Simultaneous approximation of functions by Féjer-type operators in a generalized Hölder norm

Borislav R. Draganov*

Abstract

The purpose of the paper is to give an upper estimate of the rate of the simultaneous approximation of Féjer-type operators in the L_p-norm and in a generalized Hölder L_p-norm. The estimates involve moduli of smoothness of second order. A sufficient condition for the optimal order of approximation is established.

AMS classification: 41A28, 41A25, 41A35, 42A45.

Key words and phrases: Féjer-type operators, generalized Hölder norm, rate of convergence, simultaneous approximation, modulus of smoothness.

1 Féjer-type operators

Let $\lambda > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x)/(1 + x^2)$ is summable on $\mathbb{R}$. The Féjer-type operators are defined by (see e.g. [1, Section 62])

$$F_\lambda f(x) = \lambda \int_{\mathbb{R}} \mathcal{K}(\lambda(x - y)) f(y)\, dy, \tag{1.1}$$

where the measurable kernel $\mathcal{K}$ satisfies the conditions:

a) $\mathcal{K}(x)$ is even,

b) $\mathcal{K}(x)$ is bounded on $[-1, 1]$,

c) $x^2\mathcal{K}(x)$ is bounded on $\mathbb{R}$,

d) $\int_{\mathbb{R}} \mathcal{K}(x)\, dx = 1$.

*Partially supported by grant No. 105-2008 of the National Science Fund of the Ministry of Education and Science to University of Sofia.

As it is known (see e.g. [1, Section 62]), if f is continuous on $\mathbb{R}$, then $F_\lambda f(x)$ converges to $f(x)$ for every $x \in \mathbb{R}$, as the convergence is uniform on every finite interval on the real line. It is useful to determine the rate of this convergence. In Section 2 we estimate it in the L_p-norm. These results must have already been verified but we could not find any references and since they have short and standard proofs we present them. In Section 3 we establish our main result concerning estimates of the error of Féjer-type operators in simultaneous approximation in a generalized Hölder norm. There we give a sufficient condition under which these operators approximate the function with the generally optimal order of $1/\lambda$.

2 Simultaneous approximation by Féjer-type operators in $L_p(\mathbb{R})$

We shall consider a slight generalization of the operator F_λ, defined above, as we shall use the same notation.

Definition 2.1. Let for $\lambda > 0$ the bounded linear operator $F_\lambda : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 \le p \le \infty$, be defined by (1.1), where the measurable kernel $\mathcal{K}$ satisfies the conditions:

a) $\mathcal{K}(x)$ is even,

b) $\mathcal{K}(x)$ is summable on $[0, 1]$,

c) $x^2\mathcal{K}(x)$ is bounded on $[1, \infty)$,

d) $\int_{\mathbb{R}} \mathcal{K}(x)\, dx = 1$.

Let us note that conditions a)-c) of Definition 2.1 imply that $\mathcal{K} \in L_1(\mathbb{R})$ and, consequently by Young's inequality, we get that $\|F_\lambda f\|_p \le \|\mathcal{K}\|_1 \|f\|_p$ for every $f \in L_p(\mathbb{R})$ and $\lambda > 0$, $1 \le p \le \infty$. Here and in what follows we denote by $\|\cdot\|_p$ the usual L_p-norm taken on the real line. Let also $W_p^r(\mathbb{R})$, $r \in \mathbb{N}$, denote the Sobolev spaces on $\mathbb{R}$, i.e.

$$W_p^r(\mathbb{R}) = \{f \in L_p(\mathbb{R}) : f \in AC^{r-1},\ f^{(r)} \in L_p(\mathbb{R})\},$$

where AC^s, $s \in \mathbb{N}_0$, denotes the space of functions whose derivatives up to order s are absolutely continuous on $\mathbb{R}$. We also set $W_p^0(\mathbb{R}) = L_p(\mathbb{R})$.

