CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2005
additional information (to be provided by the publisher)

Characterizations of Weighted K-Functionals and Their Application

Borislav R. Draganov and Kamen G. Ivanov

We present a characterization of a large class of weighted K-functionals in terms of the classical fixed step moduli of smoothness and proper modifications of the underlying function. It gives new estimates of the error of various approximation processes.

1. The method

The Peetre K-functional turned out to be a very useful tool in approximation theory in estimating the error. Generally the K-functional is of the form

$$K(f,t) = K(f,t;X,Y,\mathcal{D}) = \inf\bigl\{\|f-g\|_X + t\|\mathcal{D}g\|_X \ : \ g \in Y \cap \mathcal{D}^{-1}(X)\bigr\}, \quad (1)$$

where X is a Banach space, $\mathcal{D}$ is a differential operator of the form

$$\mathcal{D}g(x) = \sum_{k=0}^{r} \varphi_k(x) g^{(k)}(x), \quad \varphi_k \in X,\ k = 0,\dots,r, \quad \varphi_r > 0 \quad a.\ e.$$

with a given $r \in \mathbb{N}$, $\mathcal{D}^{-1}(X) = \{g \in X : \mathcal{D}g \in X\}$ and $Y \cap \mathcal{D}^{-1}(X)$ is usually a dense subspace of X. But the class of functions f for which we can calculate exactly the infimum in (1) for any $t \in (0,1]$ is quite narrow. That is why it is useful to have other function characteristics – moduli of smoothness – which can be calculated for wider class of functions and are equivalent to the K-functional. Up to now several definitions of moduli of smoothness have been introduced: Ivanov [8] and [9], Ditzian and Totik [2], Ky [12], etc.

Let I be an interval and let w and φ be weights on I as follows

$I=[0,1]$	$w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$	$\varphi(x) = x^{\lambda_0}(1-x)^{\lambda_1}$
$I=\mathbb{R}_+$	$w(x) = x^{\gamma_0}(x+1)^{\gamma_\infty-\gamma_0}$	$\varphi(x) = x^{\lambda_0}(x+1)^{\lambda_\infty-\lambda_0}$
$I=\mathbb{R}$	$w(x) = \begin{cases} \lvert x\rvert^{\gamma_{-\infty}}, & x<-1,\\ 1, & -1\le x\le 1,\\ x^{\gamma_{+\infty}}, & x>1.\end{cases}$	$\varphi(x) = \begin{cases} \lvert x\rvert^{\lambda_{-\infty}}, & x<-1,\\ 1, & -1\le x\le 1,\\ x^{\lambda_{+\infty}}, & x>1.\end{cases}$

The γ's and the λ's above are arbitrary real numbers.

We denote $D = \frac{d}{dx}$ and $L_p(w)(I) = \big\{f : wf \in L_p(I)\big\}$. We shall present characterizations of the weighted K-functional

$$\begin{aligned}K(f, t^r; L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r) \\ = \inf\big\{\|w(f-g)\|_{p(I)} + t^r\|w\,\varphi^r g^{(r)}\|_{p(I)} \ : \ g \in AC_{loc}^{r-1}\big\}, \quad (2)\end{aligned}$$

by the classical fixed step moduli of smoothness as the latter are taken not on the function f itself but on a certain modification of it. These characterizations will be valid not only for the weights w and φ listed in the table but for any other weights $\tilde{w}$ and $\tilde{\varphi}$ equivalent to them on I. For treatment of weights with more general asymptotic at the end-points of the domain I see Section 6 of [4].

Since a long time mathematicians have been using the transform $f \circ \cos$ to establish a connection between the best algebraic and the best trigonometric approximations. In 1993 Mastroianni and Vértesi (see [13]) showed in fact that

$$K(f, t; C[-1,1], AC_{loc}, \varphi D) \sim \omega(f \circ \cos, t)_{\infty[0,\pi]},$$

where $\varphi(x) = \sqrt{1-x^2}$ and as usual $\psi_1(F,t) \sim \psi_2(F,t)$ means that the ratio of the functions ψ_1 and ψ_2 is bounded between two positive constants independent of F and t.

