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# On the Approximation by Convolution Operators in Homogeneous Banach Spaces of Periodic Functions[1]

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The paper is concerned with establishing direct estimates for convolution operators on homogeneous Banach spaces of periodic functions by means of appropriately defined $K$-functional. The differential operator in the $K$-functional is defined by means of strong limit and described explicitly in terms of its Fourier coefficients. The description is simple and independent of the homogeneous Banach space. Saturation of such operators is also considered.

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## 1. Convolution operators in homogeneous Banach spaces of periodic functions

Let $\mathbb{T}$ denote the circle and $L(\mathbb{T})$ be the Banach space of Lebesgue summable complex-valued functions on $\mathbb{T}$ equipped with the norm

$$\|f\|_L = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\,dx, \quad f \in L(\mathbb{T}).$$

A natural generalization of $L(\mathbb{T})$ are the so-called homogeneous Banach spaces on $\mathbb{T}$. The idea of HBS's is due to Shilov [9]. However, we notice such an abstract

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approach in the definition of almost periodic functions given by Bochner and Neumann [1, Definition 1] (see also the references cited there and [8, p. 200]. In particular, Katznelson [7, Definition I.2.10] formulates the following

**Definition 1.1.** A *homogeneous Banach space* (abbreviated *HBS*) $B$ *on* $\mathbb{T}$ is a linear subspace of $L(\mathbb{T})$ having a norm $\| \circ \|_B$ under which it is a Banach space such that

(a) The translation is an isometry of $B$ onto itself, i.e. if $f \in B$ and $t \in \mathbb{T}$, then $f_t \in B$ and $\|f_t\|_B = \|f\|_B$, where $f_t(x) = f(x-t)$;

(b) The translation is continuous on $B$, i.e. for all $f \in B$ and $t, t_0 \in \mathbb{T}$ there holds $\lim_{t \to t_0} \|f_t - f_{t_0}\|_B = 0$;

(c) $B$ is continuously embedded in $L(\mathbb{T})$, i.e. there exists an absolute constant $\alpha$ such that for all $f \in B$ there holds $\|f\|_L \le \alpha \|f\|_B$.

We will consider approximation properties of convolution operators acting on a HBS on $\mathbb{T}$. First we recall the notion of a periodic approximate identity.

**Definition 1.2.** (e.g. [2, Definitions 1.1.1 and 1.1.4] and [7, Definition I.2.2]) The family of functions $\{k_n(t)\}_{n \in \mathbb{N}}$ is called a *periodic approximate identity* if it satisfies the conditions:

(a) For all $n \in \mathbb{N}$ we have $k_n \in L(\mathbb{T})$ and

$$\int_{\mathbb{T}} k_n(t)\, dt = 2\pi;$$

(b) There exists a constant $M$ such that

$$\|k_n\|_L \le M \quad \text{for all} \quad n \in \mathbb{N};$$

(c) For each $0 < \delta < \pi$, there holds

$$\lim_{n \to \infty} \int_{\delta \le |t| \le \pi} |k_n(t)|\, dt = 0.$$

Let $B$ be a HBS on $\mathbb{T}$. Given a periodic approximate identity $\{k_n(t)\}_{n \in \mathbb{N}}$, we consider bounded linear operators of $J_n : B \to B$ defined by

$$J_n f(x) = k_n * f(x) = \frac{1}{2\pi} \int_{\mathbb{T}} k_n(x-t) f(t)\, dt, \quad x \in \mathbb{T}, \quad n \in \mathbb{N}. \tag{1.1}$$

The integral above can be interpreted either as a Lebesgue integral or, equivalently, as Bochner's generalization of the Lebesgue integral of vector-valued functions (cf. [7, Lemma I.2.4]). Therefore $k_n * f(x)$ exists almost everywhere, belongs to $B$ and

$$\|k_n * f\|_B \le \|k_n\|_L \, \|f\|_B \le M\|f\|_B \quad \forall n.$$

Moreover, as is known (see e.g. [1, Theorems 1.1.5] and [7, Theorem I.2.11]), we have

$$\lim_{n\to\infty} \|f - J_n f\|_B = 0 \quad \forall f \in B. \tag{1.2}$$

In [6] we presented a set of conditions, which allow us to characterize the rate of convergence of $J_n f$ to $f$ in (1.2) by means of a $K$-functional of the form

$$K(f,\tau; B, \mathcal{D}) = \inf\{\|f-g\|_B + \tau\,\|\mathcal{D}g\|_B \;:\; g \in \mathcal{D}^{-1}(B)\}, \tag{1.3}$$

where $f \in B$, $\tau > 0$, $\mathcal{D}$ is a "differential" operator and $\mathcal{D}^{-1}(B) = \{g \in B : \mathcal{D}g \in B\}$ is dense in $B$. These sufficient conditions are based on Fourier transform methods and have been used before though not in the context of $K$-functionals but rather in establishing the saturation property and the saturation class of convolution operators (see [2, Chapter 12, especially Section 12.6] and [6, Section 3.3]). The purpose of this paper is to show that with an appropriate and quite natural definition of the differential operator $\mathcal{D}$ those of the sufficient conditions that concern it are directly satisfied. Here, for simplicity, we treat only HBS's of *univariate* periodic functions, but the same methods are applicable to the multidimensional case. The concept of HBS's can be extended to functions defined on $\mathbb{R}$ (or $\mathbb{R}^d$) in several different ways (see [7, VI.1.14] and [8, Chapter 9]. However, a HBS on $\mathbb{R}$ (resp. $\mathbb{R}^d$) is not generally continuously embedded in $L(\mathbb{R})$ (resp. $L(\mathbb{R}^d)$) and its treatment becomes more complicated. We shall present results in this respect in another publication.

In the next section we state sufficient conditions, which imply an upper estimate of the error of a convolution operator by a $K$-functional. These results were established in [6]. In Section 3 and 4 we consider respectively the rate of approximation and the form of $\mathcal{D}$ in terms of Fourier coefficients. There we establish some of the conditions, given in Section 2. Further, in Section 5, we strengthen the criterion for direct estimates. In Section 6 we discuss another definition of the differential operator in the $K$-functional. Finally, we give a number of examples in the last section.

The results presented in this paper are either extensions of known assertions for particular HBS's or they are very similar to such. The main reference is the book of Butzer and Nessel [2] (mostly Chapters 12 and 13) as well as that of DeVore and Lorentz [4](Chapter 11, §§ 2-3).

