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A Characterization of Weighted L_p Approximations by the Gamma and the Post-Widder Operators

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We present a characterization of the approximation errors of the Post-Widder and the Gamma operators in $L_p[0,\infty)$, $1 \le p \le \infty$, with a weight x^γ for any real γ. Two types of characteristics are used – weighted K-functionals of the approximated function itself and the classical fixed step moduli of smoothness taken on a simple modification of it.

1. Introduction

The Post-Widder operator is given by

$$P_s(f,x) = \frac{1}{\Gamma(s)} \int_0^\infty f\left(\frac{xv}{s}\right) e^{-v} v^s \frac{dv}{v},$$

where f is a measurable function defined on $[0,\infty)$, Γ denotes as usual the Gamma function and s is a positive real parameter. This operator for integer s is actually the Post-Widder real inversion formula for the Laplace transform.

The Gamma operator, introduced by A. Lupas and M. Müller [7], is given by

$$G_s(f,x) = \frac{1}{\Gamma(s+1)} \int_0^\infty f\left(\frac{xs}{v}\right) e^{-v} v^{s+1} \frac{dv}{v}.$$

The two operators are closely related. If for real α we denote the power function by $\chi^\alpha(x) = x^\alpha$ for $x > 0$ and set $\tau_s(u) = \frac{s+1}{s} u$ then

$$G_s(f,x) = P_{s+1}(f \circ \chi^{-1} \circ \tau_s, \chi^{-1}(x)). \tag{1}$$

Both operators have a simple action on the power functions. Direct application of the definition of the Gamma function gives

$$P_s(\chi^\alpha) = \frac{\Gamma(s+\alpha)}{s^\alpha \Gamma(s)} \chi^\alpha, \quad G_s(\chi^\alpha) = \frac{s^\alpha \Gamma(s+1-\alpha)}{\Gamma(s+1)} \chi^\alpha.$$

These formulae show that the two operators preserve the functions $\chi^0(x) = 1$ and $\chi^1(x) = x$.

Both operators were extensively studied. Here we only discuss results on characterizing their rate of convergence in terms of proper K-functionals. In view of (1) all results formulated below for one of the operators can easily be proved for the other too.

For $r \in \mathbb{N}$, $1 \le p \le \infty$, $\gamma \in \mathbb{R}$, $D = \frac{d}{dx}$ and $\varphi = \chi$ we consider the weighted K-functionals:

$$K_\gamma^r(f, t^r)_p = K(f, t^r; L_p(\chi^\gamma)[0,\infty), AC_{loc}^{r-1}, \varphi^r D^r) = \inf\left\{ \|\chi^\gamma(f-g)\|_p + t^r \|\chi^\gamma \varphi^r D^r g\|_p : g \in AC_{loc}^{r-1} \right\}, \quad (2)$$

defined for every $f \in \pi_{r-1} + L_p(\chi^\gamma)[0,\infty)$ and $t > 0$, where π_k denotes the space of all algebraic polynomials of degree k. Note that the weight in the second term in the right-hand side of (2) is $\chi^{\gamma+r}$. We use two notations (φ and χ) for one and the same function in order to underline the different role of the two multipliers in the further discussion.

The direct theorem for the approximation error of the Gamma operator in L_p, $1 \le p \le \infty$, without weights is proved by Totik [10]:

$$\|f - G_s(f)\|_p \le c K_0^2(f, s^{-1})_p.$$

In the same article [10] a weak converse theorem of the form

$$K_0^2(f, s^{-1})_p \le c s^{-1} \left(\sum_{k=2}^{s} \|f - G_k(f)\|_p + \|f\|_p \right)$$

is obtained. Here and in the sequel we denote by c positive numbers independent of the functions f and the parameter s of the operators.

The book of Ditzian and Totik [2] extends the above direct result considering weights equivalent to $w(x) = x^{\gamma_0}(1+x)^{\gamma_\infty}$ with arbitrary real exponents γ_0, γ_∞. The converse result for the same weights is given as a statement for the equivalent rates of convergence in terms of weighted Ditzian-Totik moduli.