We shall give a characterization of this type for the K-functionals (2), namely we shall construct a linear bounded operator $\mathcal{A} : L_p(w)(I) \to L_p(I')$, where I' is an interval, such that

$$K(f, t^r; L_p(w)(I), AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p(I')}.$$

It turns out that a set of several simple relations on the linear operator $\mathcal{A}$, which maps the Banach space X_1 into the Banach space X_2, provides the equivalence

$$K(f, t; X_1, Y_1, D_1) \sim K(\mathcal{A}f, t; X_2, Y_2, D_2). \quad (3)$$

We have [5]

Theorem 1. *If there exists a linear operator $\mathcal{B} : X_2 \to X_1$, related to $\mathcal{A} : X_1 \to X_2$, and both operators satisfy the conditions:*

(a) $\|\mathcal{A}f\|_{X_2} \le C\|f\|_{X_1}$ *for any* $f \in X_1$;

(b) $\|D_2\mathcal{A}f\|_{X_2} \le C\|D_1 f\|_{X_1}$ *for any* $f \in Y_1 \cap D_1^{-1}(X_1)$;

(c) $\|\mathcal{B}F\|_{X_1} \le C\|F\|_{X_2}$ *for any* $F \in X_2$;

(d) $\|D_1\mathcal{B}F\|_{X_1} \le C\|D_2 F\|_{X_2}$ *for any* $F \in Y_2 \cap D_2^{-1}(X_2)$;

(e) $\mathcal{A}(Y_1 \cap D_1^{-1}(X_1)) \subseteq Y_2 \cap D_2^{-1}(X_2)$;

(f) $\mathcal{B}(Y_2 \cap D_2^{-1}(X_2)) \subseteq Y_1 \cap D_1^{-1}(X_1)$;

(g) $f - \mathcal{B}\mathcal{A}f \in Y_1 \cap \ker D_1$ *for any* $f \in X_1$;

(h) $F - \mathcal{A}\mathcal{B}F \in Y_2 \cap \ker D_2$ *for any* $F \in X_2$.

Then

$$K(f, t; X_1, Y_1, D_1) \sim K(\mathcal{A}f, t; X_2, Y_2, D_2)$$

and

$$K(F, t; X_2, Y_2, D_2) \sim K(\mathcal{B}F, t; X_1, Y_1, D_1).$$

Remark 1. *In some cases we can take $\mathcal{B} = \mathcal{A}^{-1}$(see[4]) but there are situations in which this is not possible and we need the more general assertion above.*

Now, if one of the K-functionals above is characterized by a modulus of smoothness, we get immediately a characterization of the other too. In particular, if $K(F, t; X_2, Y_2, D_2)$ is the classical unweighted K-functional, that is, $X_2 = L_p(I),\ Y_2 = AC_{loc}^{r-1}$ and $D_2 = D^r = \frac{d^r}{dx^r}$, we get

$$K(f, t^r; X_1, Y_1, D_1) \sim \omega_r(\mathcal{A}f, t)_{p(I)}. \tag{4}$$

In order that (4) to be effective for computations the operator $\mathcal{A}$ must have an explicit and simple form, which is easy to be calculate for a given f. In some cases we get more simple linear operators if we separate the singularities of the weights w and φ by splitting the interval I beforehand. For example, if $I = [0, 1]$ and $0 < a < b < 1$, we have

$$\begin{aligned} &K(f, t^r; L_p(\chi_0^{\gamma_0}\chi_1^{\gamma_1})[0, 1], AC_{loc}^{r-1}, \chi_0^{r\lambda_0}\chi_1^{r\lambda_1}D^r) \\ &\qquad\qquad \sim K(f, t^r; L_p(\chi_0^{\gamma_0})[0, b], AC_{loc}^{r-1}, \chi_0^{r\lambda_0}D^r) \\ &\qquad\qquad\quad + K(f, t^r; L_p(\chi_1^{\gamma_1})[a, 1], AC_{loc}^{r-1}, \chi_1^{r\lambda_1}D^r), \end{aligned} \tag{5}$$

where we have put $\chi_c(x) = |x - c|$; and similarly for $I = \mathbb{R}_+$ and $I = \mathbb{R}$.