## 2. Upper estimates of the error of convolution operators on a HBS on $\mathbb{T}$

We denote the Fourier coefficients of a function $f \in L(\mathbb{T})$ by

$$\hat{f}(m) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x)\, e^{-i\,mx}\, dx, \quad m \in \mathbb{Z}.$$

The assertion of [6, Theorem 3.6] in particular gives

**Theorem A.** *Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be given by* (1.1) *with a periodic approximate identity $\{k_n\}$. Let $\mathcal{D}$ be such that $\mathcal{D}^{-1}(L(\mathbb{T}))$ is dense in $L(\mathbb{T})$ and for $g \in \mathcal{D}^{-1}(B)$ and $\eta \in \mathcal{D}^{-1}(L(\mathbb{T}))$*

$$\mathcal{D}(\eta * g) = \mathcal{D}\eta * g = \eta * \mathcal{D}g. \tag{2.1}$$

*Let also there exist $\Phi : \mathbb{N} \to (0, \infty)$, $\Psi : \mathbb{Z} \to \mathbb{C}$, $c \in \mathbb{R}$ and $\ell_n \in L(\mathbb{T})$, $n \in \mathbb{N}$, such that*

$$\widehat{\mathcal{D}\eta}(m) = \Psi(m)\hat{\eta}(m), \quad m \in \mathbb{Z}, \quad \eta \in \mathcal{D}^{-1}(L(\mathbb{T})), \tag{2.2}$$

$$1 - \hat{k}_n(m) = \Phi(n)\, \Psi(m)\, \hat{\ell}_n(m), \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}, \tag{2.3}$$

*and*

$$\|\ell_n\|_L \le c \quad n \in \mathbb{N}. \tag{2.4}$$

*Then for all $f \in B$ and $n \in \mathbb{N}$ we have*

$$\|f - J_n f\|_B \le c\, K(f, \Phi(n); B, \mathcal{D}).$$

Above and henceforth we denote by $c$ positive constants not necessarily the same that do not depend on $f$ or $n$.

Remark 2.1. As it can be seen in the proof of [6, Theorem 3.6], it is enough to assume that instead of (2.1) we only have

$$\mathcal{D}\eta * g = \eta * \mathcal{D}g.$$

Let $L(\mathbb{R})$ denote the space of all Lebesgue summable functions on $\mathbb{R}$. As is known (see e.g. [2, Proposition 3.1.12] or [7, VI.1.15]) each $k \in L(\mathbb{R})$ with

$$\int_{\mathbb{R}} k(t)\, dt = 1 \tag{2.5}$$

generates a periodic approximate identity $\{k_n\}$ by

$$k_n(t) = 2\pi \sum_{j=-\infty}^{\infty} n\, k(n(t+2j\pi)). \tag{2.6}$$

Now, taking into account that

$$\hat{k}_n(m) = \hat{k}(m/n), \tag{2.7}$$

where

$$\hat{k}(u) = \int_{\mathbb{R}} k(x) e^{-iux}\, dx, \quad u \in \mathbb{R},$$

is the Fourier transform of $k \in L(\mathbb{R})$ (see e.g. [2, Proposition 5.1.28] or [7, VI.1.15]), we derive the following assertion from Theorem A and Remark 2.1.

**Corollary B.** *(see [6, Theorem 3.13]) Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be given by* (1.1) *with a periodic approximate identity $\{k_n\}$ defined by* (2.5)-(2.6)*. Let $\mathcal{D}$ be such that $\mathcal{D}^{-1}(L(\mathbb{T}))$ is dense in $L(\mathbb{T})$ and for $g \in \mathcal{D}^{-1}(B)$ and $\eta \in \mathcal{D}^{-1}(L(\mathbb{T}))$*

$$\mathcal{D}\eta * g = \eta * \mathcal{D}g. \tag{2.8}$$

*Let also there exist $\kappa > 0$ and $\ell \in L(\mathbb{T})$ such that*

$$\widehat{\mathcal{D}\eta}(m) = |m|^{\kappa} \hat{\eta}(m), \quad m \in \mathbb{Z}, \quad \eta \in \mathcal{D}^{-1}(L(\mathbb{T})), \tag{2.9}$$

$$1 - \hat{k}(u) = |u|^{\kappa} \hat{\ell}(u), \quad u \in \mathbb{R}. \tag{2.10}$$

*Then for all $f \in B$ and $n \in \mathbb{N}$ we have*

$$\|f - J_n f\|_B \le c\, K(f, n^{-\kappa}; B, \mathcal{D}).$$

Relation (2.9) identifies $\mathcal{D}$ as the Riesz fractional derivative of order $\kappa$ (see e.g. [2, Definition 11.5.10] and [6, Definition 5.1]Dr). In [6] we also established similar results concerning strong converse inequalities.

We shall consider two definitions of the differential operator $\mathcal{D}$ of the $K$-functional (1.3). They are equivalent under certain natural assumptions as we shall show below. Both give differential operators that satisfy conditions (2.1) and (2.2). On the other hand, relations (2.3)-(2.4) also turn out to be satisfied for certain HBS's in the setting of the approximation problem under consideration (cf. Remark 6.3 below).

**Definition 2.1.** (cf. [2, Definition 13.4.3 and (13.4.1)]) Let $B$ be a HBS on $\mathbb{T}$, the convolution operator $J_n$ be defined by (1.1) with a periodic approximate identity $\{k_n\}$ and the function $\phi : \mathbb{N} \to \mathbb{C}\backslash\{0\}$ be such that $\lim_{n\to\infty} \phi(n) = 0$. For $g \in B$ we set

$$\mathcal{D}_\phi g = \mathcal{D}_{J,\phi} g = \text{s-}\lim_{n\to\infty} \frac{J_n g - g}{\phi(n)}$$

if the limit, taken in the $B$-norm, exists.

Note that thus defined $\mathcal{D}_\phi$ commutes with the convolution, that is, it satisfies (2.1). The function $\phi$ serves to measure the rate with which $J_n g$ approximates $g$ for $g \in \mathcal{D}_\phi^{-1}(B)$. More precisely, if $\mathcal{D}_\phi g \neq 0$, then for $n$ large enough we have

$$\|g - J_n g\|_B \leq 2|\phi(n)|\|\mathcal{D}_\phi g\|_B. \tag{2.11}$$

So, the faster $\phi(n)$ tends to 0 the faster $J_n g$ strongly converges to $g$ as $n \to \infty$. Our goal is to determine how fast $\phi$ can vanish at infinity for $\mathcal{D}_\phi g \neq 0$ as well as to describe the corresponding differential operator $\mathcal{D}_\phi$.

## 3. The optimal order of $\phi$

Let us recall the notion of saturation (see e.g. [2, Definition 12.0.2] or [4, Chapter 11, § 2]). We say that the approximation process $\{J_n\}$ possesses the saturation property if there exists a function $\Phi : \mathbb{N} \to (0, \infty)$ with $\lim_{n\to\infty} \Phi(n) = 0$ such that there exists $f \in B$ with

$$0 \neq \|f - J_n f\|_B = O(\Phi(n)), \quad n \to \infty, \tag{3.1}$$

(as the inequality on the left is satisfied for at least one natural $n$) whereas

$$\|f - J_n f\|_B = o(\Phi(n)), \quad n \to \infty, \tag{3.2}$$

implies

$$J_n f = f \quad \forall n \in \mathbb{N}.$$

The function $\Phi$ gives the *optimal approximation order* of $J_n$. It is also called *saturation order*. It is clear that if (3.1)-(3.2) are satisfied by two function $\Phi_1$ and $\Phi_2$, then $\Phi_1 \sim \Phi_2$, i.e.

$$c^{-1}\Phi_1(n) \leq \Phi_2(n) \leq c\,\Phi_1(n) \quad \forall n \in \mathbb{N}$$

(see [4, Chapter 11, § 2]). In this case we shall say that $\Phi_1$ and $\Phi_2$ are *equivalent*. The functions that satisfy (3.1) comprise the *saturation* or *Favard class* of $J_n$.