To estimate the error of the operator F_λ, we shall use the modulus of smoothness of second order of the function. Generally, for $n \in \mathbb{N}$ the modulus of smoothness of order n of the function $f \in L_p(\mathbb{R})$ is defined for $t > 0$ by

$$\omega_n(f, t)_p = \sup_{0<h\le t} \|\Delta_h^n f\|_p,$$

where $\Delta_h f(x) = f(x + h/2) - f(x - h/2)$ and $\Delta_h^n f(x) = \Delta_h(\Delta_h^{n-1} f)(x)$ are the symmetric finite differences of the function f with step h.

Theorem 2.1. *Let the operator F_λ be defined by Definition 2.1, $\lambda > 1$ and $f \in W_p^r(\mathbb{R})$, $r \in \mathbb{N}_0$, $1 \le p \le \infty$. Then $F_\lambda f \in W_p^r(\mathbb{R})$ and for $k = 0, \ldots, r$ there holds*

$$\|(F_\lambda f)^{(k)} - f^{(k)}\|_p \le k_1 \left[\frac{1}{\lambda - 1} \int_1^\lambda \omega_2(f^{(k)}, 1/y)_p \, dy + \frac{\|f^{(k)}\|_p}{\lambda} \right],$$

where

$$k_1 = \int_0^1 |\mathcal{K}(x)| \, dx + 4 \sup_{x \ge 1} |x^2 \mathcal{K}(x)|.$$

Proof. First, let us note that under the conditions imposed on the kernel $\mathcal{K}$, we have $F_\lambda f \in W_p^r(\mathbb{R})$ for every $f \in W_p^r(\mathbb{R})$ and $(F_\lambda f)^{(k)} = F_\lambda f^{(k)}$. Consequently, it is sufficient to establish the assertion of the theorem only for $r = 0$.

Making the change of the variable $u = \lambda(x - y)$ and using properties a) and d) of $\mathcal{K}$, we get the representation

$$F_\lambda f(x) - f(x) = \int_0^\infty \mathcal{K}(u) \Delta^2_{u/\lambda} f(x) \, du.$$

Hence by generalized Minkowski's inequality we get

$$\|F_\lambda f - f\|_p \le \int_0^\infty |\mathcal{K}(u)| \|\Delta^2_{u/\lambda} f\|_p \, du. \tag{2.1}$$

We shall estimate the integral on the right above separately on the intervals $[0, 1]$, $[1, \lambda]$ and $[\lambda, \infty)$ (see e.g. [1, Section 62]). For the first one, in view of the fact that $\omega_2(f, t)_p$ is non-decreasing, we have

$$\begin{aligned} \int_0^1 |\mathcal{K}(u)| \|\Delta^2_{u/\lambda} f\|_p \, du &\le \int_0^1 |\mathcal{K}(u)| \, \omega_2(f, u/\lambda)_p \, du \\ &\le \int_0^1 |\mathcal{K}(u)| \, du \cdot \omega_2(f, 1/\lambda)_p \\ &\le \int_0^1 |\mathcal{K}(u)| \, du \cdot \frac{1}{\lambda - 1} \int_1^\lambda \omega_2(f, 1/y)_p \, dy. \end{aligned} \tag{2.2}$$

Next, by property c) of $\mathcal{K}$, we get

$$\begin{aligned} \int_1^\lambda |\mathcal{K}(u)| \|\Delta^2_{u/\lambda} f\|_p \, du &\le \sup_{u \ge 1} |u^2 \mathcal{K}(u)| \int_1^\lambda \frac{\omega_2(f, u/\lambda)_p}{u^2} \, du \\ &= \sup_{u \ge 1} |u^2 \mathcal{K}(u)| \cdot \frac{1}{\lambda} \int_1^\lambda \omega_2(f, 1/y)_p \, dy. \end{aligned} \tag{2.3}$$

Finally, again by property c) of $\mathcal{K}$ and the inequality $\|\Delta_h^2 f\|_p \le 4\|f\|_p$, we get

$$\begin{aligned} \int_\lambda^\infty |\mathcal{K}(u)| \|\Delta^2_{u/\lambda} f\|_p \, du &\le 4\|f\|_p \sup_{u \ge 1} |u^2 \mathcal{K}(u)| \int_\lambda^\infty \frac{1}{u^2} \, du \\ &= 4\|f\|_p \sup_{u \ge 1} |u^2 \mathcal{K}(u)| \cdot \frac{1}{\lambda}. \end{aligned} \tag{2.4}$$

