Estimating the rate of best trigonometric approximation in homogeneous Banach spaces by moduli of smoothness

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Abstract

Best trigonometric approximation in homogeneous Banach spaces of periodic functions is characterized by two moduli of smoothness, which are equivalent to zero if the function is a trigonometric polynomial of a given degree. The characterization is just similar to the one given by the classical modulus of smoothness. The moduli possesses properties similar to those of the classical one. One is based on the classical finite differences but taken on a modification of the function and the other on a modification of the finite differences but taken on the function itself.

Keywords and phrases: Best trigonometric approximation, modulus of smoothness, finite differences, K-functional, convolution operators, homogeneous Banach space.

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Dedicated to the memory of Academician Borislav Bojanov

1 The characterization of best trigonometric approximation

Let $L_p(\mathbb{T})$, $1 \le p \le \infty$, denote the space of the functions with finite L_p-norm on the circle $\mathbb{T}$, as we may actually consider $C(\mathbb{T})$ – the space of the continuous functions on $\mathbb{T}$, in the place of $L_\infty(\mathbb{T})$. Best trigonometric approximation of a function $f \in B$, where B is either $L_p(\mathbb{T})$ or $C(\mathbb{T})$, is given by

$$E_n^T(f)_B = \inf_{\tau \in T_n} \|f - \tau\|_B,$$

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as T_n denotes the set of the trigonometric polynomials of degree at most n.

The order of $E_n^T(f)_B$ is estimated by the so-called classical moduli of smoothness. To recall, the modulus of smoothness of order $r \in \mathbb{N}$ is defined by

$$\omega_r(f,t)_B = \sup_{0<h\leq t} \|\Delta_h^r f\|_B, \tag{1.1}$$

where the centred finite difference of order $r \in \mathbb{N}$ of f is given by

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r/2 - k)h). \tag{1.2}$$

The following relation between $E_n^T(f)_B$ and $\omega_r(f,t)_B$ is a classical result in approximation theory (see for example [3, Ch. 7])

$$E_n^T(f)_B \leq c\,\omega_r(f, n^{-1})_B, \tag{1.3}$$

$$\omega_r(f,t)_B \leq c\,t^r \sum_{0\leq k\leq 1/t} (k+1)^{r-1} E_k^T(f)_B, \quad 0 < t \leq t_0. \tag{1.4}$$

Above and in what follows we denote by c positive constants, which do not depend on the functions in the relations, nor on $n \in \mathbb{N}$ or $0 < t \leq t_0$.

Although (1.3) looks so nice, one is bothered by the fact that $E_n^T(f)_B$ is zero always when f is a trigonometric polynomial of degree n, whereas $\omega_r(f,t)_B$ is zero only if f is a constant. To cope with this problem we have to modify the modulus. First, in 1999 Babenko, Chernykh and Shevaldin [1] considered the modulus

$$\tilde{\omega}_r^T(f,t)_B = \sup_{0<h\leq t} \|\widetilde{\Delta}_{r,h} f\|_B,$$

as the modified finite differences $\widetilde{\Delta}_{r,h}$ are defined by

$$\widetilde{\Delta}_{r,h} f(x) = \Delta_{r-1,h} \cdots \Delta_{1,h} \Delta_h f(x), \tag{1.5}$$

where

$$\Delta_{j,h} f(x) = f(x+h) - 2\cos jh\,.\,f(x) + f(x-h), \quad j = 1, 2, \ldots$$

This modulus has the property

$$\tilde{\omega}_r^T(f,t)_B \equiv 0 \quad \Longleftrightarrow \quad f \in T_{r-1}.$$

Babenko, Chernykh and Shevaldin [1] proved (1.3) with $\tilde{\omega}_r^T(f,t)_B$ in the place of the classical modulus of smoothness for the space $B = L_2(\mathbb{T})$. Later Shevaldin [11] added the case $B = C(\mathbb{T})$ for $r = 2$. Quite recently, in [8], (1.3) and (1.4) with $\tilde{\omega}_r^T(f,t)_B$ were verified for $B = L_p(\mathbb{T})$, $1 \leq p \leq \infty$. More precisely, in [8, Theorem 1.1] it was established that

$$E_n^T(f)_B \leq c\,\tilde{\omega}_r^T(f, n^{-1})_B, \quad n \geq r-1, \tag{1.6}$$

$$\tag{1.7} \tilde{\omega}_r^T(f,t)_B \le c\, t^{2r-1} \sum_{r-1\le k\le 1/t} (k+1)^{2r-2} E_k^T(f)_B, \quad 0<t\le \frac{1}{r},$$

for all $B = L_p(\mathbb{T})$, $1 \le p \le \infty$.