Let us note that if there exists $g \in \mathcal{D}_\Phi^{-1}(B)$ such that $\mathcal{D}_\Phi g \neq 0$, then $J_n$ satisfies (3.1), whereas relation (3.2) is equivalent to $f \in \mathcal{D}_\Phi^{-1}(B)$ and $\mathcal{D}_\Phi f = 0$. Consequently, if there exists $g \in \mathcal{D}_\Phi^{-1}(B)$ such that $\mathcal{D}_\Phi g \neq 0$ and if $\mathcal{D}_\Phi g = 0$ implies $J_n g = g$ for all $n \in \mathbb{N}$, then $J_n$ is saturated with order $\Phi(n)$. Let us explicitly observe that $\mathcal{D}_\Phi g = 0$ does not necessarily imply $J_n g = g$ for all $n \in \mathbb{N}$ — cf. Example 7.4. However, if $J_n$ is saturated with order $\Phi(n)$, then it does.

Results concerning the saturation of convolution operators in $L_p(\mathbb{T})$ show that their approximation rate cannot be arbitrary high (see [2, Section 12.1] and [4, Chapter 11, §§ 2-3]). The sufficient conditions considered there (cf. also Theorem A above) reveal that the saturation (optimal) order is determined by the behaviour of $1 - \hat{k}_n(m)$ as $n \to \infty$. In this section we shall establish that the converse is also true, namely, that generally, in the setting of any HBS on $\mathbb{T}$, the optimal order of $\phi(n)$ is $|1 - \hat{k}_n(a)|$ with an appropriately fixed integer $a$.

We begin with a simple relation between the function $\phi$ of $\mathcal{D}_\phi$ and the Fourier transform of the kernel $k_n$ of the operator $J_n$.

**Proposition 3.1.** *Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be defined by* (1.1) *with a periodic approximate identity $\{k_n\}$. Let $a \in \mathbb{Z}$, $\hat{k}_n(a) \neq 1$ for $n$ large enough and there exist $g_0 \in \mathcal{D}_\phi^{-1}(B)$ such that $\hat{g}_0(a) \neq 0$. Then*

$$|\phi(n)| \neq o(|1 - \hat{k}_n(a)|), \quad n \to \infty.$$

P r o o f. Using basic properties of the Fourier transform, we get for all $n$

$$|\hat{g}_0(m) - \hat{k}_n(m)\hat{g}_0(m)| = |(g_0 - J_n g_0)\widehat{\ }(m)| \leq \|g_0 - J_n g_0\|_L, \quad m \in \mathbb{Z},$$

which, in particular, implies the estimate

$$|1 - \hat{k}_n(a)| \leq \frac{1}{|\hat{g}_0(a)|} \|g_0 - J_n g_0\|_L. \tag{3.3}$$

If $\mathcal{D}_\phi g_0 \neq 0$, then we combine (3.3), property (c) in Definition 1.1 and (2.11) to deduce that for $n$ large enough there holds

$$|1 - \hat{k}_n(a)| \leq 2\alpha \frac{\|\mathcal{D}_\phi g_0\|_B}{|\hat{g}_0(a)|} |\phi(n)|. \tag{3.4}$$

If $\mathcal{D}_\phi g_0 = 0$, then

$$\|g_0 - J_n g_0\|_B = o(|\phi(n)|),$$

which again together with (3.3) and property (c) in Definition 1.1 imply

$$|1 - \hat{k}_n(a)| = o(|\phi(n)|). \tag{3.5}$$

The assertion follows from (3.4) and (3.5). ∎

If $g = \text{const}$, then $J_n g = g$ for all $n$ and $\mathcal{D}_\phi g = 0$ for any $\phi$. Taking this into consideration, we get by means of the last proposition the following assertion, which shows that if $\phi$ tends to 0 too fast, then the corresponding differential operator is trivial.

**Corollary 3.2.** *Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be defined by (1.1) with a periodic approximate identity $\{k_n\}$. Let also for each $a \in \mathbb{Z}\backslash\{0\}$, $\hat{k}_n(a) \neq 1$ for $n$ large enough. If for all $a \in \mathbb{Z}\backslash\{0\}$*

$$|\phi(n)| = o(|1 - \hat{k}_n(a)|), \quad n \to \infty,$$

*then $\mathcal{D}_\phi g = 0$ for all $g \in \mathcal{D}_\phi^{-1}(B)$.*

R e m a r k 3.3. Considerations quite similar to those used in the proof of Proposition 3.1 show that in the hypotheses of Corollary 3.2 we actually have $\mathcal{D}_\phi^{-1}(B) \subseteq \mathbb{C}$. The too rapid decrease of $\phi$ makes the domain of $\mathcal{D}_\phi$ too narrow.

Thus, generally, in the nontrivial case, we can choose the fastest vanishing $\phi$and so acquire generally the best approximation rate in (2.11) is $\hat{k}_n(a) - 1$ with an appropriately fixed integer $a$. We set $\varphi_a(n) = \hat{k}_n(a) - 1$. The quantity $|\varphi_a(n)|$ is optimal in the sense that if $J_n$ possesses the saturation property, then its saturation order is at most $|\varphi_a(n)|$ with an appropriate $a \in \mathbb{Z}$ (see also [4, Chapter 11, Theorem 2.1 (i)]. In this respect we have the following assertion, which is a straightforward generalization of part of the basic theorem cited above.

**Theorem 3.4.** *(cf. [4, Chapter 11, Theorem 2.1]) Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be defined by (1.1) with a periodic approximate identity $\{k_n\}$.*

*(i) Assume that there exists $a \in \mathbb{Z}$ such that $\mathcal{D}_{\varphi_a}^{-1}(B) \ni g_0$ with $\mathcal{D}_{\varphi_a} g_0 \neq 0$ and also such that*

$$\liminf_{n\to\infty} \left| \frac{1 - \hat{k}_n(m)}{1 - \hat{k}_n(a)} \right| \neq 0 \tag{3.6}$$

*whenever $\hat{k}_n(m) \neq 1$ for some $n \in \mathbb{N}$. Then $J_n$ possesses the saturation property and its optimal approximation order is $|\varphi_a(n)|$.*

*(ii) Assume that $J_n$ possesses the saturation property with saturation order $\phi(n)$. Then there exists $a \in \mathbb{Z}\backslash\{0\}$ such that*

$$\phi(n) \geq c\,|\varphi_a(n)|, \quad n \in \mathbb{N}. \tag{3.7}$$

P r o o f. Indeed, (2.11) implies that

$$\|g_0 - J_n g_0\|_B = O(|\varphi_a(n)|),$$

as, moreover, the norm on the left-hand side above cannot be equal to 0 since then we would have that $\mathcal{D}_{\varphi_a} g_0 = 0$.