Combining (2.1)-(2.4) we get the assertion of the theorem for $r = 0$. $\square$

Remark 2.1. The assertion of the theorem remains valid for $\lambda = 1$ too, as $\frac{1}{\lambda-1}\int_1^\lambda \omega_2(f^{(k)}, 1/y)_p\, dy$ is defined for $\lambda = 1$ by continuity to be equal to $\omega_2(f^{(k)}, 1)_p$.

Theorem 2.1 directly yields the following sufficient conditions for the approximation rate of the operators F_λ.

Corollary 2.1. *Let the operator F_λ be defined by Definition 2.1. Let $f \in L_p(\mathbb{R})$ be such that $\omega_2(f, t)_p = O(t^\alpha)$, $0 < \alpha \le 2$. Then*

$$\|F_\lambda f - f\|_p = \begin{cases} O\left(\lambda^{-\alpha}\right) & \text{if} \quad 0 < \alpha < 1, \\ O\left(\lambda^{-1}\log\lambda\right) & \text{if} \quad \alpha = 1, \\ O\left(\lambda^{-1}\right) & \text{if} \quad 1 < \alpha \le 2. \end{cases}$$

Let us recall that generally it is not possible to approximate a function f, no matter how smooth it is, with operators of the type F_λ with a rate greater than $1/\lambda$. For example, the Féjer operator, which is defined by (1.1) with

$$\mathcal{K}(x) = \frac{2}{\pi}\left(\frac{\sin(x/2)}{x}\right)^2, \quad x \in \mathbb{R},$$

has a saturation order $1/\lambda$ (see e.g. [1, Section 61], [2, Ch. 11, Section 3] and [3, Theorem 2.2]. The saturation class of the Fejer means was determined by Alexits and Zamansky for the case of continuous 2π-periodic functions. Ditzian and Ivanov [3, Theorem 2.2] established it for a broad range of Banach spaces including Hölder spaces of 2π-periodic functions. However, if the kernel $\mathcal{K}(x)$ satisfies stronger restrictions, we get higher rates of approximation. The following theorem holds.

Theorem 2.2. *Let the operator F_λ be defined by (1.1) as the kernel $\mathcal{K}$ is of a finite support $[-\zeta, \zeta]$, it is summable on $\mathbb{R}$ and satisfies conditions a) and d) of Definition 2.1. Let also $\lambda > 0$ and $f \in W_p^r(\mathbb{R})$, $r \in \mathbb{N}_0$, $1 \le p \le \infty$. Then $F_\lambda f \in W_p^r(\mathbb{R})$ and for $k = 0, \dots, r$ there holds*

$$\|(F_\lambda f)^{(k)} - f^{(k)}\|_p \le k_2\, \omega_2(f^{(k)}, \zeta/\lambda)_p,$$

where

$$k_2 = \int_0^\zeta |\mathcal{K}(x)|\, dx.$$

Proof. Now we just have

$$\|F_\lambda f - f\|_p \le \int_0^\zeta |\mathcal{K}(u)| \|\Delta^2_{u/\lambda} f\|_p\, du \le \int_0^\zeta |\mathcal{K}(u)|\, du \cdot \omega_2(f, \zeta/\lambda)_p.$$

$\square$

3 Simultaneous approximation by Féjer-type operators in Hölder spaces

Let ω be a positive and non-decreasing function on $(0,\infty)$. We set for $f \in L_p(\mathbb{R})$, $1 \le p \le \infty$, and $n \in \mathbb{N}$

$$|f|_{p,n,\omega} = \sup_{h>0} \frac{\|\Delta_h^n f\|_p}{\omega(h)}$$

and $H_{p,n,\omega}(\mathbb{R}) = \{f \in L_p(\mathbb{R}) : |f|_{p,n,\omega} < \infty\}$. $H_{p,n,\omega}(\mathbb{R})$ is a Banach space in the generalized Hölder norm

