A new modulus of smoothness for trigonometric polynomial approximation

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Abstract. The rate of convergence of certain modified Riesz operators and their combinations is estimated. The rate of convergence in best trigonometric approximation in a homogeneous Banach space is characterized by a new K-functional. A new modulus of smoothness, connected with trigonometric approximation, is introduced.

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1 Introduction

The approximation by trigonometric polynomials is well investigated. The rate of convergence in uniform and integral norm was described by the classical modulus of smoothness due to D. Jackson, S. N. Bernstein, A. Zygmund and S. B. Stechkin (see for example [10]). This result was extended to any Banach space of 2π-periodic functions for which translation is continuous isometry by H. S. Shapiro and Z. Ditzian ([16] and [5]).

Let B be a homogeneous Banach space of 2π-periodic real-valued functions. We recall that B is a homogeneous Banach space if

$$(1.1)\quad \|f(\cdot+a)\|_B = \|f(\cdot)\|_B,\ \|f(\cdot+h)-f(\cdot)\|_B \to 0 \text{ as } h\to 0 \text{ and } \|f\|_{L_1} \le C\|f\|_B,$$

where $a \in \mathbf{R}$ and C is a constant independent of f. We denote by T_n the set of all trigonometric polynomials of degree at most n and put

$$(1.2)\qquad E_n^T(f)_B = \inf_{\tau\in T_n\cap B} \|f-\tau\|_B$$

for the best trigonometric approximation, as the trigonometric polynomials are dense in

B. It was shown by the authors mentioned above that for $f \in B$ we have

$$E_n^T(f)_B \le C_r \omega_r(f; n^{-1})_B,$$
$$\omega_r(f;t)_B \le C_r t^r \sum_{0\le k\le 1/t} (k+1)^{r-1} E_k^T(f)_B,\ 0 < t \le t_0,$$

where $\omega_r(f;t)_B$ is the classical modulus of smoothness defined as follows

$$\omega_r(f;t)_B := \sup_{0<h\le t} \|\Delta_h^r f\|_B\,,$$
$$\Delta_h^r f(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r/2 - k)h).$$

In the case of best algebraic approximation the rate of convergence has just recently been described through appropriate moduli of smoothness. Ditzian and Totik solved that problem in [7, Ch. 7] by means of a modulus with a varying step. Ivanov proposed in [8] another solution through an integral modulus of smoothness. A summary on other achievements in that area can be found in [7, Ch. 13].

Let B denote $L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $C^*[-\pi,\pi]$, where

$$L_p^*[-\pi,\pi] = \{f : \mathbf{R} \to \mathbf{R} : f(x+2\pi) = f(x) \text{ a.e., } f|_{[-\pi,\pi]} \in L_p[-\pi,\pi]\}$$
$$C^*[-\pi,\pi] = \{f \in C(\mathbf{R}) : f(x+2\pi) = f(x)\}.$$

It is interesting to construct a modulus-like function $\omega_r^T(f;t)_B$ associated with best trigonometric approximation such that for $f \in B$ we have

$$\omega_r^T(f;t)_B \equiv 0 \Longleftrightarrow f \in T_{r-1} \tag{1.3}$$

(here and further $f \in T_{r-1}$ in L_p-spaces means that f coincides a.e. with a trigonometric polynomial of degree at most $r-1$) and $\omega_r^T(f;t)_B$ characterizes the best trigonometric approximation like the classical modulus does. Thus, this new modulus of smoothness describes more precisely (in the sense of (1.3)) the rate of convergence of best trigonometric approximation than the classical one. The definition of this new modulus, as we should expect, is more complicated. We shall show that the modulus of smoothness defined by

$$\omega_r^T(f;t)_B := \sup_{0<h\le t} \|\Delta_h^{2r-1} \mathcal{F}_{r-1} f\|_B,\ r = 1, 2, \ldots, \tag{1.4}$$

where

$$\mathcal{F}_{r-1}(f,x) = f(x) + \int_0^x \mathcal{K}_{r-1}(t) f(x-t)\,dt$$

and

$$\mathcal{K}_{r-1}(t) = \sum_{j=1}^{r-1} \frac{a_j^{(r-1)}}{(2j-1)!} t^{2j-1}, \quad a_j^{(r-1)} = \sum_{1\le l_1<\cdots<l_j\le r-1} (l_1 \cdots l_j)^2,$$

satisfies (1.3) and the following theorem.

Theorem 1.1. *Let $f \in B$, where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$. Then*

$$E_n^T(f)_B \le C_r \omega_r^T(f; n^{-1})_B, \quad n \ge r-1, \tag{1.5}$$

and

$$\omega_r^T(f;t)_B \le C_r t^{2r-1} \sum_{r-1 \le k \le 1/t} (k+1)^{2r-2} E_k^T(f)_B, \quad 0 < t \le \frac{1}{r}. \tag{1.6}$$

Let us observe that although $\mathcal{F}_{r-1}f$ is not generally a 2π-periodic function $\Delta_h^{2r-1}\mathcal{F}_{r-1}f$ is. Unlike the various moduli which describe the best algebraic approximation $\omega_r^T(f;t)_B$ is based on finite differences only of an odd order. Thus $\omega_r^T(f;t)_B$ is connected with the $(2r-1)$th finite difference not the rth one. This is due to the dimensions of the spaces T_{r-1} and Π_{r-1}, respectively. To state our next main result we define the K-functional

$$K_r^T(f;t)_B := \inf_{g \in B^{2r-1}} \big\{ \|f-g\|_B + t^{2r-1} \|\widetilde{D}_r g\|_B \big\}, \tag{1.7}$$

where we have put $B^s = \{g \in B : g, g', \dots, g^{(s-1)} \in AC^*[-\pi,\pi], g^{(s)} \in B\}$, $AC^*[-\pi,\pi]$ being the set of all 2π-periodic absolutely continuous functions, $\widetilde{D}_r g = D_{r-1} \cdots D_1 g'$ and $D_j g = g'' + j^2 g$. We write $\varphi(f;t) \sim \psi(f;t)$ if and only if there exists a constant $C > 0$ independent of f and t such that $C^{-1}\varphi(f;t) \le \psi(f;t) \le C\varphi(f;t)$. We shall prove

Theorem 1.2. *For $f \in B$, where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$, we have*

$$K_r^T(f;t)_B \sim \omega_r^T(f;t)_B.$$

Hence

Theorem 1.3. *Let $f \in B$, where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$. Then*

$$E_n^T(f)_B \le C_r K_r^T(f;t)_B, \quad n \ge r-1,$$

and

$$K_r^T(f;t)_B \le C_r t^{2r-1} \sum_{r-1 \le k \le 1/t} (k+1)^{2r-2} E_k^T(f)_B, \quad 0 < t \le \frac{1}{r}.$$

Analogous problem can be posed in connection to the interpolation by trigonometric polynomials. G. P. Nevai proved in [12] the following generalization of a result of S. M. Nikolskii

$$|f(x) - t_n(f,x)| \le 2^{-r} \omega_r\Big(f; \frac{2\pi}{2n+1}\Big)_\infty \lambda_n(\bar{x}) + \mathcal{O}\big(\omega_r(f;n^{-1})_\infty\big),$$

where $t_n(f,x) \in T_n$ interpolates $f \in C[-\pi,\pi]$ in the equidistant nodes $\bar{x} = (x_{-n}, \dots, x_n)$, $x_k = 2k\pi/(2n+1)$, $k = -n, \dots, n$, and $\lambda_n(\bar{x})$ is the Lebesgue constant for trigonometric

Lagrange interpolation. Having verified (1.5), the well-known Lebesgue inequality for bounded linear operators that preserves trigonometric polynomials yields

$$|f(x) - t_n(f,x)| \le C_r(1+\lambda(\bar{x}))\,\omega_r^T(f;n^{-1})_\infty$$

for any set of interpolation nodes $\bar{x}$.

The contents of the paper are organized as follows. We investigate the rate of approximation of a certain modified Riesz operator in Section 2. Next we introduce operators that preserve the trigonometric polynomials of a given degree and characterize their rate of approximation in Section 3. In the next section we define the modulus $\omega_r^T(f;t)_B$ for L_p and C spaces of 2π-periodic functions and prove its properties. In Section 5 we characterize the rate of best trigonometric approximation in L_p and C norm through it. In the same section we point out a characterization of the rate of best trigonometric approximation in any homogeneous Banach space by an appropriate K-functional.

2 A modified Riesz operator

Let B be a Banach space of 2π-periodic functions such that

$$\|f(\cdot + a)\|_B = \|f\|_B \quad \text{and} \quad \|f\|_{L_1} \le C\|f\|_B, \tag{2.1}$$

where $a \in \mathbf{R}$ and C is a constant independent of f.

We define the differential operators

$$D_j g = g'' + j^2 g, \quad j = 1, 2, \dots \tag{2.2}$$

for $g \in B^2$, where we have put $B^s = \{g \in B : g, g', \dots, g^{(s-1)} \in AC^*[-\pi,\pi], g^{(s)} \in B\}$ and $AC^*[-\pi,\pi]$ is the set of all 2π-periodic absolutely continuous functions. We also define the K-functional

$$K_j(f;t)_B := \inf_{g\in B^2}\big\{\|f-g\|_B + t^2\|D_j g\|_B\big\}, \quad t > 0. \tag{2.3}$$

Let $A_k(x) = A_k(f,x)$ be the k^{th} term in the Fourier expansion of f:

$$\begin{aligned}
&A_0(x) = \frac{a_0}{2},\\
&A_k(x) = a_k \cos kx + b_k \sin kx,\ k = 1, 2, \dots,\\
&a_k = a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt,\ k = 0, 1, 2, \dots,\\
&b_k = b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt,\ k = 1, 2, \dots .
\end{aligned}$$

In this section we consider the approximation behaviour of the following modification of the typical Riesz means:

$$R_{j,n}(f,x) := \sum_{k=0}^{n-1}\Big(1 - \frac{k^2-j^2}{n^2-j^2}\Big)A_k(x),\ n > j,\ j = 1, 2, \dots . \tag{2.4}$$

Theorem 2.1. *For a Banach space B satisfying* (2.1)*, the K-functional $K_j(f;t)_B$ and the Riesz means $R_{j,n}(f,x)$, we have*

$$\|f - R_{j,n}f\|_B \sim K_{j,n}(f;n^{-1})_B, \quad n > j.$$

Proof. We follow the proof of similar results of Ditzian and Ivanov in [6] and of Ditzian in [4]. The statement of the theorem follows from the relations

$$\|R_{j,n}f\|_B \le M_j \|f\|_B, \quad f \in B, \ n > j, \tag{2.5}$$

$$D_j R_{j,n} g = R_{j,n} D_j g \quad \text{for } g \in B^2, n > j, \tag{2.6}$$

and

$$D_j R_{j,n} f = (j^2 - n^2) R_{j,n}(f - R_{j,n}f), \ n > j. \tag{2.7}$$

First we deal with (2.5). We observe that

$$R_{j,n}f = \frac{n^2}{n^2 - j^2} \sum_{k=0}^{n-1} \Big(1 - \frac{k^2}{n^2} A_k\Big).$$

But it was shown in [6] by means of an assertion in [9] that the operators

$$R_n f = \sum_{k=0}^{n-1} \Big(1 - \frac{k^2}{n^2} A_k\Big)$$

are uniformly bounded in n. Consequently, this holds for $R_{j,n}f$ as well.

