Exact estimates of the rate of approximation of convolution operators

Borislav R. Draganov*

Abstract

The paper presents a method for establishing direct and strong converse inequalities in terms of K-functionals for convolution operators acting in homogeneous Banach spaces of multivariate functions. The method is based on the behaviour of the Fourier transform of the kernel of the convolution operator.

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1 Introduction

Let X be a Banach space and $\mathcal{L}_\rho : X \to X$, $\rho \in \mathbb{I}$, be a family of uniformly bounded linear operators, which approximates each $f \in X$, i.e.

$$\|f - \mathcal{L}_\rho f\|_X \to 0,$$

as the multi-index ρ tends to infinity in a certain sense. It is important to find out how fast $\{\mathcal{L}_\rho f\}$ approximates f. The K-functionals are especially useful in solving this problem. The K-functional is defined for $f \in X$ and $\tau > 0$ by

$$K(f, \tau; X, Y, \mathcal{D}) = \inf_{g \in Y} \{\|f - g\|_X + \tau \, \|\mathcal{D} g\|_X\},$$

where $\mathcal{D} : Y \to X$ is a (differential) operator and $Y \subseteq X$. To describe the rate of approximation of $\mathcal{L}_\rho$ we use two types of estimates: direct and converse. The simplest type of the former in terms of a K-functional is

$$\|f - \mathcal{L}_\rho f\|_X \le c\, K(f, \varphi(\rho); X, Y, \mathcal{D}) \tag{1.1}$$

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with appropriate $\mathcal{D}$, Y and $\varphi : \mathbb{I} \to \mathbb{R}_+$ such that $\lim \varphi(\rho) = 0$ as ρ tends to infinity in a certain sense. Here and below c denotes a positive constant (not necessarily the same) whose value does not depend on f and ρ. Note that

$$K(f, \tau; X, Y, \mathcal{D}) \to 0 \quad \text{as} \quad \tau \to 0 + 0 \quad \forall f \in X$$

if and only if Y is dense in X.

The converse estimates are of various types and they serve to show how precise the direct one is. Below we shall consider the strongest inverse form of (1.1), namely

$$K(f, \varphi(\rho); X, Y, \mathcal{D}) \leq c \, \|f - \mathcal{L}_\rho f\|_X.$$

For a classification of the converse inequalities and a quite general method for their verification we refer the reader to the important paper by Ditzian and Ivanov [8]. They show that a converse inequality follows from several inequalities of Bernstein and Voronovskaya-type. A similar result was established by Knoop and Zhou [13] (see also [10]). Let us also mention the geometric technique used by Totik [19] (see also [20]) and the probabilistic approach of Adell and Sangüesa [1] (see also [16]), through which they verify in the uniform norm the inequalities mentioned above. Here we present another method. It is quite simple, direct and especially useful for operators constructed by means of a convolution between the function being approximated and an appropriate kernel. We formulate this method in the two theorems below. The first deals with the direct estimate and the second with the corresponding converse one.

Theorem 1.1. *Let X be a Banach space and $\{\mathcal{L}_\rho\}_{\rho \in \mathbb{I}}$ be a family of uniformly bounded linear operators, which map X into itself. Let $\mathcal{D} : Y \to X$ be an operator as $Y \subseteq X$. Suppose that there exists a family of uniformly bounded operators $\{\mathcal{P}_\rho\}_{\rho \in \mathbb{I}}$ such that for all $g \in Y$ and $\rho \in \mathbb{I}$*

$$g - \mathcal{L}_\rho g = \varphi(\rho) \mathcal{P}_\rho \mathcal{D} g$$

with some $\varphi : \mathbb{I} \to \mathbb{R}_+$. Then for all $f \in X$ and $\rho \in \mathbb{I}$ there holds

$$\|f - \mathcal{L}_\rho f\|_X \leq c \, K(f, \varphi(\rho); X, Y, \mathcal{D}).$$

Theorem 1.2. *Let X be a Banach space and $\{\mathcal{L}_\rho\}_{\rho \in \mathbb{I}}$ be a family of uniformly bounded linear operators, which map X into itself. Let $\mathcal{D} : Y \to X$ be an operator as $Y \subseteq X$. Suppose that there exist $m \in \mathbb{N}$ and a family of uniformly bounded operators $\{\mathcal{Q}_\rho\}_{\rho \in \mathbb{I}}$ such that for all $f \in X$ and $\rho \in \mathbb{I}$*

$$\mathcal{L}_\rho^m f \in Y$$

and

$$\varphi(\rho) \, \mathcal{D} \mathcal{L}_\rho^m f = \mathcal{Q}_\rho (f - \mathcal{L}_\rho f)$$

with some $\varphi : \mathbb{I} \to \mathbb{R}_+$. Then for all $f \in X$ and $\rho \in \mathbb{I}$ there holds

$$K(f, \varphi(\rho); X, Y, \mathcal{D}) \leq c \, \|f - \mathcal{L}_\rho f\|_X.$$

Section 2 contains the proof of the two theorems above. In Section 3 we formulate their analogues particularly for convolution operators defined on homogeneous Banach spaces of multivariate functions. There we also introduce the terminology and make a brief account of similar results achieved before. In Sections 4 and 5 we consider applications in estimating the rate of convergence of convolution operators of functions defined respectively on $\mathbb{R}^d$ and the circle.

2 Proof of Theorems 1.1 and 1.2

The assertions of Theorems 1.1 and 1.2 are rather direct. So are their proofs, but for the sake of completeness we give them.

Proof of Theorem 1.1. We simply have for every $f \in X$, $g \in Y$ and $\rho \in \mathbb{I}$

$$\begin{aligned} \|f - \mathcal{L}_\rho f\|_X &\le \|f - g\|_X + \|g - \mathcal{L}_\rho g\|_X + \|\mathcal{L}_\rho (f-g)\|_X \\ &\le (1 + \|\mathcal{L}_\rho\|)\, \|f - g\|_X + \varphi(\rho)\, \|\mathcal{P}_\rho\|\, \|\mathcal{D} g\|_X \\ &\le c(\|f-g\|_X + \varphi(\rho)\, \|\mathcal{D} g\|_X), \end{aligned}$$

as c is a constant independent of f, g and ρ. Now, taking an infimum over $g \in Y$ we get the assertion of the theorem. □

Proof of Theorem 1.2. Again we follow a standard argument. On the one hand, we get

$$\begin{aligned} \|f - \mathcal{L}_\rho^m f\|_X &= \|(\mathcal{L}_\rho^{m-1} + \cdots + \mathcal{L}_\rho + \mathcal{I})(f - \mathcal{L}_\rho f)\|_X \\ &\le \big(\|\mathcal{L}_\rho\|^{m-1} + \cdots + \|\mathcal{L}_\rho\| + 1\big)\, \|f - \mathcal{L}_\rho f\|_X \\ &\le c\, \|f - \mathcal{L}_\rho f\|_X. \end{aligned} \tag{2.1}$$

Above $\mathcal{I}$ denotes the identity.

On the other hand, by the hypotheses of the theorem we have

$$\begin{aligned} \varphi(\rho)\, \|\mathcal{D}\mathcal{L}_\rho^m f\|_X &\le \|\mathcal{Q}_\rho\|\, \|f - \mathcal{L}_\rho f\|_X \\ &\le c\, \|f - \mathcal{L}_\rho f\|_X \end{aligned} \tag{2.2}$$

for every $f \in X$ and $\rho \in \mathbb{I}$. Relations (2.1), (2.2) and $\mathcal{L}_\rho^m f \in Y$ for all $f \in X$ and $\rho \in \mathbb{I}$ yield

$$\begin{aligned} K(f, \varphi(\rho); X, Y, \mathcal{D}) &\le \|f - \mathcal{L}_\rho^m f\|_X + \varphi(\rho)\, \|\mathcal{D}\mathcal{L}_\rho^m f\|_X \\ &\le c\, \|f - \mathcal{L}_\rho f\|_X, \end{aligned}$$

which completes the proof of the theorem. □

3 Direct and converse estimates of the rate of approximation of convolution operators

3.1 Basic definitions and notations

We shall consider a rather wide class of Banach spaces of real or complex-valued functions of one or several real variables. It includes the Lebesgue spaces $L_p(\mathbb{R}^d)$, $1 \leq p < \infty$, the space of uniformly continuous and bounded functions on $\mathbb{R}^d$, Lipschitz (Hölder) and Besov spaces on $\mathbb{R}^d$ as well as their analogues for functions which are 2π-periodic in each variable.

First, let us introduce a number of basic notations. Throughout the paper $\mathbb{A}$ is either $\mathbb{R}^d$ or $\mathbb{T}^d$ – the dth dimensional torus, $d \in \mathbb{N}$. We denote the elements of $\mathbb{A}$ by $x = (x_1, \ldots, x_d)$, the multiplication of a vector $x \in \mathbb{R}^d$ with a scalar $\rho \in \mathbb{R}$ by $\rho x = (\rho x_1, \ldots, \rho x_d)$ and the dot product of $x, y \in \mathbb{A}$ by $x \cdot y = x_1 y_1 + \cdots + x_d y_d$. For $x \in \mathbb{A}$ we also set $|x| = \sqrt{x \cdot x}$ (or any other norm in $\mathbb{A}$) and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_j > 0,\ j = 1, \ldots, d\}$. We denote the Banach space of all functions summable in the Lebesgue sense on $\mathbb{A}$ by $L(\mathbb{A})$ with the norm

$$\|f\|_L = \int_{\mathbb{A}} |f(x)|\, dx, \quad f \in L(\mathbb{A}).$$

Definition 3.1. (Katznelson [12, Definition I.2.10] and Shapiro [18, Definition 9.3.1.1]) A *homogeneous Banach space* (abbreviated *HBS*) $B(\mathbb{A})$ on $\mathbb{A}$ is a Banach space of Lebesgue measurable functions on $\mathbb{A}$, satisfying the conditions:

(a) The translation is an isometry of $B(\mathbb{A})$ onto itself, i.e. if $f \in B(\mathbb{A})$ and $t \in \mathbb{A}$, then $f_t \in B(\mathbb{A})$ and $\|f_t\|_B = \|f\|_B$, where $f_t(x) = f(x - t)$;

(b) The translation is continuous on $B(\mathbb{A})$, i.e. for all $f \in B(\mathbb{A})$ and $t, t_0 \in \mathbb{A}$ there holds $\lim_{t \to t_0} \|f_t - f_{t_0}\|_B = 0$;

(c) The functions of $B(\mathbb{A})$ are uniformly locally integrable, i.e. there exists a constant c such that for all $f \in B(\mathbb{A})$ and $t \in \mathbb{A}$ there holds

$$\int_{C^d} |f(x - t)|\, dx \leq c\, \|f\|_B,$$

where C^d denotes the unit cube in $\mathbb{R}^d$ $\{x \in \mathbb{R}^d : 0 \leq x_j \leq 1,\ j = 1, \ldots, d\}$.

Two functions in $B(\mathbb{A})$ are considered equivalent if they coincide almost everywhere in the Lebesgue sense.

Remark 3.1. This definition allows HBS of functions, which are 2π-periodic in each variable, and with the same norm to be considered either as defined on $\mathbb{T}^d$ or $\mathbb{R}^d$, but this ambiguity is harmless because in this case both spaces are isomorphic. Also let us note that any HBS of periodic functions is continuously embedded in $L(\mathbb{T}^d)$.