Meanwhile, the author introduced another modulus in [5]. It uses the classical finite differences (1.2) but they are taken on a suitable continuous linear transform of the function f. Namely, we set

$$\mathcal{F}_{r-1}f(x) = f(x) + \sum_{j=1}^{r-1} \frac{a_{r-1,j}}{(2j-1)!} \int_0^x (x-t)^{2j-1} f(t)\, dt,$$

where $a_{r-1,j}$ are given by the Stirling numbers of the first kind by

$$a_{r-1,j} = \sum_{m=1}^{2r-2j-1} (-1)^{r-j-m} s(r,m)\, s(r, 2r-2j-m).$$

Now, we define the modulus by

$$\omega_r^T(f,t)_B = \sup_{0<h\le t} \|\Delta_h^{2r-1} \mathcal{F}_{r-1} f\|_B.$$

This modulus also has the property

$$\omega_r^T(f,t)_B \equiv 0 \quad \Longleftrightarrow \quad f \in T_{r-1}$$

and in [5, Theorem 1.1] it was shown that it characterizes $E_n^T(f)_B$ just as in (1.6)-(1.7) for $B = L_p(\mathbb{T})$, $1 \le p \le \infty$. Let us explicitly point out that, though $\mathcal{F}_{r-1}f$ is not generally 2π-periodic, $\Delta_h^{2r-1}\mathcal{F}_{r-1}f$ is.

The purpose of this note is to extend the characterization of best trigonometric approximation by the moduli $\omega_r^T(f,t)_B$ and $\tilde{\omega}_r^T(f,t)_B$ to any homogeneous Banach space of periodic functions. Let us recall (see [9, Definition I.2.10]) that a *homogeneous Banach space* (abbreviated *HBS*) B *on* $\mathbb{T}$ is a linear subspace of $L_1(\mathbb{T})$, having a norm $\|\circ\|_B$, under which it is a Banach space such that

(a) The translation is an isometry of B onto itself, i.e. if $f \in B$ and $t \in \mathbb{T}$, then $f_t \in B$ and $\|f_t\|_B = \|f\|_B$, where $f_t(x) = f(x-t)$;

(b) The translation is continuous on B, i.e. for all $f \in B$ and $t, t_0 \in \mathbb{T}$ there holds $\lim_{t\to t_0} \|f_t - f_{t_0}\|_B = 0$;

(c) B is continuously embedded in $L_1(\mathbb{T})$, i.e. there exists an absolute constant α such that for all $f \in B$ there holds $\|f\|_{L_1(\mathbb{T})} \le \alpha \|f\|_B$.

$L_p(\mathbb{T})$ for $1 \le p < \infty$ and $C(\mathbb{T})$ as well as some Lipschitz (Hölder) spaces are HBS on $\mathbb{T}$. Let us also recall that (1.3)-(1.4) have been extended to abstract Banach spaces and, in particular, to any HBS on $\mathbb{T}$ (cf. [4], [10, Ch. 9] and [12]).

The concept of HBS's was introduced by Shilov [13]. However, we can observe a similar abstract approach in the definition of almost periodic functions by Bochner and Neumann [2, Definition 1] (see also the references cited there and [10, p. 200]).

The modulus $\tilde{\omega}_r^T(f,t)_B$ is well defined in the setting of an arbitrary HBS on $\mathbb{T}$. However, the operator $\mathcal{F}_{r-1}$ in the definition of $\omega_r^T(f,t)_B$ has to be modified a little to ensure that its image is again in B. For $L_p(\mathbb{T})$ this was done in [5] by adding an appropriate algebraic polynomial operator of degree $2r-2$ to $\mathcal{F}_{r-1}$ to get that the image is again 2π-periodic. However, this construction does not make it evident that the image is again in B in the case of an *arbitrary* HBS. We can settle this general case if we succeed to modify $\mathcal{F}_{r-1}$ so that it becomes a convolution operator. Below we give the details.

Let B be a HBS on $\mathbb{T}$, $f \in B$ and $\mathcal{K} \in L_1(\mathbb{T})$. Then the convolution between $\mathcal{K}$ and f

$$\mathcal{K} * f(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{K}(x-y)f(y)\,dy$$

is an element of B and

$$\|\mathcal{K} * f\|_B \le \|\mathcal{K}\|_{L_1(\mathbb{T})}\,\|f\|_B \tag{1.8}$$

(see [9, Problem I.2.13]).