$$\|f\|_{p,n,\omega} = \|f\|_p + |f|_{p,n,\omega},$$

as $|\cdot|_{p,n,\omega}$ is a semi-norm in $H_{p,n,\omega}(\mathbb{R})$. Assuming that $\omega(h)$ is bounded does not affect $H_{p,n,\omega}(\mathbb{R})$. More precisely, if we set $\bar{\omega}(h) = \min\{\omega(h), 1\}$, then $H_{p,n,\bar{\omega}}(\mathbb{R}) = H_{p,n,\omega}(\mathbb{R})$. What matters is whether and how fast $\omega(h)$ tends to 0 as $h \to 0$. Moreover, if $\lim_{h\to 0}\omega(h) \ne 0$, then $H_{p,n,\omega}(\mathbb{R}) = L_p(\mathbb{R})$, and if there exists a sequence $\{h_i\}$ such that $\lim h_i = 0$ and $\lim h_i^{-n}\omega(h_i) = 0$, then $f \in H_{p,n,\omega}(\mathbb{R})$ implies $f = 0$ for $1 \le p < \infty$ and $f = const$ for $p = \infty$ (as it follows from [2, Ch. 2, Proposition 7.1]).

An important example of Hölder spaces $H_{p,n,\omega}(\mathbb{R})$ are the ones with $\omega(h) = h^\alpha$, $0 < \alpha \le n$. They are also called Lipschitz spaces (see e.g [2, Ch. 2, Section 9]).

Let us observe that for $m, n \in \mathbb{N}$ with $m < n$ the semi-norm $|\cdot|_{p,m,\omega}$ is generally larger than the semi-norm $|\cdot|_{p,n,\omega}$ in the sense that $|f|_{p,n,\omega} \le 2^{n-m}|f|_{p,m,\omega}$ for every $f \in H_{p,m,\omega}(\mathbb{R})$. Hence $\|f\|_{p,n,\omega} \le 2^{n-m}\|f\|_{p,m,\omega}$ and $H_{p,m,\omega}(\mathbb{R}) \subset H_{p,n,\omega}(\mathbb{R})$. However, under certain conditions on the function ω these semi-norms, and hence the norms $\|\cdot\|_{p,m,\omega}$ and $\|\cdot\|_{p,n,\omega}$ are equivalent. The following assertion holds.

Proposition 3.1. *Let $m, n \in \mathbb{N}$ as $m < n$. Let also ω_1 and ω_2 be positive non-decreasing functions on $(0,\infty)$ such that*

$$\sup_{h>0} \frac{h^m}{\omega_2(h)} \int_h^\infty \frac{\omega_1(y)}{y^{m+1}}\, dy < \infty.$$

Then there exists a positive constant c such that for every $f \in H_{p,n,\omega_1}(\mathbb{R})$ there holds

$$|f|_{p,m,\omega_2} \le c\,|f|_{p,n,\omega_1}.$$

Proof. First, let us note that

$$|f|_{p,n,\omega} = \sup_{h>0} \frac{\omega_n(f,h)_p}{\omega(h)}$$

for every $f \in H_{p,n,\omega}(\mathbb{R})$, $n \in \mathbb{N}$, and a positive non-decreasing function ω. By the Marchaud inequality (see e.g. [2, Ch. 2, Theorem 8.1]), we have for $f \in H_{p,n,\omega}(\mathbb{R})$ and $h > 0$

$$\omega_m(f,h)_p \le c\,h^m \int_h^\infty \frac{\omega_n(f,y)_p}{y^{m+1}}\, dy \ \le c\,h^m |f|_{p,n,\omega_1} \int_h^\infty \frac{\omega_1(y)}{y^{m+1}}\, dy,$$

where c is a positive constant, which depends on n but not on f. Hence the assertion of the proposition follows. □

In view of the trivial relation $|f|_{p,n,\omega} \leq 2^{n-m}|f|_{p,m,\omega}$, mentioned above, the last proposition yields the following result concerning the identity of Hölder spaces and, in particular, some Lipschitz spaces.