Let $B(\mathbb{A})$ be a HBS on $\mathbb{A}$ and $M(\mathbb{A})$ denote the space of all finite Borel measures μ on $\mathbb{A}$ with the norm

$$\|\mu\|_M = \int_{\mathbb{A}} |d\mu|.$$

The convolution of a function $f \in B(\mathbb{A})$ and a measure $\mu \in M(\mathbb{A})$ is defined by

$$f * d\mu(x) = \int_{\mathbb{A}} f(x-t)\, d\mu(t),$$

as the integral is the Lebesgue-Stieltjes one. As it is known (cf. [18, Theorem 9.3.2.3]), $f * d\mu(x)$ exists almost everywhere, belongs to $B(\mathbb{A})$ and

$$\|f * d\mu\|_B \le \|\mu\|_M \, \|f\|_B. \tag{3.1}$$

In particular, for a measure $d\mu(t) = k(t)\, dt$ with $k \in L(\mathbb{A})$ we have

$$k * f(x) = \int_{\mathbb{A}} k(t) f(x-t)\, dt$$

and

$$\|k * f\|_B \le \|k\|_L \, \|f\|_B. \tag{3.2}$$

Equivalently we can define the convolution by means of Bochner's generalization of the Lebesgue-Stieltjes integral of vector-valued functions (cf. [18, Lemma 9.3.2.2]).

Next we recall several notions through which we shall define the operators whose rate of approximation we shall study.

Definition 3.2. (e.g. [3, Definitions 1.1.1, 1.1.4, 3.1.1 and 3.1.4] and [25, Section 1.3]) Let $d \in \mathbb{N}$ and $\rho \in \mathbb{R}_+^d$. The family $\{k_\rho(t)\}_{\rho\in\mathbb{R}_+^d}$ is called an *approximate identity on* $\mathbb{A}$ if it satisfies the conditions:

(a) For all $\rho \in \mathbb{R}_+^d$ we have $k_\rho \in L(\mathbb{A})$ and

$$\int_{\mathbb{A}} k_\rho(t)\, dt = 1;$$

(b) There exists a constant c such that

$$\|k_\rho\|_L \le c \quad \text{for all} \quad \rho \in \mathbb{R}_+^d;$$

(c) For each $\delta > 0$, there holds

$$\lim_{\min \rho_j \to \infty} \int_{\mathbb{A}\cap\{|t|\ge\delta\}} |k_\rho(t)|\, dt = 0.$$

The function $k_\rho(t)$ is called a *kernel*. The condition $\rho \in \mathbb{R}_+^d$ can be replaced with $\rho \in \mathbb{I}$, where the set $\mathbb{I} \subseteq \mathbb{R}_+^d$ is unbounded on each coordinate.

Remark 3.2. Important examples of approximate identities on $\mathbb{R}^d$ are given by the families $\{\rho^d k(\rho t)\}$ with $\rho > 0$, or more generally $\{\rho_1 \cdots \rho_d k(\rho_1 t_1, \ldots, \rho_d t_d)\}$ with $\rho_j > 0$, where

$$\int_{\mathbb{R}^d} k(t)\, dt = 1.$$

Let us also note that if $\{k_{1,\rho}(t)\}, \ldots, \{k_{d,\rho}(t)\}$ are arbitrary approximate identities on $\mathbb{R}$, respectively on $\mathbb{T}$, then $\{k_\rho(t)\}$, where $\rho = (\rho_1, \ldots, \rho_d)$, $t = (t_1, \ldots, t_d)$ and $k_\rho(t) = k_{1,\rho_1}(t_1) \cdots k_{d,\rho_d}(t_d)$, is an approximate identity on $\mathbb{R}^d$, respectively on $\mathbb{T}^d$.

Definition 3.3. Let $B(\mathbb{A})$ be a HBS on $\mathbb{A}$, $\rho \in \mathbb{R}_+^d$ and $\{k_\rho(t)\}_{\rho \in \mathbb{R}_+^d}$ be an approximate identity on $\mathbb{A}$ given by Definition 3.2. We define the bounded linear operator $J_\rho : B(\mathbb{A}) \to B(\mathbb{A})$ by

$$J_\rho f(x) = k_\rho * f(x) = \int_{\mathbb{A}} k_\rho(t) f(x-t)\, dt, \quad x \in \mathbb{A}.$$

As it is known (cf. [3, Theorems 1.1.5 and 3.1.6], [12, Lemma I.2.2 and Theorem I.2.11], [18, Corollary 9.2.4.1 and Section 9.3.3] and [25, Section 1.3]),

$$\lim_{\min \rho_j \to \infty} \|f - J_\rho f\|_B = 0 \quad \forall f \in B(\mathbb{A}). \tag{3.3}$$

Various upper estimates of the error of J_ρ can be established by means of unweighted moduli of smoothness (cf. [3, Sections 1.5, 1.6, 3.3 and 3.4] and [18, Lemma 9.2.4]). We shall evaluate the rate of convergence of the convolution operator J_ρ by means of the K-functional defined for $f \in B(\mathbb{A})$ and $\tau > 0$ by

$$K(f, \tau; B(\mathbb{A}), \mathcal{D}) = \inf\{\|f-g\|_B + \tau \|\mathcal{D}g\|_B \ : \ g \in \mathcal{D}^{-1}(B(\mathbb{A}))\},$$

where $\mathcal{D}$ is a linear operator and $\mathcal{D}^{-1}(B(\mathbb{A})) = \{g \in B(\mathbb{A}) : \mathcal{D}g \in B(\mathbb{A})\}$ is dense in $B(\mathbb{A})$. We shall assume that $\mathcal{D}$ satisfies several conditions. The first and most important is that it commutes with the convolution:

(i) *Let for $k \in L(\mathbb{A})$ and $g \in B(\mathbb{A})$*

$$\mathcal{D}(k*g) = \begin{cases} \mathcal{D}k * g, & k \in \mathcal{D}^{-1}(L(\mathbb{A})), \\ k * \mathcal{D}g, & g \in \mathcal{D}^{-1}(B(\mathbb{A})), \end{cases}$$

as either $k \in \mathcal{D}^{-1}(L(\mathbb{A}))$ or $g \in \mathcal{D}^{-1}(B(\mathbb{A}))$ implies $k*g \in \mathcal{D}^{-1}(B(\mathbb{A}))$ as well.

If $B(\mathbb{A}) \neq L(\mathbb{A})$, above and throughout the paper we assume that with every operator $\mathcal{D}$ defined on a subset of $B(\mathbb{A})$ another operator is associated, which is denoted by the same letter and is defined on a subset of $L(\mathbb{A})$ in such a way that the first case of condition (i) holds. In the applications $\mathcal{D}$ is defined on every HBS $B(\mathbb{A})$ by one and the same operation. If $\mathcal{D} : \mathcal{D}^{-1}(L(\mathbb{A})) \to L(\mathbb{A})$ is closed, then it satisfies condition (i) with $k \in \mathcal{D}^{-1}(L(\mathbb{A}))$ and $g \in B(\mathbb{A})$; and if $\mathcal{D} : \mathcal{D}^{-1}(B(\mathbb{A})) \to B(\mathbb{A})$ is closed, then it satisfies condition (i) with $k \in L(\mathbb{A})$ and $g \in \mathcal{D}^{-1}(B(\mathbb{A}))$ (cf. [11, Theorem 3.7.12]). However, condition (i) is verified more easily in the setting of the present research.

We denote the Fourier transform of a function $f \in L(\mathbb{A})$ by $\hat{f}$, more precisely, we set

$$\hat{f}(u) = \int_{\mathbb{A}} f(x)\, e^{-i\, u \cdot x}\, dx, \quad u \in \widehat{\mathbb{A}},$$

where

$$\widehat{\mathbb{A}} = \begin{cases} \mathbb{R}^d & \text{if} \quad \mathbb{A} = \mathbb{R}^d, \\ \mathbb{Z}^d & \text{if} \quad \mathbb{A} = \mathbb{T}^d. \end{cases}$$

The Fourier-Stieltjes transform $\widehat{d\mu}$ of a measure $\mu \in M(\mathbb{A})$ is defined by

$$\widehat{d\mu}(u) = \int_{\mathbb{A}} e^{-i\, u \cdot x}\, d\mu(x), \quad u \in \widehat{\mathbb{A}}.$$

3.2 The characterization

The construction considered in Theorems 1.1 and 1.2 is applied in a natural and easy way in the case of convolution operators $\mathcal{L}_\rho$. Then the operators $\mathcal{P}_\rho$ and $\mathcal{Q}_\rho$ are also of this type and can be directly identified by means of the Fourier transform because it turns the convolution into multiplication. It was H. S. Shapiro, who first stated such a method for convolution operators ([17], [18, Section 9.4], or [3, Section 13.3]). The theorem below contains a version of his comparison principle.

Theorem A. (H. S. Shapiro) *Let $B(\mathbb{R}^d)$ be a HBS on $\mathbb{R}^d$. Let $k, \ell \in L(\mathbb{R}^d)$ be such that*

$$\int_{\mathbb{R}^d} k(t)\, dt = \int_{\mathbb{R}^d} \ell(t)\, dt = 1$$

and

$$1 - \hat{k}(u) = (1 - \hat{\ell}(u))\, \widehat{d\mu}(u), \quad u \in \mathbb{R}^d,$$

with some $\mu \in M(\mathbb{R}^d)$. Set for $f \in B(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\rho > 0$

$$\mathcal{K}_\rho f(x) = \rho^d \int_{\mathbb{R}^d} k(\rho t) f(x-t)\, dt \quad \text{and} \quad \mathcal{L}_\rho f(x) = \rho^d \int_{\mathbb{R}^d} \ell(\rho t) f(x-t)\, dt.$$

Then for $f \in B(\mathbb{R}^d)$ and $\rho > 0$ we have

$$\|f - \mathcal{K}_\rho f\|_B \le \|\mu\|_M\, \|f - \mathcal{L}_\rho f\|_B.$$

There is a similar comparison principle in the periodic case (see [25, Sections 6.4.2 and 7.1.12], [21], [22], [24], etc.)

Now, let us explicitly formulate Theorems 1.1 and 1.2 for convolution operators.

Theorem 3.1. *Let $B(\mathbb{A})$ be a HBS on $\mathbb{A}$ and J_ρ be given by Definition 3.3. Let $\mathcal{D}$ satisfy condition (i) and $\mathcal{D}^{-1}(L(\mathbb{A}))$ be dense in $L(\mathbb{A})$. Let also there exist $\varphi : \mathbb{R}_+^d \to \mathbb{R}_+$, $\psi : \widehat{\mathbb{A}} \to \mathbb{C}$, $c \in \mathbb{R}$ and $\lambda_\rho \in M(\mathbb{A})$, $\rho \in \mathbb{R}_+^d$, such that*

$$\widehat{\mathcal{D}\eta}(u) = \psi(u)\hat{\eta}(u), \quad u \in \widehat{\mathbb{A}}, \quad \eta \in \mathcal{D}^{-1}(L(\mathbb{A})), \tag{3.4}$$

$$1 - \hat{k}_\rho(u) = \varphi(\rho)\, \psi(u)\, \widehat{d\lambda}_\rho(u), \quad u \in \widehat{\mathbb{A}}, \quad \forall \rho \in \mathbb{R}_+^d \tag{3.5}$$

and

$$\|\lambda_\rho\|_M \leq c \quad \forall \rho \in \mathbb{R}_+^d. \tag{3.6}$$

Then for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}_+^d$ we have

$$\|f - J_\rho f\|_B \leq c\, K(f, \varphi(\rho); B(\mathbb{A}), \mathcal{D}).$$

Remark 3.3. Relation (3.5) is typical in this context – cf. [3, Chapter 12].