2. A construction of the operator $\mathcal{A}$

In many cases an operator which satisfies the conditions of Theorem 1 can be constructed on the basis of the condition

$$D_2(\mathcal{A}g) = D_1 g \quad \text{for} \quad g \in Y_1 \cap D_1^{-1}(X_1).$$

In this way we got the A and B-operators presented in [4]. More precisely, first let us put

$$\alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!}\binom{r-1}{k-1}\prod_{\nu=0}^{r-1}(\rho + r - k - \nu)$$

and

$$\beta_{r,k}(\sigma) = \frac{(-1)^{r-k}}{(r-2)!}\binom{r-2}{k-2}\prod_{i=1}^{r-1}(k-1-i\sigma).$$

Next, as we have shown in [4], the bounded linear operator $A_0(\gamma_0;\xi)$: $L_p(\chi_0^{\gamma_0}\chi_1^{\gamma_1})[0,1] \to L_p(\chi_1^{\gamma_1})[0,1]$, $\gamma_0,\gamma_1 > -1/p$, $\xi \in (0,1)$, defined by

$$(A_0(\gamma_0;\xi)f)(x) = x^{\gamma_0}f(x) + \sum_{k=1}^{r}\alpha_{r,k}(\gamma_0)x^{k-1}\int_{\xi}^{x} y^{-k+\gamma_0}f(y)dy$$

possesses the property

$$(A_0(\gamma_0;\xi)g)^{(r)}(x) = x^{\gamma_0}g^{(r)}(x) \quad \text{a.e.}$$

for any $g \in AC_{loc}^{r-1}$. Thus A_0 helps us clear the weight $\chi_0^{\gamma_0}$, common to both terms of the K-functional. To clear the singularity at the point 0 in the weight φ for $\lambda_0 < 1$ we can use the bounded linear operator $B_0(\sigma;\xi)$: $L_p(\chi_0^{\gamma_0-(\gamma_0+1/p)\lambda_0}\chi_1^{\gamma_1})[0,1] \to L_p(\chi_0^{\gamma_0}\chi_1^{\gamma_1})[0,1]$, $\gamma_0 > -1-1/p$ and $\gamma_1 > -1/p$, defined by

$$(B_0(\sigma;\xi)f)(x) = f(x^{\sigma}) + \sum_{k=2}^{r}\beta_{r,k}(\sigma)x^{k-1}\int_{\xi}^{x} y^{-k}f(y^{\sigma})\,dy, \quad \sigma = \frac{1}{1-\lambda_0}.$$

It possesses the property

$$(B_0(\sigma;\xi)g)^{(r)}(x) = \sigma^r(x^{\sigma})^{r\lambda}g^{(r)}(x^{\sigma}) \quad \text{a.e.}$$

for any $g \in AC_{loc}^{r-1}$. The transform $x \mapsto 1-x$ gives the modifications of A_0 and B_0 through which we can treat the singularities at 1 of the weights w and φ (with $\lambda_1 < 1$):

$$(A_1(\rho;\xi)f)(x) = (1-x)^{\rho}f(x) - \sum_{k=1}^{r}\alpha_{r,k}(\rho)(1-x)^{k-1}\int_{\xi}^{x}(1-y)^{-k+\rho}f(y)dy,$$

and

$$\begin{aligned}(B_1(\sigma;\xi)f)(x) &= f(1-(1-x)^{\sigma})\\ &\quad - \sum_{k=2}^{r}\beta_{r,k}(\sigma)(1-x)^{k-1}\int_{\xi}^{x}(1-y)^{-k}f(1-(1-y)^{\sigma})\,dy.\end{aligned}$$

It is worthy to note that these operators treat the singularities at the ends of the interval separately.