We define the function $\mathfrak{a} \in L_1(\mathbb{T})$ by

$$\mathfrak{a}(x) = \frac{1}{2}\,|x|(2\pi - |x|), \quad x \in [-\pi,\pi]. \tag{1.9}$$

Let us denote by $\mathcal{K}^{*s}$ the convolution of $\mathcal{K} \in L_1(\mathbb{T})$ with itself $s \in \mathbb{N}$ times. We replace the operator $\mathcal{F}_{r-1}$ in the definition of $\omega_r^T(f,t)_B$ with $\mathfrak{F}_{r-1} : B \to B$, defined by

$$\mathfrak{F}_{r-1} f = f + \sum_{j=1}^{r-1} a_{r-1,j}\,\mathfrak{a}^{*j} * f \tag{1.10}$$

and set

$$\omega_r^T(f,t)_B = \sup_{0<h\le t} \|\Delta_h^{2r-1}\mathfrak{F}_{r-1} f\|_B.$$

Redefining $\omega_r^T(f,t)_B$ in this way does not give rise to any ambiguity because $\mathcal{F}_{r-1} f$ and $\mathfrak{F}_{r-1} f$ differ with an algebraic polynomial of degree $2r-2$ for any f as it follows from [5, Proposition 4.9(a)] and Proposition 2.3 below.

The kernel $\mathfrak{a}$ has very simple Fourier coefficients:

$$\widehat{\mathfrak{a}}(k) = \begin{cases} -\dfrac{1}{k^2}, & k \in \mathbb{Z}\backslash\{0\}, \\ \dfrac{\pi^2}{3}, & k = 0. \end{cases} \tag{1.11}$$

As usual we set for $f \in B$

$$\hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(y)e^{-iky}\,dy, \quad k \in \mathbb{Z}.$$

Let B be a HBS on $\mathbb{T}$. Then, as is known, the set of trigonometric polynomials in B is dense (see e. g. [9, Theorem I.2.12] or (2.2) below). We set

$$E_n^T(f)_B = \inf_{\tau \in T_n \cap B} \|f - \tau\|_B.$$

Our main result is the following characterization of $E_n^T(f)_B$ by means of moduli of smoothness that are invariant on the trigonometric polynomials of a given degree.

Theorem 1.1. *Let B be a HBS on $\mathbb{T}$ and $f \in B$. Then*

$$E_n^T(f)_B \le c\,\omega_r^T(f, n^{-1})_B, \quad n \ge r-1, \tag{1.12}$$

and

$$\omega_r^T(f,t)_B \le c\,t^{2r-1} \sum_{r-1 \le k \le 1/t} (k+1)^{2r-2} E_k^T(f)_B, \quad 0 < t \le \frac{1}{r}. \tag{1.13}$$

The same relations hold with $\tilde{\omega}_r^T(f,t)_B$ in the place of $\omega_r^T(f,t)_B$.

In the next section we shall give a proof of this theorem. There we consider some relevant properties of a modified Riesz operator, which was introduced in [5], and of $\mathfrak{F}_{r-1}$. In the third and final section we present the most important properties of the moduli $\omega_r^T(f,t)_B$ and $\tilde{\omega}_r^T(f,t)_B$.

2 Proof of the characterization

The proof of the characterization of the error $E_n^T(f)_B$ is comprised of two intertwining parts. One is related to the characterization of $E_n^T(f)_B$ by means of appropriately defined K-functionals and the other to their equivalence to the moduli.

Let B be a HBS on $\mathbb{T}$ and $r, n \in \mathbb{N}$ as $r \le n$. Let us denote by AC the set of the absolutely continuous functions on $\mathbb{T}$. We put for $s \in \mathbb{N}$

$$B^s = \{g \in B : g^{(\ell)} \in AC \cap B,\ \ell = 0, \dots, s-1,\ g^{(s)} \in B\}.$$

We shall use K-functionals of the following two types:

$$K_s(f,t)_B = \inf_{g \in B^s} \{\|f-g\|_B + t^s \|g^{(s)}\|_B\},$$
$$K_{r,\ell}^T(f,t)_B = \inf_{g \in B^{2r+\ell-1}} \{\|f-g\|_B + t^{2r-1+\ell} \|\widetilde{D}_r g^{(\ell)}\|_B\},$$

where $f \in B$, $t > 0$, $\ell \in \mathbb{N}_0$ and

$$\widetilde{D}_r g = D_{r-1} \cdots D_1\, g', \quad D_j g = g'' + j^2 g.$$

Note that

$$\widetilde{D}_r g = 0 \quad \iff \quad g \in T_{r-1}. \tag{2.1}$$

In [5, Sections 2 and 3] we introduced the following combination of modified Riesz operators $L_{r-1,n} : B \to B \cap T_{n-1}$, defined by