Proof of Theorem 3.1. Let μ_0 be the measure of mass one on $\mathbb{A}$ concentrated at $x = 0$ and let $d\mu_\rho = k_\rho(x)\, dx$. Then for every $f \in B(\mathbb{A})$ we have

$$f - J_\rho f = f - k_\rho * f = f * d(\mu_0 - \mu_\rho). \tag{3.7}$$

By (3.4) and (3.5) we have for every $\eta \in \mathcal{D}^{-1}(L(\mathbb{A}))$

$$\begin{aligned} \hat{\eta}\, (d(\mu_0 - \mu_\rho))\hat{} &= \hat{\eta}(1 - \hat{k}_\rho) = \varphi(\rho)\, \psi\, \hat{\eta}\, \widehat{d\lambda}_\rho \\ &= \varphi(\rho)\, \widehat{\mathcal{D}\eta}\, \widehat{d\lambda}_\rho. \end{aligned}$$

Hence, by the uniqueness of the Fourier-Stieltjes transform, we get

$$\eta * d(\mu_0 - \mu_\rho) = \varphi(\rho)\, \mathcal{D}\eta * d\lambda_\rho,$$

and, consequently, by condition (i) we get for every $g \in \mathcal{D}^{-1}(B(\mathbb{A}))$ and $\eta \in \mathcal{D}^{-1}(L(\mathbb{A}))$

$$\eta * [g * d(\mu_0 - \mu_\rho)] = \eta * [\varphi(\rho)\, \mathcal{D}g * d\lambda_\rho].$$

Now, since $\mathcal{D}^{-1}(L(\mathbb{A}))$ is dense in $L(\mathbb{A})$, we get for every $g \in \mathcal{D}^{-1}(B(\mathbb{A}))$

$$g * d(\mu_0 - \mu_\rho) = \varphi(\rho)\, \mathcal{D}g * d\lambda_\rho,$$

that is,

$$g - J_\rho g = \varphi(\rho)\, \mathcal{D}g * d\lambda_\rho. \tag{3.8}$$

Now Theorem 1.1, in view of (3.1) and (3.6), implies the assertion of the theorem. $\square$

Just similarly we establish the converse inequality. Below by k^{*m} we denote the convolution of the function $k \in L(\mathbb{A})$ with itself m times.

Theorem 3.2. *Let $B(\mathbb{A})$ be a HBS on $\mathbb{A}$ and J_ρ be given by Definition 3.3. Let $\mathcal{D}$ satisfy condition (i) and there exist $\varphi : \mathbb{R}_+^d \to \mathbb{R}_+$, $c \in \mathbb{R}$, $m \in \mathbb{N}$ and $\nu_\rho \in M(\mathbb{A})$, $\rho \in \mathbb{R}_+^d$, such that*

$$k_\rho^{*m} \in \mathcal{D}^{-1}(L(\mathbb{A})) \quad \forall \rho \in \mathbb{R}_+^d, \tag{3.9}$$

$$\varphi(\rho)\, \widehat{\mathcal{D}k_\rho^{*m}}(u) = (1 - \hat{k}_\rho(u))\widehat{d\nu}_\rho(u), \quad u \in \widehat{\mathbb{A}}, \quad \forall \rho \in \mathbb{R}_+^d \tag{3.10}$$

and

$$\|\nu_\rho\|_M \le c \quad \forall \rho \in \mathbb{R}_+^d. \tag{3.11}$$

Then for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}_+^d$ we have

$$K(f, \varphi(\rho); B(\mathbb{A}), \mathcal{D}) \le c\, \|f - J_\rho f\|_B.$$

Proof. First, let us note that, in view of property (i), the condition (3.9) implies that for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}_+^d$ we have

$$k_\rho^{*m} * f \in \mathcal{D}^{-1}(B(\mathbb{A})). \tag{3.12}$$

Further, we shall establish that for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}_+^d$ we have

$$\varphi(\rho)\, \mathcal{D}J_\rho^m f = (f - J_\rho f) * d\nu_\rho. \tag{3.13}$$

Then by Theorem 1.2, using (3.1) and (3.11), we can complete the proof of the theorem.

So it remains to verify (3.13). By the uniqueness of the Fourier transform, (3.10) directly implies

$$\varphi(\rho)\, \mathcal{D}k_\rho^{*m} = d\nu_\rho - k_\rho * d\nu_\rho.$$

Hence for each $f \in B(\mathbb{A})$ we have

$$\varphi(\rho)\, \mathcal{D}k_\rho^{*m} * f = f * d\nu_\rho - k_\rho * f * d\nu_\rho,$$

and finally, by condition (i), we get (3.13). □

Remark 3.4. As a matter of fact, if λ_ρ of Theorem 3.1 satisfies certain additional assumptions, it is possible to derive a formula like (3.13) from (3.8) and hence the corresponding converse inequality. More precisely, let the hypotheses of Theorem 3.1 be satisfied with $d\lambda_\rho = \ell_\rho(t)\, dt$, $\ell_\rho \in L(\mathbb{A})$. Let also there exist $m \in \mathbb{N}$, $\upsilon_\rho \in M(\mathbb{A})$ and $c \in \mathbb{R}$ such that

$$\ell_\rho^{*m} \in \mathcal{D}^{-1}(L(\mathbb{A})) \quad \forall \rho \in \mathbb{R}_+^d, \tag{3.14}$$

$$1 - \hat{\ell}_\rho(u) = (1 - \hat{k}_\rho(u))\, \widehat{d\upsilon}_\rho(u), \quad u \in \widehat{\mathbb{A}}, \quad \forall \rho \in \mathbb{R}_+^d \tag{3.15}$$

and

$$\|\upsilon_\rho\|_M \le c \quad \forall \rho \in \mathbb{R}^d_+. \tag{3.16}$$

Then (3.14) and condition (i) imply that $\ell_\rho^{*m} * f \in \mathcal{D}^{-1}(B(\mathbb{A}))$ for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}^d_+$. Again by condition (i) relation (3.8) can be written for all $g \in \mathcal{D}^{-1}(B(\mathbb{A}))$ and $\rho \in \mathbb{R}^d_+$ in the form

$$g - J_\rho g = \varphi(\rho)\, \mathcal{D}(\ell_\rho * g).$$

The latter with $g = \ell_\rho^{*(m-1)} * f$ gives for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}^d_+$ the representation

$$\varphi(\rho)\, \mathcal{D}(\ell_\rho^{*m} * f) = \ell_\rho^{*(m-1)} * (f - J_\rho f)$$

and hence

$$\varphi(\rho)\, \|\mathcal{D}(\ell_\rho^{*m} * f)\|_B \le c\, \|f - J_\rho f\|_B. \tag{3.17}$$

On the other hand, (3.1), (3.15) and (3.16) imply (cf. Theorem A)

$$\|f - \ell_\rho^{*m} * f\|_B \le c\, \|f - \ell_\rho * f\|_B \le c\, \|f - J_\rho f\|_B \tag{3.18}$$

for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}^d_+$. Now, as in the proof of the preceding theorem, (3.17) and (3.18) imply for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}^d_+$ the converse inequality

$$K(f, \varphi(\rho); B(\mathbb{A}), \mathcal{D}) \le c\, \|f - J_\rho f\|_B.$$

Remark 3.5. Let us observe that, in the conditions of Theorems 3.1 and 3.2 in the case $\mathbb{A} = \mathbb{R}^d$, if $\psi(u) \neq 0$ a.e., then

$$\widehat{k_\rho^{*m}}(u) = \widehat{d\lambda}_\rho(u)\, \widehat{d\nu}_\rho(u), \quad u \in \mathbb{R}^d. \tag{3.19}$$

Hence $k_\rho^{*m}(t)\, dt = d\lambda_\rho * d\nu_\rho$, where the convolution of measures is defined by

$$d\lambda_\rho * d\nu_\rho(E) = \int_{\mathbb{A}} \lambda_\rho(E - t) d\nu_\rho(t)$$

for every Borel set E on $\mathbb{R}^d$. Consequently, if there exist measures λ_ρ and ν_ρ, satisfying the assumptions of Theorems 3.1 and 3.2 in the case $\mathbb{A} = \mathbb{R}^d$, then their convolution is absolutely continuous. Also, if both λ_ρ and ν_ρ are absolutely continuous as $d\lambda_\rho = \ell_\rho(t)\, dt$ and $d\nu_\rho = \upsilon_\rho(t)\, dt$ with $\ell_\rho, \upsilon_\rho \in L(\mathbb{R}^d)$, then $k_\rho^{*m} = \ell_\rho * \upsilon_\rho$, $\rho \in \mathbb{R}^d_+$.

A similar relation holds in the periodic case if $\psi(u) \neq 0$ for $u \in \mathbb{Z}^d \backslash \{0\}$. Then we get relation (3.19) for all $u \in \mathbb{Z}^d$ after correcting (if necessary) each of the measures λ_ρ and ν_ρ by adding a measure of the type $\alpha\, dt$, $\alpha \in \mathbb{C}$.

When the conclusions of Theorems 3.1 and 3.2 hold, there exists a constant $c > 0$ such that for all $f \in B(\mathbb{A})$ and $\rho \in \mathbb{R}^d_+$ we have

$$c^{-1} K(f, \varphi(\rho); B(\mathbb{A}), \mathcal{D}) \le \|f - J_\rho f\|_B \le c K(f, \varphi(\rho); B(\mathbb{A}), \mathcal{D}).$$

We shall denote a relation of this type shortly by

$$\|f - J_\rho f\|_B \sim K(f, \varphi(\rho); B(\mathbb{A}), \mathcal{D}).$$

It readily implies that the saturation rate of J_ρ is $\varphi(\rho)$ and its saturation class consists of all functions $f \in B(\mathbb{A})$ such that $K(f, \tau; B(\mathbb{A}), \mathcal{D}) = \mathcal{O}(\tau)$ as $\tau \to 0+0$. It also yields that a function $g \in B(\mathbb{A})$ belongs to the trivial class of J_ρ if and only if $g \in \mathcal{D}^{-1}(B(\mathbb{A}))$ and $\mathcal{D}g = 0$.

In passing let us note that relations (3.8) and (3.13) can be easily iterated and thus lead to the construction of operators with a greater rate of approximation (see [8, Section 10]).

3.3 A brief comparison and retrospection

Let us recall that considerations like those in the two preceding theorems have been used before to establish the saturation class of convolution operators (cf. [3, (12.2.5), (12.3.13) and (12.3.23)], [6, Sections 3.5 and 3.6] and [14, p. 100]), or direct and converse inequalities (cf. [8, Section 2] and [3, Problem 13.3.2]). Also, Trebels [22] (cf. [4] too) uses the same idea as the one in Theorems 1.1 and 1.2 to treat multiplier operators based on generalized Fourier series. As for the methods themselves, the idea of using uniformly bounded multipliers in the case of convolution integrals on the torus is due to H. Buchwalter, but was worked out fully by G. Sunouchi [3, Section 12.6/Sec. 12.2], whereas the technique based on the Fourier transform for treating convolution integrals on $\mathbb{R}^d$ was developed by P. L. Butzer [3, Section 12.6/Sec. 12.3] (cf. [15] as well). The comparison principle on which Theorems 3.1 and 3.2 are based was formulated by H. S. Shapiro (cf. Theorem A). And last but not least, the notion of a HBS is due to S. Bochner, Y. Katznelson, and H. S. Shapiro (see [18, p. 200]).

So the method formulated in Theorems 3.1 and 3.2 is not new, but, to our knowledge, until now it has not appeared in such a systematic and simple form for so general a class of Banach spaces in connection with characterizing *precisely* the error of convolution operators (by means of K-functionals). It is remarkable that we can establish the approximation rate of such an operator in any HBS only by verifying a couple of conditions in the setting of the concrete HBS $L(\mathbb{A})$ and the space $M(\mathbb{A})$. Let us emphasize that we use Fourier transforms *only* of L-functions and measures.