The operators A_0 and B_0 can be used to clear from w and φ (with $\lambda_0 < 1$) respectively weights of the type χ_0^{α} on the interval $[0,\infty)$ too. To treat only the singularity at infinity one can use respectively for w and φ (with $\lambda_\infty < 1$) the linear operators

$$(A_\infty(\rho;\xi)f)(x) = (x+1)^\rho f(x) + \sum_{k=1}^{r} \alpha_{r,k}(\rho)(x+1)^{k-1} \int_\xi^x (y+1)^{-k+\rho} f(y)dy$$

and

$$\begin{aligned}(B_\infty(\sigma;\xi)f)(x) &= f((x+1)^\sigma - 1) \\ &+ \sum_{k=2}^{r} \beta_{r,k}(\sigma)(x+1)^{k-1} \int_\xi^x (y+1)^{-k} f((y+1)^\sigma - 1)\, dy, \quad \sigma > 0.\end{aligned}$$

The B-operators, presented so far, clear the singularity in the weight φ if the exponent λ is less that 1. To settle the case $\lambda_0 > 1$ on a finite interval (similarly on a semi-infinite) we use the operator

$$\begin{aligned}&(\widetilde{B}_0(\sigma;\xi)f)(x) = f((x+1)^\sigma) \\ &+ \sum_{k=2}^{r} \beta_{r,k}(\sigma)(x+1)^{k-1} \int_\xi^x (y+1)^{-k} f((y+1)^\sigma)\, dy, \quad x \in [0,\infty), \quad \sigma < 0.\end{aligned}$$

Above $\sigma = 1/(1-\lambda)$ is negative and to a function f defined on the finite interval $(0,1]$ we relate a function defined on the semi-infinite interval $[0,\infty)$. The case $\lambda = 1$ is essentially different.

The last operators also possess properties similar to those of A_0 and B_0.

3. A characterization of the K-functional

Now we shall present a characterization of the K-functional (2) by the classical fixed step modulus of smoothness in several important for the applications cases. Here we shall consider cases when λ's $\neq 1$. More results of this kind can be found in [4] and will be given in [5] and another forthcoming paper. Several results on the case λ's $= 1$ are presented in [6] in this volume.

3.1. The finite interval $I = [0,1]$ and $\lambda_0, \lambda_1 < 1$

Theorem 2. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\lambda_0, \lambda_1 \in (-\infty, 1)$. For $p < \infty$ we assume that $\gamma_0 \neq 1-r-1/p, 2-r-1/p, \ldots, -1/p$ and $\gamma_1 > -1/p$ or vice versa. For $p = \infty$ we assume that $\gamma_0 = \gamma_1 = 0$. Set*

$$\mathcal{A} = B_1(\sigma_1;\xi)B_0(\sigma_0;\xi)A_1(\rho_1;\xi)A_0(\rho_0;\xi),$$

$$\sigma_0 = \frac{1}{1-\lambda_0}, \quad \sigma_1 = \frac{1}{1-\lambda_1}, \quad \rho_0 = \gamma_0 + \frac{\lambda_0}{p}, \quad \rho_1 = \gamma_1 + \frac{\lambda_1}{p}.$$

Then

$$K(f, t^r; L_p(w)[0,1], AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p[0,1]}.$$

3.2. The semi-infinite interval $I = [0, \infty)$ and $\lambda_0, \lambda_\infty < 1$

Theorem 3. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\lambda_0, \lambda_\infty \in (-\infty, 1)$. For $p < \infty$ we assume that $\gamma_0 > -1/p$ and $\gamma_\infty \ne 1-r-1/p, 2-r-1/p, \ldots, -1/p$. For $p = \infty$ we assume that $\gamma_0 = \gamma_\infty = 0$. Set*

$$\mathcal{A} = A_0(\rho_0'; \infty) B_0(\sigma_0; \infty) B_\infty(\sigma_\infty; \infty) A_\infty(\rho_\infty; \infty),$$

$$\sigma_0 = \frac{1}{1-\lambda_0}, \quad \sigma_\infty = \frac{1-\lambda_0}{1-\lambda_\infty},$$
$$\rho_0' = (\gamma_0 + \frac{1}{p})\sigma_0 - \frac{1}{p}, \quad \rho_\infty = \gamma_\infty - \frac{\gamma_0 + 1/p}{\sigma_\infty} + \frac{1}{p}.$$

Then

$$K(f, t^r; L_p(w)[0,\infty), AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p[0,\infty)}.$$