$$L_{r-1,n}f = f - \prod_{j=0}^{r-1}(f - R_{j,n}f),$$

where

$$R_{j,n}f(x) = \sum_{k=1-n}^{n-1}\left(1 - \frac{k^2 - j^2}{n^2 - j^2}\right)\hat{f}(k)\,e^{ikx}, \quad x \in \mathbb{T},$$

and established that (see [5, Theorem 3.1])

$$\|f - L_{r-1,n}f\|_B \sim K^T_{r,1}(f, n^{-1})_B, \quad f \in B, \quad n \in \mathbb{N}. \tag{2.2}$$

Here the relation $\psi_1(f,n) \sim \psi_2(f,n)$ means that there exists a positive constant c such that

$$c^{-1}\psi_2(f,n) \le \psi_1(f,n) \le c\,\psi_2(f,n)$$

for all f and n under consideration.

Let us note that, in view of (2.1) and (2.2) we have

$$L_{r-1,n}f = f \quad \Longleftrightarrow \quad f \in T_{r-1}.$$

Further, let us observe that

$$K_s(f,t)_B \sim \omega_s(f,t)_B, \quad f \in B, \quad t > 0. \tag{2.3}$$

This can be established just as for $L_p(\mathbb{T})$ (see e.g. [3, p. 177]), as we observe that the classical modulus of smoothness preserves its properties in any HBS on $\mathbb{T}$ and the combination of modified Steklov functions belongs to the HBS B provided that $f \in B$ because it is representable as a convolution between f and a kernel in $L_1(\mathbb{T})$ (see [3, Chapter 6, (2.12)]).

Next, we shall extend [5, Theorem 4.12 and Remark 4.13] to any HBS on $\mathbb{T}$. That is we shall prove the following assertion.

Theorem 2.1. *Let B be a HBS on $\mathbb{T}$ and $\ell \in \mathbb{N}_0$. Then*

$$K^T_{r,\ell}(f,t)_B \sim \omega_{2r+\ell-1}(\mathfrak{F}_{r-1}f, t)_B, \quad f \in B, \quad t > 0.$$

The proof of this theorem is based on the method formulated in [6, Proposition 2.1] (or see [7, Theorem 3.1 and Remark 3.1]). It allows to substitute a K-functional with a "complex" differential operator with an equivalent one with a "simple" differential operator by modifying the function. For an easier reference we state below this result particularly for the case under consideration. Before that let us mention that due to the convolution structure of $\mathfrak{F}_{r-1}$, used here, the proofs of its properties as well as those of $\omega^T_r(f,t)_B$ become much shorter and simpler than the proofs given in [5].

Theorem 2.2. *Let B be a Banach space of functions on $\mathbb{T}$, $r \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. Let $\mathcal{A} : B \to B$ and $\mathcal{B} : B \to B$ be linear operators, satisfying the conditions:*

(a) $\|\mathcal{A}f\|_B \le c\,\|f\|_B$ *for every* $f \in B$;

(b) $\mathcal{A}g \in B^{2r+\ell-1}$ *and* $\left\|(\mathcal{A}g)^{(2r+\ell-1)}\right\|_B \le c\left\|\widetilde{D}_r g^{(\ell)}\right\|_B$ *for every* $g \in B^{2r+\ell-1}$;

(c) $\|\mathcal{B}F\|_B \le c\,\|F\|_B$ *for every* $F \in B$;

(d) $\mathcal{B}G \in B^{2r+\ell-1}$ *and* $\left\|\widetilde{D}_r(\mathcal{B}G)^{(\ell)}\right\|_B \le c\left\|G^{(2r+\ell-1)}\right\|_B$ *for every* $G \in B^{2r+\ell-1}$;

(e) $f - \mathcal{B}\mathcal{A}f \in T_{r-1}$ *for every* $f \in B$;

(f) $F - \mathcal{A}\mathcal{B}F = const$ *for every* $F \in \mathcal{A}(B)$.