The characterization of the error of the various operators we consider as application have long been known at least in L_p and uniform norm and several of them in any HBS – the references are given at the appropriate places. However, we did not find any characterization in terms of K-functionals for most of the operators in an arbitrary HBS. Also, some of the proofs presented here of already known results might be new and shorter.

Theorems 1.1 and 1.2 are also applicable to convolution operators on a finite interval. In this case the operators are defined by means of the Legendre convolution (see e.g. [5, Chapter 14] for its relevant properties). Butzer, Stens and Wehrens [7] and Butzer [2] considered the saturation problem for such operators.

Finally, let us mention that the proof of the direct estimate for operators, which are not of a convolution type, also can be realized in the way stated in Theorem 1.1 (see e.g. [8, p. 87]). It is interesting to find out whether this is true for the converse one.

3.4 The univariate case

As we observed earlier, each function $k \in L(\mathbb{R})$ such that

$$\int_{\mathbb{R}} k(t)\, dt = 1$$

generates a kernel $k_\rho^{\mathbb{R}}$, $\rho > 0$, on $\mathbb{R}$ by setting

$$k_\rho^{\mathbb{R}}(t) = \rho k(\rho t). \tag{3.20}$$

Similarly, k generates a kernel $k_\rho^{\mathbb{T}}$, $\rho > 0$, on $\mathbb{T}$ defined by (see [3, Proposition 3.1.12] or [12, VI.1.15])

$$k_\rho^{\mathbb{T}}(t) = \sum_{j=-\infty}^{\infty} \rho k(\rho(t + 2j\pi)). \tag{3.21}$$

In addition to property (i), we shall also assume that the differential operator possesses also the following properties, which are typical for the applications:

(ii) *There exists* $\psi : \mathbb{R} \to \mathbb{C}$ *such that*

$$\widehat{\mathcal{D}\eta}(u) = \psi(u)\hat{\eta}(u), \quad u \in \widehat{\mathbb{A}}, \quad \eta \in \mathcal{D}^{-1}(L(\mathbb{A}));$$

(iii) *If there exists* $\xi \in L(\mathbb{A})$ *such that* $\psi(u)\hat{\eta}(u) = \hat{\xi}(u)$, $u \in \widehat{\mathbb{A}}$, *then* $\eta \in \mathcal{D}^{-1}(L(\mathbb{A}))$;

(iv) *The function* ψ *is homogeneous of order* $\kappa > 0$, *i.e.* $\psi(\rho u) = \rho^\kappa \psi(u)$ *for all* $\rho > 0$ *and* $u \in \mathbb{R}$.

In this situation Theorems 3.1 and 3.2 imply a very simple criterion. In its formulation we add the subscript $\mathbb{A}$ to the notation of the operator $\mathcal{D}$ for convenience.

Theorem 3.3. *Let* $\mathbb{A} = \mathbb{R}$ *or* $\mathbb{A} = \mathbb{T}$, $B(\mathbb{A})$ *be a HBS on* $\mathbb{A}$ *and* J_ρ *be given by Definition 3.3 with a kernel* $k_\rho = k_\rho^{\mathbb{A}}$ *defined by* (3.20) *or* (3.21) *with* $k \in L(\mathbb{R})$, *satisfying*

$$\int_{\mathbb{R}} k(t)\, dt = 1.$$

Let also $\mathcal{D}_{\mathbb{A}}$ *satisfy conditions* (i)-(iv) *and* $\mathcal{D}_{\mathbb{A}}^{-1}(L(\mathbb{A}))$ *be dense in* $L(\mathbb{A})$. *If* $\mathbb{A} = \mathbb{T}$, *assume that there exists* $\mathcal{D}_{\mathbb{R}} : \mathcal{D}_{\mathbb{R}}^{-1}(L(\mathbb{R})) \to L(\mathbb{R})$ *which satisfies condition* *(ii) for $u \in \mathbb{R}$ with the same $\psi(u)$ as $\mathcal{D}_{\mathbb{T}}$. Finally, let there exist $m \in \mathbb{N}$ and $\lambda, \nu \in M(\mathbb{R})$ such that*

$$k^{*m} \in \mathcal{D}_{\mathbb{R}}^{-1}(L(\mathbb{R})), \tag{3.22}$$

$$1 - \hat{k}(u) = \psi(u)\,\widehat{d\lambda}(u), \quad u \in \mathbb{R}, \tag{3.23}$$

and

$$\psi(u)[\hat{k}(u)]^m = (1 - \hat{k}(u))\widehat{d\nu}(u), \quad u \in \mathbb{R}. \tag{3.24}$$

Then for $f \in B(\mathbb{A})$ and $\rho > 0$ we have

$$\|f - J_\rho f\|_B \sim K(f, \rho^{-\kappa}; B(\mathbb{A}), \mathcal{D}_{\mathbb{A}}).$$

Remark 3.6. For $B(\mathbb{A}) = L(\mathbb{A})$ property (i) follows from properties (ii) and (iii).

Proof of Theorem 3.3. We shall verify consecutively the hypotheses of Theorems 3.1 and 3.2 with $d = 1$ and $\varphi(\rho) = \rho^{-\kappa}$. First, (3.4) and condition (ii) are identical.

Next, let us verify (3.9). The Fourier transform of $k_\rho^{\mathbb{A}}$ is (see e.g. [3, Proposition 5.1.28] for the periodic case)

$$\widehat{k_\rho^{\mathbb{A}}}(u) = \hat{k}(u/\rho), \quad u \in \widehat{\mathbb{A}}. \tag{3.25}$$

Set $\ell = \mathcal{D}_{\mathbb{R}}\, k^{*m} \in L(\mathbb{R})$. By condition (ii), we have $\hat{\ell}(u) = \psi(u)[\hat{k}(u)]^m$, $u \in \mathbb{R}$. Let $\ell_\rho^{\mathbb{A}} \in L(\mathbb{A})$, $\rho > 0$, be defined by the formulae (3.20) or (3.21) via ℓ. Then we have by (3.25) and property (iv)

$$\begin{aligned}\psi(u)\widehat{(k_\rho^{\mathbb{A}})^{*m}}(u) = \psi(u)[\widehat{k_\rho^{\mathbb{A}}}(u)]^m &= \rho^\kappa \psi(u/\rho)[\hat{k}(u/\rho)]^m \\ &= \rho^\kappa \hat{\ell}(u/\rho) = \widehat{\rho^\kappa \ell_\rho^{\mathbb{A}}}(u), \quad u \in \widehat{\mathbb{A}},\end{aligned}$$

which, in view of property (iii), implies $(k_\rho^{\mathbb{A}})^{*m} \in \mathcal{D}_{\mathbb{A}}^{-1}(L(\mathbb{A}))$.

Further, in the case $\mathbb{A} = \mathbb{R}$, we set $\lambda_\rho(E) = \lambda(\rho E)$ and $\nu_\rho(E) = \nu(\rho E)$ for every Borel set E on $\mathbb{R}$. Whereas for $\mathbb{A} = \mathbb{T}$, we set

$$\lambda_\rho(E) = \lambda\left(\cup_{j=-\infty}^{\infty}(\rho E + 2\rho j\pi)\right) \quad \text{and} \quad \nu_\rho(E) = \nu\left(\cup_{j=-\infty}^{\infty}(\rho E + 2\rho j\pi)\right)$$

for every Borel set E on $\mathbb{T}$. Then $\lambda_\rho, \nu_\rho \in M(\mathbb{A})$ as conditions (3.6) and (3.11) are satisfied (see e.g. [12, VI.2.5] for the periodic case). For the Fourier-Stieltjes transform of λ_ρ and ν_ρ we have respectively $\widehat{d\lambda_\rho}(u) = \widehat{d\lambda}(u/\rho)$ and $\widehat{d\nu_\rho}(u) = \widehat{d\nu}(u/\rho)$, $u \in \widehat{\mathbb{A}}$ (for the periodic case, this can be found, for example, again in [12, VI.2.5]). Now, in view of (3.25), (3.5) follows from (3.23), and (3.10) follows from condition (ii) and (3.24). $\square$

4 The rate of approximation of convolution operators on $\mathbb{R}^d$

In this and the next section we shall consider just a few examples to illustrate the effectiveness of the method stated in Theorems 3.1 and 3.2. As it is clear from these theorems, the approximation rate of a given convolution operator primarily depends on the behaviour of the Fourier transform of its kernel $\hat{k}_\rho$. That is why the method is easily applicable whenever $\hat{k}_\rho$ has a simple form. Estimates of the rate of convergence of these and other convolution operators can be found in the literature cited in the brief historical account in the previous section.

For most of the operators we shall consider we need the Riesz derivative in order to define an appropriate K-functional. Let us recall its definition. Let $B(\mathbb{R})$ be a HBS on $\mathbb{R}$ and $f \in B(\mathbb{R})$. For $0 < \alpha < 1$ and $h \in \mathbb{R}$ we set

$$n_{h,\alpha}(x) = \frac{1}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \left(\frac{\operatorname{sgn}(x+h)}{|x+h|^{1-\alpha}} - \frac{\operatorname{sgn} x}{|x|^{1-\alpha}} \right) \in L(\mathbb{R}),$$

where $\Gamma(\alpha)$ stands for the Gamma function, and also for $h \in \mathbb{R}$, $h \neq 0$, we set

$$n_h(x) = \begin{cases} \dfrac{1}{\pi x}, & |x| \geq |h|, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$n_{h,0}(x) = n_h(x+h) - n_h(x) \in L(\mathbb{R}).$$

Definition 4.1. Let $g \in B(\mathbb{R})$ and $0 < \alpha \leq 1$. If the limit

$$\lim_{h \to 0} \frac{g * n_{h,1-\alpha}}{h}$$

exists in the B-norm, then we call it the strong Riesz derivative of order α of g and denote it by $D_s^{\{\alpha\}} g$. For $\alpha > 1$ the strong Riesz derivative of order α of $g \in B(\mathbb{R})$ is defined inductively by

$$D_s^{\{\alpha\}} g = \begin{cases} D_s^{\{1\}} \left(D_s^{\{\alpha-1\}} g \right), & \alpha \in \mathbb{N}, \\ D_s^{\{\alpha-[\alpha]\}} \left(D_s^{\{[\alpha]\}} g \right), & \alpha \notin \mathbb{N}. \end{cases}$$

Above, as usual, $[\alpha]$ denotes the largest integer not greater than $\alpha \in \mathbb{R}_+$.

Since the convolution is associative and commutative and also, due to Young's inequality (3.2), it can be considered as a bounded linear operator of each of its arguments, we conclude that the operator $D_s^{\{\alpha\}}$ satisfies condition (i).

We set

$$W^{\{\alpha\}}(B(\mathbb{R})) = \{ g \in B(\mathbb{R}) : D_s^{\{\alpha\}} g \in B(\mathbb{R}) \}.$$

The following property of the Riesz derivative is crucial for its application.

Proposition 4.1. *Let $g \in L(\mathbb{R})$ and $\alpha > 0$. Then $g \in W^{\{\alpha\}}(L(\mathbb{R}))$ if and only if there exists $G \in L(\mathbb{R})$ such that $|u|^\alpha \hat{g}(u) = \widehat{G}(u)$, $u \in \mathbb{R}$. Moreover, we have*

$$(D_s^{\{\alpha\}} g)\widehat{\ }(u) = |u|^\alpha \hat{g}(u), \quad g \in W^{\{\alpha\}}(L(\mathbb{R})).$$

Remark 4.1. The definition of the Riesz derivative we adopt here is a little bit different from the usual one (see e.g. [3, Definition 11.2.5, (11.2.16) and Definition 11.2.8]). The difference concerns the part for $\alpha = 1$. Normally, in the L_p spaces, $D_s^{\{1\}} g$ is defined as the strong first derivative of the conjugate $\tilde{g}$ of g (cf. e.g. [3, (11.2.16)]). The two definitions are equivalent in the case of $L(\mathbb{R})$ as it follows from [3, Theorems 11.2.6, 11.2.7 and 11.2.9] and Proposition 4.1 above. However, Definition 4.1 allows us easily to extend the notion of the Riesz fractional derivative to any HBS as well as to verify that it commutes with the convolution.