Next (2.6) follows from $(d/dx)A_k(g,x) = A_k(g',x), \ k = 0,1,2,\dots,$ for $g \in B^1$.

It remains to verify (2.7). We shall prove just a little bit more general result which we shall need later, namely we have

$$D_i R_{j,n} f = (i^2 - n^2) R_{j,n}(f - R_{i,n}f), \quad n > i, j. \tag{2.8}$$

We have

$$\begin{aligned} D_i R_{j,n} f &= \sum_{k=0}^{n-1} (i^2 - k^2) \Big(1 - \frac{k^2 - j^2}{n^2 - j^2}\Big) A_k \\ &= (n^2 - i^2) \Bigg(\sum_{k=0}^{n-1} \Big(1 - \frac{k^2 - j^2}{n^2 - j^2}\Big)\Big(1 - \frac{k^2 - i^2}{n^2 - i^2}\Big) A_k - \sum_{k=0}^{n-1} \Big(1 - \frac{k^2 - j^2}{n^2 - j^2}\Big) A_k \Bigg) \\ &= (i^2 - n^2) R_{j,n}(f - R_{i,n}f). \end{aligned}$$

This establishes (2.8) and hence (2.7).

(a) The direct result

The fact that the trigonometric polynomials are dense in B, (2.5) and the relations

$$\lim_{n\to\infty} R_{j,n} f_k = \lim_{n\to\infty} \Big(1 - \frac{k^2 - j^2}{n^2 - j^2}\Big) f_k = f_k,$$

$$\|f - R_{j,n}f\|_B \le \|R_{j,n}(f - \tau)\|_B + \|f - \tau\|_B + \|\tau - R_{j,n}\tau\|_B,$$

where $f_k(x) = \cos kx$ or $f_k(x) = \sin kx$ and $\tau \in B$ is a trigonometric polynomial, imply

$$\lim_{n\to\infty} \|f - R_{j,n}f\|_B = 0. \tag{2.9}$$

For any $g \in B^2$ we have

$$\|f - R_{j,n}f\|_B \le (M_j + 1)\|f - g\|_B + \|g - R_{j,n}g\|_B. \tag{2.10}$$

To estimate the second term we write, using (2.5) and (2.9),

$$\begin{aligned} \|g - R_{j,n}g\|_B &\le \|R_{j,n}g - R_{j,n}^2 g\|_B + \|g - R_{j,n}^2 g\|_B \\ &\le \|R_{j,n}g - R_{j,n}^2 g\|_B + \sum_{m=n}^{\infty} \|R_{j,m}^2 g - R_{j,m+1}^2 g\|_B. \end{aligned} \tag{2.11}$$

Next (2.6) and (2.7) yield

$$\|R_{j,n}g - R_{j,n}^2 g\|_B \le \frac{M_j}{n^2 - j^2}\|D_j g\|_B. \tag{2.12}$$

Since $R_{j,m}R_{j,l} = R_{j,l}R_{j,m}$, we have

$$\|R_{j,m}^2 g - R_{j,m+1}^2 g\|_B \le \|R_{j,m}^2 g - R_{j,m+1}R_{j,m}g\|_B + \|R_{j,m+1}^2 g - R_{j,m}R_{j,m+1}g\|_B.$$

As we established (2.8) we verify the relations

$$\begin{aligned} R_{j,m}^2 g - R_{j,m+1}R_{j,m}g &= \frac{2m+1}{[(m+1)^2 - j^2](m^2 - j^2)} D_j R_{j,m} g \\ R_{j,m+1}^2 g - R_{j,m}R_{j,m+1}g &= -\frac{2m+1}{[(m+1)^2 - j^2](m^2 - j^2)} D_j R_{j,m+1} g \end{aligned}$$

which together with (2.5) and (2.6) yield

$$\begin{aligned} \sum_{m=n}^{\infty} \|R_{j,m}^2 g - R_{j,m+1}^2 g\|_B &\le C_j \|D_j g\|_B \sum_{m=n}^{\infty} \frac{1}{m^3} \\ &\le \frac{C_j}{n^2}\|D_j g\|_B. \end{aligned}$$

This and (2.12) imply from (2.11)

$$\|g - R_{j,n}g\|_B \le \frac{C_j}{n^2}\|D_j g\|_B, \quad n > j,\ g \in B^2. \tag{2.13}$$

Finally, (2.10) and the inequality above yield the direct estimate

$$\|f - R_{j,n}f\|_B \le C_j K_j(f, n^{-1})_B.$$

(*b*) The converse result

Relations (2.5) and (2.7) yield

$$\|D_j R_{j,n} f\|_B \le C_j n^2 \|f - R_{j,n}f\|_B$$

and therefore

$$K_j(f;n^{-1})_B \leq C_j \|f - R_{j,n}f\|_B,$$

where C_j is a constant independent of f and n. This completes the proof of the theorem. □

Remark 2.2. Let $B = L_p^*[-\pi,\pi]$, $1 \leq p < \infty$, the space of all 2π-periodic functions f such that

$$\|f\|_B = \left(\int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty,$$

or $B = C^*[-\pi,\pi]$, the space of all continuous 2π-periodic functions with the sup-norm

$$\|f\|_B = \sup_{x\in[-\pi,\pi]} |f(x)| \, .$$

Then it can be shown as in Section 4 that

$$\|f - R_{j,n}f\|_B \sim \sup_{0<h\leq 1/n} \|\Delta_h^2 A_j f\|_B, \; n > j, \tag{2.14}$$

where

$$\Delta_h^2 F(x) = F(x+h) - 2F(x) + F(x-h)$$

and

$$A_j(f,x) = f(x) + j^2 \int_0^x (x-t)f(t)\, dt, \; x \in \mathbf{R}. \tag{2.15}$$

In passing let us note that $A_j f$ is not 2π-periodic for every $f \in B$ but $\Delta_h^2 A_j f$ is.

3 Operators that preserve the trigonometric polynomials of a given degree

In this section we consider the linear operator $L_{r-1,n} : B \to T_{n-1}, \; r,n \in \mathbf{N}, \; 1 \leq r \leq n$, defined by

$$L_{r-1,n} = I - \prod_{i=0}^{r-1}(I - R_{j,n}) = \sum_{i=0}^{r-1}(-1)^i \sum_{0\leq j_0<\cdots<j_i\leq r-1} R_{j_0,n} \cdots R_{j_i,n}, \tag{3.1}$$

where I is the identity and $R_{j,n}$ is given in (2.4). We shall characterize the approximation behaviour of $L_{r-1,n}$ by means of the K-functional

$$K_r'(f;t)_B = \inf_{g\in B^{2r}} \left\{ \|f-g\|_B + t^{2r} \|D_r' g\|_B \right\}, \; t > 0, \tag{3.2}$$

where $D_r' = D_{r-1} \cdots D_0$ and $D_j, \; j = 0,\ldots,r-1$, are given in (2.2). The following theorem holds.

Theorem 3.1. *For a Banach space B satisfying* (2.1) *we have*

$$\|f - L_{r-1,n}f\|_B \sim K'_r(f;n^{-1})_B,$$

where $L_{r-1,n}$ and $K'_r(f;t)_B$ are given in (3.1) *and* (3.2), *respectively.*

Proof. We follow the same steps as in the proof of Theorem 2.1.

(a) The direct result

First let us note that due to (2.5) for $j = 0, \dots, r-1$ the operator $L_{r-1,n}$ is bounded:

$$\|L_{r-1,n}f\|_B \le C_r\|f\|_B, \quad f \in B,\ n \ge r. \tag{3.3}$$

Next, as $(d/dx)A_k(g,x) = A_k(g',x),\ k = 0,1,2,\dots$ for $g \in B^1$, we have for $n > j$

$$D_iR_{j,n}g = R_{j,n}D_ig, \quad g \in B^2. \tag{3.4}$$

Besides, Theorem 2.1 yields

$$\|g - R_{j,n}g\|_B \le \frac{c_j}{n^2}\|D_jg\|_B, \quad g \in B^2. \tag{3.5}$$

Therefore, applying consecutively (3.4) and (3.5) for $j = 0, \dots, r-1$, we get for $g \in B^{2r}$ and $n \ge r$

$$\begin{aligned}
\|g - L_{r-1,n}g\|_B &= \left\|\prod_{j=0}^{r-1}(I - R_{j,n})g\right\|_B \\
&\le \frac{c_0}{n^2}\left\|D_0\prod_{j=1}^{r-1}(I - R_{j,n})g\right\|_B \\
&= \frac{c_0}{n^2}\left\|\prod_{j=1}^{r-1}(I - R_{j,n})D_0g\right\|_B \\
&\dots \\
&\le \frac{c_0\dots c_{r-1}}{n^{2r}}\|D_{r-1}\cdots D_0g\|_B.
\end{aligned}$$

Thus we have got

$$\|g - L_{r-1,n}g\|_B \le \frac{C_r}{n^{2r}}\|D'_rg\|_B, \quad g \in B^{2r},\ n \ge r, \tag{3.6}$$

where C_r depends only on r. Now inequalities (3.3) and (3.6) yield

$$\|f - L_{r-1,n}f\|_B \le C_rK'_r(f;n^{-1})_B.$$

(b) The converse result

The converse inequality follows from (2.8) and (3.3) as in the proof of Theorem 2.1. We also use that $R_{j,n}R_{i,n} = R_{i,n}R_{j,n}$ and the relation (3.4). For $f \in B$ and $n \ge r$, we have

$$
\begin{aligned}
D'_r L_{r-1,n} f &= D'_r \sum_{i=0}^{r-1} (-1)^i \sum_{0\le j_0<\cdots<j_i\le r-1} R_{j_0,n}\cdots R_{j_i,n} f \\
&= D'_{r-1} \sum_{i=0}^{r-1} (-1)^i \sum_{0\le j_0<\cdots<j_i\le r-1} R_{j_0,n}\cdots R_{j_{i-1},n} D_{r-1} R_{j_i,n} f \\
&= ((r-1)^2-n^2) D'_{r-1} \sum_{i=0}^{r-1} (-1)^i \sum_{0\le j_0<\cdots<j_i\le r-1} R_{j_0,n}\cdots R_{j_i,n}(I-R_{r-1,n}) f \\
&\cdots \\
&= \prod_{l=0}^{r-1}(l^2-n^2) \sum_{i=0}^{r-1} (-1)^i \sum_{0\le j_0<\cdots<j_i\le r-1} R_{j_0,n}\cdots R_{j_i,n} \prod_{l=0}^{r-1}(I-R_{l,n}) f.
\end{aligned}
$$

So we have shown that, for $f \in B$ and $n \ge r$,

$$
D'_r L_{r-1,n} f = \Big(\prod_{l=0}^{r-1}(l^2-n^2)\Big) L_{r-1,n}(f - L_{r-1,n} f). \tag{3.7}
$$

Hence, as in the proof of the converse inequality in Theorem 2.1, we get

$$
K'_r(f, n^{-1})_B \le C_r \|f - L_{r-1,n} f\|_B.
$$

Thus the proof of Theorem 3.1 is completed. □

Remark 3.2. As $f \in T_{r-1}$ implies $D'_r f = 0$, we have $L_{r-1,n} f = f$ for $f \in T_{r-1}$, $n \ge r$.