The question for the validity of strong converse theorems (in the terminology of [1]) complementing the direct estimates remained open for a while. In 2002 Sangüesa [9] proved the strong converse theorem of type A for $\gamma = 0, p = \infty$, namely

$$K_0^2(f, s^{-1})_\infty \le c\|f - P_s(f)\|_\infty.$$

As far as we know this is the only strong converse theorem of type A for the Post-Widder or the Gamma operators proved by now. As for strong converse theorems of type B, two results were recently published. In [6] Guo, Liu, Qi and Zhang proved that for $\gamma = 0$ and $1 \le p \le \infty$ there is a constant $m > 1$ such that

$$K_0^2(f, n^{-1})_p \le c\left(\|f - G_n(f)\|_p + \|f - G_{mn}(f)\|_p \right).$$

The other result is a similar strong converse theorem of type B, proved by Qi and Guo in [8] for $-2 \le \gamma \le 0$ and $p = \infty$.

In Section 2 we show that the approximation error of the Post-Widder and the Gamma operators in $L_p(\chi^\gamma)[0,\infty)$, $\gamma \in \mathbb{R}$ and $1 \le p \le \infty$, is equivalent to the K-functionals (2) with $r = 2$. In Section 3 we characterize the K-functionals (2) in terms of the classical fixed-step moduli of smoothness acting on a proper simple modification of the underlying function. In this way we obtain a characterization of the approximation error of the Post-Widder and the Gamma operators using the classical fixed-step moduli of smoothness.

2. Equivalence of approximation errors and weighted K-functionals

In the following theorem we give a direct result complimented with a strong converse statement of type A for the Post-Widder and the Gamma operator for $\gamma \in \mathbb{R}$ and $1 \le p \le \infty$.

Theorem 1. *There are positive numbers N, M such that for every $\gamma \in \mathbb{R}$, $s \ge N(\gamma^2+1)$, $1 \le p \le \infty$ and $f \in \pi_1 + L_p(\chi^\gamma)[0,\infty)$ we have*

$$\|\chi^\gamma(f - P_s(f))\|_p \le \left(2 + M\frac{\gamma^2+1}{s}\right) K^2_\gamma(f,(4s)^{-1})_p \tag{3}$$

and

$$K^2_\gamma(f,(4s)^{-1})_p \le \left(\kappa + M\frac{1}{\sqrt{s}} + M\frac{\gamma^2+1}{s}\right) \|\chi^\gamma(f - P_s(f))\|_p \tag{4}$$

with

$$\kappa = \frac{21 - 4\sqrt{2}}{8 - 2\sqrt{2}} = 2.966824...$$

The same inequalities are true if P_s is replaced by G_s.

Inequalities like (3) are well-known. For example, they are proved in [10] and [2], but with bigger constants. The inverse inequality (4) seems to be new (except $\gamma = 0,\ p = \infty$). It comes with a very small constant κ. Thus, the ratio $\|\chi^\gamma(f - P_s(f))\|_p / K^2_\gamma(f,(4s)^{-1})_p$ is bounded between two numbers with ratio less than 6 when s is big enough!

Theorem 1 remains true (up to the value of the constants) if the weight χ^γ is replaced by any equivalent on $[0,\infty)$ weight.

The proof of Theorem 1 is based on the following inequalities.

Lemma 1. *There are positive numbers N, M such that for every $\gamma \in \mathbb{R}$, $s \ge N(\gamma^2+1)$ and $1 \le p \le \infty$ we have*

$$\|\chi^\gamma P_s(f)\|_p \le \left(1 + M\frac{1+\gamma^2}{s}\right) \|\chi^\gamma f\|_p \quad \text{if } f \in L_p(\chi^\gamma);$$

$$\|\chi^\gamma\left(P_s(g)-g\right)\|_p \le \frac{1}{s}\left(\frac{1}{2}+M\frac{1+\gamma^2}{s}\right)\|\chi^\gamma\varphi^2D^2g\|_p \quad \textit{if } \varphi^2D^2g\in L_p(\chi^\gamma);$$

$$\left\|\chi^\gamma\left(P_s(g)-g-\frac{1}{2s}\varphi^2D^2g-\frac{1}{3s^2}\varphi^3D^3g\right)\right\|_p \le \frac{1}{s^2}\left(\frac{1}{8}+M\frac{1+\gamma^2}{s}\right)\left\|\chi^\gamma\varphi^4D^4g\right\|_p \quad \textit{if } \varphi^4D^4g\in L_p(\chi^\gamma);$$

$$\|\chi^\gamma\varphi^2D^2P_s(f)\|_p \le s\left(\sqrt{2}+M\frac{1+\gamma^2}{s}\right)\|\chi^\gamma f\|_p \quad \textit{if } f\in L_p(\chi^\gamma);$$