3.3. The finite interval $I = [0, 1]$ and $\lambda_0 > 1, \lambda_1 < 1$

Theorem 4. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\lambda_0 \in (1, \infty)$, $\lambda_1 \in (-\infty, 1)$. For $p < \infty$ we assume that $\gamma_0 \ne 1-r-1/p, 2-r-1/p, \ldots, -1/p$ and $\gamma_1 > -1/p$ or vice versa. For $p = \infty$ we assume that $\gamma_0 = \gamma_1 = 0$. Set*

$$\mathcal{A} = A_\infty(\rho_p'; \infty) \widetilde{B}_0(\sigma_0; \infty) B_1(\sigma_1; \xi) A_1(\rho_1; \xi) A_0(\rho_0; \xi),$$

$$\sigma_0 = \frac{1}{1-\lambda_0}, \quad \sigma_1 = \frac{1}{1-\lambda_1},$$
$$\rho_p' = -\frac{p+2}{2p}, \quad \rho_0 = \gamma_0 + \frac{1-\lambda_0}{2} + \frac{1}{p}, \quad \rho_1 = \gamma_1 + \frac{\lambda_1}{p}.$$

For $p = \infty$ the definition of $\mathcal{A}$ reduces to $\mathcal{A} = \widetilde{B}_0(\sigma_0; \infty) B_1(\sigma_1; \xi)$. Then

$$K(f, t^r; L_p(w)[0,1], AC_{loc}^{r-1}, \varphi^r D^r) \sim \omega_r(\mathcal{A}f, t)_{p[0,\infty)}.$$

When we reverse the restrictions on γ_1 and γ_0 in Theorems 2 and 4 we may get a problem with the boundedness of A_0. But in this case there is no need to change the order of the operators A_1 and A_0 because of their commutativity proved in Proposition 5.1 in [4]. The reversing of the restrictions on γ_1 and γ_0 does not affect the B-operators. In Theorem 3 one can reverse the restrictions on γ_0 and γ_∞ with a proper modification of the definition of $\mathcal{A}$.

To get a characterization for any γ's $\ne 1-r-1/p,\ 2-r-1/p, \ldots, -1/p$, $1 \le p < \infty$, we can separate the singularities at the end-points applying formula (5). Note that the case when w and φ are equal to 1 at one of the end-points of the interval is always included in Theorems 2, 3 and 4. The K-functional of functions, defined on the whole real axis, can also be characterized using formula (5) and arriving at two K-functionals of functions defined on semi-infinite intervals.

The K-functional (2) for $\gamma = 1 - r - 1/p,\ 2 - r - 1/p, \dots, -1 - 1/p$, $1 \le p \le \infty$, can be reduced to the one for $\gamma = -1/p$ through an A-operator. We shall consider the cases $\gamma = 1 - r - 1/p,\ 2 - r - 1/p, \dots, -1/p$, $1 \le p < \infty$, and $\gamma \ne 1 - r,\ 2 - r, \dots, 0$, $p = \infty$, in a forthcoming paper.

The values $1 - r - 1/p,\ 2 - r - 1/p, \dots, -1/p$ of the exponent γ_0 (or γ_1, or γ_∞) are omitted for $p < \infty$ because the quasi-inverse operators of A_0 (see $\mathcal{B}$ in Theorem 1) are unbounded for these values. Nevertheless this case can be settled applying the same circle of ideas but varying the construction of the operator. For example, if $1 \le p < \infty$ and $\alpha, \beta \le 1 - r - 1/p$ we have

$$K(f, t; L_p(\chi_0^\alpha)[0,1], AC_{loc}^{r-1}, D^r) \sim K(\chi_0^{\alpha-\beta} f, t; L_p(\chi_0^\beta)[0,1], AC_{loc}^{r-1}, D^r) \quad (6)$$

and hence for every γ there exists a linear operator $\mathcal{A}$ such that

$$K(f, t^r; L_p(\chi_0^\gamma)[0,1], AC_{loc}^{r-1}, D^r) \sim \omega_r(\mathcal{A}f, t)_{p[0,1]}.$$