4 A new periodic modulus of smoothness

Let $[a,b]$ be an arbitrary finite subinterval of the real line such that $0 \in [a,b]$. We write $X = X[a,b]$ for any of the function spaces $L_p[a,b]$, $1 \le p < \infty$, or $C[a,b]$ and $X^r = X^r[a,b]$ for the Sobolev spaces $W_p^r[a,b]$, $1 \le p < \infty$, or $C^r[a,b]$. We also write B for $L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $C^*[-\pi,\pi]$, where, we recall,

$$
\begin{aligned}
L_p^*[-\pi,\pi] &= \{f : \mathbf{R} \to \mathbf{R} : f(x+2\pi) = f(x) \text{ a.e., } f|_{[-\pi,\pi]} \in L_p[-\pi,\pi]\}, \\
C^*[-\pi,\pi] &= \{f \in C(\mathbf{R}) : f(x+2\pi) = f(x)\}.
\end{aligned}
$$

We write B^r for $W_p^{*r}[-\pi,\pi]$, $1 \le p < \infty$ or $C^{*r}[-\pi,\pi]$, where

$$
\begin{aligned}
W_p^{*r}[-\pi,\pi] &= \{f \in L_p^*[-\pi,\pi] : f, f', \ldots, f^{(r-1)} \in AC^*[-\pi,\pi], f^{(r)} \in L_p^*[-\pi,\pi]\}, \\
C^{*r}[-\pi,\pi] &= \{f \in C^*[-\pi,\pi] : f^{(k)} \in C^*[-\pi,\pi],\ s = 1, \ldots, r\}.
\end{aligned}
$$

We define the convolutional operator known as Duhamel's convolution $\circledast : L_1[a,b] \times L_1[a,b] \to L_1[a,b]$,

$$
f \circledast g(x) := \int_0^x f(x-t) g(t)\, dt. \tag{4.1}
$$

It is easy to verify that it possesses the properties:

1. $f \circledast g = g \circledast f$;
2. $f \circledast (g+h) = f \circledast g + f \circledast h$;
3. $f \circledast (g \circledast h) = (f \circledast g) \circledast h$.

First we shall consider the linear bounded operator $A_j : X \to X$ defined by

$$A_j(f,x) = f(x) + j^2 \int_0^x (x-t) f(t)\,dt, \quad j = 1,2,\dots \tag{4.2}$$

If we put, for $f \in X$,

$$\mathcal{I}_\alpha(f,x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad x \in [a,b],\ \alpha > 0, \tag{4.3}$$

where $\Gamma(\alpha)$ is the gamma-function, then A_j can be represented in the form

$$A_j = I + j^2 \mathcal{I}_2, \tag{4.4}$$

I being the identity. Hence, it follows that $A_j A_k = A_k A_j$.

Proposition 4.1. *The bounded linear operator A_j is invertible and*

$$A_j^{-1}(g,x) = g(x) - j \int_0^x \sin j(x-t) g(t)\,dt.$$

Hence

$$A_j^{-1}(g,x) = \frac{1}{j} \int_0^x \sin j(x-t) g''(t)\,dt$$

for $g \in X^2$ with $g(0) = g'(0) = 0$.

Proof. By definition we have

$$A_j f = f + j^2 e_1 \circledast f, \quad e_1(x) = x,$$

and let

$$B_j g = g - j\varphi_j \circledast g, \quad \varphi_j(x) = \sin jx.$$

We have to show that $A_j B_j = B_j A_j = I$, where I is the identity. Using the properties of $\circledast$ we get

$$A_j(B_j g) = B_j g + j^2 e_1 \circledast B_j g = g + (j^2 e_1 - j\varphi_j - j^3 e_1 \circledast \varphi_j) \circledast g$$

and

$$B_j(A_j f) = A_j f - j\varphi_j \circledast A_j f = f + (j^2 e_1 - j\varphi_j - j^3 e_1 \circledast \varphi_j) \circledast f.$$

But since

$$j^3 e_1 \circledast \varphi_j(x) = j^3 \int_0^x (x-t)\sin jt\,dt = j^2 x - j\sin jx = j^2 e_1(x) - j\varphi_j(x),$$

we get that $j^2 e_1 - j\varphi_j - j^3 e_1 \circledast \varphi_j = 0$ and hence, $A_j B_j = B_j A_j = I$.

The second statement of the proposition follows from the first one by integration by parts. □

Next we define the bounded linear operator $\mathcal{F}_r : X \to X$ by

$$\mathcal{F}_r = A_1 \cdots A_r. \tag{4.5}$$

The following representation of $\mathcal{F}_r$ holds true.

Proposition 4.2. *The operator $\mathcal{F}_r$ can be represented in the form*

$$\mathcal{F}_r = I + \sum_{j=1}^{r} a_j^{(r)} \mathcal{I}_{2j}, \quad a_j^{(r)} = \sum_{1 \le l_1 < \cdots < l_j \le r} (l_1 \cdots l_j)^2, \tag{4.6}$$

where $\mathcal{I}_{2j}$ are defined in (4.3). *Hence, for every $f \in X$,*

$$\mathcal{F}_r(f,x) = f(x) + \int_0^x \mathcal{K}_r(t) f(x-t)\,dt, \tag{4.7}$$

where

$$\mathcal{K}_r(t) = \sum_{j=1}^{r} \frac{a_j^{(r)}}{(2j-1)!} t^{2j-1}. \tag{4.8}$$

Proof. We shall prove the assertion by induction. The representation (4.6) is trivial for $r = 1$. Let us assume that

$$\mathcal{F}_{r-1} = I + \sum_{j=1}^{r-1} a_j^{(r-1)} \mathcal{I}_{2j}.$$

Then we have, using also (4.4) and the fact that $\mathcal{I}_j \mathcal{I}_k = \mathcal{I}_{j+k}$,

$$\begin{aligned}
\mathcal{F}_r = \mathcal{F}_{r-1} A_r &= (I + \sum_{j=1}^{r-1} a_j^{(r-1)} \mathcal{I}_{2j})(I + r^2 \mathcal{I}_2) \\
&= I + \sum_{j=1}^{r-1} a_j^{(r-1)} \mathcal{I}_{2j} + r^2 \mathcal{I}_2 + r^2 \sum_{j=1}^{r-1} a_j^{(r-1)} \mathcal{I}_{2(j+1)} \\
&= I + (a_1^{(r-1)} + r^2)\mathcal{I}_2 + \sum_{j=2}^{r-1} (a_j^{(r-1)} + r^2 a_{j-1}^{(r-1)}) \mathcal{I}_{2j} + r^2 a_{r-1}^{(r-1)} \mathcal{I}_{2r},
\end{aligned}$$

To finish the proof we only need to observe that

$$a_1^{(r-1)} + r^2 = \sum_{j=1}^{r-1} j^2 + r^2 = a_1^{(r)};$$

$$\begin{aligned}
a_j^{(r-1)} + r^2 a_{j-1}^{(r-1)} &= \sum_{1\le l_1<\cdots<l_j\le r-1} (l_1\cdots l_j)^2 + r^2 \sum_{1\le l_1<\cdots<l_{j-1}\le r-1} (l_1\cdots l_{j-1})^2 \\
&= \sum_{1\le l_1<\cdots<l_j\le r-1} (l_1\cdots l_j)^2 + \sum_{1\le l_1<\cdots<l_{j-1}<r} (l_1\cdots l_{j-1} r)^2 \\
&= \sum_{1\le l_1<\cdots<l_j\le r} (l_1\cdots l_j)^2 = a_j^{(r)}
\end{aligned}$$

and

$$r^2 a_{r-1}^{(r-1)} = [(r-1)!]^2 r^2 = (r!)^2 = a_r^{(r)},$$

hence

$$\mathcal{F}_r = I + \sum_{j=1}^{r} a_j^{(r)} \mathcal{I}_{2j}.$$

The representation (4.7) follows immediately from (4.6) and the definition of $\mathcal{I}_{2j}$ in (4.3). $\square$

Next we point out some properties of the kernel $\mathcal{K}_r(t)$ defined in (4.8).

Proposition 4.3. *The following recursion relation for the kernel $\mathcal{K}_r(t)$ holds:*

$$\begin{aligned}
&\mathcal{K}_r(t) = \mathcal{K}_{r-1}(t) + r^2 \int_0^t (t-s)\mathcal{K}_{r-1}(s)\,ds + r^2 t, \\
&\mathcal{K}_1(t) = t.
\end{aligned}$$

We also have

$$\mathcal{K}_r'' = D_r\mathcal{K}_{r-1} \text{ and } \mathcal{K}_r(t) = \int_0^t (t-s) D_r \mathcal{K}_{r-1}(s)\,ds + \frac{1}{6} r(r+1)(2r+1)t,$$

hence

$$\mathcal{K}_r'' = \sum_{j=1}^{r-1} (j+1)^2 \mathcal{K}_j \text{ and } \mathcal{K}_r(t) = \sum_{j=1}^{r-1} (j+1)^2 \int_0^t (t-s)\mathcal{K}_j(s)\,ds + \frac{1}{6} r(r+1)(2r+1)t.$$

Remark 4.4. If we put $\mathcal{K}_r(t) = t\widetilde{\mathcal{K}}_r(t)$ the recursion relation above can be rewritten as

$$\begin{aligned}
&\widetilde{\mathcal{K}}_r(t) = \widetilde{\mathcal{K}}_{r-1}(t) + r^2 t^2 \int_0^1 u(1-u)\widetilde{\mathcal{K}}_{r-1}(tu)\,du + r^2, \\
&\widetilde{\mathcal{K}}_1(t) = 1.
\end{aligned}$$

Proof of Proposition 4.3. We only have to verify the first part of the statement in the proposition above. Using the notation (4.1), (4.2) and (4.7) can be written respectively as

$$A_r f = f + r^2 e_1 \circledast f, \quad e_1(x) = x \tag{4.9}$$

and

$$\mathcal{F}_r f = f + \mathcal{K}_r \circledast f. \tag{4.10}$$

Then, from the definition of $\mathcal{F}_r$ (see (4.5)) and the properties of $\circledast$, follow

$$\begin{aligned} \mathcal{F}_r f &= \mathcal{F}_{r-1} A_r f = A_r f + \mathcal{K}_{r-1} \circledast A_r f \\ &= f + (\mathcal{K}_{r-1} + r^2 e_1 \circledast \mathcal{K}_{r-1} + r^2 e_1) \circledast f. \end{aligned}$$

Comparing (4.7) and the above, we conclude that

$$(\mathcal{K}_r - \mathcal{K}_{r-1} - r^2 e_1 \circledast \mathcal{K}_{r-1} - r^2 e_1) \circledast f = 0$$

for any $f \in X$. Hence, it easily follows that

$$\mathcal{K}_r - \mathcal{K}_{r-1} - r^2 e_1 \circledast \mathcal{K}_{r-1} - r^2 e_1 = 0,$$

which is exactly the first recursion relation in Proposition 4.3. $\square$

Remark 4.5. We put

$$P_r(x) = \prod_{j=1}^{r} (x^2 + j^2) = x^{2r} + \sum_{j=1}^{r} a_j^{(r)} x^{2(r-j)}$$

and $S_r(x,t) = P_r(x) e^{xt}$. We also put $\widetilde{D}_r g = D_{r-1} \cdots D_1 g'$ for $g \in X^{2r-1}$, where D_j, $j = 1, \dots, r-1$, are defined in (2.2). Let us note that $\widetilde{D}_r = P_{r-1}(d/dx) d/dx$. Obviously,

$$\begin{aligned} S_{r+1}(x,t) &= (x^2 + (r+1)^2) S_r(x,t), \\ S_1(x,t) &= (x^2+1) e^{xt}. \end{aligned}$$

It is easy to verify the relation

$$\mathcal{K}_r(t) = \frac{1}{(2r-1)!} \left(\frac{\partial}{\partial x} \right)^{2r-1} S_r(x,t) \bigg|_{x=0}.$$

This also implies the statement of Proposition 4.3.