Let us also note that for $r \in \mathbb{N}$ we have

$$D_s^{\{r\}} g = (-1)^{[r/2]} \begin{cases} g^{(r)}, & r \text{ is even,} \\ \tilde{g}^{(r)}, & r \text{ is odd.} \end{cases}$$

Proof of Proposition 4.1. In view of Remark 4.1, we only need to verify the assertion for $\alpha = 1$. Let $g \in W^{\{1\}}(L(\mathbb{R}))$. We shall prove that $(D_s^{\{1\}} g)\widehat{\ }(u) = |u|\hat{g}(u)$. First, we shall calculate the Fourier transform of $n_{h,0}$. Let $h \in \mathbb{R}$, $h \neq 0$, be arbitrary and $\delta \in \mathbb{R}$ be such that $\delta > 2|h|$. We have for each $u \in \mathbb{R}$

$$\begin{aligned}
\int_{-\delta}^{\delta} n_h(x) e^{-iux}\, dx &= \frac{1}{\pi} \left(\int_{-\delta}^{-|h|} + \int_{|h|}^{\delta} \right) \frac{e^{-iux}}{x}\, dx \\
&= -\frac{2i}{\pi} \int_{|h|}^{\delta} \frac{\sin ux}{x}\, dx \\
&= -\frac{2}{\pi}\, i\, (\operatorname{sgn} u) \int_{|hu|}^{\delta|u|} \frac{\sin x}{x}\, dx.
\end{aligned}$$

Similarly we establish that for each $u \in \mathbb{R}$

$$\begin{aligned}
&\int_{-\delta}^{\delta} n_h(x+h) e^{-iux}\, dx = e^{iuh} \int_{-\delta+h}^{\delta+h} n_h(x) e^{-iux}\, dx \\
&\quad = e^{iuh} \int_{-(\delta+h)}^{\delta+h} n_h(x) e^{-iux}\, dx - e^{iuh} \int_{-\delta-h}^{-\delta+h} n_h(x) e^{-iux}\, dx \\
&\quad = -\frac{2}{\pi}\, i\, (\operatorname{sgn} u)\, e^{iuh} \int_{|hu|}^{(\delta+h)|u|} \frac{\sin x}{x}\, dx - \frac{e^{iuh}}{\pi} \int_{-\delta-h}^{-\delta+h} \frac{e^{-iux}}{x}\, dx.
\end{aligned}$$

Now, since $n_h(x+h) - n_h(x)$ is summable on $\mathbb{R}$, we have

$$\begin{aligned}\hat{n}_{h,0}(u) &= \int_{\mathbb{R}} [n_h(x+h) - n_h(x)] e^{-iux}\, dx \\ &= \lim_{\delta\to\infty} \int_{-\delta}^{\delta} [n_h(x+h) - n_h(x)] e^{-iux}\, dx \\ &= -\frac{2}{\pi}\, i\,(\operatorname{sgn} u)(e^{iuh} - 1) \int_{|hu|}^{\infty} \frac{\sin x}{x}\, dx.\end{aligned} \tag{4.1}$$

Above we have also used that

$$\left| e^{iuh} \int_{-\delta-h}^{-\delta+h} \frac{e^{-iux}}{x}\, dx \right| \le \frac{2|h|}{\delta - |h|} \to 0 \quad \text{as} \quad \delta \to \infty.$$

Now, (4.1) implies that for each $u \in \mathbb{R}$

$$\lim_{h\to 0} \frac{1}{h}\, \hat{n}_{h,0}(u) = |u|.$$

To complete the proof of the first part of the assertion we need to observe that for every $g \in W^{\{1\}}(L(\mathbb{R}))$ and $u \in \mathbb{R}$ we have

$$\begin{aligned}|h^{-1}\, \hat{n}_{h,0}(u)\, \hat{g}(u) - (D_s^{\{1\}} g)^{\widehat{}}(u)| &= |(h^{-1}\, n_{h,0} * g - D_s^{\{1\}} g)^{\widehat{}}(u)| \\ &\le \|h^{-1}\, n_{h,0} * g - D_s^{\{1\}} g\|_L \to 0 \quad \text{as} \quad h \to 0.\end{aligned}$$

Hence for every $g \in W^{\{1\}}(L(\mathbb{R}))$ and $u \in \mathbb{R}$ there holds

$$(D_s^{\{1\}} g)^{\widehat{}}\,(u) = \lim_{h\to 0} \frac{1}{h}\, \hat{n}_{h,0}(u) \hat{g}(u) = |u| \hat{g}(u).$$

Now, let us verify the other part of the proposition. So let $g \in L(\mathbb{R})$ be such that there exists $G \in L(\mathbb{R})$ with $|u|\hat{g}(u) = \widehat{G}(u)$, $u \in \mathbb{R}$. We shall prove that $g \in W^{\{1\}}(L(\mathbb{R}))$ as $D_s^{\{1\}} g = G$. Let $\chi_S(x)$ denote the characteristic function of the set S. Then

$$\widehat{\chi}_{[-h,0]}(u) = \frac{e^{iuh} - 1}{iu}, \quad u \in \mathbb{R}.$$

We put for $x \in \mathbb{R}$

$$\ell(x) = \frac{1}{\pi^2 x} \log \left| \frac{1+x}{1-x} \right|.$$

As it is known (see e.g. [9, p. 18, (11)] or [3, p. 492])

$$\hat{\ell}(u) = \frac{2}{\pi} \int_{|u|}^{\infty} \frac{\sin v}{v}\, dv, \quad u \in \mathbb{R}.$$

Let us also set $\ell_\rho(x) = \rho \ell(\rho x)$ for $\rho > 0$. Then $\hat{\ell}_\rho(u) = \hat{\ell}(u/\rho)$. Besides that

$$\int_{\mathbb{R}} \ell(x)\, dx = 1.$$

Hence $\{\ell_\rho\}_{\rho\in\mathbb{R}_+}$ is an approximate identity and we have by (3.3)

$$\lim_{\rho\to\infty}\|f-\ell_\rho * f\|_L = 0 \tag{4.2}$$

for all $f\in L(\mathbb{R})$. Further, by (4.1) we have for $h\neq 0$ and $u\in\mathbb{R}$

$$\begin{aligned}(n_{h,0}*g)\widehat{\ }(u) &= \frac{e^{iuh}-1}{iu}\,\frac{2}{\pi}\int_{|hu|}^{\infty}\frac{\sin x}{x}\,dx\,|u|\,\hat{g}(u)\\ &= \widehat{\chi}_{[-h,0]}(u)\,\hat{\ell}_{|h|^{-1}}(u)\,\widehat{G}(u)\\ &= (\chi_{[-h,0]}*\ell_{|h|^{-1}}*G)\widehat{\ }(u),\end{aligned}$$

which, because of the uniqueness of the Fourier transform, yields

$$\begin{aligned}n_{h,0}*g(x) = \chi_{[-h,0]}*\ell_{|h|^{-1}}*G(x) &= \int_0^h \ell_{|h|^{-1}}*G(x+y)\,dy\\ &= h\int_0^1 \ell_{|h|^{-1}}*G(x+hy)\,dy \quad \text{a.e.}\end{aligned}$$

Consequently,

$$\begin{aligned}&\|h^{-1}n_{h,0}*g-G\|_L \le \int_0^1 \|\ell_{|h|^{-1}}*G(\circ+hy)-G(\circ)\|_L\,dy\\ &\le \int_0^1 \|\ell_{|h|^{-1}}*G(\circ+hy)-G(\circ+hy)\|_L\,dy + \int_0^1 \|G(\circ+hy)-G(\circ)\|_L\,dy\\ &= \|\ell_{|h|^{-1}}*G-G\|_L + \int_0^1 \|G(\circ+hy)-G(\circ)\|_L\,dy.\end{aligned}$$

Now, (4.2) and the continuity in the mean ($L(\mathbb{R})$ is a HBS) imply that

$$\lim_{h\to 0}\|h^{-1}n_{h,0}*g-G\|_L = 0$$

and hence by Definition 4.1 $g\in W^{\{1\}}(L(\mathbb{R}))$ as $D_s^{\{1\}}g=G$. □

Proposition 4.1 implies (via a standard argument) that all C^∞ functions of rapid decrease (the Schwartz space) have Riesz derivatives of an arbitrary order in $L(\mathbb{R})$. Hence $W^{\{\alpha\}}(L(\mathbb{R}))$ is dense in $L(\mathbb{R})$ for any $\alpha>0$. Let us also recall that if $g\in W^{\{\alpha\}}(L(\mathbb{R}))$, then $g\in W^{\{\beta\}}(L(\mathbb{R}))$ for every $0<\beta<\alpha$ (cf. [3, Theorems 6.3.14, 11.2.6, 11.2.7 and 11.2.9]).

4.1 The generalized singular integral of Picard

The generalized univariate singular integral of Picard of the function $f\in B(\mathbb{R})$ is defined by

$$C_{\kappa,\rho}f(x) = \rho\int_{\mathbb{R}} c_\kappa(\rho t)f(x-t)\,dt, \quad x\in\mathbb{R},$$

where the kernel c_κ, $\kappa > 0$, is given by its Fourier transform

$$\hat{c}_\kappa(u) = (1 + |u|^\kappa)^{-1}. \tag{4.3}$$

In particular, for $\kappa = 2$ we get the classical singular integral of Picard. In this case we have $c_2(t) = (1/2)\exp(-|t|)$.

Theorem 4.2. *Let $f \in B(\mathbb{R})$ and $\rho > 0$. Then*

$$\|f - C_{\kappa,\rho} f\|_B \sim K(f, \rho^{-\kappa}; B(\mathbb{R}), D_s^{\{\kappa\}}).$$

Proof. For each $\kappa > 0$ the operator $C_{\kappa,\rho}$ is given by Definition 3.3 by means of the kernel $k_\rho(t) = \rho c_\kappa(\rho t)$. We apply Theorem 3.3 with $k(t) = c_\kappa(t)$, $\mathcal{D}_\mathbb{R} = D_s^{\{\kappa\}}$ and $\psi(u) = |u|^\kappa$. Condition (ii) follows from Proposition 4.1. Relation (3.23) takes the form

$$\frac{|u|^\kappa}{1 + |u|^\kappa} = |u|^\kappa\, \widehat{d\lambda}(u)$$

and, in view of (4.3), it is satisfied with $d\lambda = c_\kappa(t)\, dt$.

Next, we shall show that if $m \in \mathbb{N}$ is such that $2(\kappa m - \bar{\kappa}) > 1$, where $\bar{\kappa} = \max\{\kappa, 1\}$, then $c_\kappa^{*m} \in W^{\{\kappa\}}(L(\mathbb{R}))$. To this end, we observe that the function $|u|^{\bar{\kappa}} \widehat{c_\kappa^{*m}}(u) = |u|^{\bar{\kappa}} \left(\hat{c}_\kappa(u)\right)^m$ is locally absolutely continuous and together with its first derivative belongs to $L_2(\mathbb{R})$. Consequently, the function $|u|^{\bar{\kappa}} \widehat{c_\kappa^{*m}}(u)$ is the Fourier transform of a function in $L(\mathbb{R})$ (see e.g. [3, Proposition 6.3.10]). Then, since $c_\kappa^{*m} \in L(\mathbb{R})$, we get that $c_\kappa^{*m} \in W^{\{\bar{\kappa}\}}(L(\mathbb{R})) \subseteq W^{\{\kappa\}}(L(\mathbb{R}))$.