$$\|\chi^\gamma\varphi^4D^4P_s(g)\|_p \le s\left(\sqrt{2}+M\frac{1+\gamma^2}{s}\right)\|\chi^\gamma\varphi^2D^2g\|_p \quad \textit{if } \varphi^2D^2g\in L_p(\chi^\gamma);$$

$$\|\chi^\gamma\varphi^3D^3P_s(g)\|_p \le \sqrt{s}\left(1+M\frac{1+\gamma^2}{s}\right)\|\chi^\gamma\varphi^2D^2g\|_p \quad \textit{if } \varphi^2D^2g\in L_p(\chi^\gamma).$$

Following the main idea from [1] for establishing strong converse inequalities of type A and using the above inequalities one gets

Lemma 2. *There are positive numbers N, M such that for every $\gamma\in\mathbb{R}$, $s\ge N(\gamma^2+1)$, $1\le p\le\infty$ and $f\in\pi_1+L_p(\chi^\gamma)[0,\infty)$ we have*

$$\frac{1}{4s}\|\chi^\gamma\varphi^2D^2P_s^2f\|_p \le \left(\frac{5}{8-2\sqrt{2}}+M\frac{1}{\sqrt{s}}+M\frac{\gamma^2+1}{s}\right)\|\chi^\gamma(f-P_sf)\|_p.$$

This lemma is the main step in the proof of (4). The small constant (asymptotically less than 1) in the above lemma is obtained by virtue of the constant $\frac{1}{8}$ in the third inequality of Lemma 1. This inequality represents a so-called "generalized Voronovskaya" inequality in a more complicated form than usual. The complication in the form leads us to the small constant in (4).

3. A characterization of the weighted K-functionals in terms of the classical fixed-step moduli

We present two characterizations of the weighted K-functional $K_\gamma^r(f,t^r)_p$ by the classical fixed step moduli of smoothness. The proofs are based mainly on several imbedding inequalities, among which those of Hardy, and some combinatorial identities. We denote by exp the exponential function, i.e. $\exp(x)=e^x$.

In this section we assume that $f\in L_p(\chi^\gamma)[0,\infty)$. If $f\in\pi_1+L_p(\chi^\gamma)[0,\infty)$ as in Section 2 then in all theorems below f is to be replaced by f_0 such that $f_0\in L_p(\chi^\gamma)$ and $f-f_0\in\pi_1$.

Theorem 2. *Let $\gamma \in \mathbb{R}$, $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le t_0$ and $f \in L_p(\chi^\gamma)[0,\infty)$.*

a) If $\gamma \ne 1-r-1/p, 2-r-1/p, \dots, -1-1/p, -1/p$, then

$$K_\gamma^r(f,t^r)_p \sim \omega_r((\chi^{\gamma+1/p} f)\circ \exp, t)_{p(\mathbb{R})} + t^r \, \|(\chi^{\gamma+1/p} f)\circ \exp\|_{p(\mathbb{R})}.$$

b) If $\gamma = 1-r-1/p, 2-r-1/p, \dots, -1-1/p, -1/p$, then

$$K_\gamma^r(f,t^r)_p \sim \omega_r((\chi^{\gamma+1/p} f)\circ \exp, t)_{p(\mathbb{R})} + t^{r-1}\omega_1((\chi^{\gamma+1/p} f)\circ \exp, t)_{p(\mathbb{R})}.$$

The first terms on the right-hand side of Theorem 2 are bigger than the second ones for functions with small smoothness. With the following examples we show that the second terms cannot be either dropped or replaced by higher or lower order moduli. Let $\psi \in C^r(\mathbb{R})$ be with a finite support. Set $F_n(x) = \psi(n^{-1}x)$, $n \in \mathbb{N}$. Then $\omega_k(F_n,t)_{p(\mathbb{R})} \sim n^{-k+1/p} t^k$ and

$$\omega_r(F_n,t)_{p(\mathbb{R})} + t^{r-k}\omega_k(F_n,t)_{p(\mathbb{R})} \sim n^{-k+1/p} t^r, \quad k = 0,1,\dots,r,$$

where $\omega_0(F,t)_{p(\mathbb{R})}$ means $\|F\|_{p(\mathbb{R})}$. Hence, any two of the above quantities are not equivalent with constants independent of n and $t \in (0,1]$.

The proof of Theorem 2 will be given in [3].