On the other hand, if for some $f \in B$

$$\|f - J_n f\|_B = o(|\varphi_a(n)|),$$

then, as in the proof of Proposition 3.1, we get for each $m \in \mathbb{Z}$ such that $\hat{k}_n(m) \neq 1$ for some $n \in \mathbb{N}$

$$\left| \frac{1 - \hat{k}_n(m)}{\varphi_a(n)} \hat{f}(m) \right| \leq \alpha \frac{\|f - J_n f\|_B}{|\varphi_a(n)|} \to 0, \quad n \to \infty.$$

Now, taking into account (3.6), we arrive at $\hat{f}(m) = 0$ for each $m \in \mathbb{Z}$ such that $\hat{k}_n(m) \neq 1$ for some $n \in \mathbb{N}$. Consequently, for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have

$$\hat{k}_n(m)\hat{f}(m) = \hat{f}(m),$$

which, in view of the uniqueness of the Fourier coefficients, yields $J_n f = f$ for all $n \in \mathbb{N}$. Thus $J_n$ possesses the saturation property and its optimal approximation order is $|\varphi_a(n)|$.

To establish (ii) we proceed in a similar way. Suppose that (3.7) is not valid. Then for each fixed $a \in \mathbb{Z}\backslash\{0\}$ there exists a sequence $\{n_j\}_{j=1}^{\infty}$ of naturals such that $|\varphi_a(n_j)|/\phi(n_j) \to \infty$ as $j \to \infty$. Now, let $f \in B$ satisfy (3.1). Then there exists a positive constant $C$ such that for all $j$

$$\left| \frac{\varphi_a(n_j)}{\phi(n_j)} \hat{f}(a) \right| \leq \frac{\|f - J_{n_j} f\|_L}{\phi(n_j)} \leq \alpha \frac{\|f - J_{n_j} f\|_B}{\phi(n_j)} < C.$$

Consequently, $\hat{f}(a) = 0$ for each $a \in \mathbb{Z}\backslash\{0\}$. Hence $f = \text{const}$ and then $J_n f = f$ for all $n$. This contradiction shows that our assumption was false and assertion (ii) is valid. ■

R e m a r k 3.5. 1) Let us observe that under certain assumptions condition (3.6) is also necessary in order that the corresponding convolution operator possess the saturation property (cf. [4, Chapter 11, Theorem 2.1 (i)] and Example 7.5). 2) Assertion (ii) cannot generally be strengthen. It is not true that for any non-zero integer $a$ there exists a constant $c$ such that (3.7) holds as Examples 7.4 and 7.5 will show.

Another similar result is the following

**Theorem 3.6.** *Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be defined by* (1.1) *with a periodic approximate identity $\{k_n\}$, $\phi : \mathbb{N} \to \mathbb{C}\backslash\{0\}$ and $\Phi : \mathbb{N} \to (0, \infty)$. Let there exist $g_0 \in \mathcal{D}_\phi^{-1}(B)$ such that $\mathcal{D}_\phi g_0 \neq 0$. Finally, let for all $f \in B$ and $n \in \mathbb{N}$ there holds*

$$\|f - J_n f\|_B \leq c\, K(f, \Phi(n); B, \mathcal{D}_\phi). \tag{3.8}$$

*Then $J_n$ possesses the saturation property with optimal approximation order $|\phi(n)|$.*

P r o o f. As above we have

$$0 \neq \|g_0 - J_n g_0\|_B = O(|\phi(n)|).$$

Further, relation (3.8) directly implies

$$\|g - J_n g\|_B \leq c\, \Phi(n) \|\mathcal{D}_\phi g\|_B, \quad g \in \mathcal{D}_\phi^{-1}(B), \quad n \in \mathbb{N}. \tag{3.9}$$

Now, if $f \in B$ is such that

$$\|f - J_n f\|_B = o(|\phi(n)|),$$

then $f \in \mathcal{D}_\phi^{-1}(B)$ and $\mathcal{D}_\phi f = 0$, which, in view of (3.9), yields $J_n f = f$ for all naturals $n$.

This completes the proof of the proposition. ■

R e m a r k 3.7. Let the convolution operator $J_n$ possesses the saturation property. The last proposition shows that an upper estimate like (3.8) by a $K$-functional with a differential operator given by Definition 2.1 is possible only with a function $\phi$ which is equivalent in absolute value to the saturation order of $J_n$. Let us also observe that in the conditions of the preceding theorem $\Phi(n) \geq c|\phi(n)|$ for all $n$ as otherwise (3.9) would imply that $\mathcal{D}_\phi g_0 = 0$.

## 4. The differential operator $\mathcal{D}_\phi$

Our next objective is to describe the differential operator given in Definition 2.1 in any HBS on $\mathbb{T}$. It turns out that $\mathcal{D}_\phi$ satisfies condition (2.2) as well. Let us denote by $T$ the set of all trigonometric polynomials. The following result similar to the one given in [2, pp. 436-437] is valid (cf. also Theorem 12.1.4 as well as Proposition 12.1.1 and its proof in [2]).

**Theorem 4.1.** *Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be defined by* (1.1) *with a periodic approximate identity $\{k_n\}$. Then the limit*

$$\lim_{n\to\infty} \frac{\hat{k}_n(m) - 1}{\phi(n)}$$

*exists (as a finite number) for each $m \in \mathbb{Z}$ for which there exists $g \in \mathcal{D}_\phi^{-1}(B)$ such that $\hat{g}(m) \neq 0$. Further, for all $g \in \mathcal{D}_\phi^{-1}(B)$ there holds $\widehat{\mathcal{D}_\phi g}(m) = \psi(m)\hat{g}(m)$, $m \in \mathbb{Z}$, where*

$$\psi(m) = \lim_{n\to\infty} \frac{\hat{k}_n(m) - 1}{\phi(n)}$$

*if the right-hand side exists, and $\psi(m)$ is assigned an arbitrary value, otherwise.*

Proof. The proof is quite similar to that of Proposition 3.1. Using again basic properties of the Fourier transform and condition (c) of Definition 1.1, we get for each $g \in \mathcal{D}_\phi^{-1}(B)$ and $m \in \mathbb{Z}$

$$\begin{aligned}\left| \frac{\hat{k}_n(m) - 1}{\phi(n)}\, \hat{g}(m) - \widehat{\mathcal{D}_\phi g}(m) \right| &\leq \left\| \frac{J_n g - g}{\phi(n)} - \mathcal{D}_\phi g \right\|_L \\ &\leq \alpha \left\| \frac{J_n g - g}{\phi(n)} - \mathcal{D}_\phi g \right\|_B \to 0, \quad n \to \infty.\end{aligned} \tag{4.1}$$

Set $\mathbb{M} = \{m \in \mathbb{Z} \ : \ \hat{g}(m) = 0 \ \forall g \in \mathcal{D}_\phi^{-1}(B)\}$. For each integer $m \in \mathbb{Z}\backslash\mathbb{M}$ we fix $g \in \mathcal{D}_\phi^{-1}(B)$ with $\hat{g}(m) \neq 0$ in (4.1) and derive that the limit

$$\lim_{n\to\infty} \frac{\hat{k}_n(m) - 1}{\phi(n)}$$

exists. Then, again using (4.1), we establish that $\widehat{\mathcal{D}_\phi g}(m) = \psi(m)\hat{g}(m)$ for all $g \in \mathcal{D}_\phi^{-1}(B)$ and $m \in \mathbb{Z}\backslash\mathbb{M}$.

As for $m \in \mathbb{M}$ we get by (4.1) that $\widehat{\mathcal{D}_\phi g}(m) = 0 = \psi(m)\hat{g}(m)$ for all $g \in \mathcal{D}_\phi^{-1}(B)$.