Next we consider the inverse operator of $\mathcal{F}_r$, $\mathcal{F}_r^{-1} = A_1^{-1} \cdots A_r^{-1}$. The properties stated in the next proposition were pointed to the author by K. G. Ivanov.

Proposition 4.6. *For $A_j^{-1} : X \to X$, the inverse operator of A_j, we have* *(a)* $A_k^{-1}A_j^{-1} = \dfrac{j^2}{j^2-k^2}A_j^{-1} + \dfrac{k^2}{k^2-j^2}A_k^{-1}, \quad k \neq j;$

(b) $A_1^{-1}\cdots A_r^{-1} = 2\displaystyle\sum_{j=1}^{r}\frac{j^{2r-1}}{\omega_r'(j)}A_j^{-1}$, *where* $\omega_r(x) = \displaystyle\prod_{k=1}^{r}(x^2-k^2)$;

(c) $\mathcal{F}_r^{-1}(g,x) = g(x) - \displaystyle\int_0^x \mathcal{L}_r(t)g(x-t)\,dt,$

where

$$\mathcal{L}_r(t) = 2\sum_{j=1}^{r}\frac{j^{2r}}{\omega_r'(j)}\sin jt. \tag{4.11}$$

Proof. To verify (a) we use that $A_j^{-1}g = g - j\varphi_j \circledast g$, where $\varphi_j(x) = \sin jx$, and write

$$\begin{aligned} A_k^{-1}A_j^{-1}g &= A_j^{-1}g - k\varphi_k \circledast A_j^{-1}g \\ &= g - j\varphi_j \circledast g - k\varphi_k \circledast g + kj\varphi_j \circledast \varphi_k \circledast g. \end{aligned}$$

Then, as $\varphi_j \circledast \varphi_k = (k^2-j^2)^{-1}(k\varphi_j - j\varphi_k)$, we get

$$\begin{aligned} A_k^{-1}A_j^{-1}g &= \Big(\frac{j^2}{j^2-k^2} + \frac{k^2}{k^2-j^2}\Big)g - \frac{j^2}{j^2-k^2}j\varphi_j \circledast g - \frac{k^2}{k^2-j^2}k\varphi_k \circledast g \\ &= \frac{j^2}{j^2-k^2}(g - j\varphi_j \circledast g) + \frac{k^2}{k^2-j^2}(g - k\varphi_k \circledast g). \end{aligned}$$

Thus (a) is established.

We prove (b) by induction in r but first we need to observe that

$$2\sum_{j=1}^{r}\frac{j^{2r-1}}{\omega_r'(j)} = 1. \tag{4.12}$$

The equality follows from the fact that the left-hand side is the divided difference of the function $f(x) = x^{2r-1}$ at the points $-r,\ldots,-1,1,\ldots,r$. Now the assertion in (b) follows easily by induction. Indeed, for $r=1$ it is trivial. Let us assume that

$$A_1^{-1}\cdots A_{r-1}^{-1} = 2\sum_{j=1}^{r-1}\frac{j^{2r-3}}{\omega_{r-1}'(j)}A_j^{-1}.$$

Then, using the induction hypothesis and (a), we have

$$\begin{aligned} A_1^{-1}\cdots A_r^{-1} &= 2\sum_{j=1}^{r-1}\frac{j^{2r-3}}{\omega_{r-1}'(j)}A_j^{-1}A_r^{-1} \\ &= 2\sum_{j=1}^{r-1}\frac{j^{2r-3}}{\omega_{r-1}'(j)}\Big(\frac{j^2}{j^2-r^2}A_j^{-1} + \frac{r^2}{r^2-j^2}A_r^{-1}\Big) \\ &= 2\sum_{j=1}^{r-1}\frac{j^{2r-1}}{\omega_{r-1}'(j)(j^2-r^2)}A_j^{-1} + 2\sum_{j=1}^{r-1}\frac{j^{2r-3}}{\omega_{r-1}'(j)}\frac{r^2}{r^2-j^2}A_r^{-1}. \end{aligned}$$

Next we just observe that $\omega_r'(j) = \omega_{r-1}'(j)(j^2 - r^2)$ and

$$2\sum_{j=1}^{r-1} \frac{j^{2r-3}}{\omega_{r-1}'(j)} \frac{r^2}{r^2 - j^2} = 2\sum_{j=1}^{r-1} \frac{j^{2r-3}}{\omega_{r-1}'(j)} \Big(1 - \frac{j^2}{j^2 - r^2}\Big)$$
$$= \Big(1 - 2\sum_{j=1}^{r-1} \frac{j^{2r-1}}{\omega_r'(j)}\Big) = 2\frac{r^{2r-1}}{\omega_r'(j)},$$

where at the last two steps we have used (4.12). Therefore

$$A_1^{-1} \cdots A_r^{-1} = 2\sum_{j=1}^{r-1} \frac{j^{2r-1}}{\omega_r'(j)} A_j^{-1} + 2\frac{r^{2r-1}}{\omega_r'(j)} A_r^{-1} = 2\sum_{j=1}^{r} \frac{j^{2r-1}}{\omega_r'(j)} A_j^{-1}.$$

Assertion (c) follows immediately from (b) , Proposition 4.1 and (4.12). The proof is complete. □

Similarly to Proposition 4.3 one can show

Proposition 4.7. *The kernel $\mathcal{L}_r(t)$, defined in* (4.11)*, satisfies the recursion relation*

$$\mathcal{L}_r(t) = \mathcal{L}_{r-1}(t) - r\int_0^t \sin r(t-s)\mathcal{L}_{r-1}(t)\, ds + r \sin rt$$
$$\mathcal{L}_1(t) = \sin t.$$

Next we introduce the differential operator

$$\widetilde{D}_r g = D_{r-1} \cdots D_1 g', \quad g \in X^{2r-1}, \tag{4.13}$$

where $D_j,\ j = 1, \ldots, r-1$, are defined in (2.2). The following fact is known.

Proposition 4.8. *We have $\widetilde{D}_r g = 0,\ g \in X^{2r-1}$, if and only if $g \in T_{r-1}$.*

Proposition 4.9. *For $\mathcal{F}_r$, defined in* (4.5)*, and $\widetilde{D}_r$, defined in* (4.13)*, we have*

(a) $(\mathcal{F}_r g)^{(2r+1)} = \widetilde{D}_{r+1} g, \quad g \in X^{2r+1}$;

(b) $\mathcal{F}_r \tau \in \Pi_{2r},\ \tau \in T_r$;

(c) $\mathcal{F}_r^{-1} P \in T_r,\ P \in \Pi_{2r}$.

Proof. To show (a) we only have to observe that

$$(A_j g)'' = D_j g, \quad g \in X^2 \tag{4.14}$$

and $(D_j g)' = D_j g'$. Using that we have

$$(\mathcal{F}_r g)^{(2r+1)} = (A_r \mathcal{F}_{r-1} g)^{(2r+1)} = (D_r \mathcal{F}_{r-1} g)^{(2r-1)}$$
$$= D_r(\mathcal{F}_{r-1} g)^{(2r-1)}$$
$$= \ldots$$
$$= D_r \cdots D_1 g' = \widetilde{D}_{r+1} g.$$

Assertions (b) and (c) follow immediately from (a) and Proposition 4.8. □

The next observation is due to K. G. Ivanov.

Proposition 4.10. *For $\mathcal{F}_r$, defined in* (4.5)*, and its inverse $\mathcal{F}_r^{-1}$, we have:*

(a) *If $\tau \in T_N$, $N > r$, then $\mathcal{F}_r(\tau, x) = P(x) + \sum_{k=r+1}^{N} (a_k \cos kx + b_k \sin kx)$, where $P \in \Pi_{2r}$ and a_k, $b_k \in \mathbf{R}$ depend on τ and r.*

(b) *If $P \in \Pi_N$, $N \geq 2r$, then $\mathcal{F}_r^{-1}(P, x) = P(x) + \tau(x)$, where $P \in \Pi_{N-2r}$ and $\tau \in T_r$ depend on P and n.*

Proof. The validity of (a) follows from Proposition 4.9, (a) and (b) and

$$D_r \cos kx = (r^2 - k^2) \cos kx \text{ and } D_r \sin kx = (r^2 - k^2) \sin kx. \tag{4.15}$$

Assertion (b) follows from Proposition 4.9, (a) and (c) and the observation that for any $P_s \in \Pi_s$ there exists $Q_{s+1} \in \Pi_{s+1}$ such that $\widetilde{D}_{r+1} Q_{s+1} = P_s$. To prove the latter it is enough to show that there exists $q_s \in \Pi_s$ such that $D_j q_s(x) = x^s$, $s = 0, 1, \ldots$ We can put $q_0(x) = j^{-2}$ for $s = 0$ and

$$q_s(x) = \sum_{l=0}^{[s/2]} (-1)^l j^{-2(l+1)} s(s-1) \cdots (s - 2l + 1) x^{s-2l}$$

for $s > 0$. □

After these preliminaries we can formulate and prove the main result of the section. First we introduce the bounded linear operator $\widetilde{\mathcal{F}}_{r-1} : B \to B$, $r \geq 2$, by

$$\widetilde{\mathcal{F}}_{r-1}(f, x) := \mathcal{F}_{r-1}(f, x) + P_{2r-2}(f, x), \tag{4.16}$$

where $\mathcal{F}_{r-1}$ is defined in (4.5) and $P_{2r-2}(f, x) = -\sum_{k=1}^{2r-2} \alpha_k (x + \pi)^k / k!$ is the unique algebraic polynomial of degree $2r - 2$ whose coefficients $\alpha_k = \alpha_k(f)$ depend on f and are the solution of the linear system

$$\left| \begin{array}{l} \displaystyle\sum_{k=1}^{2r-2} \frac{(2\pi)^k}{k!} \alpha_k = \int_{-\pi}^{\pi} \mathcal{K}_{r-1}(t) f(\pi - t)\, dt \\ \displaystyle\sum_{k=2}^{2r-2} \frac{(2\pi)^{k-1}}{(k-1)!} \alpha_k = \int_{-\pi}^{\pi} \mathcal{K}'_{r-1}(t) f(\pi - t)\, dt \\ \cdots \\ \displaystyle 2\pi \alpha_{2r-2} = \int_{-\pi}^{\pi} \mathcal{K}_{r-1}^{(2r-3)}(t) f(\pi - t)\, dt. \end{array} \right. \tag{4.17}$$