From Theorem 1 and Theorem 2 we immediately get

Corollary 1. *Let $\gamma \in \mathbb{R}$, $1 \le p \le \infty$, $f \in L_p(\chi^\gamma)[0,\infty)$ and $s \ge N(\gamma^2+1)$, where N is from Theorem 1.*

a) If $\gamma \ne -1-1/p, -1/p$, then

$$\begin{aligned}\|\chi^\gamma(f - P_s(f))\|_{p[0,\infty)} &\sim \|\chi^\gamma(f - G_s(f))\|_{p[0,\infty)} \\ &\sim \omega_2((\chi^{\gamma+1/p} f)\circ \exp, s^{-1/2})_{p(\mathbb{R})} + s^{-1} \, \|(\chi^{\gamma+1/p} f)\circ \exp\|_{p(\mathbb{R})}.\end{aligned}$$

b) If $\gamma = -1-1/p, -1/p$, then

$$\begin{aligned}&\|\chi^\gamma(f - P_s(f))\|_{p[0,\infty)} \sim \|\chi^\gamma(f - G_s(f))\|_{p[0,\infty)} \\ &\sim \omega_2((\chi^{\gamma+1/p} f)\circ \exp, s^{-1/2})_{p(\mathbb{R})} + s^{-1/2}\omega_1((\chi^{\gamma+1/p} f)\circ \exp, s^{-1/2})_{p(\mathbb{R})}.\end{aligned}$$

In particular, for the case $\gamma = 0, p = \infty$ (considered by Sangüesa [9]) we obtain

$$\|f - P_s(f)\|_{\infty[0,\infty)} \sim \omega_2(f\circ \exp, s^{-1/2})_{\infty(\mathbb{R})} + s^{-1/2}\omega_1(f\circ \exp, s^{-1/2})_{\infty(\mathbb{R})}.$$

In [5] we shall prove the following one-term characterization of the K-functional under consideration.

Theorem 3. *Let* $\gamma \in \mathbb{R}$, $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le t_0$ *and* $f \in L_p(\chi^\gamma)[0,\infty)$. *Set for* $x \in (-\infty,\infty)$

$$(Bf)(x) = f(e^x) + \sum_{i=1}^{r-1} (-1)^i \frac{m_{r,i}}{(i-1)!} \int_0^x (x-y)^{i-1} f(e^y)\, dy$$

with coefficients $m_{r,i}$ *given by*

$$\prod_{j=1}^{r-1} (z+j) = z^{r-1} + \sum_{i=1}^{r-1} m_{r,i} z^{r-i-1},$$

and

$$\begin{aligned}(A_\gamma F)(x) &= e^{(\gamma+1/p)x} F(x) \\ &\quad + \sum_{k=1}^{r} (-1)^k \binom{r}{k} \frac{(\gamma+1/p)^k}{(k-1)!} \int_0^x (x-y)^{k-1} e^{(\gamma+1/p)y} F(y)\, dy.\end{aligned}$$

a) If $\gamma \ne 1-r-1/p, 2-r-1/p, \dots, -1-1/p$, *then*

$$K_\gamma^r(f,t^r)_p \sim \omega_r(A_\gamma Bf, t)_{p(\mathbb{R})}.$$

b) If $\gamma = 1-r-1/p, 2-r-1/p, \dots, -1-1/p, -1/p$, *then*

$$K_\gamma^r(f,t^r)_p \sim \omega_r(B(\chi^{\gamma+1/p} f), t)_{p(\mathbb{R})}.$$

Note that the statement in case a) is valid for $\gamma = -1/p$. The reason is that $A_\gamma Bf = Bf = B(\chi^{\gamma+1/p} f)$ when $\gamma = -1/p$. Thus, we have a one-type characterization of the K-functional for $\gamma > -1-1/p$. In particular, for $\gamma = 0$ it gives common characterizations for all $1 \le p \le \infty$ unlike in Theorem 2.