Corollary 3.1. *Let $m, n \in \mathbb{N}$ as $m < n$. If ω is a positive non-decreasing function on $(0, \infty)$ such that*

$$\sup_{h>0} \frac{h^m}{\omega(h)} \int_h^\infty \frac{\omega(y)}{y^{m+1}}\, dy < \infty,$$

then $H_{p,m,\omega}(\mathbb{R}) = H_{p,n,\omega}(\mathbb{R})$ with equivalent norms. In particular, if $\omega(h) = h^\alpha$ with $0 < \alpha < m$, then $H_{p,n,\omega}(\mathbb{R}) = H_{p,m,\omega}(\mathbb{R})$ with equivalent norms for every $n > m$.

We now proceed to estimating the approximation rate of the operators F_λ in Hölder spaces. Let ω_1 and ω_2 be positive non-decreasing functions on $(0, \infty)$ such that there exist constants $c > 0$ and $0 \leq \gamma \leq 1$ so that

$$\omega_1^\gamma(h) \leq c\, \omega_2(h). \tag{3.1}$$

This inequality implies that $H_{p,n,\omega_1}(\mathbb{R}) \subseteq H_{p,n,\omega_2}(\mathbb{R})$.

Lasuriya [4, Theorem 1] proved that for any fixed $f \in H_{\infty,1,\omega_1}(\mathbb{R})$ and $\lambda_0 > 1$ there exists a positive constant C such that for any $\lambda \geq \lambda_0$ we have

$$\|F_\lambda f - f\|_{\infty,1,\omega_2} \leq C \sup_{h>0} \frac{\omega_1^\gamma(h)}{\omega_2(h)} \left[\frac{1}{\lambda} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\, dy \right], \tag{3.2}$$

as the quantity C generally depends on f and also on ω_1 and ω_2. This relation enabled Lasuriya to improve estimates of the rate of convergence of the Féjer operator in the Hölder norm. However, it cannot give a sufficient condition under which $F_\lambda f$ approximates f with the generally optimal rate of approximation $1/\lambda$ (see [4, Corollary 1]). In the next assertions, following Lasuriya's approach, we establish the analogue of (3.2) for the spaces $H_{p,2,\omega}(\mathbb{R})$, $1 \leq p \leq \infty$. This enables us to give a sufficient condition on the smoothness of the function, which yields a rate of approximation $1/\lambda$ in $H_{p,2,\omega}(\mathbb{R})$ as well as in $H_{p,1,\omega}(\mathbb{R})$. We also estimate explicitly the quantity C in terms of the functions f, ω_1, ω_2 for both $H_{p,1,\omega}(\mathbb{R})$ and $H_{p,2,\omega}(\mathbb{R})$.

Let $1 \leq p \leq \infty$, $n, r \in \mathbb{N}$ and ω be a positive and non-decreasing function on $(0, \infty)$. We set $W^r_{p,n,\omega}(\mathbb{R}) = \{f \in H_{p,n,\omega}(\mathbb{R}) : f \in AC^{r-1},\ f^{(r)} \in H_{p,n,\omega}(\mathbb{R})\}$ and $W^0_{p,n,\omega}(\mathbb{R}) = H_{p,n,\omega}(\mathbb{R})$. Let us observe that if $f \in W^r_{p,n,\omega}(\mathbb{R})$, then $f^{(k)} \in H_{p,n,\omega}(\mathbb{R})$, $k = 0, \ldots, r$.

The following results concerning the simultaneous approximation by Féjer-type operators in Hölder spaces hold.

Theorem 3.1. *Let ω_1 and ω_2 be positive non-decreasing functions on $(0,\infty)$ with (3.1) and the operator F_λ be defined by Definition 2.1. Let also $\lambda > 1$ and $f \in W^r_{p,2,\omega_1}(\mathbb{R})$, $1 \le p \le \infty$, $r \in \mathbb{N}_0$. Then $F_\lambda f \in W^r_{p,2,\omega_1}(\mathbb{R})$ and for $k = 0,\dots,r$ there holds*