The algebraic polynomial $P_{2r-2} f$ is constructed so that we may have $(\widetilde{\mathcal{F}}_{r-1} f)^{(s)}(-\pi) = (\widetilde{\mathcal{F}}_{r-1} f)^{(s)}(\pi)$, $s = 0, 1, \ldots, 2r - 3$, for any $f \in B^{2r-3}$. Indeed, one easily shows by induction that

$$\begin{aligned} \left(\frac{d}{dx}\right)^{2j} \mathcal{F}_{r-1}(f, x) &= f^{(2j)}(x) + \sum_{l=1}^{j} \mathcal{K}_{r-1}^{(2l-1)}(0) f^{(2j-2l)}(x) \\ &\quad + \int_0^x \mathcal{K}_{r-1}^{(2j)}(t) f(x - t)\, dt, \quad f \in B^{2j}, \end{aligned} \tag{4.18}$$

$$\left(\frac{d}{dx}\right)^{2j+1}\mathcal{F}_{r-1}(f,x) = f^{(2j+1)}(x) + \sum_{l=1}^{j}\mathcal{K}_{r-1}^{(2l-1)}(0)f^{(2j-2l+1)}(x) \tag{4.19}$$
$$+ \int_0^x \mathcal{K}_{r-1}^{(2j+1)}(t)f(x-t)\,dt, \quad f \in B^{2j+1},$$

where we have used (4.7) and the fact that the kernel $\mathcal{K}_{r-1}(t)$ is an odd algebraic polynomial and therefore $\mathcal{K}_{r-1}^{(2j)}(0) = 0$. Consequently, if $f \in B^{2r-3}$, then (4.17) is equivalent to $(\widetilde{\mathcal{F}}_{r-1}f)^{(s)}(-\pi) = (\widetilde{\mathcal{F}}_{r-1}f)^{(s)}(\pi)$, $s = 0, 1, \ldots, 2r-3$. Let us observe that for $s > 2r-3$ the last integral summand in (4.18) and (4.19), which is the only generally non-periodic term, vanishes. Then, if $f \in B^s$, (4.17) implies $(\widetilde{\mathcal{F}}_{r-1}f)^{(s)}(-\pi) = (\widetilde{\mathcal{F}}_{r-1}f)^{(s)}(\pi)$ for any $s \in \mathbf{N}$. What we actually do here can be expressed in another way. According to (4.6) we have

$$\mathcal{F}_{r-1} = I + \sum_{j=1}^{r-1} a_j^{(r-1)}\mathcal{I}_{2j}, \quad a_j^{(r-1)} = \sum_{1 \le l_1 < \cdots < l_j \le r-1} (l_1 \cdots l_j)^2,$$

where $\mathcal{I}_{2j}f$ is defined in (4.3). But $\mathcal{I}_{2j}f$ is not periodic for every $f \in B$. It is necessary to use an integral operator that preserves the periodicity. We put

$$S(f,x) := \int_0^x f(t)\,dt - \frac{x}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt. \tag{4.20}$$

Thus defined S is a bounded linear operator $S : B \to B$, moreover, if $f \in B^s$ then $Sf \in B^{s+1}$. We put $S_j = S^j$, $j = 1, 2, \ldots$ for the iterations of the linear operator S. We have $S_j f = \mathcal{I}_j f + \bar{P}_j f$, where $\bar{P}_j f \in \Pi_j$ and $\bar{P}_j(f, 0) = 0$. Now, if we put $\bar{Q}_{2r-2}f = \sum_{j=1}^{r-1} a_j^{(r-1)}\bar{P}_{2j}f \in \Pi_{2r-2}$ (then $\bar{Q}_{2r-2}(f,0) = 0$), we shall have

$$\begin{aligned}
\mathcal{F}_{r-1}f + \bar{Q}_{2r-2}f &= f + \sum_{j=1}^{r-1} a_j^{(r-1)}\mathcal{I}_{2j}f + \sum_{j=1}^{r-1} a_j^{(r-1)}\bar{P}_{2j}f \\
&= f + \sum_{j=1}^{r-1} a_j^{(r-1)}(\mathcal{I}_{2j}f + \bar{P}_{2j}f) \\
&= f + \sum_{j=1}^{r-1} a_j^{(r-1)} S_{2j}f.
\end{aligned}$$

Hence, if $f \in B^s$ then $\mathcal{F}_{r-1}f + \bar{Q}_{2r-2}f \in B^s$. Thus, observing that $P_{2r-2}(f,x) = \bar{Q}_{2r-2}(f,x) - \sum_{k=1}^{2r-2}\alpha_k(f)\pi^k/k!$, we get the following important property of the operator $\widetilde{\mathcal{F}}_{r-1}$:

$$\widetilde{\mathcal{F}}_{r-1}g \in B^s \text{ for any } g \in B^s,\ s \in \mathbf{N}. \tag{4.21}$$

Another substantial property of $\widetilde{\mathcal{F}}_{r-1}$ follows immediately from Proposition 4.9, (a). We have

$$(\widetilde{\mathcal{F}}_{r-1}g)^{(2r-1)} = \widetilde{D}_r g,\ g \in B^{2r-1}. \tag{4.22}$$

For consistency, we put $\mathcal{F}_0 = I$ ($\mathcal{K}_0 = 0$) and $P_0 = 0$ and then $\widetilde{\mathcal{F}}_0 = I$. We shall also need the following properties of the operator $\widetilde{\mathcal{F}}_{r-1}$:

Proposition 4.11. *For $\widetilde{\mathcal{F}}_{r-1}$ we have*

(a) $\widetilde{\mathcal{F}}_{r-1}(T_{r-1}) = \Pi_0 = T_0$.

(b) $\ker \widetilde{\mathcal{F}}_{r-1}$ *is* $(2r-2)$*-dimensional subspace of* T_{r-1}.

Proof. First, we show that $\widetilde{\mathcal{F}}_{r-1}f = const$ for any $f \in T_{r-1}$. Indeed, let $f \in T_{r-1}$. Then, as we have proved in Proposition 4.9 (b), $\mathcal{F}_{r-1}f \in \Pi_{2r-2}$. Hence this holds for $\widetilde{\mathcal{F}}_{r-1}f$ as well. On the other hand, $\widetilde{\mathcal{F}}_{r-1}f$ is 2π-periodic. Therefore $\widetilde{\mathcal{F}}_{r-1}f = const$ for any $f \in T_{r-1}$. Also one can easily see that $\widetilde{\mathcal{F}}_{r-1}(e_0, -\pi) > 0$. Thus (a) is verified. Next $\widetilde{\mathcal{F}}_{r-1}f = 0$ implies $\mathcal{F}_{r-1}f \in \Pi_{2r-2}$ (see the definition of $\widetilde{\mathcal{F}}_{r-1}$ in (4.16)), hence Proposition 4.9 (c) yields $f \in T_{r-1}$. Thus $\ker \widetilde{\mathcal{F}}_{r-1} \subseteq T_{r-1}$. Part (a) of the proposition under consideration implies that for any $f \in T_{r-1}$ there exists a constant $c = c(f)$ such that $\widetilde{\mathcal{F}}_{r-1}(f + c) = 0$. Hence $\dim \ker \widetilde{\mathcal{F}}_{r-1} = 2r - 2$. □

We introduce the following modulus of smoothness for a function $f \in B$ and $t > 0$:

$$\omega_r^T(f;t)_B = \omega_{2r-1}(\widetilde{\mathcal{F}}_{r-1}f;t)_B,\ r = 1, 2, \ldots, \tag{4.23}$$

where $\omega_{2r-1}(F;t)_B$ is the classical periodic modulus of smoothness of order $2r-1$, namely,

$$\omega_{2r-1}(F;t)_B = \sup_{0<h\le t} \|\Delta_h^{2r-1}F\|_B,$$

$$\Delta_h^{2r-1}F(x) = \sum_{k=0}^{2r-1}(-1)^k \binom{2r-1}{k} f(x + ((2r-1)/2 - k)h).$$

Let us note that $\Delta_h^{2r-1}\mathcal{F}_{r-1}f \in B$ for any $f \in B$ and

$$\omega_r^T(f;t)_B = \sup_{0<h\le t} \|\Delta_h^{2r-1}\mathcal{F}_{r-1}f\|_B,\ r = 1, 2, \ldots \tag{4.24}$$

In the definition of $\omega_r^T(f;t)_B$ the quantity $\Delta_h^{2r-1}\widetilde{\mathcal{F}}_{r-1}f(x)$ depends only on the values of f in a neighbourhood of x whose diameter diminishes with h. The point 0 in the integration limits of the integral operator used in the definition of $\widetilde{\mathcal{F}}_{r-1}$ has been chosen only for convenience – any other value can be fixed and the definition of $\omega_r^T(f;t)_B$ is invariant of this choice.

If $f \in T_{r-1}$ then $\mathcal{F}_{r-1}f$ is an algebraic polynomial of degree $2r-2$ (Proposition 4.9, (b)) and then $\Delta_h^{2r-1}\mathcal{F}_{r-1}f(x) \equiv 0$, hence $\omega_r^T(f;t)_B \equiv 0$. And vice versa, if $\omega_r^T(f;t)_B \equiv 0$ then $\Delta_h^{2r-1}\mathcal{F}_{r-1}f(x) = 0$ for $x \in [-\pi, \pi]$. Consequently, $\mathcal{F}_{r-1}f$ is an algebraic polynomial of degree $2r-2$ and then $f \in T_{r-1}$ (Proposition 4.9, (c)). So $\omega_r^T(f;t)_B \equiv 0$ if and only if $f \in T_{r-1}$.

Next we define the K-functional

$$K_r^T(f;t)_B = \inf_{g \in B^{2r-1}} \left\{ \|f - g\|_B + t^{2r-1}\|\widetilde{D}_r g\|_B \right\} \tag{4.25}$$

for $f \in B,\ t > 0$ and $\widetilde{D}_r$ defined in (4.13). We shall use the notation $K_s^c(F;t)_B$ for the classical K-functional

$$K_s^c(F;t)_B = \inf_{G\in B^s}\{\|F-G\|_B + t^s\|G^{(s)}\|_B.\} \tag{4.26}$$

The following equivalence result holds.

Theorem 4.12. *For $f \in B$, where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$, we have*

$$K_r^T(f;t)_B \sim K_{2r-1}^c(\widetilde{\mathcal{F}}_{r-1}f;t)_B,$$

where $K_r^T(f;t)_B$ and $K_{2r-1}^c(F;t)_B$ are defined in (4.25) *and* (4.26)*, respectively.*

Proof. To show that there exists a positive constant C_r, independent of f and t, such that $K_{2r-1}^c(\widetilde{\mathcal{F}}_{r-1}f;t)_B \le C_r K_r^T(f;t)_B$, we just write for $G = \widetilde{\mathcal{F}}_{r-1}g \in B^{2r-1}$, provided that $g \in B^{2r-1}$,

$$\begin{aligned}
K_{2r-1}^c(\widetilde{\mathcal{F}}_{r-1}f;t)_B &= \inf_{G\in B^{2r-1}}\big\{\|\widetilde{\mathcal{F}}_{r-1}f - G\|_B + t^{2r-1}\|G^{(2r-1)}\|_B\big\}\\
&\le \inf_{g\in B^{2r-1}}\big\{\|\widetilde{\mathcal{F}}_{r-1}f - \widetilde{\mathcal{F}}_{r-1}g\|_B + t^{2r-1}\|\widetilde{D}_r g\|_B\big\}\\
&\le \inf_{g\in B^{2r-1}}\big\{\|\widetilde{\mathcal{F}}_{r-1}\|\,\|f-g\|_B + t^{2r-1}\|\widetilde{D}_r g\|_B\big\}\\
&\le C_r K_r^T(f;t)_B.
\end{aligned}$$