Further, (3.24) takes the form

$$|u|^\kappa \frac{1}{(1 + |u|^\kappa)^m} = \frac{|u|^\kappa}{1 + |u|^\kappa}\, \widehat{d\nu}(u).$$

Hence it is satisfied with $d\nu = c_\kappa^{*(m-1)}(t)\, dt$ $(m > 1)$.

Now, Theorem 3.3 implies the assertion. □

Remark 4.2. It is worth noting that by (3.8) we have the following representation for the generalized singular integral of Picard (cf. [3, (12.4.11)])

$$g - C_{\kappa,\rho} g = \rho^{-\kappa}\, C_{\kappa,\rho} D_s^{\{\kappa\}} g, \quad g \in W^{\{\kappa\}}(B(\mathbb{R})), \quad \rho > 0. \tag{4.4}$$

In view of Remark 3.4, the converse estimate can be derived via (4.4).

4.2 The Riesz means

For $f \in B(\mathbb{R})$ the Riesz means is given by

$$R_{\kappa,\rho} f(x) = \rho \int_\mathbb{R} r_\kappa(\rho t) f(x - t)\, dt, \quad x \in \mathbb{R},$$

where the kernel r_κ, $\kappa > 0$, is defined by its Fourier transform

$$\hat{r}_\kappa(u) = \begin{cases} 1 - |u|^\kappa, & u \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $R_{1,\rho}$ is called the singular integral of Fejér. In this case, we have

$$r_1(t) = \frac{2}{\pi}\,\frac{\sin^2(t/2)}{t^2}.$$

Theorem 4.3. *Let $f \in B(\mathbb{R})$ and $\rho > 0$. Then*

$$\|f - R_{\kappa,\rho} f\|_B \sim K(f, \rho^{-\kappa}; B(\mathbb{R}), D_s^{\{\kappa\}}).$$

Proof. For each $\kappa > 0$ the operator $R_{\kappa,\rho}$ is given by Definition 3.3 by means of the kernel $k_\rho(t) = \rho\, r_\kappa(\rho t)$. Again, we apply Theorem 3.3. In view of Proposition 4.1, relation condition (ii) is satisfied with $\mathcal{D}_{\mathbb{R}} = D_s^{\{\kappa\}}$ and $\psi(u) = |u|^\kappa$. Relation (3.23) takes the form

$$|u|^\kappa\, \widehat{d\lambda}(u) = \begin{cases} |u|^\kappa, & u \in [-1,1], \\ 1, & \text{otherwise;} \end{cases}$$

that is,

$$\widehat{d\lambda}(u) = \begin{cases} 1, & u \in [-1,1], \\ |u|^{-\kappa}, & \text{otherwise.} \end{cases}$$

By [3, Theorem 6.3.11 and Problem 6.3.6] it is shown that the function on the right-hand side above is the Fourier transform of a function $\ell \in L(\mathbb{R})$. Then $d\lambda = \ell(t)\, dt$. For $\kappa \in \mathbb{N}$ the function $\ell(t)$ is given by $\ell(t) = \ell_1^{*\kappa}(t)$, where

$$\ell_1(t) = \frac{1}{\pi} \int_{|t|}^{\infty} \frac{\sin\tau}{\tau^2}\, d\tau$$

(see [3, p. 516]).

Next, we verify that $r_\kappa \in W^{\{\kappa\}}(L(\mathbb{R}))$ similarly to the case of the Picard operator. So we can set $m = 1$ in (3.22). Finally, we observe that (3.24) is satisfied with the measure $d\nu = r_\kappa(t)\, dt$. $\square$

Remark 4.3. It is good to point out that (3.13) yields the following identity for the Riesz means

$$\rho^{-\kappa}\, D_s^{\{\kappa\}} R_{\kappa,\rho} f = R_{\kappa,\rho}(f - R_{\kappa,\rho} f).$$

Similarly, we can characterize the error of the Riesz means with the kernel $r_{\kappa,\theta}$, $\kappa, \theta > 0$, defined by its Fourier transform

$$\hat{r}_{\kappa,\theta}(u) = \begin{cases} (1 - |u|^\kappa)^\theta, & u \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

4.3 The generalized singular integral of Weierstrass

The generalized univariate singular integral of Weierstrass of the function $f \in B(\mathbb{R})$ is defined by

$$W_{\kappa,\rho} f(x) = \rho \int_{\mathbb{R}} w_\kappa(\rho t) f(x-t)\, dt, \quad x \in \mathbb{R},$$

where the kernel w_κ, $\kappa > 0$, is given by its Fourier transform

$$\hat{w}_\kappa(u) = e^{-|u|^\kappa}.$$

For $\kappa = 1$ the convolution operator above is also called the singular integral of Cauchy-Poisson, whereas for $\kappa = 2$ the singular integral of Gauss-Weierstrass. In these two cases w_κ has explicit forms:

$$w_1(t) = \frac{1}{\pi} \frac{1}{1+t^2}, \quad w_2(t) = \frac{1}{2\sqrt{\pi}}\, e^{-t^2/4}.$$

Just similarly to the results in the previous two subsections we get for all $f \in B(\mathbb{R})$ and $\rho > 0$ the characterization

$$\|f - W_{\kappa,\rho} f\|_B \sim K(f, \rho^{-\kappa}; B(\mathbb{R}), D_s^{\{\kappa\}}).$$

In respect to the direct estimate (more precisely, relations (3.23)) we refer to [3, Section 12.4.3]. To establish the converse inequality we apply Theorem 3.2 (Theorem 3.3) with $m = 1$.

4.4 The multivariate singular integral of Gauss-Weierstrass

As an example of a multidimensional convolution operator, let us consider the multivariate Gauss-Weierstrass singular integral, which is defined for $f \in B(\mathbb{R}^d)$ and $\rho > 0$ by

$$W_\rho f(x) = \rho^d \int_{\mathbb{R}^d} w(\rho t) f(x-t)\, dt, \quad x \in \mathbb{R}^d,$$

with a kernel

$$w(t) = \frac{e^{-\frac{1}{4}\, t \cdot t}}{(4\pi)^{d/2}}.$$

The differential operator associated with the multivariate singular integral of Gauss-Weierstrass is given by

$$\Delta g = g''_{x_1 x_1} + \cdots + g''_{x_d x_d},$$

as each partial derivative is taken in the strong sense (in the B-norm). Let us denote by $\Delta^{-1}(B(\mathbb{R}^d))$ the set of all functions $g \in B(\mathbb{R}^d)$ with the latter property. For $\mathcal{D} = \Delta$, (3.4) holds with $\psi(u) = -u \cdot u$.

Theorem 4.4. *(see [8, Theorem 5.4]) Let $f \in B(\mathbb{R})$ and $\rho > 0$. Then*

$$\|f - W_\rho f\|_B \sim K(f, \rho^{-2}; B(\mathbb{R}^d), \Delta).$$

Proof. Clearly, we have $\hat{w}(u) = \hat{w}_2(u_1) \cdots \hat{w}_2(u_n) = \exp(-u \cdot u)$. Starting from the identity (cf. [3, Section 12.4.3])

$$1 - e^{-v} = v \int_0^1 e^{-v\tau}\, d\tau,$$

we consecutively get

$$\begin{aligned} 1 - \hat{w}(u) &= 1 - e^{-u \cdot u} = u \cdot u \int_0^1 e^{-\tau\, u \cdot u}\, d\tau \\ &= u \cdot u \int_0^1 \hat{w}(\tau^{1/2} u)\, d\tau = u \cdot u \int_0^1 (\tau^{-d/2} w(\tau^{-1/2} \circ))\widehat{\ }(u)\, d\tau \\ &= u \cdot u \left(\int_0^1 \tau^{-d/2} w(\tau^{-1/2} \circ)\, d\tau \right)^{\widehat{}} (u), \end{aligned}$$

as at the last step we have applied Fubini's theorem. Consequently, relation (3.5) holds with $k_\rho(t) = \rho^d w(\rho t)$, $\varphi(\rho) = \rho^{-2}$, $\psi(u) = -u \cdot u$ and $d\lambda_\rho = \rho^d \ell(\rho t)\, dt$, where

$$\ell(t) = - \int_0^1 \tau^{-d/2} w(t \tau^{-1/2})\, d\tau.$$

Now, Theorem 3.1 implies that there exists an absolute constant c such that for all $f \in B(\mathbb{R}^d)$ and $\rho > 0$ there holds the upper estimate

$$\|f - W_\rho f\|_B \le c\, K(f, \rho^{-2}; B(\mathbb{R}^d), \Delta).$$

As for the converse inequality, it readily follows from Theorem 3.2 with $m = 1$ as we take into consideration that the function

$$\tau(u) = - \frac{u \cdot u\ e^{-u \cdot u}}{1 - e^{-u \cdot u}}$$

belongs to the Schwartz space of C^∞ functions of d variables of rapid decrease and hence it is the Fourier transform of a function of the same class (which, on its part, is contained in $L(\mathbb{R}^d)$). More precisely, we have $\hat{v}(u) = \tau(u)$ for $v(x) = (2\pi)^{-d} \hat{\tau}(-x)$ and then we set $d\nu_\rho = \rho^d v(\rho t)\, dt$. □

Remark 4.4. In passing, let us note that by (3.8) we have the following functional equation for the multivariate Gauss-Weierstrass singular integral (cf. e.g. [3, (12.4.16)])

$$W_\rho g(x) - g(x) = \rho^{-2} \int_0^1 W_{\rho \tau^{-1/2}} \Delta g(x)\, d\tau$$

for $g \in \Delta^{-1}(B(\mathbb{R}^d))$ and $\rho > 0$.

5 The rate of approximation of convolution operators on $\mathbb{T}$

Let $B(\mathbb{T})$ be a HBS on $\mathbb{T}$. Again we shall first define a fractional derivative with appropriate properties. Since in the case of $\mathbb{T}$ we have that $B(\mathbb{T}) \subseteq L(\mathbb{T})$, each element of $B(\mathbb{T})$ has a Fourier transform in the classical sense. This allows us to introduce a fractional derivative in such a way that we have the analogue of Proposition 4.1 simply by definition and preserve the commutativity condition (i).

Definition 5.1. (cf. [3, Definition 11.5.10]) Let $g \in B(\mathbb{T})$ and $\alpha > 0$. If there exists $G \in B(\mathbb{T})$ such that $|k|^\alpha \hat{g}(k) = \widehat{G}(k)$, $k \in \mathbb{Z}$, then G is called the Riesz derivative of g of order α and is denoted by $D^{\{\alpha\}}g$.

For an equivalent definition of the Riesz derivative in the periodic case by means of a convolution we refer the reader to [3, Theorem 11.5.4]

It is directly checked that $D^{\{\alpha\}}$ satisfies condition (i). Also, let us explicitly note that each trigonometric polynomial has a Riesz derivative of an arbitrary order and the set of trigonometric polynomials is dense in $L(\mathbb{T})$.

Finally, let us recall a criterion by which we can verify that a given function on $\mathbb{Z}$ is the Fourier transform of a summable periodic function (see e.g. [3, Corollary 6.3.9], or [12, Theorem I.4.1 and its proof]). We set for a function $\mathrm{v}(k)$, defined on $\mathbb{Z}$, $\Delta^2\mathrm{v}(k) = \mathrm{v}(k+1) - 2\mathrm{v}(k) + \mathrm{v}(k-1)$ for $k \in \mathbb{Z}$.