Then for all $f \in B$ and $t > 0$ there holds

$$K^T_{r,\ell}(f,t)_B \sim K_{2r+\ell-1}(\mathcal{A}f,t)_B.$$

Below we shall verify that $\mathfrak{F}_{r-1}$, defined in (1.10), possesses the properties of the operator $\mathcal{A}$ of the theorem above. We begin with the construction of the corresponding operator $\mathcal{B}$, which we call *a quasi-inverse of $\mathcal{A}$*. We set for $F \in B$, $j \in \mathbb{N}$ and $x \in \mathbb{T}$

$$\mathfrak{B}_j F(x) = F(x) + \mathfrak{b}_j * F(x),$$

where $\mathfrak{b}_j$ is a function on $\mathbb{T}$ such that

$$\mathfrak{b}_j(x) = j(|x| - \pi)\sin|jx|, \quad x \in [-\pi, \pi].$$

Further, we put

$$\mathfrak{E}_{r-1} = \mathfrak{B}_{r-1}\cdots\mathfrak{B}_1. \tag{2.4}$$

We shall show that the operators $\mathfrak{F}_{r-1}$ and $\mathfrak{E}_{r-1}$ satisfy the hypotheses of Theorem 2.2.

Proposition 2.3. *Let B be a HBS on $\mathbb{T}$, $r \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. For $g \in B^{2r+\ell-1}$ we have $\mathfrak{F}_{r-1}g \in B^{2r+\ell-1}$ and*

$$(\mathfrak{F}_{r-1}g)^{(2r+\ell-1)} = \widetilde{D}_r g^{(\ell)}.$$

Proof. It is clear that $g \in B^{2r+\ell-1}$ yields $\mathfrak{F}_{r-1}g \in B^{2r+\ell-1}$ as well. We set for $f \in B$

$$\mathfrak{A}_j f = f + j^2\mathfrak{a} * f, \quad j = 1, 2, \ldots,$$

where $\mathfrak{a}$ is given in (1.9). The operator $\mathfrak{F}_{r-1}$ has been constructed as a composition of the operators $\mathfrak{A}_j$ for $j = 1, 2, \ldots, r-1$:

$$\mathfrak{F}_{r-1} = \mathfrak{A}_{r-1}\cdots\mathfrak{A}_1. \tag{2.5}$$

Indeed, denoting by δ the Dirac delta function, we have

$$\begin{aligned}\mathfrak{A}_{r-1}\cdots\mathfrak{A}_1 &= (\delta+(r-1)^2\mathfrak{a})*\cdots*(\delta+\mathfrak{a})*f\\ &= f+\sum_{j=1}^{r-1}\left(\sum_{1\le \ell_1<\cdots<\ell_j\le r-1}(\ell_1\cdots\ell_j)^2\right)\mathfrak{a}^{*j}*f.\end{aligned}$$

On the other hand, there holds

$$(\mathfrak{A}_j g)'' = D_j g - j^2\hat{g}(0), \quad g\in B^2. \tag{2.6}$$

To verify this relation, we just calculate the Fourier coefficients of $\mathfrak{A}_j g$. We have by (1.11) for $k\neq 0$

$$\widehat{\mathfrak{A}_j g}(k) = (1+j^2\widehat{\mathfrak{a}}(k))\,\hat{g}(k) = \left(1-\frac{j^2}{k^2}\right)\hat{g}(k).$$

Consequently, we have for $k\neq 0$

$$\widehat{(\mathfrak{A}_j g)''}(k) = -k^2\,\widehat{\mathfrak{A}_j g}(k) = (-k^2+j^2)\hat{g}(k) = \widehat{g''}(k)+j^2\hat{g}(k) = \widehat{D_j g}(k),$$

and hence the Fourier coefficients of the left and right sides of (2.6) are equal for $k\neq 0$. Also, it is clear that their Fourier coefficients at $k=0$ are both equal to 0. Therefore, in view of the uniqueness of the Fourier transform, we get (2.6).

Now, combining (2.5) and (2.6), we get the assertion of the proposition. $\square$

Proposition 2.4. *Let B be a HBS on $\mathbb{T}$ and $r\in\mathbb{N}$.*

(i) *We have $f-\mathfrak{E}_{r-1}\mathfrak{F}_{r-1}f\in T_{r-1}$ for all $f\in B$.*

(ii) *We have $F-\mathfrak{F}_{r-1}\mathfrak{E}_{r-1}=const$ for all $F\in\mathfrak{F}_{r-1}(B)$.*