The verifying of the converse inequality takes more effort. We assume that $r \ge 2$, since for $r = 1$ the assertion is trivial. Let $h_1,\dots,h_{2r-2}$, $h_i \in T_{r-1}$, $i = 1,\dots,2r-2$, be linearly independent elements of $\ker\widetilde{\mathcal{F}}_{r-1}$. The matrix

$$M = \left(\int_{-\pi}^{\pi}\mathcal{K}_{r-1}^{(s)}(t)h_k(\pi-t)\,dt\right)_{s=0,k=1}^{2r-3,2r-2}$$

is of maximal rank (namely, $2r-2$). Indeed, assuming the contrary, we get that there exists $h \in \ker\widetilde{\mathcal{F}}_{r-1}$, $h \ne 0$, such that $\int_{-\pi}^{\pi}\mathcal{K}_{r-1}^{(s)}(t)h(\pi-t)\,dt = 0$ for $s = 0,1,\dots,2r-3$. Consequently, $P_{2r-2}h = 0$, hence $\mathcal{F}_{r-1}h = 0$, and then $h = 0$. So the assumption is wrong and M is of maximal rank. Then the linear system

$$\sum_{k=1}^{2r-2}\int_{-\pi}^{\pi}\mathcal{K}_{r-1}^{(s)}(t)h_k(\pi-t)\,dt\,.\,c_k = -\int_{-\pi}^{\pi}\mathcal{K}_{r-1}^{(s)}(t)f(\pi-t)\,dt,\ s = 0,1,\dots,2r-3 \tag{4.27}$$

has a (unique) solution for any $f \in B$. Thus, for any $f \in B$, there exists a trigonometric polynomial $h = h(f) = c_1h_1 + \cdots + c_{2r-2}h_{2r-2} \in \ker\widetilde{\mathcal{F}}_{r-1}$ of degree at most $r-1$ such that $P_{2r-2}(f+h) = 0$. Then we have $\widetilde{\mathcal{F}}_{r-1}f = \widetilde{\mathcal{F}}_{r-1}(f+h) = \mathcal{F}_{r-1}(f+h)$. So we can consider without loss of generality only functions f such that $P_{2r-2}f = 0$ and then $\widetilde{\mathcal{F}}_{r-1}(f,x) = \mathcal{F}_{r-1}(f,x)$.

We define the linear operator $\mathcal{E}_{r-1} : B \to B$ by

$$\mathcal{E}_{r-1}(F,x) = \mathcal{F}_{r-1}^{-1}(F,x) + Q_{2r}(F,x),\ x \in [-\pi,\pi], \tag{4.28}$$

where $Q_{2r}(F,x)$ is an algebraic polynomial of the form $Q_{2r}(F,x) = \sum_{k=1}^{2r} \beta_k (x+\pi)^k/k!$, depending on F, such that $\mathcal{E}_{r-1}F \in B^s$ for any $F \in B^s$ for $s = 0, 1, \dots, 2r-1$. The coefficients $\{\beta_k\}$ are the solution of a linear system similar to (4.17). More precisely, the system is

$$
(4.29) \qquad \left|\begin{array}{l}
\displaystyle\sum_{k=1}^{2r} \frac{(2\pi)^k}{k!}\beta_k = \int_{-\pi}^{\pi} \mathcal{L}_{r-1}(t)F(\pi - t)\,dt \\
\displaystyle\sum_{k=2}^{2r} \frac{(2\pi)^{k-1}}{(k-1)!}\beta_k = \int_{-\pi}^{\pi} \mathcal{L}'_{r-1}(t)F(\pi - t)\,dt \\
\dots \\
\displaystyle 2\pi\beta_{2r} = \int_{-\pi}^{\pi} \mathcal{L}_{r-1}^{(2r-1)}(t)F(\pi - t)\,dt.
\end{array}\right.
$$

As in the case of the definition of $P_{2r-2}f$, this is equivalent to $(\mathcal{E}_{r-1}F)^{(s)}(-\pi) = (\mathcal{E}_{r-1}F)^{(s)}(\pi)$, $s = 0, 1, \dots, 2r-1$, for any $F \in B^{2r-1}$ ($\mathcal{L}_{r-1}(t)$ is an odd function alike $\mathcal{K}_{r-1}$, (4.11)) and $\mathcal{E}_{r-1}G \in B^s$ for any $G \in B^s$, $s = 0, 1, \dots, 2r-1$. Next we write

$$
(4.30) \qquad \begin{aligned}
\mathcal{E}_{r-1}(\mathcal{F}_{r-1}f) &= \mathcal{F}_{r-1}^{-1}(\mathcal{F}_{r-1}f) + Q_{2r}(\mathcal{F}_{r-1}f) \\
&= f + Q_{2r}\mathcal{F}_{r-1}f.
\end{aligned}
$$

Let us consider the operator $Q_{2r}\mathcal{F}_{r-1} : B_0 \to \Pi_{2r}$, where $B_0 = \{f \in B : P_{2r-2}f = 0\}$ and Π_{2r} is normed by the uniform norm over the interval $[-\pi, \pi]$. Obviously the operator $Q_{2r} : B \to \Pi_{2r}$ is bounded, hence $Q_{2r}\mathcal{F}_{r-1}$ is bounded too. Let $B_0^s = B_0 \cap B^s$. The set B_0^s is dense in B_0. Indeed, $B_0 = \{f + h(f) : f \in B\}$ and $B_0^s = \{g + h(g) : g \in B^s\}$, where $h : B \to T_{r-1}$, is defined by (4.27). Let T_{r-1} be normed with the same norm as B. Then Crammer's formulae yield

$$
|c_k| \le C_r \max_{s=\overline{0,2r-3}} \left| \int_{-\pi}^{\pi} \mathcal{K}_{r-1}^{(s)}(t) f(\pi - t)\,dt \right|, \quad k = 1, \dots, 2r-2,
$$

where C_r is a positive constant independent of f. Consequently,

$$
|c_k| \le C_r \|f\|_{L_1} \le C_r \|f\|_B, \quad k = 1, \dots, 2r-2.
$$

Hence

$$
\|h(f)\|_B \le C_r \max_{k=\overline{1,2k-2}} |c_k| \le C_r \|f\|_B,
$$

where C_r is a positive constant independent of f. So $h : B \to T_{r-1}$ is a bounded linear operator and for $f \in B$ and $g \in B^s$, we have

$$
\|f + h(f) - (g + h(g))\|_B \le \|f - g\|_B + \|h(f-g)\|_B \le (1 + \|h\|)\|f - g\|_B.
$$

As B^s is dense in B, then the above implies that so is B_0^s in B_0. Having verified that, let $g \in B_0^{2r-1}$. Then $\mathcal{F}_{r-1}g \in B^{2r-1}$ and then $\mathcal{E}_{r-1}(\mathcal{F}_{r-1}g) \in B^{2r-1}$. Therefore, bearing in mind the definition of Q_{2r} and (4.30), we conclude that $Q_{2r}\mathcal{F}_{r-1}g = 0$ for every $g \in B_0^{2r-1}$. But as B_0^{2r-1} is dense in B_0 and the operator $Q_{2r}\mathcal{F}_{r-1}$ is bounded, we get

$Q_{2r}\mathcal{F}_{r-1} = 0$ in B_0. Thus we have shown that $\mathcal{E}_{r-1}\mathcal{F}_{r-1}f = f$ for $f \in B_0$. Now let $G \in B^{2r-1}$. Then $g = \mathcal{E}_{r-1}G \in B^{2r-1}$. We have

$$\begin{aligned} \|f-g\|_B &= \|\mathcal{E}_{r-1}\mathcal{F}_{r-1}f - \mathcal{E}_{r-1}G\|_B \\ &\le \|\mathcal{E}_{r-1}\|\,\|\mathcal{F}_{r-1}f - G\|_B. \end{aligned} \tag{4.31}$$

Further we need to estimate $\|\widetilde{D}_r g\|_B$. We have

$$\begin{aligned} \widetilde{D}_r g &= \Big(\frac{d}{dx}\Big)^{2r-1}\mathcal{F}_{r-1}g = \Big(\frac{d}{dx}\Big)^{2r-1}\mathcal{F}_{r-1}\mathcal{E}_{r-1}G \\ &= \Big(\frac{d}{dx}\Big)^{2r-1}\mathcal{F}_{r-1}(\mathcal{F}_{r-1}^{-1}G + Q_{2r}G) \\ &= G^{(2r-1)} + \Big(\frac{d}{dx}\Big)^{2r-1}\mathcal{F}_{r-1}Q_{2r}G. \end{aligned}$$

We observe that $\mathcal{F}_{r-1}Q_{2r}G$ is an algebraic polynomial of degree at most $(2r-3)+2r+1 = 4r-2$ and then, obviously, there exists a constant C_r which depends on r but not on G such that

$$\left\|\Big(\frac{d}{dx}\Big)^{2r-1}\mathcal{F}_{r-1}Q_{2r}G\right\|_B \le C_r \max_{k=\overline{1,2r}} |\beta_k|\,. \tag{4.32}$$

Next, since $\{\beta_k\}$ is the solution of the linear system (4.29), then there exists a constant C_r which depends on r but not on G such that

$$|\beta_k| \le C_r \max_{s=\overline{0,2r-1}} \left|\int_{-\pi}^{\pi} \mathcal{L}_{r-1}^{(s)}(t)G(\pi-t)\,dt\right|, \quad k=1,\dots,2r. \tag{4.33}$$

Relation (4.11) shows the kernel $\mathcal{L}_{r-1}(t)$ is a trigonometric polynomial such that $a_0(\mathcal{L}_{r-1}) = 0$. For every trigonometric polynomial T with $a_0(T) = 0$ and any $G \in B^{2r-1}$ an integration by parts yields the relation

$$\int_{-\pi}^{\pi} T(t)G(t)\,dt = -\int_{-\pi}^{\pi} S(t)G^{(2r-1)}(t)\,dt,$$

where S is a trigonometric polynomial such that $S^{(2r-1)} = T$. Consequently,

$$|\beta_k| \le C_r\|G^{(2r-1)}\|_{L_1} \le C_r\|G^{(2r-1)}\|_B, \quad k=1,\dots,2r. \tag{4.34}$$

Inequalities (4.32) and (4.34) imply

$$\left\|\Big(\frac{d}{dx}\Big)^{2r-1}\mathcal{F}_{r-1}Q_{2r}G\right\|_B \le C_r\|G^{(2r-1)}\|_B$$

and then

$$\|\widetilde{D}_r g\|_B \le C_r\|G^{(2r-1)}\|_B. \tag{4.35}$$

To finish the proof we only have to observe that (4.31) and (4.35) imply that for any $G \in B^{2r-1}$ there exists $g \in B^{2r-1}$ such that

$$\|f-g\|_B + t^{2r-1}\|\widetilde{D}_r g\|_B \le C_r\big(\|\mathcal{F}_{r-1}f - G\|_B + t^{2r-1}\|G^{(2r-1)}\|_B\big).$$

Now taking infimum first over $g \in B^{2r-1}$ and then over $G \in B^{2r-1}$, we get

$$K_r^T(f;t)_B \le C_r K_{2r-1}^c(\widetilde{\mathcal{F}}_{r-1}f;t)_B,$$

where C_r depends on r but not on f and t. This completes the proof of the theorem. $\square$

Now Theorem 1.2 follows from Theorem 4.12, the definition of $\omega_r^T(f;t)_B$ and the well-known characterization

$$K_s^c(F;t)_B \sim \omega_s(F;t)_B,\ F \in B.$$

Remark 4.13. In a similar way it is shown that for any $l \in \mathbf{N}$ we have

$$\inf_{g\in B^{2r+l-1}} \big\{\|f-g\|_B + t^{2r+l-1}\|(d/dx)^l \widetilde{D}_r g\|_B\big\} \sim \omega_{2r+l-1}(\widetilde{\mathcal{F}}_{r-1}f;t)_B.$$

Actually the proof is shorter as we do not need $a_0(\mathcal{L}_{r-1}) = 0$ for $l > 0$.