Theorem B. *If* v_n*,* $n \in \mathbb{N}$*, are even functions on* $\mathbb{Z}$ *such that* $\lim_{k\to\infty} \mathrm{v}_n(k) = 0$ *and*

$$\sum_{k=1}^{\infty} k|\Delta^2 \mathrm{v}_n(k)|$$

is uniformly bounded for $n \in \mathbb{N}$*, then there exist (even) functions* $v_n \in L(\mathbb{T})$ *such that* $\hat{v}_n = \mathrm{v}_n$ *and* $\|v_n\|_L$ *is uniformly bounded for* $n \in \mathbb{N}$*.*

5.1 The Riesz typical means

As a first example let us consider the typical means of the Fourier series of $f \in B(\mathbb{T})$. It is defined for $f \in B(\mathbb{T})$ and $n \in \mathbb{N}_0$ by

$$R_{\kappa,n} f(x) = \int_{-\pi}^{\pi} r_{\kappa,n}(t) f(x-t)\, dt, \quad x \in \mathbb{T},$$

where the kernel $r_{\kappa,n}$, $\kappa > 0$, is given by

$$r_{\kappa,n}(t) = \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \left|\frac{k}{n+1}\right|^\kappa\right) e^{ikt}.$$

Since $r_{\kappa,n}$ is generated by r_κ (see Section 4.2) through formula (3.21) with $\rho = n+1$, we can derive the following characterization of the error of the Riesz typical means on $\mathbb{T}$ from Theorem 3.3.

Theorem 5.1. *(see [8, Theorem 2.2 and Remark 2.4]) Let $f \in B(\mathbb{T})$ and $n \in \mathbb{N}$. Then*

$$\|f - R_{\kappa,n} f\|_B \sim K(f, n^{-\kappa}; B(\mathbb{T}), D^{\{\kappa\}}). \tag{5.1}$$

Proof. We apply Theorem 3.3 with $\mathbb{A} = \mathbb{T}$, $k = r_\kappa$, $\mathcal{D}_{\mathbb{T}} = D^{\{\kappa\}}$, $\mathcal{D}_{\mathbb{R}} = D_s^{\{\kappa\}}$, $\psi(u) = |u|^\kappa$ and $m = 1$. Conditions (i)-(iv) with $\mathbb{A} = \mathbb{T}$ are trivially satisfied, condition (ii) with $\mathbb{A} = \mathbb{R}$ follows from Proposition 4.1 and (3.22)-(3.24) have been checked in the proof of Theorem 4.3. □

Remark 5.1. Alternatively, this characterization can be established directly by means of Theorems 3.1 and 3.2 without resorting to the connection between the two versions of the Riesz means. Actually, the method of these theorems is based on the same ideas as the proof of (5.1), given by Z. Ditzian and K. Ivanov, as we have already pointed out.

To get the direct estimate we can apply Theorem 3.1 with $\mathcal{D} = D^{\{\kappa\}}$, $\psi(k) = |k|^\kappa$, $k \in \mathbb{Z}$, and $\varphi(n) = (n+1)^{-\kappa}$. For the Fourier transform of the trigonometric polynomial $r_{\kappa,n}$ we have

$$\hat{r}_{\kappa,n}(k) = \begin{cases} 1 - \left|\dfrac{k}{n+1}\right|^\kappa, & |k| \le n, \\ 0, & |k| > n. \end{cases}$$

Then it is enough to show that there exist functions $\ell_n \in L(\mathbb{T})$ with uniformly bounded norms such that

$$\hat{\ell}_n(k) = \begin{cases} 1, & |k| \le n, \\ \left|\dfrac{n+1}{k}\right|^\kappa, & |k| > n. \end{cases}$$

This follows by means of Theorem B and was verified in this particular case by DeVore [6, pp. 67-68] (see also [8, p. 68]).

The converse estimate follows likewise from Theorem 3.2 with $m = 1$ and $d\nu_n = r_{\kappa,n}(t)\, dt$ since

$$\frac{\varphi(n)|k|^\kappa\, \hat{r}_{\kappa,n}(k)}{1 - \hat{r}_{\kappa,n}(k)} = \hat{r}_{\kappa,n}(k), \quad k \in \mathbb{Z}. \tag{5.2}$$

Remark 5.2. Let us explicitly note that relation (5.2) implies (cf. [8, (2.13)])

$$(n+1)^{-\kappa}\, D^{\{\kappa\}} R_{\kappa,n} f = R_{\kappa,n}(f - R_{\kappa,n} f)$$

for all $f \in B(\mathbb{T})$ and $n \in \mathbb{N}_0$.

5.2 The singular integrals of Jackson and of Jackson-de la Vallée Poussin

The well-known Jackson operator is defined for $f \in B(\mathbb{T})$ and $n \in \mathbb{N}$ by

$$\mathcal{J}_n f(x) = \int_{-\pi}^{\pi} j_n(t) f(x-t)\, dt, \quad x \in \mathbb{T},$$

where

$$j_n(t) = \frac{3}{2\pi n(2n^2+1)} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^4.$$

Similar to it (in definition and properties) is the Jackson-de la Vallée Poussin operator. It is given by

$$V_n f(x) = \int_{-\pi}^{\pi} \varrho_n(t) f(x-t)\, dt, \quad x \in \mathbb{T},$$

where

$$\varrho_n(t) = \frac{2+\cos t}{4\pi n^3} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^4.$$

The kernels j_n and ϱ_n are trigonometric polynomials with real coefficients.

Theorem 5.2. *Let $f \in B(\mathbb{T})$ and $n \in \mathbb{N}$. Then*

$$\|f - \mathcal{J}_n f\|_B \sim \|f - V_n f\|_B \sim K(f, n^{-2}; B(\mathbb{T}), (d/dx)_s^2).$$

Above $(d/dx)_s^2$ denotes the strong second derivative.

Proof of Theorem 5.2. We apply Theorems 3.1 and 3.2 with $\mathcal{D}g = g''$ (in the strong sense), $\psi(k) = -k^2$, $\varphi(n) = n^{-2}$ and $m = 1$. Conditions (i), (3.4) and (3.9) are directly checked.

The Fourier transforms of the kernels are (see e.g. [3, p. 517])

$$\widehat{j}_n(k) = \frac{1}{2n(2n^2+1)} \begin{cases} 3|k|^3 - 6nk^2 - 3|k| + 4n^3 + 2n, & |k| \le n, \\ -|k|^3 + 6nk^2 - (12n^2-1)|k| & \\ \qquad\qquad\qquad + 8n^3 - 2n, & n \le |k| \le 2n-1, \\ 0, & |k| \ge 2n-1 \end{cases}$$

and

$$\hat{\varrho}_n(k) = \begin{cases} 1 - \dfrac{3}{2}\left(\dfrac{k}{n}\right)^2 + \dfrac{3}{4}\left|\dfrac{k}{n}\right|^3, & |k| \le n, \\ \dfrac{1}{4}\left(2 - \left|\dfrac{k}{n}\right|\right)^3, & n \le |k| \le 2n, \\ 0, & |k| \ge 2n. \end{cases}$$

To establish (3.5)-(3.6) and (3.10)-(3.11) for each of the two operators, we shall apply Theorem B. Throughout the proof c denotes absolute positive constants. Let us put for $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$

$$\mathrm{v}_{1,n}(k) = \begin{cases} \left(\frac{n}{k}\right)^2 (\widehat{\jmath}_n(k) - 1), & k > 0, \\ -2, & k = 0, \end{cases}$$

$$\mathrm{v}_{2,n}(k) = \begin{cases} \left(\frac{n}{k}\right)^2 (\hat{\varrho}_n(k) - 1), & k > 0, \\ -\frac{3}{2}, & k = 0, \end{cases}$$

$$\mathrm{v}_{3,n}(k) = \begin{cases} \left(\frac{k}{n}\right)^2 \frac{\widehat{\jmath}_n(k)}{\widehat{\jmath}_n(k) - 1}, & k > 0, \\ 0, & k = 0, \end{cases}$$

$$\mathrm{v}_{4,n}(k) = \begin{cases} \left(\frac{k}{n}\right)^2 \frac{\hat{\varrho}_n(k)}{\hat{\varrho}_n(k) - 1}, & k > 0, \\ -\frac{2}{3}, & k = 0. \end{cases}$$

Note that the value of $\mathrm{v}_{j,n}(k)$ at $k = 0$ is immaterial (provided that it is finite) because $1 - \widehat{\jmath}_n(0) = 1 - \hat{\varrho}_n(0) = \psi(0) = 0$ for all $n \in \mathbb{N}$. We defined $\mathrm{v}_{j,n}(0)$ in the way above for convenience.

Obviously the functions $\mathrm{v}_{j,n}(k)$ are even and tend to 0 with $k \to \infty$. It remains to show that

$$\sum_{k=1}^{\infty} k|\Delta^2 \mathrm{v}_{j,n}(k)| \le c \quad \forall n \in \mathbb{N}, \quad j = 1, 2, 3, 4. \tag{5.3}$$

Before proceeding to the verification of these relations, we recall that if

$$\mathrm{v}_n(\ell) = \mathrm{u}(\ell/n), \quad \ell = k-1, k, k+1$$

with some $\mathrm{u} \in W^2_\infty[(k-1)/n, (k+1)/n]$, then

$$\begin{aligned} |\Delta^2 \mathrm{v}_n(k)| &= \left| \mathrm{u}\left(\frac{k}{n} + \frac{1}{n}\right) - 2\mathrm{u}\left(\frac{k}{n}\right) + \mathrm{u}\left(\frac{k}{n} - \frac{1}{n}\right) \right| \\ &\le \frac{1}{n^2} \|\mathrm{u}''\|_{L_\infty[\frac{k-1}{n}, \frac{k+1}{n}]}. \end{aligned} \tag{5.4}$$

If also $\mathrm{u}''(x) \ge 0$ a.e. on $[(k-1)/n, (k+1)/n]$, then $\Delta^2 \mathrm{v}_n(k) \ge 0$ too.