This completes the proof of the theorem. ■

Remark 4.2. Let $k \in L(\mathbb{R})$ be even and satisfies (2.5). Let us also assume that $\hat{k}(u) \neq 1$ for $u \neq 0$ in a neighbourhood of 0. Consider the periodic approximate identity $\{k_n\}$ defined by (2.6) and let $J_n$ be the corresponding convolution operator. Set $\varphi(n) = \hat{k}(1/n) - 1$ and assume that $T \subset \mathcal{D}_\varphi^{-1}(B)$. Then the limit

$$\psi(m) = \lim_{n\to\infty} \frac{1 - \hat{k}_n(m)}{1 - \hat{k}_n(1)} = \lim_{n\to\infty} \frac{1 - \hat{k}(m/n)}{1 - \hat{k}(1/n)}$$

exists for each $m \in \mathbb{Z}$. Moreover, we have for all $m_1, m_2 \in \mathbb{N}$

$$\psi(m_1 m_2) = \lim_{n\to\infty} \frac{1 - \hat{k}(m_1 m_2/n)}{1 - \hat{k}(1/n)} = \lim_{n\to\infty} \frac{1 - \hat{k}(m_1 m_2/(m_2 n))}{1 - \hat{k}(1/(m_2 n))}$$

$$= \lim_{n\to\infty} \frac{1-\hat{k}(m_1/n)}{1-\hat{k}(1/n)} \lim_{n\to\infty} \frac{1-\hat{k}(m_2/(m_2 n))}{1-\hat{k}(1/(m_2 n))}$$
$$= \psi(m_1)\psi(m_2).$$

Taking into account that $k$ is even, we get that $\hat{k}$ and hence $\psi$ are even too. Consequently,

$$\psi(m_1 m_2) = \psi(m_1)\psi(m_2), \quad m_1, m_2 \in \mathbb{Z}. \tag{4.2}$$

Also let us note that since $k$ is even then $\hat{k}$ is real-valued and hence so is $\psi$. It is known that if a function $\Psi : [0,\infty) \to \mathbb{R}$ is continuous and

$$\Psi(u_1 u_2) = \Psi(u_1)\Psi(u_2), \quad u_1, u_2 \geq 0, \tag{4.3}$$

as $\Psi \neq 0, 1$, then there exists $\kappa > 0$ such that $\Psi(x) = x^\kappa$, $x \geq 0$. If $\Psi$ is even, then $\Psi(x) = |x|^\kappa$, $x \in \mathbb{R}$. Thus, if further the limit

$$\Psi(u) = \lim_{n\to\infty} \frac{1-\hat{k}(u/n)}{1-\hat{k}(1/n)} \tag{4.4}$$

exists for all real $u$ as the convergence is uniform on the compact intervals, then $\Psi$ is continuous and even on $\mathbb{R}$ and similarly to (4.2) one verifies that it satisfies (4.3). Consequently, $\widehat{\mathcal{D}_\varphi g}(m) = |m|^\kappa \hat{g}(m)$, $m \in \mathbb{Z}$, and $\mathcal{D}_\varphi$ is the Riesz fractional derivative of order $\kappa$.

In passing, let us note that the existence of the uniform on the compact intervals limit (4.4) can be easily (and most naturally) established by means of the analogue of (4.1) for convolution operators on $L(\mathbb{R})$, but this is beyond the scope of this paper.

## 5. Direct estimates revisited

A straightforward observation leads us to the following assertion, which connects in a very natural way notions of saturation and upper error estimates by $K$-functionals.

**Theorem 5.1.** *Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be defined by* (1.1) *with a periodic approximate identity $\{k_n\}$. Let $T \subset \mathcal{D}_\phi^{-1}(B)$. Let us set*

$$\psi(m) = \lim_{n\to\infty} \frac{\hat{k}_n(m) - 1}{\phi(n)}, \quad m \in \mathbb{Z},$$

*and assume that $\psi(m) \neq 0$ for $m \neq 0$. Let also there exist a function $f_\psi \in L(\mathbb{T})$ such that*

$$\hat{f}_\psi(m) = \begin{cases} \frac{1}{\psi(m)}, & m \neq 0 \\ 0, & m = 0, \end{cases}$$

*as moreover,*

$$\|f_\psi - J_n f_\psi\|_L = O(|\phi(n)|). \tag{5.1}$$

*Then for all* $f \in B$ *and* $n \in \mathbb{N}$ *we have*

$$\|f - J_n f\|_B \le c\, K(f, |\phi(n)|; B, \mathcal{D}_\phi). \tag{5.2}$$

*Moreover,* $J_n$ *is saturated with optimal order* $|\phi(n)|$ *and* $f_\psi$ *belongs to its saturation class.*

R e m a r k 5.2. 1) The limit which defines the function $\psi$ exists in view of Theorem 4.1. 2) In most classical applications $\psi(m) = |m|^\kappa$, $\kappa > 0$. Then the function $f_\psi$ is given by (see e.g. [2, Problem 6.3.1 (iii)])

$$f_\psi(x) = 2 \sum_{m=1}^{\infty} m^{-\kappa} \cos mx.$$

P r o o f. (Theorem 5.1) To establish (5.2) we shall show that the hypotheses of Theorem A are fulfilled with $\mathcal{D} = \mathcal{D}_\phi$, $\Phi = \phi$ and $\Psi = \psi$.

Since $B$ is continuously embedded into $L(\mathbb{T})$, $T \subset \mathcal{D}_\phi^{-1}(B) \subset \mathcal{D}_\phi^{-1}(L(\mathbb{T}))$ and hence $\mathcal{D}_\phi^{-1}(L(\mathbb{T}))$ is dense in $L(\mathbb{T})$. Recall that $\mathcal{D}_\phi$ satisfies condition (2.1) of Theorem A. Further, by Theorem 4.1, applied for the HBS $L(\mathbb{T})$, we have $\widehat{\mathcal{D}_\phi \eta}(m) = \psi(m)\hat{\eta}(m)$, $m \in \mathbb{Z}$, for all $\eta \in \mathcal{D}_\phi^{-1}(L(\mathbb{T}))$, which verifies (2.2) of Theorem A.

Finally, set

$$\ell_n(x) = \frac{f_\psi(x) - J_n f_\psi(x)}{\phi(n)}.$$

Thus defined $\ell_n$ satisfies (2.3). Finally, taking into account (5.1) we establish that the family $\{\ell_n\}$ also possesses property (2.4). This completes the proof of (5.2).

Next, note that $f_\psi \notin \mathcal{D}_\phi^{-1}(B)$ as otherwise Theorem 4.1 would give $\widehat{\mathcal{D}_\phi f_\psi}(m) = \psi(m)\hat{f}_\psi = 1$ for $m \neq 0$, which contradicts the Riemann-Lebesgue lemma. Further, $f_\psi \notin \mathcal{D}_\phi^{-1}(B)$ yields $J_n f_\psi \neq f_\psi$ for some $n$. Thus we have (3.1) with $f = f_\psi$ and $\Phi(n) = |\phi(n)|$. On the other hand, (5.2) implies (3.2) just as in the proof of Theorem 3.6. ∎

Similarly, we get by means of Corollary B the following criterion.