Moduli which contain integrals of f have been defined before. In Section 1 we described briefly the modulus of Ivanov. The modulus $\omega_\varphi^{*r}(f;t)_p$ of Ditzian and Totik (see [7, Ch. 2, Sec. 2.2]) also is an integral modulus of smoothness. Potapov (see [14] and [15]) and Butzer, Stens and Wehrens (see [2], [3] and also [1] and the references cited in those papers) introduce moduli based on generalized translation and integral transforms. However, the modulus $\omega_r^T(f;t)_B$ is different in construction and is based on a different idea. It is actually the classical periodic modulus of smoothness taken not on f but on its image under a certain linear mapping. This linear mapping is closely connected with the differential operator which characterizes the approximating space T_{r-1}.

It is easy to verify, using the definition of $\omega_r^T(f;t)_B$ and some properties of the operator $\widetilde{\mathcal{F}}_{r-1}$, that $\omega_r^T(f;t)_B$ possesses the properties:

1. $\omega_r^T(f+g;t)_B \le \omega_r^T(f;t)_B + \omega_r^T(g;t)_B$ for $f,g \in B$;
2. $\omega_r^T(cf;t)_B = |c|\,\omega_r^T(f;t)_B,\ c$ is a constant;
3. $\omega_r^T(f;t)_B \le \omega_r^T(f;t')_B,\ t \le t'$;
4. $\omega_r^T(f;t)_B \to 0$ as $t \to 0$;
5. $\omega_r^T(f;t)_B \le (4+(r-1)^2t^2)\,\omega_{r-1}^T(f;t)_B,\ r \ge 2$;
6. $\omega_1^T(f;t)_B \le 2\|f\|_B$ and $\omega_1^T(f;t)_B \le t\|f'\|_B,\ f \in B^1$ ($\omega_1^T(f;t)_B$ coincides with the ordinary modulus of continuity);
7. $\omega_r^T(f;\lambda t)_B \le (\lambda+1)^{2r-1}\omega_r^T(f;t)_B$;
8. $\omega_r^T(f;t)_B \le t^2\omega_{r-1}^T(D_{r-1}f;t)_B,\ f \in B^2;\ r \ge 2$;
9. Marchaud inequality

$$\omega_r^T(f;t)_B \le C_r t^{2r-1}\left(\int_t^c \frac{\omega_{r+1}^T(f;u)_B}{u^{2r}}\,du + \|f\|_B\right),\ 0 < t \le c,$$

where c is any fixed positive constant.

Only the proof of relations 5, 8 and 9 need somewhat more considerations. We shall derive 9 later as a corollary from the relation between $E_n^T(f)_B$ and $\omega_r^T(f;t)_B$.

Proof of Property 5. We have for $f \in B$

$$
\begin{aligned}
\widetilde{\mathcal{F}}_{r-1}(f,x) &= \mathcal{F}_{r-1}(f,x) + P_{2r-2}(f,x) \\
&= A_{r-1}(\mathcal{F}_{r-2}f,x) + P_{2r-2}(f,x) \\
&= \mathcal{F}_{r-2}(f,x) + (r-1)^2 \int_{-\pi}^{x} (x-t)\mathcal{F}_{r-2}(f,t)\,dt + P_{2r-2}(f,x) \\
&= \mathcal{F}_{r-2}(f,x) + P_{2r-4}(f,x) \\
&\quad + (r-1)^2 \left(\int_{-\pi}^{x} (x-t)\mathcal{F}_{r-2}(f,t)\,dt + \frac{1}{(r-1)^2}\big(P_{2r-2}(f,x) - P_{2r-4}(f,x)\big) \right) \\
&= \widetilde{\mathcal{F}}_{r-2}(f,x) + (r-1)^2 \widetilde{E}(f,x),
\end{aligned}
$$

where we have denoted by $\widetilde{E}(f,x)$ the term in the parentheses above. For any $f \in B^s$, $s \in \mathbf{N}$, we have $\widetilde{\mathcal{F}}_{r-2}f$, $\widetilde{\mathcal{F}}_{r-1}f \in B^s$. Therefore, for any $f \in B^s$, $s \in \mathbf{N}$, we have $\widetilde{E}f \in B^s$ and then $(d/dx)^2\widetilde{E}f = \mathcal{F}_{r-2}f + \widetilde{P}_{2r-4}f \in B^{s-2}$, where $\widetilde{P}_{2r-4}f \in \Pi_{2r-4}$ depends on f. Let $r \geq 3$. By definition $P_{2r-4}f$ is the unique algebraic polynomial of degree at most $2r-4$ up to an additive constant such that $\mathcal{F}_{r-2}f + P_{2r-4}f \in B^{2r-5}$ for any $f \in B^{2r-5}$. This and the above for $s \geq 2r-3$ imply $(d/dx)^2\widetilde{E}f = \widetilde{\mathcal{F}}_{r-2}f + const$, $r \geq 3$. For $r = 2$ we have $(d/dx)^2\widetilde{E}f = f + const = \widetilde{F}_0 f + const$. Now, using some well-known properties of the classical modulus of smoothness, we get

$$
\begin{aligned}
\omega_r^T(f,t)_B &= \omega_{2r-1}(\widetilde{\mathcal{F}}_{r-1}f;t)_B \leq \omega_{2r-1}(\widetilde{\mathcal{F}}_{r-2}f;t)_B + (r-1)^2\omega_{2r-1}(\widetilde{E}f;t)_B \\
&\leq 4\omega_{2r-3}(\widetilde{\mathcal{F}}_{r-2}f;t)_B + (r-1)^2t^2\omega_{2r-3}(\widetilde{\mathcal{F}}_{r-2}f;t)_B \\
&= (4+(r-1)^2t^2)\omega_{r-1}^T(f;t)_B.
\end{aligned}
$$

□

Proof of Property 8. Similar considerations make up the proof of this property. Straightforward calculations and the fact that $\mathcal{K}''_{r-1} = D_{r-1}\mathcal{K}_{r-2}$ (see Proposition 4.3) yield for $f \in B^2$ and $r \geq 3$

$$
\begin{aligned}
\left(\frac{d}{dx}\right)^2 \widetilde{\mathcal{F}}_{r-1}(f,x) &= D_{r-1}f(x) + \int_0^x \mathcal{K}_{r-2}(x-t)D_{r-1}f(t)\,dt + \widetilde{P}_{2r-4}(f,x) \\
&= \mathcal{F}_{r-2}(D_{r-1}f,x) + \widetilde{P}_{2r-4}(f,x),
\end{aligned}
$$

where $\widetilde{P}_{2r-4}f \in \Pi_{2r-4}$ depends on f. Now, as in the proof of Property 5, we get that

$$
\left(\frac{d}{dx}\right)^2 \widetilde{\mathcal{F}}_{r-1}f = \widetilde{\mathcal{F}}_{r-2}D_{r-1}f + const, \quad f \in B^2,\ r \geq 3.
$$

If $r = 2$ we just get $(d/dx)^2\widetilde{F}_1(f,x) = (d/dx)^2A_1(f,x) + const = D_1f(x) + const = \widetilde{F}_0(D_1f,x) + const$. Hence, in both cases we have

$$
\begin{aligned}
\omega_r^T(f,t)_B &= \omega_{2r-1}(\widetilde{\mathcal{F}}_{r-1}f;t)_B \leq t^2\omega_{2r-3}(\widetilde{\mathcal{F}}_{r-2}D_{r-1}f;t)_B \\
&= t^2\omega_{r-1}^T(D_{r-1}f;t)_B.
\end{aligned}
$$

$\square$

We also get the following theorem as a direct corollary from the properties of the classical modulus of smoothness and the operators $\widetilde{\mathcal{F}}_{r-1}$ and $\widetilde{D}_r$.

Theorem 4.14. *For $\omega_r^T(f;t)_B$, where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$, we have:*

(i) If $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, then $\omega_r^T(f;t)_B = o(t^{2r-1})$ implies that f is equivalent to a trigonometric polynomial of degree at most $r-1$ and f is equivalent to a trigonometric polynomial of degree at most $r-1$ implies $\omega_r^T(f;t)_B \equiv 0$. If $B = C^*[-\pi,\pi]$, then $\omega_r^T(f;t)_B = o(t^{2r-1})$ implies $f \in T_{r-1}$ and $f \in T_{r-1}$ implies $\omega_r^T(f;t)_B \equiv 0$.

(ii) For $B = L_p^*[-\pi,\pi]$, $1 < p < \infty$, and $B = C^*[-\pi,\pi]$, we have $\omega_r^T(f;t)_B = \mathcal{O}(t^{2r-1})$ if and only if $f \in W_p^{*2r-1}[-\pi,\pi]$, $1 < p \le \infty$.

(iii) For $B = L_1^*[-\pi,\pi]$, we have $\omega_r^T(f;t)_B = \mathcal{O}(t^{2r-1})$ if and only if $f^{(2r-3)} \in AC^*[-\pi,\pi]$ and $f^{(2r-2)}$ is equivalent to a function of bounded variation.

5 Best trigonometric approximation

Now we can prove our main result concerning the rate of best trigonometric approximation in $L_p^*[-\pi,\pi]$, $1 \le p < \infty$, and $C^*[-\pi,\pi]$.

Proof of Theorem 1.1. First we deal with (1.5). Clearly, for $n \ge r-1$,

$$\begin{aligned} E_n^T(f)_B &\le \|f - L_{r-1,n+1}f\|_B \\ &\le C_r K_r'(f;(n+1)^{-1})_B \le C_r K_r'(f;n^{-1})_B \\ &\le C_r \omega_{2r}(\widetilde{\mathcal{F}}_{r-1}f;n^{-1})_B \le C_r \omega_{2r-1}(\widetilde{\mathcal{F}}_{r-1}f;n^{-1})_B \\ &= C_r \omega_r^T(f;n^{-1})_B, \end{aligned}$$

where we have used Theorem 3.1 and Remark 4.13 for $l = 1$.