To establish (5.3) for $j = 1$, we set

$$\mathrm{u}_{1,n}(x) = \begin{cases} \dfrac{3(x^2 - 2x - n^{-2})}{2(2 + n^{-2})x}, & 1/n \le x \le 1, \\ \dfrac{-x^3 + 6x^2 - 12x + n^{-2}x + 4 - 4n^{-2}}{2(2 + n^{-2})x^2}, & 1 < x \le 2 - 1/n, \\ -\dfrac{1}{x^2}, & x > 2 - 1/n. \end{cases}$$

Then we have

$$\mathrm{v}_{1,n}(k) = \mathrm{u}_{1,n}(k/n), \quad k \in \mathbb{N}.$$

We have $\mathrm{u}_{1,n} \in W^2_\infty$ on each of the intervals $[1/n, 1]$, $[1, 2-1/n]$ and $[2-1/n, \infty)$ as its second derivative is

$$\mathrm{u}''_{1,n}(x) = \begin{cases} -\dfrac{3}{(2n^2 + 1)x^3}, & 1/n < x < 1, \\ -\dfrac{12n^2(x-1) + 12 - x}{(2n^2 + 1)x^4}, & 1 < x < 2 - 1/n, \\ -\dfrac{6}{x^4}, & x > 2 - 1/n. \end{cases}$$

Thus $\mathrm{u}''_{1,n}(x) \le 0$ a.e. and, consequently, $\Delta^2 \mathrm{v}_{1,n}(k) \le 0$ for all k such that $2 \le k \le n-1$, or $n+1 \le k \le 2n-2$, or $k \ge 2n$. Further, we calculate

$$\begin{aligned} \Delta^2 \mathrm{v}_{1,n}(1) &= -\frac{4n^2 - 9n + 8}{4(2n^2 + 1)} < 0, \quad n \ge 1, \\ \Delta^2 \mathrm{v}_{1,n}(n) &= -\frac{3}{(2n^2 + 1)(n^2 - 1)} < 0, \quad n \ge 2, \\ \Delta^2 \mathrm{v}_{1,n}(2n-1) &= -\frac{12n^3 - 12n^2 + 8n - 1}{4(2n^2 + 1)(n-1)(2n-1)^2} < 0, \quad n \ge 2. \end{aligned}$$

Thus, for each fixed $n \in \mathbb{N}$ and every $N \in \mathbb{N}$ such that $N \ge 2n-1$, we arrive at

$$\begin{aligned} \sum_{k=1}^{\infty} k |\Delta^2 \mathrm{v}_{1,n}(k)| &= -\lim_{N \to \infty} \sum_{k=1}^{N} k\, \Delta^2 \mathrm{v}_{1,n}(k) \\ &= \lim_{N \to \infty} (-\mathrm{v}_{1,n}(0) + (N+1)\mathrm{v}_{1,n}(N) - N\mathrm{v}_{1,n}(N+1)) \\ &= 2 - n^2 \lim_{N \to \infty} \left(\frac{N+1}{N^2} - \frac{N}{(N+1)^2} \right) = 2. \end{aligned}$$

Analogously, to prove (5.3) for $j = 2$, we note that

$$\mathrm{v}_{2,n}(k) = \mathrm{u}_2(k/n), \quad k \in \mathbb{N}_0,$$

with

$$\mathrm{u}_2(x) = \begin{cases} -\dfrac{3}{4}(2-x), & 0 \le x \le 1, \\ -\dfrac{4-(2-x)^3}{4x^2}, & 1 < x \le 2, \\ -\dfrac{1}{x^2}, & x > 2. \end{cases}$$

We have $\mathrm{u}_2 \in W^2_\infty(\mathbb{R}_+)$, as

$$\mathrm{u}_2''(x) = \begin{cases} 0, & 0 \le x \le 1, \\ -\dfrac{6(x-1)}{x^4}, & 1 < x \le 2, \\ -\dfrac{6}{x^4}, & x > 2. \end{cases}$$

Consequently, $\mathrm{u}_2''(x) \le 0$ for $x \ge 0$, which implies that $\Delta^2 \mathrm{v}_{2,n}(k) \le 0$ for all $k \in \mathbb{N}$ (actually $\Delta^2 \mathrm{v}_{2,n}(k) = 0$ for $1 \le k \le n-1$). This enable us to complete the proof of (5.3) for $j = 2$ just as in the previous case.

Alternatively, we can use (5.4) and get

$$\begin{aligned} \sum_{k=1}^{\infty} k|\Delta^2 \mathrm{v}_{2,n}(k)| &= \sum_{k=1}^{2n-1} k|\Delta^2 \mathrm{v}_{2,n}(k)| + \sum_{k=2n}^{\infty} k|\Delta^2 \mathrm{v}_{2,n}(k)| \\ &\le \frac{1}{n^2} \|\mathrm{u}_2''\|_{L_\infty[0,2]} \sum_{k=1}^{2n-1} k + \frac{1}{n^2} \sum_{k=2n}^{\infty} k \, \|\mathrm{u}_2''\|_{L_\infty[\frac{k-1}{n}, \frac{k+1}{n}]} \\ &\le \frac{c}{n^2} \sum_{k=1}^{2n-1} k + c\, n^2 \sum_{k=2n}^{\infty} k^{-3} \le c \quad \forall n \in \mathbb{N}. \end{aligned}$$

Now, let $j = 3$. We have

$$\mathrm{v}_{3,n}(k) = \mathrm{u}_{3,n}(k/n), \quad k \in \mathbb{N}_0,$$

where

$$\mathrm{u}_{3,n}(x) = \begin{cases} x^2 + \dfrac{2(2+n^{-2})x}{3(x^2-2x-n^{-2})}, & 0 \le x \le 1, \\ x^2 - \dfrac{2(2+n^{-2})x^2}{x^3-6x^2+12x-4-n^{-2}x+4n^{-2}}, & 1 < x \le 2 - 1/n, \\ 0, & x > 2 - 1/n. \end{cases}$$

We have $\mathrm{u}_{3,n} \in W^2_\infty[0,1]$, $n \in \mathbb{N}$, as for $x \in [0,1]$ we calculate

$$\mathrm{u}_{3,n}'(x) = 2x - \frac{2(2n^2+1)(n^2x^2+1)}{3(n^2x(2-x)+1)^2},$$

$$\mathrm{u}''_{3,n}(x) = 2 - \frac{4n^2(2n^2+1)(n^2x^3+3x-2)}{3(n^2x(2-x)+1)^3}.$$

Since for $x \in [0,1]$ and $n \in \mathbb{N}$

$$\mathrm{u}'''_{3,n}(x) = -\frac{4n^2(2n^2+1)(n^4x^4+n^2(6x^2-8x+4)+1)}{(n^2x(2-x)+1)^4} < 0,$$

and for $n \in \mathbb{N}$

$$\mathrm{u}''_{3,n}(1/2) = \frac{2(49n^6+436n^4+496n^2+192)}{3(3n^2+4)^3} > 0,$$

we get that $\mathrm{u}''_{3,n}(x) \geq 0$ for $x \in [0,1/2]$. Then we have for $n \geq 4$

$$\Delta^2 \mathrm{v}_{3,n}(k) \geq 0, \quad 1 \leq k \leq [n/2]-1,$$

which, on its part, implies for $n \geq 4$

$$\begin{aligned}
\sum_{k=1}^{[n/2]-1} k\,|\Delta^2 \mathrm{v}_{3,n}(k)| &= \sum_{k=1}^{[n/2]-1} k\,\Delta^2 \mathrm{v}_{3,n}(k) \\
&= [n/2](\mathrm{v}_{3,n}([n/2]) - \mathrm{v}_{3,n}([n/2]-1)) - \mathrm{v}_{3,n}([n/2]) \\
&= [n/2]\left[\mathrm{u}_{3,n}\left(\frac{[n/2]}{n}\right) - \mathrm{u}_{3,n}\left(\frac{[n/2]}{n} - \frac{1}{n}\right)\right] - \mathrm{u}_{3,n}\left(\frac{[n/2]}{n}\right) \\
&= \frac{[n/2]}{n}\,\mathrm{u}'_{3,n}(\xi_n) - \mathrm{u}_{3,n}([n/2]/n),
\end{aligned}$$

where $\xi_n \in [([n/2]-1)/n, [n/2]/n]$. We have $\xi_n \sim [n/2]/n \sim 1$ for $n \geq 2$. Therefore for $n \geq 4$

$$\sum_{k=1}^{[n/2]-1} k\,|\Delta^2 \mathrm{v}_{3,n}(k)| \leq c. \tag{5.5}$$

Next, since

$$\|\mathrm{u}''_{3,n}\|_{L_\infty[1/4,1]} \leq c \quad \forall n \in \mathbb{N}.$$

we get for $n \geq 4$

$$|\Delta^2 \mathrm{v}_{3,n}(k)| \leq \frac{c}{n^2}, \quad [n/2] \leq k \leq n-1.$$

Consequently, for all $n \geq 2$ there holds

$$\sum_{k=[n/2]}^{n-1} k\,|\Delta^2 \mathrm{v}_{3,n}(k)| \leq \frac{c}{n^2}\sum_{k=1}^{n-1} k \leq c. \tag{5.6}$$

Further, it is clear that $\mathrm{u}_{3,n} \in W^2_\infty[1, 2-1/n]$, $n \geq 2$, as moreover

$$\|\mathrm{u}''_{3,n}\|_{L_\infty[1,2-1/n]} \leq c \quad \forall n \geq 2,$$

hence again

$$|\Delta^2 \mathrm{v}_{3,n}(k)| \leq \frac{c}{n^2}, \quad n+1 \leq k \leq 2n-2,$$

which, as in (5.6), yields for all $n \geq 3$ that

$$\sum_{k=n+1}^{2n-2} k\,|\Delta^2 \mathrm{v}_{3,n}(k)| \leq c. \tag{5.7}$$

Finally, we calculate for $n \in \mathbb{N}$

$$\begin{aligned} &|\Delta^2 \mathrm{v}_{3,n}(n)| = \frac{2(n^2-1)}{3n^2(n^2+1)} \leq \frac{1}{n^2}, \\ &|\Delta^2 \mathrm{v}_{3,n}(2n-1)| = \frac{12(n-1)}{n^2(2n^2+2n+3)} \leq \frac{c}{n^3}, \\ &\Delta^2 \mathrm{v}_{3,n}(k) = 0, \quad k \geq 2n. \end{aligned} \tag{5.8}$$

Now, (5.5)-(5.8) imply (5.3) for $j = 3$.

Finally, to prove (5.3) for $j = 4$, we observe that

$$\mathrm{v}_{4,n}(k) = \mathrm{u}_4(k/n)$$

with

$$\mathrm{u}_4(x) = \begin{cases} \dfrac{3x^3 - 6x^2 + 4}{3(x-2)}, & 0 \leq x \leq 1, \\ \dfrac{x^2(x-2)^3}{(x-2)^3 + 4}, & 1 < x \leq 2, \\ 0, & x > 2. \end{cases}$$

We have $\mathrm{u}_4 \in W^2_\infty(\mathbb{R}_+)$. Hence, by (5.4), we get

$$|\Delta^2 \mathrm{v}_{4,n}(k)| \leq \frac{1}{n^2}\, \|\mathrm{u}''\|_{L_\infty(\mathbb{R}_+)}, \quad k \in \mathbb{N}_0,$$

which along with $\Delta^2 \mathrm{v}_{4,n}(k) = 0$ for $k > 2n$, implies (5.3) for $j = 4$ as in the previous case.

The proof of the theorem is completed. □

Remark 5.3. The direct inequalities can also be established easily by means of standard techniques based on Taylor's formula, or moduli of smoothness (see e.g. [14, Section 4.2]). In this respect, let us mention that in view of condition (c) of Definition 3.1, if $g \in B(\mathbb{T})$ has a strong derivative of order $r \in \mathbb{N}$ in the B-norm, then it has such in the L-norm and they are equal. Hence $g^{(l)}$, $l = 0, 1, \ldots, r-1$, are absolutely continuous on $\mathbb{T}$ (see e.g. [3, Theorem 10.1.12]).

Remark 5.4. The converse estimate for the Jackson operator was verified first by Hecker, Knoop and Zhou [10] (for $L_p(\mathbb{T})$, $1 \leq p < \infty$, and $C(\mathbb{T})$) and independently by Trigub [24] (for $C(\mathbb{T})$).

Remark 5.5. Let us mention that sometimes instead of Theorem B we can use similar assertions like those that can be found in [3, Chapter 6], [14, p. 108, Note 3], [4], [23], etc.

In all the examples we have considered the measures λ_ρ and ν_ρ of Theorems 3.1 and 3.2 are absolutely continuous, which simplifies the method and its application. However, there are instances when the more general assertion of these theorems is useful – e.g. the singular integral of Bochner-Riesz (cf. [3, Section 12.4.4]).

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Borislav R. Draganov
Dept. of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
Bulgaria
bdraganov@fmi.uni-sofia.bg

Inst. of Mathematics and Informatics
Bulgarian Academy of Science
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
Bulgaria