Proof. We proceed similarly to the proof of the previous proposition. We shall show that

$$\mathfrak{E}_{r-1}\mathfrak{F}_{r-1}f = f+\tau_{r-1}*f \tag{2.7}$$

with

$$\tau_{r-1}(x) = -1-2\sum_{j=1}^{r-1}\cos jx. \tag{2.8}$$

To this end, we shall first establish that

$$\mathfrak{B}_j\mathfrak{A}_j f = f+\eta_j*f, \quad f\in B, \quad j\in\mathbb{N}, \tag{2.9}$$

where

$$\eta_j(x) = -1-2\cos jx.$$

We calculate the Fourier coefficients of the kernel $\mathfrak{b}_j$ to be

$$\hat{\mathfrak{b}}_j(k) = \begin{cases} \dfrac{j^2}{k^2 - j^2}, & k \in \mathbb{Z}\backslash\{\pm j\}, \\ -\dfrac{1}{4}, & k = \pm j. \end{cases} \tag{2.10}$$

The definition of the operators $\mathfrak{A}_j$ and $\mathfrak{B}_j$ gives

$$\mathfrak{B}_j\mathfrak{A}_j f = f + (j^2\mathfrak{a} + \mathfrak{b}_j + j^2\mathfrak{a} * \mathfrak{b}_j) * f.$$

Thus, in view of (1.11) and (2.10) we get for its Fourier transform

$$\begin{aligned}(\mathfrak{B}_j\mathfrak{A}_j f)\widehat{\ }(k) &= (1 + j^2\hat{\mathfrak{a}}(k) + \hat{\mathfrak{b}}_j(k) + j^2\hat{\mathfrak{a}}(k)\hat{\mathfrak{b}}_j(k))\,\hat{f}(k) \\ &= \begin{cases} \hat{f}(k), & k \in \mathbb{Z}\backslash\{0, \pm j\}, \\ 0, & k = 0, \pm j. \end{cases}\end{aligned}$$

The right hand side is exactly the Fourier transform of $f + \eta_j * f$; hence (2.9) follows.

Now, using (2.9) and the fact that $\mathfrak{A}_{j'}$, $\mathfrak{A}_{j''}$, $\mathfrak{B}_{j'}$ and $\mathfrak{B}_{j''}$ commute for all naturals j', j'', we arrive at

$$\begin{aligned}\mathfrak{E}_{r-1}\mathfrak{F}_{r-1} f &= \mathfrak{B}_{r-1}\cdots\mathfrak{B}_1\mathfrak{A}_{r-1}\cdots\mathfrak{A}_1 f \\ &= \mathfrak{B}_{r-1}\mathfrak{A}_{r-1}\cdots\mathfrak{B}_1\mathfrak{A}_1 f \\ &= (\delta + \eta_{r-1}) * \cdots * (\delta + \eta_1) * f.\end{aligned}$$

Applying the Fourier transform to both sides of the last relation, we get

$$\begin{aligned}(\mathfrak{E}_{r-1}\mathfrak{F}_{r-1} f)\widehat{\ } &= (1 + \hat{\eta}_{r-1})\cdots(1 + \hat{\eta}_1)\hat{f} \\ &= (1 + \hat{\tau}_{r-1})\hat{f},\end{aligned}$$

which verifies (2.7) and completes the proof of assertion (i).

Proceeding to (ii), we have for $F = \mathfrak{F}_{r-1} f$, $f \in B$, by means of (2.7) that

$$\begin{aligned}\mathfrak{F}_{r-1}\mathfrak{E}_{r-1} F &= \mathfrak{F}_{r-1}\mathfrak{E}_{r-1}\mathfrak{F}_{r-1} f \\ &= \mathfrak{F}_{r-1}(f + \tau_{r-1} * f) = F + \mathfrak{F}_{r-1}(\tau_{r-1} * f).\end{aligned}$$

But $\tau_{r-1} * f \in B^{2r-1} \cap T_{r-1}$ and by Proposition 2.3 we get that $(\mathfrak{F}_{r-1}(\tau_{r-1} * f))^{(2r-1)} = \widetilde{D}_r(\tau_{r-1} * f) = 0$. Consequently, $\mathfrak{F}_{r-1}(\tau_{r-1} * f)$ is an algebraic polynomial of degree not greater than $2r-2$. On the other hand, $\mathfrak{F}_{r-1}(\tau_{r-1} * f)$ is in B and thus 2π-periodic. Therefore it is a constant. Assertion (ii) is established. $\square$

Corollary 2.5. *Let B be a HBS on $\mathbb{T}$, $r \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. For $G \in B^{2r+\ell-1}$ we have $\mathfrak{E}_{r-1}G \in B^{2r+\ell-1}$ and*

$$\widetilde{D}_r(\mathfrak{E}_{r-1}G)^{(\ell)} = G^{(2r+\ell-1)} + \tau_{r-1} * G^{(2r+\ell-1)}.$$

where τ_{r-1} is given in (2.8).