$$\begin{aligned} &|(F_\lambda f)^{(k)} - f^{(k)}|_{p,2,\omega_2} \\ &\le 4k_1 \sup_{h>0} \frac{\omega_1^\gamma(h)}{\omega_2(h)} \left[\frac{|f^{(k)}|_{p,2,\omega_1}}{\lambda-1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\,dy + \frac{|f^{(k)}|^\gamma_{p,2,\omega_1} \|f^{(k)}\|_p^{1-\gamma}}{\lambda} \right], \end{aligned}$$

where k_1 is given in Theorem 2.1. If also $\inf_{0<h\le 1} h^{-2}\omega_1(h) > 0$, then for $k = 0,\dots,r$ there holds

$$\begin{aligned} &|(F_\lambda f)^{(k)} - f^{(k)}|_{p,2,\omega_2} \\ &\le 4k_1 \sup_{h>0} \frac{\omega_1^\gamma(h)}{\omega_2(h)} \left[1 + \left(\inf_{0<h\le 1} \frac{\omega_1(h)}{h^2} \right)^{\gamma-1} \right] \|f^{(k)}\|_{p,2,\omega_1} \frac{1}{\lambda-1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\,dy. \end{aligned}$$

Remark 3.1. Recall that if $\inf_{0<h\le 1} h^{-2}\omega_1(h) = 0$, i.e. there exists a sequence $\{h_i\}$ such that $\lim h_i = 0$ and $\lim h_i^{-2}\omega_1(h_i) = 0$, then $f = 0$ for $p < \infty$ and $f = \mathit{const}$ for $p = \infty$ and the assertions of the theorem are trivial.

Proof of Theorem 3.1. To prove the first assertion of the theorem, it is sufficient to consider it only for $r = 0$.

Since $\Delta_h^2(F_\lambda f - f) = F_\lambda(\Delta_h^2 f) - \Delta_h^2 f$, we get by Theorem 2.1 with $r = 0$ that for every $h > 0$

$$\|\Delta_h^2(F_\lambda f - f)\|_p \le k_1 \left[\frac{1}{\lambda-1} \int_1^\lambda \omega_2(\Delta_h^2 f, 1/y)_p\,dy + \frac{\|\Delta_h^2 f\|_p}{\lambda} \right].$$

Further, since $\omega_2(\Delta_h^2 f, t)_p \le 4\|\Delta_h^2 f\|_p$, $\omega_2(\Delta_h^2 f, t)_p \le 4\omega_2(f,t)_p$ and $\|\Delta_h^2 f\|_p \le 4\|f\|_p$, we get for every $h > 0$

$$\begin{aligned} \|\Delta_h^2(F_\lambda f - f)\|_p &\le k_1\, \|\Delta_h^2 f\|_p^\gamma \left[\frac{4}{\lambda-1} \int_1^\lambda \omega_2^{1-\gamma}(f,1/y)_p\,dy + \frac{(4\|f\|_p)^{1-\gamma}}{\lambda} \right] \\ &\le 4k_1\, \omega_1^\gamma(h)\, |f|^\gamma_{p,2,\omega_1} \left[\frac{|f|^{1-\gamma}_{p,2,\omega_1}}{\lambda-1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)_p\,dy + \frac{\|f\|_p^{1-\gamma}}{\lambda} \right]. \end{aligned}$$

Hence the first assertion of the theorem follows. To derive the second one from it, it is sufficient to observe that for $\lambda > 1$ we have

$$\begin{aligned} \frac{1}{\lambda-1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\,dy &\ge \left(\inf_{0<h\le 1} \frac{\omega_1(h)}{h^2} \right)^{1-\gamma} \frac{1}{\lambda-1} \int_1^\lambda y^{2(\gamma-1)}\,dy \\ &\ge \left(\inf_{0<h\le 1} \frac{\omega_1(h)}{h^2} \right)^{1-\gamma} \frac{1}{\lambda}, \end{aligned}$$

as the second estimate is verified by direct calculations. $\square$

Combining Theorem 2.1 and Theorem 3.1 we get the following estimate of the error of F_λ in the generalized Hölder norm.