We prove (1.6) using some classical methods (see for example [10, Ch. 4, Sec. 4] and [7, Ch. 7]). First we observe that the following Bernstein-type inequality holds:

$$\|\widetilde{D}_r g\|_B \le 2^{r-1} n^{2r-1} \|g\|_B,\ g \in T_n. \tag{5.1}$$

Indeed, using the classical Bernstein inequality

$$\|g'\|_B \le n\|g\|_B,\ g \in T_n$$

and the definition of D_j (see (2.2)), we get for $g \in T_n$

$$\begin{aligned}
\|\widetilde{D}_r g\|_B &\le \|(\widetilde{D}_{r-1} g)''\|_B + (r-1)^2 \|\widetilde{D}_{r-1} g\|_B \\
&\le (n^2 + (r-1)^2)\|\widetilde{D}_{r-1} g\|_B \\
&\dots \\
&\le \prod_{l=1}^{r-1} (n^2 + l^2)\|g'\|_B \\
&\le 2^{r-1} n^{2r-1}\|g\|_B.
\end{aligned}$$

Let $Q_j \in T_j$, $j = 0, 1, 2, \dots$, is such that $E_j^T(f)_B = \|f - Q_j\|_B$; let also $l = \max\{k : 2^k \le 1/t\}$ and $s = \min\{k : 2^k \ge r-1\}$ (we suppose that $0 < t \le 1/r$). We have, for $t > 0$,

$$\omega_r^T(f;t)_B \le C_r K_r^T(f;t)_B \le C_r\big(\|f - Q_{2^l}\|_B + t^{2r-1}\|\widetilde{D}_r Q_{2^l}\|_B\big). \tag{5.2}$$

Using the representation $\widetilde{D}_r Q_{2^l} = \sum_{k=s}^{l-1} \widetilde{D}_r(Q_{2^{k+1}} - Q_{2^k}) + \widetilde{D}_r(Q_{2^s} - Q_{r-1})$ and (5.1), we get the estimate

$$\begin{aligned}
\|\widetilde{D}_r Q_{2^l}\|_B &\le \sum_{k=s}^{l-1} \|\widetilde{D}_r(Q_{2^{k+1}} - Q_{2^k})\|_B + \|\widetilde{D}_r(Q_{2^s} - Q_{r-1})\|_B \\
&\le 2^{r-1} \sum_{k=s}^{l-1} 2^{(2r-1)(k+1)} \|Q_{2^{k+1}} - Q_{2^k}\|_B + 2^{r-1} 2^{(2r-1)s} \|Q_{2^s} - Q_{r-1}\|_B \\
&\le 2^{r-1} \sum_{k=s}^{l-1} 2^{(2r-1)(k+1)} \big(E_{2^k}^T(f)_B + E_{2^{k+1}}^T(f)_B\big) \\
&\qquad\qquad + 2^{r-1} 2^{(2r-1)s} \big(E_{2^s}^T(f)_B + E_{r-1}^T(f)_B\big) \\
&\le 2^r \sum_{k=s}^{l} 2^{(2r-1)(k+1)} E_{2^k}^T(f)_B + 2^{r+(2r-1)s} E_{r-1}^T(f)_B.
\end{aligned}$$

Next, as $E_{2^k}^T(f)_B \le 2^{-k+1} \sum_{j=2^{k-1}+1}^{2^k} E_j^T(f)_B$, $k = s+1, \dots, l$, we have

$$\begin{aligned}
\sum_{k=s+1}^{l} 2^{(2r-1)(k+1)} E_{2^k}^T(f)_B &\le \sum_{k=s}^{l} 2^{(2r-1)(k+1)} 2^{-k+1} \sum_{j=2^{k-1}+1}^{2^k} E_j^T(f)_B \\
&= 2^{4r-2} \sum_{k=s+1}^{l} 2^{2(r-1)(k-1)} \sum_{j=2^{k-1}+1}^{2^k} E_j^T(f)_B \le 2^{4r-2} \sum_{j=2^s+1}^{2^l} (j+1)^{2r-2} E_j^T(f)_B.
\end{aligned}$$

This estimate and the trivial ones $2^{r+(2r-1)(s+1)} \le 2^{4r-1}(2^s+1)^{2r-2}$ and $2^{r+(2r-1)s} \le 2^{4r-1} r^{2r-2}$ imply

$$2^r \sum_{k=s}^{l} 2^{(2r-1)(k+1)} E_{2^k}^T(f)_B + 2^{r+(2r-1)s} E_{r-1}^T(f)_B \le 2^{5r-2} \sum_{j=r-1}^{2^l} (j+1)^{2r-2} E_j^T(f)_B.$$

Thus, for $0 < t \le 1/r$, we have

$$\|\widetilde{D}_{r-1}Q_{2^l}\|_B \le C_r \sum_{r-1\le k\le 1/t} (k+1)^{2r-2}E_k^T(f)_B. \tag{5.3}$$

Now (5.2) and (5.3) yield (1.6). $\square$

Theorem 1.2 and Theorem 1.1 immediately yield Theorem 1.3. From Theorem 1.1, using again some classical methods, we get the following.

Corollary 5.1 (Marchaud inequality). *For $f \in B$, where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$, and $\omega_r^T(f;t)_B$ defined in (4.23), we have*

$$\omega_r^T(f;t)_B \le C_r t^{2r-1}\left(\int_t^c \frac{\omega_{r+1}^T(f;u)_B}{u^{2r}}\,du + \|f\|_B\right),\ 0<t\le c,$$

where c is any fixed positive constant.

Proof. Using (1.5) and (1.6) we have for $0 < t \le 1/r$

$$\begin{aligned}\omega_r^T(f;t)_B &\le C_r t^{2r-1}\sum_{r-1\le k\le 1/t}(k+1)^{2r-2}E_k^T(f)_B\\ &\le C_r t^{2r-1}\Big(\sum_{r\le k\le 1/t}(k+1)^{2r-2}\omega_{r+1}^T(f;k^{-1})_B + E_{r-1}^T(f)_B\Big)\\ &\le C_r t^{2r-1}\left(\int_t^c \frac{\omega_{r+1}^T(f;u)_B}{u^{2r}}\,du + \|f\|_B\right).\end{aligned}$$

$\square$

Remark 5.2. By means of considerations similar to those used in the proof of the results stated in Theorem 1.1 we can easily get the following generalization of (1.5) and (1.6):

$$E_n^T(f)_B \le C_{r,l}\,\omega_l(\widetilde{\mathcal{F}}_{r-1}f;n^{-1})_B,\ n\ge r-1,$$

and

$$\omega_l(\widetilde{\mathcal{F}}_{r-1}f;t)_B \le C_{r,l}t^l\sum_{r-1\le k\le 1/t}(k+1)^{l-1}E_k^T(f)_B,\ 0<t\le\frac{1}{r},\ l\ge 2r-1,$$

where $\widetilde{\mathcal{F}}_{r-1}f$ is defined in (4.16) and $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$ or $B = C^*[-\pi,\pi]$.

Remark 5.3. Let $L_n : B \to B$, where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$, be a bounded linear operator that preserves the trigonometric polynomials of degree n. Then the well-known Lebesgue inequality

$$\|f - L_n f\|_B \le (1 + \|L_n\|_B)E_n^T(f)_B$$

and (1.5) imply

$$\|f - L_n f\|_B \le C_r(1 + \|L_n\|_B)\, \omega_r^T(f, n^{-1})_B, \quad n \ge r-1. \tag{5.4}$$

In particular, for the partial sums of Fourier expansion $S_n(f,x) = \sum_{k=0}^{n} A_k(f,x)$ we have

$$\|f - S_n f\|_B \le C_r(1 + \|S_n\|_B)\, \omega_r^T(f, n^{-1})_B, \; n \ge r-1. \tag{5.5}$$

For $B = L_p^*[-\pi,\pi],\ 1 < p < \infty$, as it is known $\|S_n\|_p \le const\, p^2/(p-1)$, and then (5.5) reads

$$\|f - S_n f\|_p \le C_{r,p}\omega_r^T(f, n^{-1})_p, \; n \ge r-1.$$

For $B = L_1^*[-\pi,\pi]$ and $B = C^*[-\pi,\pi]$ Fejer's inequality

$$\|\mathcal{D}_n\|_1 \le \frac{4}{\pi^2}\ln n + const,$$

where $\mathcal{D}_n$ is Dirichlet's kernel, yields for $p = 1$ and $p = \infty$

$$\|f - S_n f\|_p \le C_r \ln n\, \omega_r^T(f, n^{-1})_p, \; n \ge r-1.$$

For other estimates in uniform norm the interested reader can refer to [11] and [13].

At the end of this section we shall discuss briefly another K-functional, which describes the rate of best trigonometric approximation in any homogeneous Banach space B. We define the K-functional

$$\widetilde{K}_r^T(f;t)_B = \inf_{\tilde{g} \in B^{2r-1}} \left\{ \|f - g\|_B + t^{2r-1}\|\widetilde{D}_r \tilde{g}\|_B \right\}, \tag{5.6}$$

where $\tilde{g}$ is the conjugate function of g (one can refer to [10, Ch. 7, Sec. 4] and [6] for its definition). Next we define the trigonometric operator $L'_{r-1,n} : B \to T_{n-1},\ r, n \in \mathbf{N},\ 1 \le r \le n$,

$$L'_{r-1,n} = I - (I - \sigma_n) \prod_{j=1}^{r-1} (I - R_{j,n}), \tag{5.7}$$

where $\sigma_n(f,x) = \sum_{k=0}^{n-1}\left(1 - \frac{k}{n}\right)A_k(x)$ is the Fejer operator. Ditzian and Ivanov generalized the Alexits-Zamanski saturation theorem in [6]. They proved

$$\|f - \sigma_n f\|_B \sim \widetilde{K}_1^T(f; n^{-1})_B,$$

where $\widetilde{K}_1^T(f;t)_B$ is defined in (5.6). Using this result, Theorem 2.1 and the method, demonstrated in Section 3, we prove

Theorem 5.4. *For a Banach space B satisfying* (2.1) *we have*

$$\|f - L'_{r-1,n} f\|_B \sim \widetilde{K}_r^T(f; n^{-1})_B,$$

where $\widetilde{K}_r^T(f;t)_B$ and $L'_{r-1,n}$ are given in (5.6) *and* (5.7)*, respectively.*

As an immediate corollary we get the following properties of $\widetilde{K}_r^T(f;t)_B$.

Corollary 5.5. *There exists a constant C_r which depends on $r \geq 2$ but not on $f \in B$ or $0 < t \leq 1$ such that*

(a) $\widetilde{K}_r^T(f;t)_B \leq C_r \widetilde{K}_{r-1}^T(f;t)_B$;

(b) $\widetilde{K}_r^T(f;t)_B \leq C_r t^{2r-1} \left(\int_t^1 \frac{\widetilde{K}_{r+1}^T(f;t)_B}{u^{2r}}\, du + \|f\|_B \right)$;

(c) $\widetilde{K}_r^T(f;t)_B \leq C_r t^2 \widetilde{K}_{r-1}^T(D_{r-1}f;t)_B,\ f \in B^2$.

Proof. The proof of (a) and (b) follows step by step the one given by Ditzian in [4] of a similar statement. An analogous argument implies (c) as well. Using Theorem 5.4, the commutativity of the operators $R_{j,n}$ for different j and $R_{j,n}$ and σ_n and also (3.4), which holds for σ_n too, we get for $f \in B^2$ and $r \geq 2$

$$\begin{aligned}
\widetilde{K}_r^T(f;n^{-1})_B &\leq C_r \|(I - L'_{r-1,n})f\|_B = \|(I - R_{r-1,n})(I - L'_{r-2,n})f\|_B \\
&\leq C_r n^{-2} \|D_{r-1}(I - L'_{r-2,n})f\|_B \\
&= C_r n^{-2} \|(I - L'_{r-2,n}) D_{r-1} f\|_B \\
&\leq C_r n^{-2} \widetilde{K}_{r-1}^T(D_{r-1}f;n^{-1})_B.
\end{aligned}$$

To get the assertion for any $0 < t \leq 1$ we just have to put $n = [1/t]$ and use that then $1/(n+1) < t \leq 1/n$. □

As at the beginning of this section, making use at a certain step of the Bernstein inequality, extended to any homogeneous Banach space by Ditzian in [5], we get the following characterization of the rate of best trigonometric approximation in any homogeneous Banach space B.

Theorem 5.6. *Let $f \in B$. Then*

$$E_n^T(f)_B \leq C_r \widetilde{K}_r^T(f;t)_B,\ n \geq r-1,$$

and

$$\widetilde{K}_r^T(f;t)_B \leq C_r t^{2r-1} \sum_{r-1 \leq k \leq 1/t} (k+1)^{2r-2} E_k^T(f)_B,\ 0 < t \leq \frac{1}{r}.$$

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