Proof. The formula follows directly from Proposition 2.3 and relation (2.7) as we also take into account that $\mathfrak{F}_{r-1}$ and $\mathfrak{E}_{r-1}$, being convolution operators, commute. Indeed, we just have

$$\begin{aligned}\widetilde{D}_r(\mathfrak{E}_{r-1}G)^{(\ell)} &= (\mathfrak{F}_{r-1}(\mathfrak{E}_{r-1}G))^{(2r+\ell-1)}\\ &= (\mathfrak{E}_{r-1}(\mathfrak{F}_{r-1}G))^{(2r+\ell-1)}\\ &= (G+\tau_{r-1}*G)^{(2r+\ell-1)}\\ &= G^{(2r+\ell-1)}+\tau_{r-1}*G^{(2r+\ell-1)}.\end{aligned}$$

□

Now we are ready establish Theorem 2.1.

Proof of Theorem 2.1. In view of (2.3) it is enough to show that

$$K^T_{r,\ell}(f,t)_B \sim K_{2r+\ell-1}(\mathfrak{F}_{r-1}f,t)_B, \quad f\in B, \quad t>0. \tag{2.11}$$

To this end, we apply Theorem 2.2 with $\mathcal{A}=\mathfrak{F}_{r-1}$ and $\mathcal{B}=\mathfrak{E}_{r-1}$. Conditions (a) and (c) follow from (1.8), (b) from Proposition 2.3, (d) from Corollary 2.5, and (e) and (f) are established in Proposition 2.4. □

We have all the ingredients we need to give a proof of the characterization of best trigonometric approximation in any HBS of periodic functions.

Proof of Theorem 1.1. Relation (2.2) and Theorem 2.1 for $\ell=0,1$ enable us to follow verbatim the proof of [5, Theorem 1.1 in Section 5] in any HBS on $\mathbb{T}$ and establish (1.12).

The weak converse inequality (1.13) is again verified as in [5], taking into account that the classical Bernstein inequality for trigonometric polynomials is valid in any HBS on $\mathbb{T}$. Indeed, its proof in $L_p(\mathbb{T})$, based on the Riesz interpolation formula for trigonometric polynomials θ_n of degree at most n

$$\theta_n'(x)=\frac{1}{4n}\sum_{\ell=0}^{2n-1}\frac{(-1)^\ell}{\sin^2\frac{x_\ell}{2}}\,\theta_n(x+x_\ell), \quad x_\ell=\frac{2\ell+1}{2n}\,\pi,$$

(see e.g. [14, Section 4.7.1 (3)]) is directly extendable to any normed space, in which translation is an isometry (cf. also [4, p. 569, Corollary]).

Thus the inequalities in the theorem are verified for $\omega^T_r(f,t)_B$. In view of the equivalence between this modulus and the K-functional $K^T_r(f,t)_B$, to complete the proof of the theorem for the other trigonometric modulus $\tilde{\omega}^T_r(f,t)_B$, it is sufficient to establish that it too is equivalent to $K^T_r(f,t)_B$. But this can be done just as in the proof of [8, Theorem 4.2] because all is expressed via convolutions.

□

3 Properties of $\omega_r^T(f,t)_B$ and $\tilde{\omega}_r^T(f,t)_B$

Both moduli retain the properties of the classical modulus $\omega_s(f,t)_B$. Let B be a HBS on $\mathbb{T}$. There hold:

1. $\omega_r^T(f+g,t)_B \le \omega_r^T(f,t)_B + \omega_r^T(g,t)_B$;
2. $\omega_r^T(cf,t)_B = |c|\,\omega_r^T(f,t)_B,\ c$ is a constant;
3. $\omega_r^T(f,t)_B \le \omega_r^T(f,t')_B,\ t \le t'$;
4. $\omega_r^T(f,\lambda t)_B \le (\lambda+1)^{2r-1}\omega_r^T(f,t)_B$;
5. $\omega_r^T(f,t)_B \to 0$ as $t \to 0$;
6. $\omega_1^T(f,t)_B \le 2\|f\|_B$ and $\omega_1^T(f,t)_B \le t\|f'\|_B,\ f \in B^1$ ($\omega_1^T(f,t)_B$ coincides with the ordinary modulus of continuity);
7. $\omega_r^T(f,t)_B \le (4+(r-1)^2t^2)\,\omega_{r-1}^T(f,t)_B,\ r \ge 2$;
8. $\omega_r^T(f,t)_B \le t^2\omega_{r-1}^T(D_{r-1}f,t)_B,\ f \in B^2;\ r \ge 2$;
9. Marchaud inequality
$$\omega_r^T(f,t)_B \le c\,t^{2r-1}\left(\int_t^{t_0} \frac{\omega_{r+1}^T(f,u)_B}{u^{2r}}\,du + \|f\|_B\right),\ 0 < t \le t_0.$$

The other modulus $\tilde{\omega}_r^T(f,t)_B$ possesses identical properties as property 7 adopts the stronger form

7′. $\tilde{\omega}_r^T(f,t)_B \le 4\,\tilde{\omega}_{r-1}^T(f,t)_B,\ r \ge 2$.