For $p = \infty$ and $\gamma \ne 1 - r, 2 - r, \dots, 0$ there does not exist an operator $\mathcal{A} : C(\chi_0^\gamma)[0,1] \to C[0,1]$ such that relation (4) holds for every $f \in C(\chi_0^\gamma)[0,1]$. The counter-example is given by the function $f(x) = x^{-\gamma} \notin \pi_{r-1}$. In fact for every $f \in C(\chi_0^\gamma)[0,1]$ such that $x^\gamma f(x)$ does not tend to 0 when $t \to 0+0$ we have

$$K(f, t^r; C(\chi_0^\gamma)[0,1], AC_{loc}^{r-1}, D^r) \not\to 0, \quad t \to 0+0.$$

For the set $C_0(\chi_0^\gamma)[0,1] = \{f \in C(\chi_0^\gamma)[0,1] : \lim_{x\to 0+0} x^\gamma f(x) = 0\}$ we can construct a bounded linear operator $\mathcal{A}$ such that

$$K(f, t^r; C(\chi_0^\gamma)[0,1], AC_{loc}^{r-1}, D^r) \sim \omega_r(\mathcal{A}f, t)_{\infty[0,1]} \quad \text{for every} \quad f \in C_0(\chi_0^\gamma)[0,1].$$

Finally, let us mention that in each of the intervals $(-\infty, 1 - r - 1/p), (1 - r - 1/p, 2 - r - 1/p), \dots, (-1/p, \infty)$ the operator A_0 has a different quasi-inverse operator.

4. The role of the integral summands

The operators $\mathcal{A}$ consist of a leading term and several additional terms. The leading term is obtained from the function f by a proper change of the variable and multiplication by a weight. The additional terms are anti-derivatives of the leading term multiplied by low degree polynomials. So we call them "integral summands". The smoothness of the integral summands is higher than the smoothness of the leading term inside I. Near the end-points of the interval both parts may have approximately one and the same smoothness. As a consequence, the modulus of smoothness of $\mathcal{A}f$ may deviate (in both directions) with logarithmic factors from the same modulus taken on the leading term. In other cases the rates of convergence may differ a lot more. We illustrate this property with the following examples.

Consider $I = [0,1]$ with $\varphi = \chi_0^\lambda$, $\lambda < 1$, $w = \chi_0^\gamma$, $\gamma \in \mathbb{R}$ and $r = 2$. According to Theorem 2 the operator

$$(\mathcal{A}f)(x) = x^\rho f(x^\sigma) + \int_1^x [(\rho-1)(\sigma+\rho-1)xy^{\rho-2} - \rho(\sigma+\rho)y^{\rho-1}]f(y^\sigma)\,dy$$

with $\sigma = 1/(1-\lambda)$, $\rho = \sigma(\gamma+1/p) - 1/p$ can be used in the characterization

$$K(f,t^2; L_p(w)[0,1], AC_{loc}^1, \varphi^2 D^2) \sim \omega_2(\mathcal{A}f,t)_{p[0,1]},$$

valid for $\gamma \neq -1-1/p, -1/p$ if $p < \infty$ or for $\gamma = 0$ (and $\gamma = -1$) if $p = \infty$.

The leading term of $\mathcal{A}f$ is $\mathcal{A}^*f = \chi_0^\rho(f \circ \chi_0^\sigma)$. Let us compare for different f's the rates of convergence of $\omega_2(\mathcal{A}f,t)_p$ and $\omega_2(\mathcal{A}^*f,t)_p$ for $1 \le p \le \infty$ and some ranges for λ and γ.

Set $f_1(x) = |\log x/2|^\alpha$, $\alpha \in \mathbb{R}$. For $\alpha \neq 0$ we have

$$\omega_2(\mathcal{A}f_1,t)_p \sim t^{\rho+1/p}|\log t|^{\alpha-1} \quad \text{if} \quad 0 < \gamma+1/p < 2/\sigma,$$

$$\omega_2(\mathcal{A}^*f_1,t)_p \sim t^{\rho+1/p}|\log t|^{\alpha} \quad \text{if} \quad 0 < \gamma+1/p < 2/\sigma,\ \rho \neq 0,\ 1.$$

For $\alpha = 0$ we have $\omega_2(\mathcal{A}f_1,t)_p = 0$ for all values of γ, λ and p, while

$$\omega_2(\mathcal{A}^*f_1,t)_p \sim t^{\rho+1/p} \quad \text{if} \quad 0 < \gamma+1/p < 2/\sigma,\ \rho \neq 0,\ 1.$$