Using the same ideas as in the proof of Theorems 2 and 3 we can characterize the analogue of $K_\gamma^r(f,t^r)_p$ on the finite interval $[0,1]$, given by

$$\begin{aligned}\widetilde{K}_\gamma^r(f,t^r)_p &= K(f,t^r; L_p(\chi^\gamma)[0,1], AC_{loc}^{r-1}, \varphi^r D^r) = \\ &\qquad \inf\left\{ \|\chi^\gamma (f-g)\|_{p[0,1]} + t^r \|\chi^\gamma \varphi^r D^r g\|_{p[0,1]} : g \in AC_{loc}^{r-1} \right\},\end{aligned}$$

for $f \in \pi_{r-1} + L_p(\chi^\gamma)[0,1]$ and $t > 0$. Here we shall consider the case $\gamma \ge -1/p$. Let us set for $f \in L_p(\chi^\gamma)[0,1]$

$$\begin{aligned}(\mathcal{A}f)(x) &= f(e^{-x}) - L_r'(\textstyle\int f, e^{-x}), \quad 1 \le p \le \infty; \\ (\tilde{\mathcal{A}}f)(x) &= f(e^{-x}) - L_{r-1}(f, e^{-x}), \quad p = \infty,\end{aligned}$$

where $L_r(F,x)$ is the Lagrange interpolation polynomial of degree r for the function F on some fixed nodes $0 < x_0 < x_1 < \cdots < x_r \le 1$, $L_r'(F,x)$ is its derivative and $\int f$ denotes any fixed antiderivative of f. We have the characterization

Theorem 4. *Let $\gamma \geq -1/p$, $r \in \mathbb{N}$, $1 \leq p \leq \infty$, $0 < t \leq t_0$ and $f \in L_p(\chi^\gamma)[0,1]$.*

a) If $\gamma > -1/p$, then

$$\widetilde{K}^r_\gamma(f,t^r)_p \sim \omega_r(\exp^{-\gamma-1/p} \cdot \mathcal{A}f, t)_{p[0,\infty)} + t^r \| \exp^{-\gamma-1/p} \cdot \mathcal{A}f\|_{p[0,\infty)}.$$

b) If $\gamma = -1/p$, then

$$\widetilde{K}^r_\gamma(f,t^r)_p \sim \omega_r(\mathcal{A}f, t)_{p[0,\infty)} + t^{r-1}\omega_1(\mathcal{A}f, t)_{p[0,\infty)}.$$

In the case $p = \infty$ the operator $\mathcal{A}$ can be replaced by the more simple $\tilde{\mathcal{A}}$.

The two characterizations in Theorem 4 extend respectively to the cases $\gamma \neq 1 - r - 1/p, 2 - r - 1/p, \dots, -1 - 1/p, -1/p$ and $\gamma = 1 - r - 1/p, 2 - r - 1/p, \dots, -1 - 1/p, -1/p$ with appropriate modifications of $\mathcal{A}$. Moreover, the two terms in the right-hand side of the equivalences in Theorem 4 can be reduced to a single modulus using an additional operator as in Theorem 3. The same ideas can be applied for characterizing weighted K-functionals like $K^r_\gamma(f,t^r)_p$ on $[1,\infty)$.

Characterizations analogous to those in Theorem 2 and Theorem 4 for L_p spaces with more general weights will be given in [4].

References

[1] Z. Ditzian and K. G. Ivanov, Strong converse inequalities, *J. d'Analyse Mathematique* **61** (1993), 61-111.

[2] Z. Ditzian and V. Totik, Moduli of Smoothness, SSCM 9, Springer, New York, 1987.

[3] B. R. Draganov and K. G. Ivanov, A characterization of weighted approximations by the Post-Widder and the Gamma operators, *manuscript*, 2005.

[4] B. R. Draganov and K. G. Ivanov, A characterization of the K-functional associated with the Post-Widder and the Gamma operators, *manuscript*, 2005.

[5] B. R. Draganov and K. G. Ivanov, Another characterization of the K- functional associated with the Post-Widder and the Gamma operators, *manuscript*, 2005.

[6] S. Guo, L. Liu, Q. Qi and G. Zhang, A strong converse inequality for left gamma quasi-interpolants in L_p-spaces, *Acta Math. Hungar.*, **105**, 1-2 (2004), 17-26.

[7] A. Lupas and M. Müller, Approximationseigenschaften der Gammaoperatoren, *Math. Z.*, **98** (1967), 208-226.

[8] Q.-L. Qi and S.-S. Guo, Strong converse inequality for left gamma quasi-interpolants, *Acta Mathematicae Applicatae Sinica*, **21**, 1 (2005) 115-124.

[9] C. Sangüesa, Lower estimates for centered Bernstein-type operators, *Constr. Approx.*, **18** (2002), 145-159.

[10] V. Totik, The gammaoperators in L^p spaces, *Publ. Math.* (Debrecen), **32** (1985), 43-55.

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