**Theorem 5.3.** *Let* $B$ *be a HBS on* $\mathbb{T}$ *and* $J_n$ *be given by* (1.1) *with a periodic approximate identity* $\{k_n\}$ *defined by* (2.5)-(2.6)*, where* $k$ *satisfies*

(2.10) *with* $\ell \in L(\mathbb{R})$ *such that* $\hat{\ell}(0) \neq 0$. *Set* $\phi(n) = -\hat{\ell}(0)n^{-\kappa}$. *Then for all* $f \in B$ *and* $n \in \mathbb{N}$ *we have*

$$\|f - J_n f\|_B \leq c\, K(f, |\phi(n)|; B, \mathcal{D}_\phi).$$

Proof. The assertion follows from Corollary B with $\mathcal{D} = \mathcal{D}_\phi$. Proposition 6.4 below for $B = L(\mathbb{T})$ implies that $T \subseteq \mathcal{D}_\phi^{-1}(L(\mathbb{T}))$. Consequently, $\mathcal{D}_\phi^{-1}(L(\mathbb{T}))$ is dense in $L(\mathbb{T})$.

The operator $\mathcal{D}_\phi$ satisfies condition (2.8). Further, since $\hat{k}_n(m) = \hat{k}(m/n)$, we get by (2.10) that

$$\psi(m) = \lim_{n\to\infty} \frac{1 - \hat{k}(m/n)}{\phi(n)} = \lim_{n\to\infty} \frac{|m/n|^\kappa \hat{\ell}(m/n)}{\hat{\ell}(0)n^{-\kappa}} = |m|^\kappa$$

and Theorem 4.1, applied for the HBS $L(\mathbb{T})$, yields (2.9). Now, Corollary B implies the assertion of the theorem. ∎

## 6. Another definition of the differential operator $\mathcal{D}$

Let us point out an alternative definition of the differential operator in the $K$-functional. It is constructed by means of the Fourier transform.

**Definition 6.1.** (cf. [2, Definitions 11.5.10 and 13.1.4] and [6, Definition 5.1]) Let $B$ be a HBS on $\mathbb{T}$ and $\Psi : \mathbb{Z} \to \mathbb{C}$. If for $g \in B$ there exists $G \in B$ such that $\Psi(m)\hat{g}(m) = \widehat{G}(m)$, $m \in \mathbb{Z}$, then we set

$$\widetilde{\mathcal{D}}_\Psi g = G.$$

Defined in such a way, the differential operator directly satisfies (2.1) and (2.2). As far as the upper estimate for convolution operators is concerned, Definitions 2.1 and 6.1 give equivalent differential operators. More precisely, there holds

**Proposition 6.2** *Let* $B$ *be a HBS on* $\mathbb{T}$, $\{k_n\}$ *be a periodic approximate identity and* $\phi : \mathbb{N} \to \mathbb{C}\backslash\{0\}$ *be such that* $\lim_{n\to\infty} \phi(n) = 0$. *Let the limit*

$$\psi(m) = \lim_{n\to\infty} \frac{\hat{k}_n(m) - 1}{\phi(n)}$$

*exist for each* $m \in \mathbb{Z}$.

(i) *Then for all* $g \in \mathcal{D}_\phi^{-1}(B)$ *we have* $\widetilde{\mathcal{D}}_\psi g = \mathcal{D}_\phi g$.

*(ii) Let conditions* (2.3)-(2.4) *be satisfied with* $\Phi = \phi$ *and* $\Psi = \psi$. *Let also there exist an absolute constant* $\beta$ *such that*

$$\|f\|_B \leq \beta \|f\|_{L_\infty}, \quad f \in B \cap L_\infty(\mathbb{T}). \tag{6.1}$$

*Then for all* $g \in \widetilde{\mathcal{D}}_\psi^{-1}(B)$ *we have* $\mathcal{D}_\phi g = \widetilde{\mathcal{D}}_\psi g$.

Above we have denoted by $L_\infty(\mathbb{T})$ the space of the essentially bounded $2\pi$-periodic function with the usual sup-norm $\| \circ \|_{L_\infty}$.

Proof. (Proposition 6.2) Assertion (i) follows from (4.1).

To verify (ii) we follow a standard argument (see e.g. the proof of [2, Theorem 11.2.6]). Let $g \in \widetilde{\mathcal{D}}_\psi^{-1}(B)$. Then there exists $G \in B$ such that $\widehat{G} = \psi \hat{g}$ and by (2.3) we get

$$\left(\frac{J_n g - g}{\phi(n)}\right)^{\widehat{}}(m) = (-\ell_n * G)^{\widehat{}}(m), \quad m \in \mathbb{Z}.$$

Hence, by the uniqueness of the Fourier transform, we infer that

$$\frac{J_n g - g}{\phi(n)} = -\ell_n * G.$$

Consequently, to verify (ii) it remains to show that

$$\|\ell_n * G + G\|_B \to 0 \quad \text{as} \quad n \to \infty. \tag{6.2}$$

Moreover, since the trigonometric polynomials in $B$ are dense (see [7, Theorem I.2.12]), it is sufficient to establish (6.2) only for them. To this end, we observe that for any trigonometric polynomial $\tau(x) = \sum_{m=m_1}^{m_2} c_m e^{imx}$ we have

$$\ell_n * \tau(x) + \tau(x) = \sum_{m=m_1}^{m_2} (\hat{\ell}_n(m) + 1) c_m e^{imx};$$

and hence by (6.1) we get for $\tau \in T \cap B$

$$\|\ell_n * \tau + \tau\|_B \leq c \max_{m_1 \leq m \leq m_2} |c_m| \max_{m_1 \leq m \leq m_2} |\hat{\ell}_n(m) + 1|. \tag{6.3}$$

Since $\ell_n$ satisfies (2.3) with $\Phi = \phi$ and $\Psi = \psi$, we get

$$\lim_{n \to \infty} \hat{\ell}_n(m) = -1 \tag{6.4}$$

for all $m \in \mathbb{Z}$ such that $\psi(m) \neq 0$. As for those integers $m$ for which $\psi(m) = 0$, we have that $\hat{k}_1(m) = 1$, which in view of the Riemann-Lebesgue lemma, yields that these $m$'s are finite in number and consequently, we may assume that $\hat{\ell}_n(m) = -1$ without loss of generality as we can correct all $\ell_n$ by adding to them a single suitable trigonometric polynomial. Thus we have (6.4) for all integers $m$ and (6.3) implies that

$$\|\ell_n * \tau + \tau\|_B \to 0 \quad \text{as} \quad n \to \infty \tag{6.5}$$

for all $\tau \in T \cap B$. This completes the proof of the proposition. ∎

Remark 6.3. In [2, Propositions 6.5.3 and 12.2.4 and Problem 5.5.1 (iii)] and [4, Theorem 2.1 (i)] it has been shown that for certain HBS on $\mathbb{T}$ and $\psi(m) = |m|^\kappa$, $\kappa > 0$, conditions like (2.3) and (2.4) are also necessary for assertion (ii) to hold.

For periodic approximate identities generated by a function $k \in L(\mathbb{R})$ assertion (ii) of the last proposition can be strengthen.