Set $f_2(x) = x|\log x/2|^\alpha$, $\alpha \in \mathbb{R}$. For $\alpha \neq 0$ we have

$$\omega_2(\mathcal{A}f_2,t)_p \sim t^{\rho+\sigma+1/p}|\log t|^{\alpha-1} \quad \text{if} \quad 0 < \gamma+1+1/p < 2/\sigma,$$

$$\omega_2(\mathcal{A}^*f_2,t)_p \sim t^{\rho+\sigma+1/p}|\log t|^{\alpha} \quad \text{if} \quad 0 < \gamma+1+1/p < 2/\sigma,\ \rho+\sigma \neq 0,\ 1.$$

For $\alpha = 0$ we have $\omega_2(\mathcal{A}f_2,t)_p = 0$ for all values of γ, λ and p and

$$\omega_2(\mathcal{A}^*f_2,t)_p \sim t^{\rho+\sigma+1/p} \quad \text{if} \quad 0 < \gamma+1+1/p < 2/\sigma,\ \rho+\sigma \neq 0,\ 1.$$

Set $f_3(x) = x^{-\rho/\sigma}|\log x/2|^\alpha$, $\alpha \in \mathbb{R}$. For $\rho \neq 0$ and $\rho+\sigma \neq 0$ we have

$$\omega_2(\mathcal{A}f_3,t)_p \sim t^{1/p}|\log t|^{\alpha} \quad \text{if} \quad p < \infty,$$

$$\omega_2(\mathcal{A}^*f_3,t)_p \sim \begin{cases} t^{1/p}|\log t|^{\alpha-1}, & p < \infty,\ \alpha \neq 0; \\ 0, & \alpha = 0. \end{cases}$$

Set $f_4(x) = x^{(1-\rho)/\sigma}|\log x/2|^\alpha$, $\alpha \in \mathbb{R}$. For $\rho \neq 1$ and $\rho+\sigma \neq 1$ we have

$$\omega_2(\mathcal{A}f_4,t)_p \sim t^{1+1/p}|\log t|^{\alpha} \quad \text{if} \quad p > 1,$$

$$\omega_2(\mathcal{A}^*f_4,t)_p \sim \begin{cases} t^{1+1/p}|\log t|^{\alpha-1}, & p > 1,\ \alpha \neq 0; \\ 0, & \alpha = 0. \end{cases}$$

Let us emphasize that $\omega_2(\mathcal{A}f,t)_p$ and $\omega_2(\mathcal{A}^*f,t)_p$ have different rates of convergence even for small orders and not only for orders close to the saturation as in the Marchaud type inequalities.

According to the examples above one cannot expect in general close behaviour between the moduli of $\mathcal{A}f$ and $\mathcal{A}^*f$. But for some ranges of the parameters there is such kind of relation. One example is equivalence (6) above. Other examples are given in Ditzian and Totik [3], where relations between the weighted K-functional (2) and the weighted modulus of smoothness with unvarying step for $1 \le p < \infty$ are established. For example, one of their results implies that for $0 \le \lambda < 1$, $\sigma = 1/(1-\lambda)$ the following equivalence is valid for every $f \in L_p(\chi_0^\gamma)(I)$

$$\begin{aligned} &K(f\circ\chi_0^\sigma, t^r; L_p(\chi_0^{\gamma\sigma+(\sigma-1)/p})(I), AC_{loc}^{r-1}, D^r) + t^r\|\chi_0^{\gamma\sigma+(\sigma-1)/p}(f\circ\chi_0^\sigma)\|_p \\ &\qquad\qquad \sim K(f, t^r; L_p(\chi_0^\gamma)(I), AC_{loc}^{r-1}, \chi_0^{r\lambda}D^r) + t^r\|\chi_0^\gamma f\|_p. \end{aligned}$$

The above equivalence is true when γ is bigger than a number depending on the other parameters p, r, λ and not valid otherwise. So we can clear the weight $\varphi = \chi_0^\lambda$, preserving the O-order of the K-functional, replacing the function f by its simple modification $f\circ\chi_0^\sigma$ and the weight $w = \chi_0^\gamma$ by another.