Theorem 3.2. *Let ω_1 and ω_2 be positive non-decreasing functions on $(0,\infty)$ with (3.1) and $\inf_{0<h\le 1} h^{-2}\omega_1(h) > 0$. Let also the operator F_λ be defined by Definition 2.1, $\lambda > 1$ and $f \in W^r_{p,2,\omega_1}(\mathbb{R})$, $1 \le p \le \infty$, $r \in \mathbb{N}_0$. Then $F_\lambda f \in W^r_{p,2,\omega_1}(\mathbb{R})$ and for $k = 0,\dots,r$ there holds*

$$\|(F_\lambda f)^{(k)} - f^{(k)}\|_{p,2,\omega_2} \le k_1 \left[1 + 4\sup_{h>0} \frac{\omega_1^\gamma(h)}{\omega_2(h)}\right] \left[4^\gamma + \left(\inf_{0<h\le 1} \frac{\omega_1(h)}{h^2}\right)^{\gamma-1}\right] \times \frac{\|f^{(k)}\|_{p,2,\omega_1}}{\lambda - 1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\,dy,$$

where k_1 is given in Theorem 2.1.

Proof. As in the proof of Theorem 3.1 we derive from Theorem 2.1 the estimate

$$\|(F_\lambda f)^{(k)} - f^{(k)}\|_p \le k_1 \left[4^\gamma + \left(\inf_{0<h\le 1} \frac{\omega_1(h)}{h^2}\right)^{\gamma-1}\right] \|f^{(k)}\|_{p,2,\omega_1} \frac{1}{\lambda - 1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\,dy,$$

where $k = 0,\dots,r$. This estimate and the second one in Theorem 3.1 imply the assertion of the theorem. □

In particular, we have in the case of Lipschitz spaces the following direct estimates of the error of F_λ.

Corollary 3.2. *Let the operator F_λ be defined by Definition 2.1. Let also $\omega_1(h) = h^\alpha$ and $\omega_2(h) = h^\beta$ with $0 < \beta < \alpha \le 2$. For $f \in H_{p,2,\omega_1}(\mathbb{R})$, $1 \le p \le \infty$, we have*

$$\|F_\lambda f - f\|_{p,2,\omega_2} = \begin{cases} O\left(\lambda^{\beta-\alpha}\right) & \text{if } \alpha - \beta < 1, \\ O\left(\lambda^{-1}\log\lambda\right) & \text{if } \alpha - \beta = 1, \\ O\left(\lambda^{-1}\right) & \text{if } \alpha - \beta > 1. \end{cases}$$

Remark 3.2. Let ω_1 and ω_2 be positive non-decreasing functions on $(0,\infty)$ with (3.1) and $\inf_{0<h\le 1} h^{-1}\omega_1(h) > 0$. Let $\lambda > 1$ and $f \in W^r_{p,1,\omega_1}(\mathbb{R})$, $1 \le p \le \infty$, $r \in \mathbb{N}_0$. Following the methods used in the proofs of Theorem 2.1 and Theorem 3.1 one can establish for $k = 0,\dots,r$ the following estimates

$$\|(F_\lambda f)^{(k)} - f^{(k)}\|_p \le k_3 \left[2^\gamma + \left(\inf_{0<h\le 1} \frac{\omega_1(h)}{h}\right)^{\gamma-1}\right] \|f^{(k)}\|_{p,1,\omega_1} \frac{1}{\lambda - 1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\,dy$$

and

$$|(F_\lambda f)^{(k)} - f^{(k)}|_{p,1,\omega_2}$$
$$\leq 2k_3 \sup_{h>0} \frac{\omega_1^\gamma(h)}{\omega_2(h)} \left[1 + \left(\inf_{0<h\leq 1} \frac{\omega_1(h)}{h}\right)^{\gamma-1}\right] \|f^{(k)}\|_{p,1,\omega_1} \frac{1}{\lambda-1} \int_1^\lambda \omega_1^{1-\gamma}(1/y)\, dy,$$

where
$$k_3 = 2\int_0^1 |\mathcal{K}(x)|\, dx + 4 \sup_{x\geq 1} |x^2 \mathcal{K}(x)|.$$

Combining them we get the analogue of Theorem 3.2 in $H_{p,1,\omega}(\mathbb{R})$.