**Proposition 6.4** *Let $B$ be a HBS on $\mathbb{T}$ and $J_n$ be given by* (1.1) *with a periodic approximate identity $\{k_n\}$ defined by* (2.5)-(2.6)*, where $k$ satisfies* (2.10) *with $\ell \in L(\mathbb{R})$ such that $\hat{\ell}(0) \neq 0$. Set $\psi(m) = |m|^\kappa$ and $\phi(n) = -\hat{\ell}(0)n^{-\kappa}$. Then for all $g \in \widetilde{\mathcal{D}}_\psi^{-1}(B)$ we have $\mathcal{D}_\phi g = \widetilde{\mathcal{D}}_\psi g$.*

Proof. Let $\ell_n$ be defined by (2.6) by means of $\gamma\ell \in L(\mathbb{R})$, $\gamma = 1/\hat{\ell}(0)$. Then $\hat{\ell}_n(m) = \gamma\hat{\ell}(m/n)$, $m \in \mathbb{Z}$. As in the proof of Proposition 6.2 (ii) we establish that

$$\frac{J_n g - g}{\phi(n)} = \ell_n * G$$

for any $g \in \widetilde{\mathcal{D}}_\psi^{-1}(B)$ and $G = \widetilde{\mathcal{D}}_\psi g$. Moreover, we have $\hat{\ell}_n(0) = \gamma\hat{\ell}(0) = 1$ for all $n$. Hence $\{\ell_n\}$ is a periodic approximate identity and

$$\left\| \frac{J_n g - g}{\phi(n)} - G \right\|_B = \|\ell_n * G - G\|_B \to 0 \quad \text{as} \quad n \to \infty,$$

that is, $\mathcal{D}_\phi g = G$. ∎

## 7. Examples

In this section we shall calculate $\varphi(n) = \varphi_1(n)$ and $\psi(m)$ for several well-known convolution operators and hence get the explicit form of $\mathcal{D}_\varphi$ in any HBS on $\mathbb{T}$. We also present an application of Theorem 5.1. We have checked conditions (2.3)-(2.4) (or (2.10)) for several operators in [6, Section 5]. They furnish examples for the application of Theorem 5.3.

Example 7.1. Let the one-sided averages be defined for $f \in B$ by

$$A_n f(x) = 2\pi n \chi_{[0,1/n]} * f(x) = n \int_0^{1/n} f(x-t)\,dt, \quad x \in \mathbb{T}.$$

Above the kernel $k_n(t) = 2\pi n \chi_{[0,1/n]}(t) = 2\pi n \chi_{[0,1]}(nt)$ is considered on $\mathbb{T}$. We have for its Fourier transform

$$\hat{k}_n(m) = \widehat{\chi_{[0,1]}}(m/n),$$

as on the right-hand side the transform is taken in the sense of $L(\mathbb{R})$ functions.

Consequently,

$$\hat{k}_n(m) = \begin{cases} \dfrac{n}{m} \sin \dfrac{m}{n} - i\, \dfrac{n}{m} \left(1 - \cos \dfrac{m}{n}\right), & m \neq 0 \\ 1, & m = 0, \end{cases}$$

and then

$$\varphi(n) = \hat{k}_n(1) - 1 = \left(n \sin \frac{1}{n} - 1\right) - i\, n \left(1 - \cos \frac{1}{n}\right).$$

Therefore the optimal approximation order of $A_n$ is

$$|\varphi(n)| \sim \frac{1}{n},$$

as moreover,

$$\lim_{n \to \infty} n\, \varphi(n) = -\frac{i}{2}.$$

As for the corresponding $\psi$ we have that

$$\psi(m) = \lim_{n \to \infty} \frac{1 - \hat{k}_n(m)}{1 - \hat{k}_n(1)} = m.$$

Consequently, for any $g \in \mathcal{D}_\varphi^{-1}(B)$ we have $\widehat{\mathcal{D}_\varphi g}(m) = m\hat{g}(m)$, $m \in \mathbb{Z}$; hence $\mathcal{D}_\varphi g = -ig'$.

Example 7.2. The Jackson operator is defined for $f \in B$ by

$$\mathcal{J}_n f(x) = j_n * f(x),$$

where

$$j_n(t) = \frac{3}{n(2n^2+1)} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}}\right)^4.$$

For the Fourier coefficients of $j_n \in T$ we have (see e.g. [ 2, p. 517])

$$\widehat{j}_n(m) = \frac{1}{2n(2n^2+1)} \begin{cases} 3|m|^3 - 6nm^2 - 3|m| + 4n^3 + 2n, & |m| \le n, \\ -|m|^3 + 6nm^2 - (12n^2 - 1)|m| & \\ \qquad\qquad +8n^3 - 2n, & n \le |m| \le 2n-1, \\ 0, & |m| \ge 2n-1. \end{cases}$$

Next, we calculate

$$\varphi(n) = \hat{j}_n(1) - 1 = -\frac{3}{2n^2+1}, \quad |\varphi(n)| \sim \frac{1}{n^2}$$

and

$$\psi(m) = \lim_{n\to\infty} \frac{1-\hat{j}_n(m)}{1-\hat{j}_n(1)} = \lim_{n\to\infty} \frac{-|m|^3 + 2nm^2 + |m|}{2n} = m^2.$$

Thus for any $g \in \mathcal{D}_\varphi^{-1}(B)$ we have $\widehat{\mathcal{D}_\varphi g}(m) = m^2 \hat{g}(m)$, $m \in \mathbb{Z}$; hence $\mathcal{D}_\varphi g = -g''$.

Example 7.3. The Riesz typical means of the Fourier series of $f \in B$ are defined by

$$R_{\kappa,n} f(x) = r_{\kappa,n} * f(x),$$

where the kernel $r_{\kappa,n}$, $\kappa > 0$, is given by

$$r_{\kappa,n}(t) = \sum_{m=-n}^{n} \left(1 - \left|\frac{m}{n+1}\right|^\kappa\right) e^{imt}.$$

The Fourier coefficients of $r_{\kappa,n} \in T$ are

$$\hat{r}_{\kappa,n}(m) = \begin{cases} 1 - \left|\dfrac{m}{n+1}\right|^\kappa, & |m| \le n, \\ 0, & |m| > n. \end{cases}$$

Consequently,

$$\varphi(n) = -\frac{1}{(n+1)^\kappa}, \quad |\varphi(n)| \sim \frac{1}{n^\kappa}, \quad \text{and} \quad \psi(m) = |m|^\kappa.$$

The last relation above implies that in this case $\mathcal{D}_\varphi$ is the Riesz fractional derivative of order $\kappa$.

For the Riesz means Theorem 5.1 with $\phi = \varphi$ gives for all $f \in B$ and $n \in \mathbb{N}$

$$\|f - R_{\kappa,n} f\|_B \le c\, K(f, n^{-\kappa}; B, \mathcal{D}_\varphi).$$

Indeed, it only remains to verify condition (5.1), which means, in this particular case, to establish that the family of function $\{l_n\}$ with Fourier coefficients

$$\hat{l}_n(m) = \begin{cases} 1, & |m| \le n, \\ \left|\frac{n+1}{m}\right|^\kappa, & |m| > n, \end{cases}$$

is uniformly bounded in $L$-norm. This was verified by DeVore [3, pp. 67-68] (or see [5, p. 68]).