5. Application

The approximation error of many operators has already been characterized by an appropriate weighted K-functional (or equivalent to it weighted modulus of smoothness). Based on these results, the error can be estimated by the classical fixed step modulus of smoothness, taken not on the function but on a certain linear transform of it. This linear transform can be constructed by the operators $\mathcal{A}$ presented so far. Below we give a list of some well-known operators, which have been studied extensively.

- The best weighted algebraic approximation in $L_p(w)[0,1]$, $1 \le p \le \infty$ (see [2]): $\lambda_0 = \lambda_1 = 1/2$, $r \in \mathbb{N}$;
- The Bernstein operator (see [11] and [14]): $I = [0,1]$, $\lambda_0 = \lambda_1 = 1/2$, $r = 2$;
- The Kantorovich and the Durrmeyer operators (see [1], [7] and [10]): $I = [0,1]$, $\lambda_0 = \lambda_1 = 1/2$, $r = 2$, different kind of differential operator;
- The Szász-Mirakjan operator (see [14]): $I = [0,\infty)$, $\lambda_0 = \lambda_\infty = 1/2$, $r = 2$;
- The Baskakov operator (see [14]): $I = [0,\infty)$, $\lambda_0 = 1/2$, $\lambda_\infty = 1$, $r = 2$;
- The Gamma and the Post-Widder operators (see [6]): $I = [0,\infty)$, $\lambda_0 = \lambda_\infty = 1$, $r = 2$.

References

[1] W. Chen, Z. Ditzian and K.G. Ivanov, Strong converse inequality for the Bernstein-Durrmeyer operator, *J. Approx. Theory* **75** (1993), 25-43.

[2] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer Verlag, New York, 1987.

[3] Z. Ditzian and V. Totik, K-functionals and weighted moduli of smoothness, *J. Approx. Theory* **63** (1990), 3–29.

[4] B.R. Draganov and K.G. Ivanov, A new characterization of weighted Peetre K-functionals, *Constr. Approx.* **21** (2005), 113–148.

[5] B.R. Draganov and K.G. Ivanov, A new characterization of weighted Peetre K-functionals (II), *manuscript*, 2003.

[6] B.R. Draganov and K.G. Ivanov, A characterization of weighted L_p approximations by the Gamma and the Post-Widder operators, *this volume*, 2005.

[7] H. Gonska and X.-l Zhou, The strong converse inequality for Bernstein-Kantorovich operators, *Computers Math. Applic.* **30** (1995), No. 3-6, 103–128.

[8] K. G. Ivanov, Some characterizations of the best algebraic approximation in $L_p[-1,1]$ $(1 \le p \le \infty)$, *C. R. Acad. Bulgare Sci.*, **34** (1981), 1229–1232.

[9] K. G. Ivanov, A characterization of weighted Peetre K-functionals, *J. Approx. Theory* **56** (1989), 185–211.

[10] K. G. Ivanov, A characterization theorem for the K-functional for Kantorovich and Durrmeyer operators, *in* Approximation Theory: A volume dedicated to Borislav Bojanov (D. K. Dimitrov, G. Nikolov and R. Uluchev, Eds.), pp. 112–125, Marin Drinov Academic Publishing House, Sofia, 2004.

[11] H.-B. Knoop and X.-l. Zhou, The lower estimate for linear positive operators, II, *Results in Math.* **25** (1994), 315–330.

[12] N. X. Ky, A method for characterization of weighted K-functionals, *Annales Univ. Sci. Budapest* **38** (1995), 147–152.

[13] G. Mastroianni and P. Vértesi, Approximation by some operators using certain new moduli of continuity, *Suppl. Rend. Math. Palermo*, ser. II, **33** (1993), 387–397.

[14] V. Totik, Approximation by Bernstein polynomials, *Amer. J. Math.* **116** (1994), 995–1018.

B. R. Draganov

Department of Mathematics and Informatics
The Sofia University
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
E-mail: bdraganov@fmi.uni-sofia.bg

K. G. Ivanov

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
BULGARIA
E-mail: kamen@math.bas.bg