The estimates presented in the last remark do not give a sufficient condition on the structural properties of the function in terms of the space $H_{p,1,\omega}(\mathbb{R})$, which yields an order of approximation of $1/\lambda$ in the norm of $H_{p,1,\omega}(\mathbb{R})$. However, such a condition can be formulated by means of the norm of the space $H_{p,2,\omega}(\mathbb{R})$. The result below extends [4, Corollary 1] and also gives a sufficient condition under which F_λ achieves its optimal order of approximation in the norm $\|\cdot\|_{p,1,\omega}$.

Corollary 3.3. *Let the operator F_λ be defined by Definition 2.1. Let also $\omega_1(h) = h^\alpha$ and $\omega_2(h) = h^\beta$ with $0 < \beta \leq 1$ and $\beta < \alpha \leq 2$. For $f \in H_{p,2,\omega_1}(\mathbb{R})$, $1 \leq p \leq \infty$, we have*

$$\|F_\lambda f - f\|_{p,1,\omega_2} = \begin{cases} O\left(\lambda^{\beta-\alpha}\right) & \text{if} \quad \alpha-\beta < 1, \\ O\left(\lambda^{-1}\log\lambda\right) & \text{if} \quad \alpha-\beta = 1, \\ O\left(\lambda^{-1}\right) & \text{if} \quad \alpha-\beta > 1. \end{cases}$$

Proof. For $0 < \beta < 1$ the result follows from Corollary 3.1 with $\omega(h) = \omega_2(h)$ and Corollary 3.2.

Let $\beta = 1$. Since $f \in H_{p,2,\omega_1}(\mathbb{R})$, then $\omega_2(f,h)_p = O(h^\alpha)$ as $\alpha > 1$. This implies that $f \in W_p^1(\mathbb{R})$ and $\omega_1(f',h)_p = O(h^{\alpha-1})$ (see e.g. [2, Ch. 6, Theorem 3.1]). Hence $\omega_2(f',h)_p = O(h^{\alpha-1})$ and by Corollary 2.1 we get

$$\|F_\lambda f' - f'\|_p = \begin{cases} O\left(\lambda^{1-\alpha}\right) & \text{if} \quad \alpha < 2, \\ O\left(\lambda^{-1}\log\lambda\right) & \text{if} \quad \alpha = 2. \end{cases}$$

Thus, in view of $\|\Delta_h(F_\lambda f - f)\|_p \leq h\,\|(F_\lambda f - f)'\|_p = h\,\|F_\lambda f' - f'\|_p$, we get

$$|F_\lambda f - f|_{p,1,\omega_2} = \begin{cases} O\left(\lambda^{1-\alpha}\right) & \text{if} \quad \alpha < 2, \\ O\left(\lambda^{-1}\log\lambda\right) & \text{if} \quad \alpha = 2. \end{cases} \tag{3.3}$$

On the other hand, since $\omega_2(f,h)_p = O(h^\alpha)$ with $\alpha > 1$, we get by Corollary 2.1 the estimate $\|F_\lambda f - f\|_p = O(\lambda^{-1})$, which together with (3.3) implies the assertion of the corollary for $\beta = 1$. The proof is completed. □

Acknowledgments. The author is thankful to the referee for the suggestions and corrections.

References

[1] N. I. Akhiezer, Lectures in Approximation Theory, Gostehizdat, Moskow, 1947 (in Russian); translation: Theory of Approximation, Ungar, New York, 1956.

[2] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.

[3] Z. Ditzian, K. Ivanov, Strong converse inequalities, *J. Anal. Math.* **61** (1993), 61–111.

[4] R. A. Lasuriya, On the approximation of functions defined on the real axis by Féjer-type operators in the generalized Hölder metric, *Mat. Zametki* **81** (2007), no. 4, 547–552 (in Russian); translation in *Math. Notes* **81** (2007), no. 3-4, 483–488.

Borislav R. Draganov
Dept. of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
Bulgaria
bdraganov@fmi.uni-sofia.bg

Inst. of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
Bulgaria