Example 7.4. We shall present a periodic approximate identity $\{k_n\}$ for which the limit

$$\lim_{n\to\infty} \frac{1-\hat{k}_n(m)}{1-\hat{k}_n(1/n)}$$

does not exist for all integers $m$ and hence the corresponding convolution operator does not fulfil the condition that $T \subset \mathcal{D}_\varphi^{-1}(B)$. Let $k \in L(\mathbb{R})$ be given by its Fourier transform

$$\hat{k}(u) = \begin{cases} 1-e^{-u^{-2}}, & u \neq 0, \\ 1 & u = 0. \end{cases}$$

Note that the function on the right-hand side above belongs to $L(\mathbb{R})$, it and its first derivative are absolutely continuous on $\mathbb{R}$ and its second derivative belongs to $L(\mathbb{R})$; hence it is the Fourier transform of a function $k \in L(\mathbb{R})$ with (2.5). Let $k_n$ be defined via (2.6). Then for $\varphi$ and $\psi$ we have respectively

$$\varphi(n) = \hat{k}(1/n) - 1 = -e^{-n^2}$$

and

$$\lim_{n\to\infty} \frac{1-\hat{k}(m/n)}{1-\hat{k}(1/n)} = \lim_{n\to\infty} e^{n^2\left(1-\frac{1}{m^2}\right)}$$

does not exist as a finite number for $|m| > 1$. Actually, for a trigonometric polynomial $\tau(x) = \sum_{m=m_1}^{m_2} c_m e^{imx} \in B$ with a coefficient $c_m \neq 0$ for some $|m| > 1$, there holds

$$\begin{aligned} \left\| \frac{k_n * \tau - \tau}{\varphi(n)} \right\|_B &\geq \frac{1}{\alpha} \max_{m_1\leq m\leq m_2} |c_m| \left| \frac{\hat{k}_n(m)-1}{\varphi(n)} \right| \\ &= \frac{1}{\alpha} \max_{m_1\leq m\leq m_2} |c_m|\, e^{n^2\left(1-\frac{1}{m^2}\right)} \to \infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

Consequently, $\mathcal{D}_\varphi^{-1}(B)$ does not contain trigonometric polynomials of exact degree greater than 1. However, if $B$ contains an element whose Fourier coefficient at $m = 1$ or $m = -1$ is not zero, the convolution operator possesses the saturation property with optimal approximation order $e^{-n^2}$. Indeed, $k_n$ satisfies the assumptions of Theorem 3.4 with $a = 1$. Also, for any $\tau(x) = c_{-1}e^{-ix} + c_0 + c_1 e^{ix} \in B$ we have for all $n$

$$\frac{k_n * \tau(x) - \tau(x)}{\varphi(n)} = e^{n^2} \sum_{m=-1}^{1} (1-\hat{k}_n(m)) c_m e^{imx} = \tau(x) - c_0.$$

Consequently, $\mathcal{D}_\varphi \tau = \tau - c_0$. And now if $f \in B$ is such that either $\hat{f}(1) \neq 0$ or $\hat{f}(-1) \neq 0$, then the trigonometric polynomial $R_{1,1}f$ belongs to $B$ and is of exact degree 1. Therefore $\mathcal{D}_\varphi R_{1,1} f \neq 0$. Now, Theorem 3.4 implies that $J_n$ is saturated with order $e^{-n^2}$.

Let us also observe that for a positive integer $a$ we have

$$\varphi_a(n) = \hat{k}(a/n) - 1 = -e^{-n^2/a^2}$$

and

$$\lim_{n\to\infty} \frac{1-\hat{k}(m/n)}{1-\hat{k}(a/n)} = \begin{cases} 0, & |m| < a, \\ 1, & |m| = a, \\ \infty, & |m| > a. \end{cases}$$

Hence, if (6.1) holds, then $\mathcal{D}_{\varphi_a}\tau = 0$ for any trigonometric polynomial of degree $a-1$ whereas $J_n f = f$ for all $n$ iff $f = \mathrm{const}$. Here we have $|\varphi(n)| = o(|\varphi_a(n)|)$ for $a > 1$.

Note that the kernel of Example 7.4 does not satisfy the standard condition [2, (12.1.1)] and hence [2, Theorem 12.1.3] is not applicable. Neither is [6, Theorem 3.13] nor Theorem 5.1. Also, this example shows that it is really essential for the domain of $\mathcal{D}_\phi$ to contain a sufficiently large set of trigonometric polynomials in order to have assertions like the one in [2, pp. 436-437].

Example 7.5. Let $B \nsubseteq \mathbb{C}$. Then there exists $a \in \mathbb{N}$ such that either $\hat{f}(a) \neq 0$ or $\hat{f}(-a) \neq 0$ for some $f \in B$. Set

$$k_n(x) = 1 + 2\sum_{m=1}^{a-1} \cos mx + 2\sum_{m=a}^{\infty} \left(1 - e^{-(n/m)^2}\right) \cos mx.$$

It is easy to prove that $\{k_n\}$ is a periodic approximate identity. Let $J_n$ be the corresponding convolution operator. As in the previous example it is verified that $J_n$ is saturated with order $1 - \hat{k}_n(a/n) = e^{-n^2/a^2}$ as $J_n f = f$ iff $f$ is a trigonometric polynomial of degree $a-1$. For $|m| < a$ we have $\hat{k}_n(m) = 1$ for all $n$ and

$$\lim_{n\to\infty} \frac{1-\hat{k}_n(m)}{1-\hat{k}_n(a)} = \begin{cases} 0, & |m| < a, \\ 1, & |m| = a, \\ \infty, & |m| > a. \end{cases}$$

Example 7.6. An example of a convolution operator that does not possesses the saturation property can be found in [2, pp. 476-478].

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**References**

[1] S. Bochner, J. von Neumann Almost periodic functions in groups II, *Trans. Amer. Math. Soc.* **37**, 1935, No. 1, 21-50.

[2] P.L. Butzer, R.J. Nessel. Fourier Analysis and Approximation, Birkhäser Verlag, Basel, 1971.

[3] R.A. DeVore. The approximation of continuous functions by positive linear operators, *Lecture Notes in Math.* **293**, Springer-Verlag, Berlin, 1972.

[4] R.A. DeVore, G.G. Lorentz. Constructive Approximation, Springer Verlag, Berlin, 1993.

[5] Z. Ditzian, K.G. Ivanov. Strong converse inequalities, *J. D'Analyse Math.* **61**, 1993, 61-111.

[6] B.R. Draganov. Exact estimates of the rate of approximation of convolution operators, *J. Approx. Theory* **162**, 2010, 952-979.

[7] Y. Katznelson. An Introduction to Harmonic Analysis, John Wiley & Sons, Inc., 1968.

[8] H.S. Shapiro. Topics in approximation theory, *Lecture Notes in Math.* **187**, Springer-Verlag, Berlin, 1971.

[9] G.E. Shilov. Homogeneous rings of functions, *Uspekhi Mat. Nauk* **41**, 1951, 91-137 (in Russian); English translation: *Amer. Math. Soc. Transl.* **8**, 1954, 393-455.

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