Proof of the properties. All the properties of $\tilde{\omega}_r^T(f,t)_B$ are established just as in the case $B = L_p(\mathbb{T})$ considered in [8].

Properties 1-6 of $\omega_r^T(f,t)_B$ follow directly from the corresponding properties of the classical modulus $\omega_s(f,t)_B$. Property 9 follows from Theorem 1.1 by means of a standard argument (e.g. [3, p. 210]).

To establish Property 7 we first note that

$$\mathfrak{F}_{r-1}f = \mathfrak{A}_{r-1}\mathfrak{F}_{r-2}f = \mathfrak{F}_{r-2}f + (r-1)^2\mathfrak{a} * \mathfrak{F}_{r-2}f. \tag{3.1}$$

Moreover, $\mathfrak{a} * f \in B^2$ for any $f \in B$ and

$$(\mathfrak{a} * f)''(x) = f(x) - \hat{f}(0). \tag{3.2}$$

(Relation (3.2) gives another proof of (2.6).) To verify (3.2), we observe that

$$\begin{aligned}(\mathfrak{a} * f)''(x) &= (\mathfrak{a}' * f)'(x)\\ &= \frac{1}{2\pi}\frac{d}{dx}\left(-\int_{-\pi}^{0}(\pi+y)f(x-y)\,dy + \int_0^{\pi}(\pi-y)f(x-y)\,dy\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\,\frac{d}{dx}\left(-\int_{x}^{x+\pi}(\pi+x-y)f(y)\,dy+\int_{x-\pi}^{x}(\pi-x+y)f(y)\,dy\right)\\
&= f(x)-\frac{1}{2\pi}\int_{x-\pi}^{x+\pi}f(y)\,dy\\
&= f(x)-\hat{f}(0).
\end{aligned}
$$

Now, using the properties of the classical modulus of smoothness, (3.1) and (3.2) with f replaced with $\mathfrak{F}_{r-2}f$, we arrive at the inequality in Property 7:

$$
\begin{aligned}
\omega_r^T(f,t)_B &= \omega_{2r-1}(\mathfrak{F}_{r-1}f,t)_B\\
&\le \omega_{2r-1}(\mathfrak{F}_{r-2}f,t)_B+(r-1)^2\omega_{2r-1}(\mathfrak{a}*\mathfrak{F}_{r-2}f,t)_B\\
&\le 4\omega_{2r-3}(\mathfrak{F}_{r-2}f,t)_B+(r-1)^2t^2\omega_{2r-3}((\mathfrak{a}*\mathfrak{F}_{r-2}f)'',t)_B\\
&= 4\omega_{2r-3}(\mathfrak{F}_{r-2}f,t)_B+(r-1)^2t^2\omega_{2r-3}(\mathfrak{F}_{r-2}f,t)_B\\
&= (4+(r-1)^2t^2)\,\omega_{r-1}^T(f,t)_B.
\end{aligned}
$$

It remains to verify Property 8. Since

$$
\mathfrak{F}_{r-1}f=\mathfrak{A}_{r-1}\mathfrak{F}_{r-2}f,
$$

relation (2.6) implies that

$$
(\mathfrak{F}_{r-1}f)''=D_{r-1}(\mathfrak{F}_{r-2}f)-(r-1)^2\,\widehat{\mathfrak{F}_{r-2}f}(0)=\mathfrak{F}_{r-2}(D_{r-1}f)-(r-1)^2\,\widehat{\mathfrak{F}_{r-2}f}(0).
$$

Consequently, by the corresponding property of the classical modulus, we derive

$$
\begin{aligned}
\omega_r^T(f,t)_B &= \omega_{2r-1}(\mathfrak{F}_{r-1}f,t)_B\le t^2\omega_{2r-3}((\mathfrak{F}_{r-1}f)'',t)_B\\
&= t^2\omega_{2r-3}(\mathfrak{F}_{r-2}(D_{r-1}f),t)_B=t^2\omega_{r-1}^T(f,t)_B.
\end{aligned}
$$

The proof of the properties of the moduli is completed. □

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