1}{t_2}\right\} K(f,t_2), \tag{5.15}$$

where $K(f,t)$ stands for any of the considered here K-functionals.

Let us briefly show the validity of Remark 1.4. In view of what already has been established, it is enough to demonstrate that

$$K_{1,1}(f,t)_{w,\infty} \le c\,(K_{2,\varphi}(f,t)_{w,\infty} + K_1(f,t)_{w,\infty})$$

for $\gamma_0 > 0$ and $\gamma_1 = 0$. To this end we shall apply a well-known patching technique (see e.g. [4, p. 176]). By (2.6) with $p = \infty$, $3/4$ instead of $1/2$, g' in place of g, $\gamma_{1,0} = \gamma_0$,$\gamma_{2,0} = \gamma_0 + 1$, $j = 1$ and $m = 2$, we get

$$\|\chi^{\gamma_0}g''\|_{\infty[0,3/4]} \le c\left(\|\chi^{\gamma_0}g'\|_{\infty[0,3/4]} + \|\chi^{\gamma_0+1}g'''\|_{\infty[0,3/4]}\right).$$

Then, just as above, we derive that

$$t\,\|\chi^{\gamma_0}g''\|_{\infty[0,3/4]} \\ \le c\left(\|\chi^{\gamma_0}(f-g')\|_{\infty[0,3/4]}+t\,\|\chi^{\gamma_0+1}g'''\|_{\infty[0,3/4]}\right)+c\,K_1(f,t)_{w,\infty}.$$

Using the last inequality and (5.2) with $r=1$, we deduce that there exists $\tilde{g}_t\in C^3[0,3/4]$ such that

$$\|\chi^{\gamma_0}(f-\tilde{g}_t')\|_{\infty[0,3/4]}+t^2\|\chi^{\gamma_0}\tilde{g}_t''\|_{\infty[0,3/4]}+t^2\|\chi^{\gamma_0+1}\tilde{g}_t'''\|_{\infty[0,3/4]} \\ \le c\left(K_{2,\varphi}(f,t^2)_{w,\infty}+K_1(f,t^2)_{w,\infty}\right). \tag{5.16}$$

Further, let $\widetilde{Q}(f)=\widetilde{Q}_{\widetilde{T}}(f)$ be the quasi-interpolant considered above for $r=1$ and modified for the interval $[1/5,1]$. We set $\tilde{\varphi}(x)=\sqrt{(x-1/5)(1-x)}$ and denote by $K(f,t)_{\infty(J)}$ the modification of the K-functional $K(f,t)_\infty$, in which the sup-norm is taken on the interval J instead of $[0,1]$. Then (cf. (5.14)) we have

$$\begin{aligned}
&\|f-\widetilde{Q}(f)\|_{\infty[1/4,1]}\le\|f-\widetilde{Q}(f)\|_{\infty[1/5,1]}\\
&\qquad\le c\,K_{2,\tilde{\varphi}}(f,t^2)_{\infty[1/5,1]}\le c\,K_{2,\varphi}(f,t^2)_{w,\infty},\\
&t^2\|\varphi^2\widetilde{Q}(f)''\|_{\infty[1/4,1]}\le c\,t^2\|\tilde{\varphi}^2\widetilde{Q}(f)''\|_{\infty[1/5,1]}\\
&\qquad\le c\,K_{2,\tilde{\varphi}}(f,t^2)_{\infty[1/5,1]}\le c\,K_{2,\varphi}(f,t^2)_{w,\infty},\\
&t^2\|\widetilde{Q}(f)'\|_{\infty[1/4,1]}\le t^2\|\widetilde{Q}(f)'\|_{\infty[1/5,1]}\\
&\qquad\le c\,K_1(f,t^2)_{\infty[1/5,1]}\le c\,K_1(f,t^2)_{w,\infty}
\end{aligned} \tag{5.17}$$

for $t=1/m$, $m\in\mathbb{N}$ and $m\ge m_0$ with some fixed $m_0\in\mathbb{N}$.

Let the function $g_t\in C^3[0,1]$ be such that $g_t'=(1-\psi)\tilde{g}_t'+\psi\widetilde{Q}(f)$, where $\psi\in C^\infty(\mathbb{R})$, $\psi(x)=0$ for $x\le 1/4$ and $\psi(x)=1$ for $x\ge 3/4$. It can be shown by (4.84) with $r=s=1$ and $p=\infty$, (5.16) and (5.17) that (see [4, p. 176])

$$\|w(f-g_t')\|_\infty+t^2\|w(Dg_t)'\|_\infty\le c\left(K_{2,\varphi}(f,t^2)_{w,\infty}+K_1(f,t^2)_{w,\infty}\right)$$

for $t=1/m$, $m\in\mathbb{N}$ and $m\ge m_0$. In view of property (5.15) that completes the proof of Remark 1.4. $\square$

Let us explicitly note that the characterization of the weighted simultaneous approximation by $\mathcal{B}_{r,n}$ in terms of the K-functionals $K_{2r,\varphi}(f,t)_{w,p}$ and $K_j(f,t)_{w,p}$ can be directly derived from Proposition 4.1, Corollaries 4.7 and 4.11, Propositions 4.13(b) and 4.14(b) by means of [7, Theorems 3.2 and 3.4].

At the end we include a few remarks about the proof of Corollaries 1.5 and 1.6.

Proof of Corollaries 1.5 and 1.6. As it is shown in [8, Theorem 6.1.1], there exists t_0 such that $K_{2r,\varphi}(f,t^{2r})_{w,p}\le c\,\omega_\varphi^{2r}(f,t)_{w,p}$ and $K_j(f,t^j)_{w,p}\le c\,\omega_j(f,t)_{w,p}$

for $0 < t \le t_0$. Hence the assertions of the corollaries follow for $n \ge n_0$ with some $n_0 \in \mathbb{N}$ from Theorems 1.1-1.3. For $n < n_0$ we apply Proposition 4.1 to get

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_p \le \frac{c}{n^r}\,\|wf^{(s)}\|_p, \tag{5.18}$$

which completes the proof for $s \ge 2$. For $s = 1$ we use that $\mathcal{B}_{r,n}f$ preserves the linear functions to deduce from (5.18) the estimate

$$\|w(\mathcal{B}_{r,n}f - f)'\|_p \le \frac{c}{n^r}\,E_0(f')_{w,p}, \quad n < n_0.$$

Then we apply (5.9) with f' in place of f and the relation $K_1(f',t)_{w,p} \le c\,\omega_1(f',t)_{w,p}$, $0 < t \le 1$. $\square$

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Borislav R. Draganov
Dept. of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
Bulgaria
bdraganov@fmi.uni-sofia.bg

Inst. of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
Bulgaria